X-640-71-396
PREPRINT

NASA TM X- 65710

AVERAGING METHOD FOR THE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC NONHARMONIC SOLUTIONS

FERDINAND F. CAP

SEPTEMBER 1971

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

X-640-71-396

AVERAGING METHOD FOR THE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC NONHARMONIC SOLUTIONS

Ferdinand F. Cap
Theoretical Studies Branch

September 1971

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

AVERAGING METHOD FOR THE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC NONHARMONIC SOLUTIONS

Ferdinand F. Cap*
Theoretical Studies Branch
Goddard Space Flight Center
Greenbelt, Maryland 20771

ABSTRACT

While Krylov and Bogolyubov used harmonic functions in their averaging method for the approximate solution of weak nonlinear differential equations with oscillatory solution, we apply a similar averaging technique using Jacobi elliptic functions. These functions are also periodic and are exact solutions of strong nonlinear differential equations. The method is used to solve nonlinear differential equations with linear and nonlinear small dissipative terms and/or with time dependent parameters. It is also shown that quite general dissipative terms can be transformed into time-dependent parameters.

As a special example, the Langevin (collisional) equation of motion of electrons in a neutralizing ion background under the influence of a time and space-dependent electric field is presented. The method may also be used for nonlinear control theory, dynamic and parametric stabilization of nonlinear oscillations in plasma physics, etc.

*On leave of absence Institute for Theoretical Physics, University of Innsbruck, Innsbruck, Austria.

AVERAGING METHOD FOR THE SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC NONHARMONIC SOLUTIONS

1. INTRODUCTION

In these days, nonlinear differential equations are solved mainly numerically by using digital computers. This is an efficient and well-paved way to obtain a particular solution. However, if the system of differential equations is the Lagrange subsidiary system of characteristic equations for a first-order partial differential equation such as the Vlasov equation or another more sophisticated kinetic equation (in which the author was originally interested in connection with collisional Landau damping and the saturation of the two-stream instability) or if the influence of the parameters of the equation and/or of the initial or boundary conditions on the solution are of interest, the computer solution can be very expensive or even impossible if the computing time exceeds reasonable time intervals.

Actually, some authors have recently proposed analytic methods for the solution of nonlinear, but nearly linear, differential equations. Krylov and Bogolyubov [1], [2] investigated equations of the type

$$\ddot{x} + \omega^2 x = \epsilon F(x, \dot{x}) \tag{1}$$

where ϵ is a small parameter. The methods starts from the so-called generating solution

$$x = A \sin(\omega t + \varphi), \qquad \dot{x} = A \omega \cos(\omega t + \varphi) \tag{2}$$

which satisfies (1) exactly in zero order ($\epsilon = 0$).

In order to solve (1), it is assumed that the constants of integration A and φ depend on time, so that in (2) $A \rightarrow A(t)$, $\varphi \rightarrow \varphi(t)$. Expressing $F(x, \dot{x})$ into a Fourier series in φ and assuming that the parameter ϵ is small, so that amplitude A and phase φ change very slowly during one period of the oscillation; i.e., that

$$\dot{A}/A \ll \omega, \qquad \dot{\varphi}/\varphi \ll \omega \tag{3}$$

one obtains in first order of ϵ by averaging over one period

$$\left\langle\frac{dA}{dt}\right\rangle = -\frac{\epsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi} F(A\sin\varphi,\ A\omega\cos\varphi)\cos\varphi\, d\varphi \tag{4}$$

$$\left\langle\frac{d\varphi}{dt}\right\rangle = \frac{\epsilon}{A\omega}\frac{1}{2\pi}\int_0^{2\pi} F(A\sin\varphi,\ A\omega\cos\varphi)\sin\varphi\, d\varphi \tag{5}$$

where A and φ are assumed to be time independent under the integrals. Also higher order solutions can be obtained [22].

This method which is, however, restricted to equations of the type (1) (i.e., to nearly linear equations) has been used extensively in plasma physics, theory of oscillations, control theory, etc., [3] to [7]. The method was also used to solve partial differential equations, [3] and [4].

Kruskal [8] extended the Krylov-Bogolyubov method to solve equations of the type

$$\dot{\vec{x}} = F(\vec{x}, \epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\left[\frac{d^n F(\vec{x}, \epsilon)}{d\epsilon^n}\right]_{\epsilon=0} \tag{6}$$

or

$$\ddot{x} = F(x, \dot{x}, \epsilon)$$

The solutions of these fully nonlinear equations are based on recurrent solutions and are given in the form of power series of the smallness parameter ϵ.

In this paper, we are investigating an averaging method for the solution of equations of the type

$$\ddot{x} + \omega^2 f(x) = \epsilon F(x, \dot{x}) \tag{7}$$

using elliptic functions. We are starting with the exact solution of the fully nonlinear equation (unperturbed equation)

$$\ddot{x} + \omega^2 f(x) = 0 \tag{8}$$

without developing f (x) into a power series with respect to the smallness parameter ϵ. Actually, one could bring (7) into the form (8) and apply Kruskal's method. This paper may therefore be considered as a special example of Kruskal's theory, using not a transformation technique but a direct Krylov-Bogolyubov type technique, averaging over one period of elliptic functions and giving concrete examples. Furthermore, a generalization to time dependent parameters is given.

2. ELLIPTIC FUNCTIONS AS PERIODIC NONHARMONIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

The Krylov-Bogolyubov technique is based on harmonic solutions. If, e.g., one considers Duffing's equation

$$\ddot{x} + \omega^2 x = \epsilon x^3 \tag{9}$$

or Einstein's equation for the perihelion shift

$$\ddot{x} + \omega^2 x = \epsilon x^2 + a \tag{10}$$

then the Krylov-Bogolyubov technique starts from (2) and is valid only for $\epsilon << 1$. One obtains from (4) the result A = const and the frequency modification ("amplitude dispersion") is contained in (5). It is, however, possible to solve (9) exactly and any other equation of type (8) for any ϵ. By multiplying (8) by x and integrating twice, one obtains

$$t - t_0 = \int_0^x \frac{dx}{\sqrt{2\int_0^x f(x)\,dx - 2E}} \tag{11}$$

where t_0 and E are integration constants. If f (x) is a polynomial of degree up to three or a simple harmonic function like $\sin \ell x$, then (11) is an elliptic integral and its inverse function may be expressed by a Jacobi elliptic function [9] to [12].

We are now going to consider the Langevin equation of motion of electrons in a periodic space-dependent electric field [13]. This equation is of importance in plasma physics, in kinetic theory (as Lagrange characteristic equation of collisional kinetic equations), etc., but also as a form of the Froude equation for rolling ships or the damped pendulum. It reads (ω, 1, ϵ are given constants)

$$\ddot{x} + \omega^2 \cdot \sin \ell x = \epsilon F(x, \dot{x}) \tag{12}$$

where ϵ is a collision frequency (e.g., electrons in a neutralizing ion background). The solution for $\epsilon = 0$ (unperturbed equation, generating solution in the sense of Krylov-Bogolyubov) is

$$x(t) = \frac{2}{\ell} \arcsin \left[k \, \text{sn}(\sqrt{\ell}\psi, k) \right] \tag{13}$$

where sn = the Jacobi elliptic sine function, $\psi = \omega t + \varphi$ and k, the modulus of the elliptic function (amplitude of the oscillation), and φ are integration constants.

3. AVERAGING METHOD WITH ELLIPTIC FUNCTIONS

In order to solve (12), we now replace according to Krylov-Bogolyubov [1]

$$k \to k(t), \quad \varphi \to \varphi(t) \tag{14}$$

and set up as generating solution (13) and

$$\dot{x} = \frac{2k\omega}{\sqrt{\ell}} \, \text{cn}(\sqrt{\ell}\,\psi, k) \tag{15}$$

which may be obtained by deriving (13) with k and φ kept constant and then applying (14). Also the relations

$$1 - k^2 \, \text{sn}^2 = \text{dn}^2, \quad \frac{\partial \, \text{sn}}{\partial \psi} = \sqrt{\ell} \, \text{cn dn} \tag{16}$$

were used [9]. Differentiating now (13) and observing (14), then equating it to (15) gives

$$\dot{\varphi} = - \frac{\dot{k}}{k} \, \frac{\text{sn} + k \dfrac{d\,\text{sn}}{dk}}{\sqrt{\ell} \, \text{cn dn}} \tag{17}$$

Since $\varphi = \varphi(k)$ for Jacobi functions [9], we have to use the total derivative with respect to κ when using d/dt. Equation (17) is independent of the form of $F(x, \dot{x})$.

Differentiating (15) and substituting into (12), we obtain

$$k\ \mathrm{cn} - k\sqrt{\ell}\ \mathrm{sn}\ \mathrm{dn}\cdot\dot{\varphi} + k\,\dot{k}\,\frac{d\ \mathrm{cn}}{dk} = -\frac{\epsilon\sqrt{\ell}}{2\omega}F(x,\dot{x}). \tag{18}$$

Here we used

$$\frac{\partial\ \mathrm{cn}}{\partial\psi} = \sqrt{\ell}\ \mathrm{sn}\,\mathrm{dn} \quad\text{and}\quad \sin\ell x = 2\sin\frac{\ell x}{2}\cos\frac{\ell x}{2} =$$

$$= 2k\,\mathrm{sn}\cos\arcsin[\,] = 2k\,\mathrm{sn}\cos\arccos\sqrt{1-[\,]^2} \tag{19}$$

and (16).

Solving (17) and (18) for $\dot{k}$ and $\dot{\varphi}$, we receive

$$\frac{\dot{k}}{k} = \frac{\epsilon\sqrt{\ell}}{2\omega}F(x,\dot{x})\ \mathrm{cn}(\sqrt{\ell}\,\psi,k) \tag{20}$$

Here use has been made of

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1,\quad \mathrm{sn}\frac{d\ \mathrm{sn}}{dk} = \frac{d}{dk}\frac{\mathrm{sn}^2}{2} \tag{21}$$

Furthermore, we have

$$\dot{\varphi} = -\frac{\epsilon F(x,\dot{x})}{2\omega\,\mathrm{dn}}\left(\mathrm{sn} + k\frac{d\ \mathrm{sn}}{dk}\right). \tag{22}$$

Since the Jacobi functions sn, cn, and dn are periodic with the period 4 K where the quarter period [9]

$$K(k) = \int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \tag{23}$$

is the complete elliptic integral of the first kind, we may average (20) and (21) without any Fourier series expansion. Defining $u = \sqrt{\ell}\ \varphi$ and

$$\langle \cdot\cdot \rangle = \frac{1}{4K} \int_0^{4K} \cdot\cdot \ d\varphi , \tag{23}$$

we now calculate $\langle \dot{k} \rangle$ and $\langle \dot{\varphi} \rangle$ in analogous manner to (4) and (5). We then have from (20)

$$\left\langle \frac{d \ln k}{dt} \right\rangle = \frac{\epsilon}{\omega} \cdot \frac{1}{8K} \int_0^{4K} F(\mathrm{sn}\, u, \mathrm{cn}\, u)\ \mathrm{cn}\, u\, du \tag{24}$$

where again φ and k are considered to be constant under the integral. Furthermore, from (22)

$$\left\langle \frac{d\varphi}{dt} \right\rangle = -\frac{\epsilon}{\omega} \frac{1}{8K} \int_0^{4K} F(\mathrm{sn}\, u, \mathrm{cn}\, u) \frac{\mathrm{sn}\, u + k \dfrac{d\,\mathrm{sn}}{dk}}{\mathrm{dn}\, u}\, du . \tag{25}$$

We consider now some special cases.

a. $F(x, \dot{x}) = F(x) = F(\mathrm{sn}\ u)$

This case is of no interest, since (12) is then of the form (8) and can be integrated exactly. (See (11).) From (24), we actually get $\dot{k} = 0$, since $\mathrm{cn}\ u\ du = d\ \mathrm{sn}\ u / \mathrm{dn}\ u$ or with $\mathrm{sn}\ u = y$, using (16) we have

$$\int F(y) \frac{1}{\sqrt{1 - k^2 y^2}}\, dy = \phi(m u) \Big|_0^{4K} = \phi(m\, 4K) - \phi(m\, 0) = 0 \tag{26}$$

b. $F(x, \dot{x}) = F(\dot{x}) = F(\mathrm{cn}\ u)$

We then have from (25) using $\mathrm{cn}\ u = y$ and

$$\mathrm{dn}^2 = (1 - k^2)\, k^2\, \mathrm{cn}^2, \qquad \frac{du}{\mathrm{dn}\, u} = -\frac{d\, \mathrm{cn}\, u}{\mathrm{sn}\, u\, \mathrm{dn}^2 u} \tag{27}$$

$$\left\langle \frac{d\varphi}{dt} \right\rangle = -\frac{\epsilon}{\omega}\,\frac{1}{8K}\int_0^{4K} G(y)\,dy = 0 \tag{28}$$

so that the theorem by Krylov and Bogolyubov that dissipative terms in first order do not modify the phase (frequency) is also valid here.

4. SOME SPECIAL EXAMPLES

We now present some applications of the method.

a. Linear Damping

We have

$$F(\dot{x}) = -\dot{x} = -\frac{2k\omega}{\sqrt{\ell}}\,\mathrm{cn}(u,k) \tag{29}$$

Then $<\dot{\phi}> = 0$ from (28) and (25), (24) give [9]

$$\left\langle \frac{1}{k}\,K(k)\,\frac{d\ln k}{dt} \right\rangle = -\frac{\epsilon}{4\sqrt{\ell}}\int_0^{4K} \mathrm{cn}^2 u\,du = -\frac{\epsilon}{\sqrt{\ell}}\cdot\frac{1}{k^2}\,[E(k) - (1 - k^2/K(k)] \tag{30}$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\,\mathrm{sn}^2\vartheta}\;d\vartheta \tag{31}$$

is the complete elliptic integral of the second kind. Using the identity [9]

$$kK = \frac{d}{dk}\,[E(k) - (1 - k^2)\,K(k)] \equiv H'(k) \tag{32}$$

we then have

$$-\frac{\epsilon}{\sqrt{\ell}}\,(t - t_0) = \int \frac{H'(k)}{H(k)}\,dk = \ln\,[E(k) - (1 - k^2)\,K(k)] \tag{33}$$

which gives $k = k(t)$.

b. Quadratic Damping

We have

$$F(\dot{x}) = -\dot{x}^2 = -\frac{4k^2\omega^2}{\ell} cn^2 \tag{34}$$

and from (24) we get $\dot{k} = 0$. This is understandable because, from physical considerations, the damping function should be an odd function of x; e.g., $F(\dot{x}) = |\dot{x}|\dot{x}$. See also [2] or $F = \dot{x} + \dot{x}^2$. However, variable damping is of more interest.

c. Van der Pol Damping

We generalize the Van der Pol equation

$$\ddot{x} + x - \epsilon(1 - x^2)\dot{x} = 0 \tag{35}$$

to the form (12); i.e.,

$$\ddot{x} + \omega^2 \sin \ell x = -\epsilon(x^2 - 1)\dot{x} \equiv \epsilon F \tag{36}$$

This would represent, for example, a bounded inhomogeneous plasma model. See Lashinsky [3].

From (24), we again get exactly (30) and (33).

A warning might be useful: Before applying the method described here, one has to determine if (8) has a periodic solution at all. There are cases with "big" ϵ in which (8) has a periodic solution, but (7) does not have. An example is equation [14]

$$\ddot{x} - ax + cx^3 = -\epsilon\dot{x} \qquad (36)$$

For $\epsilon = 0$, an elliptic function is the solution. For $\epsilon^2 < 8a$, we have a stable focus and a damped oscillation and the method of Section 3 can be applied. For $\epsilon^2 > 8a$, we have a stable node and no oscillation at all.

5. GENERALIZATION FOR NONELLIPTIC PERIODIC FUNCTIONS AND FOR TIME-DEPENDENT PARAMETERS

If f (x) in (7) is such that the solution of (8) is not an elliptic function (it could be a hyperelliptic or an Abelian function with more than four periods), then a similar method can be devised if, and only if, x (t) is periodic.

In some other cases, the parameter ω in (12) is time dependent; e.g., in the investigation of nonlinear Landau damping [15] and [16]. On the other hand, a dissipative equation of the type (7) in the special form x

$$\ddot{x} + \omega^2 f(x) = \epsilon\, g(\dot{x})\, h(x) \tag{37}$$

(here ω is constant) with generalized Rayleigh damping

$$g(\dot{x}) = \alpha \dot{x}^3 + \beta \dot{x}^2 + \gamma \dot{x} + \delta \tag{38}$$

and arbitrary h (x), $x = x(\xi)$, $\xi(\tau)$ can be transformed (see Appendix) into

$$\xi'' + \frac{1}{\tau^2} P(\xi) = 0 \tag{39}$$

so that even nonlinear and variable dissipation terms and therefore in special cases (7) and (12) can also be reduced to a "dissipationless" nonlinear oscillation equation with time-dependent parameter of the form

$$\ddot{x} + \omega^2(t) H(x) = 0. \tag{40}$$

It does not appear to be possible to solve this equation exactly by quadratures [23].

f (x) of (37) is now no more restricted to a polynomial of 3rd degree and ω (t) is given either by the physical problem (e.g., [15] to [16]) or from the transformation (e.g., $\omega = t^{-1}$).

If, however, $\omega = \omega(t)$ is given and dissipation is present (e.g., in the Langevin equation, see [13], [17] and Section 6), then we have instead of (7), (12) resp (40), the more general equation

$$\ddot{x} + \omega^2(t)\, f(x) = \epsilon F(x, \dot{x}) \tag{41}$$

which cannot be transformed into (40). (See Appendix.)

In order to solve (41), we first solve (8) for ω = const. Let the solution be

$$x(t) = y(\psi, k) \tag{42}$$

where

$$\psi(t) = \omega t + \varphi \tag{43}$$

and where φ and k are integration constants. Using (14), we then set up as generating solution

$$x(t) = y\big(\psi\ t, k(t)\ , k(t)\big) \tag{44}$$

$$\dot{x}(t) = \omega(t)\ \frac{dy}{d\psi} \equiv \omega(t) \cdot y_\psi \tag{45}$$

Since the argument ψ of Jacobi functions depends on k, we have to assume $\partial \varphi / \partial k \neq 0$ also in the more general case. Equating (45) with dx/dt of (44) gives $\dot{\varphi}$ analogous to (17). Differentiation of (45) and substitution into (41) gives $\dot{k}$. Solving for $\dot{\varphi}$ and $\dot{k}$ gives

$$\dot{\varphi} = \frac{\frac{dy}{dk}\ (\epsilon F - y_\psi \dot{\omega})}{\omega N} - \dot{\omega} t \tag{46}$$

$$\dot{k} = -\frac{y_\psi (\epsilon F - y_\psi \dot{\omega})}{\omega N} \tag{47}$$

where

$$N = y_{\psi\psi}\, y_k - y_\psi\, y_{\psi k} \tag{48}$$

Using the identity

$$\frac{dN}{d\psi} = y_{\psi\psi\psi} \frac{dy}{dk} - y_{\psi} \frac{dy_{\psi\psi}}{dk} \tag{49}$$

and differentiating $y_{\psi\psi} + f = 0$ with respect to ψ, respective with respect to k and subtracting, one may show that $dN/d\psi = 0$ (i.e., N is independent of ψ, so that it can be put before the integral $\int .. d\psi$.)

We now assume that the time-dependent parameter ω varies slowly with time in (41). For (40), this assumption cannot be made if $\omega(t)$ stems from dissipative terms. As a measure of this slowness, we introduce another small parameter μ and define [18] as a stretched time variable

$$\vartheta = \mu t, \qquad \dot{\omega} = \frac{d\omega}{dt} = \mu \frac{d\omega}{d\vartheta} \tag{50}$$

so that two scales defined by ϵ and μ are now involved.

Assuming

$$\psi(0) \approx \psi(T), \qquad \frac{\dot{\varphi}}{\varphi} \ll \frac{1}{T}, \qquad \frac{\dot{k}}{k} \ll \frac{1}{T}, \qquad \frac{1}{\omega}\frac{d\omega}{d\vartheta} \ll \frac{1}{\mu T} \tag{51}$$

where T is the smallest period of the periodic function (42), we now average (46) and (47) over one period assuming that φ, ψ, k, ω remain constant during T. We then have

$$\langle\dot{\varphi}\rangle = \frac{\epsilon}{\omega N T} \int_0^T \frac{dy}{dk} F(y, \dot{y}) \, d\psi, \qquad \langle\dot{\psi}\rangle = \dot{\omega} + \dot{\varphi} \tag{52}$$

$$\langle\dot{k}\rangle = - \frac{\epsilon}{\omega N T} \int_0^T F(y, \dot{y}) \, dy + \frac{\dot{\omega}}{\omega N T} \int_0^T y_{\psi}^2 \, d\psi \tag{53}$$

which are the equivalents of (25) and (24). In the derivation, the relations

$$dy = y_{\psi} \, d\psi, \qquad t = \frac{\psi - \varphi}{\omega}$$

where φ, are constant under the integral and

$$\int_0^T \frac{d\,y}{d\,k}\, d\,y = \frac{d}{d\,k} \int_0^T y\, d\,y = \frac{d}{d\,k}\, [H(y)]_0^T = 0$$

were used. For $F(y, \dot{y}) = F(y)$, the first r.h.s. term in (53) vanishes. For $F(y, \dot{y}) = F(\dot{y})$, no general conclusion could be reached.

Sometimes the integrals in (52) and (53) are difficult to obtain. It might then be useful to start with the differential equations for the function t (x), which is the inverse function of the function x (t) solving (41). Before the averaging process, it is necessary to return to the original functions.

6. THE LANGEVIN EQUATION OF MOTION

We are now applying the method described on the Langevin equation of motion of an electron. The equation reads [13]

$$\ddot{x} + \omega^2(t) \sin \ell\, x = -\epsilon\, \dot{x} \tag{54}$$

Here, ω (t) is given from either the Maxwell equations or from the energy transfer rate between electrons and the electromagnetic field, [15] to [16]. Using (13) and (29) as generating solution, we get from (52) and (53) the equations

$$<\dot{\varphi}> = \omega \tag{55}$$

$$<\dot{k}> = -\left(\epsilon + \frac{\dot{\omega}}{\omega}\right) k\, cn^2 (\sqrt{\ell}\;\psi, k) \tag{56}$$

so that

$$\omega(t)\ [E(k) - K(k)\ (1 - k^2)] = \text{const.}\ e^{-\epsilon t} \tag{58}$$

determines now k (t). With this result, Denavit's calculations on the collisionless Landau damping [6] could be extended to collisional damping [19] to [20].

ACKNOWLEDGEMENT

This work was done while the author was a Senior Postgraduate Research Associate at the Goddard Space Flight Center, National Aeronautical and Space Administration. This grant was administrated by the National Academy of Sciences, National Research Council. The support of these two bodies is gratefully acknowledged.

REFERENCES

1. K. Krylov, B. Bogolyubov, Introduction to Nonlinear Mechanics, Princeton University Press, 1947

2. N. Minorsky, Nonlinear Mechanics, J. Edwards, Ann Arbor, 1947

3. H. Lashinsky, Mathematical Models for Nonlinear Mode Interactions in Bounded Plasmas, 2nd Orsay Summer Institute on Plasma Physics, Paris 1966, in M. Feix, G. Kalman, Nonlinear Effects in Plasmas, Gordon and Breach, New York 1969

4. F. Cap, Amplitude Dispersion and Stability of Dissipative Weakly Nonlinear Waves, May 1971, Report X-640-71-179, NASA Goddard Space Flight Center, Greenbelt, Md. (will be published)

5. J. Hale, Oscillations in Nonlinear Systems, McGraw Hill, New York, 1963

6. W. Cunningham, Introduction to Nonlinear Analysis, McGraw Hill, New York, 1958

7. H. Davis, Introduction to Nonlinear Differential and Integral Equations, Dover, New York, 1962

8. M. Kruskal, Asymptotic Theory of Hamiltonian and other Systems with all Solutions Nearly Periodic, J. Math. Phys. 3 806-828 (1962)

9. P. Byrd, M. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists, Springer, Berlin, 1954

10. A. Greenhill, The Applications of Elliptic Functions, Dover, New York, 1959

11. F. Bowman, Introduction to Elliptic Functions with Applications, Dover, New York, 1961

12. E. Neville, Jacobian Elliptic Functions, Clarendon Press, Oxford, 1951

13. e.g. W. Allis, Handbuch der Physik (Encyclopedia of Physics) 21 383-444 (1956), Springer, or F. Cap, Einfuehrung in die Plasmaphysik, p 173, Vieweg, Braunschweig, 1970, see also F. Cap, Characteristics and Constants of Motion Method for Collisional Kinetic Equations, April 1971, Report X-640-71-123, NASA Goddard Space Flight Center, Greenbelt, Md., will be published)

14. H. Lashinsky, private communication August 1971

15. T. O'Neil, Collisionless Damping of Nonlinear Plasma Oscillations, Phys. Fl. 8 2255-2262 (1965)

16. V. Bailey, J. Denavit, Nonlinear Oscillations in a Collisionless Plasma, Phys. Fl. 13 451-458 (1970)

17. F. Cap, Langevin Equation of Motion for Electrons in a Nonhomogeneous Plasma (in preparation)

18. C. Tam, Amplitude Dispersion and Nonlinear Instability of Whistlers, Phys. Fl. 12 1028-1035 (1969)

19. V. Zakharov, V. Karpman, On the Nonlinear Theory of the Damping of Plasma Waves, Sov. Phys. JETP 16 351-357 (1963) (Zh. Eks. Teor. Fiz 43, 490-499 (1962))

20. J. Denavit, B. Doyle, R. Hirsch, Nonlinear and Collisional Effects on Landau Damping, Phys. Fl. 11 2241-2250 (1968)

21. E. Kamke, Differentialgleichungen. Loesungsmethoden und Loesungen. Band 1 Gewoehnliche Differentialgleichungen, Chelsea Publishing Co., New York, 1959

22. P. Musen, On the High Order Effects in the Methods of Krylov-Bogolyubov and Poincaré, J. Astronautical Sciences 12 129-134 (1965)

23. E. Milne, Bulletin Amer. Math. Soc. 28 102-104 (1922)

APPENDIX

We are presenting here the transformation of (37) into (39). The method is due to Abel and may be found in Kamke [21]. The consecutive transformations are

$$\dot{x}(t) = u(t) = u(x), \quad \ddot{x} = u'(x) \cdot u(x), \quad u' = \frac{du}{dx}, \quad u = \frac{1}{v} \tag{59}$$

which give

$$-v' + v^3 \omega^2 f(x) = \epsilon \, g\left(\frac{1}{v}\right) h(x) \, v^3 \tag{60}$$

where g (v) is given by (38). Equation (60) is an Abelian equation. Using

$$v(x) = w(x) \cdot \eta(\xi), \qquad \xi(x) = -\int w(x) \, \epsilon \, \gamma \, h(x) \, dx,$$

$$w(x) = \exp\left[-\int \epsilon \, \beta \, h(x) \, dx\right], \tag{61}$$

we obtain for $\alpha = 0$ (for $\alpha \neq 0$ see Kamke [21] for a more general transformation into another equation than (39)) the Abel equation in standard form

$$\frac{d\eta}{d\xi} = -\eta^3 P(\xi) + \eta^2 \tag{62}$$

where $P(x(\xi))$ is given by

$$P(x) = \frac{\omega^2 f(x) - \epsilon \, \delta \, h(x)}{\epsilon \, \gamma \, h(x)} \, e^{-\epsilon \beta \int h(x) dx} \tag{63}$$

$x(\xi)$ is given by (61). Using then

$$\frac{d\xi}{d\tau} \equiv \xi'(\tau) = -\frac{1}{\tau \, \eta(\xi)} \tag{64}$$

which defines τ, we obtain (39).