NASA TECHNICAL MEMORANDUM

NASA TM X-67944

NASA TM X-67944

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TECHNICAL PAPER proposed for presentation at
Fall Meeting of the Acoustical Society of America
Denver, Colorado, October 19-22, 1971

E-3748

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ABSTRACT

Propagation of waves in curved ducts and pipes belong to the class of motion which is characterized by wave patterns totally different from those known in straight ducts or in unlimited space. The curvillinear boundaries are responsible for the appearance of a continuous standing radial wave which in turn affects the transmitted tangential waves. The purpose of this paper is to solve the problem of propagation of long acoustic waves in slightly and sharply bend ducts. This problem has been only partially analyzed by various authors. In this study two acoustic systems are considered which allow determination of the basic modes of motion and describe the transition and distortion of plane waves as they propagate down the curved channel. A detailed study of the behavior of waves in junctions between straight and curved ducts is also given. Solutions and expressions for principal modes of the wave are obtained by using the linearized equation of motion solved for its characteristic values. This original approach required a novel use of Bessel functions to determine the characteristic values of the steady and the decaying fields of motion.

INTRODUCTION

The propagation in curvilinear waveguides or bent ducts is difficult to analyze, and mathematical models developed to date are complicated. Perhaps that is one of the reasons why relatively few papers are available on this subject.

The first recognition of the problem of propagation of pressure waves in a curved conduit, as distinct from the motion in a straight line, was formulated by Rayleigh (Ref. 1). In a short, brilliant expose he demonstrated that long waves in a curved pipe of infinitesimal cross section behave exactly as in a straight pipe. The curvature of the pipe is of no importance. His analysis is based on the linearized equation of motion and conclusions are not valid if the cross section of the pipe is finite.

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Subsequent research of wave motion in curvilinear ducts is almost exclusively analytical and the majority of papers treat the propagation of electromagnetic waves in curved ducts. Only a few discuss the propagation of sound waves. Interestingly enough, along with analytical formulation of the behavior of waves in bends there appeared a series of papers dealing with mathematics needed to solve the physical problem of waveguides. This parallel effort indicates that the solution of the problem of bent waveguides requires mathematical formulations and techniques not generally available. A number of papers published between 1939 and 1969 did contribute to the problem. Buchholz (Ref. 2), using separation of variables obtains a solution for propagation of electromagnetic waves in slightly bent waveguides of infinite length. He calculates an expression for a wave propagation constant and draws several general conclusions about behavior of waves in bends. Buchholz's paper is the first contribution to the problem of progressive waves. The problem of the infinitely long bend was also treated by Krasnushkin (Ref. 3), Grigor'yan (Ref. 4), and others. Krasnushkin approaches the problem by the method of separation of variables but in view of mathematical difficulties proposes a perturbation method and treats the simplified case of the slightly bent tubes. Grigor'yan solves the differential equation by expansion of the cross product of Bessel functions into a Taylor series. He tries to obtain an algorithm of sufficient generality to be applicable to all possible impedances of the waveguide walls. The method is only partially successful. Grigor'yan obtains correct general information on amplitude and distribution of the radial oscillations but his basic mode wave number does not verify the differential equation except for the Raleigh case of very narrow pipe.

Among the few treatises on propagation of sound in bends are those by Miles (Refs. 5 and 6) in which he establishes an analogy between propagation of sound and an electrical transmission line are most important. The method is then applied to a right angle joint of rectangular tubes.

The work of Miles was checked by Lippert (Refs. 7 to 9). Lippert presents an experimental study of sound wave propagation in mitered bends of various angles. The experiments were conducted over a wide range of frequencies and show that long waves in mitered bends propagate with insignificant reflections.

The present analysis considers propagation of long acoustic waves in a circular bend after Rostafinski (Ref. 10). Established will be the radial and the angular distribution of the vibrational velocities. The mathematical treatment of the problem utilizes the method of separation of variables. Solutions and expressions for principal modes of the wave are obtained by using the linearized equation of motion solved for its characteristic values.

GENERAL SOLUTION

Determination of Eigenvalues

The linearized wave equation in cylindrical coordinates is known to be separable, and since the superposition of solutions is allowed the general solution may be written

$$\phi = \int_C e^{i(\omega t+\alpha)}(a_\nu \cos \nu\theta + \sin \nu\theta)\left[A_\nu J_\nu(kr) + B_\nu Y_\nu(kr)\right]d\nu$$

$$+ e^{i(\omega t+\alpha)}(c\theta + d)\left[A_0 J_0(kr) + B_0 Y_0(kr)\right] \qquad (1)$$

where C is a set of points in the complex plane, to be determined in order to satisfy the boundary conditions.

The solution for $\nu = 0$ might have been included in the integral terms but in order to show more explicitly the linear dependence on θ of this solution, it has been written separately.

To satisfy the partial differential equation and the boundary conditions for perfectly rigid circular walls, a characteristic equation will be found whose roots will be the characteristic values of the problem: a set of values of the separation parameter ν which will yield a non-trivial solution of the problem. Differentiating equation (1) with respect to r and equating to zero we obtain for the two circular boundaries R_1 and R_2

$$\left.\begin{aligned} A_\nu J'_\nu(kR_1) + B_\nu Y'_\nu(kR_1) = 0 \\ A_\nu J'_\nu(kR_2) + B_\nu Y'_\nu(kR_2) = 0 \end{aligned}\right\} \quad \nu \neq 0 \qquad (2a)$$

$$\left.\begin{aligned} A_0 J'_0(kR_1) + B_0 Y'_0(kR_1) = 0 \\ A_0 J'_0(kR_2) + B_0 Y'_0(kR_2) = 0 \end{aligned}\right\} \quad \nu = 0 \qquad (2b)$$

In the present case where $0 < (kr) < (kR_2) << 1$, with the steep slope of $Y_1(kr)$ and relatively moderate increase of $J_1(kr)$ over the range (kR_1) to (kR_2), there is no solution in the range under consideration.

Consequently, the solution $\nu = 0$ cannot be considered. To evaluate the ν_m's we expand the J'_ν and $J'_{-\nu}$ in terms of increasing powers of the argument $(akR_1) = (kR_2)$ and (kR_1). If we limit the expansion in the first approximation to the first term we obtain

$$\frac{\nu^2}{\sin \pi\nu\Gamma(\nu + 1)(1 - \nu)} \left[(kR_1)^{-\nu-1} (kaR_1)^{\nu-1} - (kR_1)^{\nu-1} (kaR_1)^{-\nu-1} \right] = 0$$

and finally

$$\frac{\nu}{\pi(kR_1)^2} (a^{\nu-1} - a^{-\nu-1}) = 0 \qquad (3)$$

Solution $\nu = 0$ has been already rejected. Therefore, the only acceptable solution must satisfy the equation

$$a^{\nu-1} = a^{-\nu-1}$$

which may be put in the form

$$a^{2\nu} = 1 \quad \text{or} \quad e^{2\nu \ln a} = 1$$

Hence, $2\nu \ln a = 2m\pi i$; that is

$$\nu_m = \frac{m\pi i}{\ln a} \qquad (m = 1, 2, 3 \ldots) \qquad (4)$$

Better approximations will be given by the second and following terms of expansion of J'_ν and $J'_{-\nu}$. The result is

$$\nu_m = i \left\{ \frac{m\pi}{\ln a} - \frac{(kR_1)^2(a^2 - 1)\left[2 + \frac{m^2\pi^2}{(\ln a)^2}\right]}{4m\pi \left[1 + \frac{m^2\pi^2}{(\ln a)^2}\right]} \right\} \qquad (5)$$

and for $m = 0$ the transformation gives

$$\nu_o = \sqrt{\frac{2\,\frac{a^2 - 1}{\ln a}}{4(kR_1)^{-2} + a^2 + 1 + \frac{a^2 - 1}{\ln a}}} \qquad (6)$$

There is an infinite set of pure imaginary roots ν_m given by equation (5) and one single real root. The obtained eigenvalues allow us to write a complete solution in which only the integration coefficients in m are to be determined by proper boundary condition at the inlet and outlet of the duct. This determination is a relatively simple matter.

TYPICAL PHYSICAL SYSTEMS

Infinite Bend

Suppose an infinitely long circular duct (Fig. 1a) for which the closest physical example is a tightly wound coil of which the pitch is negligible compared to the radius of curvature of the duct. The far end boundary condition for an infinite duct is that no reflection of waves must be considered.

The integrated equation for the velocity potential $\phi(r,\theta,t)$ for an infinite bend may be written directly by substitution of the derived expressions for the integration constants

$$\phi = -i\frac{v_o R_1}{2\nu_o} e^{i(\omega t-\nu_o\theta)} \frac{a-1}{\ln a}\left[-2 + (kr)^2/2 + 2\nu_o^2 - 2(kR_1)^2 - (kR_1)^2 \ln\frac{r}{R_1} - \nu_o^2\left(\ln\frac{r}{R}\right)\right] + \frac{v_o R_1}{4\pi} e^{i\omega t}\sum_{m=1}^{\infty} e^{-\mu_m\theta}\frac{a\cos m \;\; - 1}{m\mu_m(1+\mu_m^2)^2}\left\{-2\mu_m\left[4(1+\mu_m^2) + (kR_1)^2 + (kr)^2\right]\cos\left(\mu_m \ln\frac{r}{R_1}\right) + 4(kR_1)^2 + 2\mu_m^2(kR_1)^2 - 2\mu_m^2(kr)^2 \sin\mu_m\left(\ln\frac{r}{R_1}\right)\right\} \quad (7)$$

and without terms small of the second order

$$\phi(r,\theta,t) = i\frac{v_o}{\nu_o}R_1\frac{a-1}{\ln a}e^{i(\omega t-\nu_o\theta)} - 2\frac{v_o R_1}{\pi}e^{i\omega t}\sum_{m=1}^{\infty}\frac{a\cos(m\pi)-1}{m(1+\mu_m^2)}e^{-\mu_m\theta}\cos\left(\mu_m \ln\frac{r}{R_1}\right)$$

The propagation of waves in a curved duct is profoundly influenced by the curvature of the conduit. The amplitude of vibrational velocities is a function of both inner radius R_1 and of the radii ratio $a = R_2/R_1$. The tangential vibrational velocities are almost exactly inversely proportional to radius, that is, velocities follow the distribution of a potential vortex. The radial vibrational velocities are approximately two orders of magnitude smaller than the tangential velocities.

The phase velocity of waves is also affected by bending of ducts. For a bend, the phase velocity is $\dot{\theta} = \omega/\nu_o$, as obtained from $(\omega t - \nu_o \theta)$, while for a straight duct it is $\dot{x} = \omega/k$, as given by $(\omega t - kx)$. To compare the two velocities we average the tangential phase velocity, $\dot{\theta} r = \dot{s}$ over the duct width and obtain $|\dot{s}| = \omega/(\nu_o)R_{mean}$. The ratio of the two velocities is

$$\frac{|\dot{s}|}{\dot{x}} = \frac{\omega}{c\,\nu_o} R_m = \frac{kR_{mean}}{\nu_o}$$

Clearly, the phase velocity in bends is always higher than in straight ducts.

The attenuated tangential vibrations which characterize change from motion of plane waves in straight duct to motion in a curved duct are examined in the next two figures. Figure 2 gives results of a sample calculation illustrating the behavior of those oscillations for a duct of radii ratio $a = 2$. The vibrations are basically of low amplitude. Even close to the piston, at $\theta = \pi/16$, they are one order of magnitude smaller than vibrations of the nondamped, propagating wave. The radial distribution of these oscillations changes significantly with wave angular position in the duct. At $\theta = \pi/4$ these oscillations are reduced to a very low level and are nearly uniform across the width of the duct. Figure 3 shows the same oscillations calculated for three different duct widths but with a single angular position of $\theta = \pi/4$. The curve for $a = 2$ was taken directly from figure 2 for comparison purposes. The two other curves indicate that the decaying oscillations are much more pronounced and extend farther when induced in wider ducts.

The radial vibrational velocities, which characterize motion of waves in curved ducts, for long waves, are of low amplitude. The permanent, standing oscillations are shown in figure 4. They are calculated for duct radii ratios of 2, 3, and 4. Generally the amplitude of these oscillations is low, approximately two orders of magnitude smaller than the tangential velocities. The radial distribution is characterized by the lack of symmetry. The maxima of curves are shifted toward bend's inner wall.

This phenomenon is even more pronounced in the case of the nonpropagating, damped, radial oscillations at the curved duct inlet. The amplitude and radial distribution of these oscillations is shown on figure 5 for three values of parameter a and $\theta = \pi/4$. For $a = 2$ the permanent and the vanishing oscillations are of approximately the same amplitude. For $a = 3$ and 4 the decaying oscillations are about twice as large as the radial oscillation of the propagating wave. Therefore, the process of decay is much slower when ducts are wide.

θ_2 Radians Bend Followed by a Straight Infinite Duct

To analyze the motion in the system bend-straight duct, it is necessary to solve

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial x} \quad \text{and} \quad \frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial y} \qquad \text{at} \begin{cases} \theta = \theta_2 \\ x = 0 \end{cases}$$

that is, the continuity conditions at the junction bend-straight tube, of a system illustrated on figure 1b.

Solving the differential equations and evaluating the constants of integration we obtain the velocity potential

$$\phi(r,\theta,t) = i\,\frac{v_o R_1}{\nu_o}\,\frac{a-1}{\ln a}\,e^{i(\omega t - \nu_o \theta)}$$

$$+ e^{i\omega t}\,\frac{v_o R_1}{\pi} \sum_{m=1}^{\infty} \frac{a\cos(m\pi) - 1}{m(1+\mu_m^2)} \cos\left(\mu_m \ln \frac{r}{R_1}\right)$$

$$\times \left[-2c^{-\mu_m\theta} + \beta_m\, e^{-\nu_o\theta_2}\, \frac{\left(e^{\mu_m\theta} + e^{-\mu_m\theta}\right)}{e^{\mu_m\theta_2}} \right] \qquad (8)$$

where the term $e^{-i\nu_o\theta_2}$ results from evaluation of a set of integration constants at $x = 0$ which is at $\theta = \theta_2$ and not at $\theta = 0$ and could be avoided should the coordinate θ of the bend be counted clockwise from the junction bend-straight duct. β_m is a set of coefficients resulting from matching of vibrational velocities at the junction. β_m are smaller than one.

The derived solution is very general. It applies to radii ratios of the bend in the range from 1 to 10; it has no limitations as to the length of the bend which may range from a small fraction of π to an infinite coil. This wide range of applicability was obtained by retaining in the solution, terms which are two orders of magnitude smaller than the principal terms. The first two terms of equation (8) are those of equation (7) of the wave motion in an infinite bend. The first term is a simple wave, the second the attenuated vibrations near the inlet section of the bend. The third term represents attenuated vibrations in the negative direction of θ, an adjustment of wave motion to discontinuity at the junction bend-straight duct. This term is identically zero at $\theta = 0$ and increases to a maximum at $\theta = \theta_2$.

The velocity potential in the straight, infinite duct is approximately

$$\phi(y,x,t) = e^{i(\omega t - v_o \theta_2)} \; i \frac{v_o}{k} e^{-ikx}$$

$$+ \frac{2v_o R_1}{\pi^2} \sum_{n=1}^{\infty} \frac{\zeta_n}{n} \cos\left[\zeta_n (y + L)\right] e^{-x\sqrt{\zeta_n^2 + k^2}} \sum_{m=1}^{\infty} \Lambda_{mn} \frac{a \cos(m\pi) - 1}{(1 + \mu_m^2)} \beta_m$$

where Λ_{mn} is a set of integrals in m and n.

The calculation of the vibrational velocities in the bend and in the straight duct that follows the 90 degree bend is straightforward. Results are reported on figure 6.

In the straight duct the waves are straightening out relatively fast. By $x/2L = 1$ there is already an almost straight wave similar to that which was generated at the bend inlet. Essentially, the process of straightening out of the wave is confined to the straight duct.

The verification of the simultaneous solution of the equations of motion for the bend and for the straight duct is shown on figure 7. The calculated tangential vibrational velocities for $r = R_1$ and $r = R_2$ for positions starting at the vibrating piston, through the bend and in the straight duct are taken from figures 11 and 12. The rapid changes in velocities by the bend exit are well illustrated. In spite of this, the values calculated for the bend match very well the values calculated for the straight duct.

CONCLUDING REMARKS

Propagation of long waves in a two-dimensional system has been analyzed. The acoustic approximation has been used and the Helmholz equation was integrated for two sets of boundary conditions. The two physical systems taken into consideration are: an infinite bend approximating a coil and a 90° bend followed by a straight, infinite duct approximating a typical industrial piping system.

The results of the analysis indicate that bending of a straight duct profoundly modifies the propagation of waves in that duct. The bend engenders the following phenomena:

1. a set of attenuated axial and radial waves which modify the plane wave generaged at the duct's inlet
2. a continuous radial, standing wave whose radial vibrations are sustained by the curvature of the bend. In a straight duct these vibrations would be quickly attenuated
3. a vortex-type distribution of the tangential vibrational velocities.

The presented analysis is not directly applicable to real flows because it is based on a linearized equation valid for acoustical waves in stationary medium only. It gives, however, an idealized picture of wave's behavior in bends, in general.

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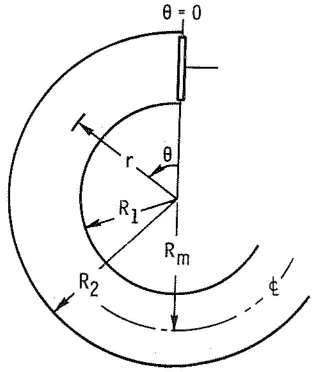
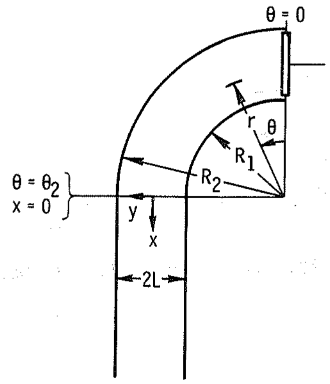


(a) Infinite bend.



(b) 90° Bend followed by an infinite straight duct.

Figure 1. - The two physical systems considered.

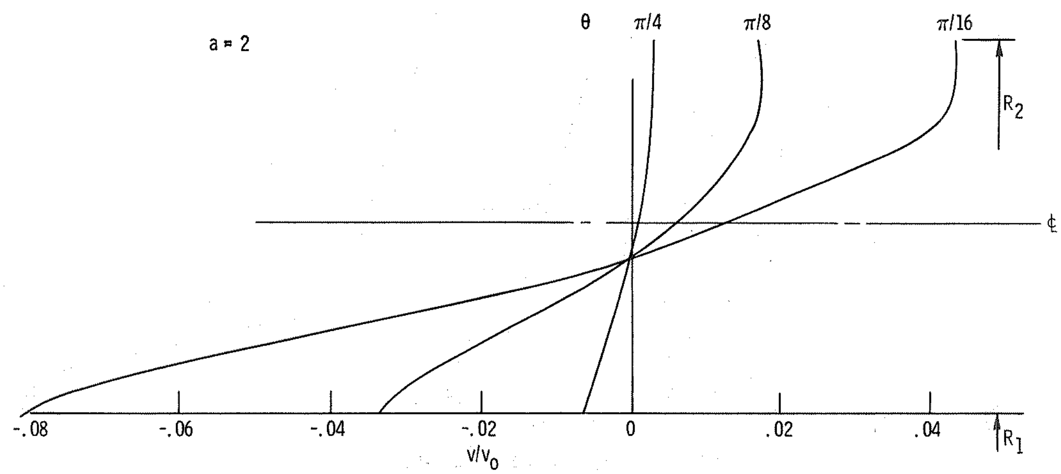


Figure 2. - Attenuated tangential vibrational velocities for three angular positions in a bend.

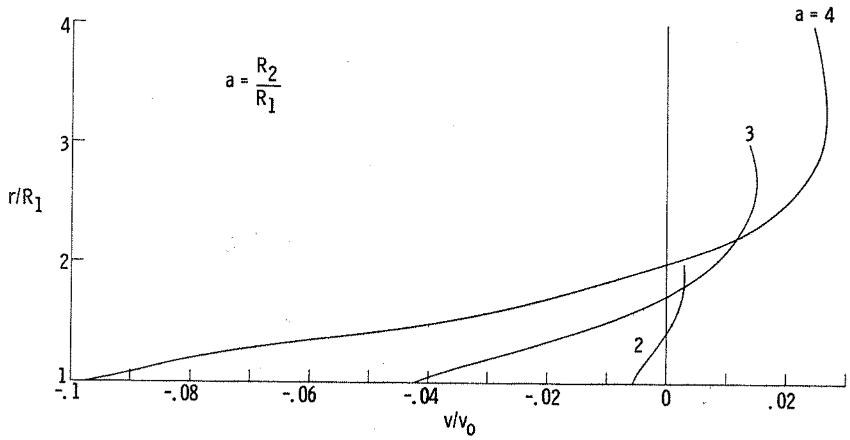


Figure 3. - Attenuated tangential vibrational velocities for three bends at $\theta = \pi/4$.

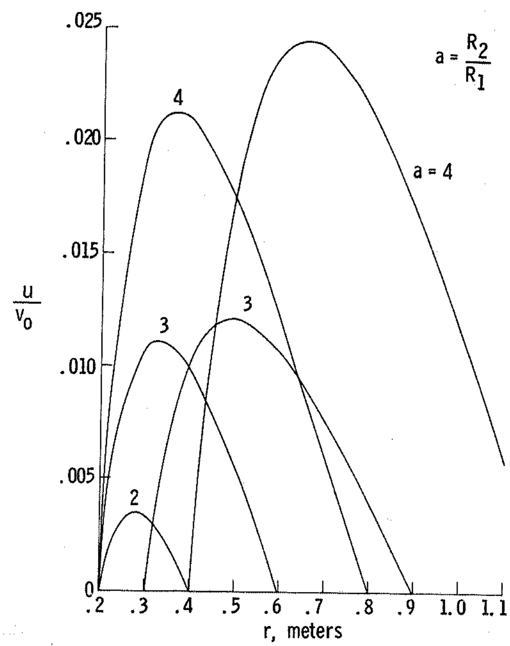


Figure 4. - Standing radial vibrational velocities in bends for three R_1 and three values of a.

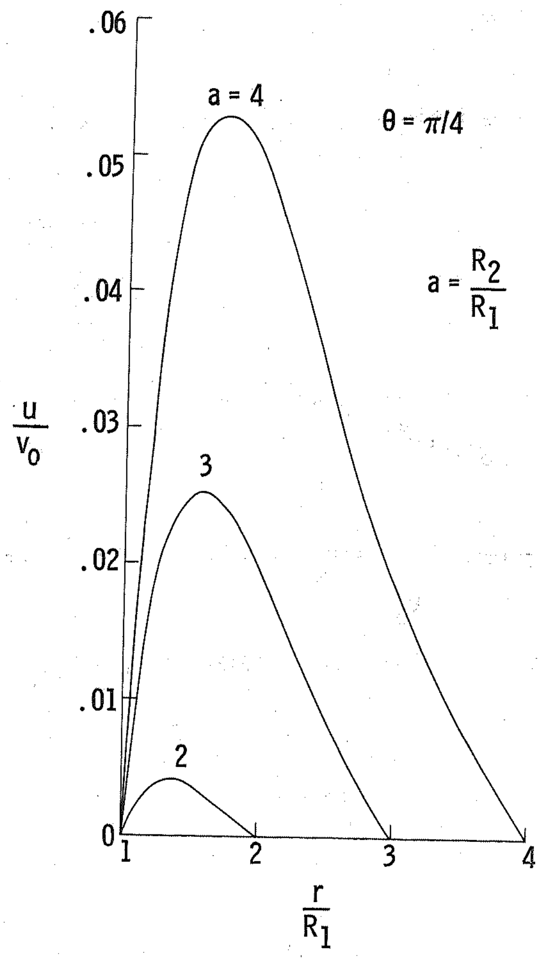
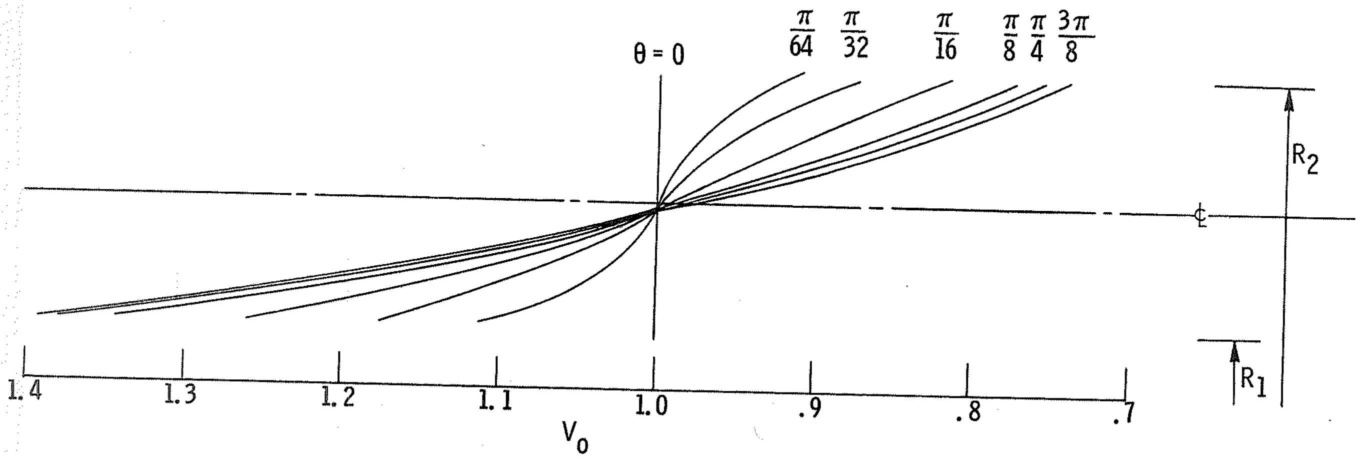
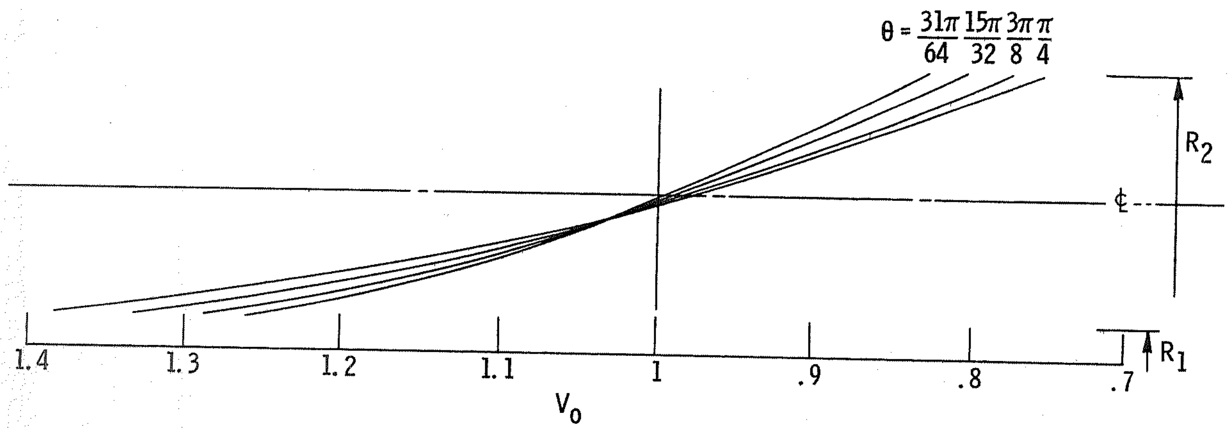


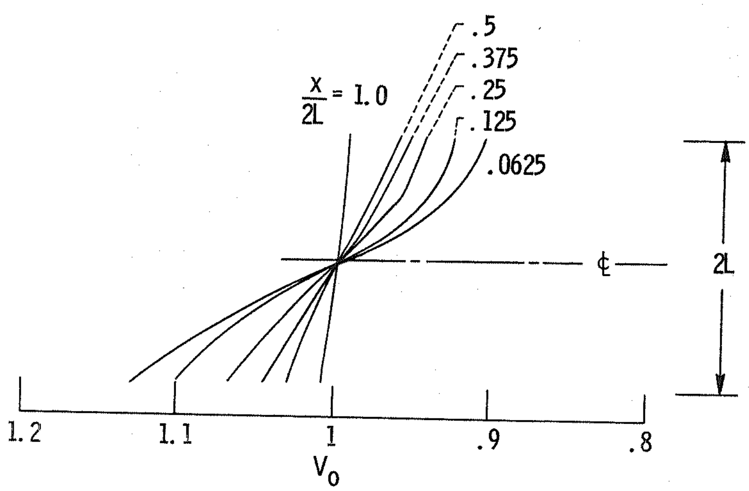
Figure 5. - Attenuated radial vibrational velocities at bend's inlet.



(a) Tangential vibrational velocity distribution near inlet of an infinite coil.



(b) Distribution of tangential vibrational velocities at bend outlet.



(c) Straight duct inlet. Distribution of axial vibrational velocities.

Figure 6. - 90° Bend followed by a straight duct. Distribution of vibrational velocities.

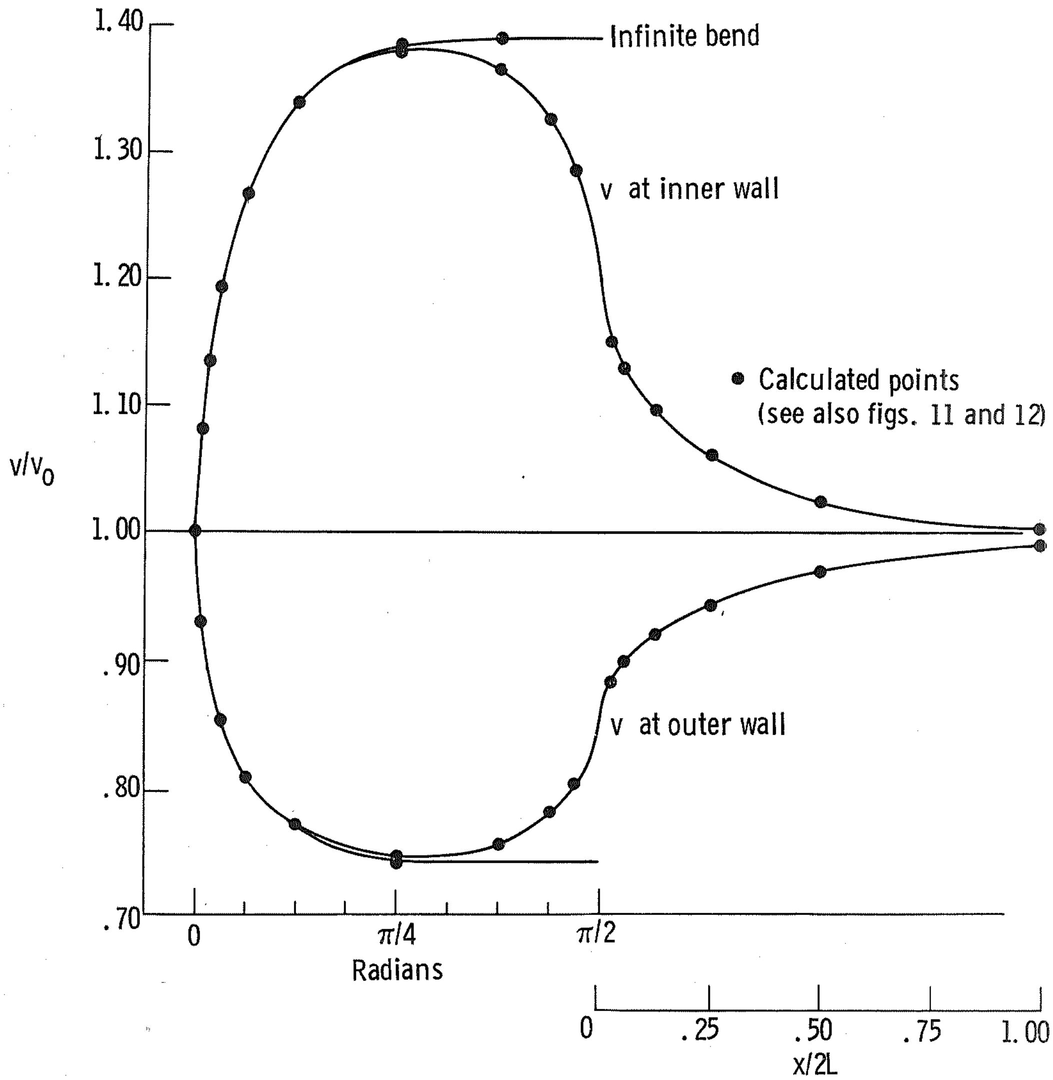


Figure 7. - Propagation in bend-straight duct system. Tangential vibrational velocities at bend's curved walls, at $r = R_1$ and $r = R_2$.