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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

*Technical Memorandum 33-496*

*A General Algorithm for the Solution  
of Kepler's Equation for Elliptic Orbits*

*Edward W. Ng*



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**JET PROPULSION LABORATORY  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
PASADENA, CALIFORNIA**

September 1, 1971

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## PREFACE

The work described in this report was performed by the Data Systems Division of the Jet Propulsion Laboratory.

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## ABSTRACT

An efficient algorithm and subroutine is presented for the solution of Kepler's equation  $f(E) = E - M - e \sin E = 0$ , where  $e$  is the eccentricity,  $M$  the mean anomaly and  $E$  the eccentric anomaly. This algorithm is based on simple initial approximations that are cubics of  $M$ , and an iterative scheme that is a slight generalization of the Newton-Raphson method. An extensive testing involving 20,000 pairs of values of  $e$  and  $M$  show that for single precision ( $\sim 10^{-8}$ ) 42.0% of the cases require one iteration, 57.8% two, and 0.2% three. Both single and double precision FORTRAN V subroutine listings are provided for the UNIVAC 1108 computer, the D. P. one requiring one additional iteration.

A GENERAL ALGORITHM FOR THE SOLUTION OF  
KEPLER'S EQUATION FOR ELLIPTIC ORBITS

Edward W. Ng

Problem Defined

The classical Kepler's equation for elliptic motions is well-known in astronomical applications. Stated simply, given a mean anomaly  $M$  in radians, and an eccentricity  $0 \leq e < 1$ , it is desired to find an eccentric anomaly  $E$  which satisfies the equation,

$$f(E) \equiv E - M - e \sin E = 0 \quad (1)$$

For a description of origin and usage of this equation, the reader is referred to Brouwer and Clemence (1961) or Danby (1962).

Stated above  $M$  may be given any real value, but the nature of equation (1) allows us to readily reduce the problem to an equivalent one for a given  $M$  restricted in  $[0, \pi]$ . First, the terms  $E$ ,  $M$  and  $\sin E$  will be positive or negative together. Therefore for a given  $M < 0$ , one may solve equation (1) for  $|M|$  and take the negative of the result as the right answer. For positive  $M > \pi$  we can define  $M = 2n\pi + \hat{M}$  and  $E = 2n\pi + \hat{E}$ , where  $\hat{M}$  and  $\hat{E}$  have values in  $[-\pi, \pi]$ . Equation (1) can then be solved for  $\hat{E}$ . With these preliminaries, we are ready to address to our main problem of solving Equation (1) for  $0 \leq M \leq \pi$ .

## Method of Solution

### 1. Iterative Scheme

Various methods have been suggested for the solution of non-linear equations of the form  $f(E) = 0$ , where  $f$  is given and it is desired to find  $E$ . Probably the most well-known is the Newton-Raphson iteration method, (cf. Traub, 1964) which possesses a local quadratic convergence property. To our knowledge most people today use this method for the solution of Kepler's equation. A well-known higher order iterative scheme, one with cubic convergence property, may be written as follows: Given  $f(E) = 0$  and an initial estimate  $E_0$ ,

$$E_{k+1} = E_k - \frac{f}{f'} \left[ 1 + \frac{ff''}{2f'^2} \right] \quad k=0,1,2,\dots \quad (2)$$

where  $f$ ,  $f'$  and  $f''$  are the function  $f(E)$  and its first two derivatives evaluated at  $E = E_k$  (cf. Traub, 1964, P. 81, Schroder's formula). To our knowledge R. Broucke first applied this scheme to the solution of Kepler's equation. If the second term in the bracket is neglected, eq. (2) becomes the Newton-Raphson scheme. Whereas the third order scheme possesses a stronger local convergence property, compared to Newton-Raphson it does so at the cost of (i) computing the term  $\frac{1}{2}ff''/f'^2$  per iteration and (ii) more sensitive dependence on the initial estimate  $E_0$ . For our particular problem on hand, since  $e \sin E = f''(E)$  is needed in the computation of  $f(E)$  anyway, the computation of the extra term costs only three multiplications and one division - a very minor additional cost. Thus the important thing is to insure that fairly good initial estimates are provided.

### 2. Initial Estimates

Rearrange eq. (1) as follows:

$$y \equiv \frac{E-M}{e} = \sin(ey+M) \quad (3)$$



and consider the function and derivative values  $y(M,e)$  and  $y'(M,e) = \frac{d}{dM}y(M,e)$  on the  $M$ - $e$  plane; we have,

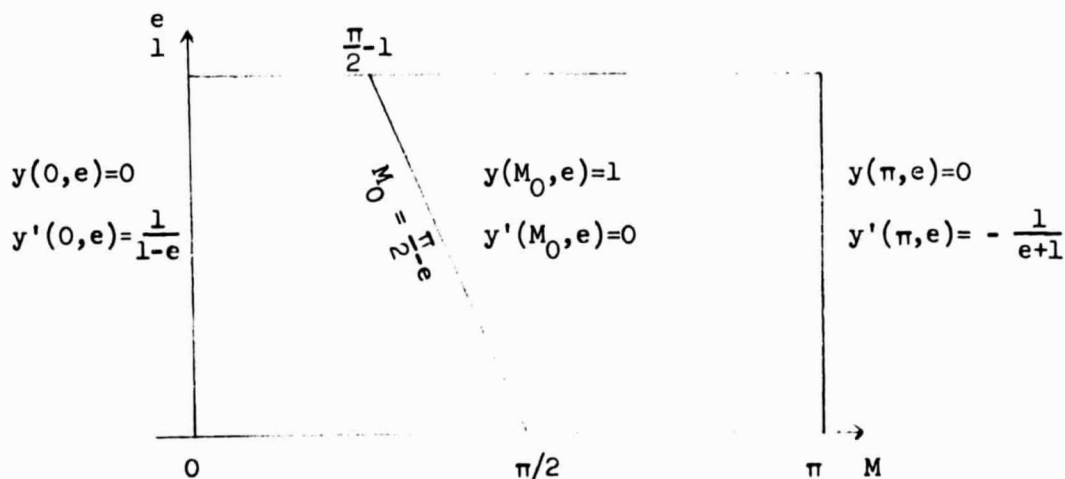


FIGURE 1

Function and Derivative Values of  $y(M,e)$  at  
Three Critical Lines on the  $M$ - $e$  Plane

The value  $M_0 = \frac{\pi}{2} - e$  has a special significance: the function  $y(M,e)$  increases monotonically from  $M = 0$  to a maximum at  $M_0$  and then decreases monotonically from  $M = M_0$  to  $M = \pi$ .

In many practical applications, the initial estimate is taken to be  $E_0 = M$ , or  $y_0 = 0$ . Whereas this approximation is inexpensive to compute, it is inaccurate for numerous values of  $e$  and  $M$ , thereby requiring many iterations for those cases. In particular, this initial estimate will actually lead to divergence [for large eccentricity, say,  $0.93 < e \leq 1$ ,] of either Newton-Raphson or other iterative schemes. Another proposed initial estimate is to use the first few terms of a Fourier series expansion of  $E$  in terms of  $M$  (cf. Brouwer and Clemence, P. 76, eqs. 70 and 71). Rearrangement of terms up to  $e^3$  in eq. 71 yields

$$y_0 \cong \frac{\sin M}{1 - \cos M} \left[ 1 - \frac{e^2}{2} \frac{\sin M}{(1 - \cos M)} \right] \quad (4)$$

Whereas this estimate is good for small eccentricity, it also leads to divergence for large  $e$ , and it further costs as much as one iteration as defined in eq. (2).

An attractive moderate alternative to the above two would be to find simple quadratic or cubic functions of  $M$  to match both function and derivative values at  $M$  equal to 0,  $M_0$  and  $\pi$ . The following two cubics match the conditions shown on Figure 1:

$$A(M,e) = 1 - \left(1 - \frac{M}{M_0}\right)^2 \left[1 + \frac{2M}{M_0} - \frac{M}{1-e}\right] \quad 0 \leq M \leq M_0 \quad (5)$$

$$B(M,e) = 1 - \left(1 - \frac{\pi-M}{\pi-M_0}\right)^2 \left[1 + \frac{2(\pi-M)}{\pi-M_0} - \frac{(\pi-M)}{1+e}\right] \quad M_0 \leq M \leq \pi \quad (6)$$

The cubic (6) is a fairly good estimate for all  $e$ , with the largest absolute error of the order of  $10^{-2}$ ; but the cubic (5) becomes very poor as  $e \rightarrow 1$ . This poor estimate stems from the matching of large derivative at  $M = 0$ , thereby causing an "overshoot". One way to remedy this difficulty is to relax the requirement of matching  $y'(0,e)$  and look for higher degree polynomials that are monotonic in  $[0, M_0]$  - a property intrinsic to  $y(M,e)$ . Keeping in mind that we are only interested in very simple initial estimates, we propose to investigate the following

$$C_k(M,e) = 1 - (-1)^k Q^k, \quad Q = \left(1 - \frac{M}{M_0}\right), \quad 0 \leq M \leq M_0, \quad k=2,3,\dots \quad (7)$$

The sequence  $\{C_k(M,e)\}$  yields the correct values of  $y(0,e)$ ,  $y(M_0,e)$  and  $y'(M_0,e)$  at  $M=0, M_0$  but for each  $k$   $C'_k(0,e)$  matches  $y'(0,e)$  at only one value of  $e$ , as shown in the table below:

$$C'_k(0,e) = y'(0,e) = (1-e)^{-1}$$

k	e
2	0.43
3	0.72
4	0.81
5	0.86
6	0.89
7	0.91
8	0.92

This suggests that we should choose different values of  $k$  for different ranges of  $e$ . Moreover, from the viewpoint of computational efficiency, it is best to increase  $k$  geometrically for increasing  $e$ . Since the quantity  $Q^2$  is needed in  $A(M,e)$ , it leads us to choose  $k=4$ , for  $e > 0.7$  and  $k=8$  for  $e > 0.85$ , respectively.

We could have used a higher degree  $C_k$  as  $e \rightarrow 1$ , say,  $C_{16}$  for  $e > 0.99$ ; but we expect it to fail for some very small region in the left upper corner of Figure 1. The reason can be seen as follows. As  $M \rightarrow 0$  and therefore  $E \rightarrow 0$ , eq. (1) can be written as

$$E - M - e(E - \frac{E^3}{6}) \approx 0 \quad (8)$$

There is some combination of  $e$  and  $M$  such that the  $E^3$  term in eq. (8) dominates so that  $E$  behaves like the cube root of  $M$ , which has very different functional properties than a polynomial. Therefore it is best for such cases to find an estimate that contains the cube root behavior. The most obvious choice would be a real solution of eq. (8): (AMS 55, P. 17)

$$D(M, e) = \left[ r + (q^3 + r^2)^{\frac{1}{2}} \right]^{\frac{1}{3}} + \left[ r - (q^3 + r^2)^{\frac{1}{2}} \right]^{\frac{1}{3}}, \quad (9)$$

$$\text{where, } r = \frac{3M}{e}, \quad q = \frac{2}{e}(1-e).$$

This initial estimate costs about as much as one iteration, but will certainly save more than one iteration in the small region in question. Therefore it is well worth the effort. Based on some empirical results we have chosen this small region to be  $0.96 \leq e \leq 1$ ,  $0 \leq M \leq 0.02$ .

In summary, we first reduce the problem to the range  $0 \leq e \leq 1$ ,  $0 \leq M \leq \pi$  and then use as initial estimates the 4 approximations in various regions as indicated diagrammatically in Figure 2.

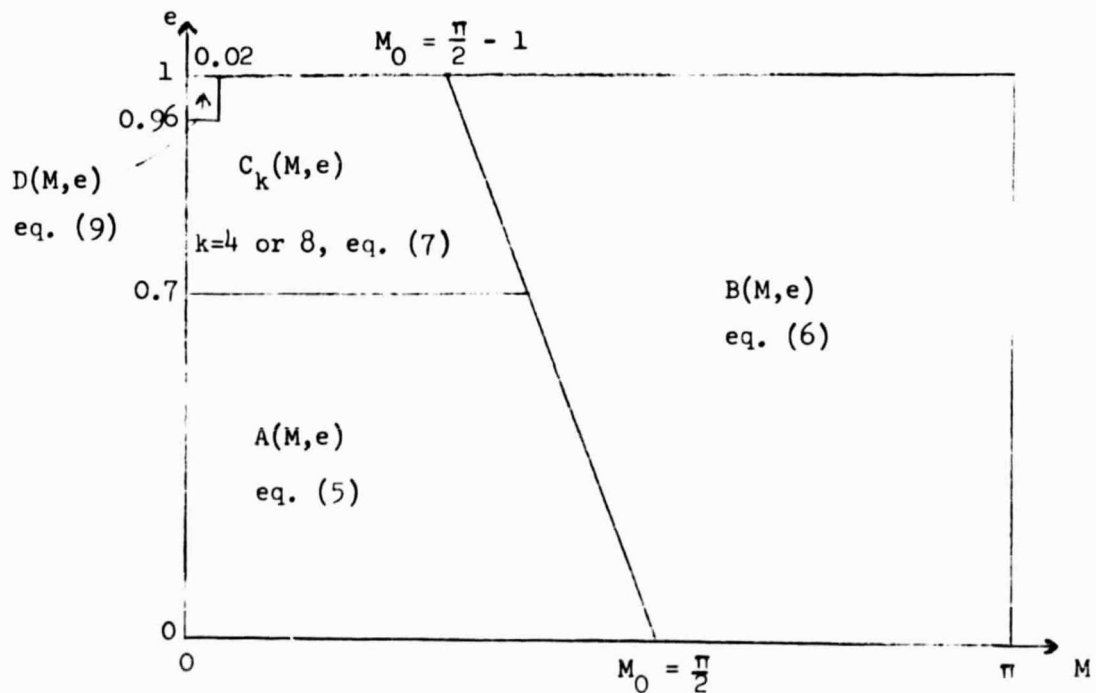


FIGURE 2

Initial Approximations for Different Regions

These approximations are all fairly good in the sense that with them usually only one or two iterations are required to achieve an accuracy of  $10^{-8}$ .

### Subroutine Described

FORTTRAN V subroutines have been written for single and double precisions for the UNIVAC 1108 computer. These programs take M and e as input and return E together with the number of iterations. In the subroutine are used standard FORTRAN features except for a cube root function which may not be part of other FORTRAN compilers. In such case one should use the operation  $**(1./3.)$  to replace the cube root function. The double precision subroutine calls the single precision one and does one iteration. Certain computational details are described as follows.

#### 1. Summary of Computing Procedure

- (a) Take input e and M; ascertain that  $0 \leq e \leq 1$ ;
- (b) Reduce the problem to  $\hat{M}$  in  $[0, \pi]$  as described on p. 1;
- (c) Choose appropriate initial approximation;
- (d) Go to iterative scheme.
- (e) When convergence is achieved add on the proper multiple of  $\pi$  and/or sign to bring E into the same revolution as M.

It should be noted that step (c) is not as complicated as it appears at first glance. For  $\hat{M} > M_0$ , only one logical test is required. For  $\hat{M} < M_0$  and  $e \leq 0.7$ , two are required. Even in the worst case only four tests are required. Such overhead is negligible compared to the computation of several to ten additional iterations with poor initial estimate.

#### 2. Testing of Subroutine

- (a) A preliminary testing is performed for selected values of eccentricity  $e = 0.1, 0.2, \dots, 0.9, 0.93, 0.96, 0.97, 0.98, 0.99, 0.995, 0.999, 1.0$ . For each e we take the mean anomalies  $M=0, 0.001, 0.01, 0.1, 0.2, 0.8, 1.4, 2.0, 2.6, 3.14, 3.14159265$ . In these preliminary tests we perform the iterations in double precision, ( $\sim 10^{-18}$ ) focussing our attention to the truncation error as a function of the difference between two iterates  $|E_{k+1} - E_k|$  in eq. (2). O. K. Smith (1961)

showed that for the Newton-Raphson iteration, a requirement  $|E_{k+1} - E_k| < 10^{-5}$  will guarantee that the truncation error is less than  $10^{-8}$  for any  $e \leq 0.99$ . Eq. (2), having a stronger convergence property, will certainly possess a more liberal requirement. Moreover, theoretical bounds tend to be too conservative. Empirical investigations of values of  $e$  and  $M$  mentioned above show that the requirement  $|E_{k+1} - E_k| < 2.5 \times 10^{-3}$  is sufficient to obtain a solution with truncation error less than  $10^{-8}$ .

- (b) An intensive testing is performed for 1000 subintervals in  $0 \leq M \leq \pi$  and 20 in  $0 \leq e \leq 1$ . A pseudorandom pair of  $(e, M)$  is computed for each of the 20,000 little rectangles.

For each such pair the input  $M$  is compared with a computed  $M_c = E_c - e \sin E_c$  from the single precision subroutine, and the difference  $\Delta M = M_c - M$  gives an indication of the error  $\Delta E$  in  $E$ , viz.,

$$\begin{aligned} M + \Delta M &= E + \Delta E - e \sin(E + \Delta E) \\ &= E + \Delta E - e \sin E \cos \Delta E - e \cos E \sin \Delta E \\ &\approx (E - e \sin E) + \Delta E - (e \cos E) \Delta E \end{aligned}$$

$$\text{or maximum } \Delta M = \Delta E(1 + e \cos E) \leq 2\Delta E.$$

The maximum and root-mean-square absolute "errors" for  $\Delta M$  are given in Table 1. The number of iterations required should serve as a very accurate indicator of the efficiency of the subroutine, since, as mentioned before, the cost of initial estimate is substantially smaller than that of an iteration (except for the region  $0 \leq M \leq 0.02$  and  $0.96 \leq e \leq 1$ ). For the 20,000 cases reported, 42.0% take one iteration, 57.8% take two and 0.2% take three. The detailed breakdown is also recorded in Table 1.

- (c) Tests of the reduction of range of  $M$  to  $[0, \pi]$  are conducted for the value of  $e$  and  $\pm 100$  times the values of  $M$  as defined in (a). Again the magnitudes of  $\Delta M$  are studied to ascertain proper performance of the subroutine.

TABLE 1

## PERFORMANCE STATISTICS OF SUBROUTINE KEPSP

Interval of e	Max AM	RMS AM	Percent of Cases With n Iterations (for e from 0 up to largest value of specified interval)		
	$\times 10^{-8}$	$\times 10^{-8}$	n=1	n=2	n=3
(0, .05)	3.0	0.2	100	0	0
(.05, .1)	4.5	0.6	100	0	0
(.1, .15)	6.0	0.8	98.1	1.9	0
(.15, .20)	6.0	1.7	93.3	6.7	0
(.20, .25)	6.0	1.7	89.6	10.4	0
(.25, .30)	6.0	1.7	86.2	13.8	0
(.30, .35)	6.0	1.8	80.6	19.4	0
(.35, .40)	6.0	1.7	75.2	24.8	0
(.40, .45)	6.0	1.8	70.2	29.8	0
(.45, .50)	6.0	1.8	65.9	34.1	0
(.50, .55)	6.0	1.9	62.1	37.9	0
(.55, .60)	6.0	1.9	58.8	41.2	0
(.60, .65)	6.0	2.0	56.8	43.2	0
(.65, .70)	6.0	1.9	53.9	46.1	0
(.70, .75)	6.0	1.9	51.3	48.7	0
(.75, .80)	6.0	1.9	49.2	50.8	0
(.80, .85)	6.0	2.0	47.0	53.0	0
(.85, .90)	6.0	2.0	45.1	54.6	0.3
(.90, .95)	6.0	2.0	43.5	56.3	0.2
(.95, 1)	6.0	2.0	42.0	57.8	0.2

- (d) The double precision subroutine just calls the single precision one and performs one additional iteration. It is tested for the values of  $(M,e)$  as given in both (a) and (c). The maximum  $\Delta M$  for this subroutine is  $7 \times 10^{-18}$ .

### 3. Usage of Subroutines

(a) Single precision version

CALL KEPSP(FM,ECC,ECAN,ITKEP)

FM = input mean anomaly

ECC = input eccentricity

ECAN = output eccentricity

ITKEP = an output number to indicate the number of iterations required.

Types of arguments: Real FM, ECC, ECAN

Integer ITKEP

(b) Double precision version

CALL KEPDP(DFM,DECC,DECAN,ITKEP)

The first three arguments are obviously D.P. generalizations of the above.

(c) Restrictions

The input FM (or DFM) is any real (or double precision) argument permissible on the machine. ECC (or DECC) must be restricted in  $[0,1]$ , otherwise an error message will be printed.

If the S.P. subroutine requires more than five iterations then a message is returned stating that there exists some problem.

### 4. Listings of subroutines are given in an appendix.

#### Acknowledgement

I had numerous helpful discussions with Drs. R. Broucke, F. Krogh, and C. Lawson. In particular, R. Broucke first introduced me to the initial approximations in Brouwer and Clemence and supplied me two D.P. subroutines with quadratic and cubic convergence. In addition, he also forewarned me of anticipating difficulty for the region of parameter values specified by the left upper corner of diag. 2. C. Lawson first suggested to me the idea of polynomial approximations.



# APPENDIX

```

- FOR, SI KEPDP . D.P. SOLUTION OF KEPLER-S EQ. FOR ELLIPTIC MOTIONS
  SUBROUTINE KEPDP(DFM, DECC, DECAN, ITKEP)
C
C TYPES MUST BE DOUBLE PRECISION DFM, DECC, DECAN
C      INTEGER ITKEP
C
C METHOD... CALL KEPSP TO OBTAIN RESULT IN SINGLE PRECISION, THEN 1
C      ITERATION WILL YIELD DOUBLE PRECISION RESULT, USING AN ITER-
C      ATION SCHEME WITH CUBIC CONVERGENCE PROPERTY
C      SEE COMMENTS IN KEPSP FOR DETAILS OF ITERATION SCHEME
C
      DOUBLE PRECISION DFM, DECC, DECAN, FP, FPP, U, FML, FMM, PI
      DATA PI / 3.14159265358979324D0 /
C
C TEST FOR RANGE OF ECCENTRICITY
C
      IF (DECC .GE. 0.D0 .AND. DECC .LE. 1.D0) GO TO 10
      WRITE (6,5) DFM, DECC
5  FORMAT(1X, -ECCENTRICITY OUT OF RANGE...-.5X, -MEAN ANOMALY=-, 1PD10.
      * 3, 10X, -ECCENTRICITY=-, 1PD10.3)
      RETURN
10 CONTINUE
C
      IFM = 1
      IF (DFM .GE. 0.D0) GO TO 8
      DFM = -DFM
      IFM = -1
8  CONTINUE
C      REDUCE RANGE OF PROB. TO (-PI, PI)
      ML = IDINT(DFM/PI)
      FML = (ML + MOD(ML, 2))
      FML = FML * PI
      FMM = DFM - FML
C
      IF ML = EVEN, LET FM = ML * PI + FMM, WHERE FMM IS IN (0, PI)
      IF ML = ODD, LET FM = (ML + 1) * PI + FMM, WHERE FMM IS IN (-PI, 0)
      FOR THE ODD CASE, FIRST COMPUTE SOLN FOR ABS(FMM) AND THEN
      TAKE NEGATIVE OF RESULT SUBSEQUENTLY
C
      IF (FMM .LT. 0.D0) FMM = -FMM
      FM = FMM
      ECC = DECC
      CALL KEPSP(FM, ECC, E0, ISP)
      DECAN = E0
      FPP = DECC * DSIN(DECAN)
      FP = 1.D0 - DECC * DCOS(DECAN)
      IF (FP .LT. 1.D-16) FP = FP + 1.D-16
      U = (DECAN - FMM - FPP) / FP
      DECAN = DECAN - U * (1.D0 + 0.5D0 * FPP * U / FP)
C
      IF (DFM .LT. FML) DECAN = -DECAN
      DECAN = FML + DECAN
      ITKEP = ISP + 1
      IF (IFM .GE. 0) RETURN
C      A NEGATIVE VALUE FOR IFM RECORDS THAT THE INPUT FM LT 0
      DECAN = -DECAN
      DFM = -DFM
C
      RETURN
      END

```

```

- FOR,SI KEPSP
  SUBROUTINE KEPSP (FM,ECC,ECAN,ITKEP)
C COMPUTES SOLUTION FOR KEPLER-S EQ.  $F(ECAN)=ECAN-FM-ECC*\sin(ECAN)=0$ 
C FOR ELLIPTIC ORBITS, I.E.,  $0 \leq ECC \leq 1$ 
C
C INPUT...
C     FM=MEAN ANOMALY IN RADIANS
C     ECC=ECCENTRICITY
C
C OUTPUT...
C     ECAN=ECCENTRIC ANOMALY
C     ITKEP=NO. OF ITERATIONS
C
C TYPES MUST BE REAL FM,ECC,ECAN, INTEGER ITKEP
C
C METHOD... ITERATION SCHEME WITH LOCAL CUBIC CONVERGENCE
C
C      $ECAN(N+1)=ECAN(N)-F/FP*(1-F*FPP/(FP*FP*2))$ , WHERE
C         F=F(ECAN) AS DEFINED ABOVE
C         FP=1ST DERIVATIVE OF F =  $1-ECC*\cos(ECAN)$ 
C         FPP=2ND DERIVATIVE OF F =  $ECC*\sin(ECAN)$ 
C
C     USE POLYNOMIALS TO APPROXIMATE ECAN(0)
C
C THIS PROGRAM SHOULD ACHIEVE COVERGENCE IN 2 ITERATIONS AT MOST
C
  DATA HALFPI/1.57079633/ ,ONE/1./,PI/3.14159265/
C
C TEST FOR PROPER RANGE OF ECCENTRICITY
C IF(ECC .LT. 0. .OR. ECC .GT. 1.) GO TO 40
C
  IFM=1
  IF(FM .GE. 0.) GO TO 3
  FM=-FM
  IFM=-1
3 CONTINUE
C     REDUCE RANGE OF PROB. TO  $(-PI,PI)$ 
  ML=INT(FM/PI)
  FML=(ML+MOD(ML,2))*PI
  FMM=FM-FML
C
C IF ML=EVEN, LET FM=ML*PI+FMM, WHERE FMM IS IN  $(0,PI)$ 
C IF ML=ODD, LET FM=(ML+1)*PI+FMM, WHERE FMM IS IN  $(-PI,0)$ 
C FOR THE ODD CASE, FIRST COMPUTE SOLN FOR ABS(FMM) AND THEN
C TAKE NEGATIVE OF RESULT SUBSEQUENTLY
C
  IF(FMM .LT. 0.) FMM=-FMM
C
  FMO=HALFPI-ECC
  IF(FMM .GE. FMO) GO TO 15
C
C INITIAL APPROX FOR CASES FM .LT. FMO
C
  R=FMM/FMO
  FK=(ONE-R)*(ONE-R)
  IF(ECC .GT. 0.7) GO TO 8
  FK=FK*(ONE+2.*R-FMM/(ONE-ECC))
  GO TO 10
8 CONTINUE
  FK=FK*FK

```

```

      IF (ECC .LT. 0.85) GO TO 10
      IF (FMM .GT. .05 .OR. ECC .LT. .96) GO TO 5
      Q=2.*(1.-ECC)/ECC
      R=3.*FMM/ECC
      G=SQRT(Q*Q+R*R)
      FO=CBRT(R+Q)+CBRT(R-Q)
      GO TO 25
5  CONTINUE
      FK=FK*FK
10  CONTINUE
      EI=1.-FK
      GO TO 20
15  CONTINUE
C
C  INITIAL APPROX FOR CASES FM .GT. FMO
C
      Q=FMM-PI
      R=HALFPI+ECC
      FI=ONE-(ONE+Q/R)**2*(ONE+Q*(HALFPI-ECC-2.)/(ECC+ONE)*R))
20  CONTINUE
C
      FO=EI*ECC+FMM
25  CONTINUE
      ITKEP=0
30  CONTINUE
      FPP=ECC*SIN(E0)
      FP=ONE-ECC*COS(E0)
      IF (FP .LT. 1.E-8) FP=FP+1.E-8
      U=(E0-FMM-FPP)/FP
      DE=U*(ONE+0.5*FPP*U/FP)
      FPP=E0-FMM-FPP
      ECAN=E0-DE
      ITKEP=ITKEP+1
      IF (ABS(DE) .LT. 2.5E-3) GO TO 35
      E0=ECAN
      IF (ITKEP- 5) 30,40,40
35  CONTINUE
C  ODD CASE WITH NEGATIVE RESULT
      IF (FM .LT. FML) ECAN=-ECAN
      ECAN=FML+ECAN
      IF (IFM .GE. 0) RETURN
C  A NEGATIVE VALUE FOR IFM RECORDS THAT THE INPUT FM LT 0
      ECAN=-ECAN
      FM=-FM
      RETURN
40  CONTINUE
      WRITE (6,45) ECC,FM
45  FORMAT(1X,-ERROR IN KEPCUB, NO. OF ITER GT 5-,10X, -ECCENTRICITY=-
      *      ,1P E8.2,10X,-MEAN ANOMALY=-,1PE8.2)
      RETURN
      FND

```

## References

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