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ABSTRACT

The object of this note is to study the planetary or test particle orbits in the Einstein space-time. After discussing the geometrical properties of the spherical space and its connection with the Einstein space-time, it is shown that the orbits in the Einstein space-time are the great circles of the spherical space, and are therefore periodics. It is also proved that the velocity of motion along the great circles is constant.

SUMMARY

There are several theories of gravitation that provide theories of models of the universe. The investigation in this paper is limited to a model of the universe based on general relativity. The main assumptions made are that the model has constant space-time curvature, and is a zero-pressure static model (the Einstein static model).

After a brief outline of relativistic model universes, the geometrical properties of the spherical space and its connection with Einstein universe are described. The variational principle, giving the ordinary geodesies of this universe, is derived. It is shown that the orbits in this model universe are the great circles of the spherical space and are therefore periodics. Finally it is shown that the velocity of the motion along the great circles is constant.

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1. INTRODUCTION

The field equations of general relativity indicate that a uniform cosmological metric is produced by a perfect fluid which follows the fundamental world-lines and has uniform proper density and uniform pressure. Mathematically a uniform cosmological metric is defined by the space-time of canonical form:

$$ds^2 = dt^2 - S^2(t)\left[dr^2 + \sigma_k^2(r)\, d\Omega^2\right] \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{2}$$

and

$$\sigma_k(r) = \begin{cases} \sin r & k = 1 \\ r & k = 0 \\ \sin hr & k = -1 \end{cases} \tag{3}$$

$S(t) = R(t)/c$, where $R(t)$, called the scale-factor, is an unknown function of the cosmic time, and c is the local speed of light. The behavior of the scale-factor $R(t)$ and its physical dimension in connection with the red-shift have been discussed by McVittie [1, 2]. Using the concept of metric automorphisms in space-time Robertson and Noonan [3] showed that for space-time of constant curvature K, the function $S(t)$ satisfies the differential equation

$$\dot{S}^2(t) + k + KS^2(t) = 0, \tag{4}$$

dot indicates derivative with respect to time, $k = 0$, or ± 1 is the constant Riemannian (spatial) curvature and K the constant Gaussian (space-time) curvature which may be classified in the three categories $K > 0$, $K = 0$, and $K < 0$.

If $\dot{S} \neq 0$, then it follows from (4) that $\ddot{S} = -KS$.

If $\dot{S} \equiv 0$, then both k and K are zero. Equations (4) and (5) are the necessary and sufficient conditions that space-time have constant curvature [3]. Translating the definition of a stationary universe into the concept of automorphism Robertson [3] showed that only stationary universes satisfy the perfect cosmological principle, and therefore they satisfy the following system of two equations

$$\begin{cases} \dot{S}/S = \text{const}, \\ k\dot{S} = 0 \end{cases} \tag{6}$$

Equations (6) have the following four solutions [3]

(a) $k = 1$ $\quad S = \text{const.}$

(b) $k = 0,$ $\quad S = \text{const.}$

(c) $k = -1$ $\quad S = \text{const.}$

(d) $k = 0$ $\quad \dot{S}/S = \text{const} \neq 0$

The cases (b) and (d) have been studied in details. In fact, the study of the static Minkovski universe ($k = 0, \dot{S} = 0$) as well as that of the expanding

Minkovski universe ($k = -1, K = 0$ special case of Equation (4)) have been made and can be found in most texts on general relativistic cosmology.

The de Sitter non-static universe (case d) is being discussed by McVittie [1, 2] in connection with the red-shift, and Robertson [3] studied in detail as special simple cosmological model. The de Sitter static universe had been found a long time ago by Chazy [4].

The purpose of this paper is to consider the case (a): $k = 1$, $S =$const., which is called the Einstein static universe (or Einstein cylindrical universe). It is preposed first to study the goemetrical properties of this universe and then to investigate the motion of planets in the same universe.

2. THE GEOMETRY OF EINSTEIN UNIVERSE

Let us consider the Einstein static universe, which is the case (a): $k = +1$, $S = \text{const} \neq 0$ or $R = \text{const}$, with the cosmical constant $\Lambda = S^{-2} > 0$. This universe is a zero-pressure static universe indicated by the space-time

$$ds^2 = dt^2 - S^2 \left[d\omega^2 + \sin^2\omega \, (d\theta^2 + \sin^2\theta \, d\phi^2)\right] \tag{7}$$

One can notice at once a three-dimensional manifold with the line-element (spatial) metric

$$d\ell^2 = S^2 \left[d\omega^2 + \sin^2\omega \, (d\theta^2 + \sin^2\theta \, d\phi^2)\right] \tag{8}$$

immbeded in a four-dimensional Euclidean space. Let the four-dimensional space be described by means of the cartesian coordinates x_1, x_2, x_3, x_4 or the polar coordinates S, ω, θ and ϕ given by

$$\begin{cases} x_1 = S \cos \omega \\ x_2 = S \sin \omega \cos \theta \\ x_3 = S \sin \omega \sin \theta \cos \phi \\ x_4 = S \sin \omega \sin \theta \sin \phi \end{cases} \tag{9}$$

where

$$0 \leq \omega \leq \pi, \qquad 0 \leq \theta \leq \pi, \qquad 0 \leq \phi \leq 2\pi$$

One deduces from (9)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = S^2 \tag{10}$$

which represents a hypersphere of radius S. The line-element in this space is

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 = S^2\left[d\omega^2 + \sin^2\omega \,(d\theta^2 + \sin^2\theta \, d\phi^2)\right] \tag{11}$$

which has the form of the line-element on three-dimensional surface of a four-dimensional sphere. This space is called Spherical Space of radius S. Therefore there is a close connection between spherical space and Einstein space-time (7).

Let us study the geodesics of the spherical space (8). These geodesics are given by the condition:

$$\delta \int S^2 \left[\left(\frac{d\omega}{ds} \right)^2 + \sin^2\omega \left(\frac{d\theta}{ds} \right)^2 + \sin^2\omega \sin^2\theta \left(\frac{d\phi}{ds} \right)^2 \right] ds = 0 \tag{12}$$

where s indicates the proper time.

The Euler-Lagrange equation for ϕ gives the first integral

$$\sin^2\omega \sin^2\theta \frac{d\phi}{ds} = \text{const.} \tag{13}$$

The section of spherical space by the surface $\phi = 0$ or hyperplane $x_4 = 0$ is according to formulae (9) a sphere Σ of radius S. On the sphere Σ the condition (12) will be reduced to the condition

$$\delta \int \left[\left(\frac{d\omega}{ds} \right)^2 + \sin^2\omega \left(\frac{d\theta}{ds} \right)^2 \right] ds = 0 \tag{14}$$

which defines the geodesics i.e., the Great Circles. Therefore the geodesics of the spherical space are the great circles of this space, each defined by the two equations

$$\phi = 0, \qquad \tan\omega \cos\theta = \text{const.}$$

By a rotation of the axes x_1, x_2, and x_3 each great circle of the spherical space may be defined by the two simple equations

$$\phi = 0, \qquad \theta = 0.$$

Therefore the great circles have the length $2\pi S$ and the shortest distance between two points on a spherical space is less than or equal to πS.

3. ORBITS IN THE EINSTEIN SPACE-TIME

The motion of a particle whose mass is negligible with respect of the sun is given by the equations of the ordinary geodesics of the space-time (7). These equations, in terms of the proper time s, are given by

$$\delta \int \left\{ \left(\frac{dt}{ds}\right)^2 - S^2 \left[\left(\frac{d\omega}{ds}\right)^2 + \sin^2\omega \left(\frac{d\theta}{ds}\right)^2 + \sin^2\omega \, \sin^2\theta \left(\frac{d\phi}{ds}\right)^2 \right] \right\} ds = 0 \quad (15)$$

Euler-Lagrange equation for t gives the first integral

$$\frac{dt}{ds} = \text{const.} \quad (16)$$

Taking into account of the above first integral and the metric (7), the three Euler-Lagrange equations for ω, θ and ϕ give relations similar to the condition (12) and the spatial metric (8). Therefore, the orbits in Einstein Universe are the great circles of the spherical space.

The combination of the first integral (16) and the metric (7) gives the new first integral

$$\left(\frac{d\omega}{dt}\right)^2 + \sin^2\omega \left(\frac{d\theta}{dt}\right)^2 + \sin^2\omega \, \sin^2\theta \left(\frac{d\phi}{dt}\right)^2 = \text{const.} \quad (17)$$

Therefore, in Einstein Universe the velocity of each particle is constant, a property which belongs to all geodesics of a static metric along which the metric is positive or zero. One can conclude that a particle, in Einstein universe, moves indefinitely along a great circle of the spherical space, that is to say, its motion is periodic.

The motion of light rays in the same universe will be the subject of another paper.

REFERENCES

1. McVittie, G. C., General Relativity and Cosmology, The University of Illinois Press, Urbana, Ill., 1965.

2. McVittie, G. C., Le décalage vers le rouge et la vitesse de recession. Ann. Inst. Henri Poincaré, Vol. V., No. 4, 1960.

3. Robertson, H. P., and Noonan, T.W., Relativity and Cosmology, W. B. Saunders Company, 1968.

4. Chazy, J., La Théorie de la Relativité et la Mécanique Cèleste, t.II, 1930, Gauthier-Villars, Paris.