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DECOUPLING AND POLE PLACEMENT VIA TRANSFER MATRIX SYNTHESIS

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September 1971

The University of Connecticut

This work has been sponsored by the
National Aeronautics and Space Administration
Research Grant NGL 07-002-002

ABSTRACT

A transfer function matrix synthesis technique is applied to the problems of decoupling and pole placement in linear multivariable systems. It is shown that either decoupling with prefilter augmentation or arbitrary pole placement can be achieved directly by inspection and elementary matrix algebra. Previously stated conditions are confirmed. Examples are included to demonstrate the efficacy of the approach.

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I. INTRODUCTION

Recently considerable attention has been given to the problems of decoupling and pole placement in linear multivariable control systems. Among the more significant proposed solutions of these problems have been those presented by Falb and Wolovich [1], Gilbert [2], Wonham [3], Wonham and Morse [4,5], Pearson and his students [6,7,8], and Jordan and Merriam [9]. For the most part the results obtained in [1]-[9], whether arrived at algebraically or geometrically, reflect the basic structure of the linear multivariable system and thus have many points of similarity. This paper is intended to demonstrate an approach to decoupling and pole placement which, while yielding like results, is easier to apply than previous approaches. In addition, modifications to be used in the presence of system irregularities or constraints are suggested.

The system that will be considered is defined by the state equations

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x}. \end{aligned} \tag{1}$$

At this point it is assumed that A,B,C are given and that B and C are full rank. However, the possibility of selecting C advantageously will be considered later. The vectors $\underline{x}$, $\underline{y}$, and $\underline{u}$ are of dimension n, m, and m, respectively. The transfer function matrix between $\underline{y}(s)$ and $\underline{u}(s)$ is given by

$$T(s) = C(sI-A)^{-1} B.$$

The following well-known expansion of T(s) should be noted:

$$T(s)=\frac{\sum_{j=0}^{p-1} K_j\ s^j}{\sum_{i=0}^{p} \alpha_i\ s^i} \tag{2}$$

where

$$K_j = \sum_{i=1+j}^{p} \alpha_i\ CA^{i-j-1}\ B \tag{3}$$

and

$$P_m(s) = \sum_{i=0}^{p} \alpha_i\ s^i$$

is the minimal polynomial of A.

II. DECOUPLING

In [9] it was shown that if a desired transfer function matrix, $T_D(s)$, has an inverse of the form

$$T_D^{-1}(s) = \sum_{k=0}^{K} P_k\ s^k$$

and if the matrix E given by

$$E = \lim_{|s|\to\infty} T_D^{-1}(s)\ T(s) \tag{4}$$

exists as a finite, non-singular matrix, then the closed loop system defined by

$$\underline{u} = E^{-1}(\underline{w} - H\underline{x}), \tag{5}$$

where

$$H = \sum_{k=0}^{K} P_k\ CA^k \tag{6}$$

and $\underline{w}$ is an m-dimensional command vector, satisfies the relationship

$$\underline{y}(s) = T_D(s)\ \underline{w}(s).$$

In order to attempt to decouple the system (1), it is necessary to choose $T_D(s)$ as a diagonal matrix, of which the i-th diagonal element is a constant divided by a monic polynomial in s of degree d_i. The numbers, $\{d_i\}$, are defined by Gilbert [2] as:

$$d_i = \max \{\ell: \text{ such that } C_i A^{\ell-1} B=0\} \tag{7}$$

where C_i is the i-th row of C. The roots of the denominator polynomials of $T_D(s)$ become closed loop poles. In general, the number of assigned poles is less than n,

$$\text{number of poles} = \sum_{i=1}^{m} d_i \leq n \tag{8}$$

Thus, decoupling often results in an incomplete specification of poles.

In the case of decoupling, E, when it exists, is identical to Falb and Wolovich's B*[1] and Gilbert's D[2], all of which must be non-singular. The requirement that E be non-singular is quite restrictive and severely limits the number of systems that can be directly decoupled. However, if the system can be augmented by adding low-pass filters in series with selected control lines, then the class of decoupleable systems will grow to include all but a relatively small number of "pathological" systems. To illustrate this, let us introduce matrices that define the behavior of T(s) as s approaches infinity. Let d_{ij} be defined by

$$d_{ij}=\max \{\ell: \text{ such that } \lim_{|s|\to\infty} |s^{\ell}\, t_{ij}(s)| <\infty\} \tag{9}$$
$$\text{for } i,j = 1,2,\ldots,m$$

where

$$T(s) = [t_{ij}(s)].$$

Also let

$$t^*_{ij} = \lim_{|s|\to\infty} s^{d^*_{ij}}\; t_{ij}(s) \tag{10}$$

where

$$d^*_{ij} = d_i + \min_{k=1,m} (d_{kj}-d_k) \tag{11}$$

Then $T^* = [t^*_{ij}]$ is a matrix of selected high frequency gains of $T(s)$, and $D = [d_{ij}]$ and $D^* = [d^*_{ij}]$ are matrices of positive integers. If the definition of strong inherent coupling given by Gilbert in [2] is changed to

<u>Definition:</u> A system is <u>strongly inherently coupled</u> if the matrix T^* is singular.

then the necessary and sufficient condition for decoupling with prefilter augmentation is given by

<u>Theorm I:</u> A system of the form (1) can be decoupled with prefilter augmentation and state feedback if and only if it is <u>not</u> strongly inherently coupled.

<u>Proof:</u> Let T^*_D be given by

$$[t^*_D]_{ij} = \lim_{|s|\to\infty} s^{d_i} [t_D(s)]_{ij}. \tag{12}$$

Then

$$E = [T^*_D]^{-1} T^* \tag{13}$$

if and only if each column of the matrix D contains a minimal element, where a minimal element is any d_{ik} such that $d_{ik} \leq d_{ij}$, j=1, 2, ..., m. Suppose that the i-th control u_i is augmented and that

$$u_i(s) = \frac{\sum_{\ell=0}^{m_i} a_\ell s^\ell}{\sum_{\ell=0}^{n_i} b_\ell s^\ell} v_i, \; a_{m_i} \neq 0, \; b_{n_i} = 1.$$

Then the i-th column of D is increased by $n_i - m_i$. By adding prefilters to appropriate control lines, the D matrix can be modified so that each column contains a minimal element. Thus $|T^*| \neq 0$ is sufficient.

If $|T^*| = 0$, then either E has at least one zero column or (13) applies. In either case, E. is singular and necessity is proved.

It is important to note that these results do not require controllability or observability, since the development has been in terms of the open loop transfer function matrix. Panda [10] has also shown an approach to decoupling with augmentation, but his method appears more difficult to apply.

Decoupling Example

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

then

$$T(s) = \begin{bmatrix} \frac{2}{(s+1)^2} & \frac{1}{(s+2)} \\ \frac{1}{(s+1)^2} & \frac{1}{(s+2)} \end{bmatrix}$$

with

$$D = D^* = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and } T^* = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

Thus the first column of D contains no minimal elements and the system is not directly decoupleable. Putting a first order filter in series with u_2, such that

$$u_2 = \frac{1}{s+3} v_2,$$

yields the same T^* and

$$D = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} .$$

Now the augmented system is decoupleable.

Suppose, instead, that C is given by

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} .$$

Then

$$T(s) = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix}$$

and

$$T^* = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Hence this system cannot be decoupled, even with augmentation. This contradicts Gilbert [2], who claims that this system is decoupleable with augmentation, since in this case $|T(s)| \neq 0$, except at $s=\infty$.

III. POLE PLACEMENT

In the past, approaches to pole placement using state feedback have been primarily concerned with showing sufficient conditions for complete pole placement. As a result of constructive proofs, methods have been devised which yield the desired pole positions. These methods are quite unwieldy in practice, requiring lengthy computation and providing little information on relative open-loop and closed-loop pole locations.

The approach to decoupling presented in Part II suggests a similar approach to pole placement. In this case, however, the output matrix, C, serves as a variable to achieve an appropriate transfer function matrix. To insure that complete pole placement is possible, it is assumed that the pair [A, B] is completely state controllable. From (7) it is apparent that complete pole placement using a transfer function approach is possible only if equality holds in (8). Thus it is necessary first to specify an appropriate matrix, C, that satisfies this condition. Then a transfer function matrix, $T_D(s)$, that exhibits the desired poles must be selected so that E is nonsingular.

Consider the scalar integer function

$$r(\nu) = \text{rank } [P(\nu)], \ \nu = 1, 2, \ldots, p \tag{14}$$

where

$$P(\nu) = [B \vdots AB \vdots --- \vdots A^{\nu-1}B]. \tag{15}$$

Through controllability, $r(p) = n$, where p is the degree of the minimal polynomial of A. Also $r(\nu)$ is monotonically non-decreasing with ν. This function is useful in formulating the following theorem that specifies d_i in terms of an arbitrary output element defined by C_i.

<u>Theorem II</u>: The maximum number of poles that can be assigned through selection of the single output $y_i = C_i\underline{x}$ is given by

$$d_{i_{max}} = \{\text{min. } \nu \text{ such that } n-r(\nu) = 0\}$$

<u>Proof</u>: Let $d_{i_{max}}$ be as defined. Then $r(d_{i_{max}}-1) < n$ and C_i can be chosen in the null space of $P(d_{i_{max}}-1)$. Thus in (2)

$$K_j = 0 \quad \text{for } j = p-1, p-2, \ldots, p-d_{i_{max}}+1$$

and the minimum pole-zero excess of $T_i(s)$ is $d_{i_{max}}$. Notice that $d_{i_{max}}+1$ poles cannot be assigned, since $r(d_{i_{max}}) = n$.

Theorem II suggests a procedure for selecting the output matrix C, composed of rows C_i, such that equality holds in (8). Before proceeding it is important to observe that if C_i specifies the i-th output, then the corresponding row of T^* is given by

$$T_i^* = C_i \, A^{d_{i_{max}}-1} \, B.$$

Thus a nonsingular E matrix depends on proper selection of C to achieve $|T^*| \neq 0$. Theorem III shows that this is always possible if the control vector is redefined

<u>Theorem III</u>: Let $\underline{v} = U\underline{u}$ be a projection of the control vector such that the dimension of $\underline{v}$ equals the minimum number of controls necessary for controllability and the pair [A, BU] is controllable. Then there exists a dim($\underline{v}$) dimensional output vector $\underline{y} = C\underline{x}$ such that

$$\sum_{i=1}^{\dim(\underline{v})} d_i = n$$

and

$$|T^*| \neq 0.$$

<u>Proof</u>: Using Wonham's development [3] with dim($\underline{v}$) = t, the output vector defined by

$$C_1 = [\ \underbrace{10{-}{-}{-}0}_{v_i} : \underbrace{0{-}{-}{-}{-}0}_{v_2} : {-}{-}{-}{-}{-}{-} : \underbrace{0{-}{-}{-}{-}0}_{v_t}\ ]\ T^{-1}$$

$$C_2 = [\ \underbrace{0{-}{-}{-}{-}0}_{v_1} : \underbrace{1{-}{-}{-}{-}0}_{v_2} : 0{-}{-}{-}{-}0 : {-}{-}{-}{-}{-}{-}\ ]\ T^{-1}$$

$$\vdots$$

$$C_t = [\ \underbrace{0{-}{-}{-}{-}0}_{v_1} : {-}{-}{-}{-}{-}{-} : {-}{-}{-}{-}{-}{-} : \underbrace{10{-}{-}{-}0}_{v_t}\ ]\ T^{-1}$$

yields

$$d_i = v_i$$

$$\sum_{i=1}^{t} d_i = \sum_{i=1}^{t} v_i = n,$$

and $T^* = I$.

In general, it is not desirable to go through the algebraic procedure of proper control redefinition. Thus a direct approach to output selection is preferred. The following procedure will yield essentially the same results as Theorem III without control projection.

A. Select C_1 as in Theorem II and form

$$T_1^* = C_1 A^{d_1-1} B$$

B. Select C_2 belonging to the null spare of $[P(\nu):C_1^T]$ for maximum ν such that $C_2 A^\nu B = T_2^*$ is linearly independent of T_1^*.

This yields $d_2 = \nu_{max}+1$

C. Continue to select C_i, i= 3, 4, ..., m in the null spare of $[P(\nu):C_1^T:---:C_{i-1}^T]$ as in (B), always maintaining linear independence of $T_1^*, T_2^*, \ldots, T_i^*$.

This constructive approach, similar to that of Theorem III, yields a nonsingular T^* matrix with $\sum_{i=1}^{m} d_i = n$.

In order to complete the process of pole placement, it is necessary to specify an appropriate matrix $T_D(s)$. The discussion in Part II demonstrates that the choice of a diagonal $T_D(s)$ matrix with inverse polynomials of degree d_i yields complete pole placement. However, in this case poles must be assigned d_i at a time. This may be objectionable as complex poles must be assigned in pairs to maintain $T_D(s)$ as a matrix having real coefficients of s. Thus in some cases it may be necessary to choose a coupled design to achieve the desired pole distribution. Whereas others have had to resort to a two step approach to achieve arbitrary pole placement, it is seen here that any set of pole locations can be achieved using a one step process. This is illustrated by the following example. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & -11 & -6 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Selecting

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

yields $d_1 = 3$, $d_2 = 1$ and

$$T^* = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} .$$

For desired pole locations of

$$s_{1,2} = 1 \pm j \ ; \ s_{3,4} = -2 \pm 2j,$$

let

$$T_D(s) = \frac{1}{(s^2+s+1)(s^2+2s+2)} \begin{bmatrix} N_{11}(s) & N_{12}(s) \\ N_{21}(s) & N_{22}(s) \end{bmatrix}$$

Then

$$T_D^{-1}(s) = \frac{(s^2+s+1)(s^2+2s+2)}{N_{11}(s)N_{22}(s)-N_{21}(s)N_{12}(s)} \begin{bmatrix} N_{22}(s) & -N_{12}(s) \\ -N_{21}(s) & N_{11}(s) \end{bmatrix}$$

For compatibility

$$\deg [N_{22}(s)] = \deg [N_{21}(s)] = 3$$

$$\deg [N_{12}(s)] = \deg [N_{11}(s)] = 1$$

and

$$N_{11}(s)N_{22}(s) - N_{21}(s)N_{12}(s) = (s^2+s+1)(s^2+2s+2) .$$

There exists an infinite number of such numerator polynomials. For example,

$$N_{11}(s) = 2s+1$$

$$N_{12}(s) = s+1$$

$$N_{21}(s) = s^3+s^2+2s-1$$

$$N_{22}(s) = s^3+2s^2+3s+1$$

yield

$$E = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

and pole placement is complete.

IV. CONCLUSION

Decoupling and pole placement are special problems that can be solved using a transfer function matrix synthesis approach. In practice, the technique presented here is quite easily applied using a combination of inspection and elementary matrix algebra. There is no requirement to transform the system to a canonical form to achieve results. The directness of the approach allows the designer to use his experience and intuition to achieve a design that not only satisfies the stated objectives but also secondary objectives, such as reduced feedback gains, pole distribution, zero locations, etc.

No mention has been made of the common problem of unavailable states. Sufficient work has been done in this area to offer good alternative designs with partial state feedback. The most straightforward approach is to use a Luenberger observer [11] to estimate the unavailable states. It has been shown by Newmann [12] and Sarma and Jayaraj [13] that with complete knowledge of initial conditions, as is the case in transfer function matrix synthesis, the resultant design is exactly the same as if all states were available. If system augmentation is to be avoided, acceptable results can sometimes be achieved using the optimization schemes of Merriam and Jordan [14] or Athans and Levine [15]. These require rephrasing the problem in terms of optimization, which can easily be done as shown in [14] or [16].

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