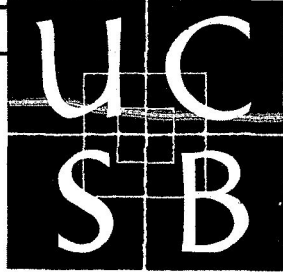
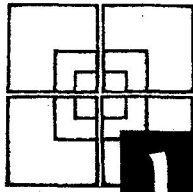


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- Part I: Librational Stability Conditions for a
Deploying Satellite
by W. R. Delameter and T. P. Mitchell
- Part II: The Influence of Corotation on Interplanetary
Dust Grain Orbits
by T. P. Mitchell
- Part III: A Note on the Rigid-Body Dynamics of the
Davis-Greenstein Effect
by T. P. Mitchell

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College of Engineering

LIBRATIONAL STABILITY CONDITIONS FOR DEPLOYING SATELLITES

by

W. R. Delameter and T. P. Mitchell

Department of Mechanical Engineering
University of California, Santa Barbara, California

The equation of planar libration of a satellite in orbit is

$$\frac{d}{dt} \left[C \left(\frac{d}{dt} \phi + \omega \right) \right] + 3\omega^2 (B - A) \sin \phi \cos \phi = 0 \quad (1)$$

where ω is the orbital angular velocity; A, B, and C are the moments of inertia of the satellite about its principal axes which are directed, when $\phi = 0$, along the radius vector of the orbit, along the transverse direction and along the normal to the orbit plane respectively; ϕ is the librational angle. (See Figure 1). For a satellite in a circular orbit ($\omega =$ positive constant), this equation may be written in the form

$$\ddot{x} + a(\tau)\dot{x} + b(\tau)\sin x = -2a(\tau) ; \quad 0 \leq \tau \leq \tau_f \quad (2a)$$

and

$$\ddot{x} + b(\tau_f)\sin x = 0 ; \quad \tau > \tau_f \quad (2b)$$

where

$$x = 2\phi$$

$$a(\tau) = \frac{\dot{C}}{C}$$

$$b(\tau) = \frac{3(B - A)}{C}$$

$$\tau = \omega t$$

and deployment is assumed to take place in the time interval $(0, t_f)$. The superscript dots denote differentiation with respect to the nondimensional time, τ . Equation (2a) is nonlinear, nonstationary (since the coefficients are nonconstant functions of time), and nonautonomous (since the right hand side is not identically zero). Very little is known concerning the solutions of such equations although G. Leitmann [1] has derived sufficient conditions for the stability of the nonlinear, nonstationary equation

$$a(t)\ddot{x} + b(t)\dot{x} + c(t) f(x) = 0 \quad (3)$$

which includes the autonomous form of equation (2).

Equation (2b) is equation (1) rewritten with the assumption that A, B, and C are constant, which they are when deployment has ended.

In this paper which is a further development and refinement of the method presented in [2], the stability of the solutions of equation (2) are studied and sufficient conditions for stability are derived where stability is defined in a physical manner, viz.:

A solution to equation (2) is said to
to be stable if the corresponding
motion of the satellite is librational
as distinct from rotational, i.e.

$$|\phi(\tau)| < \pi/2 \quad \text{for all } \tau.$$

Stability is examined about a stable equilibrium point and the coordinate axes are chosen so that this equilibrium point is in the neighborhood of $x = 0$. This implies that the coordinate axes are

chosen such that the stability requirement for three dimensional motion $C > B > A$ is satisfied. Therefore $(B - A) > 0$ and since C must be positive on physical grounds, $b(\tau) > 0$ for all τ .

It is apparent from equation (2a) and (2b) that during deployment, energy is being added and subtracted from the motion of the body, whereas after deployment ends (i.e. $\tau > \tau_f$), energy is conserved. This conservation is expressed by the first integral of equation (2b)

$$\frac{\dot{x}^2}{b_f} + 2(1 - \cos x) = \frac{\dot{x}_f^2}{b_f} + 2(1 - \cos x_f) \quad (4)$$

where the subscript f denotes the value of a quantity at $\tau = \tau_f$.

If $|x_f| < \pi$, $\dot{x}_f$ is real,

and

$$\frac{\dot{x}_f^2}{b_f} + 2(1 - \cos x_f) < 4 \quad (5)$$

it follows from equation (4) that $|x|$ will remain $< \pi$ for $\tau > \tau_f$. That is to say, the motion will remain stable.

It can therefore be asserted that a sufficient condition for stability is

$$\frac{\dot{x}^2(\tau)}{b(\tau)} + 2(1 - \cos x(\tau)) < 4 \quad ; \quad \tau > 0 \quad (6)$$

which it should be noted guarantees the inequality $|x(\tau)| < \pi$.

Equation (2a) may be written as a set of two first order differential equations

$$\dot{x} = z$$

and

$$\dot{z} = -a(\tau)z - b(\tau)\sin x - 2a(\tau) \quad (7)$$

Consider the function $V(x, z, \tau)$ defined by

$$V = 2(1 - \cos x) + \frac{1}{b} z^2 + f(\tau) \quad (8)$$

where $f(\tau)$ is to be determined later. The stability condition (5) may be expressed in terms of V as

$$V - f(\tau) < 4 \quad (9)$$

If $\frac{dV}{d\tau} \leq 0$ following the motion then $V(\tau) \leq V(\tau = 0) = V_0$, and condition (5) may be expressed in the form

$$V_0 < 4 + f(\tau) \quad (10)$$

Therefore,

The solution of the equation $\ddot{x} + a(\tau)\dot{x} + b(\tau) \sin x = -2a(\tau)$ is stable if a is finite, $b > 0$, and there exists $f(\tau)$ such that $\frac{dV}{d\tau} \leq 0$ and $V_0 < 4 + f(\tau)$ where

$$V(\tau) \equiv 2(1 - \cos x) + \frac{1}{b} z^2 + f(\tau) \quad .$$

Yet to be determined are the conditions under which $\frac{dV}{d\tau} \leq 0$.

From equations (7) and (8), $\frac{dV}{d\tau}$ following the motion is given by

$$\begin{aligned} \frac{dV}{d\tau} &= 2z \sin x + \frac{2}{b} z [-az - b \sin x - 2a] - \frac{\dot{b}}{b^2} z^2 + \dot{f} \\ &= - \left[\frac{2a}{b} + \frac{\dot{b}}{b^2} \right] z^2 - \frac{4a}{b} z + \dot{f} \\ &= Dz^2 + Ez + \dot{f} \end{aligned} \quad (11)$$

Thus $\frac{dV}{d\tau} \leq 0$ in the intervals where the quadratic polynomial in equation (11) is zero or negative. From equation (8) z may be expressed as

$$z = \pm [V - 2(1 - \cos x) - f]^{1/2} b^{1/2} \quad (12)$$

The minimum value of z accordingly satisfies the inequality

$$z_{\min} \geq -[V - f]_{\max}^{1/2} b_{\max}^{1/2} \quad (13)$$

and the maximum value of z satisfies the inequality

$$z_{\max} \leq [V - f]_{\max}^{1/2} b_{\max}^{1/2} \quad (14)$$

If condition (6) is satisfied the relations (13) and (14) may be expressed, utilizing equation (9), in the form

$$z_{\min} \geq -2b_{\max}^{1/2} \quad (15)$$

and

$$z_{\max} \leq 2b_{\max}^{1/2} \quad (16)$$

Thus, $\frac{dV}{d\tau} \leq 0$ if $\dot{f}$ is chosen so that $Dz^2 + Ez + \dot{f} \leq 0$ in the interval $[z_{\min}, z_{\max}]$. This requires that $\dot{f} < 0$ always.

If $D = 0$, the polynomial has one root at $z = -\frac{\dot{f}}{E}$ and it is required that

$$b_{\max}^{1/2} \leq \left| \frac{\dot{f}}{E} \right| \quad (17)$$

or f to be chosen as

$$f = -2b_{\max}^{1/2} \int_0^{\tau} |E| d\tau \quad (18)$$

so that condition (10) now specifies the initial conditions for stability.

If $D < 0$, the interval $[z_1, z_2]$ where

$$z_1 = \frac{-E - \sqrt{E^2 - 4D\dot{f}}}{2D},$$

and

$$z_2 = \frac{-E + \sqrt{E^2 - 4D\dot{f}}}{2D}$$

must not intersect the interval $[z_{\min}, z_{\max}]$ except at isolated points (the endpoints or at $z_1 = z_2 = -E/2D$ if this equality is satisfied) if $\frac{dV}{d\tau}$ is to be less than zero in the required interval. This condition permits $\dot{f}$ and therefore f to be specified so that condition (10) sets the permissible range of initial conditions for stability. For example, if

$$f \equiv \int_0^\tau \frac{E^2}{4D} d\tau \quad (19)$$

$\frac{dV}{d\tau} \leq 0$ over all z . However f may be chosen so that it is less restrictive on initial conditions if $\frac{dV}{d\tau}$ is allowed to become greater than zero outside the interval $[z_{\min}, z_{\max}]$.

If $D > 0$, the interval $[z_{\min}, z_{\max}]$ must be contained in the interval $[z_1, z_2]$ where z_1 and z_2 are again taken to be the roots of the quadratic equation. This again permits choice of $\dot{f}$ (and therefore of f) so that condition (10) sets the permissible range of initial conditions.

If $D = 0$ and $a = 0$ (this is the constant parameter case), $f(\tau)$ may be chosen to be identically zero and $\frac{dV}{d\tau}$ will also be identically zero. Then condition (10) becomes

$$V_0 = 2(1 - \cos x_0) + \frac{1}{b_0} \dot{x}_0^2 < 4$$

which is the usual restriction on initial conditions for stability in the constant parameter case.

The conditions under which $\frac{dV}{d\tau} \leq 0$ have now been specified in terms of restrictions on the choice of $\dot{f}$ (or of f). Therefore

The solution of the equation $\ddot{x} + a(\tau)\dot{x} + b(\tau) \sin x = -2a(\tau)$ is stable if a is finite, $b > 0$, and $V_0 \leq 4 + f(\tau)$ where $V(\tau) = 2(1 - \cos x) + \frac{1}{b} \dot{x}^2 + f(\tau)$ and $f(\tau)$ is chosen to satisfy the conditions outlined above.

In summary the conditions on f are:

i) $\dot{f} \leq 0$ for all τ ;

ii) if $D = 0$, $f = -2b_{\max}^{1/2} \int_0^\tau |E| d\tau$

iii) if $D < 0$, f must be chosen so that $[z_{\min}, z_{\max}]$ can intersect $[z_1, z_2]$ only at isolated points, f is then

chosen as $f = \int_0^\tau \dot{f} d\tau$

iv) if $D > 0$, $\dot{f}$ must be chosen so that $[z_{\min}, z_{\max}]$ is contained in the interval $[z_1, z_2]$;

where $-z_{\min} = z_{\max} = 2b_{\max}^{1/2}$ and where z_1 and z_2 are real roots of the quadratic $Dz^2 + Ez + \dot{f} \equiv -[\frac{2a}{b} + \frac{\dot{b}}{b^2}]z^2 - \frac{4a}{b}z + \dot{f}$ when these exist.

The solution to equation (2) has been derived in specific cases [3]. Comparison of the solutions obtained in [3] with the sufficient conditions for stability derived in this paper shows that they are compatible.

The case in which $D < 0$ will now be discussed in greater detail. A large variety of deployment programs is included in the following assumed forms for the moments of inertia

$$\begin{aligned} A(\tau) &= A_o + (A_f - A_o)g(\tau) \\ B(\tau) &= B_o + (B_f - B_o)g(\tau) \\ C(\tau) &= C_o + (C_f - C_o)g(\tau) \end{aligned} \tag{20}$$

where $A_o = A(o)$, $A_f = A(\tau_f)$ etc. and $g(o) = 0$, $g(\tau_f) = 1$ with $0 \leq g(\tau) \leq 1$ for $0 \leq \tau \leq \tau_f$. Furthermore the form (20) assures that the inequality $D < 0$ is satisfied. Accordingly one may choose

$$\begin{aligned} f(\tau) &= \int_o^\tau \frac{E^2}{4D} d\tau \\ &= -\alpha \ln[1 + 2\beta g(\tau)] \end{aligned} \tag{21}$$

in which the constants α and β are defined by

$$\begin{aligned} \alpha &= 2(C_f - C_o)/3(B_f - B_o - A_f + A_o) \\ \beta &= [(B_o - A_o)/(B_f - B_o - A_f + A_o) + C_o/(C_f - C_o)]^{-1} \end{aligned}$$

Two important consequences of equation (21) are firstly, $f(\tau)$ is independent of τ_f and secondly, since $g(\tau_f) = 1$, the minimum value of $f(\tau)$ is attained at $\tau = \tau_f$; in addition this minimum is independent of the details of the deployment program.

The stability condition (10) namely

$$V_o < 4 + f(\tau)$$

can now be written

$$V_O < 4 + f(\tau_f)$$

or

$$V_O < 4 - \alpha \ln[1 + 2\beta] \quad (22)$$

As an illustration of the application of this result consider the case of a satellite deploying two thin, long uniform rods in diametrically opposite directions. Assuming the mass of the rods to be much less than that of the central body of the satellite one writes

$$A(t) = A_O + \frac{2}{3} \rho \ell^3(t) \sin^2 \theta$$

$$B(t) = B_O + \frac{2}{3} \rho \ell^3(t) \cos^2 \theta$$

$$C(t) = C_O + \frac{2}{3} \rho \ell^3(t)$$

where ρ is the mass per unit length of the rods, $\ell(t)$ is the length of the rod as a function of time and θ is the angle between the A-principal axis and the direction of the rod. For constant speed, v , of deployment $\ell(t) = vt = v \tau/\omega$ one can choose $g(\tau) = (\tau/\tau_f)^3$ with

$$A_f - A_O = \frac{2}{3} \rho \sin^2 \theta (v \tau_f / \omega)^3$$

$$B_f - B_O = \frac{2}{3} \rho \cos^2 \theta (v \tau_f / \omega)^3$$

$$C_f - C_O = \frac{2}{3} \rho (v \tau_f / \omega)^3$$

Numerically e.g. one can take $\theta = 30^\circ$, $2\rho (v\tau/\omega)^3 = 3C_O$, $2A_O = C_O$, $4B_O = 3C_O$. Then it follows that $4A_f = 3C_O$, $2B_f = 3C_O$, $C_f = 2C_O$. The stability condition is accordingly

$$\frac{4}{3} \dot{x}_0^2 + 2(1 - \cos x_0) < 2.87$$

As a check a computer program was run in this case for two sets of initial conditions and parametric values $A_0 = 2.0$, $B_0 = 3.0$, $C_0 = 4.0$, $\tau_f = 3.0$. The first set of initial conditions were $x_0 = 0$, $\dot{x}_0 = -1.465$ and resulted in

$$\frac{4}{3} \dot{x}_0^2 + 2(1 - \cos x_0) = 2.865$$

$$x(\tau_f) = 2.525$$

$$\frac{4}{3} \dot{x}_f^2 + 2(1 - \cos x_f) = 3.91$$

and stable motion.

The second set employed $x_0 = 0$, $\dot{x}_0 = -1.550$ resulting in

$$\frac{4}{3} \dot{x}_0^2 + 2(1 - \cos x_0) = 3.200$$

$$x(\tau_f) = -2.778$$

$$\frac{4}{3} \dot{x}_f^2 + 2(1 - \cos x_f) = 4.35$$

and unstable motion.

Several other cases were tested on the computer, with various conditions and parametric values. No case was found to be unstable when the theory predicted stability.

References

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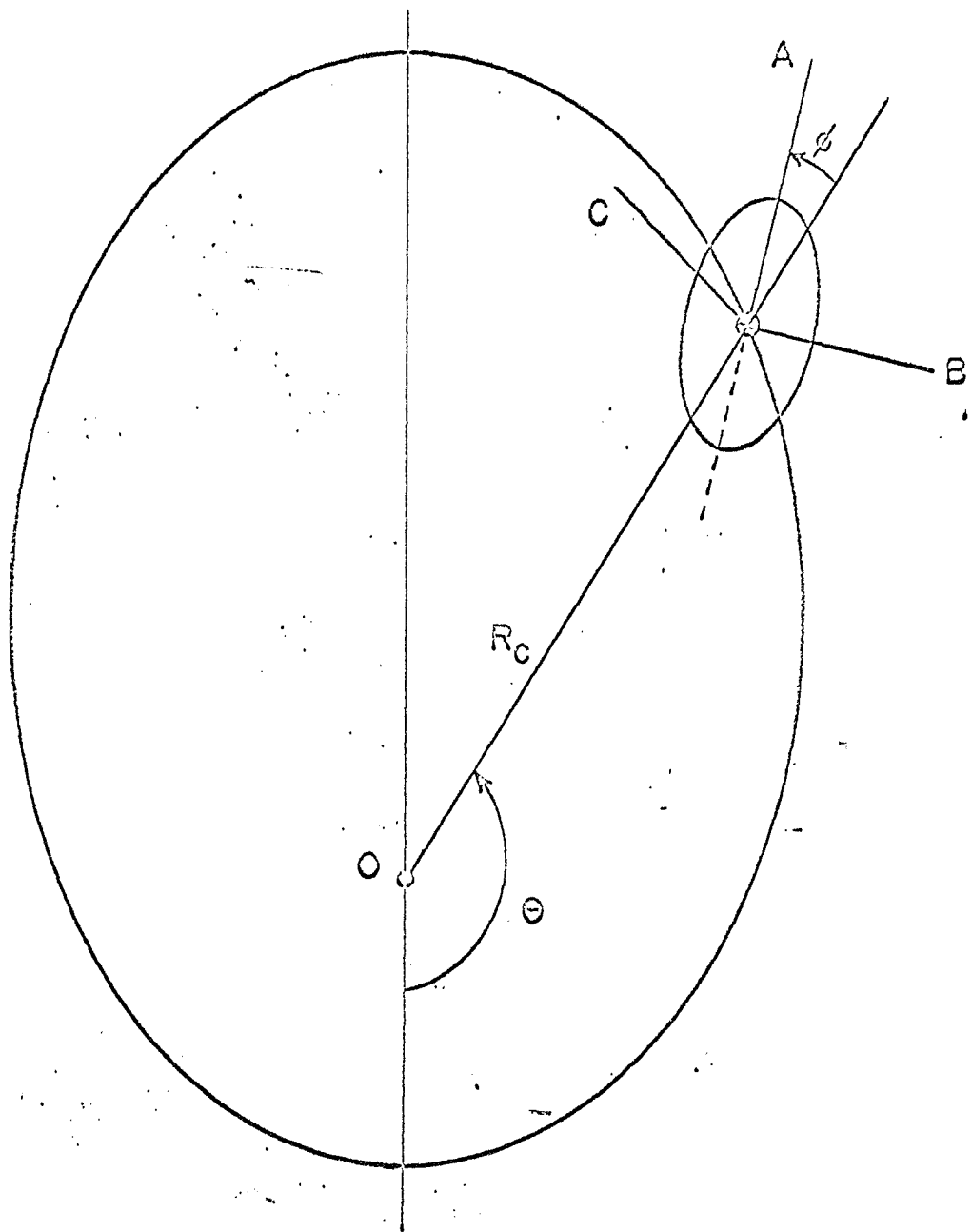


Figure 1. Angle of Libration and Principal Axes

THE INFLUENCE OF COROTATION ON INTERPLANETARY
DUST GRAIN ORBITS

THOMAS P. MITCHELL

Department of Mechanical Engineering
University of California, Santa Barbara, California

ABSTRACT

The perturbative effects of an azimuthal component of the solar wind velocity on the orbital characteristics of gravitationally bound interplanetary dust grains are analysed. The azimuthal motion of the wind is assumed to arise from corotation of the solar plasma within a certain region surrounding the sun. It is shown, in particular, that the azimuthal wind velocity, as distinct from other sources of perturbations, causes a secular decrease of the orbital inclination of the grains. The conditions under which an appreciable decrease in orbital inclination can occur, with a consequent concentration of the dust in the ecliptic, during the Poynting-Robertson lifetime of a grain are discussed.

I. INTRODUCTION

The possibility that the solar corona, and to some extent the interplanetary medium, corotate with the sun has been theoretically investigated by Axford, Dessler and Gottlieb (1963), Pneuman (1966), Weber and Davis (1967), Ferraro and Bhatia (1967). By studying observed values of plasma density and velocity, the interplanetary magnetic field and the dynamics of Type I comet tails Brandt (1967 a,b) estimates that the solar wind azimuthal

velocity at 1 AU is about 10 km/sec. This finding is strongly supported by the observations of Strong, Asbridge, Bame and Hundhausen (1967). Furthermore, Biermann and Lüst (1964), as a result of their study of Type I comet tails, conclude that the azimuthal component cannot be greater than 50 km/sec at 1 AU and is, in fact, probably less than 30 km/sec.

The extent of the region in which corotation exists has not been definitely established by either theory or observation. However, it appears to be generally accepted that it persists out to the Super-Alfvénic point i.e. the distance at which the Alfvén velocity corresponding to the meridional magnetic field equals the expansion velocity of the plasma. At greater distances the azimuthal velocity of the wind is presumably determined by the conservation of angular momentum about the solar spin axis. If one adopts this latter criterion and an azimuthal wind velocity of 10 km/sec at 1 AU corotation is found to exist out to approximately $30 R_{\odot}$. This is to be compared with the recent estimate by Brandt (1967b) of 20-40 $R_{\odot}$ for the distance to the Super-Alfvénic point.

Since dust grains in orbits of low inclination can reach high heliographic latitudes a knowledge of the solar wind behavior in regions not necessarily close to the ecliptic is required in order to carry out the aim of this paper. It is not possible, unfortunately, to infer definitely from the space probe measurements, which were made in or near the ecliptic, whether or not there is a latitudinal dependence in the wind. However, Antrack, Biermann and Lüst (1964) have found no evidence in their cometary studies, at least up to heliographic latitudes of about 30° , for such a dependence. Hence it is assumed that the solar wind characteristics are independent of heliographic latitude.

II. THE COULOMBIC AND COLLISIONAL PERTURBATIONS

In discussing the interaction of a dust grain and the solar wind the emphasis will be placed here almost exclusively on the secular behavior of the orbital inclination. It will be advantageous to adopt the standard orbital parameters a , e , f , i , and ω the semi-major axis, eccentricity, true anomaly, orbital inclination (measured from the ecliptic) and longitude of perihelion respectively in addition to the spherical polar coordinate system (r, θ, φ) . The relative velocity of the dust and the wind is then given by

$$\underline{u} = (w_r - v_r) \underline{e}_r + (w_\varphi \cos i \operatorname{Cosec} \theta - v_f) \underline{e}_f - w_\varphi \sin i \operatorname{Cosec} \theta \cos (\omega + f) \underline{e}_n \quad (1)$$

in which w is the solar wind velocity, v the Keplerian orbital velocity of the grain and the unit vector $\underline{e}_n$ is normal to the orbit plane. It is noted that, in magnitude, the orbital can be neglected relative to the wind, velocity and also that $|w_\varphi| \ll |w_r| \approx w$. Hence, one finds

$$\frac{\underline{u}}{u} = \underline{e}_r + \left(\frac{w_\varphi}{w} \cos i \operatorname{Cosec} \theta - \frac{v_f}{w} \right) \underline{e}_f - \frac{w_\varphi}{w} \sin i \operatorname{Cosec} \theta \cos (\omega + f) \underline{e}_n \quad (2)$$

where $w_\varphi = r\Omega \sin \theta$, $r \leq R_c$, $w_\varphi = R_c^2 \Omega \sin \theta / r$, $r \geq R_c$, and Ω is the solar angular velocity (2.87×10^{-6} rad/sec). The distance over which corotation exists is represented by R_c . The effects of a possible fluctuating direction of the w_φ component are not considered in this analysis.

The interaction of the wind and the grain can take either of two forms depending upon the electric charge carried by the grain.¹ Firstly, the grain may scatter the ions and electrons in the wind through Coulomb interactions and, secondly, direct collisions between the particles in the wind and the

¹ The influence of the magnetic field carried in the solar plasma is not being considered at this point.

dust grain may take place. Because of the uncertainty in the grain potential and other parameters both possibilities will be examined and compared.

a) The Coulombic Drag

To start with the Coulombic interaction of the dust and wind is considered. The problem is that of determining the rate of change of momentum of a charge (the grain) moving through a plasma (the wind) the constituents of which are assumed to have a Maxwell-Boltzmann velocity distribution. Using established formulations², Spitzer (1962), Trubnikov (1965), in conjunction with equation (2) one finds the perturbation acceleration in the e_n direction, i.e. normal to the orbit plane, to be

$$W = \frac{-8\pi q^2 e_w^2 n_w}{m_D^2} \left(1 + \frac{m_D}{m_w}\right) \left(\frac{m_w}{2kT}\right) \left[\ln(\lambda/p)\right] G(\bar{u}) \frac{w}{w} \text{Cosec } \theta \cos(\omega + f) \sin i \quad (3)$$

in which $q(\text{esu})$ denotes the charge on the grain, n_w the solar wind density, m_D the grain mass, λ the Debye length, p the critical impact parameter, e_w and m_w the charge and mass respectively of the wind constituents, T the temperature of the Maxwell-Boltzmann distribution, $\bar{u} = \sqrt{\frac{m_w}{kT}}$ and

$2\bar{u}^2 G(\bar{u}) = \text{erf}(\bar{u}) - \bar{u} \text{erf}'(\bar{u})$. It is understood that the right hand side of equation (3) is to be summed over the different constituents of the plasma which is considered to consist of electrons and hydrogen ions. The alpha particle and possibly other components are not included, their contribution being negligible. In addition, temperature anisotropies in the wind will be ignored. A possible difference between the electron and proton temperatures, Wolfe and Silva (1967) is not sufficient to change the numerical results appreciably. Hence, the electron and proton temperatures are taken to be

² A copy of the paper "Scattering Probability for Fast Test Particles in a Plasma" by Hussein & Forsen, Phys Rev.A, Nov. 1970, p2018 was not available to the author in time for consideration in this analysis.

identical. If one adopts³ a number density-position dependence of the form $n_w = n_E (R_E/r)^2$, in which n_E is the number density at $r = R_E = 1$ AU, then the total acceleration perturbation normal to the orbit plane is given by

$$- \frac{4\pi q^2 e^2 n_E R_E^2}{k T m_D a^2} \left(\frac{w}{w}\right) \left(\frac{a}{r}\right)^2 G(\bar{u}) \left(\ln \frac{r}{a} + \ln y\right) \operatorname{Cosec} \theta \cos (w + f) \sin i \quad (4)$$

in which λ_E is the Debye length at 1 AU and the parameter $y = a \lambda_E / R_E p$. For representative values of w and T viz. 5×10^7 cm/sec and 1.5×10^5 °K one finds, as expected, that the electron contribution to equation (4) dominates that of the protons by a factor of about fifty. Consequently, for the remainder of this calculation only the electron terms will be retained. The standard form of the orbital perturbation equations, Sterne (1960), gives

$$\frac{di}{dt} = \frac{rW \cos (w + f)}{\sqrt{\mu a (1 - e^2)}} \quad (5)$$

with $\mu = GM_\odot$. Interpreting as secular those terms in equation (5) which are independent of angular position in the orbit and perihelion longitude one finds that the secular rate of change, denoted by subscript S, of the orbital inclination is described by

$$\frac{di}{dt_S} = - \gamma G(u) \left[\ln y + \ln \left(\frac{1 + \sqrt{1-e^2}}{2} \right) + 1 - \sqrt{1-e^2} \right] \sin i, \quad r \leq R_c \quad (6)$$

and

$$\frac{di}{dt_S} = - \gamma \left(\frac{R_c}{a} \right)^2 (1-e^2)^{-\frac{1}{2}} G(u) \left[\ln y - \ln \left(\frac{1 + \sqrt{1-e^2}}{2} \right) + \ln (1-e^2) \right] \sin i, \quad r \geq R_c \quad (7)$$

³ For this and other assumed properties of the solar wind see the review papers by Axford, Space Science Rev. Vol 8, p331 1968 and Hundhausen, Space Science Rev. Vol 8, p.690, 1968

with $\gamma = 3n_E R_E^2 e^2 \Phi^2 / 2kTwp_s$ for a spherical grain of radius s , density ρ and potential Φ (statvolts). The reduction of equation (5) in combination with equation (4) to produce equations (6) and (7) involves the averaging over an orbit of a variety of functions of r and f . This process will not be reproduced here but it is remarked that in carrying it out w , T and q were assumed to be independent of position. Parker (1964) has discussed the perturbation of charged dust grain orbits by the magnetic field carried in the solar plasma. It can be shown that, assuming the electric field in the frame of reference of the plasma to be zero, the orbit perturbation is caused by the Lorentz force $-(q/c)w \times B$. If, following Parker⁴(1958, 1963), the field B is represented in component form by

$$B_r = B_o \left(\frac{b}{r}\right)^2 \cos \theta$$

$$B_\varphi = B_o \frac{b\Omega}{w} \frac{b}{r} \sin \theta \cos \theta$$

which corresponds to a solar dipole field, $r = b$ being the base of the corona, the rate of change of the orbital inclination is determined by

$$\frac{di}{dt} = \frac{qB_o b^2 \Omega}{2m_D c} \left[\mu a^2 (1-e^2) \right]^{\frac{1}{2}} \sin i \cos i \sin 2(\omega + f)$$

which, as distinct from the perturbations considered in this paper, does not produce any secular change of the inclination. The perturbative effect of direct collisions is examined next.

b) The Collisional Drag

It is a matter of straightforward calculation to find the drag force, due to collisions, which acts on a sphere of radius s and density ρ moving with velocity u through a Maxwell-Boltzmann distribution gas whose number

⁴ Parker in his analysis assumes the wind velocity to be radial.

density and temperature are n_w and T respectively. The details are given e.g. by Kubo (1965). The drag is

$$-\frac{3n_w kT}{4\rho_s} \left[(2\bar{u}^2 + 2 - 2\bar{u}^{-2}) \operatorname{erf}(\bar{u}) + \frac{1}{\sqrt{\pi}} (2\bar{u} + \bar{u}^{-1}) \exp(-\bar{u}^2) \right] \frac{u}{u} \quad (8)$$

where, as in equation (3), a summation over the individual components of the gas is understood. Equation (8) leads to the following two equations describing the secular rates of change of the orbital inclination

$$\frac{di}{dt}_s = - 3.5 \times 10^{-17} \frac{n_E w}{\rho_s} \left[a(1-e^2) \right]^{-\frac{1}{2}} \sin i, \quad r \leq R_c, \quad (9)$$

and

$$\frac{di}{dt}_s = - 3.5 \times 10^{-17} \frac{n_E w}{\rho_s} \left(\frac{R_c}{a} \right)^2 a^{-\frac{1}{2}} (1-e^2)^{-1} \sin i, \quad r \geq R_c \quad (10)$$

In these two equations the electron contribution has been dropped because, as is to be expected, for $w = 5 \times 10^7$ cm/sec and $T = 1.5 \times 10^5$ °K the proton terms dominate it by a factor of about 200.

III. COMPARISON WITH THE POYNTING-ROBERTSON EFFECT

The Poynting-Robertson drag on a grain is directed oppositely to the grain's orbital velocity and hence, of course, lies in its orbital plane. Accordingly, this retarding force does not affect the orbital inclination. It does, however, as was shown by Robertson (1937), secularly decrease both the major axis and the eccentricity and ultimately, in the absence of other perturbations, cause the grain to spiral into the sun. The question of interest here is whether or not significant changes can occur in the orbital

inclination, as described by equations (6), (7), (9) and (10) during the Poynting-Robertson lifetime of the grain. Before examining this question directly the relative magnitudes of the two sources of inclination decrease are compared. A short calculation based upon equations (4) and (8) shows that the ratio of the Coulombic to the collisional perturbation is, to a good approximation, $2 \times 10^{16} V^2/Tw^2$ where V (volts) is the grain potential. Hence the collisional perturbation is dominant for $w = 5 \times 10^7$ cm/sec and $T = 1.5 \times 10^5$ °K unless the potential is of the order of 100 volts. This conclusion is clearly dependent upon the choice of bulk wind velocity and temperature. Since periods of low wind speed (2×10^7 cm/sec) tend to coincide with periods of low temperature (10^4 °K) a 10 volt grain could be significantly perturbed by the charge drag during intervals of low solar activity. Wyatt (1969) estimates the grain potential at about 10 volts. Under normal conditions, however, there being no reason to suspect that the grain voltage would be high, it appears that the perturbation due to collisions is dominant and hence it alone will be compared here with the Poynting-Robertson drag. This latter perturbation results in a semi-major axis rate of change given by

$$\frac{da}{dt_s} = \frac{-3QE_{\odot}}{16\pi\rho sc^2} \frac{(2+3e^2)}{a(1-e^2)^{3/2}} \quad (11)$$

in which $E_{\odot} = 3.9 \times 10^{33}$ ergs/sec is the radiation emitted by the sun and the dependence of the effective cross-section of the grain, which is a function of its radius, composition and electrical properties, is subsumed under the one symbol Q .

That the collisional and Coulombic drag cause secular changes in the major axis is clear from the fact that they modify the orbital energy of the dust particle. Since these changes do not alter its order of magnitude, the Poynting-Robertson effect described by equation (11) is taken to be that which determines the lifetime of a grain in the solar system. On combining equations (9), (10) and (11), and confining the discussion for simplicity to circular orbits, one obtains the functional dependence of the orbital inclination on the major axis in the forms

$$\tan \frac{i}{2} = \tan \frac{i_o}{2} \exp \left[-4.5 \times 10^{-29} \frac{n_E^w}{Q} (a_o^{3/2} - a^{3/2}) \right], \quad r \leq R_c$$

(12)

and

$$\tan \frac{i}{2} = \tan \frac{i_o}{2} \exp \left[-1.3 \times 10^{-28} \frac{n_E^w}{Q} R_c^2 (a^{-1/2} - a_o^{-1/2}) \right],$$

$r \geq R_c$

(13)

The two equations (12) and (13) when applied to a grain moving under the Poynting-Robertson drag from $a_o > R_c$ to some $a < R_c$ are tantamount, assuming $w = 5 \times 10^7$ cm/sec, to the single equation

$$\tan \frac{i}{2} = \tan \frac{i_o}{2} \exp \left\{ -9 \times 10^{-21} \frac{n_E R_c^{3/2}}{Q} \left[1 - 0.7 \left(\frac{R_c}{a_o} \right)^{1/2} - 0.25 \left(\frac{a}{R_c} \right)^{3/2} \right] \right\}$$

(14)

which shows clearly that the relative importance of the secular decrease in inclination depends strongly upon the three

parameters n_E , Q and R_C . The corotation zone has been estimated to extend to $R_C \approx 30 R_\odot$ but may in fact be far less extensive. Despite such uncertainties, the implications of equation (14) can be exemplified by taking e.g. $a_0 = 1 \text{ AU}$, $n_E = 10$, $Q = 1.0$, $a = \frac{1}{2} R_C$ and R_C equal to, firstly, $30 R_\odot$ and secondly $5 R_\odot$. One finds approximately

$$\tan \frac{i}{2} = 4.5 \times 10^{-3} \tan \frac{i_0}{2}, \quad R_C = 30 R_\odot \quad (15)$$

and

$$\tan \frac{i}{2} = 5 \times 10^{-2} \tan \frac{i_0}{2}, \quad R_C = 5 R_\odot \quad (16)$$

Accordingly, if the corotation zone extends to about $1/7 \text{ AU}$ the corotation drag on the orbital inclination of a grain is considerable and leads to a concentration of dust in the ecliptic whereas if the zone extends only a few solar radii the drag effect is an order of magnitude smaller. However, in interpreting equation (16) it should be kept in mind that assuming the wind temperature and velocity to remain constant to within $2 - 3 R_\odot$ of the sun may not be valid.

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A NOTE ON THE RIGID-BODY DYNAMICS OF THE DAVIS-GREENSTEIN EFFECT

Thomas P. Mitchell
Department of Mechanical Engineering
University of California, Santa Barbara, California

In order to explain the observed polarization of starlight it is generally assumed that some physical mechanism aligns the interstellar dust. The Davis-Greenstein mechanism assumes that the dust grains are non-spherical, paramagnetic and are coupled to the interstellar magnetic field through the imaginary part of the magnetic susceptibility. This coupling, which is dissipative, has been discussed in the original paper by Davis and Greenstein (1951) and also by Davis (1958). The theoretical basis of the coupling mechanism has been further considered and developed by many authors including Jones and Spitzer (1967), Spitzer (1968) and Purcell (1969). The purpose of this note is to examine the rigid-body dynamics of an interstellar grain assuming the Davis-Greenstein mechanism is in effect and also to comment briefly on the statistical mechanics of the grains. It is shown that the dynamical motion can be described and analyzed in terms which appear more elementary and direct than those previously employed. To begin with we consider the dynamics of a grain in an ensemble of ellipsoidal grains in equilibrium at temperature T .

It is easily shown that the distributions of angular momentum $\vec{H}$ and angular velocity $\vec{\omega}$ are

$$n(\vec{H}) = (2\pi kT)^{-3/2} (I_1 I_2 I_3)^{-1/2} \exp \left\{ - \left(\frac{H_1^2}{I_1} + \frac{H_2^2}{I_2} + \frac{H_3^2}{I_3} \right) / 2kT \right\} \quad (1)$$

$$n(\vec{\omega}) = (2\pi kT)^{-3/2} (I_1 I_2 I_3)^{1/2} \exp \left\{ - (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) / 2kT \right\} \quad (2)$$

and that the average

$$\langle \omega^2 \rangle = kT \left(\frac{1}{I_1} + \frac{1}{I_2} + \frac{1}{I_3} \right) \quad (3)$$

The inertia tensor of the grain about its center of mass, is considered to be diagonal with elements I_1, I_2, I_3 . Equipartition of energy of rotation about the three axes clearly applies. Consider for definiteness a prolate ellipsoidal grain for which $I_1 = I$, $I_2 = I_3 = \gamma I$; $\gamma > 1$. Let $(\vec{i}_1, \vec{i}_2, \vec{i}_3)$ be the usual triad of orthogonal unit vectors at the center of mass with $\vec{i}_1$ pointing along the grain's axis of symmetry.

If equipartition of rotational energy applies then

$$\frac{1}{2} I \omega_1^2 = \frac{1}{2} \gamma I \omega_2^2 = \frac{1}{2} \gamma I \omega_3^2 \quad (4)$$

or

$$\omega_2 = \omega_3 = \omega_1 \gamma^{-1/2} \quad (5)$$

Hence the angular velocity is

$$\vec{\omega} = \omega_1 (\vec{i}_1 + \vec{i}_2 / \sqrt{\gamma} + \vec{i}_3 / \sqrt{\gamma}) \quad (6)$$

and the grain's angular momentum is

$$\vec{H} = I \omega_1 (\vec{i}_1 + \vec{i}_2 / \sqrt{\gamma} + \vec{i}_3 / \sqrt{\gamma}) \quad (7)$$

Accordingly for $\gamma > 1$ the angular velocity vector is biased toward the axis of symmetry while the angular momentum vector is biased away from it. In fact if ϕ is the angle between $\vec{\omega}$ and $\vec{H}$ one sees that

$$\cos\phi = 3(1 + 2\gamma)^{-1/2} (1 + 2/\gamma)^{-1/2} \quad (8)$$

and so for needle-like grains, as γ increases to large values, $\vec{\omega}$ tends to $\vec{i}_1\omega$ and ϕ tends to $\pi/2$ i.e. the grain tends to rotate about its axis of symmetry, but the angular momentum vector tends to be perpendicular to the axis of symmetry.

These observations can be extended from the single isolated grain case just discussed to the statistical situation by using the distributions (1) and (2). One can determine e.g. the distribution $f(\theta)$ of the angle θ between the angular velocity vector $\vec{\omega}$ and the axis of symmetry direction $\vec{i}_1$ as follows;

$$n(\omega_1) = (2\pi kT)^{-1/2} I^{1/2} \exp \{-I\omega_1^2/2kT\} \quad (9)$$

$$n(\omega_\perp) = 2\pi(2\pi kT)^{-1} \gamma I \omega_\perp \exp \{-\gamma I \omega_\perp^2/2kT\} \quad (10)$$

where $\omega_\perp$ is the component of $\vec{\omega}$ perpendicular to the axis of symmetry.

Thus

$$f(\theta) d\theta = \int_{-\infty}^{\infty} n(\omega_1) n(\omega_\perp) d\omega_1 d\omega_\perp \quad (11)$$

with

$$\omega_\perp = \omega_1 \tan\theta ; d\omega_\perp = \omega_1 \sec^2\theta d\theta \quad (12)$$

The result of the integration is

$$f(\theta) = \gamma \sin\theta / (\cos^2\theta + \gamma \sin^2\theta)^{3/2}$$

and hence the distribution function of θ in terms of the solid angle about the axis of symmetry is

$$F(\theta) \propto (\cos^2\theta + \gamma \sin^2\theta)^{-3/2} \quad (13)$$

This function is a maximum at $\theta = 0$ and a minimum at $\theta = \pi/2$.

Accordingly, as the single grain discussion showed, prolate ellipsoids in thermal equilibrium are most likely to be found rotating about their axes of symmetry.

We now examine the motion of a single prolate paramagnetic ellipsoid under the influence of a constant uniform magnetic field $\vec{B}$. Davis and Greenstein (1951) and Purcell (1969) showed that the torque, $\vec{L}$, which acts in the grain can be written in the form

$$\vec{L} = K V \vec{B} \times (\vec{B} \times \vec{\omega}) \quad (14)$$

where K is a positive constant related to the magnetic susceptibility, V is the volume of the grain and $\vec{\omega}$ its angular velocity. Consequently the rotational motion is governed by

$$\frac{d\vec{H}}{dt} = \frac{d\vec{H}}{dt_g} + \vec{\omega} \times \vec{H} = \alpha \vec{k} \times (\vec{k} \times \vec{\omega}) \quad (15)$$

in which the subscript g refers to time rate of change which is measured by an observer rigidly attached to the $\vec{i}_1 \vec{i}_2 \vec{i}_3$ axes in the grain and $\vec{B} = B\vec{k}$, $\vec{k}$ being a unit vector pointing in a fixed direction in space; the constant α must be positive. In discussing the motion we are confining attention here to the effects of the torque $\vec{L}$; the influence of torques produced by thermal fluctuations in the grain is neglected.

It follows directly from (15) that the rotational energy, E , of the grain satisfies

$$\frac{dE}{dt} = \vec{\omega} \cdot \vec{L} = -\alpha |\vec{k} \times \vec{\omega}|^2 \quad (16)$$

which exhibits the dissipative nature of the torque. Steady state solutions of (15) must be those for which the torque vanishes. From (14) it follows that the torque vanishes only if either $\vec{\omega}$ is parallel to $\vec{k}$ or $\vec{\omega} = 0$. The latter case is obviously not of interest. In the former the rotational energy E remains constant. But then

$$E = \frac{1}{2} \vec{H} \cdot \vec{\omega} = \frac{1}{2} (\vec{H} \cdot \vec{k}) \omega = \text{constant} \quad (17)$$

However, from (15) it follows that

$$\frac{d\vec{H}}{dt} \cdot \vec{k} = \frac{d}{dt} (\vec{H} \cdot \vec{k}) = 0 \quad (18)$$

and hence

$$\vec{H} \cdot \vec{k} = \text{constant} \quad (19)$$

is a constant of the motion in general. Thus when $\vec{\omega} = \omega \vec{k}$ it follows that the angular velocity vector is constant in magnitude and direction which implies by (15) that $\vec{\omega} \times \vec{H} = 0$. This equality can hold only if the grain is rotating about one of its principal axes of inertia. We conclude therefore that a steady state motion is one in which the grain is spinning with constant angular velocity about a principal axis and that axis is parallel to the magnetic field $\vec{B}$.

The energies of rotation in the steady states corresponding to rotation about the long axis or a short axis of the spheroid can be compared as follows. That for rotation about the long axis is

$$E_L = \frac{1}{2} I \omega_1^2 = \frac{1}{2} (\vec{H} \cdot \vec{k})^2 / I \quad (20)$$

while for the short axis one finds

$$E_s = \frac{1}{2} \gamma I \omega_2^2 = \frac{1}{2} (\vec{H} \cdot \vec{k})^2 / \gamma I \quad (21)$$

because in the steady state $\vec{\omega}$ is parallel to $\vec{k}$. Equation (19) shows that $\vec{H} \cdot \vec{k}$ is a constant of the motion. We may state therefore that for given initial angular momentum $\vec{H}$, the steady state in which the grain rotates about a short axis is one of lower rotational energy by a factor γ^{-1} than that in which rotation takes place about the long axis. Given that the rate of change of the rotational energy is negative except in the steady state configurations one expects that the ultimate configuration of a grain is that of minimum energy.

In summary, prolate spheroids in equilibrium at some temperature T , in the absence of external forces, are likely to be found spinning about their axes of symmetry with their angular momentum vectors tending to be perpendicular to those axes. On the other hand a paramagnetic prolate spheroid in the presence of a uniform constant magnetic field has as its lower energy steady state configuration that in which the angular velocity and the angular momentum are both parallel to one another, to a short axis and to the field direction. This distinction between the statistical behavior of a grain and its dynamic behavior in the presence of a magnetic field becomes the more pronounced as γ increases i.e. as the grain becomes more needle-like.

It should be understood that those conclusions are made without a detailed study of the stability of the higher energy steady state configuration or an analysis of the effects of random perturbing torques. In a further development of this note, it is hoped to examine both of these aspects of the general problem and also to estimate the pertinent relaxation times.

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