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NEW HALF-RANGE DIFFERENTIAL APPROXIMATION
FOR SPHERICALLY-SYMMETRIC RADIATIVE TRANSFER

by

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NEW HALF - RANGE DIFFERENTIAL APPROXIMATION FOR SPHERICALLY-SYMMETRIC RADIATIVE TRANSFER

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A new half-range differential approximation for radiative transfer with spherical symmetry is presented. The development is motivated by the various failures of existing approximations in determining emissive-power distributions and heat transfer for concentric-spheres problems. The new approach represents a modification of the four-moment double spherical-harmonics method, to which it reduces in the planar limit. The difference is effected by relocating the discontinuity of the assumed directional distribution of radiation intensity. The shift takes the discontinuity from precisely on the division between radially inward and radially outward, to just within the radially - outward directional half range. The method is tested on a variety of concentric spheres problems with and without internal heat sources, reproducing all the important features of the exact results. Over the range of problems treated, the new half-range method is shown to be more uniformly successful than any of the other approximations considered.

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NOMENCLATURE

B	=	Blackbody steradiancy, $\sigma T^4/\pi$
I	=	intensity of radiation
$I_n^{\pm}$	=	n^{th} $\pm$ half-range directional moments
I_n	=	n^{th} directional moment
K	=	volumetric absorption coefficient
N	=	exponent in assumed temperature distribution, Eq. (50)
q	=	Hopfs' function
Q	=	total rate of heat generation
Q_1	=	rate of heat transfer to inner wall, with internal heat generation
r	=	radial coordinate, Fig. 1
S	=	volumetric heat - generation rate
$\mathcal{S}$	=	S/K
T	=	temperature
x	=	dimensionless distance, $(r-r_a)/(r_b-r_a)$
$\underset{\sim}{y}$	=	(I_1^-, I_2^+, I_2^-)
δ_{on}	=	Kronecker delta
θ	=	azimuthal angle, Fig. 1
μ	=	$\cos\theta$
ξ	=	radius ratio, r_b/r_a
σ	=	Stefan - Boltzman constant
τ	=	optical coordinate, $Kr = [x+(\xi-1)^{-1}]\tau_L$
τ_L	=	optical thickness, $K(r_b-r_a)$

ϕ = dimensionless emissive power $(B_{wb} - B)/(B_{wb} - B_{wa})$, for no internal heat generation

ϕ_s = dimensionless emissive power $\pi B / S$, for uniform internal heat generation

SUPERSCRIPTS

(1) = first fundamental solution

(2) = second fundamental solution

(i) = inner expansion

(o) = outer expansion

SUBSCRIPTS

a = conditions at inner radius

b = conditions at outer radius

p = particular solution

w = surface condition

I. INTRODUCTION

Radiative energy transfer in spherically-symmetric media has become an increasingly studied and practical problem in recent years. Of particular interest in many applications is the determination of emissive power and radiation flux between concentric spheres, to be considered here. For this problem, with an absorbing- emitting gray gas, numerical solutions of the governing exact integral equations have been given both with internal heat generation (Sparrow, Usiskin and Hubbard[1]; Chisnell[2]), and without (Rhyming[3]; Viskanta and Crosbie[4]). Often however, the exact formulation is difficult to apply and solutions are prohibitively expensive. On the other hand, the approximate methods that have been developed have all met with various failures.

One of the earliest and perhaps best known approximations is the spherical-harmonics method, known in its lowest level as the differential approximation (Vincenti and Kruger[5]). The breakdown of the sperical--harmonics method in nonplanar geometries is well known (Chisnell[2]; Cess[6]; Dennar and Sibulkin[7]; Olfe[8]; Traugott[9]). While other approximations have been divised (Olfe[8]; Traugott[9]; Chou and Tien[10]; Hunt[11]), it appears from these references and the findings of the present study that none are uniformly successful in treating concentric -spheres problems.

From the stand point of simplicity and generality, Trougott's[9] "improved differential approximation" is of special interest. Working with the first four moment equations, Traugott retained the classical framework of the spherical-harmonics approach. In the classical approach, the directional-distribution of radiation intensity is expanded in spherical harmonics, and truncated to leave as many terms as the number of moment equations desired. This implies algebraic relations among the moments, which serve to close the system of moment equations. In Traugott's method the coefficients of the closure condition are modified for consistency with both isotropy and undirectional, radial beams (for this as well as any other closure scheme, many distributions may be consistent with the closure condition). The method avoids explicitly building in the shadow of the inner sphere, but does admit the singular distribution that occurs in the difficult transparent case with small inner sphere. For the general case, the method approximately accounts for the inner-sphere shadow. This scheme led, in Traugott's paper, to encouraging success when applied to the problem of radiation between concentric spheres without internal heat generation. In the present study, Traugott's method has been applied to problems with internal heat generation. Unfortunately, the new results, presented herein, are disappointing.

Nevertheless, the present approximation is in a certain sense in the same spirit as Traugott's. The key to the success of this new method however, is the half-range formulation, which for the cases considered also uncomplicates the boundary conditions and has other advantages. In the present paper this new differential approximation is developed, and tested on concentric-spheres problems for which exact and approximate solutions are available for comparison.

II FORMULATION

The spherically-symmetric system is shown in Fig. 1. The inner and outer walls are black, maintained at temperatures Twa and Twb, respectively. An absorbing and emitting gas fills the volume. For purposes of exposition only, the gas is assumed gray, characterized by a constant absorption coefficient K.

The intensity of radiation is a function of r and Θ only: $I = I(r,\Theta)$. Assuming local thermodynamic equilibrium and letting $\mu \equiv \cos\Theta$, the equation of transfer can be written, according to Chandrasekhar[12] (with the present sign convention),

$$\mu \frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = K\left[I(r,\theta) - B\right] \tag{1}$$

Here $B = \sigma T^4/\pi$ is the blackbody steradiancy, so the emission from the inner and outer walls is given by

$$I(r_a, \mu \le 0) = B_{wa} \quad , \quad I(r_b, \mu \ge 0) = B_{wb} \tag{2}$$

The general half - range moment of (1) is

$$(d/dr)\, I^{\pm}_{n+1} + (1/r)\left[\pm 2\pi \delta_{0n} I(r,0) - n I^{\pm}_{n-1} + (n+2) I^{\pm}_{n+1}\right] = K\left[I^{\pm}_n - (\mp 1)^n 2\pi B/(n+1)\right] \tag{3}$$

where $n \ge 0$ and the half - range directional moments are defined by

$$I^{\pm}_n(r) \equiv \mp 2\pi \int_0^{\mp 1} \mu^n\, I(r,\mu)\, d\mu \tag{4}$$

The half - range moments add directly to give the full-range moments: $I_0 = I_0^+ + I_0^-$, the average intensity; $I_1 = I_1^+ + I_1^-$, the radiation flux positive inward; and so on. The equations (3) as they stand, constitute an infinite system completely equivalent to (1). The object now is to truncate the system by writing an approximate closure condition.

Closed systems of half-range moment equations have been obtained by the so - called double spherical-harmonics method, introduced by Yvon[13] in neutron--transport analysis, and applied in radiative transfer by Le Sage[14]. Yvon's method mimics the conventional spherical-harmonics approach, but with independent expansions of the intensity on each directional half range. This allows discontinuities at $\mu=0$. The first and second approximations for $I(r,\mu)$ are among those illustrated in Fig. 2. The first is seen to be equivalent to the Schuster-Schwarzchild approximation (Chandrasekhar[15]) while the second embodies obvious first corrections. Le Sage's work[14] proved the utility of the method in planar situations; however, as Traugott[9] noted, the approximation is not useful for problems with curvature. At least part of the difficulty has to do with the symmetrical allocation of the discontinuity of $I(r,\mu)$, as shown in Fig. 2.

It will now be demonstrated that by moving the discontinuity from $\mu = 0$ to $\mu = 0-$, the structure of the truncated system can be significantly altered. The rationale of this move is that a discontinuity is known to arise on the half range $0 > \mu \geq -1$, because of the inner-sphere shadow. The move allocates the assumed discontinuity <u>fully</u> to this half range, as shown in Fig. 2.

Before proceeding, it should be mentioned that the other methods referenced in the introduction share this interest in dealing with the inner-sphere shadow. In Olfe's modified differential approximation[8], the radiation from the walls is treated exactly, while only the gas emission is handled by the differential approximation. However, this means that in problems with cold walls, the method reduces to just the differential approximation, with its attendent difficulties. The two-and three-region averaging methods developed by Chou and Tien[10], and Hunt[11], explicitly build shadows into full-range formulations. The two-region method assumes that the intensity distribution is constant on each of $1 \geq \mu > [1-(r_a/r)^2]^{1/2}$ and $[1-(r_a/r)^2]^{1/2} > \mu \geq -1$.

The direction dividing the two regions just corresponds to the edge of the inner-sphere shadow. The three-region method allows an additional discontinuity at $\mu = 0$. Despite the fact that these distributions force the exact result in the transparent limit, both methods still exhibit serious shortcomings[11]. Finally, Traugott's method and its relation to the shadow have already been discussed in the introduction.

Continuing the formulation, four moment equations will be retained in the present approximation. The assumed distribution, with the dislocated jump, is

$$I(r,\mu) = \begin{cases} A^+(r) + C^+(r)\mu & (0 > \mu \geq -1) \\ A^-(r) + C^-(r)\mu & (1 \geq \mu \geq 0) \end{cases} \qquad (5)$$

as shown in Fig. 2. Inserting (5) in (4) for $n = 0$, 1 and 2, and eliminating $A^{\pm}(r)$ and $C^{\pm}(r)$, one finds the closure

$$I_0^{\pm} = -6\left(I_2^{\pm} \pm I_1^{\pm}\right) \qquad (6)$$

and, since $A^-(r) = I(r,0)$,

$$2\pi I(r,0) = 6(3I_1^- - 4I_2^-) \tag{7}$$

In the usual double spherical-harmonics method, the discontinuity is located at $\mu = 0$ rather than 0- . This change only affects (7), which appears in the curvature term in (3). Thus in the planar limit, the present system is precisely equivalent to LeSages' "double P1 approximation". Using the above closure conditions in (3), one obtains

$$\left.\begin{aligned}
(d/dr)I_1^+ + (1/r)\left[2(I_1^+ + I_1^-) - 8(3I_2^- - 2I_1^-)\right] &= -K\left[6(I_1^+ + I_2^+) + 2\pi B\right] \\
(d/dr)I_1^- + (1/r)\left[8(3I_2^- - 2I_1^-)\right] &= K\left[6(I_1^- - I_2^-) - 2\pi B\right] \\
(d/dr)I_2^+ + (1/r)\left[3(3I_2^+ + 2I_1^+)\right] &= K\left[I_1^+ + \pi B\right] \\
(d/dr)I_2^- + (1/r)\left[3(3I_2^- - 2I_1^-)\right] &= K\left[I_1^- - \pi B\right]
\end{aligned}\right\} \tag{8}$$

Using the full-range moments and (6), the first two of (8) combine to regain the exact flux-conservation equation,

$$(1/r^2)(d/dr)(r^2 I_1) = K(I_0 - 4\pi B) \tag{9}$$

The boundary conditions for (8) are obvious, using (2) and (3):

$$I_1^+(r_a) = -\pi B_{wa} \qquad I_2^+(r_a) = \tfrac{2}{3}\pi B_{wa}$$
$$I_1^-(r_b) = \pi B_{wb} \qquad I_2^-(r_b) = \tfrac{2}{3}\pi B_{wb} \tag{10}$$

In the remainder of this paper, equations (6), (8), (9) and (10) are applied to a number of concentric - spheres problems. Along the way, Traugott's method is also applied, to obtain several new results. This is of special interest, since Traugott considered only the problem without internal heat generation. Discussion of results will mainly be deferred to Section V.

III ZERO OR UNIFORM HEAT GENERATION BETWEEN CONCENTRIC SPHERES

For a stationary gas in which heat generation proceeds at the rate S per unit volume, considering only radiative transfer, the energy balance requires

$$(1/r^2)(d/dr)(r^2 I_1) = K(I_0 - 4\pi B) = -S \tag{11}$$

The minus sign preceeding S is a result of the sign convention for I_1.
Two classes of problems will be considered:

1.) Bwb $\neq$ Bwa = 0, S = 0 (no internal heat generation)

2.) Bwb = Bwa = 0, S = Constant (uniform internal heat generation)

Because of the linearity of the governing equations, one can superpose on either case the trivial problem Bwa = Bwb $\neq$ 0 and S = 0, so there is no loss of generality in taking Bwa = 0. Defining a new independent variable $x = (r - r_a)/(r_b - r_a)$ and the parameters $\mathcal{S} \equiv S/K$, $\tau_L \equiv K(r_b - r_a)$, $\xi \equiv r_b/r_a$, an exact integral of (11) can be expressed as

$$I_1^+ + I_1^- = I_1 = \left[I_1(0) + \tfrac{1}{3}\mathcal{S}\tau_L/(\xi-1)\right]\left[(\xi-1)x+1\right]^{-2} - \left[\tfrac{1}{3}\mathcal{S}\tau_L/(\xi-1)\right]\left[(\xi-1)x+1\right] \qquad (12)$$

This replaces the first of the moment equations (8). From (11), using (6), one finds

$$\pi B = \tfrac{1}{4}(I_0 + \mathcal{S}) = \tfrac{3}{2}(I_1^- - I_2^- - I_1^+ - I_2^+) + \tfrac{1}{4}\mathcal{S} \qquad (13)$$

which can be used to eliminate πB from the three remaining moment equations:

$$\frac{dI_1^-}{dx} + 8\,\frac{3I_2^- - 2I_1^-}{x + 1/(\xi-1)} = \tau_L\left[3I_1 - \tfrac{1}{2}\mathcal{S} + 3(I_2^+ - I_2^-)\right]$$

$$\frac{dI_2^+}{dx} + 3\,\frac{3I_2^+ - 2I_1^- + 2I_1}{x + 1/(\xi-1)} = \tau_L\left[-\tfrac{1}{2}I_1 + \tfrac{1}{4}\mathcal{S} + 2I_1^- - \tfrac{3}{2}(I_2^+ + I_2^-)\right] \qquad (14)$$

$$\frac{dI_2^-}{dx} + 3\,\frac{3I_2^- - 2I_1^-}{x + 1/(\xi-1)} = \tau_L\left[\tfrac{3}{2}I_1 - \tfrac{1}{4}\mathcal{S} - 2I_1^- + \tfrac{3}{2}(I_2^+ + I_2^-)\right]$$

The boundary conditions (10) complete the specification of the problem.

In the planar limit ($\xi = 1$), closed - form solutions of Eqs. (10), (12) and (14), in terms of the radiative flux and emissive power, are found without difficulty. For $\mathcal{S} = 0$, Le Sage[14] has given

$$I_1 = \pi (B_{wb} - B_{wa}) \frac{4}{3} \left\{ \tau_L + 1 + \left[1-(3)^{-1/2}\right] \frac{1-\left[2-(3)^{1/2}\right] e^{-(12)^{1/2}\tau_L}}{1-\left[7-4(3)^{1/2}\right] e^{-(12)^{1/2}\tau_L}} \right\}^{-1} \qquad (15)$$

$$\Phi \equiv (B_{wb} - B)/(B_{wb} - B_{wa}) = P(x, \tau_L) \,/\, R(\tau_L) \qquad (16)$$

where

$$P(x, \tau_L) = \tau_L \left(x - \tfrac{1}{2}\right)\left\{1 - \left[7-4(3)^{1/2}\right] e^{-(12)^{1/2}\tau_L}\right\} + \left[1 - \tfrac{1}{2}(3)^{1/2}\right]\left[e^{-(12)^{1/2}\tau_L(1-x)} - e^{-(12)^{1/2}\tau_L x}\right]$$

$$R(\tau_L) = 2 - (3)^{-1/2} - \tau_L - \left[2 + (3)^{-1/2} + \tau_L\right]\left[7 - 4(3)^{1/2}\right] e^{-(12)^{1/2}\tau_L}$$

For $\mathcal{S} \neq 0$, the equations yield, trivially,

$$I_1 = -\tau_L \mathcal{S} \left(x - \tfrac{1}{2}\right) \qquad (17)$$

and after straight-forward but lengthy manipulation,

$$\Phi_s \equiv \frac{\pi B}{\mathcal{S}} = \frac{1}{4} + \frac{1 + 2\tau_L + 3\tau_L^2 x(1-x)}{8}$$

$$- \frac{\tau_L + 2}{8} \, \frac{3\left[\sinh (12)^{1/2}\tau_L x + \sinh (12)^{1/2}\tau_L (1-x)\right] - \sinh (12)^{1/2}\tau_L}{4 \sinh (12)^{1/2}\tau_L + (12)^{1/2}\left[\cosh (12)^{1/2}\tau_L - 1\right]} \qquad (18)$$

Next, the nonplanar case is considered. The radiative flax then varies with position between the walls for both $\mathcal{S} = 0$ and $\mathcal{S} \neq 0$. However, the dependence on position is a known function, given by Eq. (12). Thus, for purposes of comparison, only $I_1(0)$ in (12) need be presented. For the case with internal heat generation, $I_1(0)$ will be presented in terms of the fraction of internally - generated heat transferred to the inner wall, as in Ref. 7:

$$\frac{Q_1}{Q} \equiv \frac{4\pi r_a^2 I_1(0)}{\frac{4}{3}\pi(r_b^3 - r_a^3) S} = \frac{3 I_1(0)}{(\xi^2 + \xi + 1)\tau_L \mathcal{S}} \tag{19}$$

Before proceeding to the numerical treatment of Eqs. (10), (12) and (14) for the nonplanar cases, closed-form limiting solutions are considered.

Transparent limit

The solutions of Eqs. (10), (12) and (14) can be developed as power series in $\tau_L \ll 1$. Thus one obtains, without complication, for $\mathcal{S} = 0$,

$$I_1(0) = \pi(B_{wb} - B_{wa}) + O(\tau_L) \tag{20}$$

$$\phi = (3/14)(r_a/r)^2 + (2/7)(r_a/r)^9 + O(\tau_L) \tag{21}$$

and for $\delta \neq 0$,

$$Q_1/Q = \tfrac{1}{2}\left[(1-\xi^6) + (9/7)(\xi^7-1)\right]\left[\xi^6(\xi^3-1)\right]^{-1} + O(\tau_L) \tag{22}$$

$$\phi_s = \tfrac{1}{4} + O(\tau_L) \tag{23}$$

Corresponding exact results are, for $\delta = 0$ (Rhyming[3]),

$$I_1(0) = \pi(B_{wb} - B_{wa}) + O(\tau_L) \tag{24}$$

$$\phi = \tfrac{1}{2} - \tfrac{1}{2}\left[1 - (r_a/r)^2\right]^{1/2} + O(\tau_L) \tag{25}$$

and for $\delta \neq 0$ (Viskanta and Crosbie[4]),

$$Q_1/Q = \tfrac{1}{2} - \tfrac{1}{2}(\xi^2-1)^{3/2}(\xi^3-1)^{-1} + O(\tau_L) \tag{26}$$

$$\Phi_s = \tfrac{1}{4} + O(\tau_L) \tag{27}$$

Equation (26), although not given explicitly in Ref. 4, is easily extracted. One can perform the same analysis with Traugott's differential approximation. The results for $\delta = 0$, from Traugott's paper[9], are

$$I_1(0) = \pi(B_{wb} - B_{wa})\tfrac{1}{3}[(5)^{1/2}+1]\,\xi^2(\xi^2+1)(\xi^4+1)^{-1} + O(\tau_L) \tag{28}$$

$$\Phi = (\xi^4+1)^{-1}\left\{1 + \tfrac{1}{3}\xi^2[\xi^2-1][r_a/r]^2[(5)^{1/2}-1]^{-1}\right\} + O(\tau_L) \tag{29}$$

For $\delta \neq 0$, the present application of Traugott's method yields

$$\frac{Q_1}{Q} = \frac{[(5)^{1/2}-7] + 2\xi^2 + [5-(5)^{1/2}]\xi^3 - [5-(5)^{1/2}]\xi^4 + 2\xi^5 + [3-(5)^{1/2}]\xi^7}{2(5)^{1/2}[(5)^{1/2}-1](\xi^3-1)(\xi^4+1)} + O(\tau_L) \tag{30}$$

$$\Phi_s = \tfrac{1}{4} + O(\tau_L) \tag{31}$$

Opaque limit

It is also possible to develop asymptotic solutions in terms of $1/\tau_L \ll 1$, when $\tau \equiv [x + 1/(\xi - 1)]\,\tau_L$ is everywhere large. A look at the planar results, Eqs. (15)-(18), previews some of the features of the asymptotic solutions. First of all, uniformly - valid expansions of ϕ and ϕ_s in $1/\tau_L \ll 1$ are not possible beyond the first approximation because of the functional dependence on exp $[(12)^{1/2}\tau_L x]$.
This gives rise to boundary layers of $O(1/\tau_L)$- thickness at the walls.
For this same reason one finds, secondly, that simple power series in $1/\tau_L$ are not complete; the separate expansions at each wall and in the region away from the walls all exhibit exponentially - small terms, as well.
From the expression for the flux integral, (12), it is clear that a uniformly valid expansion for that quantity is possible, although again, exponentially - small terms arise.

These features carry over to the nonplanar analysis. Ignoring exponentially - small terms, application of the method of matched asymptotic expansions gives, for $\delta = 0$, the uniformly - valid result

$$I_1(0) = \pi(B_{wb} - B_{wa})\tfrac{4}{3}(1/\tau_L)[\xi - C(1+\xi^2)/\tau_L + O(1/\tau_L^2)] \quad (32)$$

and the composite solution

$$\phi = (r_a/r)[(1-x) - f(\xi, x)/\tau_L + O(1/\tau_L^2)] \quad (33)$$

where

$$C = 1 - \tfrac{1}{2}(3)^{-1/2} = .711325 \quad (34)$$

$$f = C\left\{\xi - [1+\xi]x + [1/11][9-4(3)^{1/2}][1+(\xi-1)x]\left[\xi^{-1}e^{-(12)^{1/2}\tau_L(1-x)} - \xi e^{-(12)^{1/2}\tau_L x}\right]\right\} \quad (35)$$

Emanuel's asymptotic expansion[16] of the exact integral equations gives the same form as (32) and (33), but with

$$C = q(\infty) = .710447 \quad (36)$$

$$f = C\left\{\xi - [1+\xi]x + 1 - \xi[1 - q(\tau_L x)/q(\infty)] - q[\tau_L(1-x)]/q(\infty)\right\} \quad (37)$$

where q is Hopf's function[16], which varies monotonically from $q(0) = (3)^{-1/2}$ to $q(\infty) = .710447$.

At the walls f can be expanded in $1/\tau_L$ to show that the present and exact results for ϕ agree to $O(1/\tau_L^2)$. Far from the walls the very close agreement is obvious by inspection. Results using the differential approximation are also available[16]. Expanding them, one finds again, (32) and (33), but with

$$C = \frac{2}{3} \tag{38}$$

$$f = C\left[\xi - (1+\xi)x\right] \tag{39}$$

This result for f does not exhibit the boundary - layer structure seen in either (35) or (37). All three methods however, yield identical leading terms, which constitute, as Emanuel[16] has indicated, the Rosseland approximation:

$$I_1 \approx \frac{4}{3}\pi(d/d\tau)B \tag{40}$$

For the heat-generating case, only the leading term in the expansion of the exact system has been given, in Viskanta and Crosbie[4]. Expanding the present system yields

$$\phi_s = \tau_L^2\left\{\frac{1}{8}x[1-x][1+(r_a/r)(1+\xi)] + O(1/\tau_L)\right\} \tag{41}$$

which is identical with the exact result. Again, to lowest order, the Rosseland approximation (40) holds, so the flux can be calculated exactly from (41):

$$Q_1/Q = \tfrac{1}{2}(\xi^2+\xi-2)(\xi^3-1)^{-1} + O(1/\tau_L) \qquad (42)$$

Mixed regime

A third limiting case occurs when the gas near the inner sphere is optically thin, while the volume overall is optically thick. The requisite conditions are $\xi \gg \tau_L \gg 1$. Emanuel[17] has also analyzed this situation, for $\gamma = 0$, using the exact integral equations and matched asymptotic expansions.

The present system of equations, (10), (12) and (14), written in terms of the inner variable r/r_a , exhibits one parameter, $\tau_L/(\xi-1)$. The inner expansion in $\tau_L/(\xi-1) \ll 1$ gives, independently of the outer expansion,

$$\Phi^{(i)} = (3/14)(r_a/r)^2 + (2/7)(r_a/r)^9 + O[\tau_L/(\xi-1)] \qquad (43)$$

and the uniformly-valid result

$$I_1(0) = \pi(B_{wb}-B_{wa})\{1-.259\,\tau_L/(\xi-1) + O[\tau_L^2/(\xi-1)^2]\} \qquad (44)$$

The appropriate outer variable is $X\tau_L$. The equations again exhibit the parameter $\tau_L/(\xi-1)$, but now offer little hope of yielding an analytical solution suitable for matching purposes. One can obtain part of the outer expansion, valid for large distances form the inner sphere, by integrating the optically - thick result (40), using (44):

$$\phi^{(0)} = \frac{3}{4}\frac{r_a}{r}\frac{\tau_L}{\xi-1}\left\{1 + O\left(\frac{\tau_L}{\xi-1}\right)\right\} + O\left(\frac{1}{X\tau_L}\right) \qquad (45)$$

Equations (43), (44) and (45) are identical with the exact results, with the exception[17] that in the exact version of Eq. (44), .259 is replaced by 1/3.

Numerical solution

Apart from extereme - parameter cases, Eqs. (10), (12) and (14) are easily integrated numerically, using the Adams - Moulten scheme. Substituting (12) into (14), there are actually three first-order differential equations, for I_1^- , I_2^+ and I_2^- . Aside from the parameters ξ and τ_L , $I_1(0) = I_1^+(0) + I_1^-(0)$ also appears in the equations.

When $\delta = 0$, normalizing the dependent variables by $I_1(0)$, the boundary conditions (10) can be written

$$I_1^-(0) = 1 \qquad I_2^+(0) = 0$$
$$I_1^-(1) = \tfrac{3}{2} I_2^-(1) \qquad I_1^-(1) = \pi B_{wb} / I_1(0) \tag{46}$$

where $I_1^+(0) = 0$ has been rewritten in terms of $I_1^-(0)$. Letting $\underset{\sim}{y} = (I_1^-, I_2^+, I_2^-)$, two fundamental solutions, defined by

$$\underset{\sim}{y}^{(1)}(0) = (1,0,0)$$
$$\underset{\sim}{y}^{(2)}(0) = (0,0,1) \tag{47}$$

are required, Superposing, $\underset{\sim}{y} = \underset{\sim}{y}^{(1)} + D\underset{\sim}{y}^{(2)}$ satisfies the conditions at $X = 0$. Then D and $I_1(0)$ are determined by the conditions at $X = 1$. Carrying out the superposition and multiplying through by $I_1(0)$ completes the solution.

When $\delta \neq 0$, the boundary conditions are

$$I_1^+(0) = I_2^+(0) = I_1^-(1) = I_2^-(1) = 0 \tag{48}$$

A particular integral for the problem is defined by the initial conditions $\underset{\sim}{y}_p(0) = \underset{\sim}{0}$, which with (48) imply $I_1(0)_p = 0$. In order to satisfy the outer - wall conditions, a solution of the homogeneous system ($\delta = 0$), satisfying

$$I_1^+(0) = I_2^+(0) = 0$$

$$I_1^-(1) = -I_1^-(1)_p \qquad (49)$$

$$I_2^-(1) = -I_2^-(1)_p$$

is superposed on the particular integral. This complementary solution is obtained in the manner described in the preceding paragraph.

IV FIXED TEMPERATURE DISTRIBUTION BETWEEN CONCENTRIC SPHERES

Another problem in which internal heat generation occurs is posed by prescribing the temperature distribution, instead of the heat-source distribution. Chisnell[2] has treated this problem both exactly and approximately, with the object of determining the inner-sphere heat transfer for cold, black walls He prescribed (N = 0, 1, 2)

$$B = x^N \tag{50}$$

The present, approximate system, Eqs. (8) and (30), partially decouple into two pairs of equations, only one pair of which are needed to compute the inner-sphere flux. This pair is

$$\begin{aligned} \frac{dI_1^-}{dx} + 8\,\frac{3I_2^- - 2I_1^-}{x + 1/(\xi - 1)} &= \tau_L\left[6(I_1^- - I_2^-) - 2x^N/\pi\right] \\ \frac{dI_2^-}{dx} + 3\,\frac{3I_2^- - 2I_1^-}{x + 1/(\xi - 1)} &= \tau_L\left[I_1^- - x^N/\pi\right] \end{aligned} \tag{51}$$

The boundary conditions, from (10), are:

$$I_1^-(1) = I_2^-(1) = 0 \tag{52}$$

In the planar case $(\xi = 1)$, the solution of this system, for $N \geq 0$ gives

$$I_1(0)/\pi = a_0 + \left\{ \frac{1}{(3)^{1/2}} + \frac{1}{2}\sum_{j=0}^{N} \left[\frac{j}{(3)^{1/2}\tau_L} - (3)^{1/2} - 1 \right] a_j \right\} \left\{ e^{-[3-(3)^{1/2}]\tau_L} - e^{[3+(3)^{1/2}]\tau_L} \right\}$$

$$- \sum_{j=0}^{N} a_j e^{-[3+(3)^{1/2}]\tau_L} \qquad (53)$$

$$a_N = 1 \, , \quad a_{N-1} = \tfrac{2}{3} N/\tau_L \; ; \quad a_j = [(j+1)/\tau_L][a_{j+1} - \tfrac{1}{6}(j+2)a_{j+2}/\tau_L] \, ,$$

$$j \leq N-2$$

In the general case $(\xi \geq 1)$ the equations (31) are reducible to the confluent hypergeometric equation. However, the coefficients in the equation are not convenient numerical values. On the other hand, for the range of parameters of interest here, this initial-value problem can be numerically integrated with dispatch.

Closed - form solutions of Traugott's system for Chisnell's problem can be found, although the algebra gets tedious. Only the case N = 0 has been solved here, and the result for the inner-wall flux is too involved to present, except in an important limiting form, as $\xi \to \infty$:

$$I_1(0)/\pi \sim \tfrac{1}{6}\xi^2 [1+(5)^{1/2}] \; Y(\tau_L) / Z(\tau_L)$$

$$Y(\tau_L) = 25[1-(5)^{1/2}]\tau_L^2 e^{\tau_L} + \{6 - [15 - 9(5)^{1/2}]\tau_L\} e^{-(5)^{1/2}\tau_L}$$

$$- \{6 - [15 + 3(5)^{1/2}]\tau_L + 5[(5)^{1/2} - 1]\tau_L^2\} e^{(5)^{1/2}\tau_L} \qquad (54)$$

$$Z(\tau_L) = [3 - 4(5)^{1/2}\tau_L + 10\tau_L^2] e^{[1+(5)^{1/2}]\tau_L} - [3 + 2(5)^{1/2}\tau_L] e^{-[(5)^{1/2}-1]\tau_L}$$

V DISCUSSION

In this section, comparisons are made between the exact results and the results of the present and various other approximate methods. Those cases where differences are sharpest are emphasized. Additional results are available from the author.

No heat generation

First of all, the results of all the methods considered agree in the opaque limit. Some differences in first corrections for finite opacity, given in Eqs. (32) - (39), have already been discussed.

In the important transparent limit, a single curve describes the exact variation of emissive power with r/r_a , as shown in Fig. 3. Coincident with this are the limiting results from Olfe's modified differential approximation[8], and from the regional averaging methods[10,11]. Traugott's result, Eq. (29), appears in Fig. 3 as a family of curves, due to a spurious dependence on radius ratio ξ . Also dependent on ξ , and much less accurate, are the spherical-harmonics results, not shown. Finally, the limiting result of the present method, Eq. (21), is plotted in Fig. 3. While not exact, it <u>is</u> a single curve, with correct inner-wall temperature slip, and reasonable accuracy elsewhere.

The heat flux at the inner sphere is just $\pi(B_{wb} - B_{wa})$, in the exact transparent limit. Again, not all of the approximations yield the correct limiting forms. Traugott's result, Eq. (28), is in error by a factor ranging from $\frac{1}{3}[1+(5)^{1/2}]$ to $\frac{1}{2}[1+(2)^{1/2}]$. In the spherical - harmonics result[9], the factor can exceed 2. The regional averaging methods[10,11] give the exact result, but the present result, Eq.(20), is also exact, and without having built in an inner-sphere shadow.

Figures 4 and 5 display some numerical results that indicate the behavior of the present approximation for intermediate values of opacity and radius ratio.

Uniform heat generation

For this case also, the important differences among results are in the optically - thin regime. The transparent limit of the heat-transfer ratio Q_1 / Q, is shown in Fig. 6. Here, in contrast with the case of no internal heat generation, it can be seen that none of the approximations yield the exact limit. A surprising result is the performance of Traugott's method, Eq. (30): as the radius ratio tends to infinity, the heat-transfer ratio tends to a finite value, .138, instead of the correct result, which is zero. The differential approximation, to which Olfe's method reduces here, commits errors of to 50 per cent[11].

The two -region method, not shown, as much as doubles this error[11]. The three-region method breaks down in the planar limit, as discussed by Hunt[11]. Finally, the limiting result from the present method, Eq. (22) is plotted in Fig. 6, indicating a clear improvement in the prediction of Q_1/Q. This improvement, it turns out, is independent of the form assumed for I on $0 > \mu \geq -1$, once continuity has been assumed over $1 \geq \mu \geq 0$. The same is true of the transparent equilibrium heat-transfer result. For the other quantities, the present method might be reconstructed to achieve somewhat better accuracy, by explicitly building in the inner-sphere shadow. The price of course would be loss of generality.

Some numerical results for $\tau_b = 1$ are also presented in Fig. 6, to illustrate the trend with increasing opacity.

The dimensionless emissive power ϕ_s, in the transparent limit, is just the trivial result $\frac{1}{4}$, by all the methods. For finite opacity, one useful plot comparing some approximate and exact results can be found in Hunt's paper[11]. These are partially repeated here in Fig. 7, to which the present results have been added. The comparison favors the 3-region method near the inner sphere, and the present method near the outer sphere. A more detailed comparison shows the advantage going to the present method as the geometry becomes more planar.

Fixed temperature distribution

Chisnell[2] has compared the exact and two approximate predictions for inner-sphere flux, for various fixed temperature distributions and radius ratios. One of his approximations was simply the planar integral formula. The other method was the differential approximation, to which Olfe's method[8] again reduces here.

In Fig. 84, Chisnell's results are presented for the large radius-ratio case ($\xi = 11$) with uniform temperature. The present results are also shown, and clearly are superior to those of the approximations considered by Chisnell. The order of occuracy indicated was found for all the cases considered by Chisnell. The comments concerning the unimportance of the assumed form for I on $-1 \leq \mu < 0$, which can be found in the discussion of the transparent limit of Q_1/Q, also hold for Chisnell's problem.

In order to plot the closed-form result found using Traugott's approximation, the exact and present results are repeated on a reduced scale in Fig. 8b. The unexpected breakdown of Traugott's method, again for a problem with internal heat generation, is clearly illustrated.

CONCLUSION

Motivated by the various failures of existing approximations in concentric - spheres problems, a new differential approximation for radiative transfer with spherical symmetry has been presented. The approximation is closely related to the double P1 (spherical harmonics) method, described by LeSage[14], to which it reduces in the planar limit. The difference between the two is the result of a single, crucial change. In the new method, the discontinuity in the assumed directional-distribution of intensity is shifted, to lie just within the radially outward directional half range.

The new approximation has been applied to a variety of concentric-spheres problems with and without internal heat generation, reproducing all the essential behavior of the exact solutions. Along the way, Traugott's approximation has been applied for the first time to cases with internal heat generation, uncovering important defects in that method. Over the range of problems treated, the new method has been more consistently satisfactory than any of the other approximations considered.

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CAPTIONS

Fig. 1	Concentric - spheres geometry
Fig 2.	Directional distributions of intenstity of radiation
Fig. 3	Variation of emissive power with radius. Transparent limit, no heat generation.
Fig. 4	Variation of emissive power with position between spheres. No heat generation, optical thicknesses and radius ratios as indicated.
Fig. 5	Variation of inner-sphere heat flux with optical thickness. No heat generation, radius ratios as indicated.
Fig. 6	Variation of inner-sphere heat flux with radius ratio. Uniform heat generation, transparent limit except where indicated.
Fig. 7	Variation of emissive power with radius. Uniform heat generation, optical thickness $\tau_L = .095$.
Fig. 8a	Variation of inner-sphere heat flux with optical thickness. Uniform temperature (N=0), radius ratio $\xi = 11$.
Fig. 8b	Reduction of Fig. 8a, in order to include the result from Traugott's approximation.

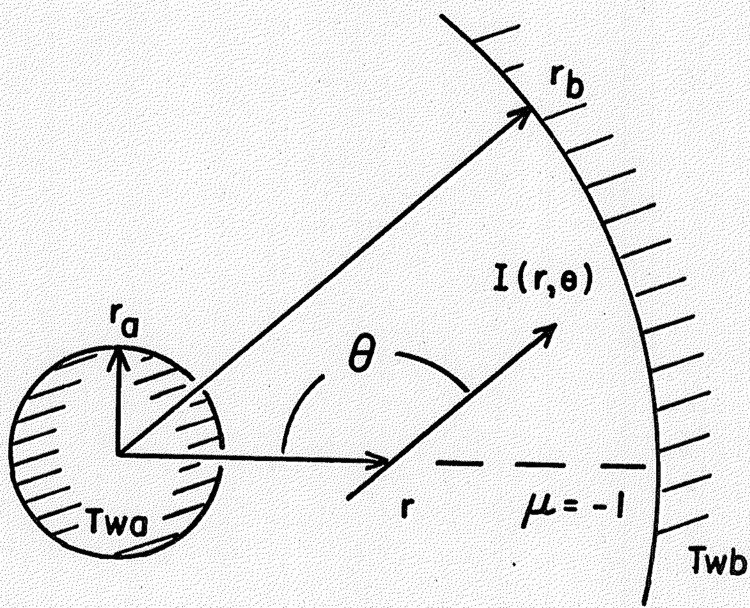


Figure 1

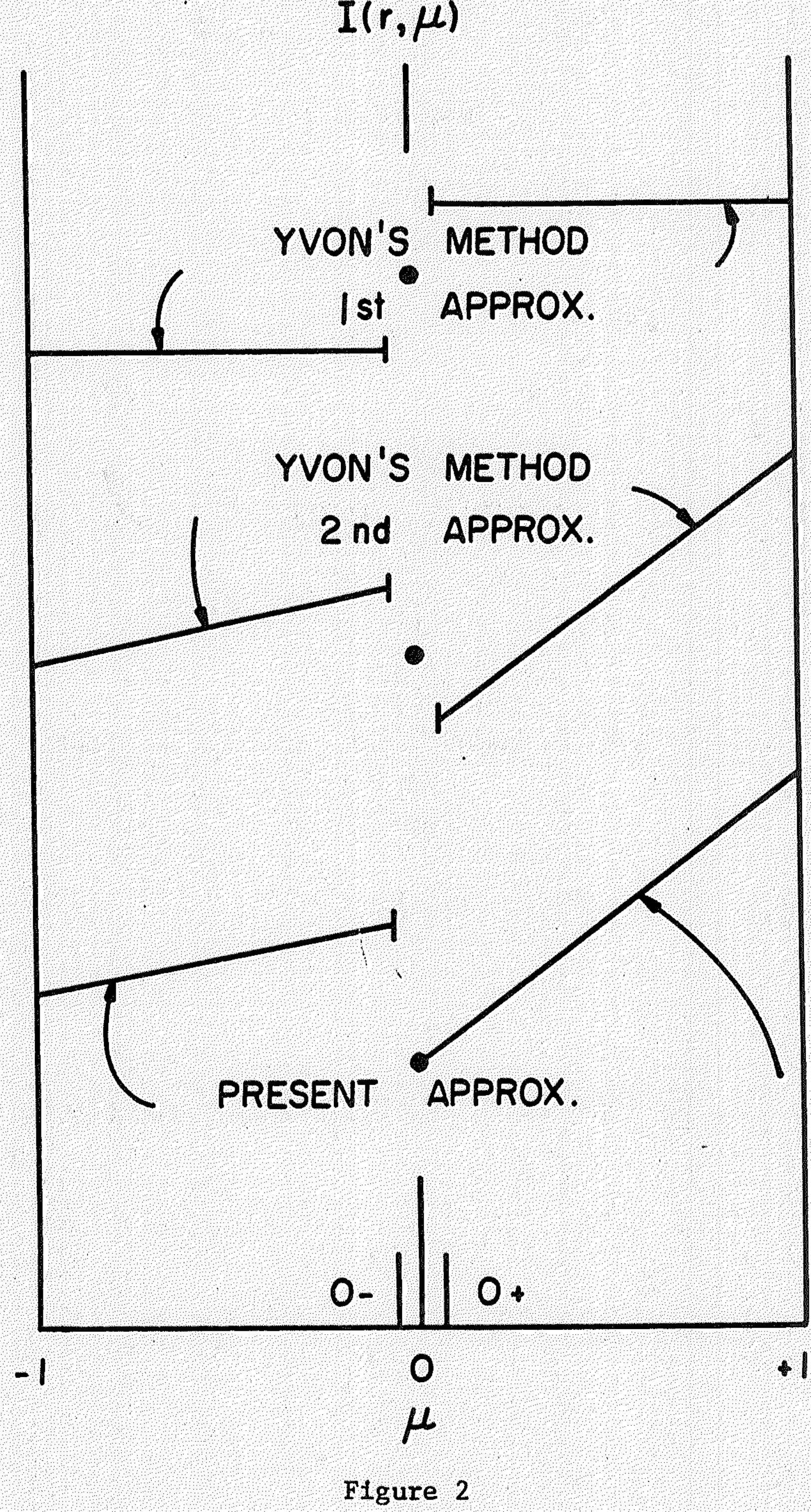
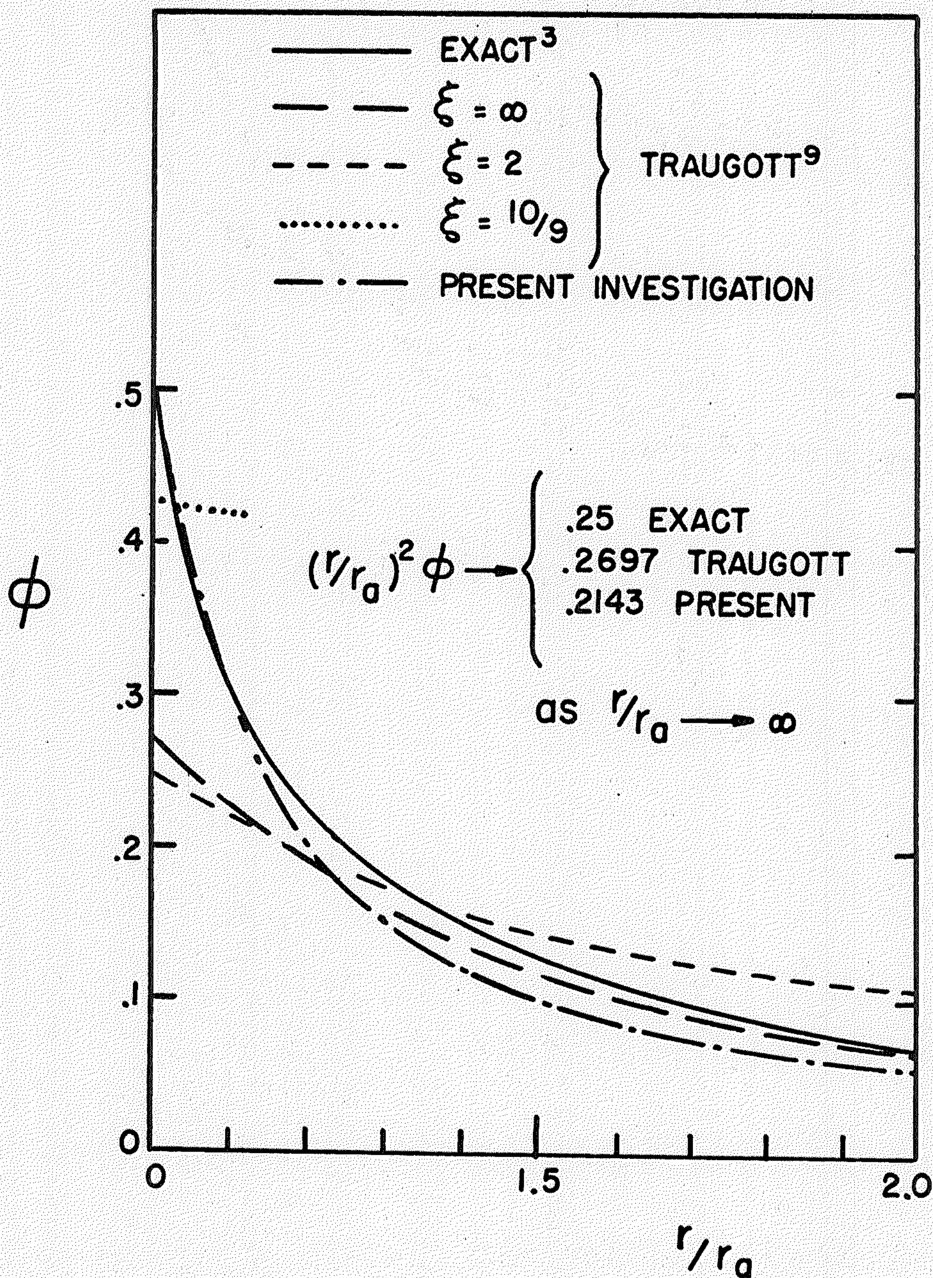

I(r, μ)
YVON'S METHOD
1st APPROX.
YVON'S METHOD
2nd APPROX.
PRESENT APPROX.
0-
0+
-1
0
+1
μ

Figure 2



EXACT3
ξ = ∞
ξ = 2
ξ = 10/9
TRAUGOTT9
PRESENT INVESTIGATION
$(r/r_a)^2 \phi \rightarrow$
.25 EXACT
.2697 TRAUGOTT
.2143 PRESENT
as $r/r_a \rightarrow \infty$
φ
.5
.4
.3
.2
.1
0
0
1.5
2.0
r/r_a

Figure 3

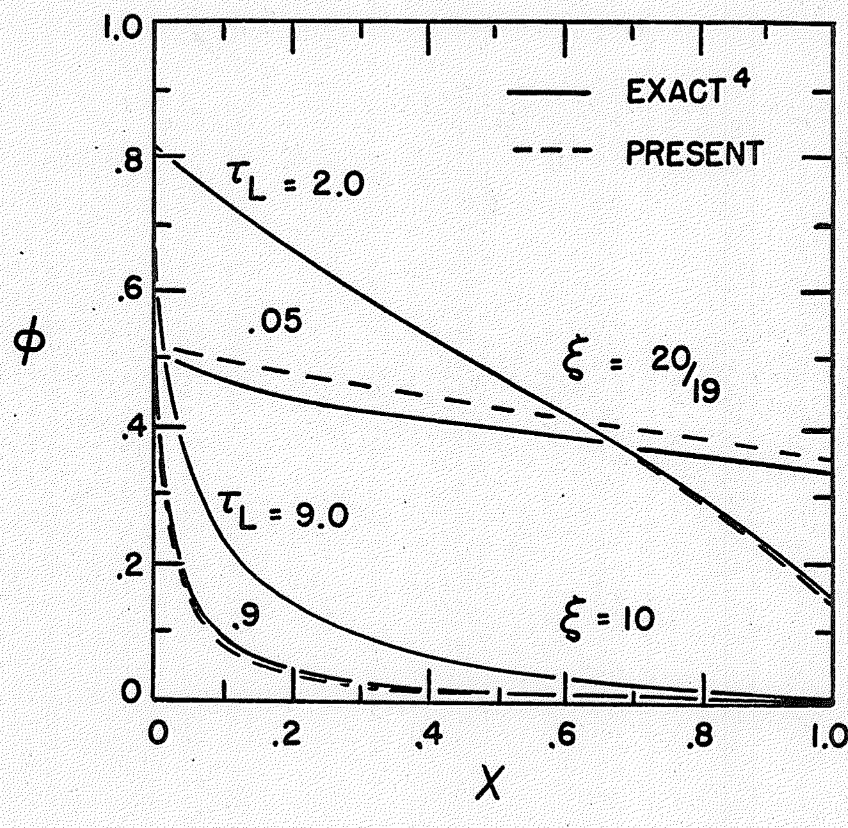


Figure 4

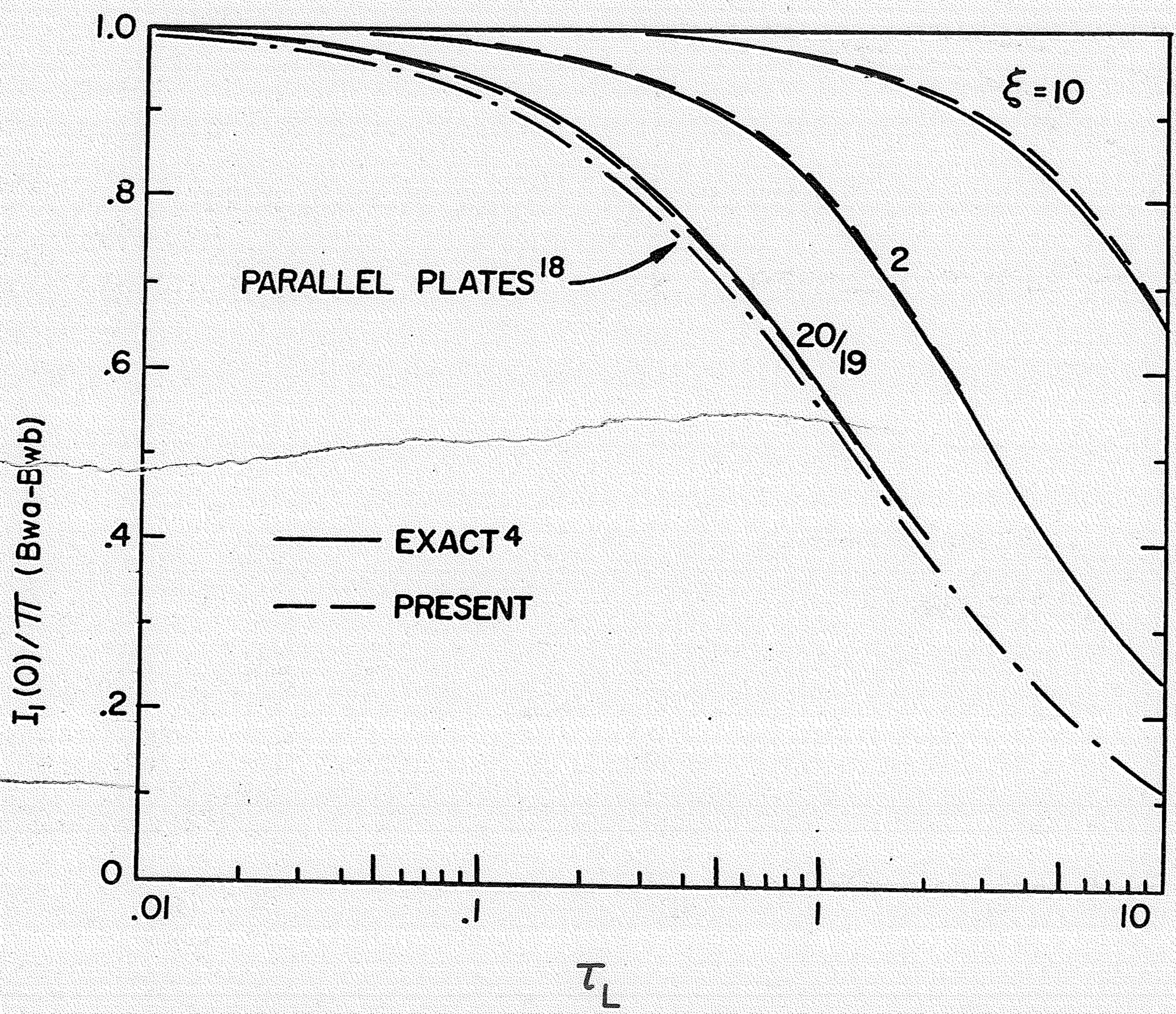

1.0
.8
.6
.4
.2
0
.01
.1
1
10
ξ = 10
2
20/19
PARALLEL PLATES[18]
EXACT[4]
PRESENT
$I_1(0)/\pi$ (Bwa-Bwb)
τ_L

Figure 5

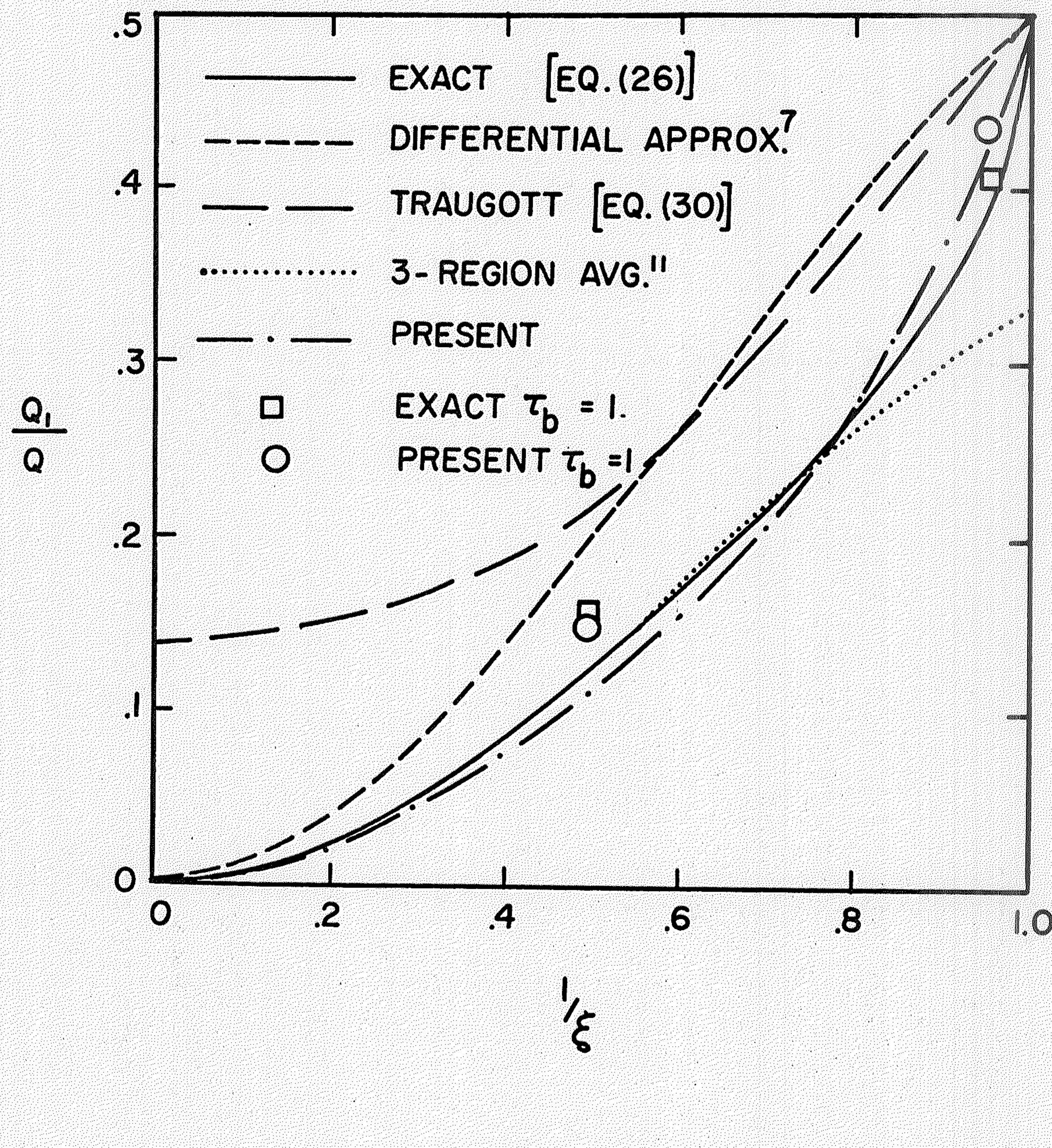
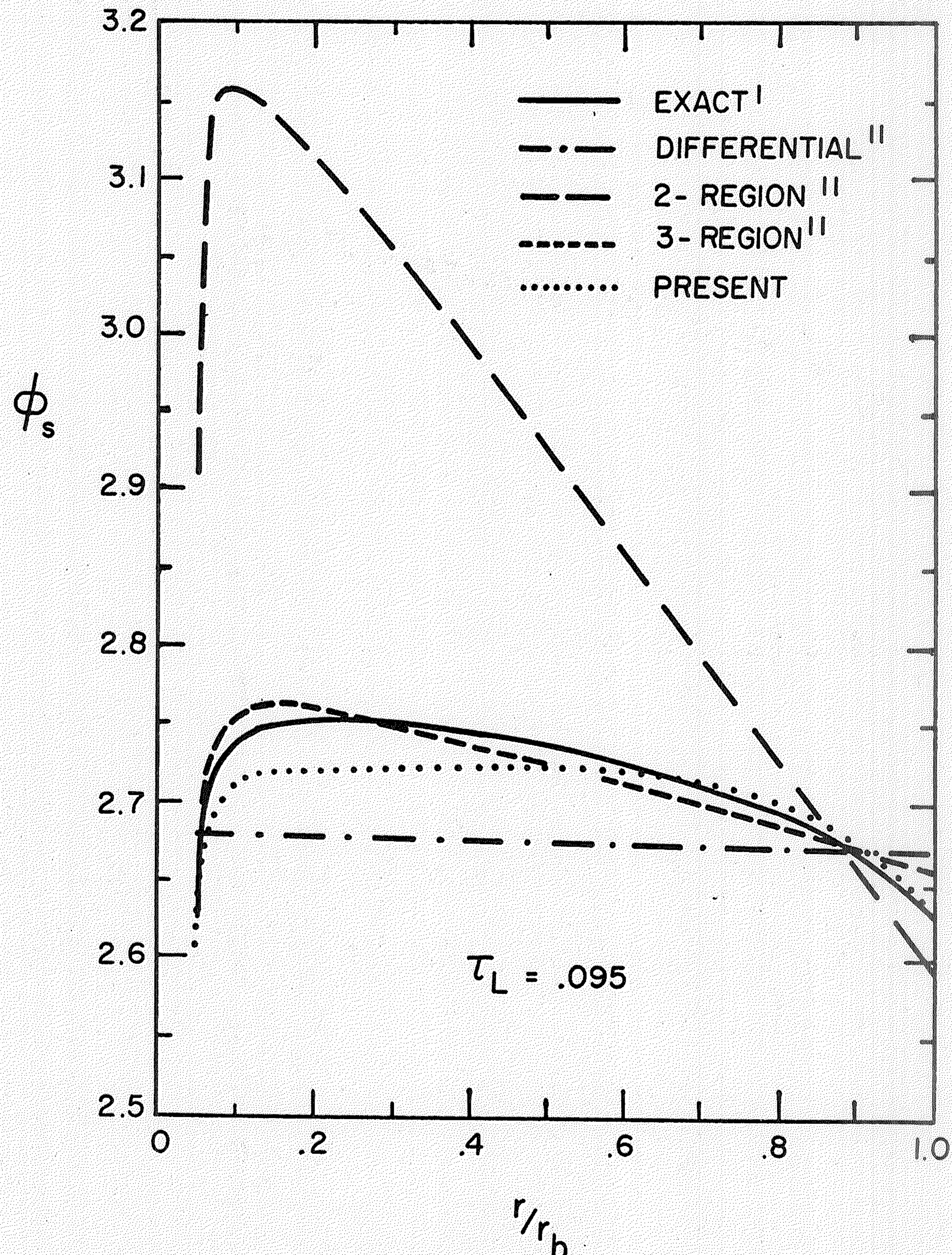

EXACT [EQ. (26)]
DIFFERENTIAL APPROX.[7]
TRAUGOTT [EQ. (30)]
3-REGION AVG.[11]
PRESENT
EXACT τ_b = 1.
PRESENT τ_b = 1
$\frac{Q_1}{Q}$
$1/\xi$
.5
.4
.3
.2
.1
0
0
.2
.4
.6
.8
1.0

Figure 6



3.2
3.1
3.0
2.9
2.8
2.7
2.6
2.5
ϕ_s
EXACT I
DIFFERENTIAL II
2- REGION II
3- REGION II
PRESENT
τ_L = .095
0
.2
.4
.6
.8
1.0
r/r_b

Figure 7

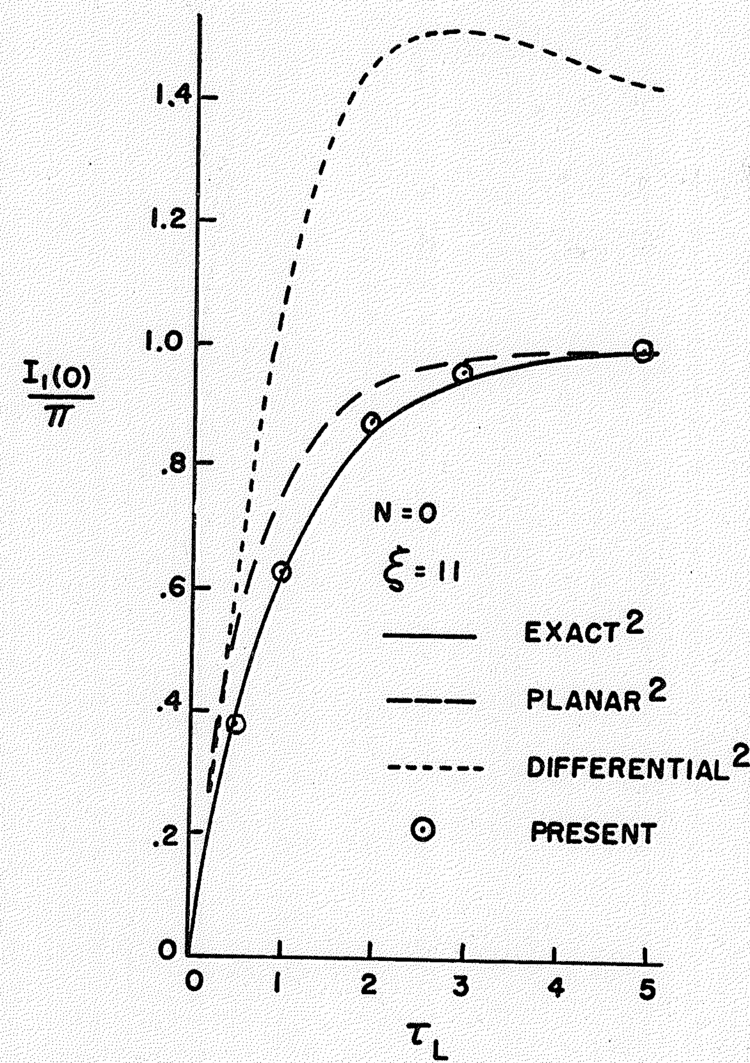
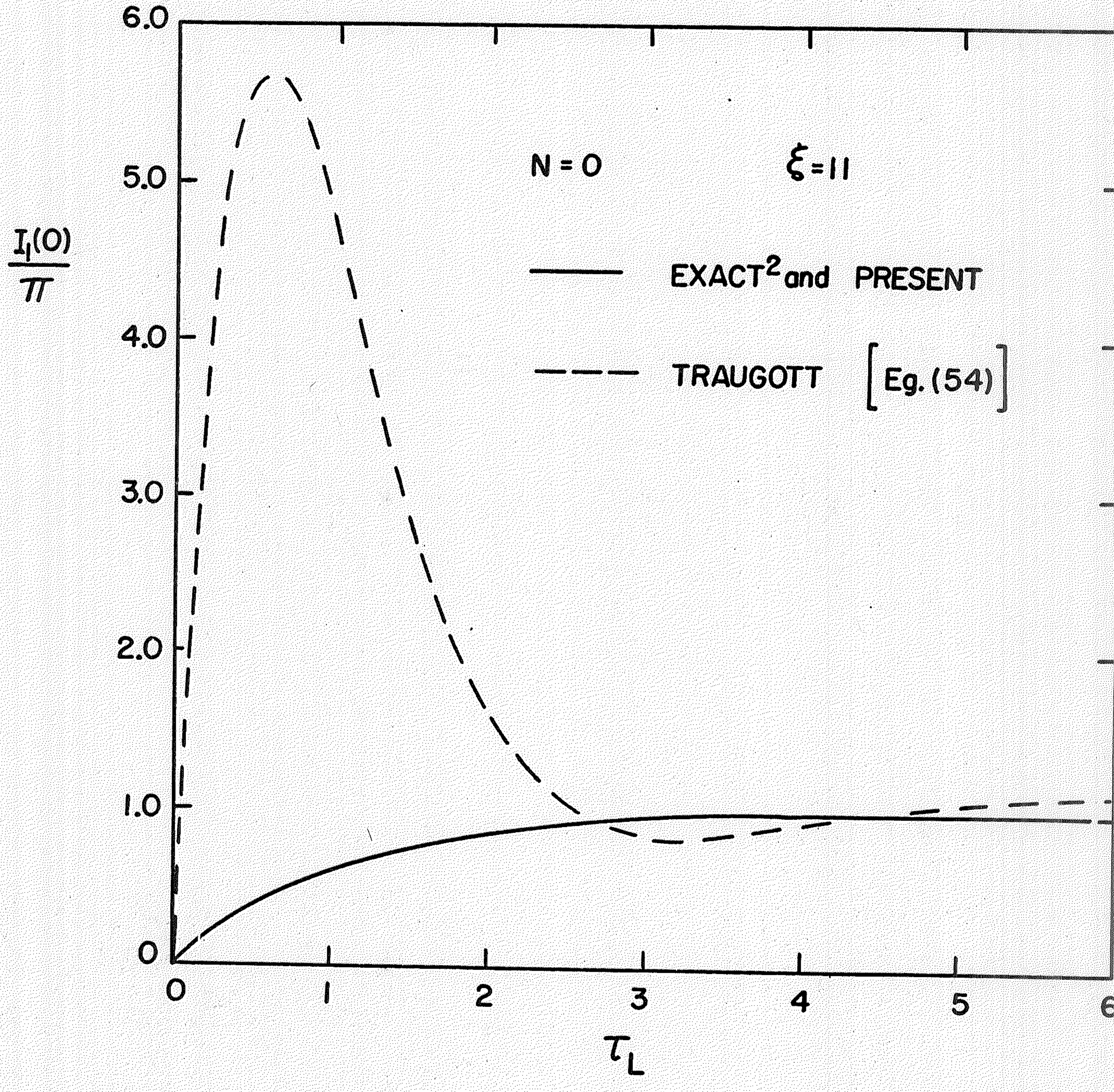

1.4
1.2
1.0
.8
.6
.4
.2
0
$\frac{I_1(0)}{\pi}$
0
1
2
3
4
5
τ_L
N = 0
ξ = 11
EXACT 2
PLANAR 2
DIFFERENTIAL 2
PRESENT

Figure 8a



6.0
5.0
4.0
3.0
2.0
1.0
0
0
1
2
3
4
5
6
$\frac{I_1(0)}{\pi}$
τ_L
N = 0
$\xi = 11$
EXACT[2] and PRESENT
TRAUGOTT [Eg. (54)]

Figure 8b