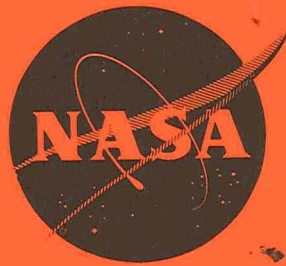


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Theoretical Investigation on Transverse Mode Combustion Instability for Liquid Propellant Rockets

by

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PRINCETON UNIVERSITY

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16. Abstract A mathematical approach is described which provides an analytical solution to the problem of nonlinear transverse combustion instability in a thin annular liquid propellant rocket chamber. Finite chamber lengths are considered as well as a time dependency on the droplet lifetime. Boundary conditions existing at the injector and nozzle eliminate the restrictions of holding mass, energy, and momentum constant with time. A numerical integration of the waveshape equation and employing the Priem-Heidmann droplet evaporation model results in a shock-type periodic solution. Threshold limits corresponding to triggering instability as well as limiting-cycle behavior are found using this approach. To design for rocket operation as far as possible from the triggering limit this analysis indicates that (1) the combustion should either be spread over the entire length of the combustor or completed in less than 20% of chamber length for the example cited, (2) the chamber length should be as small as possible compared to the mean annulus circumference and (3) the ratio of nozzle entrance to injected velocity should be maximized by injecting the propellant at the lowest velocity and increasing the chamber Mach number.					
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FOREWORD

The research described in this report was supported by NASA Grant NGL 31-001-155 and monitored by Mr. Marcus Heidmann. Early theoretical efforts were performed by Professor Crocco both at Princeton University and while on leave. The work was completed by Mr. Jean Lorenzetto under Professor Crocco's direction. Assistance in the preparation of this report was provided by Mr. David T. Harrje, Senior Research Engineer and Lecturer at Princeton University.

SUMMARY

A quantitative understanding of the combustion process associated with transverse mode combustion instability in an annular liquid rocket motor is the objective of this theoretical research. Emphasis is on the nonlinear aspects of the phenomena. The gas dynamic equations for the annular motor are developed and represent a considerable simplification when compared to the cylindrical chamber. The equations are solved by the technique of expansion in powers of a small parameter related to the perturbation amplitude. Since the source term is strongly dependent on the comparative magnitude of the transverse perturbation v and the steady-state relative axial velocity $u_z - \bar{u}$, different ranges of these velocities are considered and conditions for the existence of shock-type periodic solutions is determined.

This theoretical study confirmed that shock-type waves are characteristic of transverse mode instability in an annular chamber as shown experimentally. The design requirements to minimize the occurrence of such instability (apart from requiring solutions such as baffles and acoustic liners) are that: (1) the combustion should either be spread over the entire length of the combustor or completed in less than 20% of chamber length for the example cited, (2) the chamber length should be as small as possible compared to the mean annulus circumference and (3) the ratio of nozzle entrance to injected velocity should be maximized by injecting the propellant at the lowest velocity and increasing the chamber Mach number.

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NOMENCLATURE

A_i	=	injection port area
A	=	function of the parameters: $\bar{u}_n$, l and γ
a	=	sonic velocity
B	=	function of the parameters: $\bar{u}_n$, l and γ
b	=	station in the combustion chamber wherein steady state, $\bar{v} = 0$
C	=	constant
C_1	=	constant (see Eq.(18))
C_2	=	constant (see Eq.(18))
c_r	=	reference sonic velocity
D	=	function of the parameters: β , τ , l and γ
f	=	frequency
F	=	general quantity
ΔF	=	jump of the quantity F through the shock
g	=	linear integral function of θ
G	=	integral function of $g(\theta)$
K	=	function of β
l	=	combustion chamber length
M	=	function of θ
$\dot{m}_i$	=	injection flux per unit area
N^d	=	droplet number density
$O()$	=	order of
p	=	pressure
Pr	=	Prandtl number
Q	=	mass source (or evaporation rate)
q	=	gas velocity

q_l	=	liquid propellant velocity
q_{rel}	=	relative velocity
r_i	=	droplet radius at injection
Re	=	Reynolds number
t	=	time
u	=	longitudinal gas velocity
u_l	=	longitudinal liquid velocity
u_n	=	gas velocity at nozzle entrance
V	=	droplet volume (relative to injection droplet volume)
v	=	circumferential gas velocity
x	=	longitudinal coordinate
y	=	circumferential coordinate
$\frac{D}{Dt}$	=	substantial derivative

Greek Symbols

α	=	circumferential coordinate defined as: $ft-y$
β	=	ratio of the gas velocity at nozzle entrance to the liquid propellant velocity.
γ	=	ratio of specific heats
δ	=	time elapsed from the injection of the droplet
φ	=	arbitrary function of
ξ	=	stretching of the longitudinal coordinate
μ	=	nondimensional injection flux
ν	=	square root of the relative droplet volume
ρ	=	gas density
ρ_r	=	reference gas density
ρ_L	=	liquid propellant density

- ρ_{e_i} = amount of propellant per unit chamber volume at injection
 τ = droplet lifetime
 ζ = dummy variable
 ϕ = guess function of θ to start the iteration procedure
 σ = nondimensional entropy change
 θ = stretching of the circumferential coordinate
 θ^* = value of θ for which the transversal velocity $v(\theta)$ takes on the mean value v_m

Subscripts

- 0 = zeroth-order quantity
 1 = first-order quantity
 2 = second-order quantity
 m = corresponding to a mean value along the shock
 n = nozzle
 α = partial derivative with respect to α
 φ = partial derivative with respect to φ
 θ = partial derivative with respect to θ
 x = partial derivative with respect to x
 t = partial derivative with respect to t
 bar = vector
 asterisk = related to the station in the combustion chamber where the steady-state gas velocity is equal to the liquid propellant velocity.

Superscripts

- asterisk = dimensional value
 bar = steady-state value
 prime = perturbation

SECTION I

Introduction

In a liquid propellant rocket motor, the combustion process is never entirely smooth. During the steady-state period between starting and cutoff, fluctuations occur in all important properties (pressure, temperature, velocity, etc.) around the desired operating values. The amplitude of such fluctuations can vary over a wide range, from motor to motor and in one motor for different operating conditions. When the fluctuations are completely random, the operation is classified as "rough" combustion. Combustion instability, on the other hand, consists of organized oscillations, which are maintained and amplified by the combustion process itself.

Theoretical explanations of the causes of combustion instability date back to Rayleigh's analysis. Since that time a number of concepts have been advanced.¹⁻¹² One example is the constant combustion time lag which has proven to be useful in explaining low frequency instability (i.e., interaction between the feed system and combustion chamber). Later Crocco introduced the time varying combustion lag in his analysis of high frequency instability. That approach involved both insensitive and sensitive time lags, the later responding to fluctuations in the chamber conditions. Sirignano analyzed longitudinal mode, nonlinear combustion instability for a combustion time negligible compared to wave travel time in the chamber and for combustion times of the same order of magnitude. Unstable operation was shown possible for both conditions with triggering action initiated whenever a phase existed between energy addition and pressure. Later theoretical studies at Princeton dealt with transverse modes and with concentrated combustion.

A mechanistic approach to the origins of combustion instability was investigated by Priem and Guentert⁹ at Lewis Research Center. The approach used the Priem-Heidmann¹³ drop-let evaporation model as the basic for energy generation. The assumptions used in that mechanistic approach may be

briefly stated as:

- 1) the burning rate was equal to the vaporization rate,
- 2) the combustion chamber was composed of toroids of a very small thickness, Δr , and length, Δz ,
- 3) and the total mass, momentum and energy in the toroids were constant.

In the present work, which is the subject of this report, the theoretical model of the combustion chamber is an annular geometry and use has been made of the same droplet evaporation model. However, the differences are in the assumptions in that:

- 1) instead of a one-dimensional toroidal section with a very small thickness and length, here we consider a two-dimensional combustion with a finite length l ,
- 2) in the present model a time dependency for the droplet has been introduced ($\tau = b/u_l$, where b is the station at which the droplet vanishes and u_l is the constant liquid velocity),
- 3) restrictions as to mass, energy and momentum constant with time are no longer necessary because of the boundary conditions existing in the present model at the injector and nozzle entrance planes.

As liquid rocket designs have changed with time one observes that tangential modes have surpassed longitudinal modes in importance, shock waves and other nonlinear behavior have become increasingly important. The purpose of this report is to focus attention on the combustion behavior of an annular chamber to illustrate how the basic mechanism of droplet evaporation together with the chamber gas dynamics can result in combustion instability. Triggering behavior and limit cycle operation are traced to the effect of the pressure oscillations on the combustion processes where the amount of energy feedback is sufficient to balance the energy absorbed by dissipative or other damping processes.

SECTION II

Annular Motor

The mathematical approach which was developed by Crocco for nonlinear transverse mode instability is described in this section.

2.1 Theoretical Model and Assumptions

A combustion chamber geometry corresponding to a constant thin annular cross section followed by a large number of individual small nozzles (Fig. 1) has been chosen. The dependence on the radial coordinate can be disregarded and the only relevant independent variables are the time t , the axial coordinate x and the circumferential coordinate y . The reference length will be chosen equal to the circumferential development of the annulus; the reference velocity is the sonic velocity in the reference state; and the reference time is the ratio of the reference length to the velocity. All other quantities are normalized accordingly.

The fundamental assumptions are the following:

- a) The gaseous material contained in the chamber is in the form of gases of perfect combustion. The volume occupied by liquid or vaporized propellants and intermediate combustion products is negligible.
- b) The combustion gases are assumed to be thermally and calorically perfect and homocompositional. Viscous effects and heat conduction are disregarded. The combustion immediately follows the propellant evaporation.
- c) The droplets have no drag, so that their velocity u_d is constant in space and time.
- d) A combustion model has been retained such that the combustion rate which has been assumed to be equal to the droplet evaporation rate, has a law of proportionality with respect to the square root of the Reynolds number¹³.

2.2 The Combustion Chamber Equations

Following Crocco's mathematical derivation, we first write the conservation equations for the combustion chamber:

$$\rho_t + \nabla \cdot (\rho \underline{q}) = Q$$

$$\rho \left[\underline{q}_t + (\underline{q} \cdot \nabla) \underline{q} \right] + \frac{\nabla p}{\gamma} = -Q (\underline{q} - \underline{q}_e)$$

$$\frac{p_t}{\gamma} + \nabla \cdot (p \underline{q}) + \frac{\gamma-1}{2} \left[(\rho q^2)_t + \nabla \cdot (\rho q^2 \underline{q}) \right] = Q$$

The right-hand sides of the three equations represent the mass, momentum and energy sources corresponding to the rate of gasification (or combustion rate) Q of the liquid propellants. The expressions for the first and the third actually coincide only because in the nondimensionalization the energy content of any mass generated is taken as unity.

The conservation equations can be transformed to the variables p , $\underline{q}$ and σ (representing a nondimensional entropy change from the reference conditions) by taking $\rho = p^{1/\gamma} e^{-\sigma}$. After rearranging, the new equations can be written in the convenient form:

$$\frac{1}{\gamma} p_t + \frac{1}{\gamma} \underline{q} \cdot \nabla p + p \nabla \cdot \underline{q} = Q \left[1 - \frac{\gamma-1}{2} (2 \underline{q}_e \cdot \underline{q} - q^2) \right]$$

$$\underline{q}_t + (\underline{q} \cdot \nabla) \underline{q} + p^{-1/\gamma} e^{\sigma} \frac{\nabla p}{\gamma} = -p^{-1/\gamma} e^{\sigma} Q (\underline{q} - \underline{q}_e)$$

$$\sigma_t + \underline{q} \cdot \nabla \sigma = Q \left[\frac{1}{p} - p^{-1/\gamma} e^{\sigma} - \frac{\gamma-1}{2p} (2 \underline{q}_e \cdot \underline{q} - q^2) \right]$$

The only relevant independent variables are t , x and y . The reference length will be chosen equal to the circumferential development of the annulus. Hence, all physical quantities must

be periodic in y with unitary period. The equations for the annular geometry become:

$$\frac{p_t}{\gamma} + u \frac{p_x}{\gamma} + v \frac{p_y}{\gamma} + p(u_x + v_y) = Q \left[1 - \frac{\gamma-1}{2} (2u_\ell u - u^2 - v^2) \right]$$

$$u_t + u u_x + v u_y + p^{-1/\gamma} e^\sigma \frac{p_x}{\gamma} = -p^{-1/\gamma} e^\sigma Q (u - u_\ell)$$

$$v_t + u v_x + v v_y + p^{-1/\gamma} e^\sigma \frac{p_y}{\gamma} = -p^{-1/\gamma} e^\sigma Q v$$

$$\sigma_t + u \sigma_x + v \sigma_y = Q \left[\frac{1}{p} - p^{-1/\gamma} e^\sigma - \frac{\gamma-1}{2p} (2u_\ell u - u^2 - v^2) \right]$$

For the time being we have assumed that the velocity u_ℓ of the liquid propellants is undisturbed, which means that the droplet drag is vanishingly small. This produces a substantial simplification in the treatment because if the injection velocity is constant and axial, so the droplet velocity will remain at every location. Hence u_ℓ represents the constant value of the (axial) droplet velocity.

2.3 Steady-State Equations

Equations (1) can be applied, of course, also in steady state, or the corresponding solution can be obtained directly from the conservation equations written in finite form

$$p^{-1/\gamma} e^{-\bar{\sigma}} \bar{u} = \bar{\gamma} \bar{u} = \int_0^x \bar{Q} dx$$

$$\frac{\bar{p}-1}{\gamma} = \bar{p}^{1/\gamma} e^{-\bar{\sigma}} \bar{u} (u_\ell - \bar{u}) = \bar{\gamma} \bar{u} (u_\ell - \bar{u}) \quad (1)$$

$$\bar{p}^{\frac{\gamma-1}{\gamma}} e^{\bar{\sigma}} = 1 - \frac{\gamma-1}{2} \bar{u}^2 = \frac{\bar{p}}{\bar{\gamma}}$$

2.4 Purely Tangential Waves; First Order Equations

By definition the nondimensional values of steady-state pressure and temperature have been taken as unity at the injector

and the corresponding sonic velocity has been used as reference velocity. Only two cases have been considered, the "purely" axial and the "purely" transversal cases. The purely axial case was developed by Mitchell, so here we will present only the transversal case. We assume that we have a spinning type of instability, traveling in the positive y direction. Then all quantities must be functions of

$$\alpha = ft - y$$

where f is the still unknown frequency. We see that any of the dependent variables must be periodic in α with period 1.

In the new variables the equations can be written

$$\begin{aligned} (f-v) \frac{p_\alpha}{\gamma} + u \frac{p_x}{\delta} + p(u_x - v_\alpha) &= Q \left[1 - \frac{\gamma-1}{2} (2u_\ell u - u^2 - v^2) \right] \\ (f-v) u_\alpha + u u_x + p^{-1/\gamma} e^\sigma \frac{p_x}{\gamma} &= -p^{-1/\gamma} e^\sigma Q(u - u_\ell) \\ (f-v) v_\alpha + u v_x - p^{-1/\gamma} e^\sigma \frac{p_\alpha}{\gamma} &= -p^{-1/\gamma} e^\sigma Q v \\ (f-v) \sigma_\alpha + u \sigma_x &= Q \left[\frac{1}{p} - p^{-1/\gamma} e^\sigma - \frac{\gamma-1}{2p} (2u_\ell u - u^2 - v^2) \right] \end{aligned} \quad (2)$$

The boundary conditions at the injector and at the nozzle are at $x = 0$, $u = 0$ (3a)

at $x = l$, $M = \frac{u}{a} = \text{const.} \left(1 + \frac{\gamma-1}{2} \frac{u^2 + v^2}{a^2} \right)^{\frac{1}{2} \frac{\gamma+1}{\gamma-1}}$ (3b)

where

$$a = \left(\frac{p}{\rho} \right)^{1/2} = p^{\frac{\gamma-1}{2\gamma}} e^{\sigma/2} \quad (4)$$

represents the local sonic velocity. The nozzle boundary condition holds for a very short, multi-orifice nozzle, and was derived by Crocco and Sirignano.¹⁴

To these x -boundary conditions one has to add those of periodicity in the α variable. In the case where the spinning

wave includes a shock, the conservation equations through the shock have also to be satisfied. We will expand all quantities in powers of a small parameter expressing the injection flux. Indeed, if the injection flux reduces to zero, the steady-state pressure becomes constant and uniform throughout the chamber. Hence any deviation from uniformity in space or from constancy in time should be related to the injection flux and vanish with it. Similarly, the axial velocity u reduces to zero if there is no combustion. Again both the steady value of u or its oscillations, or the oscillations of v and σ (their steady-state values are zero) should be related to the injection flux and vanish with it. Hence, calling μ the injection flux, we will assume the following expansions.

$$p = 1 + \mu p_1 + \mu^2 p_2 + O(\mu^3) \quad (5a)$$

$$u = \mu u_1 + \mu^2 u_2 + O(\mu^3) \quad (5b)$$

$$v = \mu v_1 + \mu^2 v_2 + O(\mu^3) \quad (5c)$$

$$\sigma = \mu \sigma_1 + \mu^2 \sigma_2 + O(\mu^3) \quad (5d)$$

$$Q = \mu Q_1 + \mu^2 Q_2 + O(\mu^3) \quad (5e)$$

We have taken $p_0 = 1$ in agreement with the choice of the reference conditions for the steady-state solution.

The injection flux can be written as

$$\mu = \rho_{li} u_l$$

where $\rho_{li} = \rho_L A_i$ is the product of the actual liquid density times the injection port area A_i per unit area, and represents the actual droplet mass per unit chamber volume at injection. The injection flux, for a given propellant, can vary as a result of varying A_i and u_l . Here we shall assume that A_i is kept constant, in which case we can write

$$u_l = \mu u_{l1} \quad (\rho_{li} u_{l1} = 1) \quad (5f)$$

Observe that in steady state (assuming the combustion to be terminated before the nozzle entrance) we must have

$$\mu = \int_0^l \bar{Q} dx$$

Or, by virtue of Eq. (5e)

$$\int_0^l \bar{Q}_1 dx = 1 \quad ; \quad \int_0^l \bar{Q}_2 dx = 0 \quad ; \quad \dots \quad (6)$$

Insertion of the expressions (5) in the equations (2) will provide, after separation of the terms in different powers of μ , the equations to be solved. We notice that each one of equations (2) contains a mean convective term due to the steady-state gas velocity $\bar{u}$. The presence of the convective terms results in inconsistencies which we shall not discuss here, since they affect only terms in higher powers of μ than those considered in the present study. It is sufficient to say that the inconsistencies even in the higher order terms can probably be suppressed by using a double scale of axial length, according to a recently established technique for the treatment of certain nonlinear problems.

Another technique which is found useful in nonlinear problems is that of the coordinate stretching. Here we have already introduced a time stretching through the use of the frequency f , which can also be expanded as

$$f = 1 + \mu f_1 + \mu^2 f_2 + O(\mu^3) \quad (7)$$

where the constants $f_1, f_2, \dots$ are to be determined and, of course, the frequency reduces to unity for $\mu = 0$.

In addition to this time stretching, it is convenient to introduce also a stretching of the variables α and x ,* in the form of a series

$$\begin{aligned} \theta &= \alpha + \mu \theta_1(\theta, \xi) + \mu^2 \theta_2(\theta, \xi) + O(\mu^3) \\ \xi &= x + \mu \xi_1(\theta, \xi) + \mu^2 \xi_2(\theta, \xi) + O(\mu^3) \end{aligned} \quad (8)$$

*Observe that instead of introducing f , and stretching a and x , one could just stretch the original variables t, y , and x , with the same end results. The mixed procedure we are following here allows faster derivations.

where the functions $\theta_1, \xi_1, \theta_2, \xi_2, \dots$ are to be determined. All dependent variables are functions of θ, ξ ; they must be periodic in θ with period 1. If there is a traveling transversal shock, we choose to stretch α in such a way that the shock corresponds to $\theta = \text{const}$. In particular, if we place the shock at $\theta = 0$, because of the periodicity we must find again the shock at $\theta = 1$ and at any other integral value of θ .

The derivatives of a general quantity

$$F = F_0 + \mu F_1 + \mu^2 F_2 + O(\mu^3)$$

with $F_0 = \text{const}$ (as it is in all expansions (5)) can then be written as

$$F_\alpha = \mu F_{1\theta} + \mu^2 (F_{2\theta} + F_{1\theta} \theta_{1\theta} + F_{1\xi} \xi_{1\theta}) + O(\mu^3)$$

$$F_x = \mu F_{1\xi} + \mu^2 (F_{2\xi} + F_{1\theta} \theta_{1\xi} + F_{1\xi} \xi_{1\xi}) + O(\mu^3)$$

Introducing all the expansions above in Eq. (2) and separating the powers of μ we obtain the following equations for the first order quantities

$$\frac{p_{1\theta}}{\gamma} + u_{1\xi} - v_{1\theta} = Q_1 \quad (9a)$$

$$u_{1\theta} + \frac{p_{1\xi}}{\gamma} = 0 \quad (9b)$$

$$v_{1\theta} - \frac{p_{1\theta}}{\gamma} = 0 \quad (9c)$$

$$\sigma_{1\theta} = 0 \quad (9d)$$

2.5 Second Order Equations

The following are the second order equations :

$$\frac{p_{2\theta}}{\gamma} + \frac{p_{1\theta}}{\gamma} \theta_{1\theta} + \frac{p_{1\xi}}{\gamma} \xi_{1\theta} + (f_1 - v_1) \frac{p_{1\theta}}{\gamma} + u_1 \frac{p_{1\xi}}{\gamma} + u_{2\xi} + u_{1\theta} \theta_{1\xi} + \quad (10a)$$

$$u_{1\xi} \xi_{1\xi} - v_{2\theta} - v_{1\theta} \theta_{1\theta} - v_{1\xi} \xi_{1\theta} + p_1 (u_{1\xi} - v_{1\theta}) = Q_2$$

$$u_{2\theta} + u_{1\theta} \theta_{1\theta} + u_{1\zeta} \zeta_{1\theta} + (p_1 - \sigma_1) u_{1\theta} + u_1 u_{1\zeta} + \frac{p_{2\theta}}{\gamma} + \frac{p_{1\theta}}{\gamma} \theta_{1\zeta} + \quad (10b)$$

$$\frac{p_{1\zeta}}{\gamma} \zeta_{1\zeta} - \left(\frac{p_1}{\gamma} - \sigma_1 \right) \frac{p_{1\zeta}}{\gamma} = -Q_1 (u_1 - u_{1,1})$$

$$v_{2\theta} + v_{1\theta} \theta_{1\theta} + v_{1\zeta} \zeta_{1\theta} + (p_1 - \sigma_1) v_{1\theta} + u_1 v_{1\zeta} - \frac{p_{2\theta}}{\gamma} - \frac{p_{1\theta}}{\gamma} - \quad (10c)$$

$$\frac{p_{1\zeta}}{\gamma} \zeta_{1\theta} + \left(\frac{p_1}{\gamma} - \sigma_1 \right) \frac{p_{1\theta}}{\gamma} = -Q_1 v_1$$

$$\sigma_{2\theta} + \sigma_{1\theta} \theta_{1\theta} + \sigma_{1\zeta} \zeta_{1\theta} + (p_1 - \sigma_1) \sigma_{1\theta} + u_1 \sigma_{1\zeta} = -Q_1 \left(\frac{\gamma-1}{\gamma} p_1 + \sigma_1 \right) \quad (10d)$$

The injector and nozzle conditions (3a) and (3b) are also expanded with the help of (4). They show that at the injector we must have

$$u'_1(\theta, 0) = u'_2(\theta, 0) = 0 \quad (11a)$$

and at the nozzle*

$$u'_1(\theta, l) = 0$$

$$u'_2(\theta, l) + u'_{1\zeta}(\theta, l) \zeta_{1\theta}(\theta, l) = \bar{u}_1(l) a'_1(\theta, l) = \bar{u}_1(l) \left[\frac{\gamma-1}{2\gamma} p'_1(\theta, l) + \frac{\sigma'_1(\theta, l)}{2} \right] \quad (11b)$$

On the other hand, if there are shocks at $\theta = 0$ and 1 and we indicate with ΔF the jump of the quantity F through the shock, this is obviously a function of x only, and can be expanded as

$$\begin{aligned} \Delta F(x) &= \mu \left[F_1(0, \zeta) - F_1(1, \zeta) \right]_{\zeta=x} + \mu^2 \left[F_2(0, \zeta) - F_2(1, \zeta) + \right. \\ &\quad \left. \zeta_{1\theta}(0, \zeta) F_{1\zeta}(0, \zeta) - \zeta_{1\theta}(1, \zeta) F_{1\zeta}(1, \zeta) \right]_{\zeta=x} + O(\mu^3) \\ &= \mu (\Delta F_1)_{\zeta=x} + \mu^2 (\Delta F_2 + \zeta_{1,1m} \Delta F_{1\zeta} + F_{1\zeta m})_{\zeta=x} + O(\mu^3) \end{aligned}$$

where we have defined for the general quantity $g(\theta, \zeta)$ the two quantities

$$\begin{aligned} \Delta g &= g(0, \zeta) - g(1, \zeta) \\ g_m &= \frac{1}{2} [g(0, \zeta) + g(1, \zeta)] \end{aligned}$$

*The assumption is made that combustion terminated at the nozzle entrance is steady state, so that $\bar{u}_{1\zeta}(l) = 0$.

The equation of the shock located at $\Theta = 0$ is given by (8) as

$$\alpha = ft - y = -\mu \theta_1(0, \xi) - \mu^2 \theta_2(0, \xi) + O(\mu^3) \quad (12a)$$

Hence the local angle between the shock and the axial direction is, to first order, given by

$$\beta = \left(\frac{dy}{dx} \right)_{t=\text{const}} = \mu \theta_{1\xi}(0, \xi) = \mu \beta_1 \quad (12b)$$

Applying the conservation equations through the shock one obtains the first and second order shock conditions in the form

$$\Delta u'_1 = 0 ; \quad \Delta \sigma'_1 = 0 ; \quad \Delta \left(v'_1 - \frac{p'_1}{\gamma} \right) = 0 \quad (13)$$

$$\begin{aligned} \Delta u'_2 + \xi_{1m} \Delta u'_{1\xi} + u_{1\xi m} \Delta \xi_1 &= -\beta_1 \Delta v'_1 ; \quad \Delta \sigma'_2 + \xi_{1m} \Delta \sigma'_{1\xi} + \sigma_{1\xi m} \Delta \xi_1 = 0 \\ \Delta \left(v'_2 - \frac{p'_2}{\gamma} \right) + \xi_{1m} \Delta \left(v'_{1\xi} - \frac{p'_{1\xi}}{\gamma} \right) + \left(v'_{1\xi m} - \frac{p'_{1\xi m}}{\gamma} \right) \Delta \xi_1 &= \left(-\frac{\gamma+1}{2} \frac{p'_1}{\gamma} + \frac{\sigma'_1}{2} \right) \frac{\Delta p'_1}{\gamma} \end{aligned} \quad (14)$$

The first equation of (13) and (14) represents the continuity of the tangential velocity component. The second equation in each group expresses that up to second order the shock is isentropic. The remaining equations express the invariance through the shock of the appropriate Riemann invariant $\sigma - \frac{2}{\gamma-1} a$. This invariance holds indeed to second order for the oblique shock under consideration as for a normal shock. Observe that the second term of each Equation (14) vanishes because of Equations (13). Finally, it is evident that the shock velocity in the y direction is given by f . For the purposes of this treatment we need only the first order coefficient f_1 of the expansion (7). It is given by the same relation as if the shock were normal, and hence by the mean value of $v_1' + a_1'$ across the shock, that is by

$$f_1 = v'_{1m} + \frac{\gamma-1}{2\gamma} p'_{1m} + \frac{\sigma'_1}{2} \quad (15)$$

2.6 First Order Solution

The steady-state solution can be obtained either from Equations (9), setting all θ -derivatives equal to zero, or, more directly, from expansions of the Equation (1) already in finite form. The result is

$$\bar{u}_{1\frac{\xi}{\gamma}} = \bar{Q}_1 ; \quad \bar{u}_1(\frac{\xi}{\gamma}) = \int_0^{\frac{\xi}{\gamma}} \bar{Q}_1 d\frac{\xi}{\gamma} ; \quad \bar{p}_1 = \bar{\sigma}_1 = 0$$

Comparing with (6) we see that

$$\bar{u}_1(l) = 1 \quad (16)$$

For the unsteady components we obtain from Equation (9)

$$u'_{1\frac{\xi}{\gamma}} = Q'_1 ; \quad \frac{p'_1}{\gamma} = -u'_{1\theta} ; \quad v'_{1\theta} = \frac{p'_{1\theta}}{\gamma} ; \quad \sigma'_{1\theta} = 0 \quad (17)$$

For many combustion models (such as that considered later) the effect of the perturbed conditions on the combustion rate Q is not felt before the second order term Q_2' . Hence $Q_1' = 0$, and $u_1' = \text{const} = 0$ consistently with the injector and nozzle conditions (11a and b). This result is, of course, due to the assumption of purely transversal waves, as shown by the second Equation (17) which shows that p_1' must be a function of θ alone. Hence the other two equations (17) are satisfied by

$$v'_1 = \varphi(\theta) ; \quad \frac{p'_1}{\gamma} = \varphi(\theta) + C_1 ; \quad \sigma'_1 = C_2 \quad (18)$$

where C_1 and C_2 are constants to be determined,* and $\varphi(\theta)$ is an arbitrary function. It is clear that this solution satisfies the first order shock conditions (13).

2.7 Second Order Solution

The second order solution in steady state can again be obtained either from the Eq. (10) or directly from the Equation (1). If we assume $\bar{Q}(\frac{\xi}{\gamma}) = \mu \bar{Q}_1(\frac{\xi}{\gamma})$ for all values of μ , then $\bar{Q}_2 = 0$, and one gets

$$\bar{u}_2 = 0 ; \quad \bar{p}_2 = \gamma \bar{u}_1(u_{l_1} - \bar{u}_1) ; \quad \bar{\sigma}_2 = -(\gamma-1) \bar{u}_1(u_{l_1} - \frac{\bar{u}_1}{2}) \quad (19)$$

*Actually C_2 could be an arbitrary function of $\frac{\xi}{\gamma}$, however, in the way C_1 and C_2 will be determined, it can be shown that C_2 is independent of $\frac{\xi}{\gamma}$. For simplicity, we assume from the beginning that it is constant.

The equations for the unsteady components are then, using the relation $\bar{Q}_1(\xi) = \bar{u}_{1\xi}$

$$\frac{p'_{2\theta}}{\gamma} + u'_{2\xi} - v'_{2\theta} - \left[(\gamma+1)\varphi - f_1 + \gamma C_1 \right] \varphi_\theta + \bar{u}_{1\xi} (\gamma\varphi + \gamma C_1 + \xi_{1\xi}) = Q'_2 \quad (20a)$$

$$u'_{2\theta} + p'_{2\xi}/\gamma + \theta_{1\xi} \varphi_\theta + \bar{u}_{1\xi} \xi_{1\theta} = 0 \quad (20b)$$

$$v'_{2\theta} - \frac{p'_{2\theta}}{\gamma} + (f_1 + C_1 - C_2) \varphi_\theta = -\bar{u}_{1\xi} \varphi \quad (20c)$$

$$\sigma'_{2\theta} = -\bar{u}_{1\xi} \left[(\gamma-1)\varphi + (\gamma-1)C_1 + C_2 \right] \quad (20d)$$

2.8 Wave Shape Equation in the Most General Case

It is assumed that the constants C_1 and C_2 of Eq. (18) to be zero and this is generally the right value with some exceptions. We can then rewrite Eq. (18) so that:

$$v'_1 = \varphi(\theta) \quad ; \quad \frac{p'_1}{\gamma} = \varphi(\theta) \quad \text{and} \quad \sigma'_1 = 0 \quad (21)$$

represent a possible first order solution (purely transverse wave). From Equation (15) we have (having taken $\xi_1 = 0$):

$$f_1 = \frac{\gamma+1}{2} \varphi_m$$

which results in the following equation

$$u'_{2\xi} - (\gamma+1)(\varphi - \varphi_m) \varphi_\theta + \bar{u}_{1\xi} (\gamma+1)\varphi = Q'_2 \quad (22)$$

We also have:

$$u_{1\xi} = \bar{u}_{1\xi} = Q_1 = \bar{Q}_1 \quad (23)$$

Multiplying Eq. (22) by μ^2 and Eq. (23) by μ , adding them up and recalling the expansions (5), we find the equation:

$$u_{\xi} = (\gamma+1)(v-v_m)v_{\theta} - \bar{u}_{\xi}(\gamma+1)v + Q$$

between the unexpanded quantities u , v , Q . This equation applies for the case when all quantities are periodic, and hence can be expressed as functions of $\theta = ft - y$ and ξ alone. If the solution is not periodic, every quantity should be considered to still depend on t , independently on the θ dependence. In this case an additional term should appear on the left hand side of the equation which writes:

$$u_{\xi} + 2v_{\xi} = (\gamma+1)(v-v_m)v_{\theta} - \bar{u}_{\xi}(\gamma+1)v + Q$$

In reality this equation holds only if $Q_1' = 0$, and if μ (and hence v) are small as well as when the Q perturbation is of order μ^2 while the steady state Q is of order μ . However, we shall suppose its approximate validity as well as that of Eq. (21) even in the case when Q' is of order μ , provided $v \ll 1$. Integrating from 0 to l we obtain

$$u(l) = u_m = (\gamma+1)l(v-v_m)v_{\theta} - (\gamma+1)\bar{u}_m v + \int_0^l Q(\xi) d\xi$$

From the expanded nozzle condition, Eq. (11b), we can again (multiplying the first by μ and the second by μ^2 and adding up) synthesize the result:

$$u_m = \bar{u}_m \left(1 + \frac{\gamma-1}{2} v \right)$$

and hence we obtain:

$$(\gamma+1)l(v-v_m)v_{\theta} = \bar{u}_m \left(1 + \frac{3\gamma+1}{2} v \right) - \int_0^l Q(\xi) d\xi \quad (24)$$

The corresponding equation in the non-periodic case is,

$$-2lv_{\xi} + (\gamma+1)l(v-v_m)v_{\theta} = \bar{u}_m \left(1 + \frac{3\gamma+1}{2} v \right) - \int_0^l Q(\xi) d\xi \quad (24a)$$

When the relationship between Q and v is known, Equation (24) (or 24a) represents a nonlinear integro-differential equation for v .

2.9 Droplet Evaporation Model

The quasi-steady evaporation rate, under certain assumptions, is defined by the relation:

$$\frac{D_e r^*}{D t^*} = \frac{\partial r^*}{\partial t^*} + u_l^* \frac{\partial r^*}{\partial x^*} = \frac{-K}{r^*} (1 + 0.3 Re^{1/2} Pr^{1/3})$$

where stars stand for dimensional quantities, K is a constant related to the properties of the gases and of the droplets, Pr is the Prandtl number, the Reynolds number is defined as:

$$Re = 2 \frac{r^* \rho^* q_{rel}^*}{\mu^*}$$

and q_{rel}^* is the droplet velocity relative to the gas, given by:

$$q_{rel}^{*2} = (u_l^* - u^*)^2 + v^{*2}$$

Introducing the relative droplet volume

$$V = (r^* / r_i^*)^3$$

where r_i^* represents the droplet radius at injection, and non-dimensionalizing the other quantities in the usual fashion we obtain:

$$\frac{D_e V}{D t} = f V_\alpha + u_l V_x = -C (V \rho q_{rel})^{1/2} \quad (A)$$

where we have: $q_{rel}^2 = (u_l - u)^2 + v^2$

$$\text{and } C = \frac{0.9\sqrt{2} K \ell^* Pr^{1/3}}{c_r^* r_i^{*2}} \left(\frac{\rho_r^* r_i^* c_r^*}{\mu^*} \right)^{1/2}$$

contains the reference length ℓ^* (developed periphery of the annulus), the reference sonic velocity c_r^* and gas density ρ_r^* (in our case these are the corresponding values at the injector end) and the viscosity μ^* is the average gas viscosity.

The injection flux per unit area is given by:

$$\dot{m}_i^* = N^d \frac{4}{3} \pi r_i^{*3} \rho_L^* u_l^* = \rho_{L,i}^* u_l^*$$

where N^d is the droplet number density which is constant since u_l^* is constant, ρ_L^* is the liquid propellant density, and ρ_{li}^* is the amount of propellant per unit chamber volume at injection. The dimensional evaporation rate is:

$$Q^* = -N^d \frac{4}{3} \pi \rho_L^* \frac{D_e (r^{*3})}{Dt^*}$$

and its nondimensional value, $Q = \ell^* Q^* / \rho_r^* c_r^*$ is immediately proved to be:

$$Q = -\rho_{li} \frac{D_e V}{D t} \quad (B)$$

where $\rho_{li} = \rho_{li}^* / \rho_r^*$ is related to the nondimensional injection flux μ by the equation: $\mu = \rho_{li} u_e$. Equations (A) and (B) completely define the source strength Q .

2.10 Wave Shape Equation With Droplet Evaporation Model

From Equations (A) and (B) we have:

$$-\frac{Q}{\rho_{li}} = f V_\theta + u_e V_\xi = -C (V \rho q_{rel})^{1/2} \quad (25)$$

Here we have identified α with θ ($\theta_i = 0$). Hence, taking $V^{1/2} = \psi$ we can write:

$$f \psi_\theta + u_e \psi_\xi = -\frac{C}{2} (\rho q_{rel})^{1/2} \quad (26)$$

Here: $q_{rel}(\xi, \theta) = \left[(u_e - \bar{u})^2 + v^2 \right]^{1/2}$

where u has been replaced by $\bar{u}$, consistently with the approximation that, to order μ , $u' = 0$, and where within the same accuracy, we can take (see Eq. (21))

$$f \approx 1 + \mu f_1 \approx 1 + \frac{\gamma+1}{2} \sqrt{m}; \quad \rho \approx 1 + \mu \frac{p_1'}{\gamma} \approx 1 + v = \rho(\theta) \quad (27)$$

The formal solution $\psi(\xi, \theta)$ of Eq. (26) which has

$\psi(0, \theta) = 1$ (initial droplet diameter fixed) is:

$$\psi(\xi, \theta) = 1 - \frac{C}{2u_e} \int_0^\xi \left[\rho \left(\theta - f \frac{\xi - \xi'}{u_e} \right) q_{rel} \left(\xi', \theta - f \frac{\xi - \xi'}{u_e} \right) \right]^{1/2} d\xi' \quad (28)$$

after which Eq. (25) gives

$$Q(\xi, \theta) = \rho_{li} c \left[p(\theta) q_{rel}(\xi, \theta) \right]^{1/2} v(\xi, \theta)$$

We have then Eq. (24) in the form:

$$(\gamma+1) l (v - v_m) v_\theta = \bar{u}_n \left(1 + \frac{3\gamma+1}{2} v \right) - \rho_{li} c \left[p(\theta) \right]^{1/2} \int_0^l \left[q_{rel}(\xi, \theta) \right]^{1/2} v(\xi, \theta) d\xi + 2l v_\theta \quad (29)$$

the last term being present for non periodic solutions. Since Eq. (27) gives $\bar{S} = 1$ (within the present approximation) we can obtain ρ_{li} from:

$$\mu = \rho_{li} u_\ell = \bar{S}_n \bar{u}_n = \bar{u}_n \quad (30)$$

The equation for v is now entirely defined for assigned values of c , $\bar{u}_n$, u_ℓ , γ and l . It must be observed that it applies to the case of shock-type waves, and v_m represents the mean value at the shock with the frequency:

$$f = 1 + \frac{\gamma+1}{2} v_m$$

For shockless waves, of course, v_m is meaningless, but f is still meaningful, and v_m should be simply replaced by $(2/(\gamma+1))(f-1)$. In this case, instead of the unknown v_m to be determined simultaneously with the solution $v(\theta)$, we will have the unknown frequency f to be determined also in the same time as the solution $v(\theta)$.

We also have:

$$\bar{u} = \bar{u}_n (1 - V) = \bar{u}_n (1 - \bar{v}^2) \quad (31)$$

Since $\bar{S} = 1$, and (30) $\mu = \bar{u}_n$. Hence we have from Eq. (26):

$$u_\ell \bar{v}_\xi = -\frac{c}{2} |u_\ell - \bar{u}|^{1/2} = -\frac{c}{2} u_\ell^{1/2} |1 - \beta(1 - \bar{v}^2)|^{1/2} \quad (32)$$

where we have introduced $\beta = \bar{u}_n / u_\ell$

2.11 Case for $\beta < 1$

The solution of Equation (32) for $\beta < 1$, where $\bar{u} < u_e$ at all stations is:

$$\bar{v} = \left(\frac{1-\beta}{\beta} \right)^{1/2} \text{Sh} \left[K \left(1 - \frac{\delta}{\tau} \right) \right] \quad (33)$$

where

$$\delta = \frac{\xi}{u_e} \quad (34)$$

represents the time elapsed from the injection of the droplet,

$$\tau = \frac{b}{u_e} \quad (35)$$

is the droplet lifetime, b representing the station where, in steady state, $\bar{v} = 0$, and K being related to β by:

$$K(\beta) = \frac{1}{2} \ln \frac{1 + \beta^{1/2}}{1 - \beta^{1/2}} = \frac{1}{2} \ln \left| \frac{\beta^{1/2} + 1}{\beta^{1/2} - 1} \right|$$

The relation between b or τ and the constant C appearing in Eq. (25) is found to be:

$$C u_e^{1/2} \tau = \frac{C}{u_e^{1/2}} b = \frac{2K(\beta)}{\beta^{1/2}}$$

From Eqs. (31) and (32) we obtain then:

$$1 - \frac{\bar{u}}{u_e} = (1-\beta) \text{Ch}^2 \left[K \left(1 - \frac{\delta}{\tau} \right) \right]$$

Application of Eq. (28) gives then:

$$v_1(\delta, \theta) = 1 - \frac{K}{\tau \beta^{1/2}} \int_0^\delta \left\{ 1 + \frac{1}{2} v \left[\theta - f(\delta - \xi) \right] \right\} \left\{ (1-\beta)^2 \text{Ch}^4 K \left(1 - \frac{\xi}{\tau} \right) + \frac{v^2}{u_e^2} \left[\theta - f(\delta - \xi) \right] \right\}^{1/4} d\xi \quad (36)$$

2.12 Case for $\beta > 1$

In the case $\beta > 1$, $\bar{u} < u_e$ only to a certain value of δ_* (or a certain station ξ_* see Eq. (34)), where $\bar{v}$ becomes equal to a certain value $\bar{v}_*$; after this station we have $\bar{u} > u_e$.

We find for $\delta < \delta_*$, $\bar{v} > \bar{v}_*$, $\bar{u} < u_\ell$

$$\bar{v} = \bar{v}_* \operatorname{ch} \left[K - (K + \pi/2) \frac{\delta}{\tau} \right]$$

$$1 - \frac{\bar{u}}{u_\ell} = (\beta - 1) \operatorname{Sh}^2 \left[K - (K + \pi/2) \frac{\delta}{\tau} \right]$$

and for $\delta > \delta_*$, $\bar{v} < \bar{v}_*$, $\bar{u} > u_\ell$

$$\bar{v} = \bar{v}_* \cos \left[(K + \pi/2) \frac{\delta}{\tau} - K \right]$$

$$\frac{\bar{u}}{u_\ell} - 1 = (\beta - 1) \sin^2 \left[(K + \pi/2) \frac{\delta}{\tau} - K \right]$$

with: $K(\beta) = \frac{1}{2} \ln \frac{\beta^{1/2} + 1}{\beta^{1/2} - 1} = \frac{1}{2} \ln \left| \frac{\beta^{1/2} + 1}{\beta^{1/2} - 1} \right|$

$$\bar{v}_* = \left(\frac{\beta - 1}{\beta} \right)^{1/2}$$

$$c u_\ell^{1/2} \tau = 2 \int_0^1 \frac{d\bar{v}}{|1 - \beta(1 - v^2)|^{1/2}} = \frac{2}{\beta^{1/2}} (K + \pi/2)$$

$$\frac{\delta_*}{\tau} = \frac{K}{K + \pi/2}$$

For: $0 \leq \delta \leq \delta_*$ we have:

$$v_2(\delta, \theta) = 1 - \frac{K + \pi/2}{\tau \beta^{1/2}} \int_0^\delta \left\{ (\beta - 1)^2 \operatorname{Sh}^4 \left[K - (K + \pi/2) \frac{\xi}{\tau} \right] + \frac{1}{u_\ell^2} v^2 \left[\theta - f(\delta - \xi) \right] \right\}^{1/4} \left\{ 1 + \frac{1}{2} v \left[\theta - f(\delta - \xi) \right] \right\} d\xi \quad (37)$$

and for $\delta > \delta_*$ we must take:

$$\begin{aligned}
 (\delta, \theta) = 1 - \frac{\kappa + \pi/2}{\tau \beta^{1/2}} & \left(\int_0^{\delta_*} \left\{ (\beta-1)^2 \operatorname{Sh}^4 \left[\kappa - (\kappa + \frac{\pi}{2}) \frac{\xi}{\tau} \right] + \frac{1}{u_\ell^2} v^2 [\theta - f(\delta - \xi)] \right\}^{1/4} \left\{ 1 + \frac{1}{2} v [\theta - f(\delta - \xi)] \right\} d\xi + \right. \\
 & \left. + \int_{\delta_*}^{\delta} \left\{ (\beta-1)^2 \sin^4 \left[(\kappa + \frac{\pi}{2}) \frac{\xi}{\tau} - \kappa \right] + \frac{1}{u_\ell^2} v^2 [\theta - f(\delta - \xi)] \right\}^{1/4} \left\{ 1 + \frac{1}{2} v [\theta - f(\delta - \xi)] \right\} d\xi \right) \quad (38)
 \end{aligned}$$

Following Equation (27) we have taken: $\xi'^{1/2} \simeq 1 + \frac{1}{2} v$.

Finally we have all the elements to write the last term in Equation (29):

$$\begin{aligned}
 & S_{\ell i} C \xi'^{1/2} \int_0^{\ell} [q_{\text{rel}}(\xi, \theta)]^{1/2} v(\xi, \theta) d\xi = \\
 & = \frac{2 \beta^{1/2} \kappa}{\tau} u_\ell \left[1 + \frac{v(\theta)}{2} \right] \int_0^{\ell/u_\ell} \left[(1-\beta)^2 \operatorname{Ch}^4 \kappa (1 - \frac{\delta}{\tau}) + \frac{1}{u_\ell^2} v^2(\theta) \right]^{1/4} v_1(\delta, \theta) d\delta
 \end{aligned}$$

for $\beta < 1$, and:

$$\begin{aligned}
 & = 2 \frac{\beta^{1/2} (\kappa + \pi/2)}{\tau} u_\ell \left[1 + \frac{v(\theta)}{2} \right] \left(\int_0^{\delta_*} \left\{ (\beta-1)^2 \operatorname{Sh}^4 \left[\kappa - (\kappa + \frac{\pi}{2}) \frac{\delta}{\tau} \right] + \frac{1}{u_\ell^2} v^2(\theta) \right\}^{1/4} v_2(\delta, \theta) d\delta + \right. \\
 & \left. + \int_{\delta_*}^{\ell/u_\ell} \left\{ (\beta-1)^2 \sin^4 \left[(\kappa + \frac{\pi}{2}) \frac{\delta}{\tau} - \kappa \right] + \frac{1}{u_\ell^2} v^2(\theta) \right\}^{1/4} v_3(\delta, \theta) d\delta \right)
 \end{aligned}$$

for $\beta > 1$.

It is obvious that any negative value of ν resulting from Equation (36) or (38) must be replaced by 0 .

SECTION III

Numerical Analysis for Periodic Solutions

This section concerns the numerical integration of the wave equation [Eq. (29)] in the previous section. A shock-type solution is sought in the case of $\partial v / \partial t = 0$ which represents the case of a periodic solution. It is useful to recall that in a linear treatment once instability appears, the amplitude of the perturbation grows indefinitely larger with time. Of course, in reality, this is not so, because of the presence of nonlinear effects resulting in a limitation of the final amplitude.

Crocco's mathematical study of the nonlinear behavior in the case of a thin annular chamber has, indeed, the purpose of determining the limiting-cycle (periodic solution) of a linearly unstable rocket. However, an even more useful goal is the solution of the problem of nonlinear instability, that is instability which may result during an otherwise stable run, when the application of disturbances of a sufficiently high level results in amplification, while disturbances of lower-level decay. This kind of instability is said to be "triggering instability" and the corresponding "threshold" has a periodic solution which is called "triggering cycle". For this kind of analysis it is immaterial if $\beta < 1$ or $\beta > 1$.

3.1 Solution of the Nonlinear Integral-Differential Equation (29)

If the integral appearing in the last term of Eq. (29) is formally reduced to a simple function $M(\theta)$, the same equation may be written as follows:

$$(\gamma+1)\ell (v-v_m) v_\theta = \bar{u}_n \left[1 + \frac{3\gamma+1}{2} v(\theta) \right] - \frac{2\beta^{1/2}K}{\tau} u_\ell \left[1 + \frac{v(\theta)}{2} \right] M(\theta) \quad (39)$$

adding further simplifications,

$$(v-v_m) v_\theta = A + B v(\theta) - D \left[1 + \frac{v(\theta)}{2} \right] M(\theta) \quad (40)$$

where

$$A = \frac{\bar{u}_n}{(\gamma+1)l} ; B = \frac{(3\gamma+1)}{2(\gamma+1)} \frac{\bar{u}_n}{l} ; D = \frac{2\beta'^{1/2} k u_e}{\tau(\gamma+1)l} \quad (41)$$

are constants for given: $\gamma, l, u_e, \bar{u}_n$ (and hence β), and b (or $\tau = \frac{b}{u_e}$).

Let us define the right-hand side of Eq. (4) as $g(\theta)$, where g is seen to be a linear integral function. With this assumption, Eq. (40) may be written:

$$(v - v_m) v_\theta = g(\theta) \quad (42)$$

$$\text{or: } v_\theta = \frac{g(\theta)}{v - v_m} \quad (43)$$

$$\text{where: } v_m = \frac{1}{2} [v(0) + v(1)] \quad (44)$$

Since we have a periodic solution with discontinuities only at the endpoints of the interval $0 \leq \theta \leq 1$, it follows that there is some θ^* in this range, such that $v(\theta^*) = v_m$. From Eq. (43) it is seen that θ^* is a singular point and for that equation to have meaning it must be:

$$g(\theta^*) = 0 \quad (45)$$

Integration of Eq. (42) yields the formal relation

$$[v(\theta) - v_m]^2 = [v(0) - v_m]^2 + 2 \int_0^\theta g(\theta') d\theta' \quad (46)$$

Evaluation of Eq. (46) at $\theta=1$ and the definition Eq. (45) combine to yield another condition on the function $g(\theta)$, namely:

$$\int_0^1 g(\theta) d\theta = 0 \quad (47)$$

Equation (21) may be considered as a quadratic relation for $g(\theta)$ and, therefore, has two solutions. It may readily be seen that, in the case where a shock exists and so $v(1) \neq v(0)$ we must have:

$$v(\theta) = v_m + \left\{ [v(0) - v_m]^2 + 2 \int_0^\theta g d\theta' \right\}^{1/2} \quad (48a)$$

for $0 \leq \theta \leq \theta^*$, and

$$v(\theta) = v_m - \left\{ [v(0) - v_m]^2 + 2 \int_0^\theta g d\theta' \right\}^{1/2} \quad (48b)$$

for $\theta^* \leq \theta \leq 1$.

In the particular case which arises for $\theta = \theta^*$, $v(\theta^*) = v_m$ and so Eqs. (48) yield:

$$\left\{ [v(0) - v_m]^2 + 2 \int_0^{\theta^*} g d\theta' \right\}^{1/2} = 0 \quad (49)$$

In order to obtain real-valued solutions that are continuous at θ^* , it is necessary that (let us call $G(\theta) = \int_0^\theta g d\theta'$):

$$G(\theta^*) = \int_0^{\theta^*} g d\theta'$$

has a certain negative value such that the term under the square-root sign in Eq. (48) has a local minimum value (which equals zero) at $\theta = \theta^*$. This is consistent with the fact that $g = 0$ at $\theta = \theta^*$.

From Eq. (49) it follows:

$$v_m = v(0) - \sqrt{-2G(\theta^*)} \quad (50)$$

Of course, Eq. (48) is merely an implicit relation for $v(\theta)$, since $g(\theta)$ depends upon $v(\theta)$; however, the equation is in convenient form to solve by Picard's method.

The iteration procedure used begins with assuming $v(\theta) = v(0) + \phi(\theta)$ where $\phi(0) = 0$. An initial guess of $\phi(\theta)$ is made (typically, a sawtooth profile was assumed), but rather than specifying $v(0)$, Eq. (47) is used to calculate $v(0)$.

Knowing $v(0)$ and $\phi(\theta)$, $v(\theta)$ is immediately known and may be substituted into the right-hand side of Eq. (48). Then use is made of Condition (45) for which Eq. (49) is true and by means of Eq. (50), v_m may be determined. Eq. (48) is then used to calculate a new $v(\theta)$. Subtracting $v(0)$ from $v(\theta)$, a new $\phi(\theta)$ is known and then Eq. (48) is again used to calculate a new $v(0)$ and so forth, until the sequences for $v(\theta)$ and v_m converge.

The iteration was performed on an IBM 360/91 computer with the solutions converging in approximately ten to fifteen steps. In all the cases where different initial profiles for $\phi(\theta)$ were assumed, the same limiting solution was obtained. Usually the qualitative behavior is demonstrated immediately (in the case of convergence) by the first step in the iterative procedure.

3.2 Influence of β Upon the Numerical Analysis

As it was stated at the beginning of Section III, it is immaterial whether β is > 1 or < 1 as far as the iteration method is concerned. Only the function $M(\theta)$ is affected because it takes on two different forms according to the value of β compared to 1. For $\beta > 1$, a station θ^* exists in the combustion chamber where the gas velocity gets larger than u_L . This is because at the nozzle entrance $\bar{u}_h > u_L$ and thus the way to calculate v is split in two parts as shown in Eq. (38).

For information, the computer program for $\beta > 1$ has been introduced in Appendix A. The reason is that the situation $\beta > 1$ is more likely to occur in reality and another reason is that its numerical generality makes the case $\beta < 1$ a special case.

3.3 Discussion of Results for Periodic Solutions

One of the most interesting and important conclusions that can be drawn from the numerical solution of Eq. (29) for $\frac{\partial v}{\partial t} = v_t = 0$ is that a shock-type pressure distribution does exist within the annular chamber.

Depending on the magnitude of the gas velocity at the nozzle entrance compared to the constant droplet velocity (i.e., depending on whether $\beta = \bar{u}_n/u_\ell$ is ≤ 1) we can have a shock-type pressure distribution as shown in Fig. 2 or in Fig. 3. In both cases the wave decays sharply at the beginning (low values of θ) or, as to say, decays for high negative values of $d\sigma/d\theta$, especially for the case $\beta > 1$ for which larger amplitudes exist also as a general result.

The characteristic droplet lifetime τ , in the way it is defined by Eq. (35) (time elapsed from the injection of the droplet to the moment it vanishes) influences the shock amplitude and, in a more sensitive way, the energy associated with it as it will be seen later.

Fig. 4 shows how wave amplitudes are distributed for given combustion chamber length and droplet velocity for various values of τ . The wave shape for $\tau = 0.8$ is significant of a situation where for smaller values of τ the convergence towards the solution becomes more difficult to obtain until the situation is reached where no solutions exist in the sense that no convergence is obtained in the numerical iteration procedure.

The periodic solution is a condition of dynamic equilibrium and, as such, may be stable or unstable. If the amplitude is perturbed slightly from the value for a periodic solution, the perturbation may grow or decay. If both positive and negative perturbations grow in absolute magnitude, the periodic solution is unstable (triggering cycle) while if both positive and negative perturbations decay, the periodic solution is stable (limiting cycle). We are not in a position to assert that the periodic solution we obtained is the triggering cycle, and it is likely to be

such, because of small wave amplitudes involved, etc. A systematic search has been performed in order to find the limiting cycle for given values of l , u_1 , β and position at which droplets vanish (actually the parameters were $l = 0.40$, $u_1 = 0.10$, $b = 50\%$, $\beta = 1.1$ and the initial wave amplitude guess was of $O(40)$, which means of the order of 200 times more than the triggering cycle amplitude). From such an initial high value of the wave amplitude all the following wave amplitudes, during the iteration procedure, converged to lower and lower amplitudes, reaching the triggering cycle without being "caught" by any other solution. A third order, highly involved mathematical analysis, could perhaps yield a limiting cycle which certainly exists for $t \rightarrow \infty$.

A comparison can be made with Crocco's previous calculations, where the results are valid only for $\sqrt{\beta} \ll u_e$ and for $\bar{u} \ll u_e$ and no chance is given for $\bar{u}$ to become larger than u_e , as is the case in the present work where we have $\beta > 1$. There it is shown that a shock-type solution can appear only if $l u_{e1}^2 \leq 1/33.8$, i.e., only if $\beta \geq 33.8$ (in fact $u_{e1} = u_e / \mu^{1/2} = u_e / \bar{u}_n^{1/2}$). Actually this condition is quite well satisfied in the present case as long as β is small; it is possible to show that Eq. (29) yields the same conclusions mentioned above as long as $\beta \rightarrow 0$ and numerical computations have shown that it is no longer possible to obtain solutions for $\beta < 0.15$ ($l = 0.15$ and $u_e = 0.02$ have been taken).

From Figures 2 and 3 it is easy to compare the breadth of the wave profile in the two cases where the gas velocity at the nozzle entrance is smaller than u_e ($\beta = 0.8$) and larger than u_e ($\beta = 1.1$). The more β increases, the more the wave amplitude increases (for $\beta = 5$, the value of $p/\gamma = v(0)$ (at $\theta = 0$) reaches 0.46). The most significant way to look at the behavior of the shock-type pressure distribution is by studying the behavior of the energy associated with the wave, versus the droplet characteristic lifetime τ . In this study we are considering the case of a spinning wave and the energy $\mathcal{E}$ associated with it (kinetic energy + potential energy) that is proportional to $\int_0^1 v^2 d\theta = \int_0^1 (p/\gamma)^2 d\theta$. Figures 5 and 6 show the behavior of $\mathcal{E}$ versus τ for $\beta = 0.8$ and for $\beta = 1.1$ and in each case for given l and u_e .

The parameter τ has been made variable by changing the combustion distribution length (station b at which droplets vanish) and keeping ℓ and u_e constant. Considering that the energy associated with the triggering cycle should be minimum where the instability is most likely to happen, (because more energy should be put in the wave in order to drive the instability), such a condition is not reached for $\beta = 0.8$, Fig. 5 (note that the curve is smoother for $\tau = 0.5$). A common feature appears evident from Figures 5 and 6; the effect of τ upon the energy $\mathcal{E}$ gets less and less evident as long as τ increases consistent with Crocco's previous calculations. For $\beta = 1.1$, as it is shown in Fig. 6, the instability is more likely to occur for values of τ ranging from 0.5 to 1.2 (from $\ell = 0.2$ to $\ell = 0.4$). Numerical computations have shown that, at least for $u_e = 0.1$, and for values of ℓ close to 0.3, it is easier to obtain solutions for $\beta > 1$ than for $\beta < 1$. This, perhaps, is the reason why we could not reach a minimum for $\mathcal{E}$ in the case of $\beta = 0.8$. A discontinuity appears for $\tau = 1.4$, but on the ground of theoretical analysis no singularity appears in Eq. (29) to give a justification and besides pressure distributions for values of τ in the neighborhood of the discontinuity appear to be normally behaved. Nevertheless, convergence is reached faster, for $\beta < 1$ when ℓ , as well as u_e , are given lower values.

Figure 7 shows that the energy associated with the triggering cycle (everything being equal) becomes smaller when the droplet velocity increases and that it is influenced in lesser extent by τ .

A common characteristic appears from Figure 6 and that is for values of τ lower than $\tau_{\mathcal{E}_{\min}}$ (value of τ for which $\mathcal{E}$ is minimum) the tendency towards instability becomes increasingly high and at a very fast rate, up to a point where no valid solutions are obtained (dotted portions of curves for $\ell = \text{constant}$).

Crocco and Mitchell¹⁵ in their theoretical investigation on the transverse combustion instability for an annular motor, used the n- τ model and in doing so they expressed the second

order combustion rate Q'_2 as follows:

$$Q'_2(\xi, \theta) = n \bar{a}_{1\xi}(\xi) \left[p'_1(\xi, \theta) - p'_1(\xi, \theta - \tilde{\tau}) \right] \quad (51)$$

where n is the interaction index, $\tilde{\tau}$ the stretched time lag and $\theta - \tilde{\tau}$ a retarded variable.

In the present work the instantaneous combustion rate is related to other physical factors such as the droplet relative volume, the gas density and the gas relative velocity, as can be seen from Eq. (25). From Crocco's time lag theory we can assume that the physical factors are correlated and we can disregard the explicit effects of all the factors except that of the pressure and say that the rate of combustion and pressure are related together by means of the interaction index n . Under this assumption we can say that a given local and instantaneous combustion rate, at any value of the interaction index n , corresponds to a pressure disturbance p . Through the integration $\int_0^1 (p/\bar{p})^2 d\theta$ we can say, from the present work, that we still have two values of τ for any ξ in the region of ξ_{minimum} .

SECTION IV

CONCLUSIONS

The theoretical work described in this report provides conformation to the experimental observations of Harrje, Varma and others that shock-type waves are characteristic of transverse mode instability in an annular chamber.

The following is a listing of the practical considerations that one can derive from this theoretical investigation of an annular rocket motor experiencing first tangential mode combustion instability.

1) For the more realistic case of $\beta = \bar{u}_n/u_\ell > 1$ (since the gas velocity at the nozzle entrance is most likely to be larger than the liquid propellant velocity) Fig. 6 shows that in the particular case of $\beta = 1.1$ and $u_\ell = 0.10$ a combustion distribution extending to approximately 30% of the chamber length must be avoided. With such a combustion concentration, the energy feedback required to trigger combustion instability is minimized. The behavior of the energy for different values of β has not been investigated.

2) Figure 6 also shows that for a combustion distribution below 20-25% of the combustion chamber length no shock-type instability can be triggered. This would appear to indicate that these combustion lengths represent the best point to complete combustion. If the combustion distribution is required to be spread further along the chamber axis, then Fig. 6 shows that it is far better to attempt to provide very large values of τ (characteristic time required for a complete combustion). This takes advantage of the higher energies required for triggering the instability with large τ and thereby moves the operational point further from the limits of unstable combustion.

3) Again from Fig. 6 it is shown that improved stability is achieved if the combustion chamber length is taken as small as possible compared to the mean circumference of the annulus (for example, $\ell = \ell^*/\pi d^* = 0.33$ represents the length of a square

longitudinal cross section combustion chamber, i.e., where $\ell^* = d$, here d^* is the dimensional mean diameter of the annulus). For large values of τ it is far more convenient to operate with a short combustion chamber, since a larger quantity of feedback energy is then required (i.e., more stable operation).

4) It is also suggested to operate at as low an injection velocity as possible in order to increase the feedback energy required for triggering the instability. This trend is shown by Fig. 7 where for $\beta = 1.1$ and $\ell = 0.3$ larger energy requirements for instability are associated with $u_\ell = 0.10$ than with $u_\ell = 0.12$. Actually, the parameters u_ℓ , b and ℓ are not completely independent, because of the effect of u_ℓ on the combustion distribution.

5) For a given injection velocity and chamber length, another way to stabilize an engine (i.e., require more energy to drive the instability) is by increasing β which means making the gas velocity at the nozzle entrance as large as possible compared to the liquid injection velocity. This is shown by Figs. 5 and 6 for a transition from $\beta = 0.8$ to $\beta = 1.1$, where shock amplitudes and associated energies become larger and larger (for $\beta = 5$, $P'_{1/8} = 0.46$) enlarging the safety margin from the instability limit. Likewise, as has been pointed out above, the increasing of β by increasing the gas velocity at the nozzle entrance would also affect other parameters which in this specific case have been taken as constant. It should be pointed out that in this theoretical investigation the energy associated with the shock wave has been considered per unit volume of the combustion chamber. If, on the other hand, energy computations for the overall combustion chamber volume are made, that situation would lead towards reversing the relative position of the curves, i.e., the longer chamber with more volume would exhibit the larger overall energy levels (Fig. 6).

To summarize the design requirements needed to operate as far as possible from the triggering limit one should (apart from requiring solutions such as baffles and acoustic liners):

a) Either distribute the combustion over a range not exceeding

20% of the combustion chamber length, or spread the combustion over the entire length of the combustion chamber (this is to avoid the minimum energy zone of 30% of the combustion length).

- b) Keep the combustion chamber length as small as possible compared to the mean annulus circumference (πd^*).
- c) Maximize β by injecting the liquid propellant at the lowest velocity and maintaining the gas velocity at the nozzle entrance as high as possible.

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APPENDIX A
Computer Program for Periodic Solution with $\beta > 1$

FORTRAN IV G LEVEL 19

MAIN

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0001      DIMENSION F(200),G(200),Q(200),FT(200),ET(200),FINE(200),DIF(200),
1 FINE(200),TIC(200),PSI(200),V(200),CAPG(200),SMALLG(200),V1(200),
1 Q1(200),Q2(200),Q3(200),CRI(200),V2(200),V1SQ(200)
0002      READ(5,10) JDELTA,ITETA,JVNB,S,VK,BE,UL,PCNT,VINT,EN
0003      10 FORMAT(3I4,6F10.5,F8.3)
0004      READ(5,14) COEF,DVSN
0005      14 FORMAT(2F10.5)
0006      READ(5,500) (PSI(K3),K3=1,21)
0007      500 FORMAT(7(2X,F8.0),10X)
0008      WRITE(6,247) EN,COEF,DVSN
0009      247 FORMAT(/30X,6H EN = ,F10.4,8H COEF = ,F10.4,8H DVSN = ,F10.4/)
0010      WRITE(6,20)JDELTA,ITETA,JVNB,S,VK,BE,UL,PCNT
0011      20 FORMAT(10H DELTA S = ,I4,9H TETA S = ,I4,9H VK(0) = ,I4,5H L = ,
1 F7.4,15H FIRST VK(0) = ,F9.4,8H BETA = ,F7.4,
1 6H UL = ,F7.4,5H B = ,F7.4)
0012      WRITE(6,22)VINT
0013      22 FORMAT(15H DELTA VK(0) = ,F9.4)
0014      VKAPPA=VK
0015      CA=0.5*ALOG(ABS((BE**0.5+1.0)/(BE**0.5-1.0)))
0016      TAU=PCNT*S/UL
0017      R=S/UL
0018      PAI=3.141592
0019      RAT=((BE**0.5)*TAU)/(2.0*(CA+PAI*0.5))
0020      AC=(BE*UL)/(2.2*S)
0021      BC=(BE*UL)*1.044/S
0022      DC=(2.0*(BE**0.5)*(CA+PAI*0.5)*UL)/(TAU*2.2*S)
0023      DLTSTR=TAU*CA/(CA+0.5*PAI)
0024      WRITE(6,23) AC,BC,DC,DLTSTR
0025      23 FORMAT(10X,5H A = ,F7.4,5H B = ,F7.4,5H D = ,F7.4,10H DLTSTR = ,F7
1 1.4)
0026      WRITE(6,25) CA,TAU,R,RAT
0027      25 FORMAT(11H K(BETA) = ,1PE12.4,7H TAU = ,1PE12.4,
1 8H L/UL = ,1PE12.4,7H RAT = ,1PE12.4)
0028      TETA=ITETA
0029      DTETA=1.0/TETA
0030      ITETP=ITETA+1
0031      DELT=JDELTA
0032      DDELT=R/DELT
0033      JDELP=JDELTA+1
0034      LAPP=2
0035      LAST=1
0036      15 MARY=1
0037      VKINT=0.10*VK
0038      MINA=1
0039      LIRA=1
0040      LAMA=1
0041      MAMA=1
0042      MELA=1
0043      LUNA=ITETP
0044      26 DO 800 I=1,JVNB
0045      WRITE(6,12) VK
0046      12 FORMAT(/10H VK(0) I = ,F12.7)
0047      24 GO TO (27,11),MAMA
0048      27 TI=0.0
0049      KLV=1
0050      11 DO 700 K3=KLV,ITETP

```

```
0051      13 DJ=0.0
0052      DO 620 K2=1,JDELP
0053      GO TO (284,286),LAPPA
0054      284 FRK=1.0+1.1*(VK-0.5*COEF)
0055      GO TO 1416
0056      286 FRK=1.0+1.1*(VK+0.5*PSI(ITETP))
0057      1416 MAGGIO=1
0058      MAGGIO=1
0059      LAFIN=1
0060      LIMBO=1
0061      IF(DJ.GT.DLTSTR) GO TO 41
0062      40 A=TI-FRK*DJ
0063      KOV=1
0064      GO TO 42
0065      41 DJMDST=DJ-DLTSTR
0066      IF(DJMDST.LT.DDELT) GO TO 64
0067      203 A=TI-FRK*DJMDST
0068      MISA=1
0069      KOV=2
0070      LAFIN=2
0071      GO TO 66
0072      64 LIRSTK=K2
0073      SECT=DJMDST
0074      LASTK=LIRSTK-1
0075      IF(LASTK.LT.3) GO TO 242
0076      GO TO 243
0077      242 WRITE(6,244)
0078      244 FORMAT(2X,14F LASTK IS LT 3)
0079      STOP
0080      243 BEBE=DJ
0081      A=TI-FRK*DLTSTR
0082      DJ=DLTSTR
0083      66 GO TO (45,46), LAFIN
0084      45 MAGGIO=2
0085      GO TO 42
0086      46 MAGGIO=3
0087      LIMBO=2
0088      GO TO 42
0089      42 J=A
0090      251 B=J
0091      EI=A
0092      RI=B
0093      D=B-A
0094      IAC=1
0095      IF(D) 50,60,100
0096      50 IF((A.GT.0.0).AND.(A.LT.1.0)) GO TO 110
0097      ALFA=B
0098      GO TO 30
0099      60 IF(A) 70,80,90
0100      70 ALFA=B
0101      B=B+1.0
0102      GO TO 30
0103      80 ALFA=0.0
0104      B=TI
0105      GO TO 30
0106      90 IF(A.NE.1.0) GO TO 98
```

```

0107          92 ALFA=0.0
0108          B=TI
0109          GO TO 30
0110          98 ALFA=B-1.0
0111          B=B-1.0
0112          GO TO 30
0113          100 ALFA=B-1.0
0114          GO TO 30
0115          110 ALFA=0.0
0116          E=TI-A
0117          IF(E) 120,130,140
0118          120 B=TI
0119          GO TO 30
0120          130 ANU1=1.0
0121          MONA=1
0122          GO TO 600
0123          140 B=TI
0124          30 DX=(B-A)/20.0
0125          DDX=ABS(DX)
0126          X=A
0127          DO 180 L=1,21
0128          CSI=X-ALFA
0129          GO TO (392,393),LAPPA
0130          392 FUN=COEF*(2.0*(CSI**EN)-EN*(CSI**2))/(EN-2.0)
0131          GO TO 290
0132          393 TITTI=0.0
0133          DO 334 IL=1,LUNA
0134          IF(IL.EQ.1) GO TO 332
0135          ILM=IL-1
0136          TITTIM=TITTI-DTETA
0137          336 IF(CSI.GE.TITTI) GO TO 332
0138          FUN=PSI(ILM)+(CSI-TITTIM)*(PSI(IL)-PSI(ILM))/DTETA
0139          GO TO 290
0140          332 TITTI=TITTI+DTETA
0141          334 CONTINUE
0142          290 GO TO(335,48), LIMBO
0143          335 F(L)=(1.0+0.5*(VK+FUN))*(((BE-1.0)**2)*(SINH(CA-(CA+0.5*PAI)
          1*(DJ*FRK-(TI-X))/(FRK*TAU)))**4+((VK+FUN)/UL)**2)**0.25
0144          LOVA=1
0145          GO TO 52
0146          48 F(L)=(1.0+0.5*(VK+FUN))*(((BE-1.0)**2)*(SIN((CA+0.5*PAI)*(DJ*FRK-
          1TI-X))/(FRK*TAU)-CA)**4+((VK+FUN)/UL)**2)**0.25
0147          LOVA=2
0148          52 X=X+DX
0149          180 CONTINUE
0150          NSEGAP=21
0151          190 CALL QSF(DCX,F,G,NSEGAP)
0152          IF((EI.GE.0.0).AND.(EI.LE.1.0)) GO TO 400
0153          GO TO (195,210,220,230),IAC
0154          195 AINT=G(NSEGAP)
0155          200 ALFA=0.0
0156          A=0.0
0157          B=1.0
0158          IAC=2
0159          GO TO 30
0160          210 PER=G(NSEGAP)

```

```

0161      IF(EI.LT.0.0) GO TO 225
0162      BINT=(BI-1.0)*PER
0163      IAC=3
0164      A=1.0
0165      GO TO 92
0166      220 ANU1=1.0-((CA+0.5*PAI)/(FRK*TAU*(BE**0.5)))*(AINT+BINT+G(NSEGAP))
0167      MONA=2
0168      GO TO 600
0169      225 BINT=ABS(BI)*PER
0170      IAC=4
0171      A=0.0
0172      GO TO 80
0173      230 GO TO 220
0174      400 GO TO 231
0175      231 ANU1=1.0-((CA+0.5*PAI)/(FRK*TAU*(BE**0.5)))*G(NSEGAP)
0176      MONA=3
0177      GO TO (350,351),LAPPA
0178      350 RUN=COEF*(2.0*(TI**EN)-EN*(TI**2))/(EN-2.0)
0179      GO TO 600
0180      351 RUN=PSI(K3)
0181      600 GO TO (43,16,47),MAGGIO
0182      43 IF(DJ.GT.DLTSTR) GO TO 609
0183      16 Q1(K2)=((((BE-1.0)**2)*(SINH(CA-(CA+0.5*PAI)*DJ/TAU))**4+
      1 ((VK+RUN)/UL)**2)**0.25)*ANU1
0184      Q1STR=Q1(K2)
0185      252 IF((DJ-DLTSTR).EQ.0.0) GO TO 245
0186      GO TO 209
0187      245 CINTST=1.0-ANU1
0188      ANUSTR=ANU1
0189      STAR2 =((((BE-1.0)**2)*(SIN(((CA+0.5*PAI)*DJ/TAU)-CA))**4+
      1 ((VK+RUN)/UL)**2)**0.25)*ANUSTR
0190      MISA=2
0191      DJ=BERE
0192      209 IF(Q1(K2).LT.0.0) GO TO 249
0193      GO TO 609
0194      249 JL=K2-1
0195      IF(JL.EQ.1) GO TO 228
0196      GO TO 227
0197      228 G(JL)=0.0
0198      GO TO 226
0199      227 IF(JL.EQ.2) GO TO 224
0200      GO TO 223
0201      224 JLM=JL-1
0202      G(JL)=(Q1(JLM)+Q1(JL))*DDELT*0.5
0203      GO TO 226
0204      223 CALL QSF(DDELT,Q1,G,JL)
0205      226 D1=DJ-DDELT
0206      Z1=((D1)*Q1(K2)-DJ*Q1(JL))/(Q1(K2)-Q1(JL))
0207      ATRI1=0.5*(Z1-D1)*Q1(JL)
0208      FT(K3)=G(JL)+ATRI1
0209      254 GO TO (361,363),LAPPA
0210      47 ANU3=ANU1-CINTST
0211      Q2(K2)=((((BE-1.0)**2)*(SIN(((CA+0.5*PAI)*DJ/TAU)-CA))**4+
      1 ((VK+RUN)/UL)**2)**0.25)*ANU3
0212      256 IF(Q2(K2).LT.0.0) GO TO 610
0213      609 IF(DJ.LT.DLTSTR) GO TO 250

```

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0214      IF((DJ.GT.DLTSTR).AND.(DJMDST.LT.DDELT)) GO TO 206
0215      GO TO 250
0216      206 IF(MISA.EQ.2) GO TO 203
0217      250 DJ=DJ+DDELT
0218      620 CONTINUE
0219      623 WRITE(6,624)
0220      624 FORMAT(10X,29H NUI DOES NOT BECOME NEGATIVE)
0221      STOP
0222      610 KK=K2-1
0223      MILLA=1
0224      IF(DJMDST.LT.DDELT) GO TO 221
0225      IF((DJMDST.LT.(2*DDELT)).AND.(DJMDST.GT.DDELT)) GO TO 237
0226      GO TO 238
0227      237 KK=K2-1
0228      G(MILLA)=0.0
0229      CALL QSF(DDELT,Q1,G,LASTK)
0230      Q1INT=G(LASTK)
0231      GO TO 239
0232      238 IF((DJMDST.LT.(3*DDELT)).AND.(DJMDST.GT.(2*DDELT))) GO TO 241
0233      GO TO 55
0234      241 Q2INT=(Q2(LIRSTK)+Q2(KK))*DDELT*0.5
0235      CALL QSF(DDELT,Q1,G,LASTK)
0236      Q1INT=G(LASTK)
0237      GO TO 239
0238      221 Q2(KK)=STAR2
0239      236 CALL QSF(DDELT,Q1,G,LASTK)
0240      Q1INT=G(LASTK)
0241      234 AREA1=(Q1(LASTK)+Q1STR)*(DDELT-SECT)/2.0
0242      DD=DLTSTR
0243      Z=((DD)*Q2(K2)-DJ*Q2(KK))/(Q2(K2)-Q2(KK))
0244      ATRI=0.5*(Z-DD)*Q1STR
0245      AREA2=0.0
0246      G(MILLA)=0.0
0247      GO TO 631
0248      55 MILLA=KK-LASTK
0249      DO 61>NNL=1,MILLA
0250      61 Q3(NNL)=Q2(LASTK+NNL)
0251      CALL QSF(DDELT,Q1,G,LASTK)
0252      Q1INT=G(LASTK)
0253      CALL QSF(DDELT,Q3,G,MILLA)
0254      Q2INT=G(MILLA)
0255      239 AREA1=(Q1(LASTK)+Q1STR)*(DDELT-SECT)/2.0
0256      AREA2=((Q2(LIRSTK)+STAR2)*SECT)/2.0
0257      DD=DJ-DDELT
0258      Z=((DD)*Q2(K2)-DJ*Q2(KK))/(Q2(K2)-Q2(KK))
0259      ATRI=0.5*(Z-DD)*Q2(KK)
0260      631 FT(K3)=Q1INT+Q2INT+AREA1+AREA2+ATRI
0261      257 GO TO (361,363),LAPPA
0262      361 PSI(K3)=COEF*(2.0*(TI**EN)-EN*(TI**2))/(EN-2.0)
0263      363 GO TO (699,899),LIRA
0264      899 V(K3)=VK+PSI(K3)
0265      SMALLG(K3)=AC+BC*V(K3)-DC*(1.0+0.5*V(K3))*FT(K3)
0266      IF(K3.EQ.1) GO TO 903
0267      MAL=K3-1
0268      CAPG(K3)=(SMALLG(K3)+SMALLG(MAL))*DTETA/2.0+CAPG(MAL)
0269      CAPG(ITETP)=0.0

```

```
0270      GO TO 904
0271      903 CAPG(K3)=0.0
0272      GO TO 632
0273      904 PROD=SMALLG(K3)*SMALLG(MAL)
0274      998 IF(((K3.EQ.ITETP).AND.(MINA.EQ.1)) GO TO 900
0275      IF(PROD.LT.0.0) GO TO 831
0276      GO TO 632
0277      831 ZFGO=TI-ABS((DTETA*SMALLG(K3))/(ABS(SMALLG(K3))+ABS(SMALLG(MAL
      1))))
0278      STORM1=TI-DTETA
0279      ZATRI=(ZFGO-STORM1)*SMALLG(MAL)/2.0
0280      PATRI=(TI-ZFGO)*SMALLG(K3)/2.0
0281      QUAD=PATRI+ZATRI
0282      CAPG(K3)=CAPG(MAL)+QUAD
0283      WRITE(6,444) CAPG(K3)
0284      444 FORMAT(1X,8H CAPG = ,1PE16.6)
0285      GO TO (842,843,560,561,562,563),LAMA
0286      842 CGST1=CAPG(MAL)+ZATRI
0287      ZFGO1=ZFGO
0288      564 CAPGST=CGST1
0289      TISTAR=ZFGO1
0290      WRITE(6,540) ZFGO1
0291      540 FORMAT(1X,9H ZFGO1 = ,1PE16.6)
0292      LAMA=2
0293      GO TO 632
0294      843 CGST2=CAPG(MAL)+ZATRI
0295      ZFGO2=ZFGO
0296      WRITE(6,550) ZFGO2
0297      550 FORMAT(1X,9H ZFGO2 = ,1PE16.6)
0298      LAMA=3
0299      IF(CGST1.LT.CGST2) GO TO 632
0300      CAPGST=CGST2
0301      TISTAR=ZFGO2
0302      GO TO 632
0303      560 CGST3=CAPG(MAL)+ZATRI
0304      ZFGO3=ZFGO
0305      LAMA=4
0306      IF(CAPGST.GT.CGST3) GO TO 565
0307      GO TO 632
0308      565 CAPGST=CGST3
0309      TISTAR=ZFGO3
0310      GO TO 632
0311      561 CGST4=CAPG(MAL)+ZATRI
0312      ZFGO4=ZFGO
0313      LAMA=5
0314      IF(CAPGST.GT.CGST4) GO TO 566
0315      GO TO 632
0316      566 CAPGST=CGST4
0317      TISTAR=ZFGO4
0318      GO TO 632
0319      562 CGST5=CAPG(MAL)+ZATRI
0320      ZFGO5=ZFGO
0321      LAMA=6
0322      IF(CAPGST.GT.CGST5) GO TO 567
0323      GO TO 632
0324      567 CAPGST=CGST5
```

```
0325      TISTAR=ZFGC5
0326      GO TO 632
0327      563 CGST6=CAPG(MAL)+ZATRI
0328      ZFG06=ZFG0
0329      IF(CAPGST.GT.CGST6) GO TO 568
0330      GO TO 632
0331      568 CAPGST=CGST6
0332      TISTAR=ZFG06
0333      699 V(K3)=VK+PSI(K3)
0334      SMALLG(K3)=AC+BC*V(K3)-DC*(1.0+0.5*V(K3))*FT(K3)
0335      632 TI=TI+DTETA
0336      700 CONTINUE
0337      CALL QSF(DTETA,SMALLG,G,ITETP)
0338      TIC(I)=G(ITETP)
0339      WRITE(6,283) TIC(I)
0340      283 FORMAT(20X,7H TIC = ,1PE16.6)
0341      9976 GO TO (802,803),MARY
0342      802 IF(I.EQ.1) GO TO 801
0343      IM=I-1
0344      TEST=TIC(I)*TIC(IM)
0345      IF(TEST.LT.0.0) GO TO 805
0346      801 VK=VK+VINT
0347      IF((I.EQ.1).AND.(TIC(I).GT.0.0)) GO TO 248
0348      IF(I.EQ.JVNB) GO TO 248
0349      GO TO 800
0350      248 STOP
0351      800 CONTINUE
0352      805 GIC1=TIC(IM)
0353      GIC2=TIC(I)
0354      810 X1=VK-VINT
0355      X2=VK
0356      812 X=(X1*GIC2-X2*GIC1)/(GIC2-GIC1)
0357      VK=X
0358      MARY=2
0359      GO TO 26
0360      803 WRITE(6,729)
0361      729 FORMAT(50X,8H M(TETA))
0362      WRITE(6,730) (FT(K3),K3=1,ITETP)
0363      730 FORMAT(47X,1PE16.6)
0364      LIRA=2
0365      MAMA=1
0366      GO TO 26
0367      900 IF(LAMA.EQ.1) GO TO 262
0368      IF(LAMA.EQ.2) GO TO 263
0369      GO TO 264
0370      262 WRITE(6,268)
0371      268 FORMAT(/20X,26H SMALLG(K3) DOES NOT CROSS)
0372      STOP
0373      263 IF(CGST1.GT.0.0) GO TO 266
0374      GO TO 267
0375      266 WRITE(6,269)
0376      269 FORMAT(/20X,42H SMALLG(K3) CROSSES ONCE BUT CGST1 GT ZERO)
0377      267 CAPGST=CGST1
0378      TISTAR=ZFG01
0379      264 WRITE(6,571) CAPGST,TISTAR
0380      571 FORMAT(2X,10H CAPGST = ,1PE16.6,10H TISTAR = ,1PE16.6)
```

```
0381      921 IF(CAPGST.LT.0.0) GO TO 923
0382      WRITE(6,922)
0383      922 FORMAT(/50X,25H G(TETA STAR) IS POSITIVE)
0384      STOP
0385      923 VAVRG=VK-(-(2.0*CAPGST))*0.5
0386      WRITE(6,924) VAVRG
0387      924 FORMAT(1X,8H V(M) = ,1PE16.6)
0388      WRITE(6,970) TI
0389      970 FORMAT(1X,6H TI = ,F7.4)
0390      THI=0.0
0391      DO 931 LL=1,ITETP
0392      UNO=(VK-VAVRG)**2+2.0*CAPG(LL)
0393      IF(UNO.GE.0.0) GO TO 928
0394      WRITE(6,927)
0395      927 FORMAT(/20X,22H V1(TETA) IS IMAGINARY)
0396      STOP
0397      928 IF(THI.LT.TISTAR) V1(LL)=VAVRG+UNO**0.5
0398      IF(THI.GT.TISTAR) V1(LL)=VAVRG-UNO**0.5
0399      IF(LAST.EQ.1) GO TO 292
0400      DIF(LL)=ABS(V2(LL)-V1(LL))
0401      292 CRI(LL)=V1(LL)-VK
0402      PSI(LL)=(CRI(LL)+PSI(LL))/2.0
0403      V2(LL)=V1(LL)
0404      VISQ(LL)=V1(LL)*V1(LL)
0405      WRITE(6,929) V1(LL) ,LL,CRI(LL)
0406      929 FORMAT(1X,12H V1(TETA) = ,1PE16.6,2X,6H K3 = ,I4,
13X,12H PSI1(K3) = ,1PE14.6)
0407      THI=THI+DTETA
0408      931 CONTINUE
0409      IF (LAST.EQ.1) GO TO 385
0410      DO 380 NN=1,ITETP,5
0411      WRITE(6,293) DIF(NN)
0412      293 FORMAT(/10X,7H DIF = ,1PE16.6)
0413      IF(DIF(NN).LT.0.0010) GO TO 380
0414      GO TO 385
0415      380 CONTINUE
0416      CALL QSF(DTETA,VISQ,G,ITETP)
0417      ENERGY=G(ITETP)
0418      WRITE(6,3901) ENERGY
0419      3901 FORMAT(50X,10H ENERGY = ,1PE16.4)
0420      WRITE(6,390)
0421      390 FORMAT(/////50X,17H BRAVO GIOVANNINO)
0422      GO TO 399
0423      385 LAPP=2
0424      LAST=2
0425      VK=VKAPPA
0426      GO TO 15
0427      399 STOP
0428      END
```

APPENDIX B

Program Outline For Transient Solutions

The nonlinear integro-differential Eq.(29), after the formal simplifications seen in Section 3.1, will read as follows:

$$(\nu - \nu_m) \frac{\partial \nu}{\partial \theta} = A + B \nu(\theta, t) - D \left[1 + \frac{\nu(\theta, t)}{2} \right] M(\theta, t) + \frac{2}{\gamma+1} \frac{\partial \nu}{\partial t} \quad (I)$$

Let us call again:

$$A + B \nu(\theta, t) - D \left[1 + \frac{\nu(\theta, t)}{2} \right] M(\theta, t) = g(\theta, t)$$

thus, (I) becomes:

$$(\nu - \nu_m) \frac{\partial \nu}{\partial \theta} = g + \frac{2}{\gamma+1} \frac{\partial \nu}{\partial t} \quad (II)$$

The easiest way to think of a solution for this nonlinear partial integro-differential equation (any other method such as the method of characteristics, is made very hard by the unknown behavior of the rather complicated form of $g(\theta, t)$) is to write Eq. II as follows:

$$(\nu - \nu_m) \frac{\Delta \nu}{\Delta \theta} = g + \frac{2}{\gamma+1} \frac{\Delta \nu}{\Delta t} \quad (III)$$

For a given solution of the wave equation at $t = t_0$ and for some given Δt , we are able to find at $t = t_0 + \Delta t$ a new wave solution as long as we regard the equation dependent on θ only.

To give now an expression for Δt let us consider:

$$\theta = ft - y$$

The shock velocity is given by: $dy/dt = f$ or, considering finite increments, by $\Delta y/\Delta t = f$. For a period, $\Delta y = 1$ then: $\Delta t = 1/f$ where $f = 1 + \frac{\gamma+1}{2} \nu_m$. With these elements it is easy to work with the iteration procedure as it has been

done in the case of periodic solutions.

It must be understood that great care has to be taken in defining $\Delta v / \Delta t$ in that portion of the transverse velocity distribution which experiences the steepest slope (i.e., for early values of θ). It should be wise to take values of $\Delta \theta$ as small as possible to approach a smooth v_θ distribution along the period from 0 to 1.

Transverse mode combustion instability

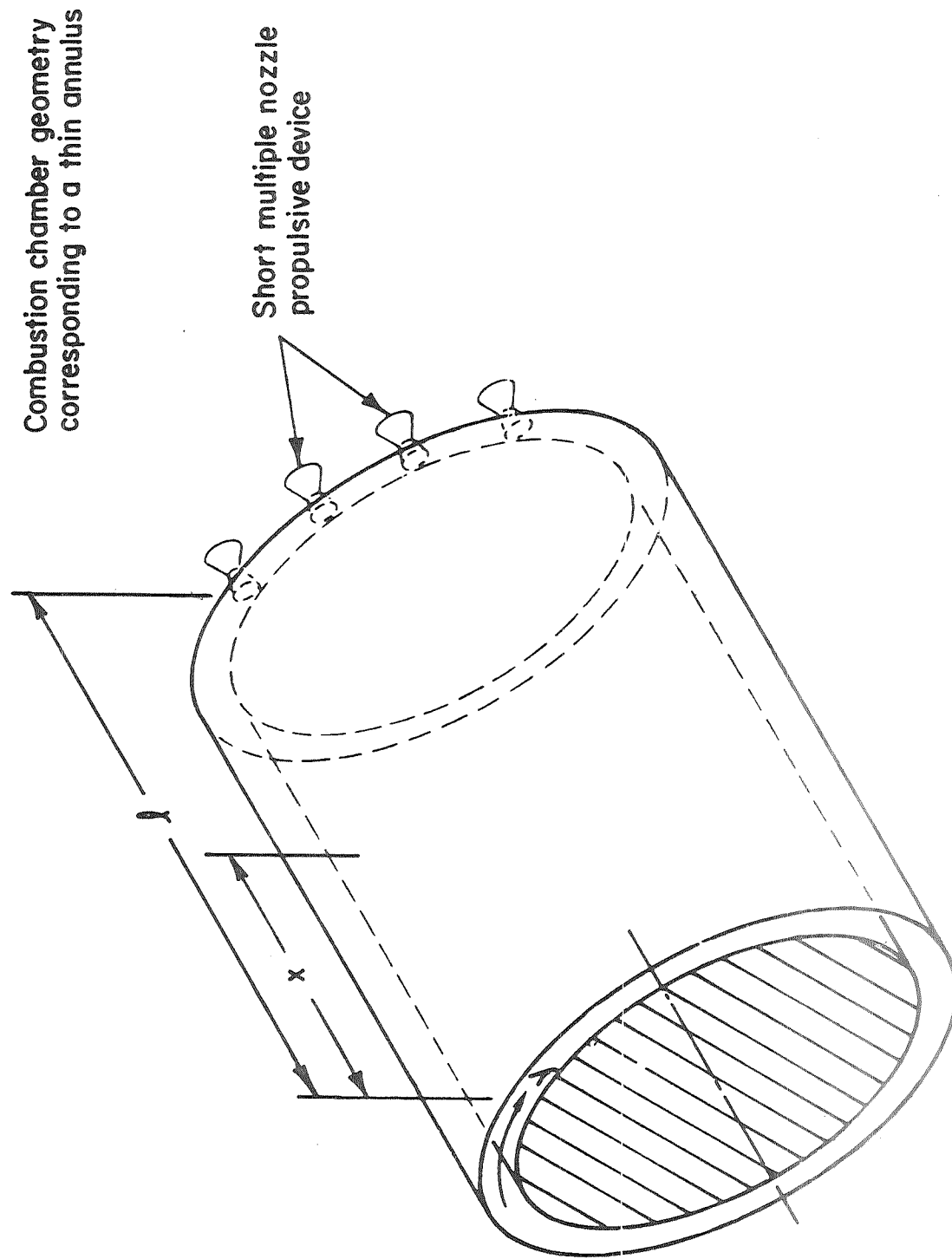
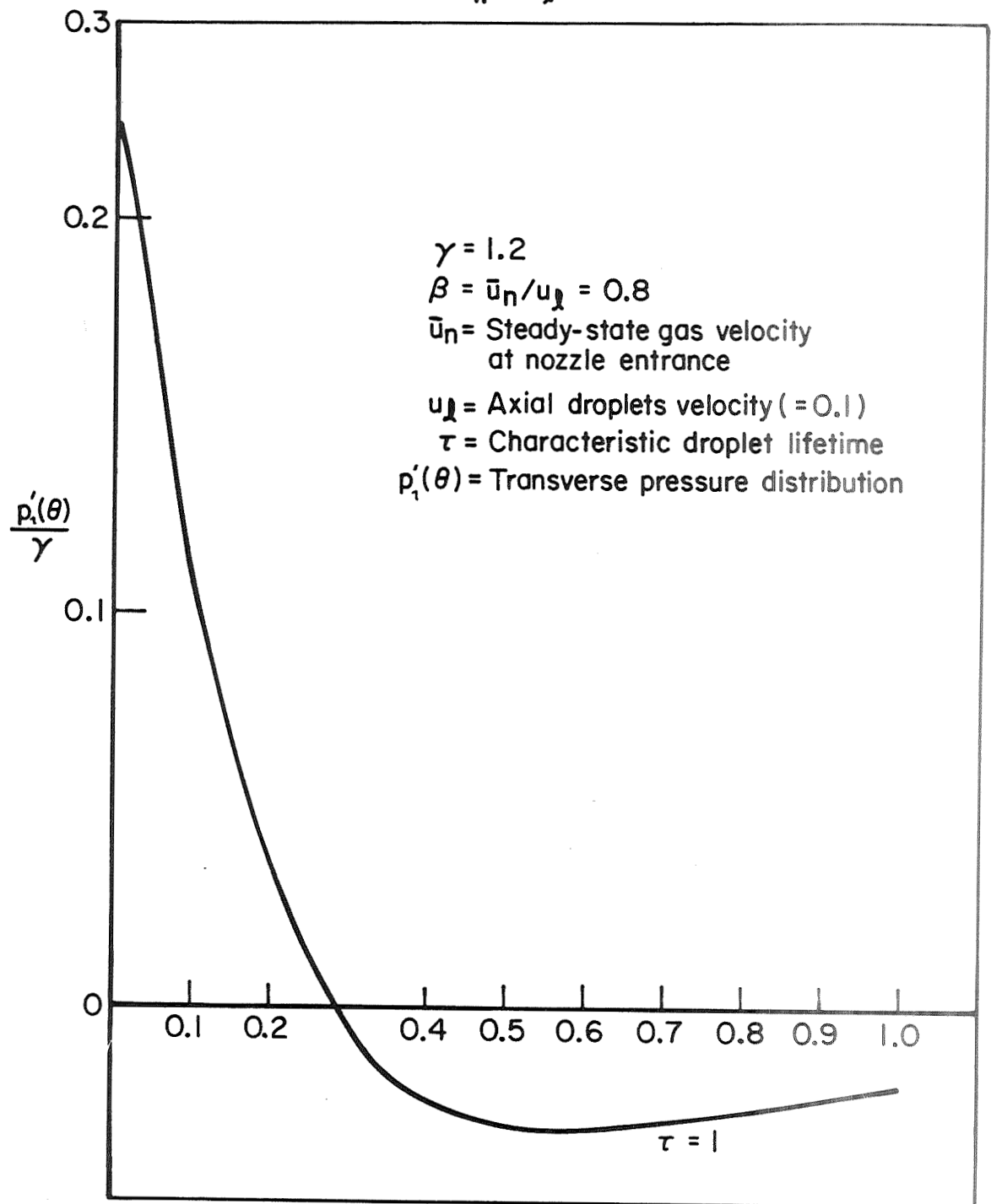


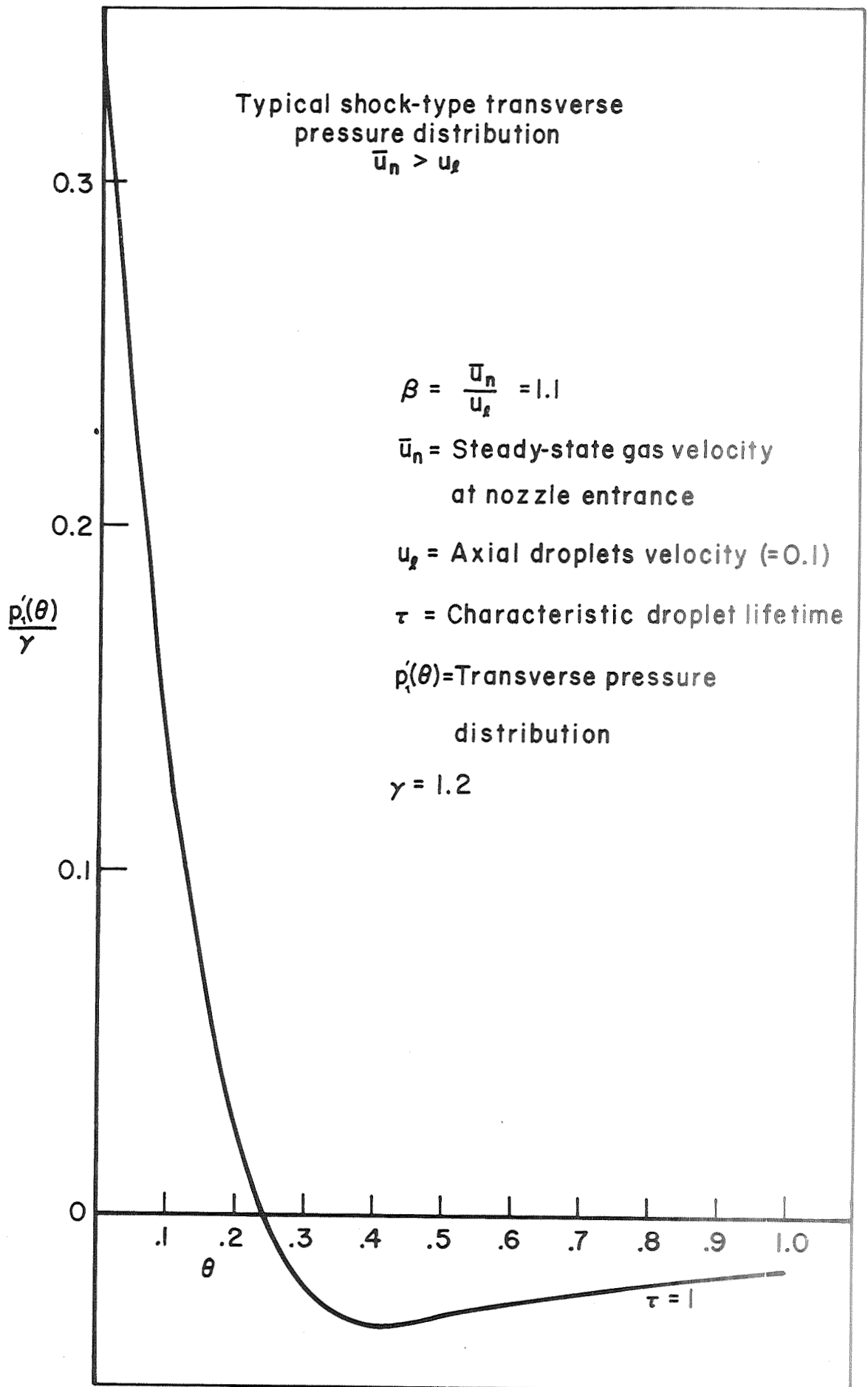
Figure 1

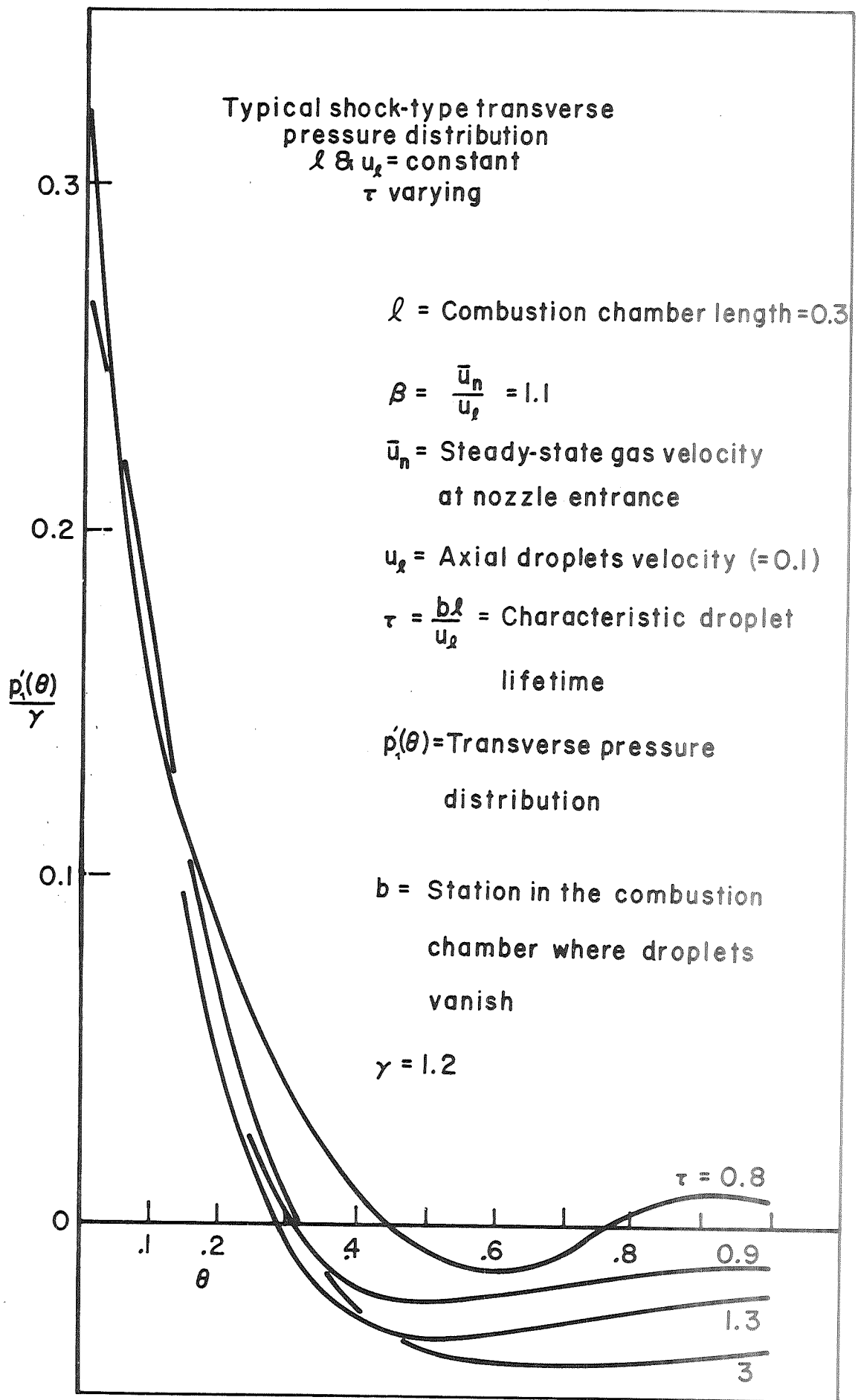
Typical shock-type transverse
pressure distribution

$$\bar{u}_n < u_d$$



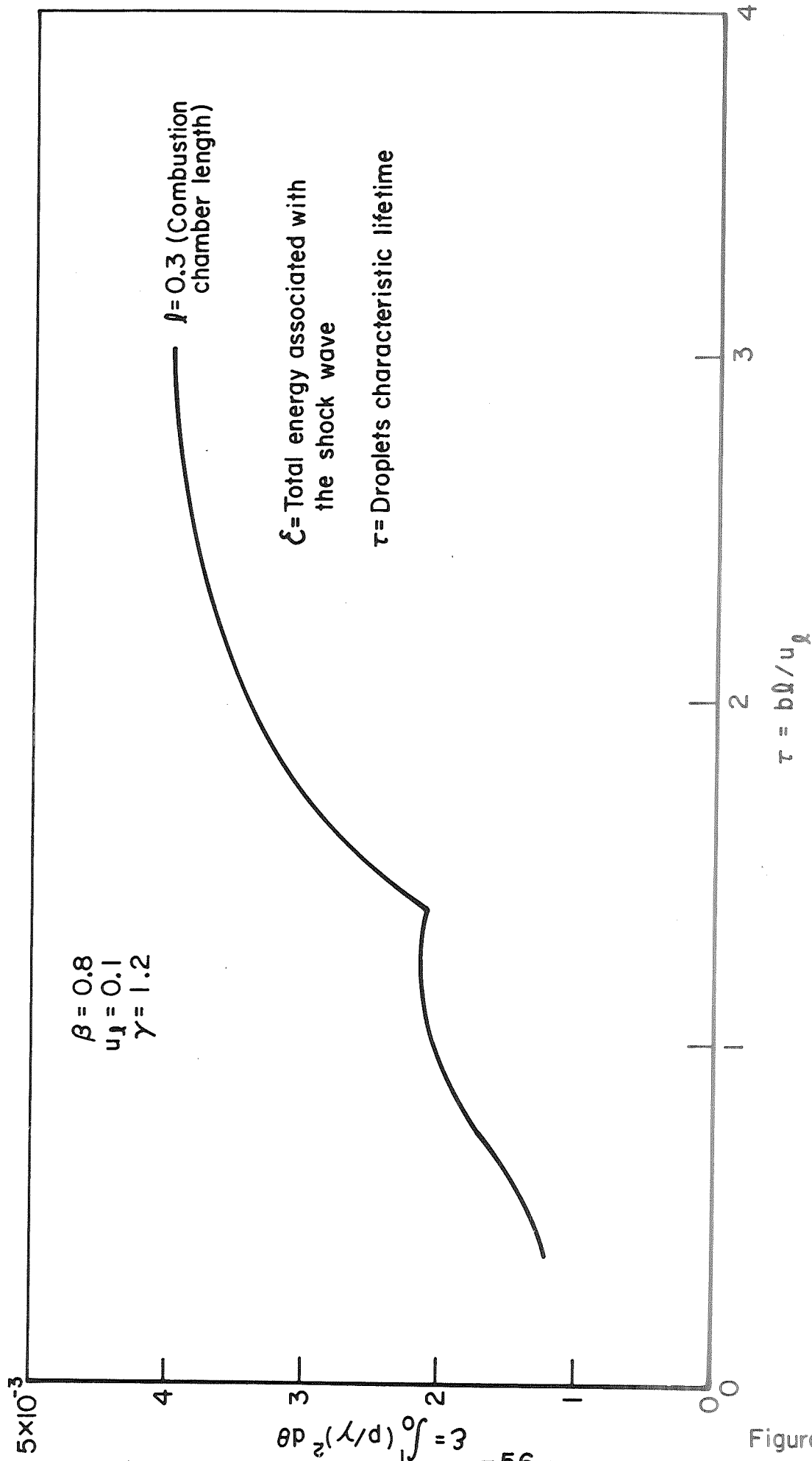
6114 2 125 71





Shock energy versus droplet lifetime

$$\bar{u}_n < u_d$$



$$\theta p_z(\lambda/d) \int_0^{\infty} = 3$$

Figure 5

Energy in transverse mode combustion instability, $\mathcal{E}$,
versus droplet lifetime with $\bar{u}_n > u_d$, ℓ varying

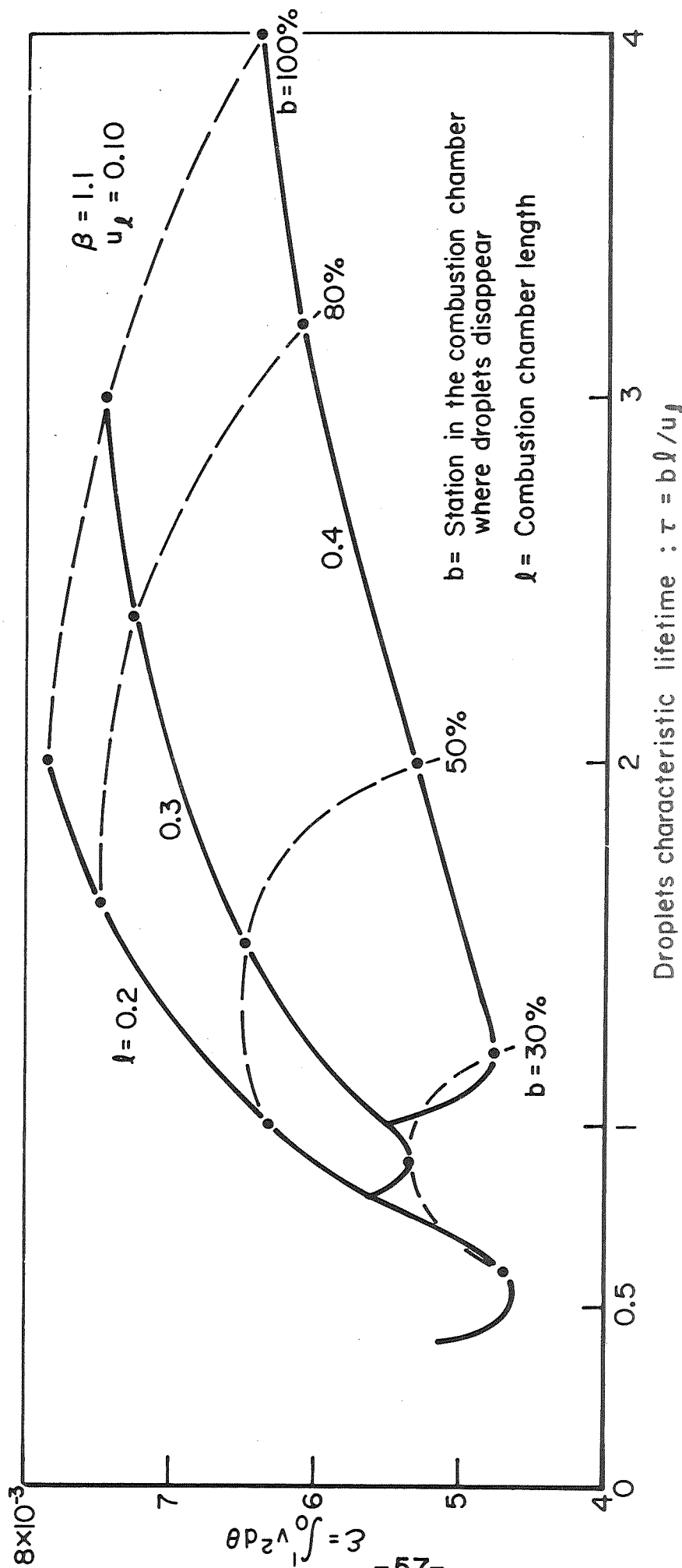


Figure 6

Energy in transverse mode combustion instability, $\mathcal{E}$,
versus droplet lifetime with ℓ and β constant, u_d varying

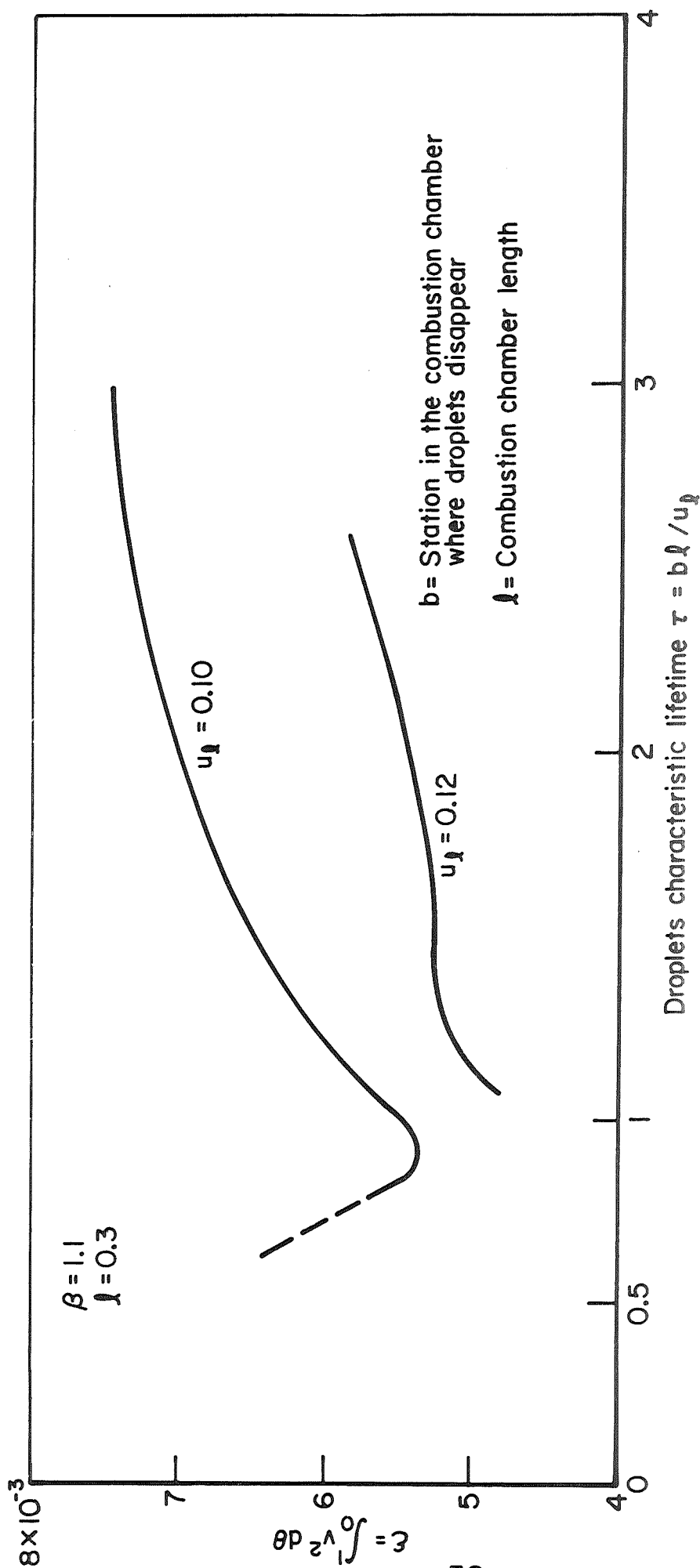


Figure 7

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