

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

*Technical Memorandum 33-494*

*Mathematical Model of the Solar Radiation Force  
and Torques Acting on the Components  
of a Spacecraft*

*R. M. Georgevic*

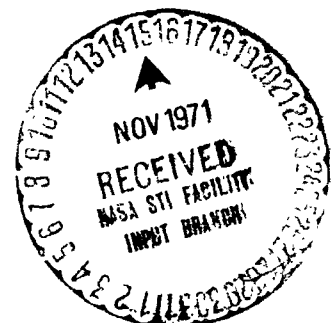
N72-11717 (NASA-CR-123361) MATHEMATICAL MODEL OF THE  
SOLAR RADIATION FORCE AND TORQUES ACTING ON  
THE COMPONENTS OF A SPACECRAFT R.M.  
Georgevic (Jet Propulsion Lab.) 1 Oct.  
08096 1971 111 p CSCL 03B G3/29

FACILITY FORM 602

(ACCESSION NUMBER)	111	(THRM)	13
(PAGES)	111	(CODE)	29
(NASA CR OR TMX OR AD NUMBER)	CR-123361	(CATEGORY)	

JET PROPULSION LABORATORY  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
PASADENA, CALIFORNIA

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PREFACE

The work described in this report was performed by the Mission Analysis Division of the Jet Propulsion Laboratory.

## ACKNOWLEDGMENT

The author wishes to express his appreciation to A. Liu for checking, both analytically and numerically, a large portion of the analytic results developed in this paper. Thanks are also due Dr. H. Lass, for discussing the material presented in the Appendix.

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## ABSTRACT

Solar radiation pressure exerts a mechanical force upon the surface of a spacecraft which intercepts the stream of photons coming from the Sun. For high-precision spacecraft attitude control and orbit determination, it is necessary to generate a precise mathematical model of the solar radiation force and the moment of that force; such a model must be more accurate than the currently used "flat surface" model, based on the radiation force on the effective cross-section area of the irradiated body.

In this report the general expressions for the solar radiation force and torques are derived in the vectorial form for any given reflecting surface, provided that the reflecting characteristics of the surface, as well as the value of the solar constant, are known. An appropriate choice of a spacecraft-fixed frame of reference leads to relatively simple expressions for the solar radiation forces and torques in terms of the functions of the Sun-spacecraft-Earth angle.

The advantage of such a model over the standard flat-surface model is obvious, and it is very easy to find the expressions for the error of the standard model for any given reflecting surface. Another advantage of the model is that it can be used for the effects of the air drag, solar wind, etc.



## I. INTRODUCTION

To derive the expression for the mechanical force exerted by radiation upon an intercepting surface, let us consider an elementary flat surface area, perpendicular to the direction of the incoming radiation. Let  $S$  (Fig. 1) be such a surface and let  $J$  be the radiant energy per unit area and per unit of time, impinging on  $S$ . One part of this energy,  $(1 - \gamma)J$ ,  $\gamma < 1$ , will be absorbed by the surface and reradiated isotropically into space. The remaining part,  $\gamma J$ , will be reflected according to a certain reflection law  $f(\theta)$ , where  $\theta$  is the angle between the direction of the reflected particle and the normal to the surface area  $S$ . If the reflection is purely specular,  $f(\theta)$  will be the two-dimensional Dirac delta function. If the reflection is totally diffuse, the reflected particles obey Lambert's reflection law (Ref. 1):

$$f(\theta) = \cos \theta.$$

The total reflected radiation is in fact the combination of these two types of reflection. If we denote by  $\sigma$  the surface area of the hemisphere shown on Fig. 1, and by  $d\omega$  the elementary solid angle

$$d\omega = \sin \theta \, d\phi \, d\theta,$$

the total reflected radiation is

$$\gamma J = \int_{\sigma} I \, d\omega$$

where  $I$  is the radiant flux per unit solid angle on the hemisphere  $\sigma$ .

Now set

$$I = I_0 f(\theta)$$

where  $I_0$  is a constant. Subsequently, integrating over the area of the hemisphere, we find

$$\gamma J = I_0 \int_0^{2\pi} d\phi \int_0^{\pi/2} f(\theta) \sin \theta d\theta = I_0 A(f),$$

where

$$A(f) = \begin{cases} 1 & \text{for specular reflection} \\ \pi & \text{for diffuse reflection.} \end{cases}$$

The momentum of the radiation is  $J/c$ , where  $c$  is the speed of light. The momentum exchange due to the reflected beam of photons in the direction of the normal to the surface area  $S$  is

$$\int_{\sigma} \frac{I}{c} \cos \theta d\omega = \frac{I_0}{c} \int_{\sigma} f(\theta) \sin \theta \cos \theta d\phi d\theta,$$

and the tangential component of the momentum is zero. After an integration over the surface of the hemisphere  $\sigma$ , the integral above becomes

$$\int_{\sigma} \frac{I}{c} \cos \theta d\omega = \frac{\gamma J}{c A(f)} \int_0^{2\pi} d\phi \int_0^{\pi/2} f(\theta) \sin \theta \cos \theta d\theta = \frac{\gamma J}{c} B(f),$$

where

$$B(f) = \begin{cases} 1 & \text{for specular reflection} \\ 2/3 & \text{for diffuse reflection.} \end{cases}$$

The total pressure is the sum of momentum exchanges due to the incident and reflected radiation

$$P = \frac{J}{c} [1 + \gamma B(f)].$$

Since  $J$  is inversely proportional to  $r_{SP}^2$  ( $r_{SP}$  being the heliocentric distance of the irradiated body) the acceleration due to the radiation pressure is

$$a_{SRP} = \frac{k}{r_{SP}^2}$$

where

$$k = \frac{J_0}{c} S \frac{1 + \gamma B(f)}{M}.$$

Here,  $J_0$  is the value of  $J$  at the distance of the Earth from the Sun (one astronomical unit), known as the solar constant:

$$J_0 = 1.353 \times 10^3 \text{ watts/meter}^2 \text{ (Ref. 2),}$$

while  $M$  is the mass of the reflecting body and  $S$  is the irradiated surface area.

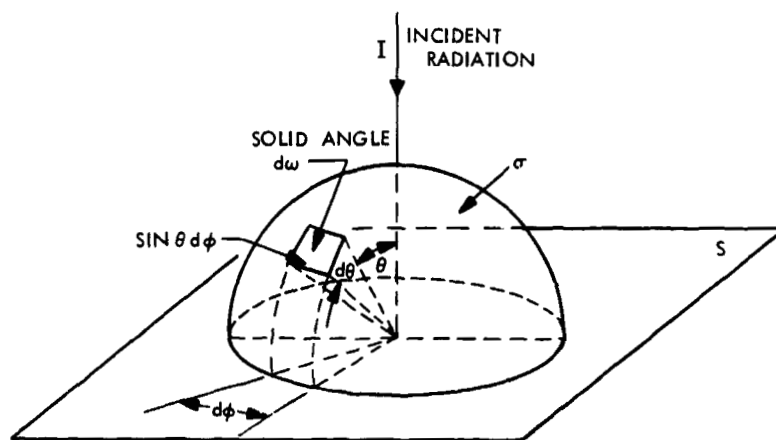


Fig. 1. Radiation reflected from an elementary surface area

## II. SOLAR RADIATION FORCE ON AN INCLINED SURFACE

Let us assume now that the direction of the incoming radiation is inclined by an angle  $\theta$  to the normal to the elementary surface area  $S$  (Fig. 2). The incident radiation produces the force

$$F_I = F_N \cos \theta$$

where  $F_N$  is the radiation force for  $\theta = 0$ , i. e., in the case when the incident radiation is perpendicular to the intercepting surface, considered in the preceding section. This force is given by

$$F_N = \frac{K_{\text{SRP}}}{r_{\text{SP}}^2} S \quad (1)$$

where  $K_{\text{SRP}}$  is the solar radiation pressure constant. The value of this constant can be easily calculated from the solar constant  $J_0$ . It is

$$K_{\text{SRP}} = \frac{(\text{AU})^2 J_0}{c} \text{ kg m/s}^2.$$

Taking for the value of the astronomical unit (Ref. 3)

$$\text{AU} = 149,597,893.0 \text{ km}$$

and for the speed of light (Ref. 3)

$$c = 299,792.5 \text{ km/s}$$

we finally compute

$$K_{\text{SRP}} = 1.0227 \times 10^{17} \text{ kg m/s}^2. \quad (2)$$

The component of the force  $F_I$  along the normal to the surface area  $S$  is the pressure force (Fig. 2)

$$P_I = F_I \cos \theta = F_N \cos^2 \theta, \quad (3)$$

and the component of the same force in the tangential plane to the surface is

$$T_I = F_I \sin \theta = F_N \sin \theta \cos \theta. \quad (4)$$

As mentioned beforehand, of all the photons impinging on the surface  $S$ , the portion of  $1 - \gamma$  remains absorbed and reradiated isotropically into the neighboring space, while the remaining portion  $\gamma$  is reflected specularly or diffusely. Let  $\beta\gamma$  be the portion of the specularly reflected photons. The force produced by the reflection, after the collision with the surface  $S$ , is

$$F_R = \beta\gamma F_I,$$

and the components of this force along the normal to the surface of collision and in the tangential plane are

$$\left. \begin{aligned} P_R &= F_R \cos \theta \\ T_R &= F_R \sin \theta \end{aligned} \right\}$$

so that, substituting  $F_R$  and  $F_I$ , we finally find, in terms of  $F_N$ ,

$$P_R = \beta\gamma F_N \cos^2 \theta \quad (5)$$

and

$$T_R = \beta\gamma F_N \sin \theta \cos \theta. \quad (6)$$

The force exerted on the surface S by the diffusely reflected photons,  $(1 - \beta)\gamma$ , is perpendicular to S. Its magnitude is

$$F_D = \frac{2\gamma}{3}(1 - \beta)F_N \cos \theta. \quad (7)$$

The total radiation force along the normal to the surface is then the sum

$$P_N = P_I + P_R + F_D,$$

and the total tangential force is

$$T = T_I - T_R.$$

Substituting  $P_I$ ,  $P_R$ ,  $F_D$ ,  $T_I$ , and  $T_R$  from Eqs. (3-7), we obtain, in terms of  $F_N$ ,

$$P_N = F_N \left[ (1 + \beta\gamma) \cos^2 \theta + \frac{2\gamma}{3}(1 - \beta) \cos \theta \right] \quad (8)$$

and

$$T = F_N(1 - \beta\gamma) \sin \theta \cos \theta. \quad (9)$$

The polygon of forces and the resultant force R are shown on Fig. 2. The magnitude of the resultant force is

$$R_N = F_N \cos \theta \sqrt{K_1 + K_2 \cos \theta + K_3 \cos 2\theta} \quad (10)$$

where

$$K_1 = 1 + \beta^2\gamma^2 + \frac{4\gamma^2}{9}(1 - \beta)^2$$

$$K_2 = \frac{4\gamma}{3}(1 - \gamma)(1 + \beta\gamma)$$

$$K_3 = 2\beta\gamma.$$

The angle between the direction of the resultant force and the normal to the surface (oriented toward the surface) is

$$\theta_R = \arctan \left[ \frac{(1 - \beta\gamma) \sin \theta}{\frac{2\gamma}{3}(1 - \beta) + (1 + \beta\gamma) \cos \theta} \right]. \quad (11)$$



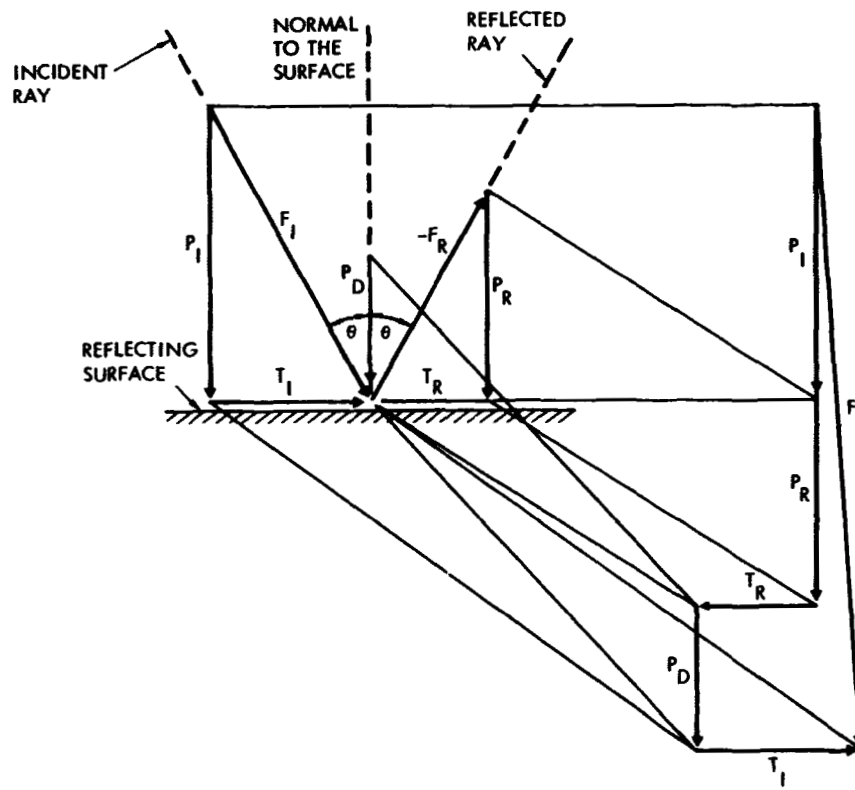


Fig. 2. Polygon of forces acting on an elementary surface area

### III. COMPUTATION OF THE SOLAR RADIATION FORCE

Consider an infinitesimally small element  $dS$  of the irradiated reflecting surface  $S$  (Fig. 3), and let  $\bar{N}^*$  be the unit vector of the normal to the surface at the point  $A(x, y, z)$  oriented in such a manner that  $0 \leq \theta \leq 90$  deg. If  $\Phi(x, y, z) = 0$  is the reflecting surface, then, obviously,

$$\bar{N}^* = \frac{\text{grad } \Phi}{|\text{grad } \Phi|}, \quad (12)$$

and the oriented vector element of surface is

$$d\bar{S} = \bar{N}^* dS = \bar{e}_1 dy dz + \bar{e}_2 dz dx + \bar{e}_3 dx dy, \quad (13)$$

where  $\bar{e}_1, \bar{e}_2, \bar{e}_3$  are the unit vectors of a reference frame  $xyz$ . Hence it is easy to see that

$$dS = \frac{dy dz}{|\bar{N}^* \cdot \bar{e}_1|} = \frac{dz dx}{|\bar{N}^* \cdot \bar{e}_2|} = \frac{dx dy}{|\bar{N}^* \cdot \bar{e}_3|}. \quad (14)$$

The unit vector of the direction of the tangential force  $\bar{T}^*$  lies in the plane containing the normal, incident and reflected rays, which pass through the point  $A$ . Also  $\bar{T}^*$  is directed toward the half of the plane in which the source of the light is, so that the angle between  $\bar{T}^*$  and the direction of the light source is acute. Denoting by  $\bar{u}$  the unit vector of the Spacecraft-Sun direction, we find, from the conditions

$$\begin{aligned} \bar{T}^* \cdot \bar{N}^* &= 0 \\ (\bar{u} \times \bar{N}^*) \cdot \bar{T}^* &= 0 \\ \bar{u} \cdot \bar{T}^* &= \sin \theta \end{aligned}$$

that  $\bar{T}^*$  is given by

$$\bar{T}^* = \frac{\bar{u}}{\sin \theta} - \bar{N}^* \cot \theta. \quad (15)$$

Denote by  $d\bar{F}$  the resultant radiation force acting upon the element of surface  $dS$ . Since

$$d\bar{F} = -\bar{N}^* dP_N - \bar{T}^* dT,$$

and since, from Eqs. (8) and (9),

$$\left. \begin{aligned} dP_N &= \left[ (1 + \beta\gamma) \cos^2 \theta + \frac{2\gamma}{3}(1 - \beta) \cos \theta \right] dF_N \\ dT &= (1 - \beta\gamma) \sin \theta \cos \theta dF_N \end{aligned} \right\} \quad (16)$$

where, from Eq. (1),

$$dF_N = \frac{K_{SRP}}{r_{SP}^2} dS = K(r_{SP}) dS, \quad (17)$$

we can write, for the elementary radiation force,

$$d\bar{F} = -K(r_{SP}) [p(\theta)\bar{N}^* + t(\theta)\bar{T}^*] dS.$$

The total solar radiation force over the whole area  $S$  is then

$$\bar{F} = -K(r_{SP}) \iint_S [p(\theta)\bar{N}^* + t(\theta)\bar{T}^*] dS, \quad (18)$$

where

$$\left. \begin{aligned} p(\theta) &= (1 + \beta\gamma) \cos^2 \theta + \frac{2\gamma}{3}(1 - \beta) \cos \theta \\ t(\theta) &= (1 - \beta\gamma) \sin \theta \cos \theta. \end{aligned} \right\} \quad (19)$$

The element of surface  $dS$  is given by Eq. (14), and the choice of the projection of the area  $S$  on the coordinate planes, over which the double integration should be performed, is completely arbitrary. One should choose, of course, the coordinate plane in which the projection of the area  $S$  is the simplest geometrical figure which renders the technically simplest integration procedure.

Replacing, in Eq. (18), vector  $\bar{T}^*$  by its value in Eq. (15), we give the expression for the total radiation force the following form

$$\bar{F} = -K \iint_S \left[ B(\theta) \bar{N}^* + \frac{t(\theta)}{\sin \theta} \bar{u} \right] dS$$

where

$$B(\theta) = p(\theta) - t(\theta) \cot \theta, \quad K = K(r_{SP}).$$

Using the notation

$$\left. \begin{aligned} C_1 &= \frac{2Y}{3} (1 - \beta) \\ C_2 &= 1 - \beta Y \\ C_3 &= \beta Y \end{aligned} \right\} \quad (20)$$

we can write the function  $B(\theta)$  in the form

$$B(\theta) = C_1 \cos \theta + 2C_3 \cos^2 \theta, \quad (21)$$

and the radiation force becomes

$$\bar{F} = -K \iint_S B(\theta) d\bar{S} - KC_2 \bar{u} \iint_S \cos \theta dS \quad (22)$$

where

$$d\bar{S} = \bar{N}^* dS.$$

The second integral in Eq. (22), because of

$$\cos \theta = \bar{u} \cdot \bar{N}^*,$$

becomes

$$\iint_S \cos \theta dS = \iint_S (\bar{u} \cdot \bar{N}^*) dS = \bar{u} \cdot \iint_S d\bar{S} = \bar{u} \cdot \bar{S}.$$

Denote by  $A_{yz}$ ,  $A_{xz}$ ,  $A_{xy}$  the projections of the total illuminated area on the coordinate planes. Subsequently,

$$\bar{S} = A_{yz} \bar{e}_1 + A_{xz} \bar{e}_2 + A_{xy} \bar{e}_3.$$

If  $\alpha$  is the angle between the direction of the incident Sun ray, defined by the unit vector  $\bar{u}$ , and the z-axis of the system, then

$$\bar{u} = \bar{e}_2 \sin \alpha + \bar{e}_3 \cos \alpha \quad (23)$$

and

$$\bar{u} \cdot \bar{S} = A_{xz} \sin \alpha + A_{xy} \cos \alpha. \quad (24)$$

The total radiation force is then

$$\bar{F} = -K \iint_S B(\theta) d\bar{S} - KC_2 (A_{xz} \sin \alpha + A_{xy} \cos \alpha) \bar{u} \quad (25)$$

where, from Eq. (21)

$$B(\theta) = \frac{2\gamma}{3} (\bar{u} \cdot \bar{N}^*) \left[ 1 - \beta + 3\beta(\bar{u} \cdot \bar{N}^*) \right], \quad (26)$$

and also where

$$\bar{N}^* = \frac{\text{grad } \Phi}{|\text{grad } \Phi|} = \frac{\nabla \Phi}{|\nabla \Phi|} = \frac{\nabla \Phi}{W}, \quad (27)$$

since  $\Phi(x, y, z) = 0$  is the equation of the illuminated surface area and  $W = |\nabla \Phi|$ .

Incidentally, for a black body,  $\gamma = 0$ ,  $\beta = 0$ ,  $C_2 = 1$ , and the radiation force (Eq. 25), is given by

$$\bar{F} = -K(A_{xz} \sin \alpha + A_{xy} \cos \alpha) \bar{u}. \quad (28)$$

The first integral in Eq. (25) in the form

$$\iint_S B(\theta) d\bar{S},$$

may be hard to evaluate since all three projections of the area  $S$  do not necessarily have to be simple geometric figures. Therefore, as was mentioned before, the best method for the practical evaluation of the double integral in Eq. (25) would be to use it in the form

$$\iint_S B(\theta) \bar{N}^* d\bar{S}$$

and to choose  $dS$  from one of the three expressions given by Eq. (14). Hence we can adopt for the total form of the total radiation pressure, the following expression:

$$\begin{aligned} \bar{F} = & -\frac{2\gamma}{3} K \iint_S \left[ 1 - \beta + 3\beta \frac{(\bar{u} \cdot \nabla\Phi)}{W} \right] \frac{(\bar{u} \cdot \nabla\Phi)}{W^2} \nabla\Phi \, dS \\ & - (1 - \beta\gamma) K \frac{(\bar{u} \cdot \nabla\Phi)}{W} (A_{xz} \sin \alpha + A_{xy} \cos \alpha) \bar{u} \end{aligned} \quad (29)$$

where, from Eq. (14),

$$dS = \frac{W \, dy \, dz}{|\nabla\Phi \cdot \bar{e}_1|} = \frac{W \, dx \, dz}{|\nabla\Phi \cdot \bar{e}_2|} = \frac{W \, dx \, dy}{|\nabla\Phi \cdot \bar{e}_3|}$$

or

$$dS = \frac{W}{\left| \frac{\partial\Phi}{\partial x} \right|} dy \, dz = \frac{W}{\left| \frac{\partial\Phi}{\partial y} \right|} dx \, dz = \frac{W}{\left| \frac{\partial\Phi}{\partial z} \right|} dx \, dy. \quad (30)$$

In the practical application of Eq. (29), one should be aware of the fact that the normal to the surface, defined by the unit vector  $\bar{N}^*$ , bisects the angle between the incident and the reflected solar rays so that  $0 \leq \theta \leq \frac{\pi}{2}$ , and, therefore,

$$\bar{N}^* \cdot \bar{u} > 0$$

or

$$\nabla\Phi \cdot \bar{u} > 0.$$

Hence, the direction of  $\nabla\Phi$  should be chosen in such a manner that the inequality above is always satisfied. At the end of this section, it should be noted that from  $\cos \theta = \bar{u} \cdot \bar{N}^*$  and Eq. (27),

$$\cos \theta = \left[ \left( \frac{\partial \Phi}{\partial y} \right) \sin \alpha + \left( \frac{\partial \Phi}{\partial z} \right) \cos \alpha \right] \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 + \left( \frac{\partial \Phi}{\partial z} \right)^2 \right]^{-1/2} .$$



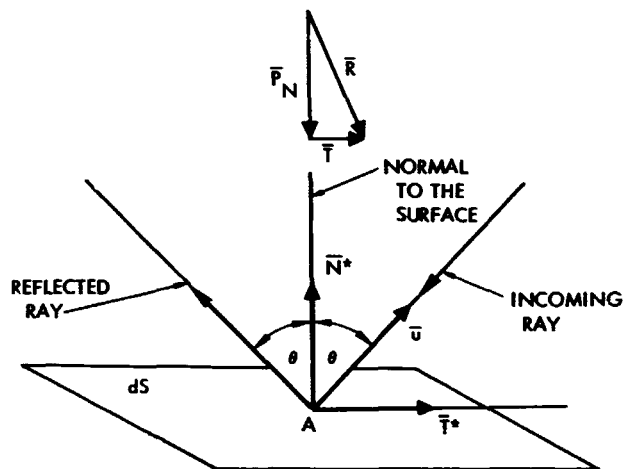


Fig. 3. Orientation of unit vectors along the tangent and normal to the reflecting surface

#### IV. MOMENT OF THE SOLAR RADIATION FORCE

The same considerations used in the preceding section can be extended to the problem of finding the moment of the solar radiation force via a double integration over the illuminated surface  $S$ . The elementary moment of the force  $d\bar{F}$ , acting upon the elementary surface area  $dS$ , is given by

$$d\bar{M} = \bar{r} \times d\bar{F}$$

where  $\bar{r}$  is the position vector of the point A (Fig. 3) in the system of reference  $xyz$ . From the expression for the elementary force in the form

$$d\bar{F} = -K \left[ B(\theta) \bar{N}^* + C_2 \bar{u} \cos \theta \right] dS$$

we find that

$$d\bar{M} = -K \left[ B(\theta) (\bar{r} \times \bar{N}^*) + C_2 (\bar{r} \times \bar{u}) \cos \theta \right] dS,$$

so that the total moment of the solar radiation force over the area  $S$  is

$$\bar{M} = -K \iint_S B(\theta) (\bar{r} \times d\bar{S}) + KC_2 \bar{u} \times \iint_S \bar{r} \cos \theta dS$$

or

$$\bar{M} = -K \iint_S B(\theta) (\bar{r} \times d\bar{S}) + KC_2 \bar{u} \times \iint_S \bar{r} (\bar{u} \cdot d\bar{S}).$$

Finally, setting

$$d\bar{S} = \frac{\nabla\Phi}{W} dS,$$

we obtain

$$\begin{aligned} \bar{M} = & \frac{2\gamma}{3} K \iint_S \left[ 1 - \beta + 3\beta \frac{(\bar{u} \cdot \nabla\Phi)}{W} \right] \frac{(\bar{u} \cdot \nabla\Phi)}{W^2} (\nabla\Phi \times \bar{r}) dS \\ & + (1 - \beta\gamma) K \bar{u} \times \iint_S \bar{r} \frac{(\bar{u} \cdot \nabla\Phi)}{W} dS. \end{aligned} \quad (31)$$

The unit vector  $\bar{u}$  is directed toward the Sun. Denoting the heliocentric position vector of the spacecraft (Sun-Probe vector) by  $\bar{r}_{SP}$  as before, we have

$$\bar{u} = -\frac{\bar{r}_{SP}}{r_{SP}}. \quad (32)$$

Equation (31) is the final form of the expression for the moment of the solar radiation force. The quantity  $A$  is, as before, the projection of the illuminated surface area on the  $xy$  coordinate plane.

Before moving to the next section, in which we shall proceed with the integration of Eqs. (29) and (31) for the spacecraft components, we must mention that the reflection does not strictly obey Lambert's law for some materials and, therefore, the function  $f(\theta)$  should be determined in such a manner that it represents the actual response of a certain material to reflection. The parameter  $B(f)$ , mentioned beforehand, can then be computed using  $f(\theta)$  for that particular material.

Another factor, which can also contribute an extra force of small magnitude, is the reradiation of the thermal energy. In Ref. 4 the author

suggests an alternative expression for Eq. (8), which accounts for the above-mentioned effects. The expression is

$$P_N = F_N \left\{ (1 + \beta\gamma) \cos^2 \theta + B(f) \left[ (1 - \beta)\gamma + k(1 - \gamma) \right] \cos \theta \right\}$$

where  $F_N$  is given by Eq. (1) and where the parameter  $k$  depends on the temperatures and emissivities of the front and back sides of the reflecting surface and on the absorbed and conducted flux. For adiabatic surfaces,  $k = 1$ .

It is obvious that with the expression above for  $P_N$ , the function  $B(\theta)$  in Eqs. (29) and (31) will have a somewhat different form. Namely, because in this case  $p(\theta)$  is given by

$$p(\theta) = (1 + \beta\gamma) \cos^2 \theta + B(f) \left[ (1 - \beta)\gamma + k(1 - \gamma) \right] \cos \theta,$$

while  $t(\theta)$  remains unchanged, and

$$B(\theta) = p(\theta) - t(\theta) \cot \theta,$$

we find that

$$B_1(\theta) = 2\beta\gamma \cos^2 \theta + B(f) \left[ (1 - \beta)\gamma + k(1 - \gamma) \right] \cos \theta.$$

Hence, we see that, using the notation

$$C'_1 = B(f) \left[ (1 - \beta)\gamma + k(1 - \gamma) \right],$$

we can write the function  $B(\theta)$  in the form

$$B_1(\theta) = C'_1 \cos \theta + 2\beta\gamma \cos^2 \theta.$$

Comparison of this expression with  $B(\theta)$  given by Eq. (21) indicates that  $B_1(\theta)$  and  $B(\theta)$  are one and the same function of  $\theta$ , differing only in the value of the constant  $C_1$ . This difference is

$$C_1' - C_1 = \left[ B(f) - \frac{2}{3} \right] \gamma(1 - \beta) + kB(f)(1 - \gamma).$$

## V. THE SOLAR RADIATION FORCE ON THE REFLECTING SURFACE OF A PARABOLIC ANTENNA REFLECTOR

The high-gain antenna of the Pioneer spacecraft has the shape of a paraboloid of revolution, with its axis of symmetry presumably pointing toward the Earth. The configuration of the spacecraft and the position of the antenna reflector relative to the other components is shown on Fig. 4. A simplified sketch of the reflector only is given on Fig. 5., displaying the dimensions of the dish: the semidiameter  $\delta$  and the depth  $\zeta$ .

Since the double integration, indicated in Eq. (29), constitutes the major technical difficulty in the derivation of the expression for the solar radiation force, it is a matter of utmost importance that the coordinate system, with respect to which the integration should be performed, be properly and adequately chosen.

The most suitable frame of reference, which would be most objective for the purpose of the double integration, provided that the roll-axis of symmetry of the spacecraft coincides with the spacecraft - Earth direction, can be defined in the following manner. Taking the  $z'$ -axis along the spacecraft - Earth direction, we find that the unit vector of the  $z'$ -axis is given by

$$\bar{k} = - \frac{\bar{r}_{EP}}{r_{EP}}$$

where  $\bar{r}_{EP}$  is the geocentric position vector of the spacecraft and  $r_{EP} = |\bar{r}_{EP}|$ .

The  $y'$ -axis of this system lies in the plane defined by the unit vector  $\bar{k}$  and the unit vector  $\bar{u}$ , defined by Eq. (32). The  $x'$ -axis is perpendicular to this plane; hence

$$\bar{i} = \frac{\bar{u} \times \bar{k}}{|\bar{u} \times \bar{k}|}$$

and

$$\bar{j} = \bar{k} \times \bar{i}.$$

Denote by  $\alpha_0$  the angle between the positive direction of the  $z'$ -axis and the direction of the unit vector  $\bar{u}$  (Earth-Probe-Sun angle). Then

$$|\bar{u} \times \bar{k}| = \sin \alpha_0.$$

Taking the expression for  $\bar{u}$  from Eq. (32), we can write explicitly

$$\bar{i} = \frac{\bar{r}_{SP} \times \bar{r}_{EP}}{r_{SP} r_{EP} \sin \alpha_0} = \frac{\bar{r}_{SP} \times \bar{r}_{EP}}{|\bar{r}_{SP} \times \bar{r}_{EP}|}$$

$$\bar{j} = \frac{\bar{r}_{EP} \times (\bar{r}_{EP} \times \bar{r}_{SP})}{r_{EP}^2 r_{SP} \sin \alpha_0}$$

$$\bar{k} = -\frac{\bar{r}_{EP}}{r_{EP}}$$

This frame of reference is noninertial since the two vectors  $\bar{r}_{SP}$  and  $\bar{r}_{EP}$  vary with time.

With the introduction of the heliocentric position vector of the Earth,

$$\bar{R}_E = \bar{r}_{SP} - \bar{r}_{EP},$$

whence

$$\bar{r}_{EP} = \bar{r}_{SP} - \bar{R}_E,$$

the formulas above can be simplified and reduced to the form

$$\bar{i} = \frac{\bar{R}_E \times \bar{r}}{r R_E \sin \psi}$$

$$\bar{j} = \bar{k} \times \bar{i}$$

$$\bar{k} = (\bar{R}_E - \bar{r}) \left[ r^2 + R_E^2 - 2rR_E \cos \psi \right]^{-1/2}$$

where  $\bar{r} = \bar{r}_{SP}$ ,  $r = |\bar{r}|$ , and  $\psi$  is the angle between the vectors  $\bar{r}$  and  $\bar{R}_E$  (Earth-Sun-Probe angle).

Denote by  $\bar{a}_1, \bar{a}_2, \bar{a}_3$  the unit vectors of direction of coordinate axes of an inertial reference frame (for example, the 1950 equatorial coordinate system), and let  $X, Y, Z$ , and  $X_E, Y_E, Z_E$  be the components of vectors  $\bar{r}$  and  $\bar{R}_E$  respectively. The instantaneous angular velocity vector  $\bar{\omega}$ , defining the rotation of the above-described rotating frame relative to the inertial equatorial frame of reference, is then

$$\bar{\omega} = \frac{1}{2} \left( \bar{i} \times \frac{d\bar{i}}{dt} + \bar{j} \times \frac{d\bar{j}}{dt} + \bar{k} \times \frac{d\bar{k}}{dt} \right).$$

The unit vectors of the noninertial system of reference can easily be expressed in terms of the coordinates of the spacecraft and the Earth and their derivatives in the inertial frame. Thus

$$\bar{i} = \frac{1}{rR_E \sin \psi} \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ X_E & Y_E & Z_E \\ X & Y & Z \end{vmatrix}$$

$$\bar{j} = \bar{k} \times \bar{i}$$

$$\bar{k} = \frac{1}{p} \begin{bmatrix} X_E - X \\ Y_E - Y \\ Z_E - Z \end{bmatrix}$$



where

$$\rho = \left[ (X - X_E)^2 + (Y - Y_E)^2 + (Z - Z_E)^2 \right]^{1/2}$$

$$r = (X^2 + Y^2 + Z^2)^{1/2}$$

$$R_E = (X_E^2 + Y_E^2 + Z_E^2)^{1/2}$$

$$\cos \psi = \frac{XX_E + YY_E + ZZ_E}{rR_E}.$$

However, due to the existence of torques, the axis of symmetry of the reflector moves relative to the above-described system of coordinates, defined by unit vectors  $\bar{i}$ ,  $\bar{j}$ ,  $\bar{k}$ . Therefore, for the computation of the radiation force from Eq. (29), by double integration, we can establish another coordinate system, defined in the following manner:

- (1) The z-axis is taken along the axis of symmetry of the reflector. The unit vector of this axis is  $\bar{e}_3$ .
- (2) The y-axis, defined by the unit vector  $\bar{e}_2$ , lies in the plane of vectors  $\bar{e}_3$  and  $\bar{u}$ .
- (3) The x-axis is perpendicular to the yz-plane, forming a right-hand-oriented triad with the first two axes. Its unit vector will be denoted by  $\bar{e}_1$ . Therefore,

$$\bar{e}_1 = \frac{\bar{u} \times \bar{e}_3}{\sin \alpha},$$

$$\bar{e}_2 = \bar{e}_3 \times \bar{e}_1,$$

where  $\alpha = \angle(\bar{u}, \bar{e}_3)$ , namely, the angle between the z-axis and the direction of the incoming radiation. The xyz-coordinate system described above is shown on Figs. 5 and 6.

The geometry of the parabolic reflector is shown on Fig. 6. Placing the origin of the above-described coordinate system at the vertex of the paraboloid, we can write for the equation of the reflecting surface

$$\Phi(x, y, z) \equiv z - \lambda(x^2 + y^2) = 0 \quad (33)$$

where  $\lambda$  is the parameter

$$\lambda = \frac{\zeta}{\delta^2}, \quad (34)$$

depending on the dimensions of the reflector only. For the Pioneer F/G spacecraft, we have, approximately,

$$\lambda = 0.202 \text{ meters}^{-1}.$$

The magnitude of the illuminated inside area of the reflector is a function of the angle  $\alpha$ . For small values of  $\alpha$ , the whole area is illuminated and the projection of this area on the  $xy$ -plane is a circle of radius  $\delta$ . When the solar ray passing through the point A on the brim of the reflector dish (Fig. 6) becomes tangent to the surface, the tangent line cuts off the segment  $\zeta$  on the negative  $z$ -axis, so that

$$\alpha = 90^\circ - \Omega$$

where

$$\tan \Omega = \frac{2\zeta}{\delta} \quad (35)$$

or

$$\tan \Omega = 2 \tan \psi, \quad (36)$$

where

$$\tan \psi = \frac{\zeta}{\delta}. \quad (37)$$

The two angles,  $\Omega$  and  $\psi$ , are shown on Fig. 6.

If  $\alpha > 90^\circ - \Omega$ , one part of the inside area of the reflector will be in the shade, and the projection area in the  $xy$ -plane will not be the full area of a circle, but a somewhat more complicated figure. To find this figure, let us start with the equation of a solar ray passing through any point  $(x_0, y_0, z_0 = \zeta)$  on the brim of the dish:

$$\frac{x - x_0}{0} = \frac{y - y_0}{\sin \alpha} = \frac{z - \zeta}{\cos \alpha} = t$$

where

$$x_0^2 + y_0^2 = \delta^2 .$$

Take  $y_0 = \delta w$ , where  $w$  is a parameter. Then, in terms of  $w$ , the equations above yield

$$\left. \begin{aligned} x &= \delta \sqrt{1 - w^2} \\ y &= w\delta + t \sin \alpha \\ z &= \zeta + t \cos \alpha . \end{aligned} \right\} \quad (38)$$

These equations represent, in the parametric form, the equation of the inclined elliptic cylinder, shown on Fig. 6. The semimajor axis of this cylinder is  $\delta$ , and its semiminor axis is  $\delta \cos \alpha$ . The line of intersection of this cylinder with the paraboloid of revolution

$$z = \lambda(x^2 + y^2)$$

occurs for the values of  $t$  which are the solutions of the quadratic equation

$$(\lambda \sin^2 \alpha)t^2 + (2\lambda\delta w \sin \alpha - \cos \alpha)t + (\lambda\delta^2 - \zeta) = 0 . \quad (39)$$

However, from Eq. (34),  $\lambda\delta^2 - \zeta^2 = 0$ , and the solutions of Eq. (39) are

$$t_1 = 0, \quad t_2 = \frac{1}{\sin\alpha} \left( \frac{\cot\alpha}{\lambda} - 2\delta w \right).$$

For  $t_1 = 0$ , we have

$$x = \delta \sqrt{1 - w^2}$$

$$y = \delta w$$

and, eliminating the parameter  $w$ , we obtain the equation of the circle

$$x^2 + y^2 = \delta^2.$$

For

$$t_2 = \frac{\delta}{\sin\alpha} (\cot\alpha \cot\psi - 2w),$$

we obtain the equation of the line of intersection in the parametric form,

$$\left. \begin{aligned} x &= \delta \sqrt{1 - w^2} \\ y &= \delta (\cot\alpha \cot\psi - w) \\ z &= \delta \tan\psi \left[ (1 - w^2) + (\cot\alpha \cot\psi - w)^2 \right] \end{aligned} \right\} \quad (40)$$

The projection of the curve given by Eq. (40) on the  $xy$ -plane is given in the parametric form by

$$x = \delta \sqrt{1 - w^2}$$

$$y = \delta (\cot\alpha \cot\psi - w).$$

Eliminating  $w$ , we find

$$x^2 + (y - y_C)^2 = \delta^2, \quad (41)$$

where

$$y_C = \delta \cot \alpha \cot \psi. \quad (42)$$

The two circles intersect if  $0 < y_C < 2\delta$  or, in other words, if

$$0 < \cot \alpha \cot \psi < 2,$$

or

$$0 < \cot \alpha < 2 \tan \psi = \tan \Omega.$$

or if

$$90^\circ - \Omega < \alpha < 90^\circ.$$

For  $\alpha < 90^\circ - \Omega$ , the whole inside area of the reflector is illuminated.

For  $\alpha = 90^\circ$ , the whole area is in the shade. The  $y$ -coordinate of the point B (Fig. 6) is

$$y_B = y_C - \delta = \delta(\cot \alpha \cot \psi - 1), \quad (43)$$

and the two circles intersect in the points with coordinates

$$\left. \begin{aligned} x_S &= \pm \frac{\delta}{2} \sqrt{4 - \cot^2 \alpha \cot^2 \psi} \\ y_S &= \frac{\delta}{2} \cot \alpha \cot \psi. \end{aligned} \right\} \quad (44)$$

The shaded area on Fig. 7 represents the projection of the illuminated inside area of the reflector on the  $xy$ -plane. The angle  $\phi_0$  is given by

$$\sin \phi_0 = \frac{y_S}{\delta} = \frac{1}{2} \cot \alpha \cot \psi = \frac{A_0}{2} = \cot \alpha \cot \Omega$$

where

$$A_0 = \cot \alpha \cot \psi = 2 \sin \phi_0 . \quad (45)$$

With this denotation, the equations of the two circles, given by  $x^2 + y^2 = \delta^2$  and Eq. (41), are

$$\left. \begin{aligned} y &= \pm \sqrt{\delta^2 - x^2} \\ y &= \delta A_0 \pm \sqrt{\delta^2 - x^2} \end{aligned} \right\} \quad (46)$$

and the parametric equation of the line of intersection (Eqs. 40) becomes

$$\left. \begin{aligned} x &= \delta \sqrt{1 - w^2} \\ y &= \delta(A_0 - w) \\ z &= \zeta \left[ (1 - w^2) + (A_0 - w)^2 \right] \\ &= \zeta \left( 1 + A_0^2 - 2A_0 w \right) . \end{aligned} \right\} \quad (47)$$

The projection of this curve on the  $xz$ -plane is, in the parametric form,

$$\begin{aligned} x &= \delta \sqrt{1 - w^2} \\ z &= \zeta \left( 1 + A_0^2 - 2A_0 w \right) , \end{aligned}$$

and, after the elimination of the parameter  $w$ , we obtain the ellipse

$$z = \zeta \left[ 1 + A_0^2 - \frac{2A_0}{\delta} \sqrt{\delta^2 - x^2} \right]$$

or

$$z = \zeta(1 + A_0^2) - 2 \cot \alpha \sqrt{\delta^2 - x^2},$$

or

$$\left(\frac{x}{\delta}\right)^2 + \left[\frac{z - \zeta(1 + A_0^2)}{2\delta \cot \alpha}\right]^2 = 1. \quad (48)$$

The projection of the same line on the yz-plane is, in parametric form,

$$y = \delta(A_0 - w)$$

$$z = \zeta(1 + A_0^2 - 2A_0w)$$

and, after the elimination of the parameter w, we obtain the equation of the straight line

$$z = 2y \cot \alpha + \zeta(1 - A_0^2). \quad (49)$$

For  $z = \zeta$ , we get

$$y = \frac{\delta A_0}{2} = y_S$$

and for the point of intersection with the parabola  $z = \lambda y^2$  (point B), from

$$\lambda y^2 - 2y \cot \alpha - \zeta(1 - A_0^2) = 0$$

we find

$$y = \delta(A_0 - 1) = y_B,$$

which is the value already obtained in Eq. (43).

The projections defined by Eqs. (48) and (49) are shown on Fig. 8 and Fig. 9, respectively.

From the equation of the reflecting surface, we find

$$\phi(x, y, z) = z - \lambda(x^2 + y^2) = 0,$$

which yields

$$\nabla\phi = -2\lambda x\bar{e}_1 - 2\lambda y\bar{e}_2 + \bar{e}_3 \quad (50)$$

and

$$W = |\nabla\phi| = \sqrt{1 + 4\lambda^2(x^2 + y^2)}, \quad (51)$$

so that

$$\bar{N}^* = \frac{1}{W}(-2\lambda x\bar{e}_1 - 2\lambda y\bar{e}_2 + \bar{e}_3). \quad (52)$$

From

$$\cos \theta = \bar{N}^* \cdot \bar{u}$$

where  $\bar{u}$  is given by Eq. (23), we find

$$\cos \theta = \frac{\cos \alpha - 2\lambda y \sin \alpha}{W} \quad (53)$$

where  $\theta$  is the angle between the incident solar ray and the normal to the surface  $\phi$  at a particular point. Therefore

$$\cos \theta > 0$$

or

$$\cos \alpha - 2\lambda y \sin \alpha > 0.$$

This condition is satisfied for

$$y < \frac{\cot \alpha}{2\lambda} = \frac{\delta A_0}{2} = y_S,$$

which is correct.



Returning to Eq. (22), we shall first find the force over the whole area of the reflector, when  $\alpha < 90^\circ - \Omega$ , that is, when the entire interior of the mirror is illuminated. Introducing the polar coordinates

$$x = \frac{r}{2\lambda} \cos \phi, \quad y = \frac{r}{2\lambda} \sin \phi, \quad dx dy = \frac{r}{4\lambda^2} dr d\phi, \quad (54)$$

we can write

$$\cos \theta = \frac{\cos \alpha - r \sin \phi \sin \alpha}{\sqrt{1 + r^2}} \quad (55)$$

and

$$dS = \frac{dx dy}{|\bar{N}^* \cdot \bar{e}_3|} = \frac{r dr d\phi}{4\lambda^2} \sqrt{1 + r^2}. \quad (56)$$

Since

$$d\bar{S} = \bar{N}^* dS = \frac{r dr d\phi}{4\lambda^2} \nabla \Phi,$$

Eq. (22) becomes

$$\bar{F} = -\frac{K}{4\lambda^2} \iint_{A_{xy}} \nabla \Phi B(\theta) r dr d\phi - \frac{KC_2}{4\lambda^2} \bar{u} \iint_{A_{xy}} \sqrt{1 + r^2} \cos \theta r dr d\phi.$$

Hence,

$$\left. \begin{aligned} F_x &= \frac{K}{4\lambda^2} \iint_{A_{xy}} B(\theta) r^2 \cos \phi dr d\phi \\ F_y &= \frac{K}{4\lambda^2} \iint_{A_{xy}} B(\theta) r^2 \sin \phi dr d\phi - KC_2 I \sin \alpha \end{aligned} \right\} \quad (57a)$$

$$\left. \begin{aligned}
 F_z &= -\frac{K}{4\lambda^2} \iint_{A_{xy}} B(\theta) r \, dr \, d\phi - KC_2 I \cos \alpha \\
 \text{where} \\
 I &= \frac{1}{4\lambda^2} \iint_{A_{xy}} \sqrt{1+r^2} \cos \theta r \, dr \, d\phi
 \end{aligned} \right\} \quad (57b)$$

with  $B(\theta)$  given by Eq. (21).

The integration of the right-hand sides of the three equations given above is rather complicated in polar coordinates, particularly when  $y_C < \delta\sqrt{2}$ , or when (Fig. 7)

$$\phi_0 < 45^\circ,$$

because the radius vector  $OS_1$  (or  $OS_2$ ) intersects the upper circle on Fig. 7 in two points. The integration is simpler in rectangular coordinates, as will be shown later.

However, before going deeper into the details of the integration over the area  $A_{xy}$ , we shall first derive the components of the radiation force over the whole inside (front) area of the parabolic reflector, when  $\alpha < 90^\circ - \Omega$ . The projection of this area on the  $xy$ -plane is the area of the circle  $x^2 + y^2 = \delta^2$ , which we shall call  $A_{xy}^0$ . The components of the force of radiation over the whole area will be denoted by

$$\bar{F}^0 = \{F_x^0, F_y^0, F_z^0\}.$$

The limits for the integration over the area  $A_{xy}^0$  are: for  $\phi$ ,  $\phi_1 = 0$ ,  $\phi_2 = 2\pi$ , and for  $r$ ,  $r_1 = 0$ ,  $r_2 = 2\lambda\delta = 2 \tan \psi = 2 \, \text{m}$ . Then

$$\begin{aligned}
F_x^0 &= \frac{KC_3}{2\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha)^2 \frac{r^2 dr}{1+r^2} \cos \phi d\phi \\
&+ \frac{KC_1}{4\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha) \frac{r^2 dr}{\sqrt{1+r^2}} \cos \phi d\phi
\end{aligned}
\tag{58}$$

$$\begin{aligned}
F_y^0 &= \frac{KC_3}{2\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha)^2 \frac{r^2 dr}{1+r^2} \sin \phi d\phi \\
&+ \frac{KC_1}{4\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha) \frac{r^2 dr}{\sqrt{1+r^2}} \sin \phi d\phi - KC_2 I \sin \alpha
\end{aligned}
\tag{59}$$

$$\begin{aligned}
F_z^0 &= -\frac{KC_3}{2\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha)^2 \frac{r dr}{1+r^2} d\phi \\
&- \frac{KC_1}{4\lambda^2} \int_0^{2\pi} \int_0^{2m} (\cos \alpha - r \sin \phi \sin \alpha) \frac{r dr}{\sqrt{1+r^2}} d\phi - KC_2 I \cos \alpha .
\end{aligned}
\tag{60}$$

It is easy to see that, from Eq. (58),

$$F_x^0 = 0 . \tag{61}$$

Equations (59) and (60) can be written in the form

$$\left. \begin{aligned} F_y^0 &= K(C_3 I_{21} + C_1 I_{22} - C_2 I \sin \alpha) \\ F_z^0 &= -K(C_3 I_{31} + C_1 I_{32} + C_2 I \cos \alpha) \end{aligned} \right\} \quad (62)$$

where, after the integration is performed,

$$I_{21} = -\frac{\sin 2\alpha}{2\lambda^2} \int_0^{2\pi} \sin^2 \phi \, d\phi \int_0^{2m} \frac{r^3 \, dr}{1+r^2} = -\pi\delta^2 \sin 2\alpha (1 + 2 \cot^2 \Omega \ln \cos \Omega), \quad (63)$$

$$I_{22} = -\frac{\sin \alpha}{4\lambda^2} \int_0^{2\pi} \sin^2 \phi \, d\phi \int_0^{2m} \frac{r^3 \, dr}{\sqrt{1+r^2}} = -\frac{\pi\delta^2}{3} \sin \alpha \tan^2 \frac{\Omega}{2} (2 + \sec \Omega), \quad (64)$$

$$\begin{aligned} I_{31} &= \frac{\cos^2 \alpha}{2\lambda^2} \int_0^{2\pi} d\phi \int_0^{2m} \frac{r \, dr}{1+r^2} + \frac{\sin^2 \alpha}{2\lambda^2} \int_0^{2\pi} \sin^2 \phi \, d\phi \int_0^{2m} \frac{r^3 \, dr}{1+r^2} \\ &= \pi\delta^2 \left[ \sin^2 \alpha + 2 \cot^2 \Omega (1 - 3 \cos^2 \alpha) \ln \cos \Omega \right] \\ &= \frac{\pi\delta^2}{2} \left[ (1 - 2 \cot^2 \Omega \ln \cos \Omega) - (1 + 6 \cot^2 \Omega \ln \cos \Omega) \cos 2\alpha \right], \quad (65) \end{aligned}$$

$$I_{32} = \frac{\cos \alpha}{4\lambda^2} \int_0^{2\pi} d\phi \int_0^{2m} \frac{r \, dr}{\sqrt{1+r^2}} = 2\pi\delta^2 \cos \alpha \frac{\cos \Omega}{1 + \cos \Omega}, \quad (66)$$

$$I = \frac{\cos \alpha}{4\lambda^2} \int_0^{2\pi} d\phi \int_0^{2m} r \, dr = \pi\delta^2 \cos \alpha. \quad (67)$$

Collecting all terms given by Eqs. (63-67) together, we can write Eqs. (62) in the following form:

$$\left. \begin{aligned} F_y^0 &= -\pi\delta^2 K(f_1 \sin \alpha + f_2 \sin 2\alpha) \\ F_z^0 &= -\pi\delta^2 K(g_0 + g_1 \cos \alpha + g_2 \cos 2\alpha) \end{aligned} \right\} \quad (68)$$

where

$$f_1 = \frac{2\gamma}{9}(1 - \beta) \frac{1 - \cos \Omega}{1 + \cos \Omega} (2 + \sec \Omega), \quad (69)$$

$$f_2 = \frac{1 + \beta\gamma}{2} + 2\beta\gamma \cot^2 \Omega \ln \cos \Omega, \quad (70)$$

$$g_0 = \frac{1}{2} - \beta\gamma \cot^2 \Omega \ln \cos \Omega, \quad (71)$$

$$g_1 = \frac{4\gamma}{3}(1 - \beta) \frac{\cos \Omega}{1 + \cos \Omega}, \quad (72)$$

$$g_2 = \frac{1}{2} - \beta\gamma(1 + 3 \cot^2 \Omega \ln \cos \Omega). \quad (73)$$

For the Pioneer spacecraft

$$\delta = 1.3716 \text{ m}$$

$$\zeta = 0.3803 \text{ m}$$

$$\pi\delta^2 = 5.9102 \text{ m}^2$$

$$m = \tan \psi = 0.277267$$

$$\psi = 15^\circ.498$$

$$\tan \Omega = 0.554535$$

$$\Omega = 29^{\circ}012$$

and

$$f_1 = 0.046754\gamma(1 - \beta)$$

$$f_2 = 0.5 - 0.371922\beta\gamma$$

$$g_0 = 0.5 + 0.435961\beta\gamma$$

$$g_1 = 0.622046\gamma(1 - \beta)$$

$$g_2 = 0.5 + 0.307883\beta\gamma$$

To compute the solar radiation force exerted upon the inside surface of the reflector using Eq. (22) or Eq. (29), when only one part of the surface is illuminated, one must select the proper projection of the surface on one of the coordinate planes which yields the simplest double integration.

From the preceding considerations, it follows that the projection in the yz-plane (Fig. 9) is the simplest geometrical figure composed of a part of the parabola  $z = \lambda y^2$ , A'OB, and the segment BS of the straight line

$$z = 2y \cot \alpha + \zeta(1 - A_0^2).$$

Since, from Eq. (14),

$$dS = \frac{dy dz}{|\bar{N}^* \cdot \bar{e}_1|} = \frac{W}{2x} dy dz = \frac{\sqrt{1 + 4\lambda z}}{\sqrt{z - \lambda y^2}} \sqrt{\lambda} dy dz,$$

Eq. (22) becomes

$$\bar{F} = -\frac{K}{2\sqrt{\lambda}} \iint_{2A_{yz}} \bar{N}^* B(\theta) \sqrt{\frac{1 + 4\lambda z}{z - \lambda y^2}} dy dz - \frac{KC_2}{2\sqrt{\lambda}} \bar{u} \iint_{2A_{yz}} W \cos \theta \frac{dy dz}{\sqrt{z - \lambda y^2}}$$

where  $2A_{yz}$  is the projection of the surface area  $S$  on the  $yz$ -plane (the area projects twice). The equation above is equivalent to

$$\bar{F} = -\frac{K}{\sqrt{\lambda}} \iint_{A_{yz}} B(\theta) \frac{\nabla \Phi}{\sqrt{z - \lambda y^2}} dy dz - \frac{KC_2}{\sqrt{\lambda}} \bar{u} \iint_{A_{yz}} W \cos \theta \frac{dy dz}{\sqrt{z - \lambda y^2}}.$$

The component of this force along the  $x$ -axis, because of the axial symmetry of the reflector, vanishes. The other two components are

$$\left. \begin{aligned} F_y &= 2K\sqrt{\lambda} \iint_{A_{yz}} \left( C_1 \cos \theta + 2C_3 \cos^2 \theta \right) \frac{y dy dz}{\sqrt{z - \lambda y^2}} - \frac{KC_2}{\sqrt{\lambda}} J \sin \alpha \\ F_z &= -\frac{K}{\sqrt{\lambda}} \iint_{A_{yz}} \left( C_1 \cos \theta + 2C_3 \cos^2 \theta \right) \frac{dy dz}{\sqrt{z - \lambda y^2}} - \frac{KC_2}{\sqrt{\lambda}} J \cos \alpha \end{aligned} \right\} \quad (74)$$

where

$$J = \iint_{A_{yz}} (\cos \alpha - 2\lambda y \sin \alpha) \frac{dy dz}{\sqrt{z - \lambda y^2}}. \quad (75)$$

Integration of Eqs. (74) and (75) cannot be performed without difficulties taking  $z = z(y)$  because for the limit  $z = \lambda y^2$  the denominators of the integrands in Eqs. (74) and (75) vanish. Therefore, the integration should be performed taking  $y$  as a function of  $z$ . Thus, the limits for integration over the part  $A'B_1BS$  are

$$y_1(z) = \sqrt{\frac{z}{\lambda}}$$

$$y_2(z) = \frac{\tan \alpha}{2} \left[ z - (1 - A_0^2) \zeta \right]$$

$$z_1 = z_B = \lambda y_B^2$$

$$z_2 = \zeta$$

and, for the integration over the remaining part  $B_1OB$ , the limits are

$$y_1'(z) = -\sqrt{\frac{z}{\lambda}}$$

$$y_2'(z) = \sqrt{\frac{z}{\lambda}}$$

$$z_1' = 0$$

$$z_2' = z_B.$$

Taking  $\cos \theta$  from Eq. (53), we find

$$\left. \begin{aligned} F_y &= 2K\sqrt{\lambda}(C_1J_{21} + 2C_3J_{22}) - \frac{KC_2}{\sqrt{\lambda}}J \sin \alpha \\ F_z &= -\frac{K}{\sqrt{\lambda}}(C_1J_{31} + 2C_3J_{32}) - \frac{KC_2}{\sqrt{\lambda}}J \cos \alpha \end{aligned} \right\} \quad (76)$$

where

$$J_{21} = \iint_{A_{yz}} \cos \theta \frac{y \, dy \, dz}{\sqrt{z - \lambda y^2}},$$

$$J_{22} = \iint_{A_{yz}} \cos^2 \theta \frac{y \, dy \, dz}{\sqrt{z - \lambda y^2}},$$



$$J_{31} = \iint_{A_{yz}} \cos \theta \frac{dy dz}{\sqrt{z - \lambda y^2}}$$

$$J_{32} = \iint_{A_{yz}} \cos^2 \theta \frac{dy dz}{\sqrt{z - \lambda y^2}}.$$

The evaluation of these four integrals, together with the fifth integral J with  $\cos \theta$  taken from Eq. (53), is somewhat complicated; it is relatively simple to reduce the double integration to the single integration with constant limits  $z_1$  and  $z_2$ , or  $z'_1$  and  $z'_2$ . However, it may be much simpler and more suitable to apply directly the numerical double integration procedure with variable limits.

The integration for  $90^\circ - \Omega < \alpha < 90^\circ$  can be performed in rectangular coordinates in the following way. Again taking the projection of the illuminated surface of the reflector on the xy-plane, we can write (see Fig. 7)

$$\iint_{A_{xy}} = \int_{-\delta}^{-x_S} \int_{Y_1(x)}^{Y_2(x)} + \int_{-x_S}^{x_S} \int_{Y_1(x)}^{Y(x)} + \int_{x_S}^{\delta} \int_{Y_1(x)}^{Y_2(x)}$$

and, because of the symmetry of the reflector with respect to the yz-plane,

$$\int_{-\delta}^{-x_S} \int_{Y_1(x)}^{Y_2(x)} = \int_{x_S}^{\delta} \int_{Y_1(x)}^{Y_2(x)}$$

and

$$\int_{-x_S}^{x_S} \int_{Y_1(x)}^{Y(x)} = 2 \int_0^{x_S} \int_{Y_1(x)}^{Y(x)}$$

so that

$$\iint_{A_{xy}} = 2 \left( \int_0^{x_S} \int_{Y_1(x)}^{Y(x)} + \int_{x_S}^{\delta} \int_{Y_1(x)}^{Y_2(x)} \right).$$

In order to avoid the double integration, an approximative method can be introduced. The method is based on the assumption that the ratio of the components of the force over the illuminated area and the components of the force over the whole inside area of the reflector are the same as the ratios of the respective projection areas. Namely, we can assume that

$$\left. \begin{aligned} F_y &= \eta(\alpha) F_y^0 \\ F_z &= \eta(\alpha) F_z^0 \end{aligned} \right\} \quad (77)$$

where, from Fig. 7,

$$\eta(\alpha) = \frac{\text{Projection of the illuminated surface}}{\text{Area of the circle}}.$$

Explicitly, this ratio is

$$\eta(\alpha) = \frac{2\phi_0 + \sin 2\phi_0}{\pi}, \quad (78)$$

where  $\phi_0$  is a function of the angle  $\alpha$ , defined by

$$\sin \phi_0 = \frac{1}{2} \cot \psi \cot \alpha = \cot \Omega \cot \alpha. \quad (79)$$

For the Pioneer spacecraft,  $\psi = 15^\circ.498$  and

$$\sin \phi_0 = 1.803 \cot \alpha.$$

For  $\alpha = 90^\circ - \Omega$ , when the whole inside area of the reflector is illuminated,  $\sin \phi_0 = 1$ ,  $\phi_0 = \pi/2$  and  $\eta(\alpha) = 1$ . Therefore,

$$\eta(\alpha) = \begin{cases} \frac{1}{\pi}(2\phi_0 + \sin 2\phi_0), & \text{for } 90^\circ - \Omega < \alpha < 90^\circ \\ 1, & \text{for } \alpha \leq 90^\circ - \Omega. \end{cases}$$

The graph of this function is shown on Fig. 10.

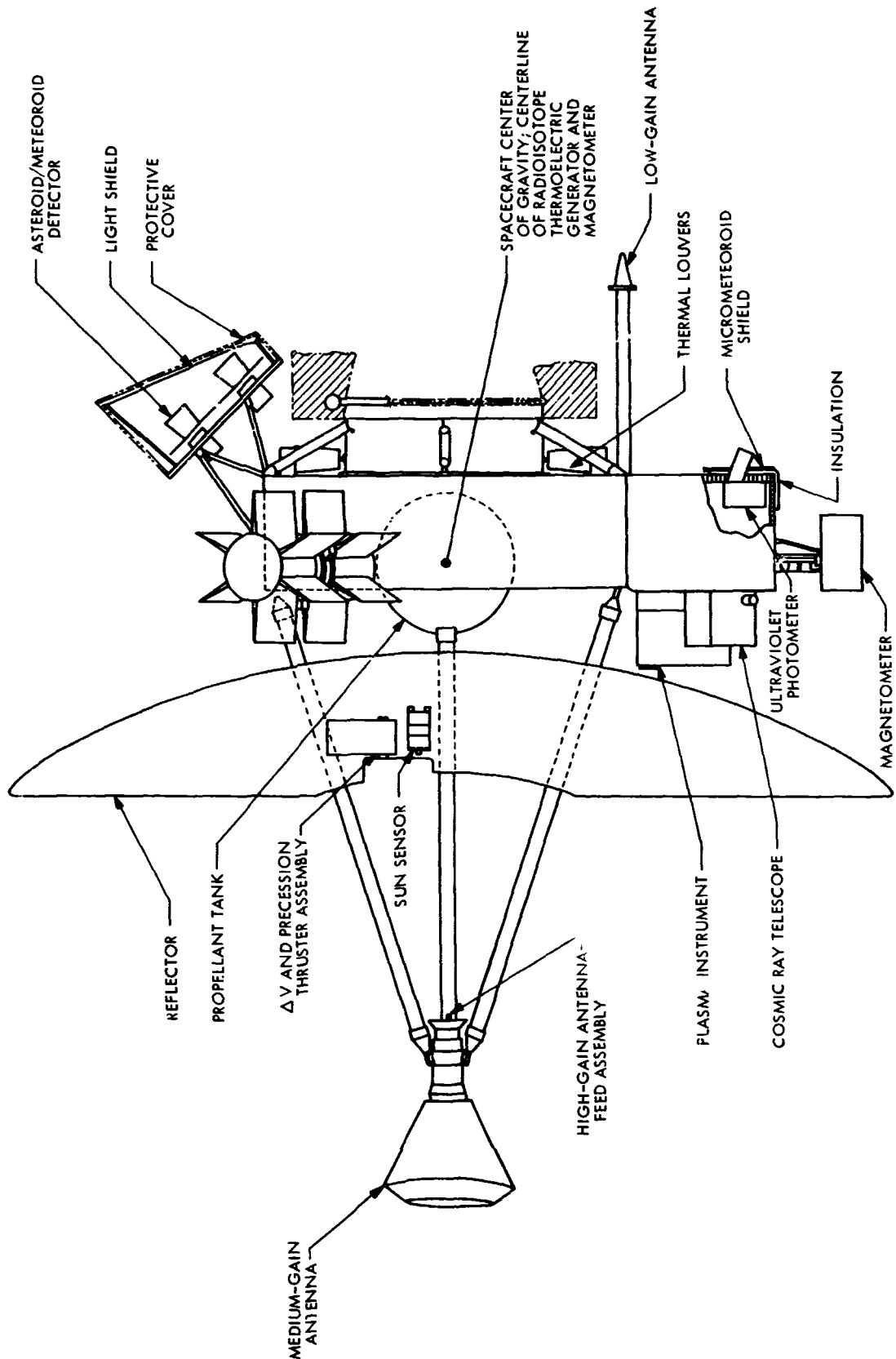


Fig. 4. Configuration of components of the Pioneer spacecraft

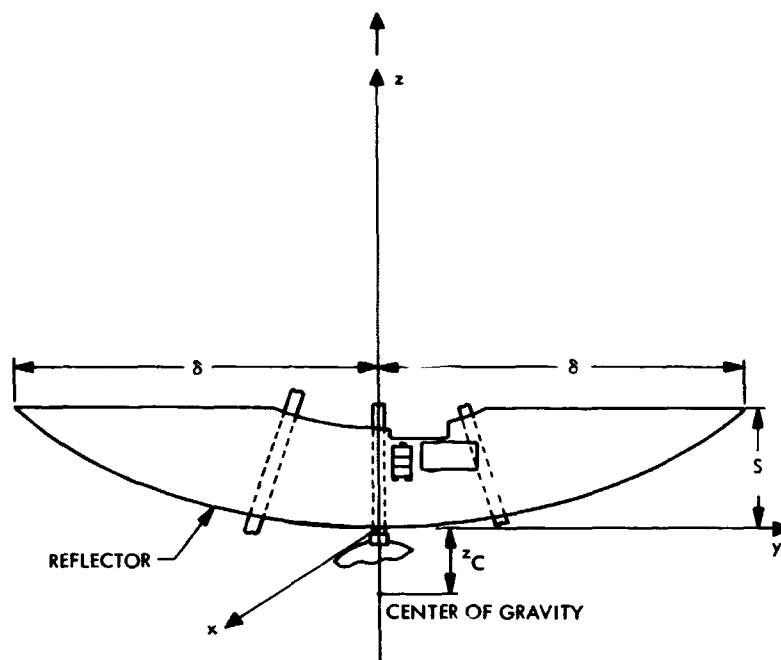


Fig. 5. High-gain antenna reflector of the Pioneer spacecraft



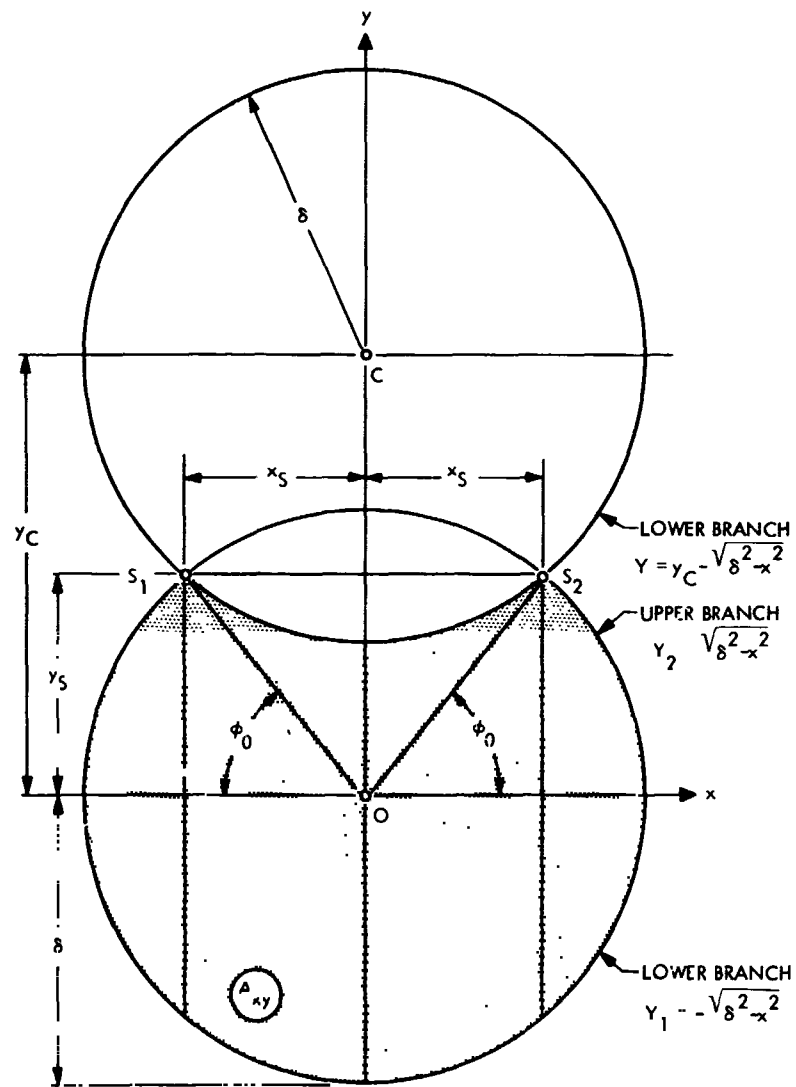


Fig. 7. Projection of the illuminated inside area of the reflector on the  $xy$ -plane of reference





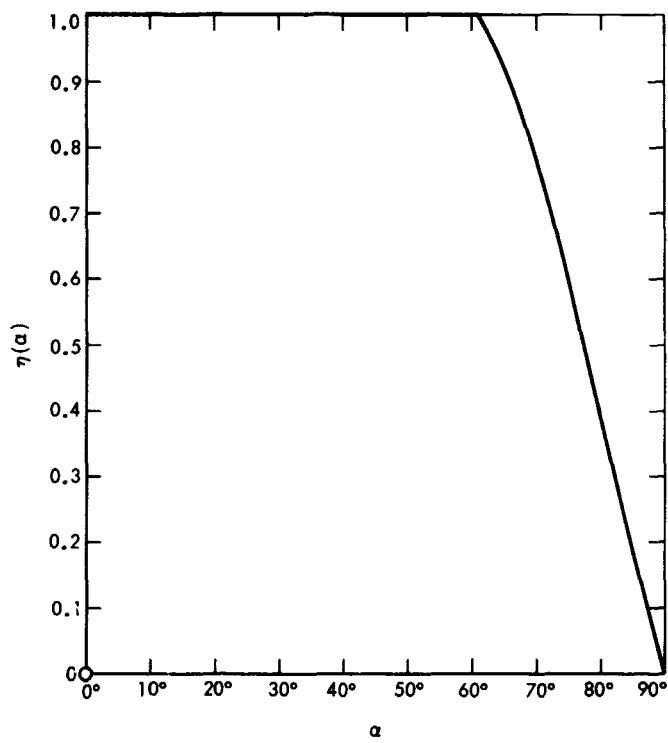


Fig. 10. Graph of the blocking function for the concave side of the parabolic reflector

## VI. THE MOMENT OF THE SOLAR RADIATION FORCE ON THE REFLECTING SURFACE OF THE PARABOLIC REFLECTOR

Equation (31) gives the final form of the expression for the moment of the solar radiation force over a surface area  $S$ . For the inside area of the parabolic reflector,

$$\bar{r} \times d\bar{S} = \bar{r} \times \bar{N}^* \frac{dx dy}{|\bar{N}^* \cdot \bar{e}_3|} = (y\bar{e}_1 - x\bar{e}_2)(1 + 2\lambda z) dx dy$$

$$\bar{u} \times \bar{r} = \bar{e}_1(z \sin \alpha - y \cos \alpha) + \bar{e}_2 x \cos \alpha - \bar{e}_3 x \sin \alpha$$

and the three components of the moment vector  $\bar{M}$  are

$$\left. \begin{aligned} M_x &= -K \iint_{A_{xy}} y B(\theta) (1 + 2\lambda z) dx dy + KC_2 \iint_{A_{xy}} W (z \sin \alpha - y \cos \alpha) \cos \theta dx dy \\ M_y &= K \iint_{A_{xy}} x B(\theta) (1 + 2\lambda z) dx dy + KC_2 \cos \alpha \iint_{A_{xy}} x W \cos \theta dx dy \\ M_z &= -KC_2 \sin \alpha \iint_{A_{xy}} x W \cos \theta dx dy. \end{aligned} \right\} (80)$$

To integrate over the whole inside area of the reflector, we must substitute  $B(\theta)$  from Eq. (21) and  $\cos \theta$  from Eq. (55) and use the polar coordinates defined by Eq. (54). Also, because of  $z = r^2/4\lambda$ ,

$$1 + 2\lambda z = \frac{2 + r^2}{2}$$

and

$$W = \sqrt{1 + r^2}.$$

Since  $\cos \theta$  contains  $\sin \phi$  only, and  $B(\theta)$  is a function of  $\cos \theta$  only, all integrals containing  $x = (r/2\lambda) \cos \phi$  (the two integrals in  $M_y$  and one in  $M_z$ ) vanish when integrated from  $\phi = 0$  to  $\phi = 2\pi$ , because

$$\int_0^{2\pi} \int_{r=0}^{2m} R(r)F(\sin \phi) \cos \phi \, dr \, d\phi = 0.$$

Therefore,

$$M_y^0 = M_z^0 = 0. \quad (81)$$

The  $M_x^0$  component has the form

$$\begin{aligned} M_x^0 = \frac{K}{16\lambda^3} & \left[ \frac{C_1}{2} \sin \alpha \int_0^{2\pi} d\phi \int_0^{2m} \frac{r^3(2+r^2)}{\sqrt{1+r^2}} dr \right. \\ & + C_3 \sin 2\alpha \int_0^{2\pi} d\phi \int_0^{2m} \frac{r^3(2+r^2)}{1+r^2} dr \\ & \left. + C_2 \sin 2\alpha \int_0^{2\pi} d\phi \int_0^{2m} r^3 dr \right], \end{aligned}$$

or, after the integration,

$$\begin{aligned}
 M_x^0 &= \frac{\pi K C_1}{80\lambda^3} \sin \alpha (4 + \sec^5 \Omega - 5 \sec \Omega) \\
 &+ \frac{\pi K C_3}{32\lambda^3} \sin 2\alpha (\sec^4 \Omega - 1 + 4 \ln \cos \Omega) \\
 &+ \frac{\pi K C_2}{32\lambda^3} \sin 2\alpha \tan^4 \Omega.
 \end{aligned}$$

Finally, substituting  $\lambda = (\tan \psi)/\delta$ , we find

$$M_x^0 = \pi \delta^2 \zeta K (h_1 \sin \alpha + h_2 \sin 2\alpha) \quad (82)$$

where

$$\left. \begin{aligned}
 h_1 &= \frac{2\gamma}{15} (1 - \beta) \cot^4 \Omega (4 + \sec^5 \Omega - 5 \sec \Omega) \\
 h_2 &= \frac{1}{2} \left[ 1 + 2\beta\gamma \cot^4 \Omega (\tan^2 \Omega + 2 \ln \cos \Omega) \right]
 \end{aligned} \right\} \quad (83)$$

For the Pioneer spacecraft, these values are

$$h_1 = 0.334912\gamma(1 - \beta)$$

$$h_2 = 0.5 + 0.416502\beta$$

and

$$\pi \delta^2 \zeta = 2.2477 \text{ m}^3.$$

For  $90^\circ - \Omega < \alpha < 90^\circ$ , when one part of the inside area of the reflector is in the shadow,

$$\left. \begin{aligned} M_x &= \eta(\alpha) M_x^0 \\ M_y &= 0 \\ M_z &= 0 \end{aligned} \right\} \quad (84)$$

where the "blocking function"  $\eta(\alpha)$  was given in Eq. (78).

## VII. THE SOLAR RADIATION FORCE ON THE BACK SIDE OF THE PARABOLIC REFLECTOR

Assuming that the thickness of the reflector is negligible, we can write the equation of the outside surface in the form

$$\Phi \equiv z - \lambda(x^2 + y^2) = 0 .$$

The normal to this surface is now pointing outwards, i. e. , the grad  $\Phi$  is now given by

$$\text{grad } \Phi = 2\lambda x \bar{e}_1 + 2\lambda y \bar{e}_2 - \bar{e}_3 \quad (85)$$

and the unit normal vector  $\bar{N}^*$  is

$$\bar{N}^* = \frac{1}{W} (2\lambda x \bar{e}_1 + 2\lambda y \bar{e}_2 - \bar{e}_3) \quad (86)$$

where  $W$  is given by Eq. (51).

The geometry of the outside (back side) area of the reflector is shown on Fig. 11. The curve which separates the illuminated part of the surface from the part in the shadow is the locus of points at which the solar rays are tangential to the surface of the paraboloid. If

$$\frac{x - x_0}{0} = \frac{y - y_0}{\sin \alpha} = \frac{z - z_0}{\cos \alpha} = t$$

is the equation of a solar ray parallel to the unit vector  $\bar{u}$ , the interception of this straight line and the surface of the paraboloid is defined by the following set of equations:

$$\left. \begin{aligned} x &= x_0 \\ y &= y_0 + t \sin \alpha \\ z &= z_0 + t \cos \alpha \\ z &= \lambda(x^2 + y^2). \end{aligned} \right\} \quad (87)$$

Here we have four equations with four parameters:  $x_0$ ,  $y_0$ ,  $z_0$ , and  $t$ . To be able to eliminate them we must introduce the condition that the straight line is tangent to the surface  $z = \lambda(x^2 + y^2)$ . Thus we have

$$z_0 + t \cos \alpha = \lambda[x_0^2 + (y_0 + t \sin \alpha)^2]$$

or

$$(\lambda \sin^2 \alpha)t^2 + (2\lambda y_0 \sin \alpha - \cos \alpha)t + [\lambda(x_0^2 + y_0^2) - z_0] = 0,$$

a quadratic equation in  $t$  which, due to the condition that the straight line has to be tangent to the paraboloid, has to have equal roots,  $t_1 = t_2$ . Since

$$t_{1/2} = \frac{\cos \alpha - 2\lambda y_0 \sin \alpha \pm \sqrt{(\cos \alpha - 2\lambda y_0 \sin \alpha)^2 - 4\lambda \sin^2 \alpha [\lambda(x_0^2 + y_0^2) - z_0]}}{2\lambda \sin^2 \alpha}$$

we have

$$t_1 = t_2 = t = \frac{\cos \alpha - 2\lambda y_0 \sin \alpha}{2\lambda \sin^2 \alpha}$$

and

$$(\cos \alpha - 2\lambda y_0 \sin \alpha)^2 = 4\lambda \sin^2 \alpha \left[ \lambda (x_0^2 + y_0^2) - z_0 \right]. \quad (88)$$

The substitution of the above-given value for  $t$  into the second of Eqs. (87) yields

$$y = \frac{\cot \alpha}{2\lambda} = \frac{\delta}{2} \cot \psi \cot \alpha = \frac{\delta A_0}{2} = y_S, \quad (89)$$

i. e., the value previously given by the second of Eqs. (44). This equation shows that the limit of the shadow is a plane curve, contained in a plane parallel to the  $z$ -axis, which projects in the  $xy$ -plane as the part of a straight line parallel to the  $x$ -axis. From the first and third of Eqs. (87) and from Eq. (88) we can derive the equation of the limiting curve in the  $xz$ -plane, eliminating  $x_0$  and  $z_0$ . Setting  $y = y_S$ , we find that the curve is the parabola

$$z = \lambda x^2 + \frac{y_S}{2} \cot \alpha, \quad ,$$

i. e., the parabola  $z = \lambda x^2$  (projection of the reflector on the  $xz$ -plane) moved translationally upward. The projection of the illuminated surface area of the reflector is shown on Fig. 12.

The gradient of the outside surface of the reflector is pointed outwards; in other words, its direction is opposite to the direction of the gradient of the inside surface of the reflector. Thus, from Eq. (85),

$$\nabla \Phi = 2\lambda x \bar{e}_1 + 2\lambda y \bar{e}_2 - \bar{e}_3, \quad ,$$

and, from Eq. (86),

$$\bar{N}^* = \frac{1}{W} (2\lambda x \bar{e}_1 + 2\lambda y \bar{e}_2 - \bar{e}_3), \quad ,$$



where  $W$  was given by Eq. (51). The cosine of the angle  $\theta$  between the normal to the surface  $\phi$  and the direction of the incoming radiation is then

$$\cos \theta = \bar{u} \cdot \bar{N}^* = \frac{2\lambda y \sin \alpha - \cos \alpha}{W}. \quad (90)$$

From what was previously said, we orient the normal to the surface in such a manner that the angle  $\theta$  lies in the first quadrant; i. e., so that

$$\cos \theta = \bar{u} \cdot \bar{N}^* > 0.$$

This condition yields

$$2\lambda y \sin \alpha - \cos \alpha > 0.$$

For  $90^\circ < \alpha < 180^\circ$ ,  $y > 0$ ,  $\sin \alpha > 0$ ,  $-\cos \alpha > 0$ , and the condition is always satisfied. For  $90^\circ - \Omega < \alpha < 90^\circ$ , the inequality above yields the constraint

$$y > \frac{\cot \alpha}{2\lambda} = y_S,$$

which is always satisfied for the illuminated portion of the back surface of the reflector.

The solar radiation force on the back surface can be found in the same way as was done for the front side of the reflector. For  $90^\circ - \Omega < \alpha < 90^\circ + \Omega$ , one part of the outside surface is illuminated, while for  $90^\circ + \Omega < \alpha < 180^\circ$ , the whole back side is lit by solar rays. The force over the whole area can be obtained from the corresponding expressions for the front side of the reflector if we note that  $\cos \theta$  from Eq. (90) is the negative of the expression for  $\cos \theta$  used for the front side, given by Eq. (53). Thus, denoting

by a prime the integrals corresponding to integrals  $I_{21}$ ,  $I_{22}$ ,  $I_{31}$ ,  $I_{32}$  and  $I$  for the front surface, we find

$$\left. \begin{aligned} I'_{21} &= I_{21} \\ I'_{22} &= -I_{22} \\ I'_{31} &= I_{31} \\ I'_{32} &= -I_{32} \\ I' &= I, \end{aligned} \right\} \quad (91)$$

and the components of the solar radiation force will be

$$\left. \begin{aligned} F'_x &= 0 \\ F'_y &= K(-C_3 I_{21} + C_1 I_{22} - C_2 I \sin \alpha) \\ F'_z &= K(C_3 I_{31} - C_1 I_{32} - C_2 I \cos \alpha) \end{aligned} \right\} \quad (92)$$

where  $I_{21}$ ,  $I_{22}$ ,  $I_{31}$ ,  $I_{32}$ , and  $I$  are given by Eqs. (63), (64), (65), (66), and (67), respectively.

Equations corresponding to Eqs. (68) for the front side are then

$$\left. \begin{aligned} F'_y &= -\pi \delta^2 K (f'_1 \sin \alpha + f'_2 \sin 2\alpha) \\ F'_z &= -\pi \delta^2 K (g'_0 + g'_1 \cos \alpha + g'_2 \cos 2\alpha) \end{aligned} \right\} \quad (93)$$

where

$$\left. \begin{aligned} f'_1 &= f_1 \\ f'_2 &= -f_2 \\ g'_0 &= -g_0 \\ g'_1 &= g_1 \\ g'_2 &= -g_2 \end{aligned} \right\} \quad (94)$$

so that

$$\left. \begin{aligned} F'_y &= -\pi\delta^2 K(f_1 \sin \alpha - f_2 \sin 2\alpha) \\ F'_z &= -\pi\delta^2 K(-g_0 + g_1 \cos \alpha - g_2 \cos 2\alpha) \end{aligned} \right\} \quad (95)$$

where  $f_1$ ,  $f_2$ ,  $g_0$ ,  $g_1$ , and  $g_2$  are given by Eqs. (69), (70), (71), (72), and (73), respectively.

Introducing again the ratio

$$\eta_1(\alpha) = \frac{\text{Projection of the illuminated area}}{\text{Area of the circle}}$$

we find, for  $90^\circ < \alpha < 90^\circ + \Omega$ , from Fig. 12,

$$\eta_1(\alpha) = \frac{1}{2} \left( 1 - \frac{2\phi_0 + \sin 2\phi_0}{\pi} \right) \quad (96)$$

so that the total force on the illuminated portion of the back side is approximately

$$\left. \begin{aligned} F'_y &= \eta_1(\alpha) F'_y{}^0 \\ F'_z &= \eta_1(\alpha) F'_z{}^0 \end{aligned} \right\} \quad (97)$$

where

$$\eta_1(\alpha) = \begin{cases} \frac{1}{2} \left( 1 - \frac{2\phi_0 + \sin 2\phi_0}{\pi} \right) & \text{for } 90^\circ - \Omega < \alpha < 90^\circ + \Omega \\ 1 & \text{for } 90^\circ < \alpha < 180^\circ. \end{cases}$$

The graph of this function, for the Pioneer Spacecraft, is shown on Fig. 13. From Eqs. (78) and (96) we can also write

$$\eta_1(\alpha) = \frac{1}{2} [1 - \eta(\alpha)] . \quad (98)$$

This method, however, does not give as good results as in the case of the front side of the reflector and can possibly be used only for the values of  $\alpha$  which are not too far below the value  $\alpha = 90^\circ + \Omega$ . For example, taking  $\alpha = 90^\circ$  and assuming black body structure of the back side of the reflector we find  $F'_y = F'_z = 0$ , which is not true. Therefore, the components of the radiation force in the whole interval  $90^\circ - \Omega < \alpha < 90^\circ + \Omega$  should be determined by the double integrations over the area of projection  $A_{xy}$ :

$$\begin{aligned} F'_x &= 0, \\ F'_y &= -\frac{K}{4\lambda^2} \iint_{A_{xy}} B(\theta) r^2 \sin \phi \, dr \, d\phi - KC_2 I \sin \alpha , \\ F'_z &= \frac{K}{4\lambda^2} \iint_{A_{xy}} B(\theta) r \, dr \, d\phi - KC_2 I \cos \alpha , \end{aligned}$$

where  $B(\theta)$  is given by Eq. (21),  $\cos \theta$  is given by Eq. (90), and

$$I = -\frac{1}{4\lambda^2} \iint_{A_{xy}} \sqrt{1+r^2} \cos \theta r dr d\phi .$$

The integration is simpler than in the case of the front side of the reflector because the projection of the illuminated area is the segment of the circle  $x^2 + y^2 = \delta^2$ . The integration limits are

$$\text{For } \phi: \phi = \phi_0 \text{ to } \phi = \pi - \phi_0$$

$$\text{For } r: r = 2m \frac{\sin \phi_0}{\sin \phi} \text{ to } r = 2m ,$$

where  $\phi_0$  is given by Eq. (90).

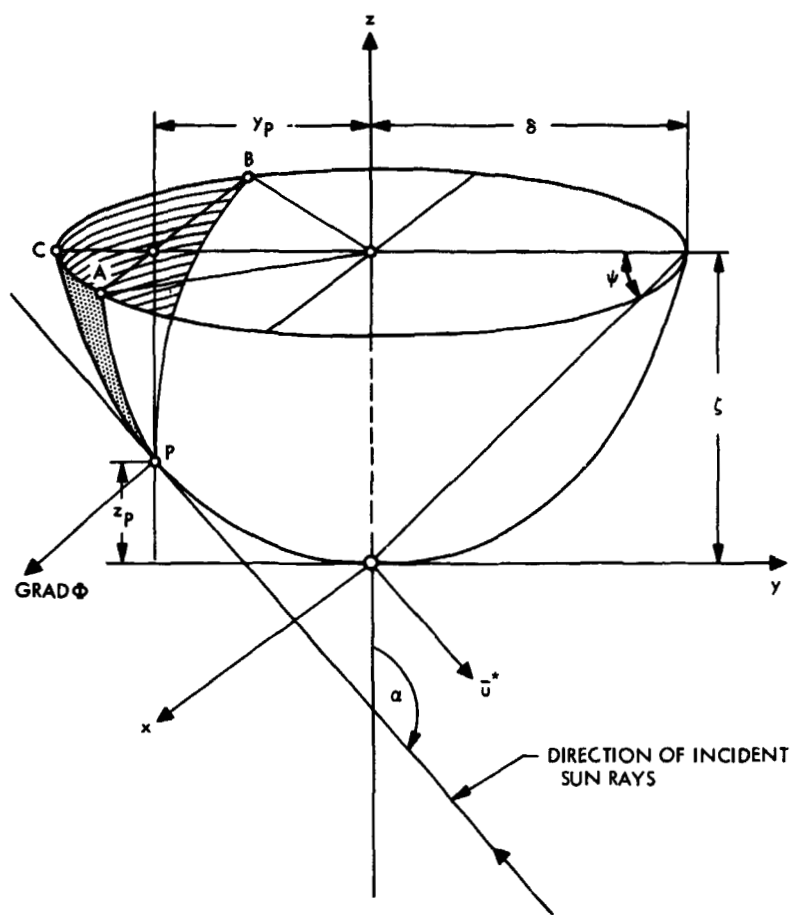


Fig. 11. Solar radiation on the convex side of the parabolic reflector

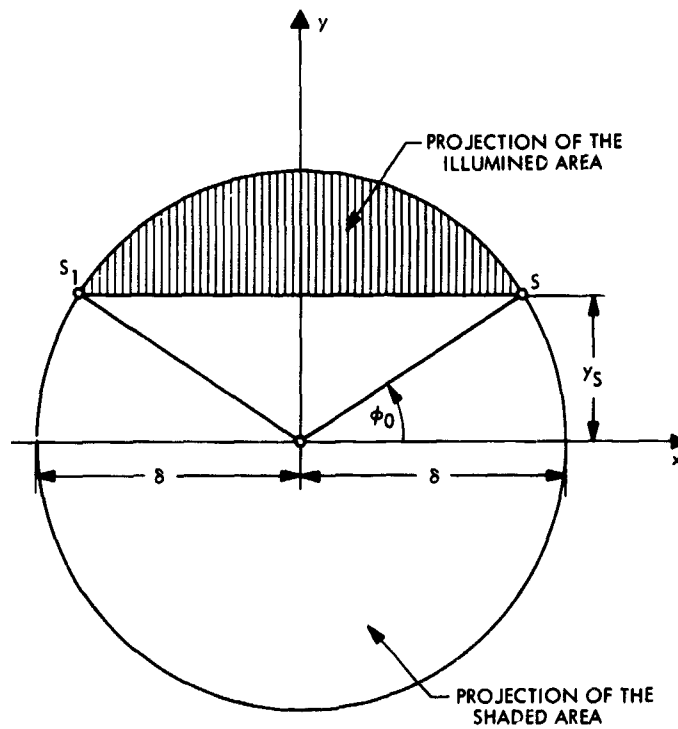


Fig. 12. Projection of the illuminated outside area of the reflector on the  $xy$ -plane of reference

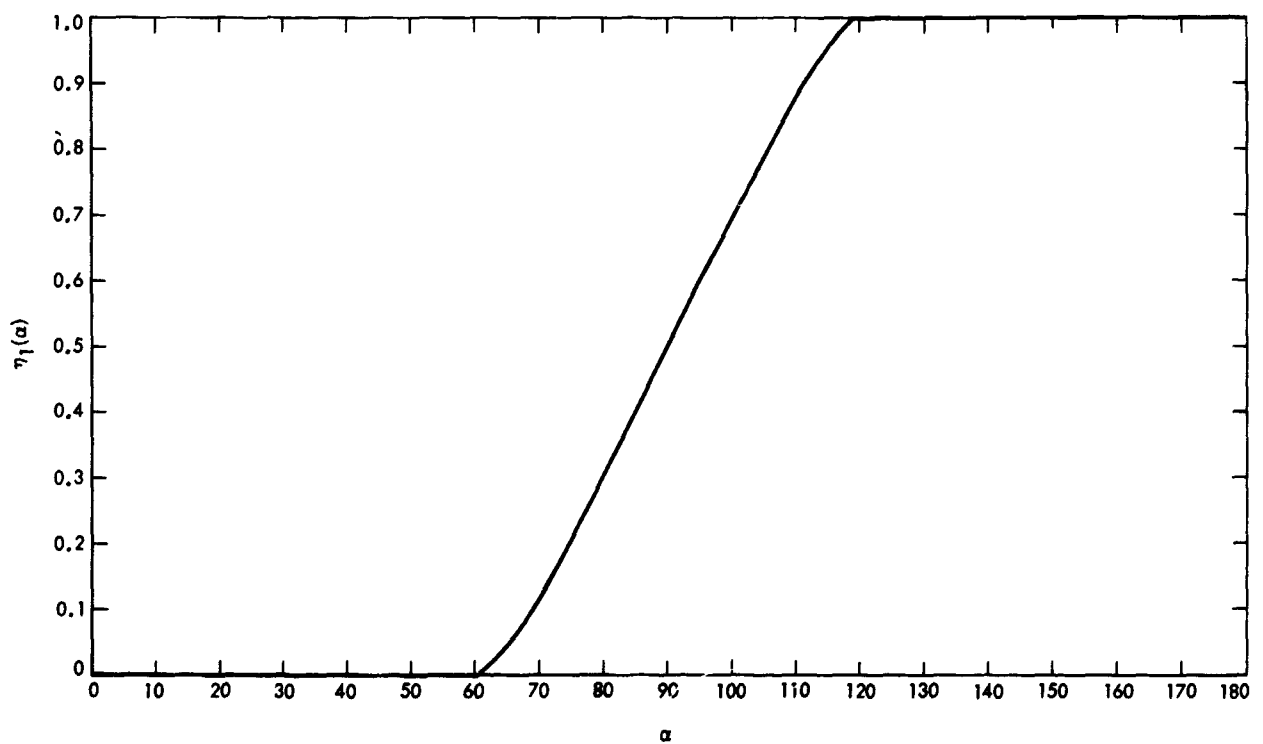


Fig. 13. Graph of the blocking function for the convex side of the parabolic reflector

### VIII. THE MOMENT OF THE SOLAR RADIATION FORCE ON THE BACK SIDE OF THE PARABOLIC REFLECTOR

In the same manner as for the inside area of the reflector, we can prove that

$$M'_y = M'_z = 0.$$

Since, for the back side,

$$\bar{\mathbf{r}} \times d\bar{\mathbf{S}} = \bar{\mathbf{r}} \times \bar{\mathbf{N}}^* \frac{dx dy}{|\bar{\mathbf{N}}^* \cdot \mathbf{e}_3|} = -(y\bar{\mathbf{e}}_1 - x\bar{\mathbf{e}}_2)(1 + 2\lambda z) dx dy,$$

and

$$\bar{\mathbf{u}} \times \bar{\mathbf{r}} = \bar{\mathbf{e}}_1(z \sin \alpha - y \cos \alpha) + \bar{\mathbf{e}}_2 x \cos \alpha - \bar{\mathbf{e}}_3 x \sin \alpha,$$

the expression for the moment, given by Eq. (31), becomes (note that for the back side  $\bar{\mathbf{r}} \times d\bar{\mathbf{S}}$  is the negative of the corresponding value for the front side)

$$M'_x = K \iint_{A_{xy}} yB(\theta)(1 + 2\lambda z) dx dy + KC_2 \iint_{A_{xy}} W(z \sin \alpha - y \cos \alpha) \cos \theta dx dy,$$

and, since  $\cos \theta$  has the opposite sign of the same expression for the front side of the reflector, integrating over the whole area of the circle, we find



$$\begin{aligned}
M_x^{10} = \frac{K}{16\lambda^3} & \left[ \frac{C_1}{2} \sin \alpha \int_0^{2\pi} d\phi \int_0^{2m} \frac{r^3(2+r^2)}{\sqrt{1+r^2}} dr \right. \\
& - C_3 \sin 2\alpha \int_0^{2\pi} d\phi \int_0^{2m} \frac{r^3(2+r^2)}{1+r^2} dr \\
& \left. - C_2 \sin 2\alpha \int_0^2 d\phi \int_0^{2m} r^3 dr \right],
\end{aligned}$$

or

$$M_x^{10} = \pi \delta^2 \zeta K (h_1 \sin \alpha - h_2 \sin 2\alpha) \tag{99}$$

where  $h_1$  and  $h_2$  are given by Eqs. (83)

IX. TOTAL SOLAR RADIATION FORCE AND TORQUE ON BOTH SIDES OF THE PARABOLIC REFLECTOR: RECAPITULATION OF FORMULAS

In the general case, the reflectivity characteristics of the front and back sides of the parabolic antenna reflector are different. We shall denote by  $\beta_F, \gamma_F$  the values of  $\beta$  and  $\gamma$  for the front side, and by  $\beta_B, \gamma_B$  the values of the same quantities for the back side of the reflector. Also, set

$$\beta\gamma = \mu \tag{100}$$

so that

$$\left. \begin{aligned} \mu_F &= \beta_F \gamma_F \\ \mu_B &= \beta_B \gamma_B \end{aligned} \right\} \tag{101}$$

Let us compute first, for given values of  $\delta$  and  $\zeta$ ,

$$\Omega = \arctan\left(\frac{2\zeta}{\delta}\right)$$

and the constants of the reflector

$$\left. \begin{aligned} a_1 &= \cot^2 \Omega \ln \cos \Omega \\ a_2 &= \frac{2}{9} \frac{1 - \cos \Omega}{1 + \cos \Omega} (2 + \sec \Omega) \\ a_3 &= \frac{4}{3} \frac{\cos \Omega}{1 + \cos \Omega} \\ b_1 &= \frac{2}{15} \frac{1 + 2 \cos \Omega + 3 \cos^2 \Omega + 4 \cos^3 \Omega}{\cos \Omega (1 + \cos \Omega)^2} \\ b_2 &= \cot^2 \Omega. \end{aligned} \right\} \tag{102}$$

Then

$$\left. \begin{aligned} f_1^F &= a_2 v_F \\ f_2^F &= \frac{1}{2} + \left(\frac{1}{2} + 2a_1\right) \mu_F \\ g_0^F &= \frac{1}{2} - a_1 \mu_F \\ g_1^F &= a_3 v_F \\ g_2^F &= \frac{1}{2} - (1 + 3a_1) \mu_F \\ h_1^F &= b_1 v_F \\ h_2^F &= \frac{1}{2} + a_1 b_2 \mu_F \end{aligned} \right\} \quad (103)$$

for the front side, and

$$\left. \begin{aligned} f_1^B &= a_2 v_B \\ f_2^B &= \frac{1}{2} + \left(\frac{1}{2} + 2a_1\right) \mu_B \\ g_0^B &= \frac{1}{2} - a_1 \mu_B \\ g_1^B &= a_3 v_B \\ g_2^B &= \frac{1}{2} - (1 + 3a_1) \mu_B \\ h_1^B &= b_1 v_B \\ h_2^B &= \frac{1}{2} + a_1 b_2 \mu_B \end{aligned} \right\} \quad (104)$$

for the back side of the reflector, where

$$\begin{aligned} \nu_F &= \gamma_F - \mu_F, \\ \nu_B &= \gamma_B - \mu_B. \end{aligned} \tag{105}$$

The components of the solar radiation force and torque are schematically shown on Fig. 14, for different intervals of the angle  $\alpha$  between 0 and 180 deg. If we redefine the functions  $\eta(\alpha)$  and  $\eta_1(\alpha)$  in the following manner

$$\eta(\alpha) = \begin{cases} 1, & \text{for } 0^\circ \leq \alpha \leq 90^\circ - \Omega \\ \frac{1}{\pi}(2\phi_0 + \sin 2\phi_0), & \text{for } 90^\circ - \Omega < \alpha < 90^\circ \\ 0, & \text{for } 90^\circ \leq \alpha \leq 180^\circ \end{cases}$$

we can write, for the components of the solar radiation force and torque, the following expressions:

$$\left. \begin{aligned} F_y^T &= \eta(\alpha)F_y^0 + \Gamma F_y' \\ F_z^T &= \eta(\alpha)F_z^0 + \Gamma F_z' \\ M_x^T &= \eta(\alpha)M_x^0 + \Gamma M_x' \end{aligned} \right\} \tag{106}$$

where

$$\Gamma = \begin{cases} 0 & \text{for } 0^\circ < \alpha \leq 90^\circ - \Omega \\ 1 & \text{for } 90^\circ - \Omega < \alpha < 180^\circ \end{cases}$$

and where, from Eqs. (68), (82), (95), (99), (103), and (104),

$$\left. \begin{aligned}
 F_y^0 &= -\pi\delta^2 K \left( f_1^F \sin \alpha + f_2^F \sin 2\alpha \right) \\
 F_z^0 &= -\pi\delta^2 K \left( g_0^F + g_1^F \cos \alpha + g_2^F \cos 2\alpha \right) \\
 F_y'^0 &= -\pi\delta^2 K \left( f_1^B \sin \alpha + f_2^B \sin 2\alpha \right) \\
 F_z'^0 &= -\pi\delta^2 K \left( -g_0^B + g_1^B \cos \alpha - g_2^B \cos 2\alpha \right) \\
 M_x^0 &= \pi\delta^2 \zeta K \left( h_1^F \sin \alpha + h_2^F \sin 2\alpha \right) \\
 M_x'^0 &= \pi\delta^2 \zeta K \left( h_1^B \sin \alpha - h_2^B \sin 2\alpha \right).
 \end{aligned} \right\} \quad (107)$$

Figure 14 shows the values of the components of the solar radiation force and moment for different values of angle  $\alpha$ , as well as the values of the total force on both sides of the parabolic reflector.

Equations (106), together with Eqs. (103), (104), and (107), represent the final form of the solution.

(a) FORCES AND TORQUES ON FRONT AND BACK SIDES

$\alpha$	FRONT SIDE			BACK SIDE			FRONT	BACK
	$F_x$	$F_y$	$F_z$	$F_x$	$F_y$	$F_z$	$M_x$	$M_x$
$[0^\circ, 90^\circ - \Omega]$	0	$F_y^0$	$F_z^0$	0	0	0	$M_x^0$	0
$(90^\circ - \Omega, 90^\circ)$	0	$F_y$	$F_z$	0	$F'_y$	$F'_z$	$M_x$	$M'_x$
$[90^\circ, 90^\circ + \Omega)$	0	0	0	0	$F'_y$	$F'_z$	0	$M'_x$
$[90^\circ + \Omega, 180^\circ]$	0	0	0	0	$F_y^0$	$F_z^0$	0	$M_x^0$

(b) TOTAL FORCE AND TORQUE

$\alpha$	$F_x$	$F_y$	$F_z$	$M_x$
$[0^\circ, 90^\circ - \Omega]$	0	$F_y^0$	$F_z^0$	$M_x^0$
$(90^\circ - \Omega, 90^\circ)$	0	$F_y + F'_y$	$F_z + F'_z$	$M_x + M'_x$
$[90^\circ, 90^\circ + \Omega)$	0	$F'_y$	$F'_z$	$M'_x$
$[90^\circ + \Omega, 180^\circ]$	0	$F_y^0$	$F_z^0$	$M_x^0$

Fig. 14. Table of the total radiation force and torques

## X. COMPARISON WITH THE FLAT-SURFACE MODEL

The so-called "effective area" of the illumination in the flat surface model is the projection of the illuminated area on the plane perpendicular to the direction of the solar radiation. To simplify the derivations of the flat model (the model used in Ref. 5) and still be able to find the order of magnitude of the deviation of the flat surface model from the real force model, we shall consider the case when  $0 \leq \alpha \leq 90^\circ - \Omega$ . We shall also assume that the values of  $\beta$  and  $\gamma$  are the same for both models. Then the components of the solar radiation force for a flat surface can be derived from Eqs. (68) by setting  $\zeta = 0$ . Then  $\Omega = 0$  and

$$f_1 = 0,$$

$$f_2 = \frac{1 + \beta\gamma}{2} + 2\beta\gamma \lim_{\Omega \rightarrow 0} \left[ \frac{\ln \cos \Omega}{\tan^2 \Omega} \right] = \frac{1 - \beta\gamma}{2},$$

$$g_0 = \frac{1 + \beta\gamma}{2},$$

$$g_1 = \frac{2\gamma}{3} (1 - \beta),$$

$$g_2 = \frac{1 + \beta\gamma}{2},$$

so that

$$F_y^0 (\text{flat}) = -\pi\delta^2 K (1 - \beta\gamma) \sin \alpha \cos \alpha,$$

$$F_z^0 (\text{flat}) = -\pi\delta^2 K \left[ \frac{2\gamma}{3} (1 - \beta) \cos \alpha + (1 + \beta\gamma) \cos^2 \alpha \right].$$

The differences in the components of the real force and the flat surface force model are

$$\Delta F_y^0 = -\pi\delta^2 K \left[ \frac{2\gamma}{9} (1 - \beta) \frac{1 - \cos \Omega}{1 + \cos \Omega} (2 + \sec \Omega) \sin \alpha \right. \\ \left. + \beta\gamma(1 + 2 \cot^2 \Omega \ln \cos \Omega) \sin 2\alpha \right]$$

$$\Delta F_z^0 = \pi\delta^2 K \left[ \frac{2\gamma}{3} (1 - \beta) \frac{1 - \cos \Omega}{1 + \cos \Omega} \cos \alpha \right. \\ \left. + \frac{\beta\gamma}{2} (1 + 3 \cos 2\alpha)(1 + 2 \cot^2 \Omega \ln \cos \Omega) \right]$$

For the Pioneer F/G spacecraft, these differences are

$$\Delta F_y^0 = -\pi\delta^2 K \left[ 0.046754 \gamma(1 - \beta) \sin \alpha + 0.128078 \beta\gamma \sin 2\alpha \right]$$

$$\Delta F_z^0 = \pi\delta^2 K \left[ 0.044621 \gamma(1 - \beta) \cos \alpha + 0.064039 \beta\gamma(1 + 3 \cos 2\alpha) \right]$$

which shows the order of magnitude of the error in the flat surface model. For example, for  $\alpha = 0$ ,  $\beta = \gamma = 1$ ,

$$\Delta F_z^0 = 0.256156 \pi\delta^2 K$$



## XI. PRINCIPAL AXES OF INERTIA; EULER'S EQUATIONS OF MOTION

In order to derive the equations of motion of the spacecraft under the action of the solar radiation force, we shall assume that the spacecraft is rotationally symmetric and that its center of gravity (point C on Fig. 15) lies on the z-axis. Now define the system of principal axes of inertia  $x_p$ ,  $y_p$ ,  $z_p$  of the body in the following manner. Because of symmetry the principal  $z_p$ -axis will coincide with the z-axis of one frame of reference and, for the same reason, the  $x_p$  and  $y_p$  axes may be chosen arbitrarily. Therefore, to simplify the problem, we shall take the  $x_p$  and  $y_p$  axes to be parallel to the x and y axes respectively. Hence,

$$x = x_p$$

$$y = y_p$$

$$z = z_p - z_C.$$

The components of the moment of the solar radiation force with respect to the point O, the origin of the system xyz, are

$$M_x^{(O)} = M_x$$

$$M_y^{(O)} = 0$$

$$M_z^{(O)} = 0.$$

If  $\bar{r}$  is the position vector of a point of the body with respect to O as the origin and  $\bar{r}^{(C)}$  is the position vector of the same point with respect to the point C as the origin, the relationship between these two vectors is

$$\bar{r}^{(C)} = \bar{r} + z_C \bar{e}_3.$$

The moments of the solar radiation force relative to the points O and C are, respectively,

$$\bar{M}^{(O)} = \iint_S \bar{r} \times d\bar{F}$$

$$\bar{M}^{(C)} = \iint_S \bar{r}^{(C)} \times d\bar{F} = \bar{M}^{(O)} + z_C \bar{e}_3 \times \bar{F}.$$

further on, because  $F_x = 0$ ,

$$\bar{e}_3 \times \bar{F} = -F_y \bar{e}_1$$

so that

$$\bar{M}^{(C)} = \bar{M} - z_C F_y \bar{e}_1$$

or, consequently,

$$\left. \begin{aligned} M_x^{(C)} &= M_x - z_C F_y \\ M_y^{(C)} &= 0 \\ M_z^{(C)} &= 0 \end{aligned} \right\} \quad (108)$$

where

$$x_p = x$$

$$y_p = y$$

$$z_p = z - z_C$$

are the principal axes.

The moments of inertia of the hollow paraboloid of revolution with respect to the axes of the system xyz are

$$I_x = I_y = \frac{M_R}{6} (2\delta^2 + 3\zeta^2)$$

$$I_z = \frac{M_R \delta^2}{3}$$

where  $M_R$  is the mass of the reflector. According to Steiner's theorem, the principal moments of inertia of the reflector are

$$\left. \begin{aligned} A = B &= M_R \left( \frac{\delta^2}{3} + \frac{\zeta^2}{2} + z_C^2 \right) \\ C &= \frac{M_R \delta^2}{3} \end{aligned} \right\} \quad (109)$$

Let  $J_x, J_y, J_z$  be the principal moments of inertia of the rest of the spacecraft. Then, using Eqs. (108) and (109), we can write Euler's equations of motion of the whole spacecraft in the form

$$\left. \begin{aligned} (A + J_x) \dot{\omega}_x - \left[ (A - C) + (J_y - J_z) \right] \omega_y \omega_z &= M_x - z_C F_y \\ (A + J_y) \dot{\omega}_y + \left[ (A - C) + (J_x - J_z) \right] \omega_z \omega_x &= 0 \\ (C + J_z) \dot{\omega}_z - (J_x - J_y) \omega_x \omega_y &= 0 \end{aligned} \right\} \quad (110)$$

Here we assume that the rest of the spacecraft is not illuminated ( $\omega_x, \omega_y, \omega_z$  are components of the angular velocity vector).

For a rotationally symmetric spacecraft,  $J_x = J_y$ , and

$$\omega_z = n$$

so that

$$\omega_x = - \frac{A + J_x}{n[(A - C) + (J_x - J_z)]} \dot{\omega}_y = - l \dot{\omega}_y$$

and the differential equation for  $\omega_y$  is

$$\frac{(A + J_x)^2}{n(A - C + J_x - J_z)} \ddot{\omega}_y + n(A - C + J_x - J_z) \dot{\omega}_y = z_C F_y - M_x. \quad (111)$$

The right-hand side of this equation can be written in another form, by means of the first of Eqs. (68) and Eq. (82), namely

$$z_C F_y - M_x = -\pi \delta^2 K \left[ (f_1 z_C + \zeta h_1) \sin \alpha + (f_2 z_C + \zeta h_2) \sin 2\alpha \right]. \quad (112)$$

From Eqs. (69), (70) and the first of Eqs. (83), we can find the values of  $f_1 z_C + \zeta h_1$  and  $f_2 z_C + \zeta h_2$  in terms of the angle  $\Omega$  of the reflector.

From

$$K = K(r_{SP}) = \frac{K_{SRP}}{r^2},$$

using the notation

$$\left. \begin{aligned} f_1 z_C + \zeta h_1 &= l_1 \\ f_2 z_C + \zeta h_2 &= l_2 \\ \pi \delta^2 K_{SRP} &= C_S \end{aligned} \right\} \quad (113)$$

we have the right-hand side of Eq. (111) in the form

$$z_C F_y - M_x = -C_S (l_1 \sin \alpha + l_2 \sin 2\alpha). \quad (114)$$

For the Pioneer spacecraft,

$$l_1 = \gamma (1 - \beta) (0.127367 + 0.046754 z_C) \text{ meters}$$

$$l_2 = \frac{1}{2} (0.3803 + z_C) + (0.158396 - 0.371922 z_C) \beta \gamma \text{ meters}$$

$$C_S = 6.0444 \times 10^{17} \text{ kilogram meter/second}^2$$

or, if the heliocentric distance of the spacecraft is given in astronomical units (AU),

$$C_S = 2.7008 \times 10^{-5} \text{ kilogram meter/second}^2$$

With a negligible loss of accuracy we can assume that the angle  $\alpha$  is equal to the Earth-Spacecraft-Sun angle. Therefore,

$$\sin \alpha = \frac{|\bar{r} \times \bar{R}_E|}{r\rho}$$

Using the notation

$$n (A - C + J_x - J_z) = q_1$$

$$(A + J_x)^2 = q_2$$

we can rewrite Eq. (111) in the form

$$\frac{q_2}{q_1} \ddot{\omega}_y + q_1 \omega_y = - \frac{C_S}{r^2} (l_1 \sin \alpha + l_2 \sin 2\alpha)$$

or

$$\ddot{\omega}_y + \Omega_0^2 \omega_y = - \epsilon_0 (1 + e \cos f)^2 (l_1 \sin \alpha + l_2 \sin 2\alpha) \quad (115)$$

where

$$\Omega_0 = \frac{n(A - C + J_x - J_z)}{A + J_x}$$

$$\epsilon_0 = \frac{C_S n}{a_0^2 (1 - e^2)^2} \frac{A - C + J_x - J_z}{(A + J_x)^2}$$

$a_0$  = semimajor axis of the spacecraft's orbit in astronomical units

$e$  = the eccentricity of the spacecraft's orbit

$f$  = true anomaly of the spacecraft.

The period of free oscillations (without the forcing function) of the rotationally symmetric spacecraft ( $J_x = J_y$ ) is given by

$$T_0 = \frac{2\pi}{n} \frac{A + J_x}{A - C + J_x - J_z} \quad (116)$$

where  $n$  is the rate of rotation of the spacecraft about its roll axis.

If there were no other parts of the spacecraft (the reflector only),

$$J_x = J_z = 0$$

$$z_C = \frac{2}{3} \zeta$$

$$A = M_R \left( \frac{\delta^2}{3} + \frac{\zeta^2}{2} - \frac{4\zeta^2}{9} \right) = M_R \left( \frac{\delta^2}{3} + \frac{\zeta^2}{18} \right)$$

$$C = M_R \frac{\delta^2}{3}$$

$$A - C = M_R \frac{\zeta^2}{18}$$

and

$$T_0 = \frac{2\pi}{n} \left[ 1 + 6 \left( \frac{\delta}{\zeta} \right)^2 \right].$$

For the Pioneer spacecraft, the rate of rotation about its roll axis is 5 revolutions/minute, and the period of spinning is thus

$$T_{\text{spin}} = 12 \text{ seconds.}$$

Therefore,

$$T_0 = T_{\text{spin}} \left[ 1 + 6 \left( \frac{\delta}{\zeta} \right)^2 \right].$$

The ratio  $\delta/\zeta$  is approximately 3.6, so that

$$T_0 = 79 T_{\text{spin}} = 948 \text{ seconds} = 15.8 \text{ minutes.}$$

The period of free oscillations about the x-axis is the same. Indeed, setting the right-hand side of the first of Eqs. (110) equal to zero, we can substitute  $\omega_y$  into the second of Eqs. (110) and obtain again Eq. (115) in the form

$$\ddot{\omega}_x + \Omega_0^2 \omega_x = 0.$$

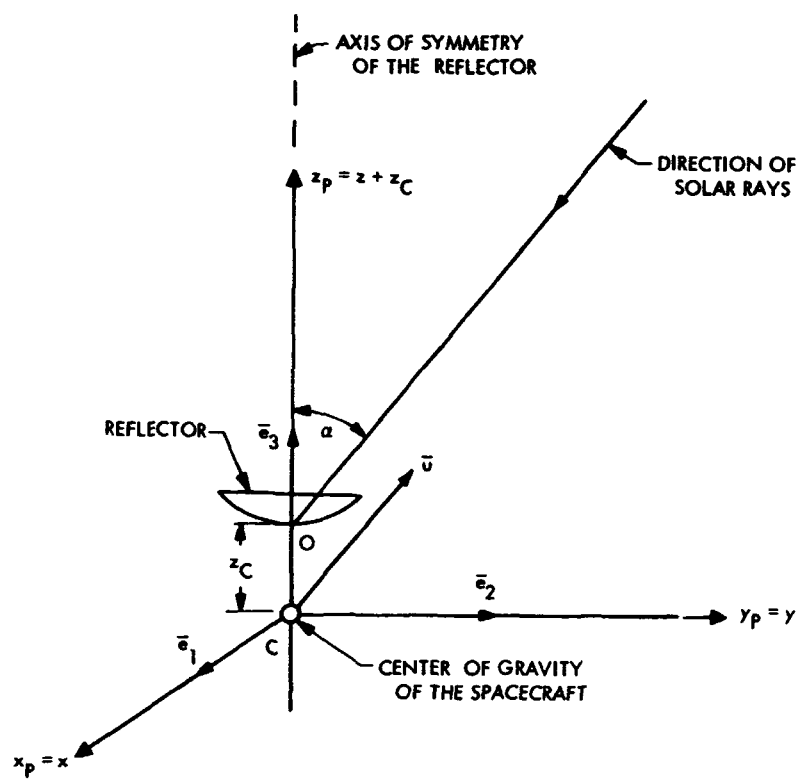


Fig. 15. Orientation of axes of the noninertial reference frame



## XII. SOLAR RADIATION FORCE IN THE INERTIAL FRAME OF REFERENCE

The components of the solar radiation force, given by the first two of Eqs. (106), are relative to the noninertial frame of reference  $xyz$ . The angular velocity vector of rotation of this system is given by

$$2\bar{\omega} = \bar{i} \times \frac{d\bar{i}}{dt} + \bar{j} \times \frac{d\bar{j}}{dt} + \bar{k} \times \frac{d\bar{k}}{dt} \quad (117)$$

where

$$\bar{i} = \frac{\bar{R}_E \times \bar{r}}{|\bar{R}_E \times \bar{r}|}$$

$$\bar{j} = \bar{k} \times \bar{i}$$

$$\bar{k} = -\frac{\bar{\rho}}{\rho}, \quad \bar{\rho} = \bar{r} - \bar{R}_E$$

Here,

$$\bar{i} \times \frac{d\bar{i}}{dt} = \frac{\bar{R}_E \times \bar{r}}{|\bar{R}_E \times \bar{r}|^2} \times \frac{d}{dt} (\bar{R}_E \times \bar{r})$$

$$\bar{j} \times \frac{d\bar{j}}{dt} = \frac{(\bar{R}_E - \bar{r}) \times (\bar{R}_E \times \bar{r})}{(\rho |\bar{R}_E \times \bar{r}|)^2} \times \frac{d}{dt} [(\bar{R}_E - \bar{r}) \times (\bar{R}_E \times \bar{r})]$$

$$\bar{k} \times \frac{d\bar{k}}{dt} = \frac{\bar{R}_E - \bar{r}}{\rho^2} \times \left( \frac{d\bar{R}_E}{dt} - \frac{d\bar{r}}{dt} \right)$$

and the equation of motion of the spacecraft in an inertial frame of reference is

$$\frac{d^2\bar{r}}{dt^2} - 2\bar{\omega} \times \frac{d\bar{r}}{dt} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) - \dot{\bar{\omega}} \times \bar{r} = \frac{\bar{F}}{m_S}$$

where  $\bar{F} = F_y \bar{j} + F_z \bar{k}$ , and  $m_S$  is the mass of the spacecraft.

### XIII. SOLAR RADIATION FORCE ON THE SURFACE OF A CIRCULAR CYLINDER

A spacecraft may carry one or more components of cylindrical shape such as fuel tanks, instruments, etc. To find the solar radiation force exerted on the surface of a cylinder of an arbitrary radius in the general case, we shall assume that the axis of symmetry of the cylinder lies along an arbitrarily chosen direction, defined by the unit vector  $\bar{n}_1$ . Let  $xyz$  be the noninertial frame of reference, the axes of which are directed along the unit vectors  $\bar{e}_1, \bar{e}_2, \bar{e}_3$  of the already described system of coordinates, translated into the center of gravity as the origin, and let the point  $O$  be the point on which the axis of symmetry of the cylinder penetrates the  $yz$ -plane of the above-described coordinate system. Taking  $O$  as the origin of another coordinate system of axes  $x', y', z'$  parallel to the axes  $x, y,$  and  $z$  respectively, we have, from Fig. 16,

$$\left. \begin{aligned} x' &= x \\ y' &= y - y_0 \\ z' &= z - z_0 \end{aligned} \right\} \quad (118)$$

where the two quantities  $y_0$  and  $z_0$ , are known. In order to bring the axes of the system  $x' y' z'$  into the position of the system  $\xi \eta \zeta$ , we must perform the following two rotations:

- (1) Rotation about the  $z'$ -axis by an angle  $\chi$ , in the positive direction; this brings the  $x'$ -axis into the position  $OL$ .
- (2) Rotation about the new  $y$ -axis by an angle  $\epsilon$ , in the negative direction. This brings the  $x'$ -axis into the position  $O\xi$ .

The equations of transformation between the  $x' y' z'$  system and  $\xi \eta \zeta$  system are

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos \epsilon & 0 & \sin \epsilon \\ 0 & 1 & 0 \\ -\sin \epsilon & 0 & \cos \epsilon \end{bmatrix} \begin{bmatrix} \cos \chi & \sin \chi & 0 \\ -\sin \chi & \cos \chi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

or

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos \chi \cos \epsilon & \sin \chi \cos \epsilon & \sin \epsilon \\ -\sin \chi & \cos \chi & 0 \\ -\cos \chi \sin \epsilon & -\sin \chi \sin \epsilon & \cos \epsilon \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (119)$$

The equation of the cylinder in the system  $\xi \eta \zeta$  is

$$\eta^2 + \zeta^2 = a^2. \quad (120)$$

The components of the unit vector

$$\bar{u} = \bar{e}_2 \sin \alpha + \bar{e}_3 \cos \alpha$$

in the  $\xi \eta \zeta$  -system are

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos \chi \cos \epsilon & \sin \chi \cos \epsilon & \sin \epsilon \\ -\sin \chi & \cos \chi & 0 \\ -\cos \chi \sin \epsilon & -\sin \chi \sin \epsilon & \cos \epsilon \end{bmatrix} \begin{bmatrix} 0 \\ \sin \alpha \\ \cos \alpha \end{bmatrix}$$

or

$$\left. \begin{aligned} u_1 &= \sin \chi \cos \epsilon \sin \alpha + \sin \epsilon \cos \alpha \\ u_2 &= \cos \chi \sin \alpha \\ u_3 &= -\sin \chi \sin \epsilon \sin \alpha + \cos \epsilon \cos \alpha \end{aligned} \right\} \quad (121)$$

and the equation of the plane parallel to unit vector  $\bar{u}$  and  $\bar{n}_1$  is

$$(\bar{r} - \bar{r}_0) \cdot (\bar{u} \times \bar{n}_1) = 0$$

or

$$\begin{vmatrix} \xi - \xi_0 & \eta - \eta_0 & \zeta - \zeta_0 \\ u_1 & u_2 & u_3 \\ 1 & 0 & 0 \end{vmatrix} = 0,$$

or

$$u_3 (\eta - \eta_0) - u_2 (\zeta - \zeta_0) = 0.$$

This equation can be written also in the form

$$u_3 \eta - u_2 \zeta + p = 0. \quad (122)$$

Out of infinitely many planes given by Eq. (122), two are tangent to the surface of the cylinder. Their traces are shown on Fig. 17. The two lines along which these planes touch the cylinder project a two points, P and Q, on Fig. 17. These two lines separate the illuminated part of the surface from the part of the surface of the cylinder which remains in the shadow. The illuminated area projects on the  $\zeta\eta$ -plane as a rectangle of length  $\overline{AB} = h$  and width  $2\eta_0$  (Fig. 17), which must be determined in the following way.

Combining the equations of the cylinder (Eq. 120) with the equation of the plane (Eq. 122), we find the interception of these two surfaces to be given by

$$\eta = \frac{u_2}{u_3} \zeta - \frac{p}{u_3}$$

$$\eta^2 + \zeta^2 = a^2.$$

Substituting the value of  $\eta$  from the first equation into the second, we obtain the the quadratic equation for  $\zeta$ :

$$\left(u_2^2 + u_3^2\right) \zeta^2 - 2p u_2 \zeta + \left(p^2 - a^2 u_3^2\right) = 0.$$

Because of the condition for tangency, the two roots of this equation must be the same. Hence, we find

$$p = \pm a \sqrt{u_2^2 + u_3^2} = \pm a \sqrt{1 - u_1^2} \quad (123)$$

because

$$u_1^2 + u_2^2 + u_3^2 = 1.$$

Therefore, the coordinates of points P and Q are

$$\zeta_{P, Q} = \pm \frac{au_2}{\sqrt{1 - u_1^2}}$$

and

$$\eta_{P, Q} = \pm \frac{au_3}{\sqrt{1 - u_1^2}}$$

so that

$$\eta_0 = \frac{au_3}{\sqrt{1 - u_1^2}} \tag{124}$$

The unit vector of the normal to the surface

$$\bar{N}^* = \frac{\nabla \Phi}{|\nabla \Phi|} = \frac{\nabla \Phi}{W}$$

lies in the  $\eta\zeta$  - plane. From the equation of the surface

$$\Phi(\xi, \eta, \zeta) = \eta^2 + \zeta^2 - a^2 = 0$$

we find

$$\frac{\partial \Phi}{\partial \xi} = 0, \quad \frac{\partial \Phi}{\partial \eta} = 2\eta, \quad \frac{\partial \Phi}{\partial \zeta} = 2\zeta,$$

so that

$$W = \sqrt{4(\eta^2 + \zeta^2)} = 2a \quad (125)$$

and

$$\bar{N}^* = \frac{\eta}{a}\bar{n}_2 + \frac{\zeta}{a}\bar{n}_3. \quad (126)$$

Also, from  $\cos \theta = \bar{u} \cdot \bar{N}^*$ , we find

$$\cos \theta = \frac{u_2\eta + u_3\zeta}{a}. \quad (127)$$

Taking the coordinate transformation

$$\eta = a \sin \phi$$

$$\zeta = a \cos \phi$$

we can write, instead of Eq. (127),

$$\cos \theta = u_2 \sin \phi + u_3 \cos \phi. \quad (128)$$

Since the  $\eta$ -coordinate of the point Q (Fig. 17) is

$$\eta_Q = \eta_0 = a \sin \Lambda$$

the integration limits for the solar radiation force in Eq. (29) will be

$$\text{for } \xi : \xi_A, \xi_A + h$$

$$\text{for } \phi : -\Lambda, +\Lambda,$$

where

$$\Lambda = \arcsin \left( \frac{u_3}{\sqrt{1-u_1^2}} \right). \quad (129)$$

The solar radiation force is then given by

$$\begin{aligned} \bar{F} &= -K \int_{\phi=-\Lambda}^{\Lambda} \int_{\xi=\xi_A}^{\xi_A+h} B(\theta) (\bar{n}_2 \sin \phi + \bar{n}_3 \cos \phi) \frac{a \cos \phi \, d\xi \, d\phi}{|\bar{n}_3 \cdot \bar{N}^*|} \\ &= -KC_2 (u_1 \bar{n}_1 + u_2 \bar{n}_2 + u_3 \bar{n}_3) \int_{\phi=-\Lambda}^{\Lambda} \int_{\xi=\xi_A}^{\xi_A+h} \cos \theta \frac{a \cos \phi \, d\xi \, d\phi}{|\bar{n}_3 \cdot \bar{N}^*|}, \end{aligned}$$

or, because

$$|\bar{n}_3 \cdot \bar{N}^*| = \frac{\zeta}{a} = \cos \phi,$$

$$\begin{aligned} \bar{F} &= -Kha \int_{-\Lambda}^{\Lambda} B(\theta) (\bar{n}_2 \sin \phi + \bar{n}_3 \cos \phi) \, d\phi \\ &= -KhC_2 a (u_1 \bar{n}_1 + u_2 \bar{n}_2 + u_3 \bar{n}_3) \int_{-\Lambda}^{\Lambda} \cos \theta \, d\phi \quad (130) \end{aligned}$$

where

$$B(\theta) = C_1 (u_2 \sin \phi + u_3 \cos \phi) + 2C_3 (u_2 \sin \phi + u_3 \cos \phi)^2.$$



The components of the solar radiation force, given by Eq. (130), along the axes of the system  $\xi\eta\zeta$ , are

$$F_{\xi} = -KhaC_2u_1 \int_{-\Lambda}^{\Lambda} (u_2 \sin \phi + u_3 \cos \phi) d\phi$$

$$F_{\eta} = -Kha \int_{-\Lambda}^{\Lambda} B(\theta) \sin \phi d\phi + \frac{u_2}{u_1} F_{\xi}$$

$$F_{\zeta} = -Kha \int_{-\Lambda}^{\Lambda} B(\theta) \cos \phi d\phi + \frac{u_3}{u_1} F_{\xi}$$

Performing the indicated integration, we finally obtain

$$F_{\xi} = -2KahC_2u_1u_3 \sin \Lambda = -2KahC_2 \frac{u_1 u_3^2}{\sqrt{1-u_1^2}} \quad (131)$$

$$F_{\eta} = -Khau_2 \left[ C_1 \arcsin \left( \frac{u_3}{\sqrt{1-u_1^2}} \right) - \frac{u_3}{1-u_1^2} \left( C_1 u_2 - \frac{8C_3}{3} \frac{u_3^3}{\sqrt{1-u_1^2}} \right) \right] + \frac{u_2}{u_1} F_{\xi} \quad (132)$$

$$F_{\zeta} = -Khau_3 \left[ C_1 \arcsin \left( \frac{u_3}{\sqrt{1-u_1^2}} \right) + \frac{u_3}{1-u_1^2} \left( C_1 u_2 + \frac{8C_3}{3} u_3 \frac{1-u_1^2+u_2^2}{\sqrt{1-u_1^2}} \right) \right] + \frac{u_3}{u_1} F_{\xi} \quad (133)$$

The components of the solar radiation force along the axes of the system  $x'y'z'$  will then be

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \cos \chi \cos \epsilon & -\sin \chi & -\cos \chi \sin \epsilon \\ \sin \chi \cos \epsilon & \cos \chi & -\sin \chi \sin \epsilon \\ \sin \epsilon & 0 & \cos \epsilon \end{bmatrix} \begin{bmatrix} F_\xi \\ F_\eta \\ F_\zeta \end{bmatrix} \quad (134)$$

Since the system  $x'y'z'$  is rotating in space, and the spacecraft also spins about its roll axis, the angle  $\chi$  is a function of time.

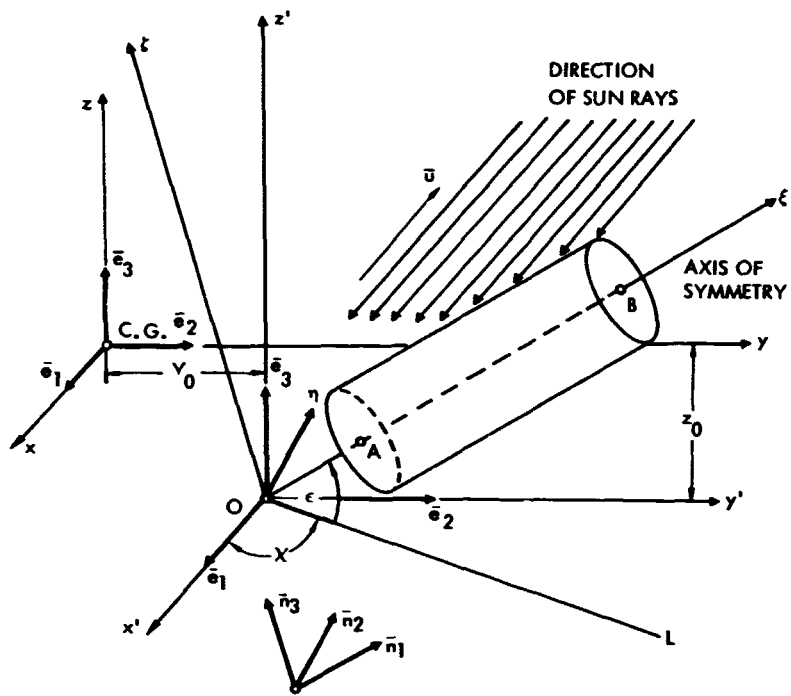


Fig. 16. Solar radiation on the curved surface of a circular cylinder

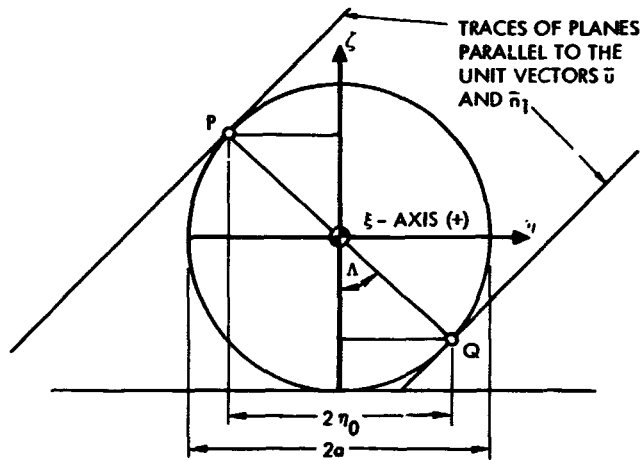


Fig. 17. Trace of the projection of the illuminated area of the cylinder on the  $xy$ -plane

#### XIV. SPECIAL CASES OF THE SOLAR RADIATION FORCE ON A CIRCULAR CYLINDER

If the circular cylinder lies in the  $x'y'$  plane,  $\epsilon = 0$ , and, from Eqs. (121),

$$u_1 = \sin \chi \sin \alpha$$

$$u_2 = \cos \chi \sin \alpha$$

$$u_3 = \cos \alpha$$

so that, from Eq. (129),

$$\sin \Lambda = \frac{\cos \alpha}{\sqrt{1 - \sin^2 \chi \sin^2 \alpha}}$$

From Eq. (131) we can derive

$$F_{\xi} = -KahC_2 \frac{\sin \chi \sin \alpha \cos^3 \alpha}{\sqrt{1 - \sin^2 \chi \sin^2 \alpha}}$$

and the other two components can be obtained by a simple substitution of the values of  $u_1$ ,  $u_2$ , and  $u_3$  into Eqs. (132) and (133).

Another interesting case is noted when the axis of symmetry of the cylinder coincides with the  $z$ -axis (axis of symmetry of the reflector). Then, the  $\xi$ -axis lies along the  $z$ -axis, the  $\eta$ -axis coincides with the  $x$ -axis, and the  $\zeta$ -axis coincides with the  $y$ -axis, so that  $\chi = -\pi/2$ , and  $\epsilon = \pi/2$ . Assuming that the cylinder hangs behind the surface of the parabolic reflector, we can take the bottom base of the cylinder as the  $\eta\zeta$ -plane, which yields  $y_0 = 0$ ,  $z_0 = h$ . Thus

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \sin \alpha \\ \cos \alpha \end{bmatrix}$$

or

$$u_1 = \cos \alpha$$

$$u_2 = 0$$

$$u_3 = \sin \alpha.$$

Neglecting the shadowing effects of other components of the spacecraft on the surface of the cylinder, we can write from Eq. (129)

$$\sin \Lambda = 1, \quad \Lambda = 90^\circ.$$

Formulas given by Eqs. (131-133) yield

$$F_\xi = -Kah(1 - \beta\gamma) \sin 2\alpha$$

$$F_\eta = 0$$

$$F_\zeta = -Kah \sin \alpha \left[ \frac{\pi Y}{3}(1 - \beta) + \frac{2}{3}(3 + \beta\gamma) \sin \alpha \right]$$

which leads to the following expressions  $(F_{\xi} = F_z, F_{\eta} = F_x, F_{\zeta} = F_y)$ .

$$\left. \begin{aligned} F_x &= 0 \\ F_y &= -Kah \sin \alpha \left[ \frac{\pi Y}{3} (1 - \beta) + \frac{2}{3} (3 + \beta\gamma) \sin \alpha \right] \\ F_z &= -Kah(1 - \beta\gamma) \sin 2\alpha. \end{aligned} \right\} \quad (135)$$

## APPENDIX

### STOKES' THEOREM; CONVERSION OF DOUBLE INTEGRALS INTO LINE INTEGRALS

The previously obtained expression for the solar radiation force, given by Eq. (22), may be written in the form

$$\bar{F} = -K \left[ \iint_S B(\theta) d\bar{S} + C_2 \bar{u} \iint_S \bar{u} \cdot d\bar{S} \right] \quad (A-1)$$

i. e. , as the sum of two double integrals. We have seen earlier that the second integral in Eq. (A-1) can be solved immediately, since, insofar as the double integration is concerned, the unit vector  $\bar{u}$  is a constant vector and, therefore,

$$\iint_S \bar{u} \cdot d\bar{S} = \bar{u} \cdot \iint_S d\bar{S} = \bar{u} \cdot \bar{S} \quad (A-2)$$

The function  $B(\theta)$  in the first double integral in Eq. (A-1) is a function of  $x$ ,  $y$ , and  $z$ . From the equation of the reflecting surface,

$$\Phi(x, y, z) = 0$$

we can obtain  $z$  as a function of  $x$  and  $y$ . With the substitution of the value of  $z$  so obtained into the function  $B(\theta)$ , it becomes a function of  $x$  and  $y$  only, i. e. ,  $B(x, y)$ . Thus we have

$$\bar{F} = -K \left[ \iint_S B(x, y) d\bar{S} + C_2 (\bar{u} \cdot \bar{S}) \bar{u} \right] \quad (A-3)$$

Now consider the double integral

$$\iint_S B(x, y) d\bar{S}. \quad (\text{A-4})$$

Since this integral physically represents one part of the solar radiation force exerted on the surface of a certain component of a spacecraft, the surface area  $S$  is always bounded by a simple closed curve or, in the general case, a closed geometrical figure bounded by several parts of different simple curves. Therefore, we can write the Stokes theorem in the form

$$\iint_S (d\bar{S} \times \nabla) \times \bar{v} = - \oint_L \bar{v} \times d\bar{r}, \quad (\text{A-5})$$

where  $L$  is the closed curve, bounding the surface  $S$ . The vector  $\bar{v}$  is an unknown vector function that must be determined from

$$(d\bar{S} \times \nabla) \times \bar{v} = B(x, y) d\bar{S}. \quad (\text{A-6})$$

Since, from what we had before,

$$d\bar{S} = \bar{N}^* dS = \frac{\nabla\phi}{W} dS,$$

we can rewrite Eq. (A-6) in the form

$$(\nabla\phi \times \nabla) \times \bar{v} = B(x, y)\nabla\phi. \quad (\text{A-7})$$

The equation of the surface  $\phi$  having been known, we can consider  $\nabla\phi$  a known vector,

$$\bar{N} = \nabla\phi$$



the components of which are  $N_i(x, y)$ ,  $i = 1, 2, 3$ . The left-hand side of Eq. (A-7) is a triple cross-product, which can be written as

$$(\bar{N} \times \nabla) \times \bar{v} = \nabla(\bar{v} \cdot \bar{N}) - \bar{N}(\nabla \cdot \bar{v}) = \text{grad}(\bar{v} \cdot \bar{N}) - \bar{N} \text{div} \bar{v} \quad (\text{A-8})$$

where the asterisk above the vector  $\bar{N}$  means that the operator  $\nabla$  is applied to vector  $\bar{v}$  only.

Substituting the value of the triple cross-product  $(\bar{N} \times \nabla) \times \bar{v}$  from Eq. (A-8) into Eq. (A-7), we find

$$\text{grad}(\bar{v} \cdot \bar{N}) - \bar{N} \text{div} \bar{v} = B(x, y)\bar{N}$$

or

$$\text{grad}(\bar{v} \cdot \bar{N}) = \bar{N}[\text{div} \bar{v} + B(x, y)] \quad (\text{A-9})$$

Using components of the vector function  $\bar{v}$ :  $v_1$ ,  $v_2$ , and  $v_3$ , Eq. (A-9) yields the following three partial differential equations:

$$\left. \begin{aligned} N_1 \frac{\partial v_1}{\partial x} + N_2 \frac{\partial v_2}{\partial x} + N_3 \frac{\partial v_3}{\partial x} &= N_1 \left[ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} + B(x, y) \right] \\ N_1 \frac{\partial v_1}{\partial y} + N_2 \frac{\partial v_2}{\partial y} + N_3 \frac{\partial v_3}{\partial y} &= N_2 \left[ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} + B(x, y) \right] \\ N_1 \frac{\partial v_1}{\partial z} + N_2 \frac{\partial v_2}{\partial z} + N_3 \frac{\partial v_3}{\partial z} &= N_3 \left[ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} + B(x, y) \right] \end{aligned} \right\} \quad (\text{A-10})$$

or

$$\begin{aligned}
N_1 \left[ \frac{\partial v_2}{\partial v} + \frac{\partial v_3}{\partial z} + B(x, y) \right] - N_2 \frac{\partial v_2}{\partial x} - N_3 \frac{\partial v_3}{\partial x} &= 0 \\
-N_1 \frac{\partial v_1}{\partial y} + N_2 \left[ \frac{\partial v_1}{\partial x} + \frac{\partial v_3}{\partial z} + B(x, y) \right] - N_3 \frac{\partial v_3}{\partial y} &= 0 \\
-N_1 \frac{\partial v_1}{\partial z} - N_2 \frac{\partial v_2}{\partial z} + N_3 \left[ \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + B(x, y) \right] &= 0
\end{aligned}$$

where  $B(x, y)$  is a known function of  $x$  and  $y$ . Since this system is homogeneous with respect to  $N_1$ ,  $N_2$ , and  $N_3$ , the determinant of the system must be zero.

We shall now consider a particular case when the right-hand side of Eq. (A-9) is zero; in other words, we shall find a particular vector function  $\bar{v}$  which makes the right-hand side of the Eq. (A-9) vanish, so that

$$\operatorname{div} \bar{v} + B(x, y) = 0. \quad (\text{A-11})$$

Since in this case the right-hand sides of the three scalar equations (Eqs. A-10) vanish, the Jacobian  $J(v_1, v_2, v_3/x, y, z)$  is equal to zero, i. e.,

$$J \left( \begin{array}{c} v_1, v_2, v_3 \\ x, y, z \end{array} \right) = \begin{vmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_2}{\partial x} & \frac{\partial v_3}{\partial x} \\ \frac{\partial v_1}{\partial y} & \frac{\partial v_2}{\partial y} & \frac{\partial v_3}{\partial y} \\ \frac{\partial v_1}{\partial z} & \frac{\partial v_2}{\partial z} & \frac{\partial v_3}{\partial z} \end{vmatrix} = 0$$

This fact implies that the three components of the vector function  $\bar{v}$  are not independent quantities. Indeed, writing the equation of the reflecting surface in the explicit form

$$\Phi(x, y, z) = f(x, y) - z = 0$$

and, therefore,

$$\left. \begin{aligned} N_1 &= \frac{\partial f}{\partial x} \\ N_2 &= \frac{\partial f}{\partial y} \\ N_3 &= -1. \end{aligned} \right\}$$

We find, from the third of Eqs. (A-10),

$$\frac{\partial}{\partial z} \left( \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 - v_3 \right) = 0.$$

and, accordingly,

$$\frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 - v_3 = \psi(x, y)$$

or

$$v_3 = \frac{\partial f}{\partial x} v_1 + \frac{\partial f}{\partial y} v_2 - \psi(x, y). \quad (\text{A-12})$$

Hence

$$\frac{\partial v_3}{\partial x} = \frac{\partial^2 f}{\partial x^2} v_1 + \frac{\partial^2 f}{\partial x \partial y} v_2 + \frac{\partial f}{\partial x} \frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial v_2}{\partial x} - \frac{\partial \psi}{\partial x}$$

$$\frac{\partial v_3}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} v_1 + \frac{\partial^2 f}{\partial y^2} v_2 + \frac{\partial f}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial v_2}{\partial y} - \frac{\partial \psi}{\partial y}$$

$$\frac{\partial v_3}{\partial z} = \frac{\partial f}{\partial x} \frac{\partial v_1}{\partial z} + \frac{\partial f}{\partial y} \frac{\partial v_2}{\partial z}$$

and, substituting these values into the left-hand sides of the first two of Eqs. (A-12), we find

$$\left. \begin{aligned} \frac{\partial^2 f}{\partial x^2} v_1 + \frac{\partial^2 f}{\partial x \partial y} v_2 &= \frac{\partial \psi}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} v_1 + \frac{\partial^2 f}{\partial y^2} v_2 &= \frac{\partial \psi}{\partial y} \end{aligned} \right\} \quad (\text{A-13})$$

Assuming that the determinant of this system does not vanish, i. e., if

$$G(x, y) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 \neq 0,$$

the two components  $v_1$  and  $v_2$  can be determined from

$$\left. \begin{aligned} v_1 &= \frac{1}{G(x, y)} \left( \frac{\partial^2 f}{\partial y^2} \frac{\partial \psi}{\partial x} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial \psi}{\partial y} \right) \\ v_2 &= \frac{1}{G(x, y)} \left( \frac{\partial^2 f}{\partial x^2} \frac{\partial \psi}{\partial y} - \frac{\partial^2 f}{\partial x \partial y} \frac{\partial \psi}{\partial x} \right) \end{aligned} \right\} \quad (\text{A-14})$$

Eqs. (A-14) show that  $v_1$  and  $v_2$  are functions of  $x$  and  $y$  only. Therefore

$$\frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} = 0$$

and, accordingly,

$$\frac{\partial v_3}{\partial z} = 0$$

The conditional scalar equation (Eq. A-11) then yields

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = -B(x, y), \quad (\text{A-15})$$

so that, differentiating the first equation (Eq. A-14) with respect to  $x$  and the second equation with respect to  $y$ , and substituting the results into Eq. (A-15), we finally arrive at a partial differential equation of the second order for the determination of the function  $\psi(x, y)$ .

To illustrate this method, we shall consider the equation of the surface of the paraboloid of revolution

$$\Phi(x, y, z) = \lambda(x^2 + y^2) - z = 0$$

where

$$\frac{\partial \Phi}{\partial x} = 2\lambda x, \quad \frac{\partial \Phi}{\partial y} = 2\lambda y$$

$$\frac{\partial^2 \Phi}{\partial x^2} = 2\lambda, \quad \frac{\partial^2 \Phi}{\partial x \partial y} = 0, \quad \frac{\partial^2 \Phi}{\partial y^2} = 2\lambda$$

$$G(x, y) = 4\lambda^2.$$

Here, from Eqs. (A-14),

$$v_1 = \frac{1}{2\lambda} \frac{\partial \psi}{\partial x}$$

$$v_2 = \frac{1}{2\lambda} \frac{\partial \psi}{\partial y}$$

and Eq. (A-15) yields the Laplace equation in two dimensions:

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = -2\lambda B(x, y).$$

## NOMENCLATURE

$a$	semidiameter of the circular cylinder
$a_0$	semin. major axis of the spacecraft's orbit
$a_1, a_2, a_3$	constants of the parabolic reflector
$\bar{a}_1, \bar{a}_2, \bar{a}_3$	unit vectors along the axes of the inertial reference frame
$a_{SRP}$	acceleration of the solar radiation force
$A$	moment of inertia of the parabolic reflector about the principal $x_p$ - axis of the spacecraft
$A_0$	parameter used in integration
$A_{xy}, A_{xz}, A_{yz}$	projections of the illuminated surface area on the three coordinate planes
$A(f)$	reflectivity function
AU	astronomical unit
$b_1, b_2$	constants of the parabolic reflector
$B$	moment of inertia of the reflector about the principal $y_p$ - axis of the spacecraft
$B(\theta)$	function of the angle between the direction of the incident radiation and normal to the reflecting surface
$B(f)$	reflectivity function
$c$	speed of light
$c_S$	modified solar constant
$C$	moment of inertia of the parabolic reflector about the principal $z_p$ - axis of the spacecraft
$C_1, C_2, C_3, C'_1$	reflectivity characteristics of a specified reflecting surface
$e$	eccentricity of the spacecraft's orbit
$\bar{e}_1, \bar{e}_2, \bar{e}_3$	unit vectors along the axes of the noninertial frame of reference xyz
$f$	true anomaly of the spacecraft in its orbit

$f_1, f_2$	constants of the parabolic reflector
$f_1^F, f_2^F$	values of constants $f_1$ and $f_2$ for the front side of the reflector
$f_1^B, f_2^B$	values of constants $f_1$ and $f_2$ for the back side of the reflector
$f(\theta)$	reflection law
$\bar{F}$	solar radiation force
$F_x, F_y, F_z$	components of the solar radiation force along the axes of the noninertial frame of reference $xyz$
$F_N$	solar radiation force exerted upon a surface perpendicular to the direction of the incoming radiation
$F_I$	part of the radiation force due to the incoming radiation
$F_R$	part of the radiation force due to the specularly reflected radiation
$F_D$	part of the radiation force due to the diffusely reflected radiation
$\bar{F}^0$	solar radiation force exerted on the whole front surface of the parabolic reflector
$F_x^0, F_y^0, F_z^0$	components of force $\bar{F}^0$ along the axes of the noninertial reference frame $xyz$
$F_x'^0, F_y'^0, F_z'^0$	components of the solar radiation force exerted on the whole back area of the reflector, along the axes of the noninertial reference frame $xyz$
$F_y^T, F_z^T$	components of the total solar radiation force in the system $xyz$
$F_\xi, F_\eta, F_\zeta$	components of the solar radiation force along the axes of the noninertial system $\xi\eta\zeta$
$g_0, g_1, g_2$	constants of the parabolic reflector
$g_0^F, g_1^F, g_2^F$	values of constants $g_0, g_1,$ and $g_2$ for the front side of the reflector
$g_0^B, g_1^B, g_2^B$	values of constant $g_0, g_1,$ and $g_2$ for the back side of the reflector



$h$	length of the circular cylinder
$h_1, h_2$	constants of the reflector; coefficients in the expression for the moment of the radiation force
$h_1^F, h_2^F$	values of constants $h_1$ and $h_2$ for the front side of the reflector
$h_1^B, h_2^B$	values of $h_1$ and $h_2$ for the back side of the reflector
$\bar{i}$	unit vector along the x-axis
$I$	radiant flux per unit solid angle on a hemisphere
$I_0$	constant, having the same dimensions as $I$
$I, I_{21}, I_{22}, I_{31}, I_{32}$	definite integrals that appear in the expressions for the components of the solar radiation force
$I_x, I_y, I_z$	moments of inertia of the reflector about the axes of the coordinate frame $xyz$
$\bar{j}$	unit vector along the y-axis
$J$	radiant energy of the incoming radiation per unit area per unit of time
$J, J_{21}, J_{22}, J_{31}, J_{32}$	definite integrals that appear in the expressions for the components of the solar radiation force
$J_0$	solar constant
$J_x, J_y, J_z$	principal moments of inertia of the spacecraft without the parabolic reflector
$\bar{k}$	unit vector along the z-axis, in the Spacecraft-Earth direction
$k$	radiation constant
$K = K(r_{SP}) = K(r)$	function of the heliocentric distance of the spacecraft
$K_1, K_2, K_3$	reflecting characteristics of a surface
$K_{SRP}$	solar radiation constant
$l_1, l_2$	auxiliary constants
$m$	constant of the reflector
$m_S$	total mass of the spacecraft

$M$	mass
$\bar{M}$	moment of the solar radiation force
$M_x, M_y, M_z$	components of the vector $\bar{M}$ along the axes of the noninertial system xyz
$M_x^{(O)}, M_y^{(O)}, M_z^{(O)}$	same as above; O is the origin of the system xyz
$M_x^{(C)}, M_y^{(C)}, M_z^{(C)}$	components of the moment of the solar radiation force along axes parallel to x, y, z-axes, centered at the center of gravity of the spacecraft
$M_x^0, M_y^0, M_z^0$	values of components $M_x, M_y, M_z$ respectively for the whole inside area of the reflector
$M_x', M_y', M_z'$	components of the moment of the solar radiation force for the back side of the reflector
$M_x'^0, M_y'^0, M_z'^0$	values of components $M_x', M_y', M_z'$ for the whole back side area of the reflector
$M_R$	mass of the parabolic reflector
$n$	rate of rotation of the spacecraft about its roll axis (z-axis)
$\bar{n}_1, \bar{n}_2, \bar{n}_3$	unit vectors along the axes of the system $\xi \eta \zeta$
$\bar{N}^*$	unit vector along the normal to the reflecting surface
$P_I$	component of force $F_I$ along the normal to the reflecting surface
$P_R$	component of force $F_R$ along the normal to the surface
$q_1, q_2$	constants
$\bar{r}$	heliocentric position vector of the spacecraft, also $\bar{r}_{SP} \cdot  \bar{r}  = r_{SP}$
$r$	dimensionless polar coordinate
$\bar{r}^{(C)}$	position vector of a point of the spacecraft relative to its center of gravity
$\bar{R}_E$	heliocentric position vector of the Earth
$\bar{S}$	oriented surface
$S$	surface area

$T_0$	period of free oscillations of the spacecraft about the x- and y-axis
$\bar{u}$	unit vector in the Spacecraft-Sun direction
$u_1, u_2, u_3$	components of the unit vector $\bar{u}$ along the axes of the coordinate system $\xi\eta\zeta$
$W$	magnitude of the gradient vector
$x, y, z$ } $x', y', z'$ }	noninertial, spacecraft-fixed coordinates
$x_p, y_p, z_p$	principal axes of inertia of the spacecraft
$X, Y, Z$	heliocentric inertial coordinates of the spacecraft
$X_E, Y_E, Z_E$	heliocentric inertial coordinates of the Earth
$\alpha$	angle between the axis of symmetry of the reflector and the direction of incoming radiation
$\alpha_0$	Earth-Spacecraft-Sun angle
$\beta$	portion of photons reflected specularly
$\gamma$	portion of reflected photons
$\delta$	semidiameter of the parabolic reflector
$\epsilon$	angle of inclination of the axis of symmetry of the circular cylinder to the xy-plane
$\zeta$	depth of the parabolic reflector. Also, coordinate in the $\xi\eta\zeta$ system of reference
$\eta$	coordinate in the $\xi\eta\zeta$ system of reference
$\eta(\alpha), \eta_1(\alpha)$	blockage (shadowing) functions for the front and back surface of the reflector, respectively
$\theta$	angle between the normal to the surface and the direction of the incident radiation
$\lambda$	constant of the parabolic reflector
$\Lambda$	polar angle used in integration
$\mu$	reflectivity characteristics of a surface
$\mu_F, \mu_B$	values of $\mu$ for the front and back surface of the reflector, respectively

$\nu$	reflectivity characteristics of a surface
$\nu_F, \nu_B$	values of $\nu$ for the front and back surface of the reflector, respectively
$\xi$	coordinate in the $\xi\eta\zeta$ reference frame
$\bar{\rho}$	geocentric position vector of the spacecraft
$\rho$	$ \bar{\rho} $
$\phi$	polar coordinate
$\Phi = \Phi(x, y, z)$	equation of the reflecting surface
$\chi$	angle between the projection of the axis of symmetry of the circular cylinder on the xy-plane and the x-axis
$\psi$	angular dimension of the parabolic reflector
$\bar{\omega}$	angular velocity vector
$\omega$	angular velocity
$\omega_x, \omega_y, \omega_z$	components of the angular velocity vector along the axes of the noninertial frame of reference xyz
$\Omega$	angular dimension of the parabolic reflector
$\Omega_0$	frequency of the free oscillations of the spacecraft about the x- and y-axis

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