

CONCEPTS FOR A THEORETICAL AND EXPERIMENTAL
STUDY OF LIFTING ROTOR RANDOM
LOADS AND VIBRATIONS

(Analysis of Gust Alleviation Methods and Rotor Dynamic Stability)

Phase V-B Report under Contract NAS2-4151

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Aerospace Engineering

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Washington University
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St. Louis, Missouri

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Prepared for the Ames Directorate, AMRDL,
at Ames Research Center, Moffett Field, California

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Scope of Contract NAS2-4151

Work under Contract NAS2-4151 started on February 1, 1967. Phase I Report of September 1967 develops analytical concepts for a random loads and vibrations analysis of lifting rotors. Phase II Report of August 1968 presents a perturbation solution method for random blade flapping. Phase III Report of June 1969 develops a more general method to include high rotor advance ratios and makes use of a specific atmospheric turbulence model. Phase IV Report of June 1970 extends the method to the computation of threshold crossing statistics for random blade flapping and introduces non-uniformity of the vertical turbulence velocity in the longitudinal direction. During FY 1971 the work was extended in three directions, resulting in 3 separate Phase V reports. Phase V-A Report covers the inclusion of blade torsional flexibility in the blade random gust response statistics. Phase V-B Report covers the analysis of lifting rotor gust alleviation methods and rotor dynamic stability. Phase V-C Report describes the efforts to develop experimental methods of substantiating the random loads and vibration analysis. The work summarized in Phase V-A Report was performed under Modification 5 to subject contract. The work summarized in Phase V-B and Phase V-C Reports was performed under Modification 6 to subject contract. A proposal has been submitted

for an extension of Contract NAS2-4151 through FY 1972 and FY 1973. The scope of the proposed extension is to remove some of the limitations of the present analytical model and at the same time simplify the method of analysis, and to conduct model tests to support the analysis.

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Abstract

Previous work under subject contract has shown that excessive blade loads must be expected when operating lifting rotors at high advance ratio in atmospheric turbulence. Therefore, the effects of various gust alleviation methods on the random blade response in flapping have been studied analytically, assuming a rigid rotor support. Since the gust alleviation methods involve feedback of blade motions into the cyclic and collective controls, dynamic instabilities as affected by the feedback gains are of primary interest. The analytical model assumes rigid flapping blades with elastic root restraints. Linearized equations approximately valid at low lift conditions have been used. Because of the interblade coupling from the feedback devices, the method of multiblade generalized coordinates was found to be most

convenient. This method, previously limited to the use of first order cyclic coordinates, is extended to include coning, differential coning and warping of the rotor. Because the theory was to cover high rotor advance ratios, the analytical method considers the periodicity of the coefficients in the multiblade dynamic rotor equations. The conventional modal columns then become complex valued periodic modal time functions. The numerical examples cover the dynamic stability characteristics as affected by feedback gains of 3, 4, 5 and 6 bladed rotors.

The number of blades was found to have large effects on the stability limits and on the modal time functions at these limits. The analytical method developed herein allows to establish constant coefficient rotor representations by merely omitting the periodic terms in the multiblade rotor representation. It was found that the constant coefficient equations yield fairly good approximations for the low frequency modes even at an advance ratio of .8, the more so the higher the blade number. In addition to determining the rotor stability limits for four selected feedback devices, the random flapping response of the blades to atmospheric turbulence was determined at 1.6 rotor advance ratio using feedback gains below the stability limit. The most effective reduction of the random blade flapping response per unit gain could be achieved with a rotor coning angle feedback.

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Notation

$[A(t)]$	characteristic matrix, periodic with T
$\{A_j(t)\}$	j-th column of $[A(t)]$
B	blade tip loss factor
$C(\psi)$	nondimensional damping coefficient
$[D(t)]$	state matrix, periodic with T
$\{F(t)\}$	excitation column matrix
$F_\beta(\psi_k)$	flapping equation for k-th blade in terms of multiblade coordinates
K_i	tilt integral feedback gain
K_o	coning feedback gain
K_p	tilt proportional feedback gain
$K_{11}, K_{12}, K_{21}, K_{22}$	gain factors for tilting feedback
$K(\psi)$	nondimensional aerodynamic spring rate
L	equivalent feedback system spring constant
L/R	ratio of turbulence scale length over rotor radius
m_{θ_o}	nondimensional coefficient for collective pitch
m_{θ_I}	nondimensional coefficient for forward longitudinal cyclic pitch
$m_{\theta_{II}}$	nondimensional coefficient for left lateral cyclic pitch
m_λ	nondimensional coefficient for inflow ratio
N	number of blades
P	blade parameter describing elastic flapping restraint

$S_{\lambda}(\omega)$	two-sided power spectral density for λ
T	nondimensional basic period of periodic coefficients
t	nondimensional time for which period of one rotor revolution is 2π
$\tan \delta_3$	flapping feedback gain
$u(t)$	unit step function
$\{x(t)\}$	state vector
$\{x_j(t)\}$	modal function column
$\{y(t)\}$	state vector obtained from the transformation $\{y(t)\} = [A(t)]^{-1}\{x(t)\}$
$\{\alpha\}$	constant column matrix
β_k	flapping angle of k-th blade, positive up
$\beta_o, \beta_d, \beta_I, \dots, \beta_{VI}$	multiblade flapping coordinates: coning, differential coning (only for even bladed rotors), tilting, warping.
$\beta_c(\omega, t), \beta_s(\omega, t)$	blade flapping response to inputs $\lambda = u(t)\cos \omega t$ and $\lambda = u(t)\sin \omega t$ respectively
$\beta_{1+}, \beta_{2+}, \dots$	Normal multiblade coordinates of progressing modes
$\beta_{1-}, \beta_{2-}, \dots$	Normal multiblade coordinates of regressing modes
γ	blade Lock number
δ_3	flapping hinge inclined angle
η_k	kth complex eigenvalue
θ_k	pitch angle of k-th blade, equal to $\theta_o - \theta_I \sin \psi_k + \theta_{II} \cos \psi_k$
$\theta_o, \theta_I, \theta_{II}$	collective, forward longitudinal, left lateral cyclic pitch respectively

Λ_k	k-th eigenvalue of Floquet state transition matrix, equal to $e^{(\lambda_k + i\omega_k)T}$
λ	inflow ratio, or real part of characteristic value
λ_k	real part of η_k
μ	rotor advance ratio
$\sigma_\beta(t)$	time variable flapping standard deviation
σ_λ	time invariable standard deviation for λ
$[\phi(t)]$	state transition matrix
ψ_k	azimuth angle of k-th blade
ω	non-dimensional circular frequency
ω_k	imaginary part of η_k
Ω	rotor angular velocity

1. Introduction

The results of the preceding work under Contract NAS2-4151 have been summarized in Phase I to Phase IV Reports and in the publications of References (1) to (3). The following report presents a summary of analytical results with respect to gust alleviation methods and rotor dynamic stability obtained during FY 1971. Part of the results have been published in Reference (4) but are repeated here for the sake of completeness.

The previous work has shown that excessive blade loads and/or vibrations must be expected when operating lifting rotors at high advance ratio in severe atmospheric turbulence. The studies so far have not included the effects of airframe motions and the rotor hub has been assumed rigid. This condition is approximately realized in convertible aircraft during forward flight conditions where the airframe is well stabilized by conventional means and where the rotor is off-loaded and idling either continuously or as a transient condition toward being stopped. In such conditions an automatic control system making use of airframe motion or attitude sensors can at best minimize the fuselage motions, but it cannot prevent excessive blade motions or blade loads. To alleviate the blade-to-hub motions, sensors for these motions must be provided which produce feedbacks into the rotor controls. A number of simple feedback systems will be examined in the

following with respect to their stability characteristics and their gust alleviation potential.

All previous work under Contract NAS2-4151 assumed the absence of coupling between the motions of the various blades of a rotor so that a single blade analysis was adequate. Due to the strong interblade coupling from the feedback systems it was found advantageous to use generalized multiblade coordinates as defined in Section 5 of Phase IV Report. For not too high rotor advance ratio such a formulation makes it possible to reduce the problem approximately to one with constant coefficients in the multiblade equations, leading to a large reduction in computational effort for the solution of the turbulence response problem as well as the stability problem. For high advance ratios the periodicity of the coefficients in the linearized multiblade equations of motion must be considered. The Floquet state transition matrix method of Reference (5) is then a convenient tool for the stability analysis. As far as the response to atmospheric turbulence is concerned, the method of Reference (2) can be extended to the multiblade representations, though requiring an increased computational effort. The comparison of a more sophisticated stochastic model with the simpler "point approximation" performed in Reference (3) has shown that the latter approximation will be adequate in many cases. Accordingly the point approximation, where the blades are assumed to

feel at any time instant the turbulence velocity actually existing only at the rotor center has been used for the multiblade random response analyses. As before, only the effect of the vertical turbulence velocity component on the blades is considered, the effects of the longitudinal and lateral turbulence components are neglected. The analytical rotor model, same as before, assumes rigid blades hinged at the rotor center and elastically restrained in flapping, while rigid in the chord-wise direction and in torsion.

Multiblade representations have been previously used for first order cyclic chord-wise and flap-wise modes in the classical papers on mechanical instability and on prop-rotor whirl flutter and also in Reference (6). However, the previous studies were for constant coefficient systems. In this report the previous studies are extended to include rotor warping modes and to include the effects of the periodicity of the coefficients.

2. Definition of Multiblade Generalized Coordinates

In the flutter analysis of fixed wings it is customary to express wing deflections by their normal mode contents whereby the normal modes are purely elastic modes without structural or aerodynamic damping. Each normal mode is represented by a factor called normal coordinate. For the multiblade stability analysis of lifting rotors one could express the blade motions in terms of the normal multiblade modes. Take for example a vibration mode of a single blade excluding mechanical damping and excluding aerodynamic forces, say a flapping or a flap-bending mode seen in a rotating frame of reference. In a multiblade mode all blades oscillate with the same frequency and the same amplitude, but with different phase angles. Depending on the phase differences between the oscillations of the various blades, there are four types of multiblade modes: collective modes where the phase angles of all blade oscillations are identical, differential collective modes, only possible for even bladed rotors where the oscillations of subsequent blades are in counter phase, regressing modes where the location of maximum blade deflection rotates in a rotor fixed reference system opposite to the direction of rotor rotation, and progressing modes where the location of maximum blade deflection rotates in a rotor fixed reference system in the direction of rotor rotation. These modes are obviously normal modes,

since they can exist independent of each other. Such multi-blade modes can be defined for any type of blade oscillation: out-of-plane, in-plane and torsion.

Here the method of multiblade generalized coordinates will be applied to blade flapping. The types of multiblade normal modes then are collective flapping or coning, differential collective flapping (only possible for even bladed rotors), and regressing and progressing flapping. Since in a N-bladed rotor one needs exactly N modes to describe the general flapping motion, one has to use the coning and differential coning modes for a 2-bladed rotor, the coning and the regressing and progressing flapping modes for a 3-bladed rotor, in addition the differential coning mode for a 4-bladed rotor; the coning mode and 2 regressing and 2 progressing flapping modes for a 5 bladed rotor, in addition the differential coning mode for a 6 bladed rotor and so on. Instead of using generalized multiblade coordinates for the regressing and progressing flapping modes one can also base the generalized coordinates on cyclic flapping modes of various orders leading to tilting or warping of the rotor tip path. The cyclic modes have the advantage of easier algebraic manipulation, however, they are not normal modes in the sense that they cannot exist independently of each other. As will be shown later, two cyclic modes can be combined to represent a normal progressing or regressing mode.

The flapping angle β_k of the k-th blade in terms of the various multiblade flapping coordinates, using cyclic modes, is

$$\begin{aligned} \beta_k = & \beta_o + \beta_d(-1)^k + \beta_I \cos \psi_k + \beta_{II} \sin \psi_k + \beta_{III} \cos 2\psi_k \\ & + \beta_{IV} \sin 2\psi_k + \beta_V \cos 3\psi_k + \beta_{VI} \sin 3\psi_k + \dots \end{aligned} \quad (1)$$

$$k = 1, 2, \dots, N$$

where $\psi_k = t + (2\pi/N)(k-1)$ is the azimuth angle of the k-th blade. For an N-bladed rotor only the first N terms of Eq. (1) are retained, whereby β_d occurs only in even bladed rotors.

By inverting the system of Eq. (1) and applying the following formulas,

$$\sum_{k=1}^N \cos m\psi_k = \begin{cases} 0, & \text{if } m \text{ is not a multiple of } N \\ N \cos mt, & \text{if } m = sN; s = 1, 2, \dots \end{cases} \quad (2)$$

$$\sum_{k=1}^N \sin m\psi_k = \begin{cases} 0, & \text{if } m \text{ is not a multiple of } N \\ N \sin mt, & \text{if } m = sN; s = 1, 2, \dots \end{cases} \quad (3)$$

one obtains the multiblade coordinates in terms of the blade flapping angles:

Collective flapping (coning)

$$\beta_o = \frac{1}{N} \sum_{k=1}^N \beta_k$$

Differential Collective flapping
(only for even-bladed rotors)

$$\beta_d = \frac{1}{N} \sum_{k=1}^N \beta_k (-1)^k$$

First order cyclic flapping
(tilting)

$$\beta_I = \frac{2}{N} \sum_{k=1}^N \beta_k \cos \psi_k,$$

$$\beta_{II} = \frac{2}{N} \sum_{k=1}^N \beta_k \sin \psi_k$$

Second order cyclic flapping
(warping)

$$\beta_{III} = \frac{2}{N} \sum_{k=1}^N \beta_k \cos 2\psi_k,$$

$$\beta_{IV} = \frac{2}{N} \sum_{k=1}^N \beta_k \sin 2\psi_k$$

Third order cyclic flapping

$$\beta_V = \frac{2}{N} \sum_{k=1}^N \beta_k \cos 3\psi_k,$$

⋮

$$\beta_{VI} = \frac{2}{N} \sum_{k=1}^N \beta_k \sin 3\psi_k$$

For a hovering lifting rotor with a rigid held hub, the coordinates β_o and β_d are normal coordinates, the cyclic coordinates are pairwise normal, however β_I , β_{II} , β_{III} , β_{IV} , etc., in themselves, as mentioned before are not normal coordinates. The various multiblade coordinates are shown in Figure 1.

It is clear from Figure 1 that the collective flapping β_0 is the coning angle, the differential collective flapping β_d is a rotating tilting for a two-bladed rotor and a rotating warping for higher even bladed rotors, the first cyclic flapping β_I , β_{II} the stationary tilting, and the higher order cyclic flapping coordinates β_{III} , β_{IV} , β_V , β_{VI} the stationary warping of the rotor tip path.

In order to correlate the cyclic multiblade coordinates with the normal modes as described previously, the motion of the k-th blade is expressed as

$$\begin{aligned} \beta_k = & \sum_{m=1}^r \beta_{m+} \cos \left[(\omega t + \phi) - m \frac{2\pi}{N} (k-1) \right] \\ & + \sum_{m=1}^r \beta_{m-} \cos \left[(\omega t + \phi) + m \frac{2\pi}{N} (k-1) \right] \end{aligned} \quad (4)$$

where $r = (N-1)/2$ for odd bladed rotors, $r = (N-2)/2$ for even bladed rotors. β_{m+} and β_{m-} represent the amplitudes of the corresponding m-th progressing and regressing modes with frequency ω in the rotor fixed reference system. Thus, $m = 1$ for three and four bladed rotors, $m = 1, 2$ for five and six bladed rotors.

The various multiblade normal modes are illustrated in Figure 2 for blade number $N = 3$ to 7. The wave lines moving to the right give the motions of the blades vs. the phasing of the blades. Blade 1 is always shown in its maximum up deflection. The remaining blades are shown at the same time in their respective phases in relation to blade 1.

Now the relation between the cyclic coordinates and the amplitudes of the normal modes at $\mu = 0$ can be derived as follows: Since

$$(\omega t + \phi) + m \frac{2\pi}{N} (k-1) = [(\omega \pm m)t + \phi] + m\psi_k \quad (5)$$

and applying the formulas of Eqs. (2) and (3), it can be easily found that the cyclic flapping coordinates are

$$\begin{aligned} \beta_I &= \frac{2}{N} \sum_{k=1}^N \beta_k \cos \psi_k = \beta_{1+} \cos [(\omega+1)t + \phi] \\ &\quad + \beta_{1-} \cos [(\omega-1)t + \phi] \end{aligned} \quad (6)$$

$$\begin{aligned} \beta_{II} &= \frac{2}{N} \sum_{k=1}^N \beta_k \sin \psi_k = \beta_{1+} \sin [(\omega+1)t + \phi] \\ &\quad - \beta_{1-} \sin [(\omega-1)t + \phi] \end{aligned} \quad (7)$$

$$\begin{aligned} \beta_{III} &= \frac{2}{N} \sum_{k=1}^N \beta_k \cos 2\psi_k = \beta_{2+} \cos [(\omega+2)t + \phi] \\ &\quad + \beta_{2-} \cos [(\omega-2)t + \phi] \end{aligned} \quad (8)$$

$$\begin{aligned} \beta_{IV} &= \frac{2}{N} \sum_{k=1}^N \beta_k \sin 2\psi_k = \beta_{2+} \sin [(\omega+2)t + \phi] \\ &\quad - \beta_{2-} \sin [(\omega-2)t + \phi] \end{aligned} \quad (9)$$

$$\begin{aligned} \beta_V &= \frac{2}{N} \sum_{k=1}^N \beta_k \cos 3\psi_k = \beta_{3+} \cos [(\omega+3)t + \phi] \\ &\quad + \beta_{3-} \cos [(\omega-3)t + \phi] \end{aligned} \quad (10)$$

$$\begin{aligned} \beta_{VI} &= \frac{2}{N} \sum_{k=1}^N \beta_k \sin 3\psi_k = \beta_{3+} \sin [(\omega+3)t + \phi] \\ &\quad - \beta_{3-} \sin [(\omega-3)t + \phi] \end{aligned} \quad (11)$$

These equations show that for zero coupling between blades tilting in the progressing mode occurs with $\omega+1$, in the regressing mode with $\omega-1$, first warping in the progressing mode occurs with $\omega+2$, in the regressing mode with $\omega-1$, etc.

3. Multiblade Flapping Equations

The linearized flapping equation of the k-th spring-restrained blade with a centrally arranged flapping hinge, neglecting reverse flow effects, reads, (7)

$$\beta_k + \left(\frac{B^4 \gamma}{8} + \frac{B^3 \gamma \mu}{6} \sin \psi_k \right) \beta_k + \left(P^2 + \frac{B^3 \gamma \mu}{6} \cos \psi_k + \frac{B^2 \gamma \mu^2}{8} \sin 2\psi_k \right) \beta_k = m_{\theta_o} \theta_o + m_{\theta_I} \theta_I + m_{\theta_{II}} \theta_{II} + m_\lambda \lambda \quad (12)$$

with

$$\theta_k = \theta_o - \theta_I \sin \psi_k + \theta_{II} \cos \psi_k \quad (13)$$

where the parameter P describes the elastic flapping restraint of the blade ($P = 1$ for a freely flapping blade), and m_{θ_o} , m_{θ_I} , $m_{\theta_{II}}$ and m_λ are functions of the blade azimuth angle ψ_k . Neglecting reverse flow effects, these functions are:

$$m_{\theta_o} = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} \right) + \frac{B^3 \gamma \mu}{3} \sin \psi_k - \frac{B^2 \gamma \mu^2}{8} \cos 2\psi_k \quad (14)$$

$$m_{\theta_I} = - \frac{B^3 \gamma \mu}{6} - \left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} \right) \sin \psi_k + \frac{B^3 \gamma \mu}{6} \cos 2\psi_k + \frac{B^2 \gamma \mu^2}{16} \sin 3\psi_k \quad (15)$$

$$m_{\theta_{II}} = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \cos \psi_k + \frac{B^3 \gamma \mu}{6} \sin 2\psi_k - \frac{B^2 \gamma \mu^2}{16} \cos 3\psi_k \quad (16)$$

$$m_\lambda = \frac{B^3 \gamma}{6} + \frac{B^2 \gamma \mu}{4} \sin \psi_k \quad (17)$$

The multiblade coordinates representations of blade flapping for a rotor with various blade numbers, neglecting reverse flow effects, are derived for blade number $N = 2$ to 6. First, corresponding to a certain blade number, the appropriate representation of the k -th blade flapping angle β_k by Eq. (1) along with its first and second time derivatives is substituted in Eq. (12) then applying Eqs. (2) and (3), the flapping equations in terms of multiblade coordinates are derived in the following.

For a two-bladed rotor, performing the transformation $\beta_k = \beta_o + \beta_d(-1)^k$ in Eq. (12), denoting the resulting flapping equation by $F_\beta(\psi_k)$, then the manipulations of

$\sum_{k=1}^2 F_\beta(\psi_k)$ and $\sum_{k=1}^2 F_\beta(\psi_k)(-1)^k$ lead to

$$\begin{aligned} \ddot{\beta}_o + \frac{B^4 \gamma}{8} \dot{\beta}_o + \left(p^2 + \frac{B^2 \gamma \mu^2}{8} \sin 2t \right) \beta_o - \frac{B^3 \gamma \mu}{6} \dot{\beta}_d \sin t - \frac{B^3 \gamma \mu}{6} \beta_d \cos t \\ = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} - \frac{B^2 \gamma \mu^2}{8} \cos 2t \right) \theta_o + \left(-\frac{B^3 \gamma \mu}{6} + \frac{B^3 \gamma \mu}{6} \cos 2t \right) \theta_I \\ + \frac{B^3 \gamma \mu}{6} \theta_{II} \sin 2t + \frac{B^3 \gamma}{6} \lambda \end{aligned} \quad (18)$$

$$\begin{aligned} - \frac{B^3 \gamma \mu}{6} \dot{\beta}_o \sin t - \frac{B^3 \gamma \mu}{6} \beta_o \cos t + \ddot{\beta}_d + \frac{B^4 \gamma}{8} \dot{\beta}_d + \left(p^2 + \frac{B^2 \gamma \mu^2}{8} \sin 2t \right) \beta_d \\ = - \frac{B^3 \gamma \mu}{3} \theta_o \sin t + \left[\left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} \right) \sin t - \frac{B^2 \gamma \mu^2}{16} \sin 3t \right] \theta_I \\ - \left[\left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \cos t - \frac{B^2 \gamma \mu^2}{16} \cos 3t \right] \theta_{II} - \frac{B^2 \gamma \mu}{4} \lambda \sin t \end{aligned} \quad (19)$$

For a three-bladed rotor, performing the transformation $\beta_k = \beta_o + \beta_I \cos \psi_k + \beta_{II} \sin \psi_k$ in Eq. (12), denoting the resulting equation by $F_\beta(\psi_k)$, then the manipulations of

$\sum_{k=1}^3 F_\beta(\psi_k)$, $\sum_{k=1}^3 F_\beta(\psi_k) \cos \psi_k$, and $\sum_{k=1}^3 F_\beta(\psi_k) \sin \psi_k$ yield

the following equations:

$$\begin{aligned} \ddot{\beta}_o + \frac{B^4 \gamma}{8} \dot{\beta}_o + P^2 \beta_o + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} - \frac{B^2 \gamma \mu^2}{16} \beta_{II} \cos 3t + \frac{B^2 \gamma \mu^2}{16} \beta_I \sin 3t \\ = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} \right) \theta_o - \left(\frac{B^3 \gamma \mu}{6} - \frac{B^2 \gamma \mu^2}{16} \sin 3t \right) \theta_I - \frac{B^2 \gamma \mu^2}{16} \theta_{II} \cos 3t \\ + \frac{B^3 \gamma}{6} \lambda \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{B^3 \gamma \mu}{6} \beta_o + \ddot{\beta}_I + \frac{B^4 \gamma}{8} \dot{\beta}_I + (P^2 - 1) \beta_I + 2 \dot{\beta}_{II} + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \beta_{II} \\ + \left(\frac{B^3 \gamma \mu}{6} \beta_I - \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} \right) \cos 3t + \left(\frac{B^2 \gamma \mu^2}{8} \beta_o + \frac{B^3 \gamma \mu}{12} \dot{\beta}_I + \frac{B^3 \gamma \mu}{6} \beta_{II} \right) \sin 3t \\ = - \frac{B^2 \gamma \mu^2}{8} \theta_o \cos 3t + \frac{B^3 \gamma \mu}{6} \theta_I \cos 3t + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} + \frac{B^3 \gamma \mu}{6} \sin 3t \right) \theta_{II} \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{B^3 \gamma \mu}{6} \dot{\beta}_o - 2 \dot{\beta}_I + \left(\frac{B^2 \gamma \mu^2}{16} - \frac{B^4 \gamma}{8} \right) \beta_I + \ddot{\beta}_{II} + \frac{B^4 \gamma}{8} \dot{\beta}_{II} + (P^2 - 1) \beta_{II} \\ + \left(\frac{B^3 \gamma \mu}{6} \beta_I - \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} \right) \sin 3t - \left(\frac{B^2 \gamma \mu^2}{8} \beta_o + \frac{B^3 \gamma \mu}{12} \dot{\beta}_I + \frac{B^3 \gamma \mu}{6} \beta_{II} \right) \cos 3t \\ = \left(\frac{B^3 \gamma \mu}{3} - \frac{B^2 \gamma \mu^2}{8} \sin 3t \right) \theta_o - \left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} - \frac{B^3 \gamma \mu}{6} \sin 3t \right) \theta_I \\ - \frac{B^3 \gamma \mu}{6} \theta_{II} \cos 3t + \frac{B^2 \gamma \mu}{4} \lambda \end{aligned} \quad (22)$$

It is seen that only the periodic coefficients $\cos 3t$ and $\sin 3t$, with a period of $2\pi/3$, remain in the above equations.

For a four-bladed rotor, performing the transformation $\beta_k = \beta_o + \beta_d(-1)^k + \beta_I \cos \psi_k + \beta_{II} \sin \psi_k$ in Eq. (12), denoting the resulting equation by $F_\beta(\psi_k)$, then the manipulations of $\sum_{k=1}^4 F_\beta(\psi_k)$, $\sum_{k=1}^4 F_\beta(\psi_k) \cos \psi_k$, $\sum_{k=1}^4 F_\beta(\psi_k) \sin \psi_k$ and $\sum_{k=1}^4 F_\beta(\psi_k)(-1)^k$ give

$$\begin{aligned} \ddot{\beta}_o + \frac{B^4 \gamma}{8} \dot{\beta}_o + P^2 \beta_o + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} - \frac{B^2 \gamma \mu^2}{8} \beta_d \sin 2t \\ = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} \right) \theta_o - \frac{B^3 \gamma \mu}{6} \theta_I + \frac{B^3 \gamma}{6} \lambda \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{B^3 \gamma \mu}{6} \beta_o + \ddot{\beta}_I + \frac{B^4 \gamma}{8} \dot{\beta}_I + (P^2 - 1) \beta_I + 2 \dot{\beta}_{II} + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \beta_{II} \\ - \frac{B^2 \gamma \mu^2}{16} \beta_{II} \cos 4t + \frac{B^2 \gamma \mu^2}{16} \beta_I \sin 4t - \frac{B^3 \gamma \mu}{6} \dot{\beta}_d \sin 2t - \frac{B^3 \gamma \mu}{6} \beta_d \cos 2t \\ = \frac{B^2 \gamma \mu^2}{16} \theta_I \sin 4t + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} - \frac{B^2 \gamma \mu^2}{16} \cos 4t \right) \theta_{II} \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{B^3 \gamma \mu}{6} \dot{\beta}_o - 2 \dot{\beta}_I + \left(\frac{B^2 \gamma \mu^2}{16} - \frac{B^4 \gamma}{8} \right) \beta_I + \ddot{\beta}_{II} + \frac{B^4 \gamma}{8} \dot{\beta}_{II} + (P^2 - 1) \beta_{II} \\ - \frac{B^2 \gamma \mu^2}{16} \beta_I \cos 4t - \frac{B^2 \gamma \mu^2}{16} \beta_{II} \sin 4t + \frac{B^3 \gamma \mu}{6} \dot{\beta}_d \cos 2t - \frac{B^3 \gamma \mu}{6} \beta_d \sin 2t \\ = \frac{B^3 \gamma \mu}{3} \theta_o - \left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} + \frac{B^2 \gamma \mu^2}{16} \cos 4t \right) \theta_I - \frac{B^2 \gamma \mu^2}{16} \theta_{II} \sin 4t \\ + \frac{B^2 \gamma \mu}{4} \lambda \end{aligned} \quad (25)$$

$$\begin{aligned} - \frac{B^2 \gamma \mu^2}{8} \beta_o \sin 2t - \frac{B^3 \gamma \mu}{12} \dot{\beta}_I \sin 2t - \frac{B^3 \gamma \mu}{6} \beta_I \cos 2t + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} \cos 2t \\ - \frac{B^3 \gamma \mu}{6} \beta_{II} \sin 2t + \ddot{\beta}_d + \frac{B^4 \gamma}{8} \dot{\beta}_d + P^2 \beta_d = \frac{B^2 \gamma \mu^2}{8} \theta_o \cos 2t \\ - \frac{B^3 \gamma \mu}{6} \theta_I \cos 2t - \frac{B^3 \gamma \mu}{6} \theta_{II} \sin 2t \end{aligned} \quad (26)$$

There are second and fourth harmonic coefficients remaining in the above equations. The period of the coefficients is π .

For a five-bladed rotor, similarly performing the transformation $\beta_k = \beta_o + \beta_I \cos \psi_k + \beta_{II} \sin \psi_k + \beta_{III} \cos 2\psi_k + \beta_{IV} \sin 2\psi_k$ in Eq. (12), denoting the resulting equation by $F_\beta(\psi_k)$, then the manipulations of $\sum_{k=1}^5 F_\beta(\psi_k)$, $\sum_{k=1}^5 F_\beta(\psi_k) \cos \psi_k$, $\sum_{k=1}^5 F_\beta(\psi_k) \sin \psi_k$, $\sum_{k=1}^5 F_\beta(\psi_k) \cos 2\psi_k$ and $\sum_{k=1}^5 F_\beta(\psi_k) \sin 2\psi_k$ lead to

$$\ddot{\beta}_o + \frac{B^4 \gamma}{8} \dot{\beta}_o + P^2 \beta_o + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} + \frac{B^2 \gamma \mu^2}{16} \beta_{IV} = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} \right) \theta_o - \frac{B^3 \gamma \mu}{6} \theta_I + \frac{B^3 \gamma}{6} \lambda \quad (27)$$

$$\begin{aligned} & \frac{B^3 \gamma \mu}{6} \beta_o + \ddot{\beta}_I + \frac{B^4 \gamma}{8} \dot{\beta}_I + (P^2 - 1) \beta_I + 2 \dot{\beta}_{II} + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \beta_{II} \\ & + \left(-\frac{B^3 \gamma \mu}{12} + \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \beta_{III} + \left(\frac{B^3 \gamma \mu}{12} \dot{\beta}_{IV} - \frac{B^2 \gamma \mu^2}{16} \beta_{IV} \cos 5t \right) \\ & = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \theta_{II} \end{aligned} \quad (28)$$

$$\begin{aligned} & \frac{B^3 \gamma \mu}{6} \dot{\beta}_o - 2 \dot{\beta}_I + \left(\frac{B^2 \gamma \mu^2}{16} - \frac{B^4 \gamma}{8} \right) \beta_I + \ddot{\beta}_{II} + \frac{B^4 \gamma}{8} \dot{\beta}_{II} + (P^2 - 1) \beta_{II} \\ & - \frac{B^3 \gamma \mu}{12} \dot{\beta}_{III} - \frac{B^2 \gamma \mu^2}{16} \beta_{III} \cos 5t - \left(\frac{B^3 \gamma \mu}{12} + \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \beta_{IV} \\ & = \frac{B^3 \gamma \mu}{3} \theta_o - \left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} \right) \theta_I + \frac{B^2 \gamma \mu}{4} \lambda \end{aligned} \quad (29)$$

$$\begin{aligned}
 & \left(\frac{B^3 \gamma \mu}{6} + \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \beta_I - \left(\frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} + \frac{B^2 \gamma \mu^2}{16} \beta_{II} \cos 5t \right) \\
 & + \ddot{\beta}_{III} + \left(\frac{B^4 \gamma}{8} + \frac{B^3 \gamma \mu}{12} \sin 5t \right) \dot{\beta}_{III} + \left(P^2 - 4 + \frac{B^3 \gamma \mu}{4} \cos 5t \right) \beta_{III} \\
 & + \left(4 - \frac{B^3 \gamma \mu}{12} \cos 5t \right) \dot{\beta}_{IV} + \left(\frac{B^4 \gamma}{4} + \frac{B^3 \gamma \mu}{4} \sin 5t \right) \beta_{IV} \\
 & = - \frac{B^2 \gamma \mu^2}{8} \theta_0 + \left(\frac{B^3 \gamma \mu}{6} + \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \theta_I - \frac{B^2 \gamma \mu^2}{16} \theta_{II} \cos 5t
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 & \frac{B^2 \gamma \mu^2}{8} \beta_0 + \frac{B^3 \gamma \mu}{12} \dot{\beta}_I - \frac{B^2 \gamma \mu^2}{16} \beta_I \cos 5t + \left(\frac{B^3 \gamma \mu}{6} - \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \beta_{II} \\
 & - \left(4 + \frac{B^3 \gamma \mu}{12} \cos 5t \right) \dot{\beta}_{III} + \left(\frac{B^3 \gamma \mu}{4} \sin 5t - \frac{B^4 \gamma}{4} \right) \beta_{III} + \ddot{\beta}_{IV} \\
 & + \left(\frac{B^4 \gamma}{8} - \frac{B^3 \gamma \mu}{12} \sin 5t \right) \dot{\beta}_{IV} + \left(P^2 - 4 - \frac{B^3 \gamma \mu}{4} \cos 5t \right) \beta_{IV} \\
 & = - \frac{B^2 \gamma \mu^2}{16} \theta_I \cos 5t + \left(\frac{B^3 \gamma \mu}{6} - \frac{B^2 \gamma \mu^2}{16} \sin 5t \right) \theta_{II}
 \end{aligned} \tag{31}$$

It is seen that only the periodic coefficients $\cos 5t$ and $\sin 5t$, with a period of $2\pi/5$, remain in the above equations.

For a six-bladed rotor, again performing the transformation $\beta_k = \beta_0 + \beta_d(-1)^k + \beta_I \cos \psi_k + \beta_{II} \sin \psi_k + \beta_{III} \cos 2\psi_k + \beta_{IV} \sin 2\psi_k$ in Eq. (12), denoting the resulting equation by $F_\beta(\psi_k)$, then the manipulations of $\sum_{k=1}^6 F_\beta(\psi_k)$, $\sum_{k=1}^6 F_\beta(\psi_k) \cos \psi_k$, $\sum_{k=1}^6 F_\beta(\psi_k) \sin \psi_k$, $\sum_{k=1}^6 F_\beta(\psi_k) \cos 2\psi_k$, $\sum_{k=1}^6 F_\beta(\psi_k) \sin 2\psi_k$ and $\sum_{k=1}^6 F_\beta(\psi_k) (-1)^k$ lead to

$$\ddot{\beta}_0 + \frac{B^4 \gamma}{8} \dot{\beta}_0 + P^2 \beta_0 + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} + \frac{B^2 \gamma \mu^2}{16} \beta_{IV} = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{8} \right) \theta_0 - \frac{B^3 \gamma \mu}{6} \theta_I + \frac{B^3 \gamma}{6} \lambda \quad (32)$$

$$\frac{B^3 \gamma \mu}{6} \beta_0 + \ddot{\beta}_I + \frac{B^4 \gamma}{8} \dot{\beta}_I + (P^2 - 1) \beta_I + 2 \dot{\beta}_{II} + \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \beta_{II} - \frac{B^3 \gamma \mu}{12} \beta_{III} + \frac{B^3 \gamma \mu}{12} \dot{\beta}_{IV} - \frac{B^2 \gamma \mu^2}{8} \beta_d \sin 3t = \left(\frac{B^4 \gamma}{8} + \frac{B^2 \gamma \mu^2}{16} \right) \theta_{II} \quad (33)$$

$$\frac{B^2 \gamma \mu}{6} \dot{\beta}_0 - 2 \dot{\beta}_I + \left(\frac{B^2 \gamma \mu^2}{16} - \frac{B^4 \gamma}{8} \right) \beta_I + \ddot{\beta}_{II} + \frac{B^4 \gamma}{8} \dot{\beta}_{II} + (P^2 - 1) \beta_{II} - \frac{B^3 \gamma \mu}{12} \dot{\beta}_{III} - \frac{B^3 \gamma \mu}{12} \beta_{IV} + \frac{B^2 \gamma \mu^2}{8} \beta_d \cos 3t = \frac{B^3 \gamma \mu}{3} \theta_0 - \left(\frac{B^4 \gamma}{8} + \frac{3B^2 \gamma \mu^2}{16} \right) \theta_I + \frac{B^2 \gamma \mu}{4} \lambda \quad (34)$$

$$\frac{B^3 \gamma \mu}{6} \beta_I - \frac{B^3 \gamma \mu}{12} \dot{\beta}_{II} + \ddot{\beta}_{III} + \frac{B^4 \gamma}{8} \dot{\beta}_{III} + \left(P^2 - 4 + \frac{B^2 \gamma \mu^2}{16} \sin 6t \right) \beta_{III} + 4 \dot{\beta}_{IV} + \left(\frac{B^4 \gamma}{4} - \frac{B^2 \gamma \mu^2}{16} \cos 6t \right) \beta_{IV} - \frac{B^3 \gamma \mu}{6} \dot{\beta}_d \sin 3t - \frac{B^3 \gamma \mu}{6} \beta_d \cos 3t = - \frac{B^2 \gamma \mu^2}{8} \theta_0 + \frac{B^3 \gamma \mu}{6} \theta_I \quad (35)$$

$$\frac{B^2 \gamma \mu^2}{8} \beta_0 + \frac{B^3 \gamma \mu}{12} \dot{\beta}_I + \frac{B^3 \gamma \mu}{6} \beta_{II} - 4 \dot{\beta}_{III} - \left(\frac{B^4 \gamma}{4} + \frac{B^2 \gamma \mu^2}{16} \cos 6t \right) \beta_{III} + \ddot{\beta}_{IV} + \frac{B^4 \gamma}{8} \dot{\beta}_{IV} + \left(P^2 - 4 - \frac{B^2 \gamma \mu^2}{16} \sin 6t \right) \beta_{IV} + \frac{B^3 \gamma \mu}{6} \dot{\beta}_d \cos 3t - \frac{B^3 \gamma \mu}{6} \beta_d \sin 3t = \frac{B^3 \gamma \mu}{6} \theta_{II} \quad (36)$$

$$\begin{aligned}
 & - \frac{B^2 \gamma \mu^2}{16} \beta_I \sin 3t + \frac{B^2 \gamma \mu^2}{16} \beta_{II} \cos 3t - \frac{B^3 \gamma \mu}{12} \left(\dot{\beta}_{III} \sin 3t + 3\beta_{III} \cos 3t \right) \\
 & + \frac{B^3 \gamma \mu}{12} \left(\dot{\beta}_{IV} \cos 3t - 3\beta_{IV} \sin 3t \right) + \ddot{\beta}_d + \frac{B^4 \gamma}{8} \dot{\beta}_d + P^2 \beta_d \\
 & = - \frac{B^2 \gamma \mu^2}{16} \theta_I \sin 3t + \frac{B^2 \gamma \mu^2}{16} \theta_{II} \cos 3t \quad (37)
 \end{aligned}$$

There are third and sixth harmonic coefficients remaining in the equations with a period $2\pi/3$.

4. Interblade Coupling Feedback Systems

A rotor blade with a Lock number of 8 and unrestrained flapping hinges becomes dynamically unstable at advance ratios of approximately $\mu = 2$. The instability is caused primarily by transient negative aerodynamic spring effects upsetting blade flapping at the front half of the rotor disk. For present operating conditions this kind of instability is of minor concern. The main problem is the deterioration of the response characteristics in the stable region. With increasing advance ratio, one of the most undesirable characteristics is such that the flapping motion becomes very sensitive to longitudinal disturbances. In other words, the rotor responds to gusts and angle of attack changes with an excessive tip path plane tilt which transfers large moments to the fuselage. This is particularly true if blades with elastic root restraint are used.

In order to reduce the sensitivity of the flapping response to external disturbances, such as atmospheric turbulence, a mechanical or electronic servo-device is required which automatically counteracts the external disturbances by proper collective or cyclic pitch input. The feedback systems studied here for their effects on rotor stability and gust response are described in the following:

4.1 Coning Feedback System

The system refers to a mechanism which applies, without time lag, a collective pitch proportional to the coning angle of the rotor disk. The feedback equation can be represented by

$$\theta_o = -K_o \beta_o \quad (38)$$

where K_o is the feedback gain factor of the system. It should be pointed out here that the mean values in the feedback equations throughout this study are omitted, because they do not affect the stability characteristics of the system.

The coning feedback system has actually been used for the McDonnell XV-1 lifting rotor which exhibited a low gust sensitivity.

4.2 Tilt Proportional Feedback System

Compensation of hub moments can be achieved by employing a cyclic pitch feedback which is displacement proportional

to the longitudinal and lateral tilting angles such that

$$\left. \begin{aligned} \theta_I &= -K_{11}\beta_I \cos \epsilon - K_{12}\beta_{II} \sin \epsilon \\ \theta_{II} &= -K_{21}\beta_I \sin \epsilon - K_{22}\beta_{II} \cos \epsilon \end{aligned} \right\} \quad (39)$$

where K_{11} , K_{12} , K_{21} , K_{22} are the gain factors, ϵ represents a shift in the phasing of the cyclic controls.

Here a cross coupling between the longitudinal and lateral components is introduced to consider the effects of phase angles on the stability margin.

As will be shown later (in Figure 18) the out-of-phase feedback ($\epsilon = 90^\circ$) gives better stability characteristics for the system than the in-phase feedback ($\epsilon = 0$), therefore the following equations are used to represent the tilt proportional feedback system.

$$\left. \begin{aligned} \theta_I &= K_p \beta_{II} \\ \theta_{II} &= -K_p \beta_I \end{aligned} \right\} \quad (40)$$

Eq. (40) is obtained by setting $K_{11} = K_{22} = -K_{12} = K_{21} = K_p$ and $\epsilon = 90^\circ$ in Eq. (39).

4.3 Flapping Feedback System

The flapping feedback system is a combination of coning feedback system with tilt proportional feedback system and can be characterized by the equations

$$\left. \begin{aligned} \theta_o &= -K_o \beta_o \\ \theta_I &= K_p \beta_{II} \\ \theta_{II} &= -K_p \beta_I \end{aligned} \right\} \quad (41)$$

It can be shown that for a three-bladed rotor, if the gain factors K_o , K_p are set equal, the flapping feedback system has actually the same function as the conventional δ_3 blade pitch-flap coupling

$$\theta_k = -\beta_k \tan \delta_3 \quad (42)$$

For blade number $N \geq 4$, the two feedback systems are not equivalent, even with $K_o = K_p$.

In this study, the flapping feedback system is applied to a three-bladed rotor, with the same gain factor value for both collective and cyclic pitch controls. Although for this special case, one needs only a single blade analysis to study the effects of flapping feedback on the rotor stability and gust response, the method of multiblade coordinates has been applied in the analysis to facilitate comparisons with the other feedback configurations.

4.4 Tilt Integral Feedback System

Compensation of hub moments can also be achieved by a cyclic pitch feedback system which is designed to be rate proportional to the longitudinal and lateral tilting angles such that

$$\left. \begin{aligned} \dot{\theta}_I + L\theta_I &= -K_{11}\beta_I \cos \epsilon - K_{12}\beta_{II} \sin \epsilon \\ \dot{\theta}_{II} + L\theta_{II} &= -K_{21}\beta_I \sin \epsilon - K_{22}\beta_{II} \cos \epsilon \end{aligned} \right\} \quad (43)$$

where L is an equivalent spring parameter of the controller used as possible insurance against dynamic instabilities, ϵ represents a shift in the phasing of the cyclic control.

The tilt integral feedback has been used for the Lockheed gyro rotor which also showed low gust sensitivity.

In the later numerical examples the phase angle $\epsilon = 0$ is assumed, which, according to Reference (8), is close to the optimal phasing. Furthermore, $K_{11} = K_{22} = -K_{12} = K_{21} = K_i$. The tilt integral feedback system is then represented by

$$\left. \begin{aligned} \dot{\theta}_I + L\theta_I &= -K_i\beta_I \\ \dot{\theta}_{II} + L\theta_{II} &= -K_i\beta_{II} \end{aligned} \right\} \quad (44)$$

For the numerical examples with rotor advance ratio $\mu = .8$ the equations (38), (40), (41), (44) are inserted in one of the set of equations (18) to (37) to obtain the stability characteristics of 3 to 6 bladed rotors having one of the 4 feedback systems. For the numerical examples with rotor advance ratio $\mu = 1.6$ and for one case of $\mu = .8$ reversed flow is considered leading to more complex multiblade equations. The periodic coefficients are then non-analytic and have been replaced by truncated Fouries series selected in such a way that the estimated truncation error is less than 1 per cent.

5. Solution Methods

5.1 STABILITY ANALYSIS FOR SYSTEMS WITH PERIODIC COEFFICIENTS

The method of the Floquet state transition matrix applied in Reference (5), has been extended to include the computation of modal function columns which are the equivalent of the modal amplitude ratios and phase angles used in describing independent real modes for constant coefficient systems.

Using the state vector $\{x\}$ to represent the state variables $\beta_o, \dot{\beta}_o, \beta_d, \dot{\beta}_d, \beta_I, \dot{\beta}_I, \dots$ etc., the free motions of the rotor system are described by

$$\{\dot{x}\} = [D(t)]\{x\} \quad (45)$$

where $[D(t)]$ is the state matrix with period T . The state transition matrix $[\phi(t)]$ is defined as the solution of the following matrix differential equation:

$$\frac{d}{dt} [\phi(t)] = [D(t)][\phi(t)], [\phi(0)] = [I] \quad (46)$$

It can be easily verified that any solution of Eq. (45) is given by

$$\{x(t)\} = [\phi(t)]\{x(0)\} \quad (47)$$

Floquet's theorem states that for the system represented by Eq. (45), if the eigenvalues Λ of $[\phi(T)]$ are all distinct, the transient solution is of the form

$$\{x(t)\} = [A(t)]\{\alpha e^{(\lambda+i\omega)t}\} \quad (48)$$

where $[A(t)]$ is the characteristic matrix, also periodic with T , and

$$\{\alpha\} = [A(0)]^{-1}\{x(0)\} \quad (49)$$

is the column of constants to be determined from the initial value column of the state variables $\{x(0)\}$. Inserting Eq. (49) into Eq. (48), it follows

$$\{x(t)\} = [A(t)][e^{(\lambda+i\omega)t}][A(0)]^{-1}\{x(0)\} \quad (50)$$

By comparison of Eqs. (47) and (50) one obtains

$$[\phi(t)] = [A(t)][e^{(\lambda+i\omega)t}][A(0)]^{-1} \quad (51)$$

or

$$[A(t)] = [\phi(t)][A(0)][e^{-(\lambda+i\omega)t}] \quad (52)$$

For $t = T$ this equation becomes, because of $[A(T)] = [A(0)]$:

$$[A(0)]^{-1}[\phi(T)][A(0)] = [e^{(\lambda+i\omega)T}] \quad (53)$$

which shows that

$$e^{(\lambda+i\omega)T} = \Lambda \quad (54)$$

are the eigenvalues of the matrix $[\phi(T)]$ and that $[A(0)]$ is the associated modal matrix. After solving the eigenvalue problem Eq. (53) the characteristic matrix $[A(t)]$ can be computed from Eq. (52).

From Eq. (47), it can be seen that the column matrix $\{x(T)\}$ corresponding to the initial conditions

$$x_j(0) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \quad (55)$$

is identical to the k -th column of $[\phi(T)]$. By integrating the equation of motion Eq. (45), using the Runge-Kutta method, through one period for each set of initial conditions defined by Eq. (55), $j = 1, N$, it is therefore possible to generate the N columns, $\{x_j(T)\}$, $j = 1, N$, corresponding to the columns of $[\phi(T)]$. In the numerical examples each column of $[\phi(t)]$ was computed using a stepsize of $\Delta t = .1$.

To find the eigenvalues Λ and the associated modal matrix $[A(0)]$ of the state transition matrix $[\phi(T)]$, a computer program has been prepared first to calculate the eigenvalues. The equation $|[\phi(T)] - \Lambda[I]| = 0$ is expanded into a characteristic polynomial based on the Leverrier-Faddeev method (9):

$$\Lambda^N - P_1 \Lambda^{N-1} - P_2 \Lambda^{N-2} - \dots - P_N = 0 \quad (56)$$

The coefficients $P_1, P_2, \dots, P_N$ are given by

$$P_N = (1/N)(\text{trace}[\phi_N(T)]) \quad (57)$$

where

$$[\phi_1(T)] = [\phi(T)] \quad (58)$$

$$[\phi_N(T)] = [\phi(T)][[\phi_{N-1}(T)] - P_{N-1}[I]]$$

The characteristic roots of Eq. (56) are solved by using the Bairstow's algorithm which is available in IBM System/360 Scientific Subroutines. With the obtained eigenvalues Λ , the corresponding eigenvectors are then calculated by solving a set of simultaneous algebraic equations. The program showed that for a 10×10 matrix $[\phi(T)]$, there was one third reduction in computing time as compared to the root squaring method used in Reference (5).

By inspecting Eq. (48), one knows that the stability of the system depends on the signs of λ . Since $\Lambda_k = e^{(\lambda_k + i\omega_k)T}$, one obtains easily that

$$\lambda_k = \frac{1}{2T} \ln[(\text{Re}\Lambda_k)^2 + (\text{Im}\Lambda_k)^2] \quad (59)$$

$$\omega_k = \frac{1}{T} \arctan (\text{Im}\Lambda_k / \text{Re}\Lambda_k) \quad (60)$$

Because the characteristic value with the largest real part λ represents the least stable one, the largest in absolute value Λ is required for stability determination. The stability limit, $\lambda_k = 0$, is reached when $|\Lambda_k| = 1$. Since $\arctan$ is a multivalued function, the ω_k determined from Eq. (60) could be a fundamental frequency plus or minus an integer multiple of $2\pi/T$. In case of a three-bladed rotor and using a non-dimensional time in which the period of rotor revolution is 2π , $T = 2\pi/3$, see Eqs. (20) to (22), so that ω_k is only determined except for the addition of an arbitrary multiple of ± 3 . In case of a four-bladed rotor the

basic period is $T = \pi$, see Eqs. (23) to (26), so that ω_k is only determined except for the addition of an arbitrary multiple of ± 2 . The same principle is also applied to the cases of five and six-bladed rotors. The indeterminacy of ω has no effect on the solution, Eq. (48), since $[A(t)]$ contains the compensating factor $e^{-i\omega t}$ as seen in Eq. (52). In the present study the ω_k values are selected in such a way that for zero gain, when the blades are uncoupled, the frequencies of the progressing and regressing multiblade modes according to Eqs. (6) to (11) are obtained.

5.2 RESPONSE ANALYSIS USING UNCOUPLED EQUATIONS

For the forced rotor dynamic system, the equations of motion can be represented by

$$\{\dot{x}\} = [D(t)]\{x\} + \{F(t)\} \quad (61)$$

where $\{F(t)\}$ is a column matrix representing the excitations. Now if the variable substitution $\{x(t)\} = [A(t)]\{y(t)\}$ is made in Eq. (61), it can be shown that the equations become uncoupled as

$$\{\dot{y}\} - [\lambda + i\omega_k]\{y\} = [A(t)]^{-1}\{F(t)\} \quad (62)$$

These uncoupled, constant coefficient equations can be used to compute the force responses of the system.

For the initial condition $\{y(0)\} = \{0\}$, the response of the system to the input $\{F(t)\}$ can be determined by the convolutional integral equations

$$\{y(t)\} = \int_0^{\infty} [e^{(\lambda+i\omega)(t-\tau)}][A(\tau)]^{-1}\{F(\tau)\}d\tau \quad (63)$$

In order to apply the above uncoupled integral equations, it is first necessary to calculate the timewise-tabulated state transition matrix $[\phi(t)]$ over the period from Eq. (47), and calculate the eigenvalues $e^{(\lambda+i\omega)T}$ and corresponding modal matrix $[A(0)]$ of the Floquet state transition matrix $[\phi(T)]$. Then $[A(t)]$, which is periodic with T , can be obtained by using Eq. (52), and the inverse of $[A(t)]$, though a complex function, is not difficult to calculate.

If the computer program based on Eq. (63) can give favorable results, it can be applied in the calculations of random gusts response and would give a large saving of computing time. The program did give accurate results for a single blade case, but unfortunately, the results were quite inaccurate for the case of three blades. This can be explained by the behavior of $[A(t)]^{-1}$. For the single blade case the components of $[A(t)]^{-1}$ vary smoothly and mildly with time. The numerical integration of Eq. (63), using a stepsize of $\Delta t = .2$, yields a very good result. For a three-bladed rotor system described in terms of multiblade coordinates,

some components of $[A(t)]^{-1}$ vary abruptly and have very high magnitudes at certain time ranges. Accordingly, numerical integration of Eq. (63), using even a smaller stepsize of $\Delta t = .1$, fails to give acceptable results. For this reason the later numerical examples of blade responses have been computed directly with the Runge-Kutta method without making use of the uncoupled equations.

5.3 MODAL FUNCTION COLUMNS

We denote two characteristic values $\lambda_j + i\omega_j$ and $\lambda_k + i\omega_k$ combinable, if by an admissible addition to ω_j or ω_k in accordance with Eq. (60) we can obtain a conjugate complex pair $\lambda_j \pm i\omega_j$. This requires $\lambda_j = \lambda_k$. It can be shown that the columns of the characteristic matrix $[A(t)]$ associated with conjugate complex characteristic values are also conjugate complex. Assuming for the two associated α -values conjugate complex numbers, for example $\alpha_j^* = \alpha_j = 1.0$, the real modal function column for the pair $\lambda_j \pm i\omega_j$ is

$$\{x_j(t)\} = \{A_j(t)\} e^{i\omega_j t} + \{A_j^*(t)\} e^{-i\omega_j t} \quad (64)$$

To obtain the actual motion this expression must be multiplied by $e^{\lambda_j t}$. For a constant coefficient system the modal columns contain constant complex numbers independent of time, and Eq. (64) gives the phase angles and amplitude ratios of

all state variables for a real mode associated with a pair of conjugate complex eigenvalues. In this case all state variables oscillate with the same frequency but with various phase angles. For a system with periodic coefficients one must know the entire time function for each state variable as expressed in Eq. (64) in order to define an independent mode. Numerical examples of such independent modes will be given later.

While for a constant coefficient system the characteristic values are either real or conjugate complex, for a system with periodic coefficients single non-combinable complex characteristic values can occur. In this case the modal function column is given by

$$\{x_j(t)\} = \{A_j(t)\} e^{i\omega_j t} \quad (65)$$

whereby $\{A_j(t)\}$ consists of real functions multiplied by $e^{-i\omega_j t}$, so that the product is real.

6. Results and Discussions

6.1 STABILITY RESULTS

The effects of various feedback systems on the rotor dynamic stability have been studied for rotor advance ratios $\mu = .8$ and 1.6 , for blade Number N varying from 3 to 6 . Reversed flow effects have been included for the cases of $\mu = 1.6$ and for one case of $\mu = .8$. In the numerical presentations a non-dimensional time has been used such that the

basic period of one rotor revolution is 2π . All numerical examples are for $\gamma = 8$, $P = 1.15$ and $B = .97$.

As mentioned before, the imaginary part of the characteristic values, because of Eq. (60), is multi-valued, and a certain selection rule must be established in order to plot stability charts like Figure 5. The obvious selection rule, as also mentioned before, is to stipulate that for zero feedback gain the frequencies should coincide with those of the various progressing and regressing modes defined in Eqs. (6) to (11) and illustrated in Figure 2. For $\gamma = 8$ and $P = 1.15$ the characteristic values for a single blade in the rotating reference system are shown in Figure 3 with rotor advance ratio μ as parameter. Since the non-dimensional frequency of the periodic coefficients is 1, the imaginary part can be shifted by $\pm n$, $n = 1, 2, 3, \dots$. For small μ one has a combinable set of characteristic values which split at an advance ratio somewhat below .4 into 2 non-combinable characteristic values. Because of the indeterminacy of the imaginary part, the "root locus" could either be as indicated by the solid lines, or as indicated by the dash lines, or as indicated by the combined solid and dash line at either $\omega = 1$ or at $\omega = -1$ or in any other fashion obtained by the shift along the vertical axis by $\pm n$, $n = 1, 2, 3, \dots$.

We select now out of all these possibilities the "root locus" as shown for $\omega = 1$. In other words we stipulate that both characteristic values for larger μ have the same imaginary

part $\omega = 1$ and that for small μ -values the combinable characteristic values have imaginary parts a little above and a little below $\omega = 1$. At $\mu = 0$ the system degenerates to a constant coefficient system with conjugate complex eigenvalues, whereby $|\omega| > 1.0$. The point $\mu = 0$ for $|\omega| < 1$ occurs only as a limiting case of a periodic coefficient system, but is not obtained for constant coefficients.

In making the selection as described one obtains root locus curves similar to the familiar curves for constant coefficient systems, however, the break-away of vertical branches from horizontal branches can now occur also for ω -values different from zero, which is not possible for constant coefficient systems.

Having now established for $\mu > .4$ and $\gamma = 8$, $P = 1.15$ that the frequency of the single blade is $\omega = 1$ with two values of the real part λ_1 , λ_2 , we now can use Eqs. (6) to (11) to select the appropriate frequency values for the uncoupled multiblade modes as shown in Figure 4. All coning modes and all differential coning modes where they exist have $\omega = 1$. Progressing and regressing tilting modes have $\omega = 2$ and 0 respectively, progressing and regressing first warping modes have $\omega = 3$ and -1 respectively, etc. Each of these modes occur with a low and a high λ -value.

In the stability charts, Figures 5 to 21 except Figure 18, the horizontal scale refers to the real part λ , the

vertical scale to the imaginary part ω of the characteristic values. The curves represent the characteristic values as a function of the gain factors for the various feedback systems as parameters. As pointed out previously, some of the characteristic values are combinable complex and some are single non-combinable complex. Curves representing combinable characteristic values are recognized by their symmetry, and curves representing single non-combinable characteristic values are neither symmetrical nor located on the $\omega = 0$ line. Just as two real characteristic values can break away to form a pair of conjugate complex values in the root-locus of a constant coefficient system, two single non-combinable complex values can also break away to form a pair of combinable complex values for the a periodic coefficient system.

The solid lines in the stability charts for $\mu = .8$, Figures 5 to 12 represent the characteristic values obtained from the complete equations with periodic coefficients, using the Floquet state transition matrix method. The dashed lines are the results obtained from the corresponding constant coefficient systems where all the periodic terms are neglected. The curves for the constant coefficient systems are drawn in the conventional root-locus manner, except only the curves corresponding to the characteristic values with positive or zero imaginary parts are shown.

Shown in Figure 5 are the characteristic curves for a three-bladed rotor with coning feedback at $\mu = .8$. The curves between $\omega = 1$ and $\omega = 2$ represent the modes with mainly the coupling between coning and progressing tilting. The characteristic values associated with the lower damping mode are a pair of combinable values for K_o up to about .5. As K_o increases, the combinable pair become two single non-combinable complex values. However, as K_o exceeds 1.3, two single complex values become a pair of combinable complex values once again. At $K_o = 3.0$, it is seen that the system is very close to the unstable boundary. The dashed lines representing the characteristic values for the constant coefficient system, although they fail to show the break-aways, indicate the instability boundary fairly accurate in this case.

Figure 6 shows the effect of tilt proportional feedback on the characteristic values for a three-bladed rotor at $\mu = .8$. At zero gain, there are two pairs of combinable complex values and two real values. As K_p increases, the real values become a pair of conjugate complex values representing a low frequency mode. The frequency of the mode increases with the increasing gain factors. At $K_p \sim 3.0$, a pair of conjugate complex values become two single non-combinable complex values with $\omega = 1.5$. No instability occurs with K_p up to 3. The constant coefficient approximation is too conservative for this case.

Figure 7 represents the characteristic values for a three-bladed rotor with flapping feedback at $\mu = .8$. Up to $\tan \delta_3 = .3$, there are two pairs of combinable characteristic values and two single real values, making for a total of four real modes. Beyond $\tan \delta_3 = .3$ there are three combinable pairs of characteristic values, with three associated real modes. Figure 7 shows no trend toward instability as δ_3 is increased. The constant coefficient system gives a good approximation for $\tan \delta_3 = 0.2$ to 0.6 . At zero gain there is a bigger difference between the characteristic values of the two systems.

Figure 8 shows the characteristic curves for a three-bladed rotor with tilt integral feedback at $\mu = .8$ and $L = .1$. At $K_i = .2$ there are four pairs of combinable characteristic values with four associated modal function columns, two of which are shown in Figures 22a and 23a. At $K_i = .8$ there are three pairs of combinable characteristic values and two single complex values with $\omega = 1.5$, one of them with positive real part. The associated modal function column is shown in Figure 24a. The four-bladed rotor with tilt integral feedback at $\mu = .8$ and $L = .1$, see Figure 9, has for $K_i = .2$ four pairs of combinable characteristic values with four associated modal function columns, two of which are shown in Figure 22b and 23b. In addition there are two single characteristic values at $\omega = 2$ with two more associated modal function columns, one of which is shown in

Figure 23c. At $K_i = .8$ there is a combinable pair of characteristic values with positive real part. The associated modal function is shown in Figure 24b. In comparing Figures 24a and 24b it is seen that instability both for the three and four-bladed rotor occurs at a gain factor of somewhat below $K_i = .8$, however with entirely different modes.

The dashed lines in Figures 8 and 9 are the characteristic curves for the corresponding constant coefficient systems. The three-bladed rotor has four modes throughout the K_i range, and the four-bladed rotor has five modes, whereby one of them is not affected by the gain factors. It is seen that the constant coefficient system is at least for the low frequency modes a fairly good approximation for gain factor values up to $K_i = .4$. Figure 8 for a three-bladed rotor shows the instability is not predicted by the constant coefficient approximation. One can then conclude that for an advance ratio of $\mu = .8$ the constant coefficient system can lead to erroneous and unconservative results.

Figures 22 to 24 show modal functions determined with Equations (64) and (65) for the conditions indicated in Figures 8 and 9. Figure 22 shows the modal functions for $K_i = .2$ the less stable of the two low frequency modes for the three and the four-bladed rotors. The lowest frequency is about .4, the corresponding period is about 16 time units of which 6.5 are shown in Figures 22 and 23. The basic low

frequency motion is superimposed by higher frequency motions. Coning is quite pronounced both for 3 and 4 blades, and differential coning of substantial magnitude occurs with 4 blades. Figure 23 shows the modal functions for $K_i = .2$ the less stable high frequency modes for 3 and 4 blades. In case of 3 blades the progressing motion is superimposed by large coning motions. In case of 4 blades there is one progressing motion with very large participation of coning and of differential coning, and another progressing motion with little coning and practically no differential coning.

Figures 24a and 24b shows for $K_i = .8$ the unstable modes for the three and four-bladed rotors. The three-bladed rotor becomes unstable in a pure progressing tilting mode with some coning superimposed. The four-bladed rotor becomes unstable in a low frequency tilting mode with much coning and differential coning present.

Figures 10 and 11 show for tilt integral feedback at $\mu = .8$ the characteristic curves for 5 and 6 blades respectively. Note that compared to the preceding figures the scale has been changed in order to accommodate ω -values down to -1 , which is the frequency for the uncoupled regressing warping mode, see Fig. 4. The progressing warping mode is at $\omega = 3$. Neither these two modes occurred for $N = 3$ and $N = 4$. The stability limit of somewhat below $K_i = .8$ is about the same for the 4, 5 and 6 bladed rotor and instability occurs at the same frequency of $\omega = .83$, while the 3 bladed rotor becomes unstable at a smaller gain and in another mode with frequency $\omega = 1.5$. The

constant coefficient root curves for the 5 and 6 bladed rotor agree very well with the exact roots, particularly for the lower frequencies. The constant coefficient roots in Fig. 10 to 12 are indicated by circles. There are 6 modes for $N = 5$ and 7 modes for $N = 6$.

Figures 25a and 25b shows the unstable modes at $K_i = .8$ for the 5 and 6 bladed rotor respectively. The regressing warping modes (β_{III} , β_{IV}) are contributing strongly, particularly for the 6-bladed rotor.

Figure 12 shows the characteristic curves of the same 6-bladed rotor of Fig. 11, however, now with reverse flow effects included. The stability limit is now higher and instability occurs at a somewhat higher frequency. Note that the range of gain factors extends in Fig. 12 to $K_i = 1.2$. The constant coefficient approximation is again very good. While for 3 and 4 bladed rotors the constant coefficient stability limit is considerably in error, it is quite exact for the 5 and 6 bladed rotor with and without reverse flow effects.

Figures 13 to 21 show the stability charts of the rotors at $\mu = 1.6$, with various blade numbers and feedback control systems. Reverse flow effects have been included in the analysis and the more complex equations have been used.

Figures 13 and 14 represent the characteristic values for a three-bladed and a four-bladed rotor system, using coning feedback. The gain factors at the stability limits are $K_i = .5$ and $K_i = .3$ for the three-bladed and four-bladed

rotors respectively. The types of instability are again different and the gain factor at the stability limit for the four-bladed rotor is lower than for the three-bladed rotor.

Figures 15 and 16 represent the characteristic values for a five-bladed and a six-bladed rotor system, using coning feedback. The types of instability and the gain factors at the stability limits, $K_i = .3$, are the same for both five-bladed and six-bladed rotor systems.

In comparing Figures 13, 14, 15 and 16 it is seen that the three-bladed rotor has the best stability characteristics, while the other three rotor systems have practically the same lower stability limit.

Figure 17 shows the stability chart of a three-bladed rotor with tilt proportional feedback. One of the modes with $\omega = 0$ becomes unstable at $K_p = .3$. It is noticed, however, as K_p exceeds 1.0, the mode becomes stable again.

Figure 18 represents the stability limits in terms of gain factors vs. the phase angles ϵ of the tilt proportional feedback, for a three-bladed rotor at $\mu = 0, .8$ and 1.6. Reverse flow effects are included in the analysis, and the gain factors in Eq. (39) are selected such that $K_{11} = K_{22} = -K_{12} = K_{21} = K_p$. Figure 18 shows that at $\epsilon = 90^\circ$, the stability limits K_p are very high for $\mu = 0$ and .8. It indicates that the stability boundary is discontinuous at $\epsilon \sim 85^\circ$ for the case of $\mu = 1.6$. From the analysis using

the method of generalized multiblade coordinates, it has been found that the discontinuity is due to the fact that there are two types of instability depending on whether the phase angle ϵ is larger or smaller than 85° . When $0^\circ < \epsilon < 85^\circ$, the unstable mode is the high frequency mode with $\omega = 1.5$, and when $85^\circ < \epsilon < 90^\circ$, the unstable mode is the low frequency mode with $\omega = 0$, as shown in Figure 17.

Figure 19 shows the results of a three-bladed rotor with flapping feedback. Up to about $\tan \delta_3 = .5$ there are two pairs of combinable characteristic values and two single real values, making for a total of four real modes. Beyond $\tan \delta_3 = .5$ there are three pairs of combinable characteristic values, with three associated modes. Figure 19 shows no trend toward instability at $\mu = 1.6$ as δ_3 as increased. It is known, however, that in combination with torsionally flexible blades δ_3 can become greatly destabilizing.

Figure 20 shows a three-bladed rotor with tilt integral feedback similar to Figure 8, except for $\mu = 1.6$ including reversed flow. Instability now occurs in the same way but at a much lower gain of $K_i \sim .28$.

Figure 21 shows a case of a four-bladed rotor with tilt integral feedback similar to Figure 9, except for $\mu = 1.6$. The gain factor at the stability limit is $K_i \sim .17$. In comparing Figures 20 and 21 it is seen that the

types of instability are again different and the gain factor at the stability limit for the four-bladed rotor is lower than for the three-bladed rotor.

6.2 RESPONSE TO RANDOM TURBULENCE

The estimation of the effects of various feedback systems on blade flapping response to atmospheric turbulence can be simplified by using the assumption that the turbulence velocity is uniform over the rotor disk and equal to the turbulence felt at the rotor center. In the analysis the turbulence velocity is treated as a part of the inflow ratio λ .

For the system represented by Eq. (12) and the associated feedback equation, Eq. (38), (40), (42) or (44), denoting $\beta_c(\omega, t)$ and $\beta_s(\omega, t)$ the blade flapping response to $m_\lambda u(t) \cos(\omega t)$ and $m_\lambda u(t) \sin(\omega t)$ respectively, the time variable standard deviation of the flapping angle β , according to the assumptions of Reference (2), is given by

$$\sigma_\beta(t) = \left[2 \int_0^\infty \{ \beta_c^2(\omega, t) + \beta_s^2(\omega, t) \} S_\lambda(\omega) d\omega \right]^{1/2} \quad (66)$$

where $S_\lambda(\omega)$ is the two-sided power spectral density function of λ .

According to Reference (2) the following expression for $S_\lambda(\omega)$ is used

$$S_{\lambda}(\omega)/\sigma_{\lambda}^2 = a/\pi(a^2 + \omega^2) \quad (67)$$

where $a = 2\mu/(L/R)$ and where σ_{λ} is the time invariable standard deviation for λ . The frequency response functions $\beta_c(\omega, t)$ and $\beta_s(\omega, t)$ used in Eq. (66) have been computed for a three-bladed rotor for 3 feedback systems for which the stability charts are shown in Figures 13, 19 and 20. The rotor with tilt proportional feedback was omitted since according to Figure 18 instability occurs at a low gain factor for $\epsilon = 90^\circ$ and $\mu = 1.6$. The phase angle should be between 70° and 80° for improved stability. First the frequency response for β_o , β_I , β_{II} was determined using the multiblade equations for a three-bladed rotor at $\mu = 1.6$, and then the individual blade frequency response was obtained from Eq. (1). Finally Eq. (66) was used to determine the time variable flapping standard deviation $\sigma_{\beta}(t)$. A Runge-Kutta routine using a stepsize of $\Delta t = .1$ was used for the integration of the multiblade equations. The results for $\sigma_{\lambda} = 1$ are shown in Figures 26, 27 and 28.

All three feedback systems result in appreciable reductions in random flapping amplitudes. A doubling of the gain shows, however, diminishing returns and in the case of Figure 28 even a reduced gust alleviation when K_i is increased from .1 to .2. The random flapping amplitudes are practically the same for the coning, flapping and tilt integral feedback systems with gain factors at $K_o = .1$, $\tan \delta_3 = .2$ and $K_i = .1$.

respectively. It is interesting to note that coning feedback needs as compared to flapping feedback only half the gain factor to accomplish the same reduction in random flapping response.

7. Summary and Conclusions

The method of multiblade coordinates has been extended in this study to include coning, differential coning and warping modes and to retain the periodic coefficients in the equations of motion for the multiblade coordinates. The modes other than tilting and the periodic terms in the multiblade equations of motion both substantially contribute to potential dynamic instabilities. The dynamic stability analysis using the method of multiblade coordinates shows that the stability limits and the types of instabilities depend on the number of rotor blades. Especially, substantial differences were found in the stability limits and types of instabilities of three and four-bladed rotors. As compared to a stability analysis using individual blade coordinates as in Reference (5), there are four main advantages in the use of multiblade coordinates. First, one obtains the constant coefficient approximation merely by omitting the periodic terms in the equations using multiblade coordinates. Second, the analysis provides modal time functions, from which it is easier to assess the types

of potential dynamic instabilities. Third, one saves considerable computation effort. For a three-bladed rotor, the use of Eqs. (20) to (22) reduces the computer time to about one third as compared to the analysis using single blade coordinates, in spite of assuming the same number of integration intervals per period. Fourth, the results of the multiblade analysis in the form of characteristic values and modal function columns can be easier interpreted with respect to effects on the fuselage and to coupling with fuselage modes, since they define the characteristics of coning or tilting or warping of the rotor. The following detailed conclusions are based on the analyses with multiblade coordinates for a rotor configuration with a blade Lock number of $\gamma = 8$, a tip loss factor of $B = .97$, with a rigid hub, and with rigid flapping blades having elastic flapping restraints such that the undamped flapping frequency is 1.15 rotor rpm.

1. The constant coefficient approximation for a three-bladed rotor at .8 with tilt integral feedback, or with coning feedback has only a moderate effect on the low frequency modes, but it removes a potential instability at 1.5 rpm, and therefore, it is unconservative and cannot be used to determine the stability limits.
2. The constant coefficient approximation has only a moderate effect on the characteristic values for a

four-bladed rotor at $\mu = .8$ with tilt integral feedback. The approximation is much better for a five and six-bladed rotor at $\mu = .8$ with tilt integral feedback.

3. Tilt proportional feedback at a rotor advance ratio of $\mu = 1.6$ results at phase angles of $\epsilon = 0$ and 90° in low stability limits. However for $\epsilon = 70^\circ$ to 80° the stability limit is high.
4. Even at a high rotor advance ratio of $\mu = 1.6$, flapping feedback for a three-bladed rotor, which is equivalent to δ_3 pitch-flap coupling, is stable up to high feedback gains. Torsional blade flexibility, not included in this analysis, is known to destabilize blades with δ_3 pitch-flap coupling.
5. Among the three, four, five and six bladed rotors with coning feedback at $\mu = 1.6$, the three-bladed rotor has the highest stability limit, while the others have practically the same but lower stability limit.
6. Tilt integral feedback at a rotor advance ratio of $\mu = 1.6$ results for a four-bladed rotor in lower stability limit than for a three-bladed rotor.
7. The reduction in blade random flapping response to atmospheric turbulence for a three-bladed rotor at $\mu = 1.6$ is particularly good for coning angle feedback. Flapping feedback with the same gain

leads to higher random blade responses, and tilt integral feedback is least effective in reducing random blade flapping.

8. The method of transforming to uncoupled equations proved to be computationally impractical for the forced response which was, therefore, obtained by integrating the coupled equation.

8. References

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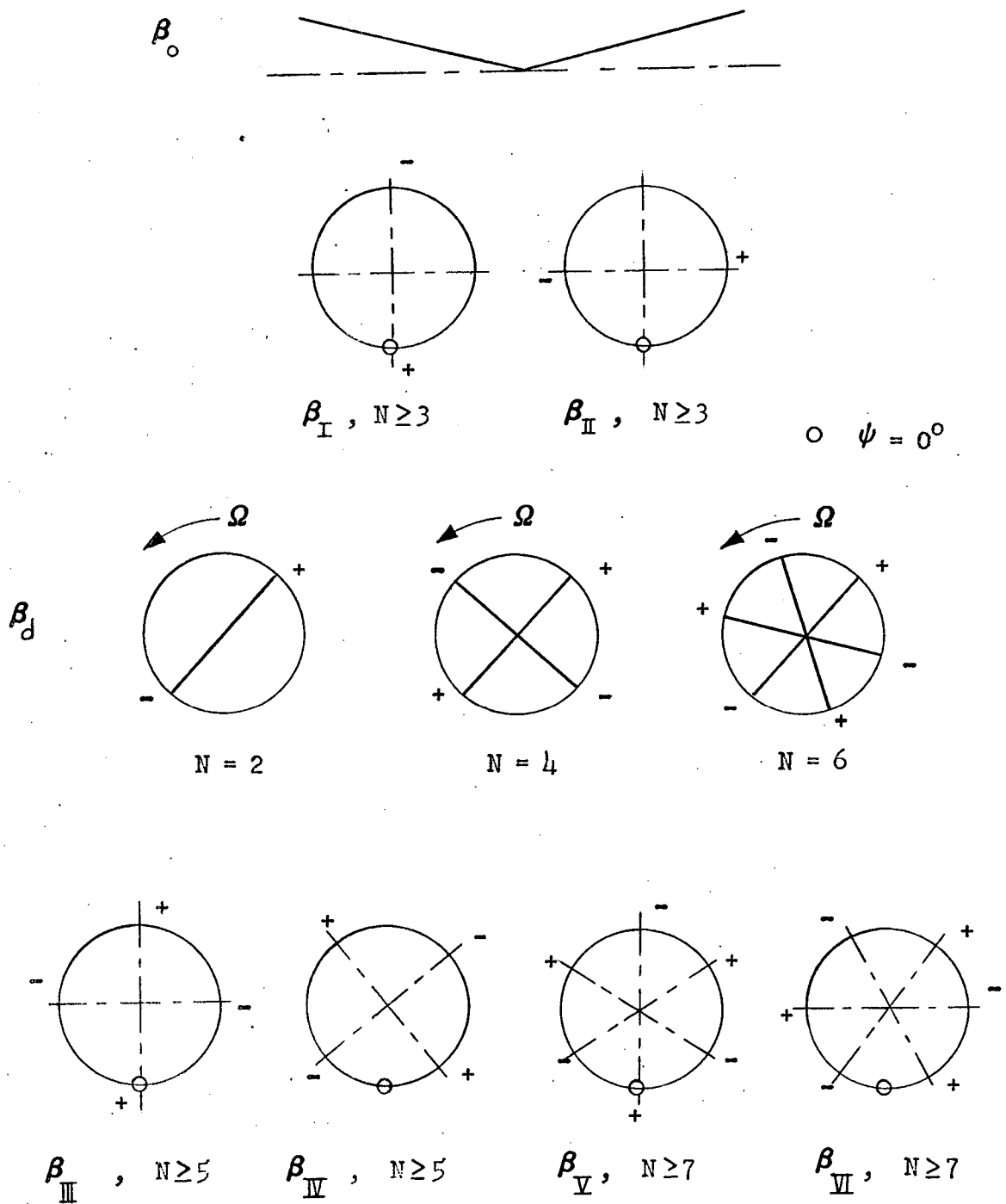
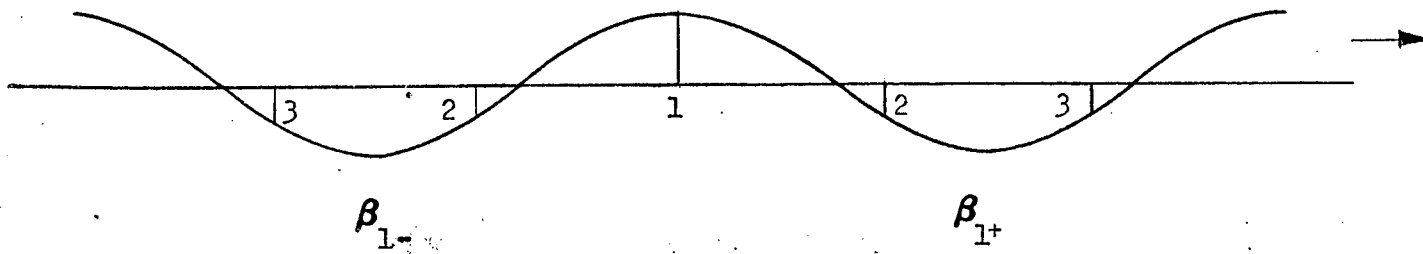
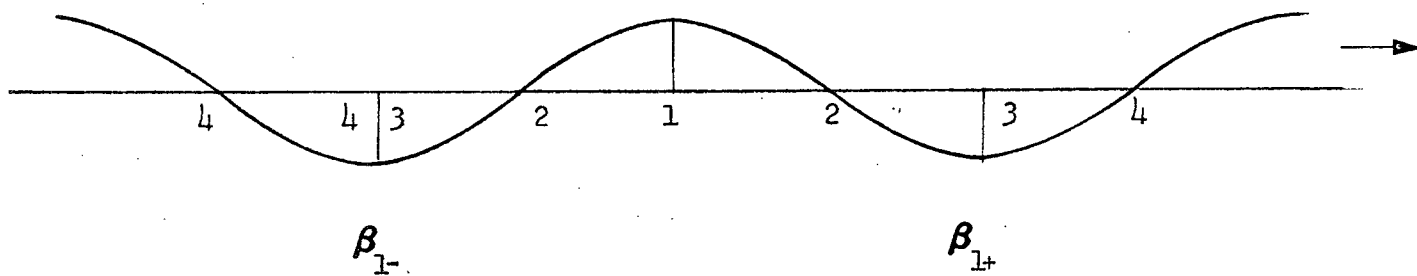


Figure 1. Definition of Multiblade Modes

$N = 3$



$N = 4$



$N = 5$

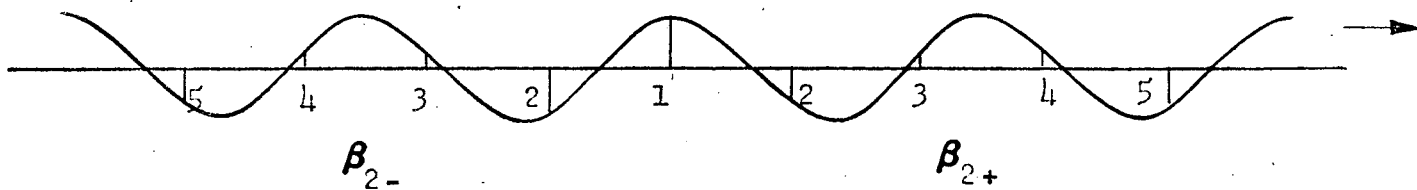
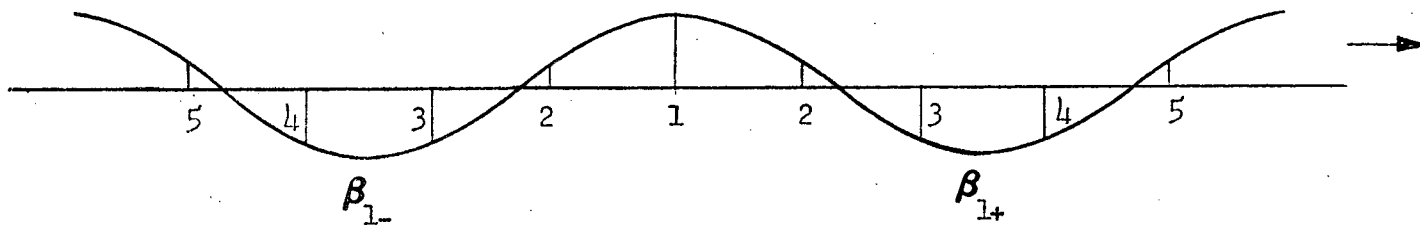
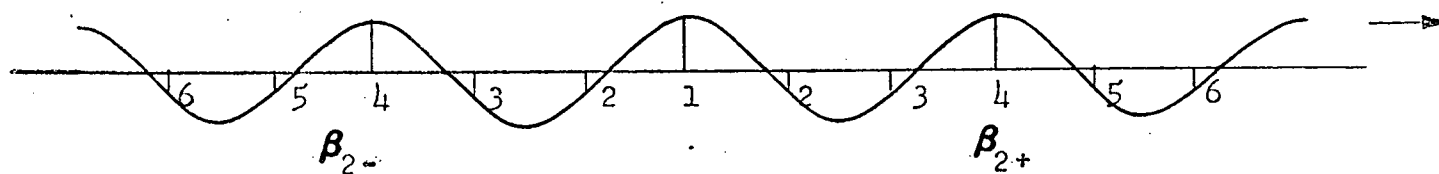
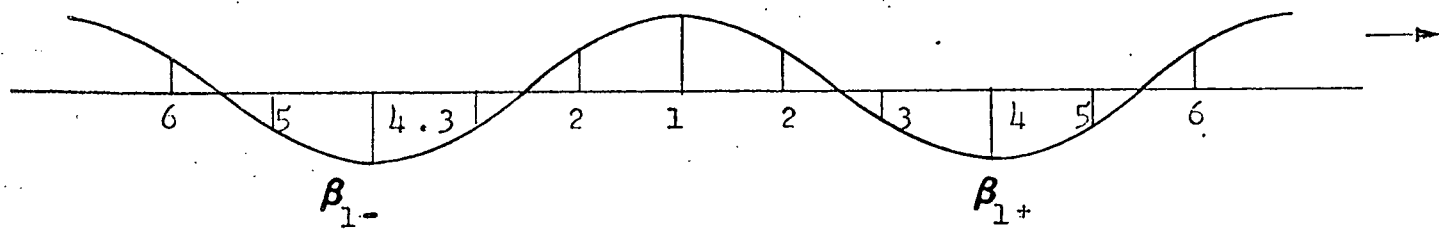


Figure 2. Blade Tip Path for Various Regressing and Progressing Modes

$N = 6$



$N = 7$

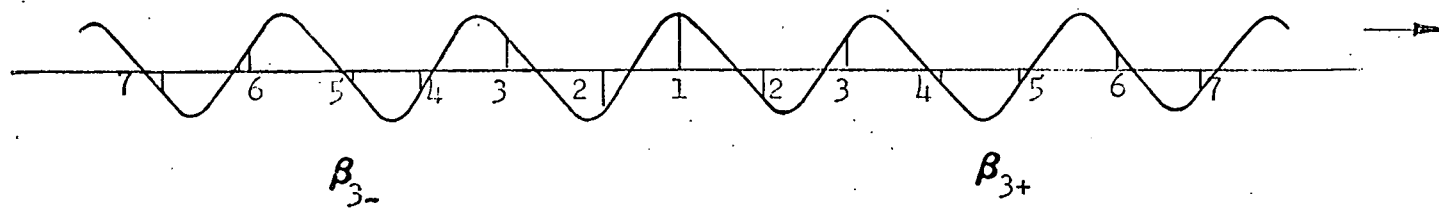
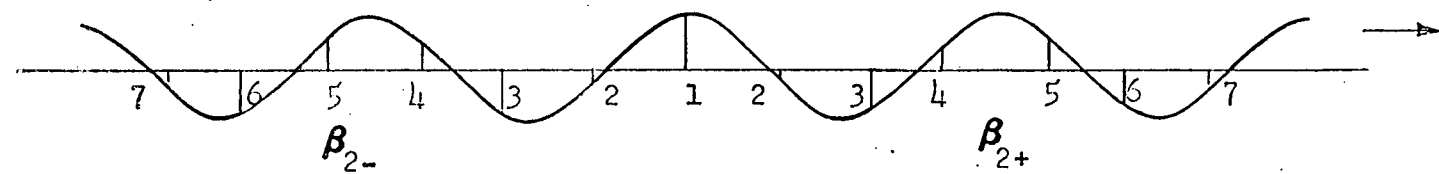
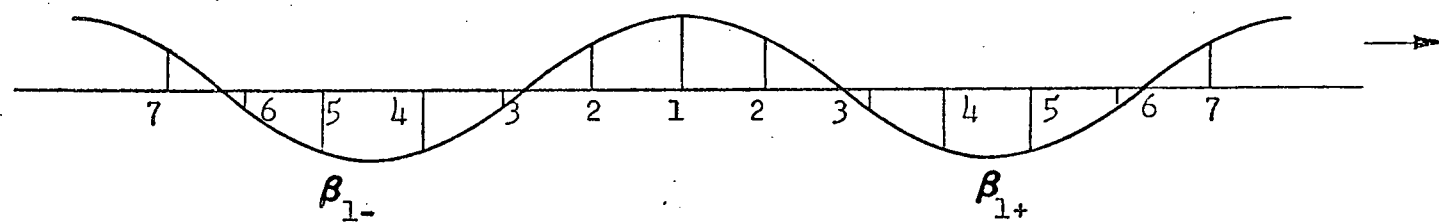


Figure 2. (Continued)

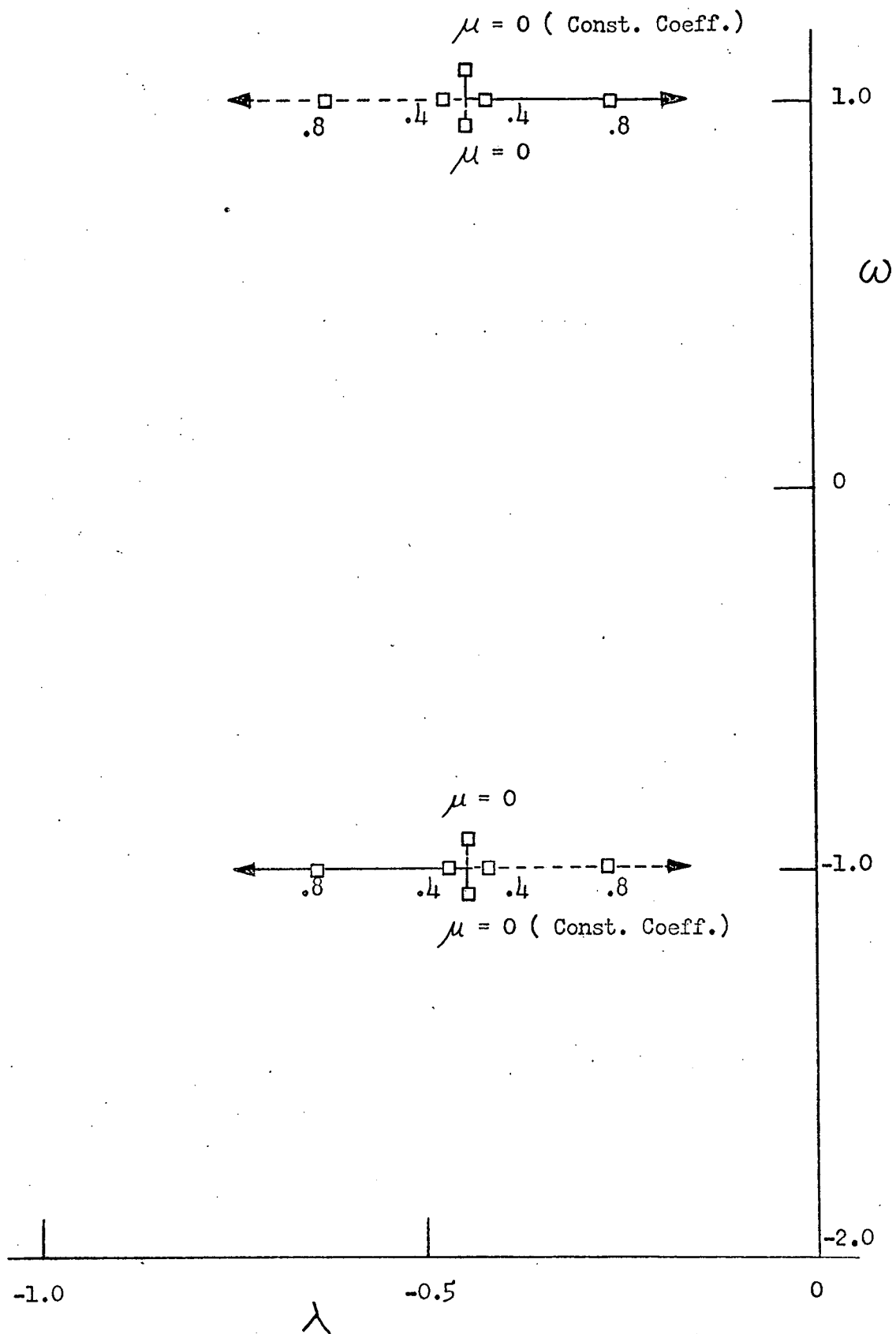


Figure 3. Characteristic Values for Single Blade Flapping Motion in Rotating Reference System, $\gamma=8$, $B=.97$

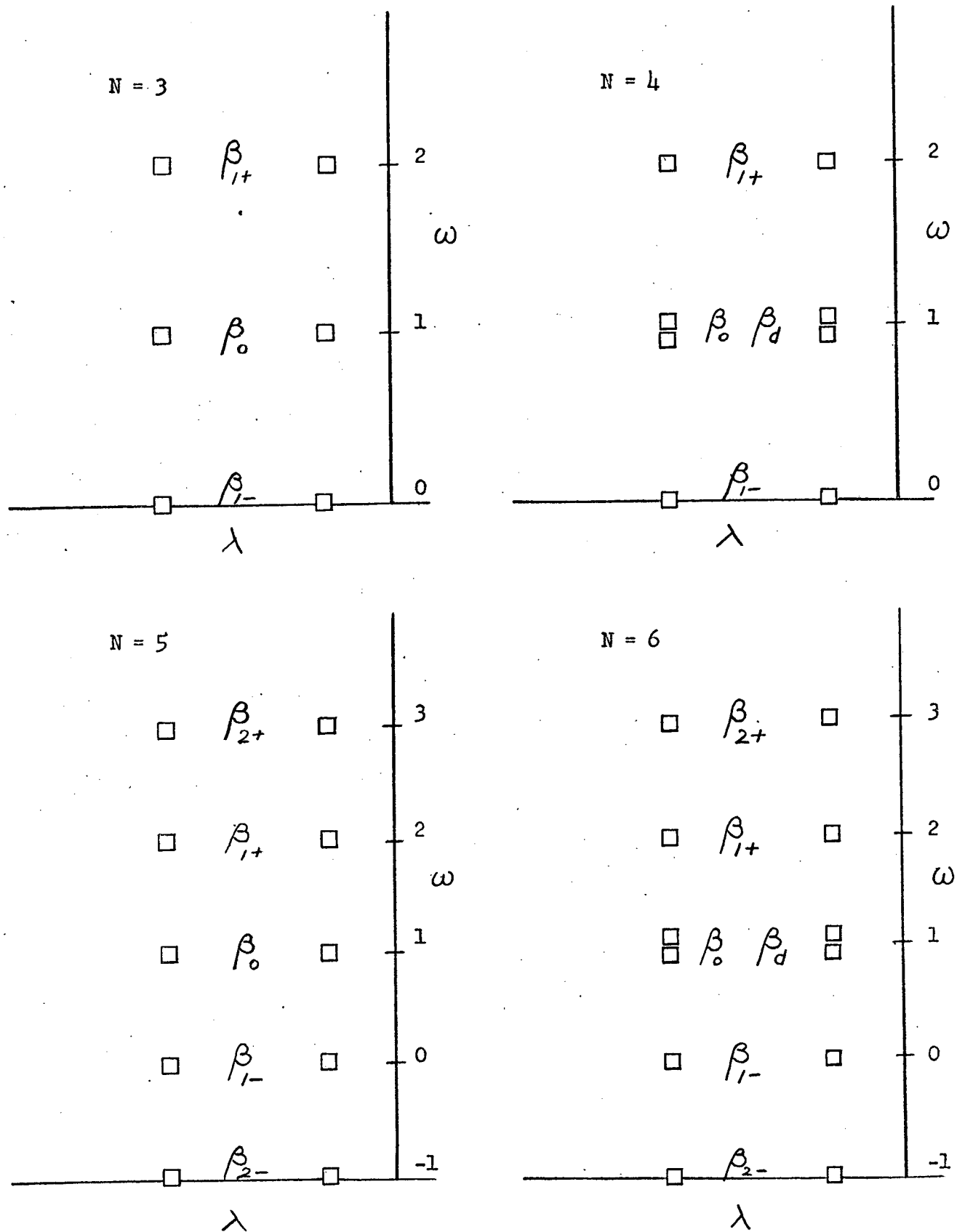


Figure 4. Arrangements of Characteristic Values at $\mu \geq .4$ without feedback for Blade Numbers $N=3$ to 6, $\gamma=8$, $B=.97$

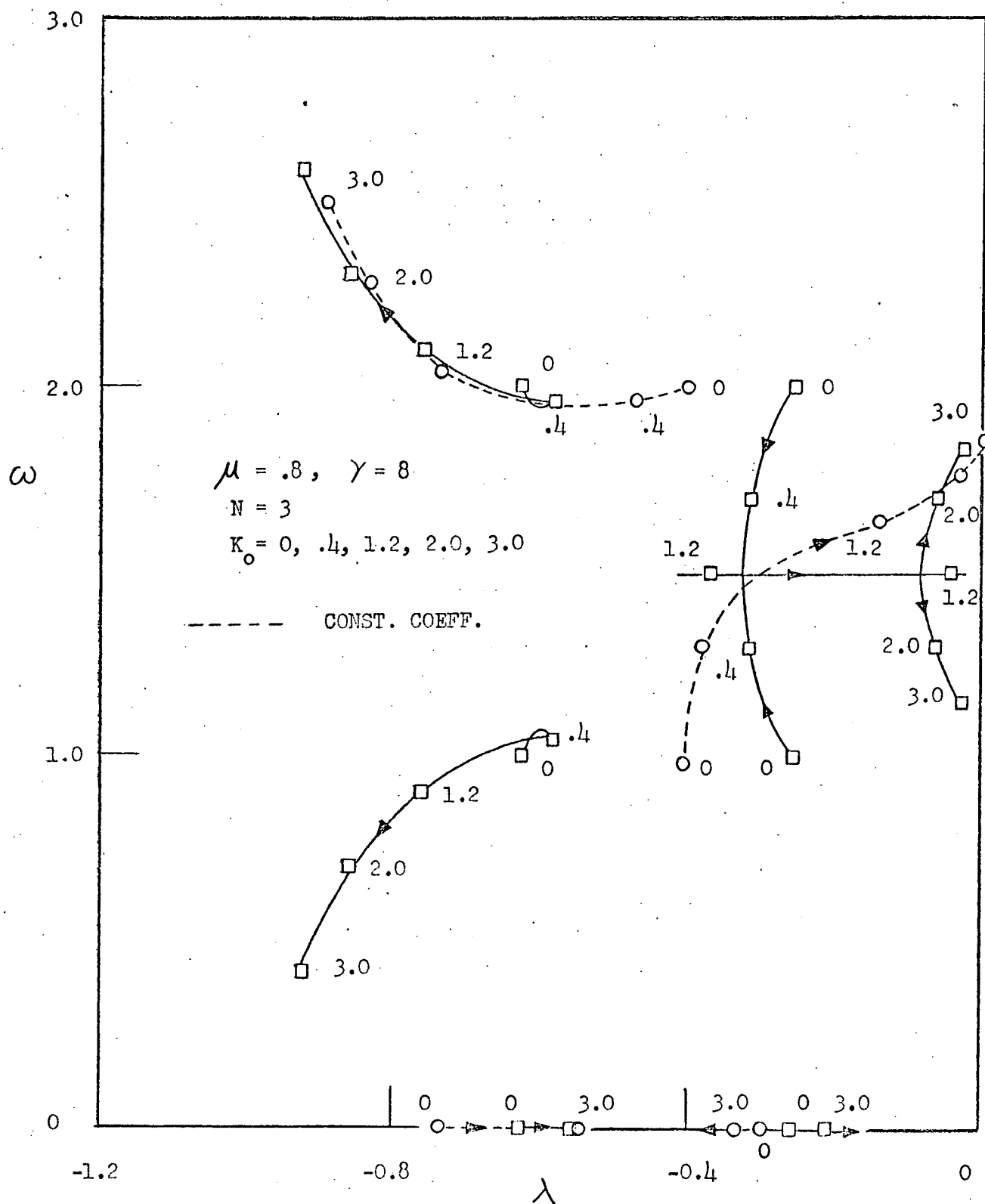


Figure 5. Stability Chart for Three-Bladed Rotor with Coning Feedback at $\mu=.8$

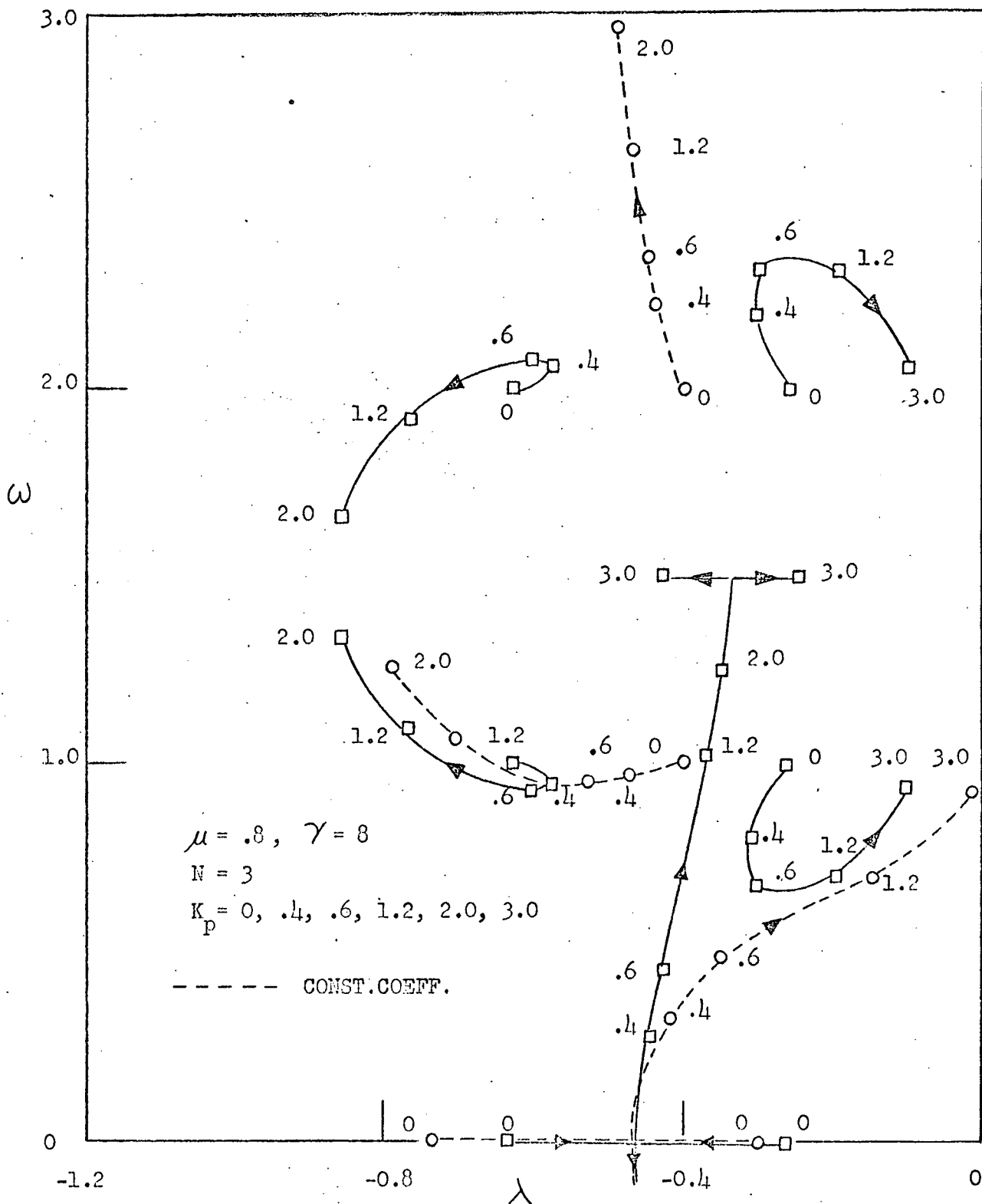


Figure 6. Stability Chart for Three-Bladed Rotor with Tilt Proportional Feedback at $\mu=.8$

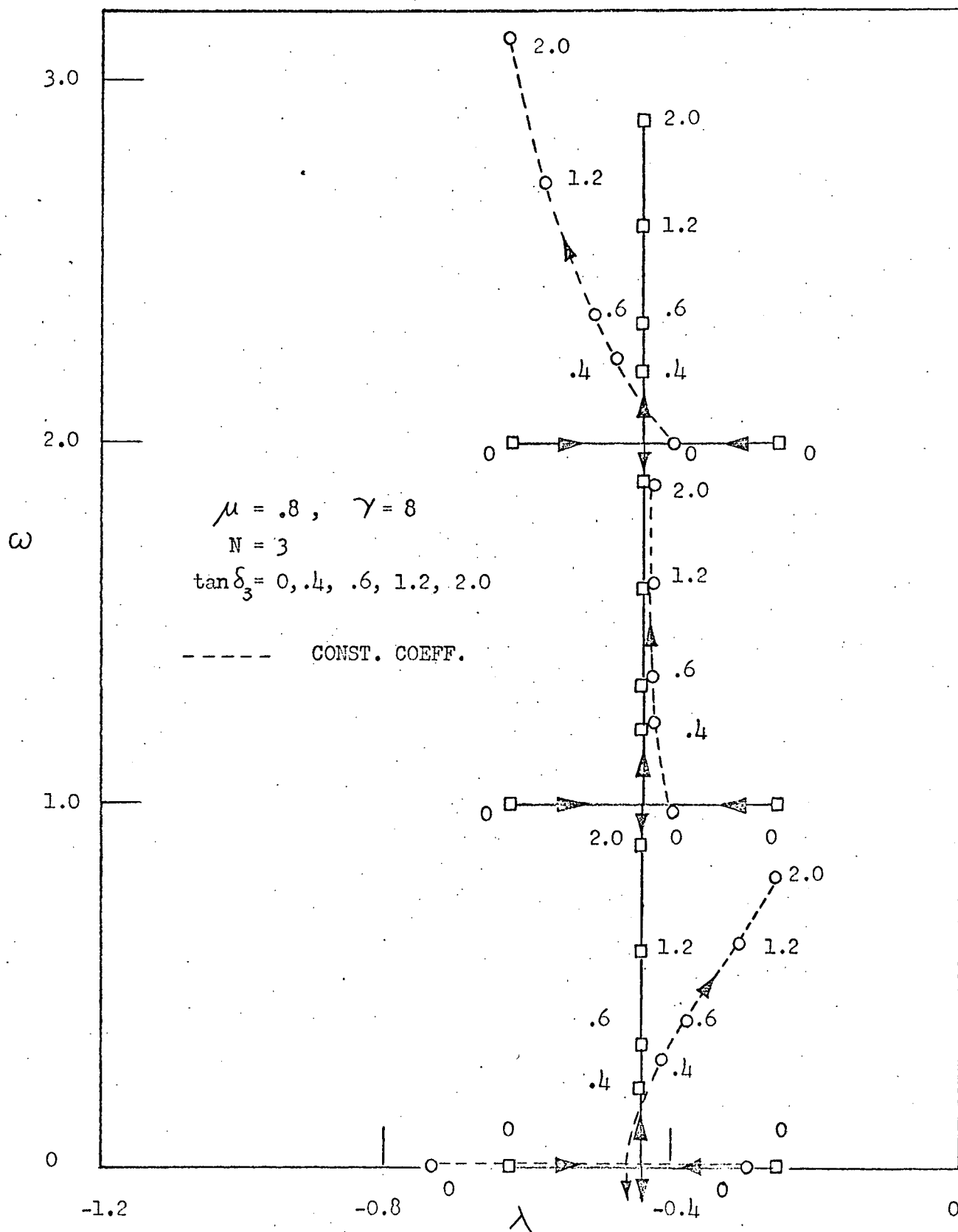


Figure 7. Stability Chart for Three-Bladed Rotor with Flapping Feedback at $\mu=.8$

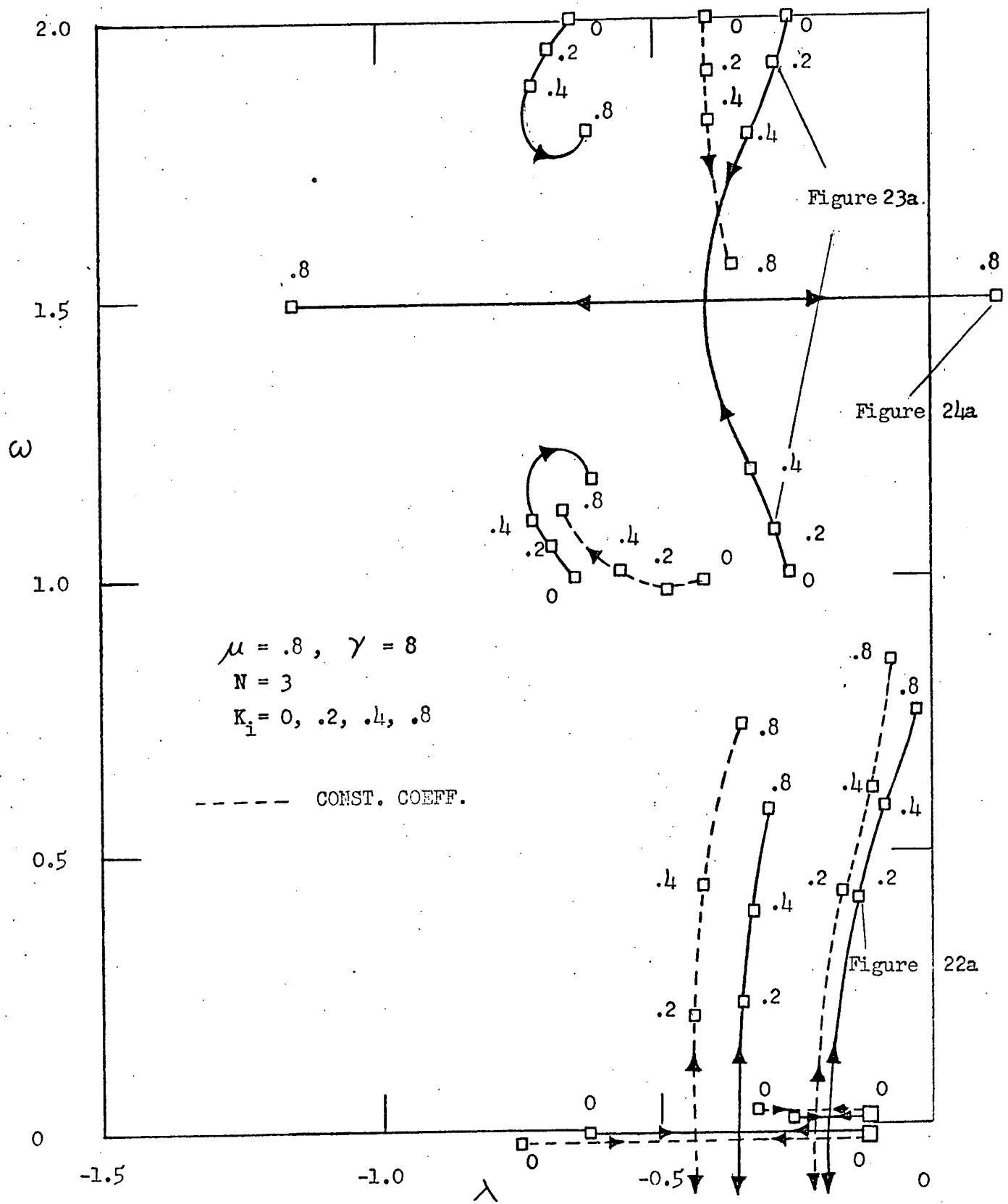


Figure 8. Stability Chart for Three-Bladed Rotor with Tilt Integral Feedback at $\mu=.8$ and $L=.1$

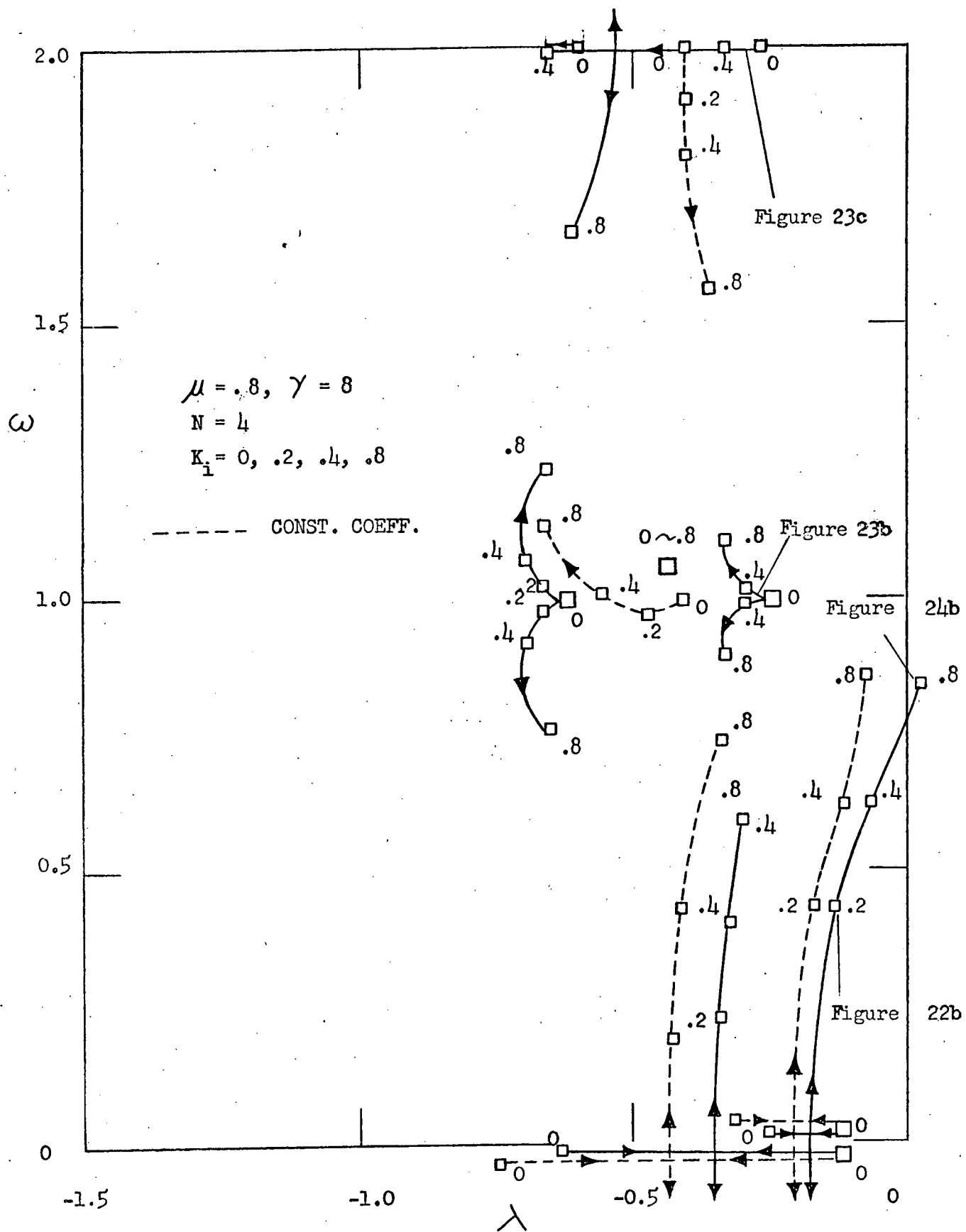


Figure 9. Stability Chart for Four-Bladed Rotor with Tilt Integral Feedback at $\mu=.8$ and $L=.1$

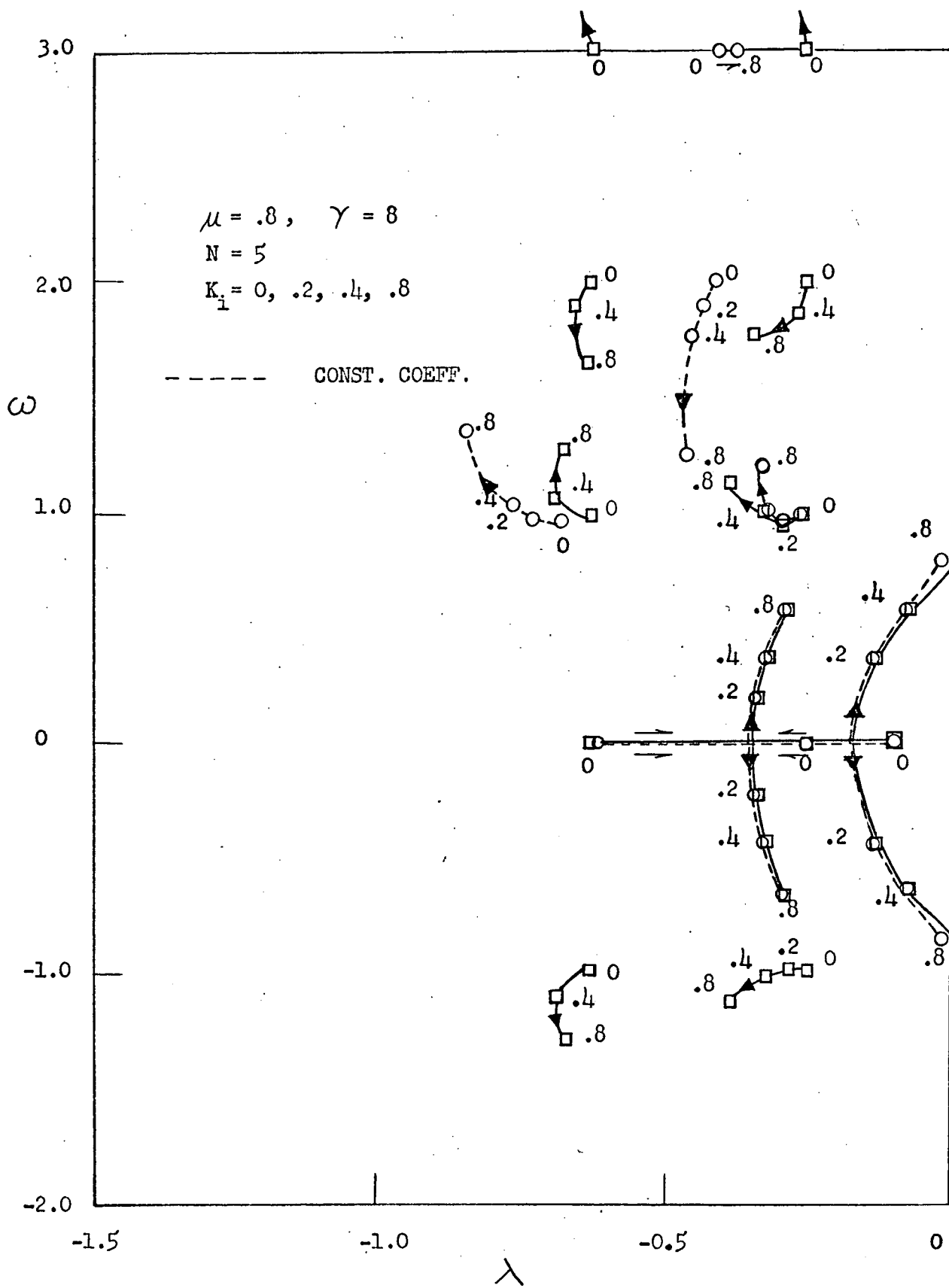


Figure 25a

Figure 10. Stability Chart for Five-Bladed
 Rotor with Tilt Integral Feedback
 at $\mu = .8$ and $L = .1$

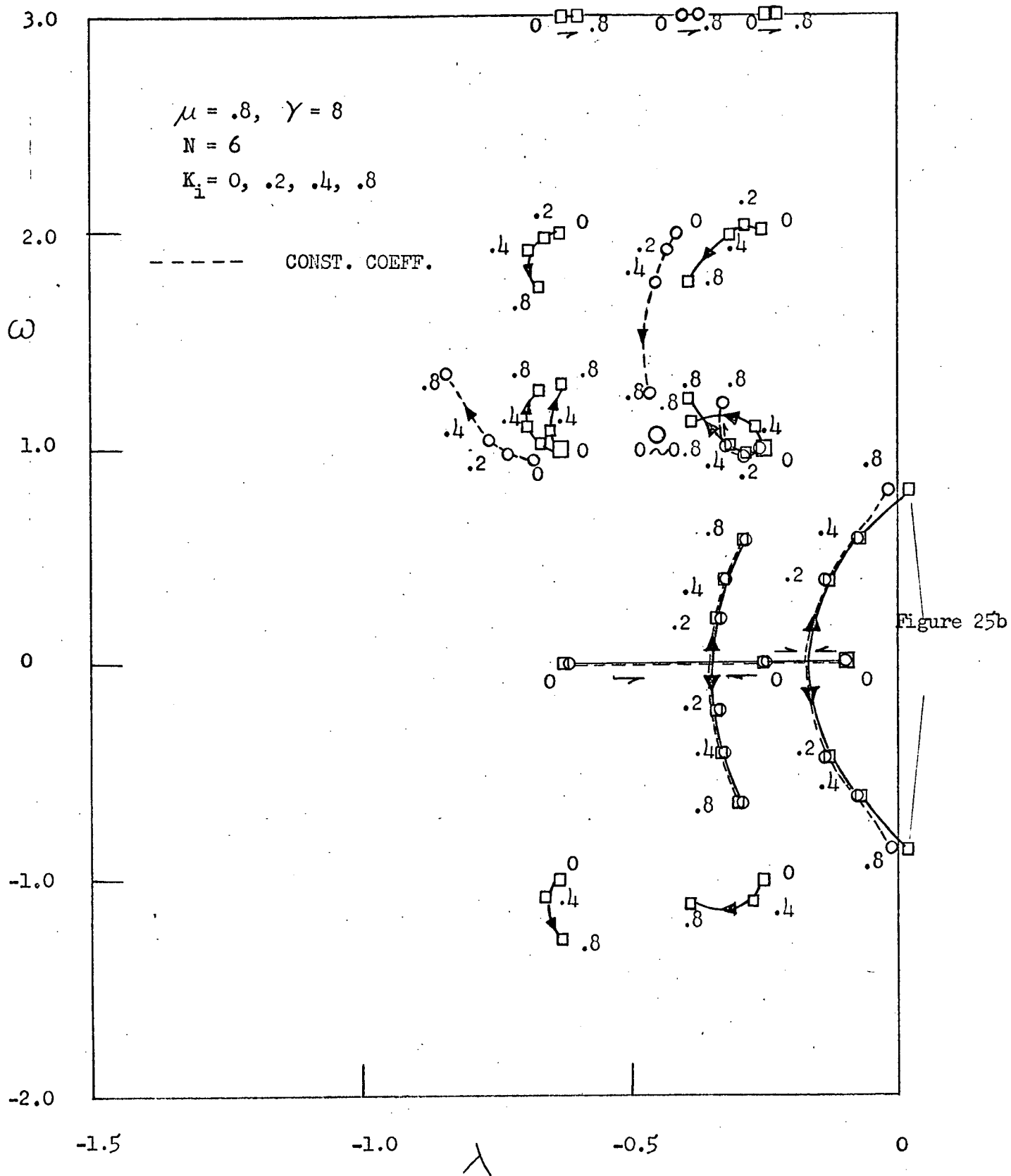


Figure 11. Stability Chart for Six-Bladed Rotor with Tilt Integral Feedback at $\mu=.8$ and $L=.1$

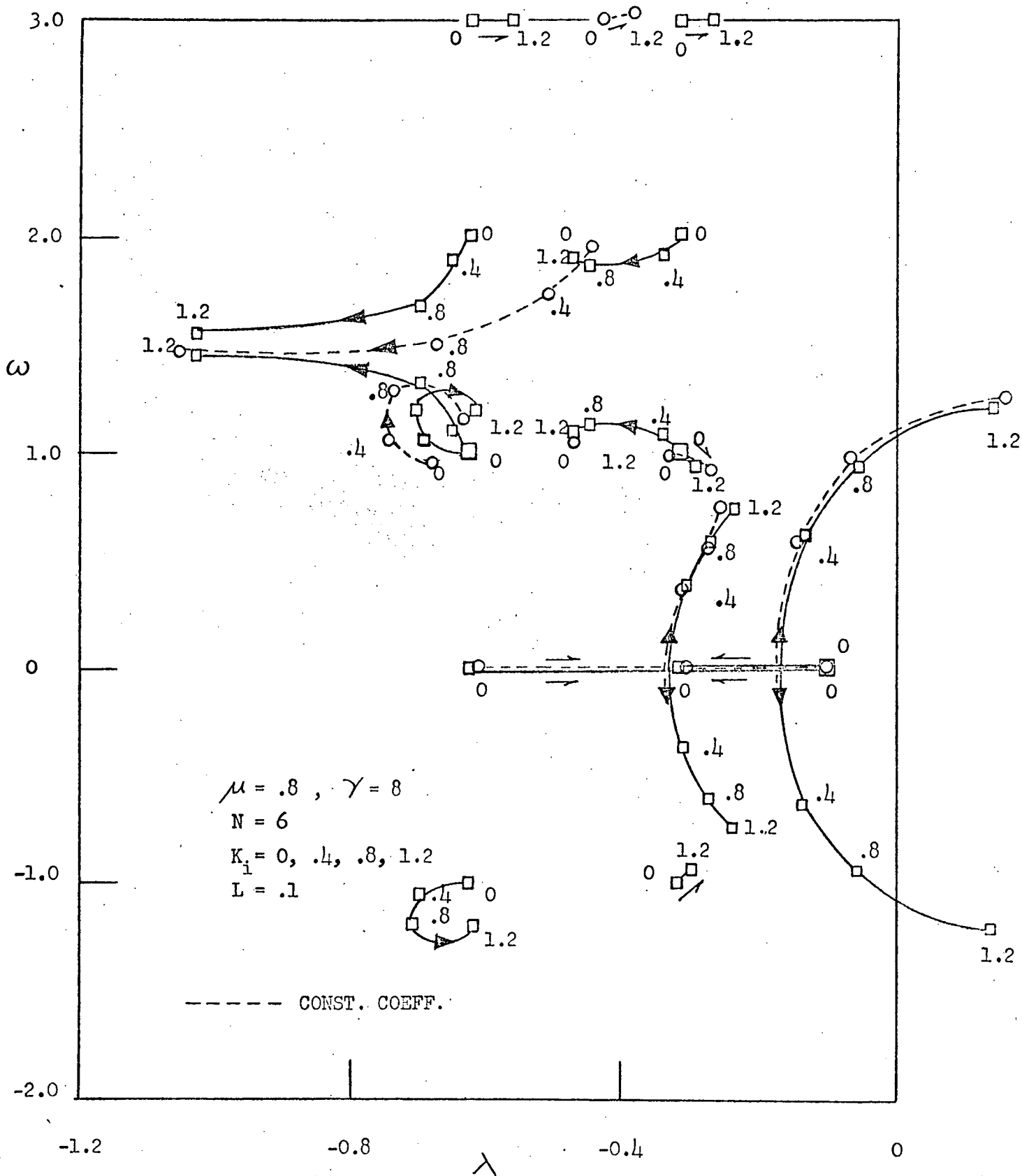


Figure 12. Stability Chart for Six-Bladed Rotor with
 Tilt Integral Feedback at $\mu=.8$ and $L=.1$
 Including Reverse Flow

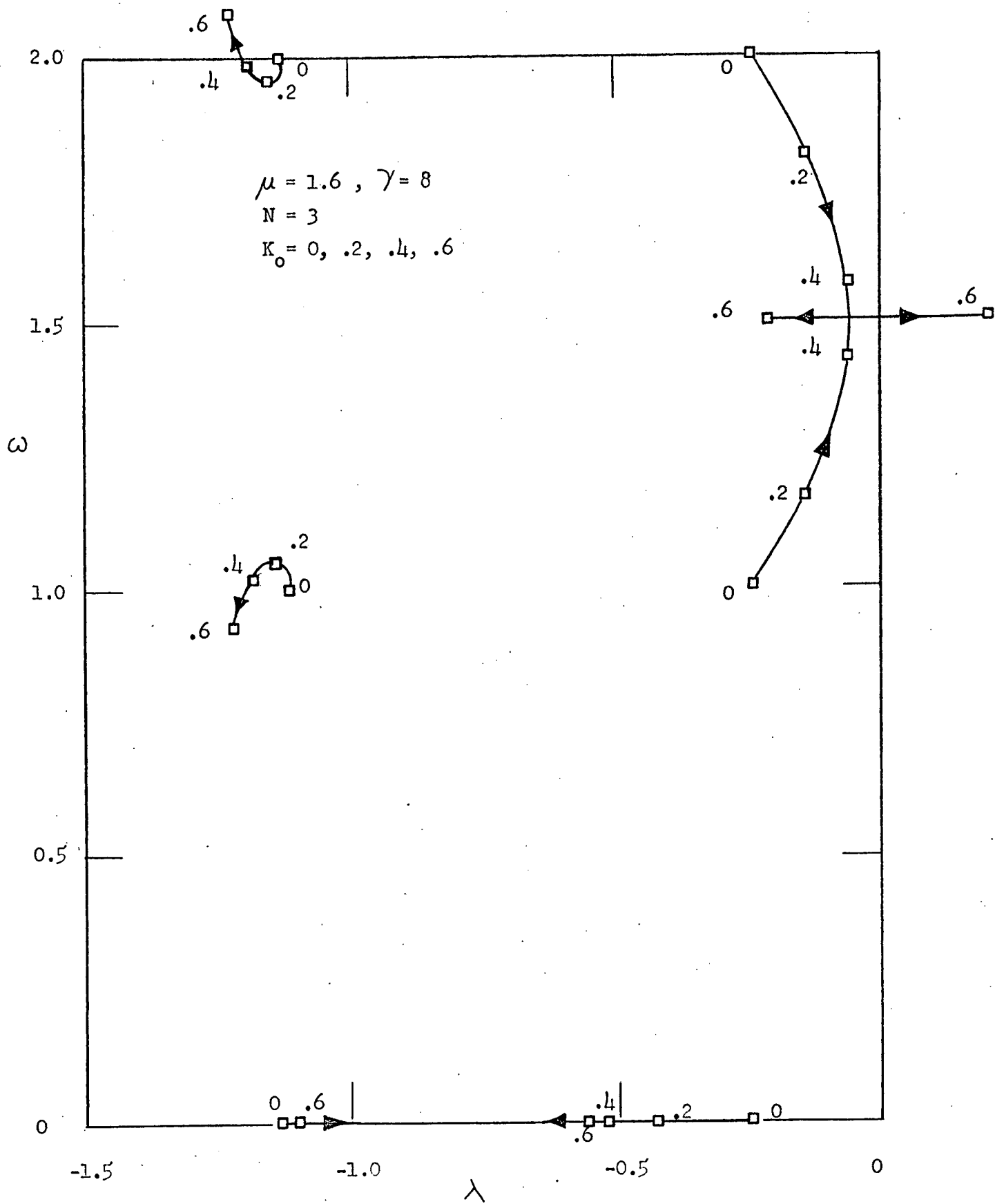


Figure 13. Stability Chart for Three-Bladed Rotor with Coning Feedback at $\mu=1.6$

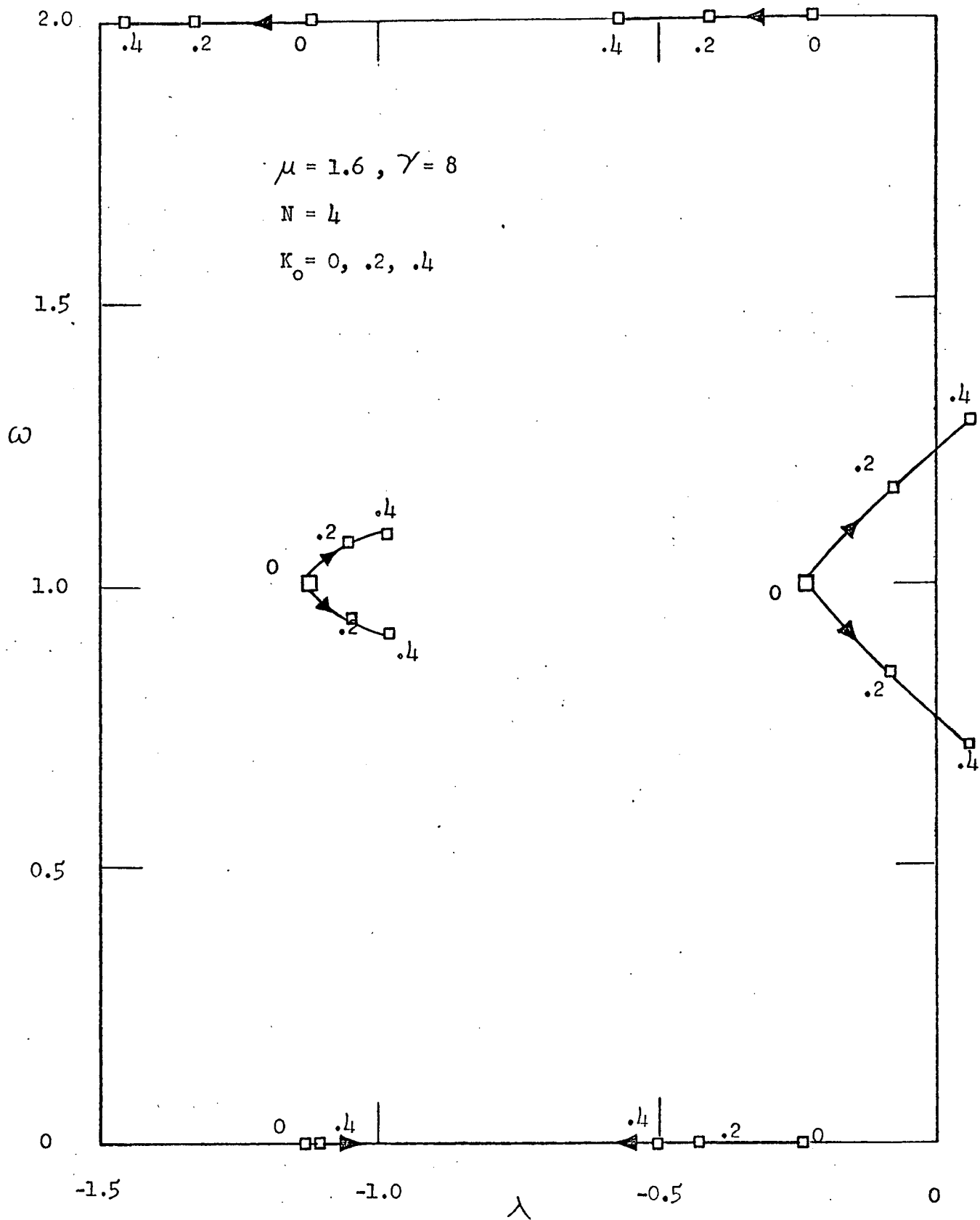


Figure 14. Stability Chart for Four-Bladed Rotor with Coning Feedback at $\mu=1.6$

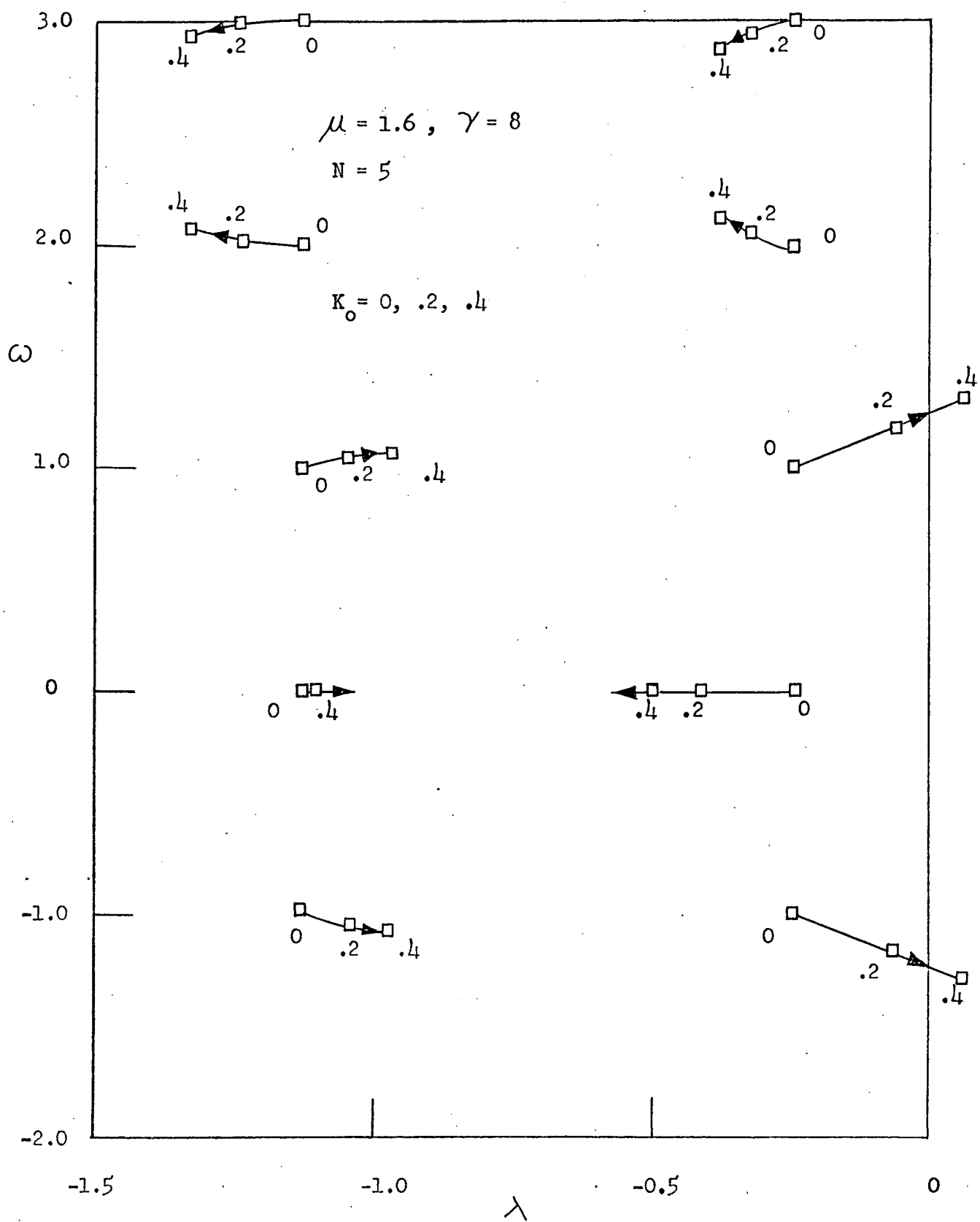


Figure 15. Stability Chart for Five-Bladed Rotor with Coning Feedback at $\mu=1.6$

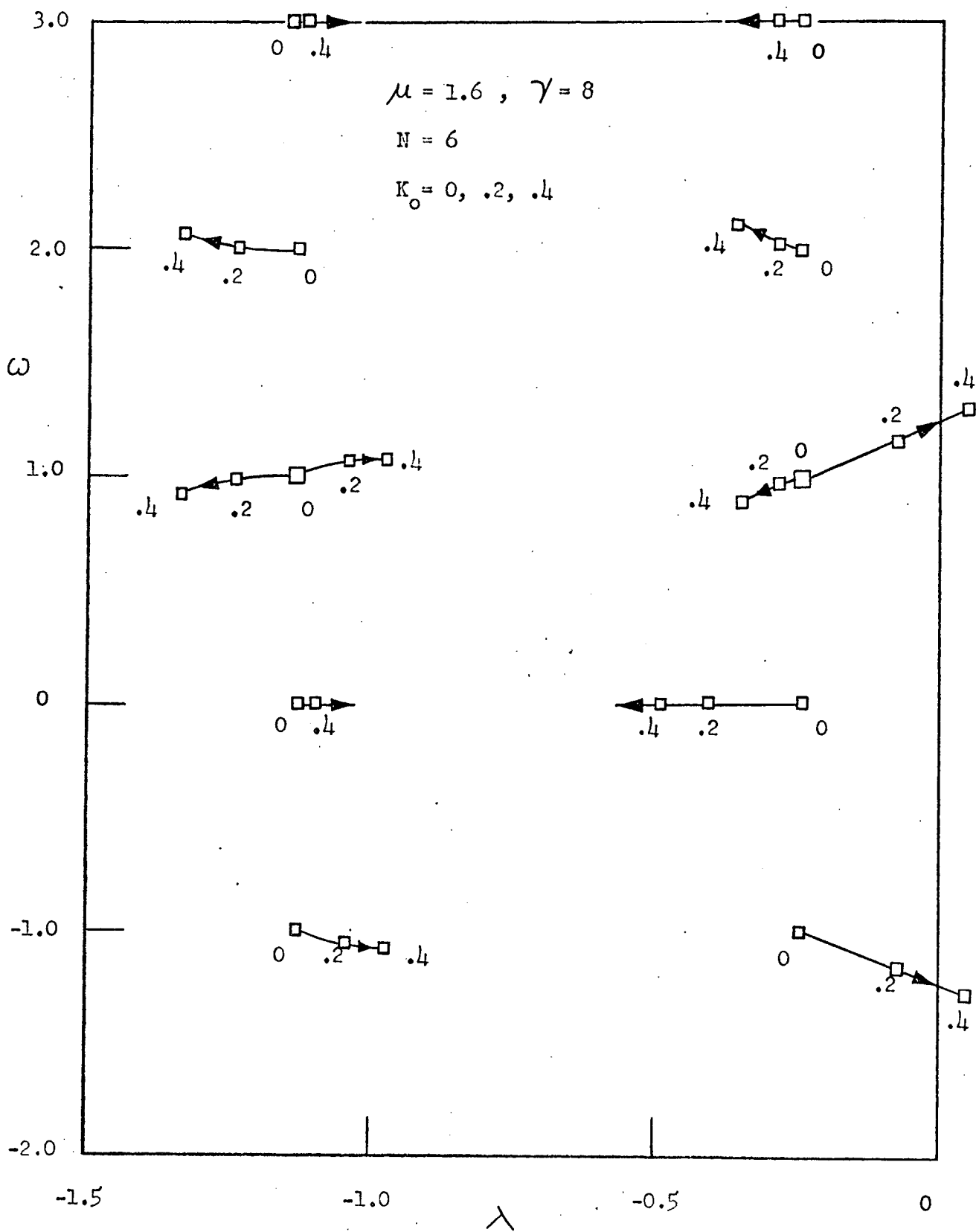


Figure 16. Stability Chart for Six-Bladed Rotor with Coning Feedback at $\mu=1.6$

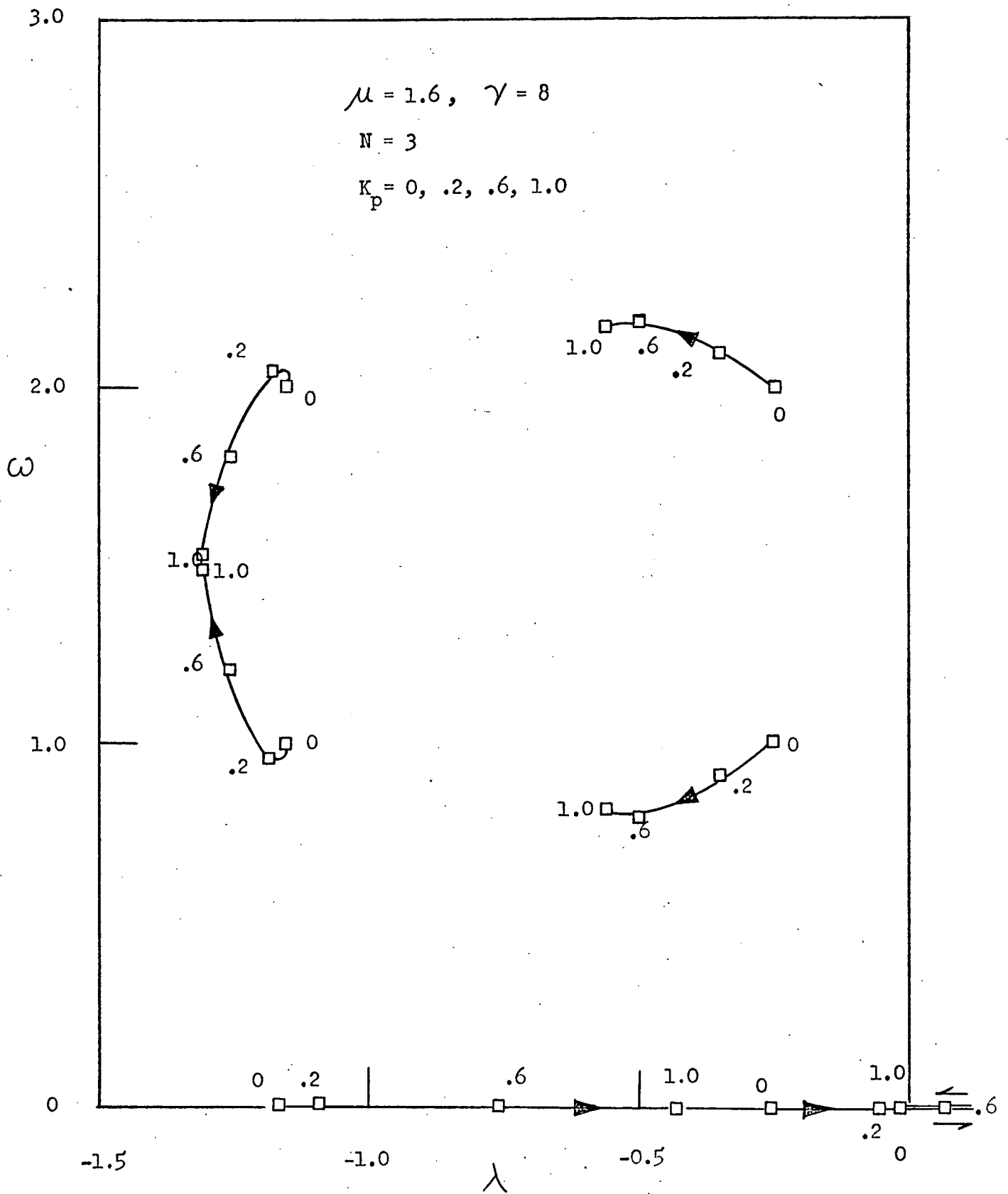


Figure 17. Stability Chart for Three-Bladed Rotor
 with Tilt Proportional Feedback at
 $\mu=1.6$

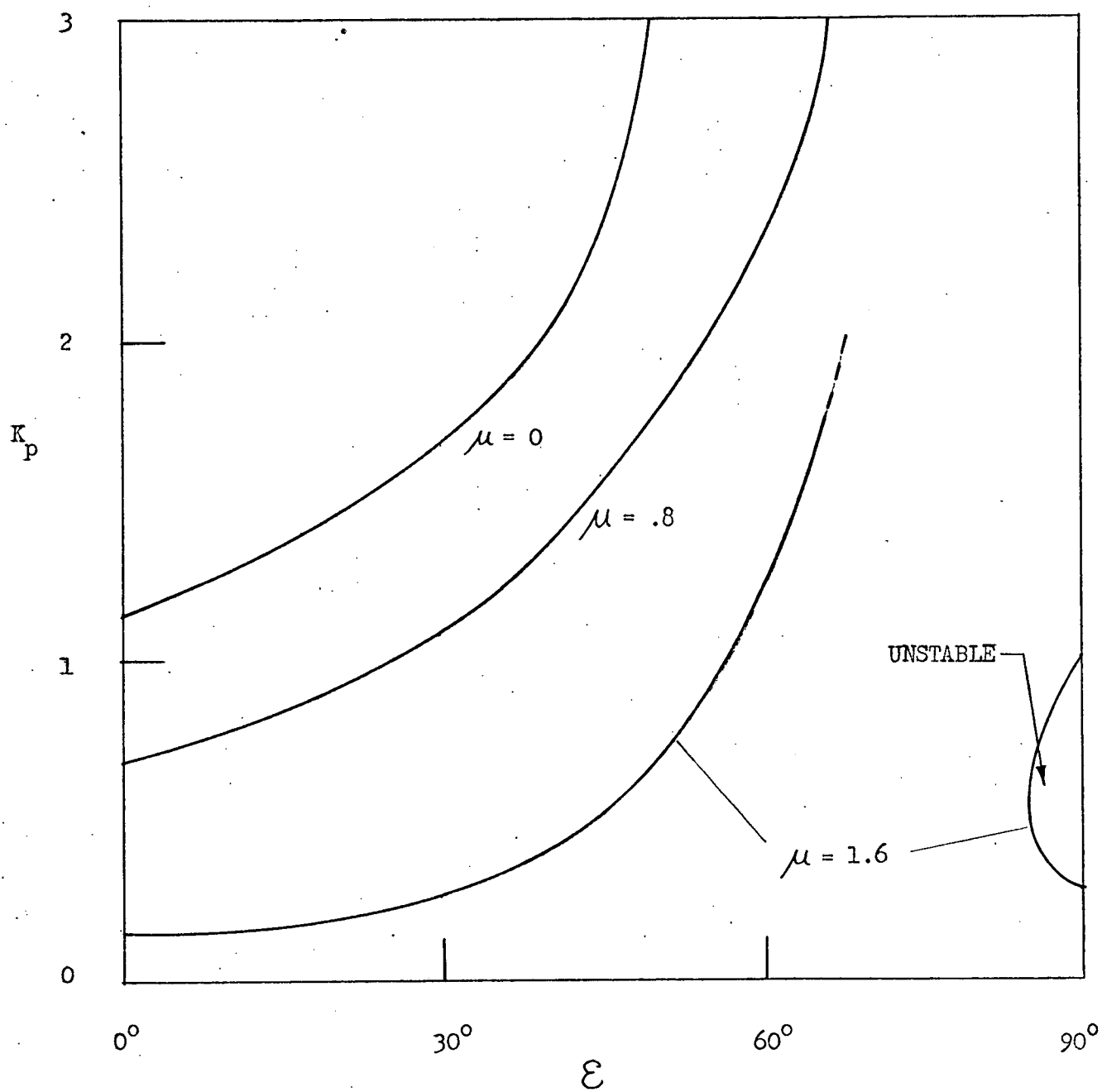


Figure 18. Effect of Phase Angle of Tilt Proportional Feedback on Stability Limit for Three-Bladed Rotor

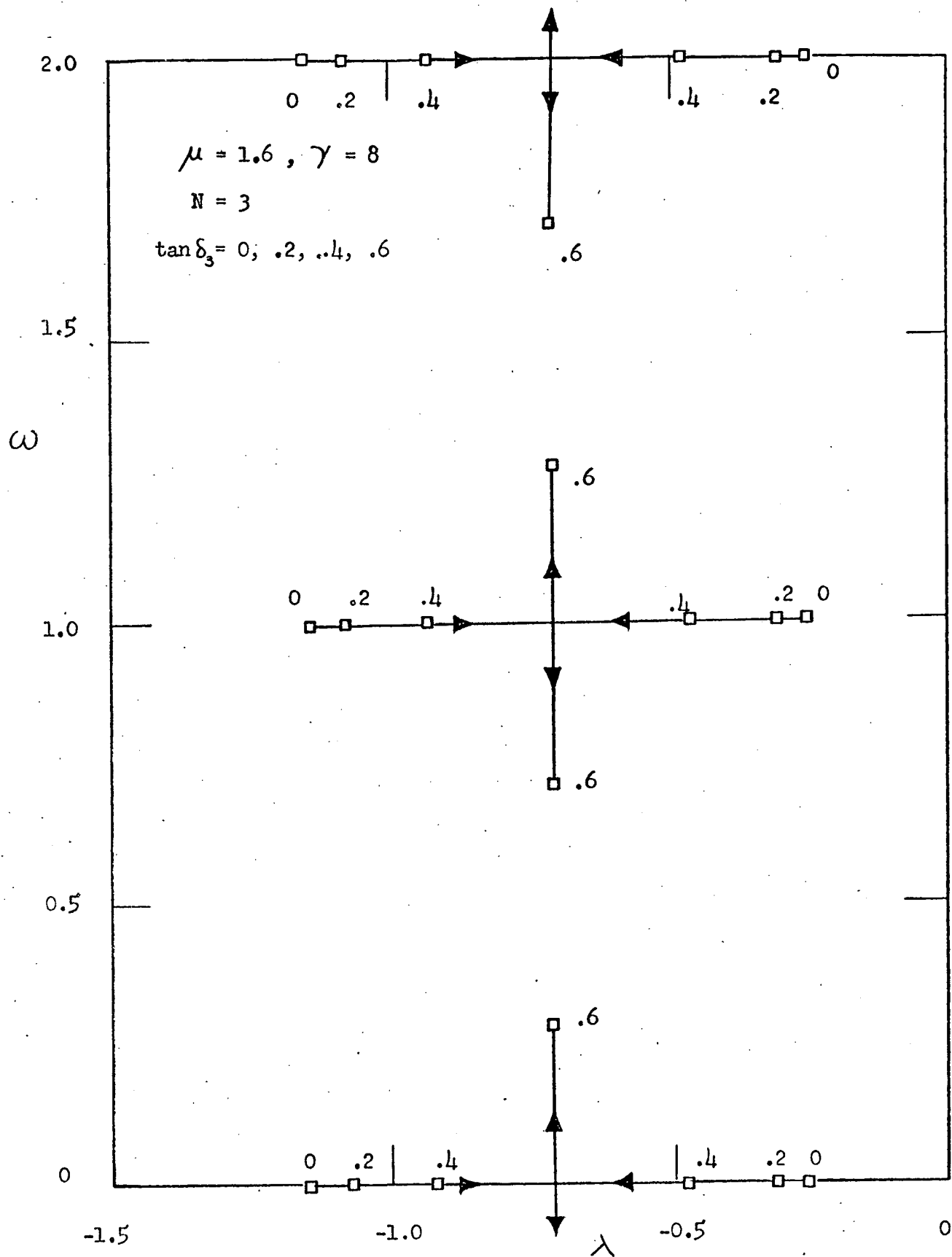


Figure 19. Stability Chart for Three-Bladed Rotor with Flapping Feedback at $\mu=1.6$

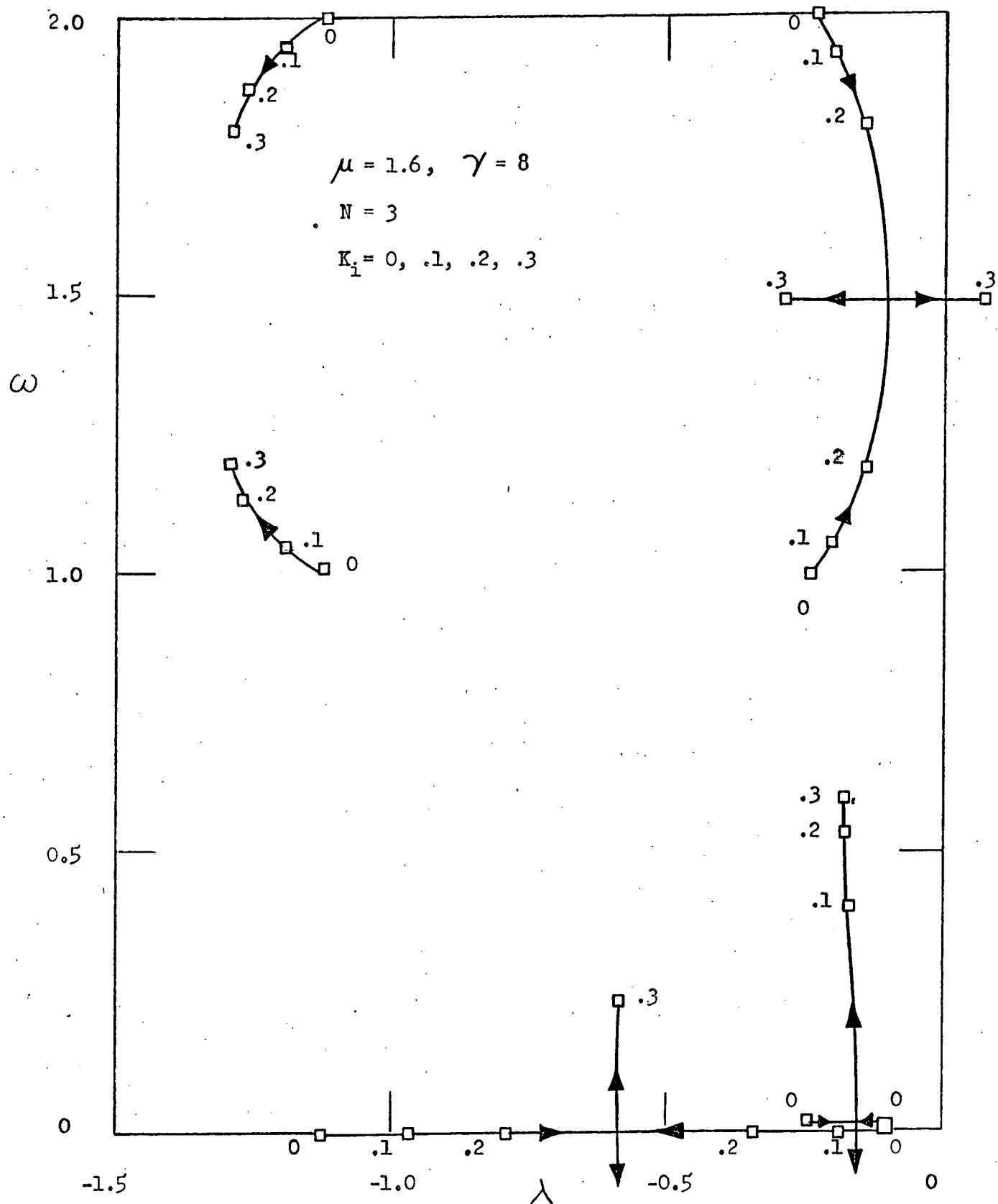


Figure 20. Stability Chart for Three-Bladed Rotor with Tilt Integral Feedback at $\mu=1.6$

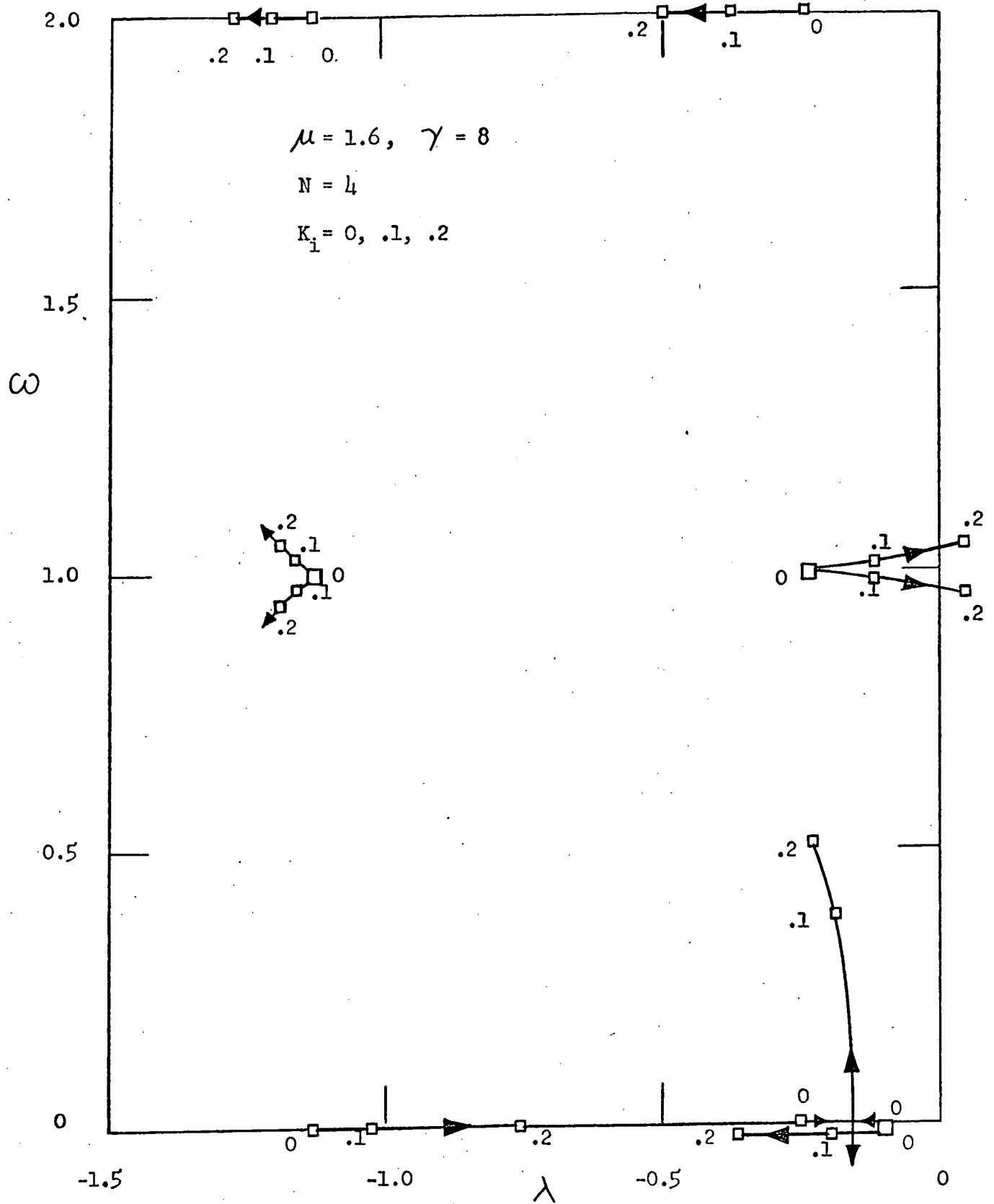


Figure 21. Stability Chart for Four-Bladed Rotor with Tilt Integral Feedback at $\mu=1.6$

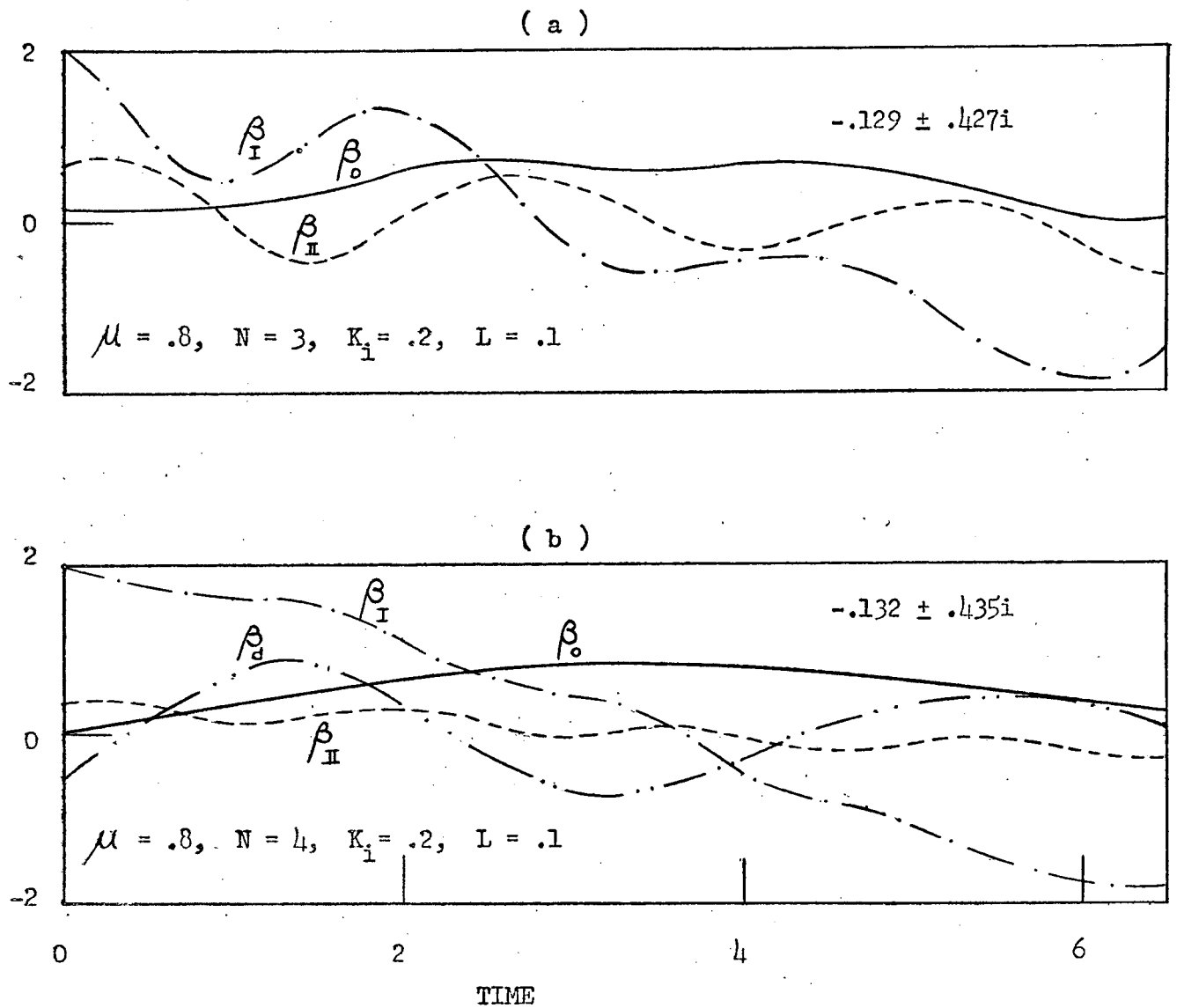


Figure 22. Comparison of Modal Functions for the Three and Four Bladed Rotors of Figures 8 and 9 --Low Frequency Modes, $K_i = .2$

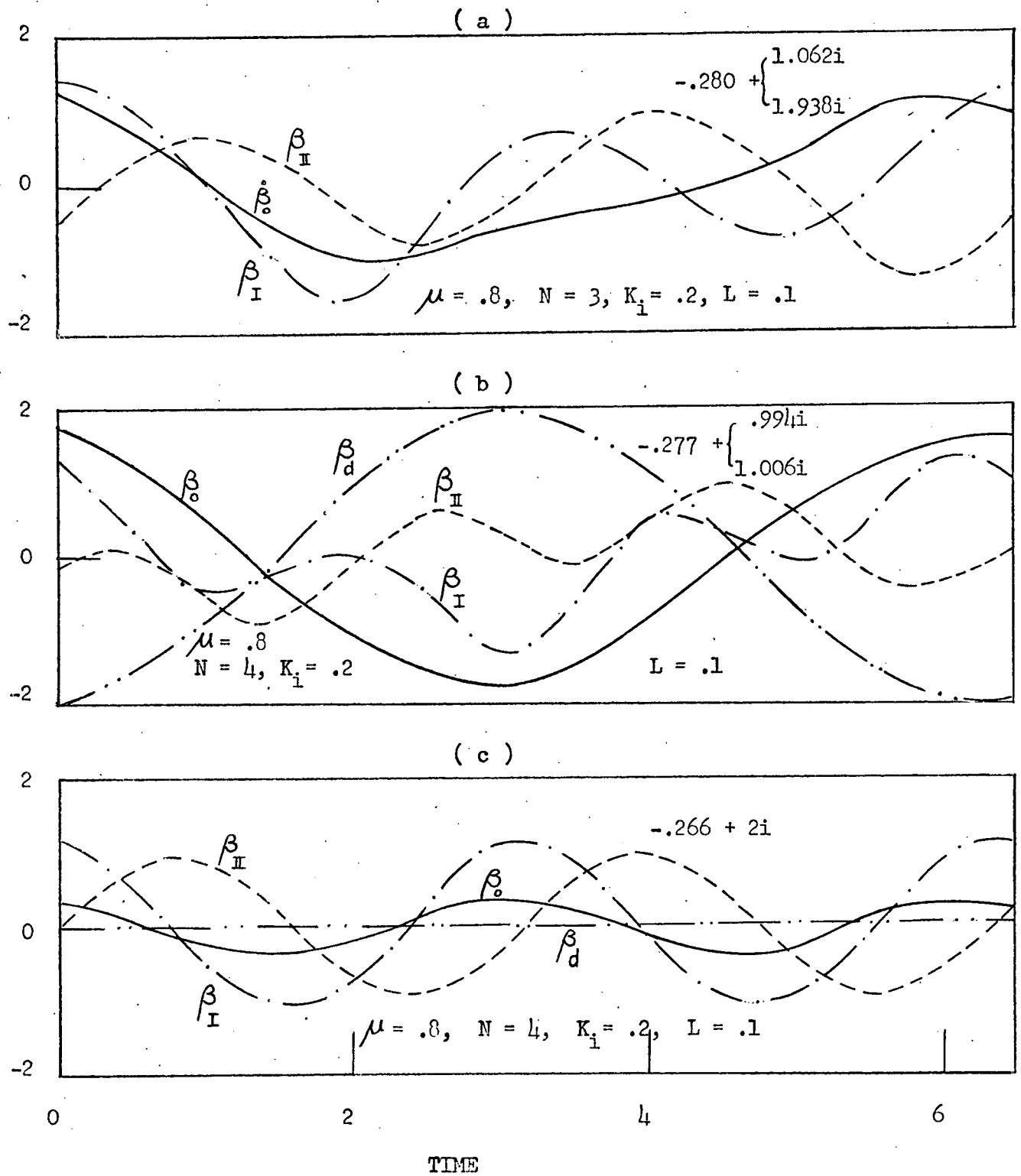


Figure 23. Comparison of Modal Functions for the Three and Four Bladed Rotors of Figures 8 and 9. -- High Frequency Modes, $K_1 = .2$

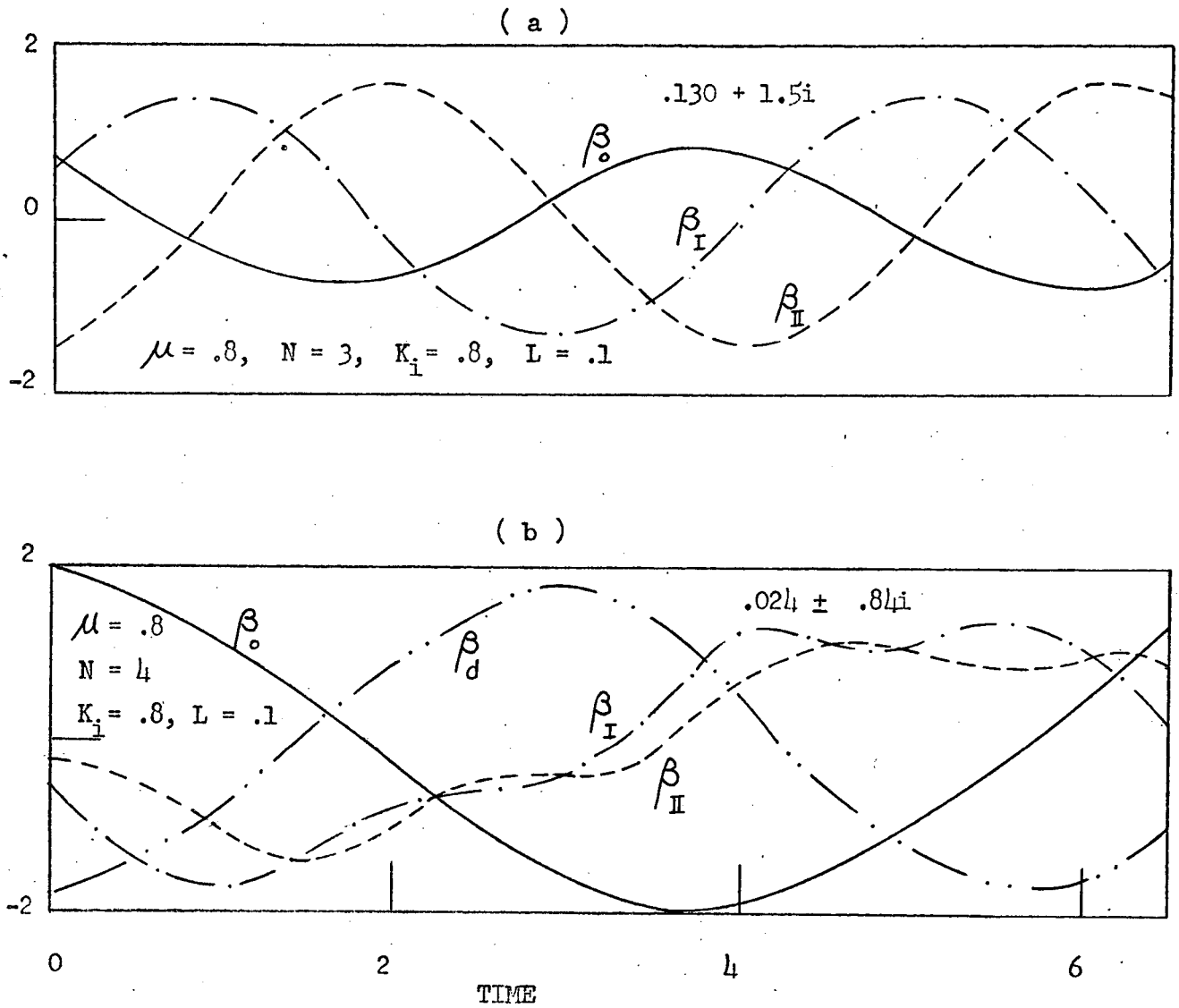


Figure 24. Comparison of Modal Functions for the Three and Four Bladed Rotors of Figures 8 and 9 -- Unstable Modes, $K_i = .8$

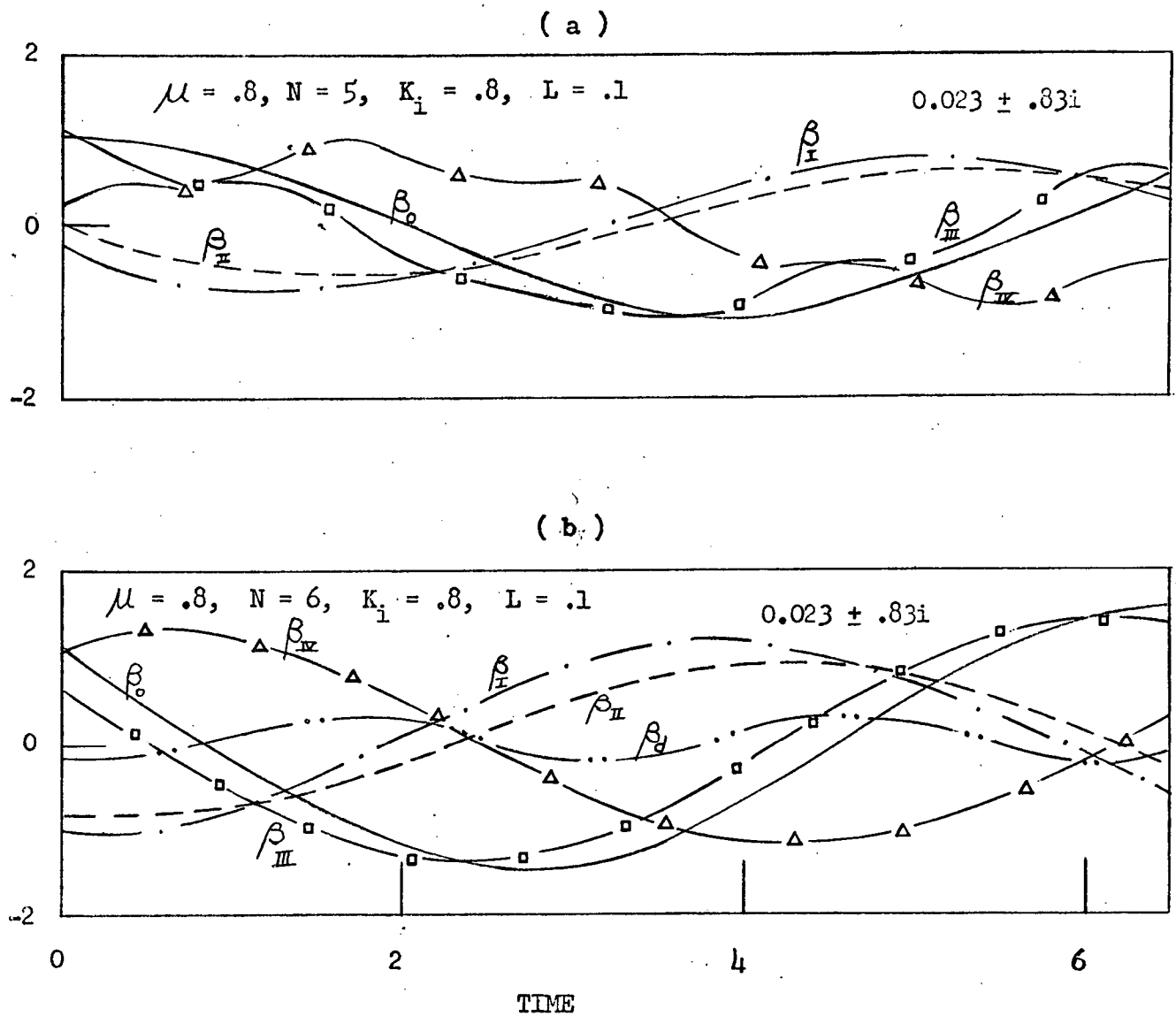


Figure 25. Comparison of Modal Functions for the Five and Six Bladed Rotors of Figures 10 and 11 - Unstable Modes, $K_i = .8$

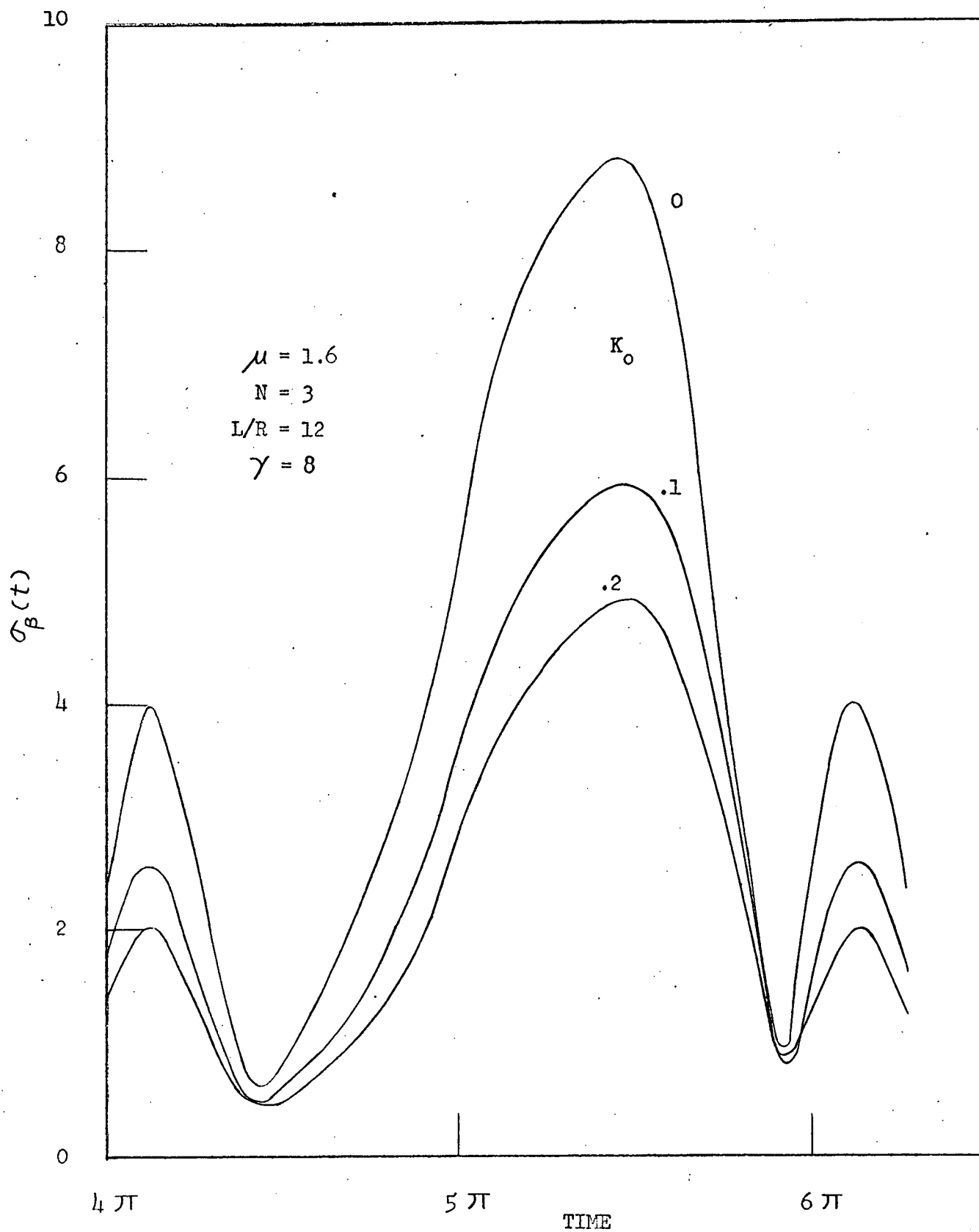


Figure 26. Effect of Coning Feedback on Random Flapping Response to Atmospheric Turbulence

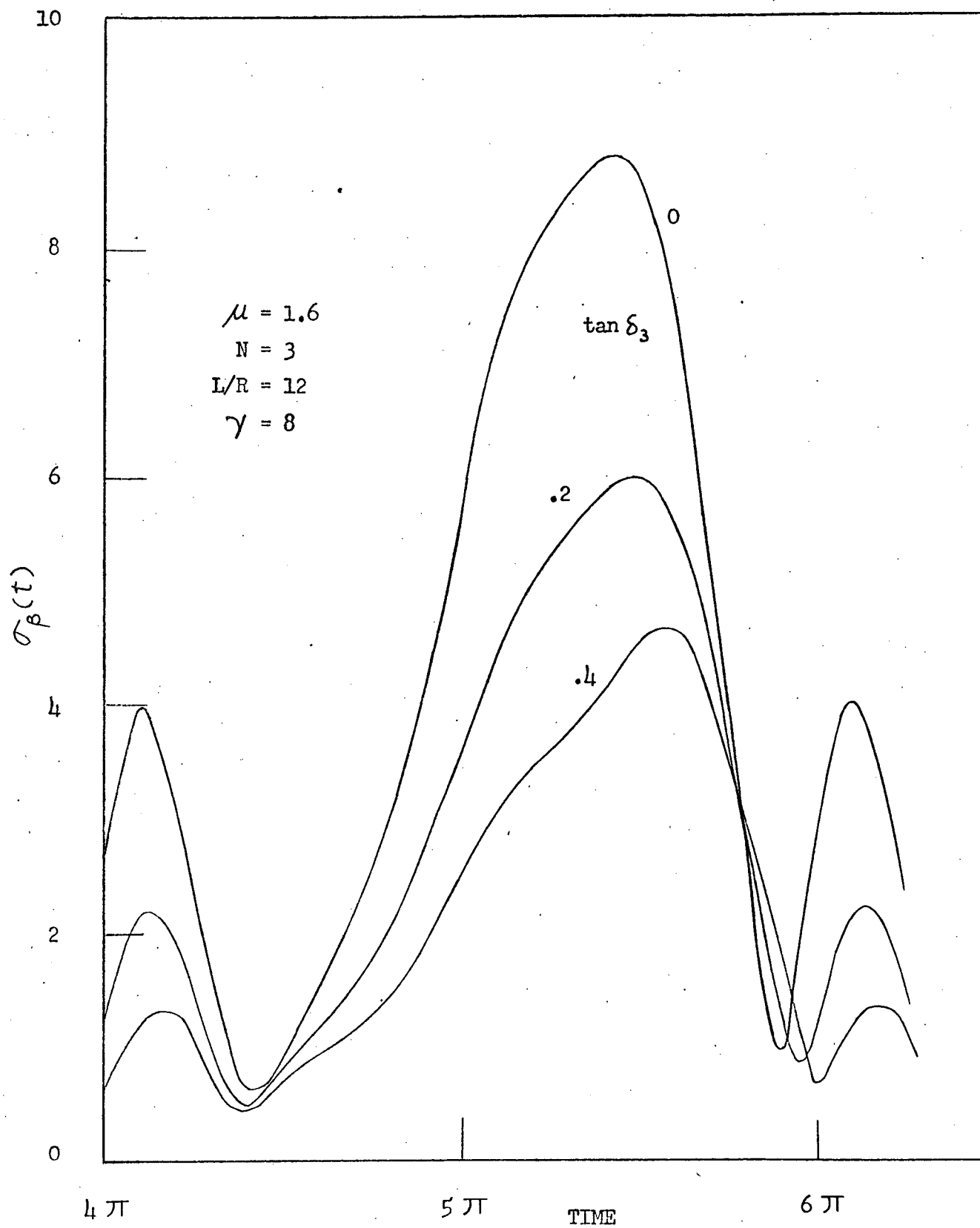


Figure 27. Effect of Flapping Feedback on Random Flapping Response to Atmospheric Turbulence

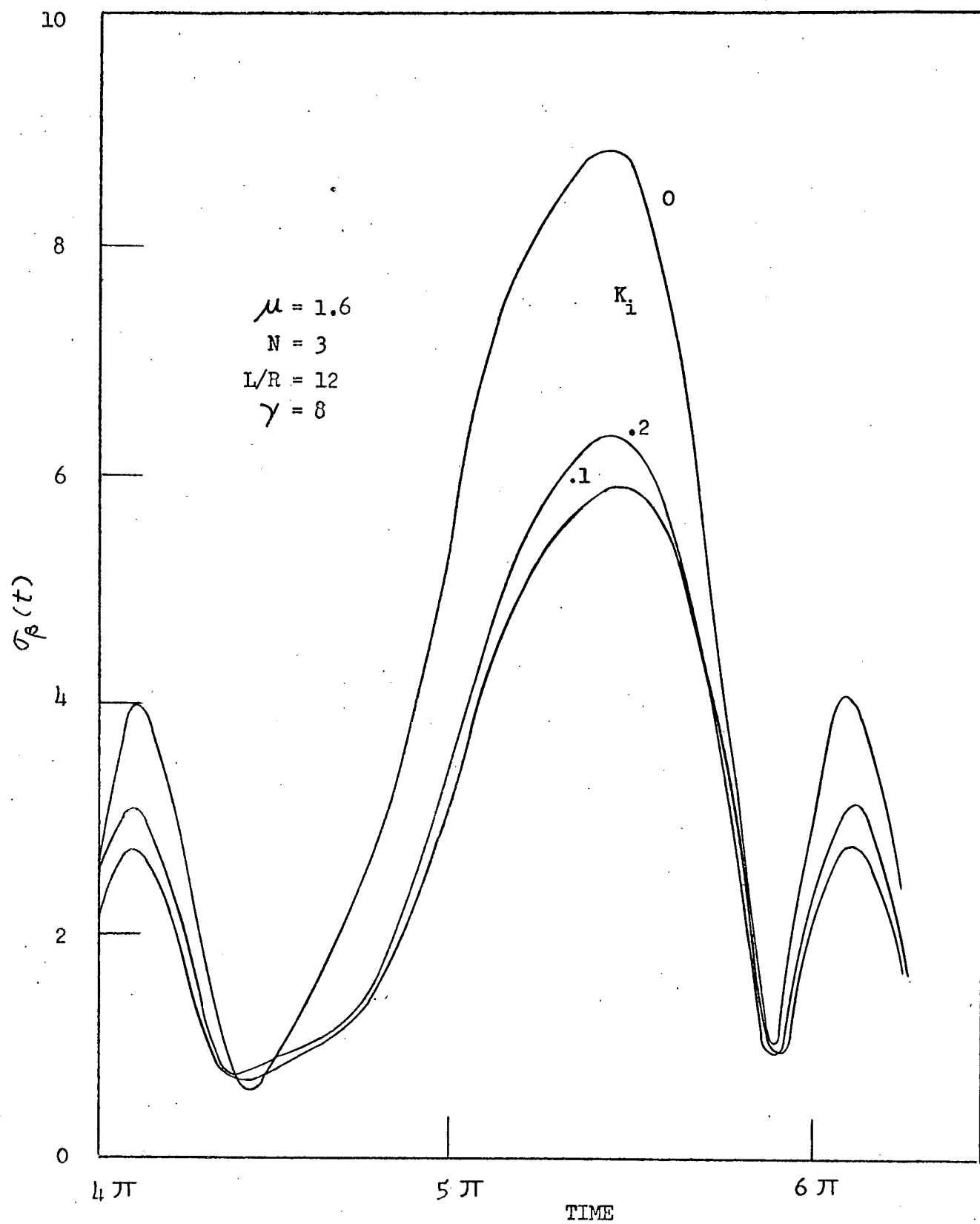


Figure 28. Effect of Tilt Integral Feedback on Random Flapping Response to Atmospheric Turbulence