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THE RELATIONSHIP BETWEEN PHASE STABILITY AND FREQUENCY STABILITY AND A METHOD OF CONVERTING BETWEEN THEM

PETER P. BOHN

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GODDARD SPACE FLIGHT CENTER
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THE RELATIONSHIP BETWEEN PHASE STABILITY AND FREQUENCY STABILITY AND A METHOD OF CONVERTING BETWEEN THEM

I. Introduction

Modern communication, navigation and tracking systems require extremely stable primary oscillators for the proper performance of their system functions.[1] Depending on the system application, the stability requirements may be specified in terms of either frequency stability or phase stability. Although both phase and frequency instabilities originate from the same physical processes, there is no algebraic relation which allows one to convert directly from a measurement of frequency stability to a number representing the phase stability. However, because it is much easier to measure time domain frequency stability (using a counter) than it is to measure the phase fluctuation spectrum (which is needed to obtain the phase stability), some method of conversion between the two stabilities would be convenient in order to determine the value of the phase stability from time domain measurements of the frequency stability.

It is the purpose of this report to present a method of obtaining the value of the phase stability from time domain frequency stability measurements; however, before one may make such a conversion, one must have a precise definition of what is meant by frequency and phase stability. These definitions are presented in Section II. Section II describes the various types of noise sources in an oscillator and how their location in the oscillator circuitry determines the resultant phase and frequency noise spectrum. With this knowledge, one may determine the type of noise spectrum from time domain frequency stability measurements. Using certain conversions, presented in Section IV, one may then obtain the total phase noise spectrum, which is integrated to obtain the phase stability. Examples of the conversion process are presented in Section V.

II. Definition of Frequency and Phase Stability

A. General Definition of Instantaneous Frequency

The instantaneous value of an arbitrary sinusoidal signal may be expressed as

$$v(t) = [V_0 + \epsilon(t)] \sin [2\pi \upsilon_0 t + \varphi(t)] \tag{1}$$

where

V_0 = nominal signal amplitude

υ = nominal signal frequency

$\epsilon(t)$ = instantaneous amplitude fluctations

$\varphi(t)$ = instantaneous phase fluctuations

The instantaneous frequency of the signal described by (1) is given by

$$\upsilon = \frac{d}{dt} \frac{1}{2\pi} [2\pi \upsilon_0 t + \varphi(t)]$$
$$= \upsilon_0 + \frac{\dot{\varphi}(t)}{2\pi} \tag{2}$$

where $\dot{\varphi}(t)$ is the time derivative of $\varphi(t)$ and is called the instantaneous frequency deviation from the nominal frequency υ_0.

If the signal v (t) is to be considered the output of a precision oscillator, the following inequalities must be satisfied.

$$\left| \frac{\epsilon(t)}{V_0} \right| << 1 \quad \text{and} \quad \left| \frac{\dot{\varphi}(t)}{2\pi\upsilon_0} \right| << 1$$

This is merely to say that the instantaneous fluctuations are small compared to their nominal values. These inequalities guarantee that the statistical processes used in characterizing the frequency and phase stability are valid.

B. Frequency Domain – Frequency Stability

By definition, let

$$y(t) = \frac{\dot{\varphi}(t)}{2\pi\upsilon_0} \tag{3}$$

As defined above, y (t) is the instantaneous fractional frequency deviation from the nominal frequency υ_0; i. e., $(\upsilon - \upsilon_0)/\upsilon_0$.

One definition of the frequency stability is the one-sided spectral density, $S_y(f)$, of the instantaneous fractional frequency fluctuations of $y(t)$. To explain what this spectrum represents consider the signal described by (1) as displayed on a spectrum analyzer. If $\epsilon(t)$ is equal to zero (i. e., no AM noise) the spectrum that one would observe would consist of a line at ν_0 (the nominal frequency) and symmetrical upper and lower sidebands representing the FM noise contribution of the function $\dot{\varphi}(t)/2\pi$. If one were to take the upper sideband, translate it to zero frequency, multiply the amplitude of this sideband by two (to account for the total power in both sidebands) and divide the amplitude by ν_0 (the normalization in (3)) then one would obtain the spectrum Sy (f).*

C. Time Domain – Frequency Stability

By definition, let

$$\bar{y}_k = \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)\,dt = \frac{\varphi(t_k+\tau) - \varphi(t_k)}{2\pi\nu_0\tau} \tag{4}$$

where $t_{k+1} = t_k + T$, $k = 1, 2, \ldots, N-1$, and T is the repetition interval for measurements of duration τ. Note that the term $[\varphi(t_k+\tau) - \varphi(t_k)]$ is the total accumulated phase in time τ, which when divided by $2\pi\tau$ gives the average frequency deviation (from ν_0) for an averaging time τ. This then is normalized to the nominal frequency ν_0. This is, in essence, the type of measurement made with frequency counters, except that the counter measures

$$\bar{\nu}_k = \frac{1}{\tau}\int \nu\,dt = \frac{2\pi\nu_0\tau + \varphi(t_k+\tau) - \varphi(t_k)}{2\pi\tau}$$

and the operator performs the mathematics to obtain $\bar{y}_k$. The repetition interval, T, of the counter is generally controlled by the operator.

The second definition of frequency stability is defined by the relation

$$\langle\sigma_y^2(N, T, \tau)\rangle = \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)^2\right\rangle \tag{5}$$

* Note that a notational distinction is being made between the instantaneous value of the signal frequency, ν, and a spectrum of Fourier frequencies, f. This distinction has been suggested by Barnes, et. al.[2]

where the brackets < > denote the infinite time average of the quantity enclosed in the brackets. Note that σ_y (N, T, τ) is the sample variance of N measurements, made at intervals of T seconds, of the τ second average fractional frequency deviation.

Since there are cases in which the result (4) may diverge with increasing N, a preferred definition is obtained by choosing $T = \tau$ and $N = 2$. This is called the Allen variance, denoted by σ_y^2 (τ), and given by

$$\sigma_y^2 (\tau) = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle . \tag{6}$$

Naturally, σ_y^2 (τ) can never be obtained exactly, since an infinite number of measurements would be required. Therefore, an estimate of σ_y^2 (τ) must be obtained by making a finite number of measurements of σ_y^2 ($2, \tau, \tau$) and averaging. Of course, the number of measurements, m, made to determine the estimate should be stated along with the estimate.

It should be obvious from the definitions of the two types of frequency stability specification that the specification which contains the most information about the FM noise in the system is the spectral density, Sy (f), definition. This is because the only physical limitation in measuring Sy (f) is the low frequency limit of the test equipment, which can be as low as hundredths of a Hertz. Also, it will be shown in Paragraph II.D. that if Sy (f) is known, $\langle \sigma_y^2 (N, T, \tau) \rangle$ may be calculated exactly. The opposite is not true. However, from a practical point of view, it is easier to make the time domain measurements on a frequency counter, than to make the FM noise spectral analysis measurements.

D. Translations Between the Time and Frequency Domains [3]

The translation between the time and frequency domain can be made by using the relation

$$\langle \sigma_y^2 (N, T, \tau) \rangle = \frac{N}{N-1} \int_0^\infty df\, S_y(f) \frac{\sin^2 (\pi f \tau)}{(\pi f \tau)^2} \left[1 - \frac{\sin^2 (N r \pi f \tau)}{N^2 \sin^2 (r \pi f \tau)} \right] \tag{7}$$

where $r = T/\tau$. For the Allen variance,

$$\sigma_y^2 (\tau) = 2 \int_0^\infty df\, S_y(f) \frac{\sin^4 (\pi f \tau)}{(\pi f \tau)^2} \tag{8}$$

E. Definition of Phase Stability

The definition of phase stability is the infinite time average of the phase variance and can be obtained from the relation

$$\langle \sigma_\varphi^2 (t) \rangle = 2 \int_0^\infty S_\varphi (f)\, df \tag{9}$$

where $S_\varphi(f)$ is the two sided spectral density of the instantaneous phase fluctuations of $\varphi(t)$. Here again, one cannot physically obtain the infinite time average of the above quantity. In practice, the integral, (8), is high and low frequency limited by the band-limit of the system and the length of the measurement time, respectively. Therefore, an estimate of the phase stability may be obtained from the definition

$$\sigma_\varphi^2 (f_L, f_H) = 2 \int_{f_L}^{f_H} S_\varphi (f)\, df \tag{10}$$

where f_L and f_H are the low and high frequency limits, respectively. Again, in specifying, or reporting, the phase stability, the values of f_L and f_H must be given for the specification to be meaningful. The accuracy of the estimate given by (10), if used in the specification of a desired system phase stability, is as good as one could desire, since f_L and f_H are functions of the system requirements. If it is necessary to extend f_L and f_H to satisfy new system requirements, one must remeasure the phase noise over the new desired bandwidth and again integrate equation (10).

F. Relations between Various Spectra[2-4]

Because several types of spectra are referred to in these definitions and in what is to follow, the definitions of these spectra and the relationships between them are given below.

1. $\mathcal{L}(f)$: the ratio of the single sideband phase noise power in a 1 - Hz. bandwidth to the signal power, as a function of the offset frequency f, and referred to a specified carrier frequency υ_0. The units are Hz^{-1}.

2. $S_\varphi(f)$: the two-sided spectral density of the instantaneous phase fluctuations of $\varphi(t)$. The units are radians squared per Hz. The positive half of the spectrum $S_\varphi(f)$ is numerically equal to $\mathcal{L}(f)$.

$$S_{\varphi}(|f|) = \mathcal{L}(f) \tag{11}$$

This difference between these two spectra is more apparent than real. $S_{\varphi}(f)$ is used mathematically by theoreticians; whereas $\mathcal{L}(f)$ is measured by experimentalists. $S_{\varphi}(f)$ is a mathematical quantity in which there is no physical significance to negative frequencies; $\mathcal{L}(f)$ is defined in terms of spectra sideband power above and below the carrier frequency and therefore, negative frequencies are meaningful.

3. $S_{\dot{\varphi}}(f)$: The two-sided spectral density of the instantaneous frequency fluctuations of $\dot{\varphi}(t)$. The units of $S_{\dot{\varphi}}(f)$ are Hz squared per Hz. The following relations hold between $\mathcal{L}(f)$, $S_{\varphi}(f)$ and $S_{\dot{\varphi}}(f)$.

$$S_{\dot{\varphi}}(f) = (2\pi f)^2 S_{\varphi}(f) \tag{12}$$

$$S_{\dot{\varphi}}(|f|) = (2\pi f)^2 \mathcal{L}(f) \tag{13}$$

4. $S_y(f)$: the one-sided spectrum of the instantaneous fractional frequency fluctuations of $y(t)$, normalized to a specified nominal frequency ν_0. The units of $S_y(f)$ are Hz^{-1}. The following relations exist between this and the other spectra.

$$S_y(f) = 2(2\pi\nu_0)^{-2} S_{\dot{\varphi}}(f) \tag{14}$$

$$S_y(f) = 2(f/\nu_0)^2 S_{\varphi}(f) \tag{15}$$

$$S_y(|f|) = (f/\nu_0)^2 \mathcal{L}(f) \tag{16}$$

III. Determination of the Spectrum of Instantaneous Phase Fluctuations From Time Domain Measurements

In order to determine the phase stability from time domain measurements, one must have a knowledge of the spectral dependence of the instantaneous phase fluctuations. A good approximation to this spectrum may be obtained by way of measurements made in the time domain if these measurements are properly interpreted. Figure 1 shows a typical plot of frequency stability, as measured in the time domain.* One can see in this figure three different regions of τ dependence: τ^0, $\tau^{-1/2}$ and τ^{-1}. The problem is to determine what type of noise spectrum results in the various dependences.

* Measurement made on Arvin Industries 5 MHz VCXO, Serial No. 701H01.

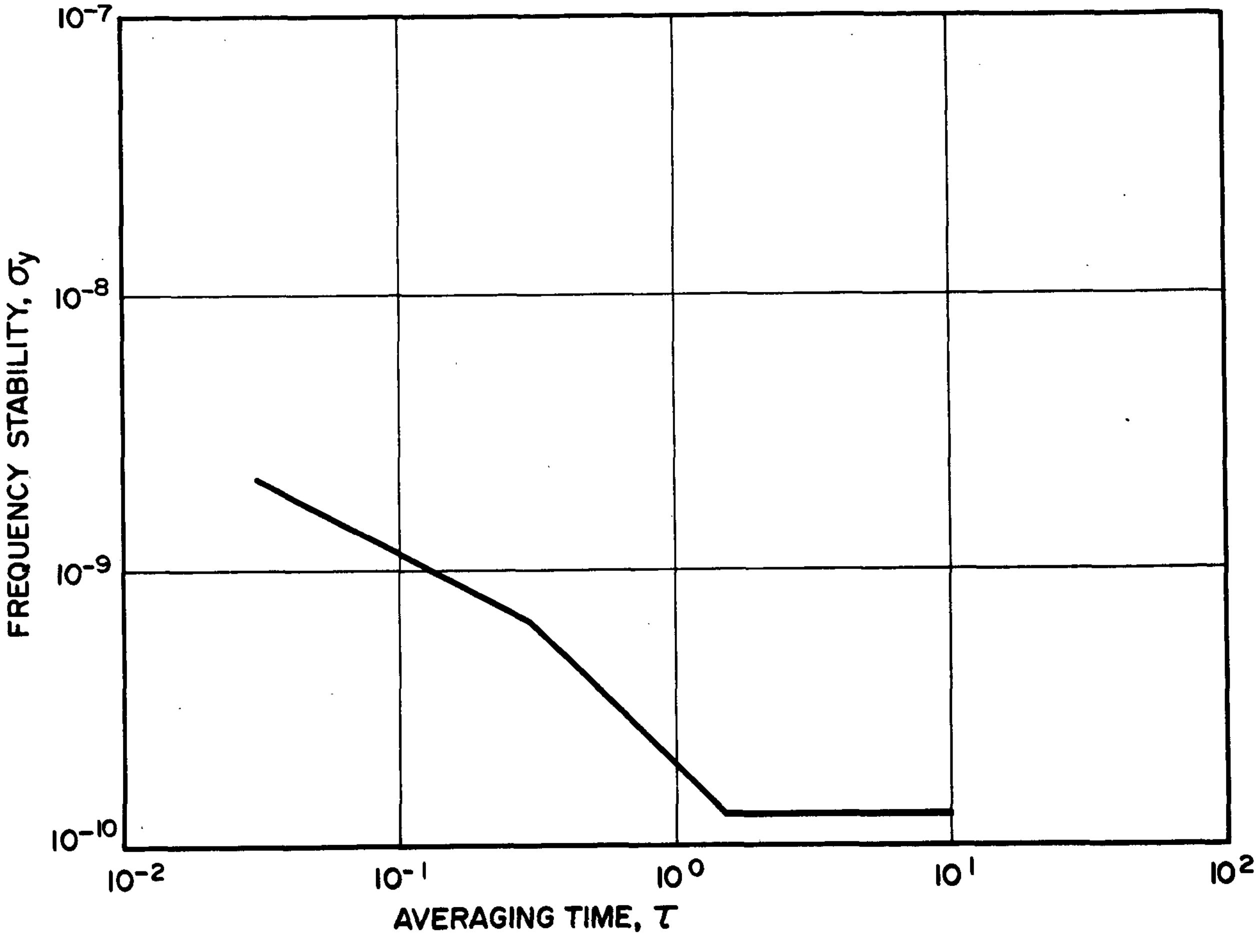

Figure 1. Time Domain Frequency Stability of a Commercial Oscillator

A. Types of Phase Fluctuation Spectral Densities[5,6]

There are four basic types of spectral densities of instantaneous phase fluctuations, each having a characteristic frequency dependence. These* four are: white noise sources outside the oscillator loop, flicker noise sources outside the loop, white noise within the loop and flicker noise within the loop.

1. White phase noise:**

$$S_\varphi(f) = K_0;\; S_{\dot\varphi}(f) = 4\pi^2 f^2 K_0;\; S_y = (f/\upsilon_0)^2 K_0$$

This spectral density results from white noise sources outside the oscillator loop. This is often called additive external noise.

2. Flicker phase noise:

$$S_\varphi(f) = K_1/f;\; S_{\dot\varphi}(f) = \pi^2 4 K_1 f;\; S_y(f) = (2f/\upsilon_0^2) K_1$$

This results from flicker (or f^{-1} type) noise outside the oscillator loop.

3. White frequency noise:

$$S_\varphi(f) = K_2/f^2;\; S_{\dot\varphi}(f) = 4\pi^2 K_2;\; S_y(f) = (2/\upsilon_0^2) K_2$$

This is the result of white noise sources within the oscillator loop. This is sometimes called internal additive noise.

4. Flicker frequency noise:

$$S_\varphi(f) = K_3/f^3;\; S_{\dot\varphi}(f) = 4\pi^2 K_3/f;\; S_y(f) = (2/f\,\upsilon_0^2) K_3$$

Flicker (or f^{-1}) noise sources within the oscillator loop result in this spectral density.

One comment should be made concerning the above before proceeding. Most circuit elements which may act as noise sources can easily be distinguished as being inside or outside the oscillator loop; however, one important exception

* These divisions are actually simplifications; however, their use is standard.

** The K_i ($i = 0, 1, 2, 3$) are constant.

to this is an element across the output of the loop, a capacitor for instance, contributes to both internal and external additive noise. The type of noise which is dominant in this case depends on the circuit.

B. Time Domain Frequency Stability for K_n/f^n Type

Phase Spectra

The following will be presented in this section*.

i. The sample variance (σ_y^2 (N, T, τ)) for each K_n/f^n, with the spectra sharply cut-off at the upper frequency f_H.

ii. The Allen variance $\sigma_y^2(\tau)$ for each K_n/f^n, with the spectra sharply cut-off at f_H.

iii. The Allen variance $\sigma_y^2(\tau)_F$ for each K_n/f^n, low pass filtered by $[1 + (f/f_H)^2]^{-1}$, i. e.

$$S_{\dot{\varphi}}(f) \rightarrow S_{\dot{\varphi}}(f)/[1 + (f/f_H)^2]$$

This term gives a somewhat more realistic description of the effects of tuned circuit filtering than the sharp cut-off description.

These are obtained from (7) and (8).

1. White phase noise ($2\pi f_H \tau >> 1$, except as noted)

$$\text{i.} \quad \langle \sigma_y^2 (N, T, \tau) \rangle = \frac{N + S_k (r - 1)}{\pi^2 N} \quad \frac{f_H}{\tau^2 \nu_0^2} K_0 \tag{17}$$

$$\text{ii.} \quad \sigma_y^2(\tau) = \frac{3 f_H}{2 \pi^2 \nu_0^2 \tau^2} K_0 \tag{18}$$

$$\text{iii.} \quad \sigma_y^2(\tau)_F = \frac{3 f_H}{2 \pi \nu_0^2 \tau^2} K_0 \qquad 2\pi f_H \tau >> 1 \tag{19}$$

$$= \left(\frac{f_H}{\nu_0}\right)^2 \frac{K_0}{\tau} \qquad 2\pi f_H \tau >> 1 \tag{20}$$

* The first two of these have been calculated by Barnes, et. al[2]

This is an important case for high quality oscillators. The break point between the two cases occurs at $\tau = 1/2\pi f_H$.

2. Flicker phase noise ($2\pi f_H \tau >> 1$).

i.
$$\langle \sigma_y^2 (N, T, \tau) \rangle = \frac{K_1}{(\pi \tau \nu_0)^2} \Big\{ 2 + \ln (2\pi f_H \tau) + N(N-1) \sum_{n=1}^{N-1} (N-n) \ln \left[\frac{n^2 r^2}{n^2 r^2 - 1} \right] \Big\} \quad (21)$$

for $r >> 1$

ii.
$$\sigma_y^2 (\tau) = \frac{K_1}{2(\pi \tau \nu_0)^2} \{3 [2 + \ln (2\pi f_H \tau)] - \ln 2\} \quad (22)$$

iii.
$$\sigma_y^2 (\tau)_F = \frac{K_1}{2(\pi \tau \nu_0)^2} \left\{ 3 [2 + \ln (2\pi f_H \tau)] + \left(\frac{16}{\pi f_H \tau} - 1 \right) \ln 2 \right\} \quad (23)$$

3. White frequency noise

i.
$$\langle \sigma_y^2 (N, T, \tau) \rangle = \frac{K_2}{\nu_0^2 \tau}; \; r \geq 1 \quad (24)$$

$$= \frac{K_2}{3 \nu_0^2 \tau} r (N+1); \; N r \leq 1 \quad (25)$$

ii.
$$\sigma_y^2 (\tau) = \frac{K_2}{\nu_0^2 \tau} \quad (26)$$

iii.
$$\sigma_y^2 (\tau)_F = \frac{K_2}{\nu_0^2 \tau}; \; 2\pi f_H \tau >> 1 \quad (27)$$

4. Flicker frequency noise

i. $$\langle \sigma_y^2 (N, T, \tau) \rangle = \frac{2 K_3}{\upsilon_0^2} \frac{1}{N (N-1)} \sum_{n=1}^{N} (N - n)$$

$$\cdot [-2 (n r)^2 \ell n (n r) + (n r + 1)^2 \ell n (n r + 1) + (n r - 1)^2 \ell n [n r - 1] \tag{28}$$

ii. $$\sigma_y^2 (\tau) = \frac{4 \ell n^2}{\upsilon_0^2} K_3 \tag{29}$$

iii. $$\sigma_y^2 (\tau)_F = \frac{4 \ell n^2}{\upsilon_0^2} K_3 \tag{30}$$

Note: Conversion from the constant h; in Barnes, et. al.[2] may be accomplished by

$$K_n = 1/2\, \upsilon_0^2\, h_{2-n}$$

The results compiled above are summarized in Figure 2. (Note that S_y (f)~ $S_{\dot{\varphi}}$ (f).) One can see that the τ dependence of σ_y (τ) is not a unique function of the spectral density involved; however, in general, there need not be any major difficulty in determining the noise spectrum resulting in a given τ dependence. The τ^0 occurs only once; therefore, there is no ambiguity as to its origin: S_φ (f) = f^{-3}. The $\tau^{-1/2}$ dependence occurs twice: once for white phase noise and once for white frequency noise. In practice one usually observes the $\tau^{-1/2}$ dependence followed by τ^{-1} dependence (see Figure 1), which identifies both regions as white phase noise. A $\tau^{-1/2}$ dependence following a τ^{-1} dependence is in general caused by white frequency noise. Measurements made on passive atomic frequency standards typically display a $\tau^{-1/2}$ dependence following a τ^0 dependence. Here, the $\tau^{-1/2}$ dependence can be identified as white frequency noise from the atomic reference.

The only confusion which may appear in attempting to identify the spectral density resulting in a particular τ dependence comes from the τ^{-1} dependence. This can result from either white phase noise or flicker phase noise. The noise source in this case can be distinguished by an additional measurement in the time domain.

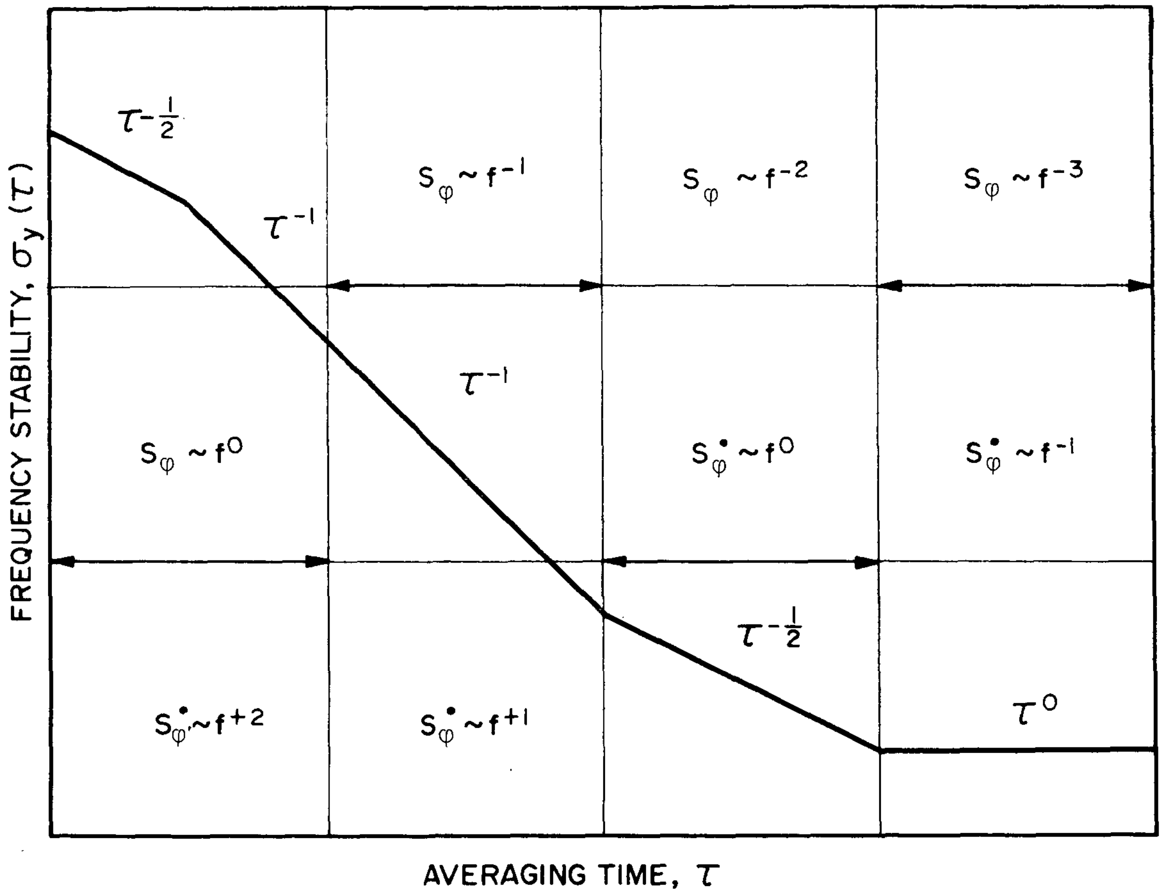

Figure 2. Example of Time Domain Frequency Stability Variations with Respect to Averaging Time for the Four Basic Types of Noise Frequency Spectra

This measurement involves remeasuring $\sigma_y(\tau)$ with a different output filter bandwidth than in the original measurement. The effects of changing the system bandwidth will determine noise source spectral density type. If $\sigma_y^2(\tau)$ goes as $\ell n(f_H)$, then the noise is flicker phase noise, and if $\sigma_y^2(\tau)$ goes as f_H, then the noise is white phase noise.

IV. Determination of Phase Stability From Time Domain Measurements of Frequency Stability

Once the noise source spectral density has been determined from the time domain measurements of frequency stability, one need only determine the frequencies at which the spectra of the various noise sources intercept each other (this may be done either graphically or mathematically), and the value of the constant multiplier in each of the spectral descriptions. The value of the constan multiplier can be found from the following relations, if the Allen variance is measured. (The Hewlett-Packard Computing Counter, HP-5360A, approximately measures the Allen variance. For this system $T \neq \tau$; however, $r = T/\tau \simeq 1$ and the error between its measure and the Allen variance is negligible.)

1. White phase noise

$$K_0 = \frac{2\pi \nu_0^2 \tau^2 \sigma_y^2}{3 f_H} \tag{31}$$

2. Flicker phase noise

$$K_1 = \frac{2\pi^2 \tau^2 \nu_0^2 \sigma_y^2}{3[2 + \ell n(2\pi f_H \tau)] - \ell n 2} \tag{32}$$

3. White frequency noise

$$K_2 = \nu_0^2 \tau \sigma_y^2 \tag{33}$$

4. Flicker frequency noise

$$K_3 = \frac{\nu_0^2 \sigma_y^2}{4 \ell n 2} \tag{34}$$

Having this information one may now plot the spectral density of the instantaneous phase fluctuations, $S_\varphi(f)$. Integration of the spectral density $S_\varphi(f)$, using (10), gives the value of the phase variance.

The analytical solutions to this integral for the four types of phase noise spectral densities are listed below:

1. White phase noise

$$\sigma_\varphi^2 (f_L, f_H) = 2 K_0 (f_H - f_L) \tag{35}$$

2. Flicker phase noise

$$\sigma_\varphi^2 (f_L, f_H) = 2 K_1 \ln\left(\frac{f_H}{f_L}\right) \tag{36}$$

3. White frequency noise

$$\sigma_\varphi^2 (f_L, f_H) = 2 K_2 \left(\frac{f_H - f_L}{f_H f_L}\right) \tag{37}$$

4. Flicker frequency noise

$$\sigma_\varphi^2 (f_L, f_H) = K_3 \left[\frac{f_H^2 - f_L^2}{(f_H f_L)^2}\right] \tag{38}$$

If the output is filtered such that

$$S_\varphi(f) \to S_\varphi(f) \Big/ \left[1 + \left(\frac{f}{f_H}\right)^2\right]$$

these results become

1. White phase noise

$$\sigma_\varphi^2 (f_L, f_H) = 2 \pi K_0 \left(f_H - \frac{f_L}{\pi}\right) \tag{39}$$

2. Flicker phase noise

$$\sigma_\varphi^2 (f_L, f_H) = K_1 \ln \left[1 + \left(\frac{f_H}{f_L}\right)^2\right] \tag{40}$$

3. White frequency noise

$$\sigma_\varphi^2 (f_L, f_H) = \begin{cases} \dfrac{2 K_2}{f_L} \left[1 + \dfrac{f_L}{f_H} \tan^{-1} \dfrac{f_L}{f_H} - \dfrac{\pi}{2} \dfrac{f_L}{f_H}\right] & (41) \\[2ex] \simeq \dfrac{2 K_2}{f_L} \qquad f_H >> f_L & (42) \end{cases}$$

4. Flicker frequency noise

$$\sigma_\varphi^2 (f_L, f_H) = K_3 \left\{\frac{1}{f_L^2} - \ln \left[1 + \left(\frac{f_H}{f_L}\right)^2\right]\right\} \tag{43}$$

$$\simeq \frac{K_3}{f_L^2} \tag{44}$$

V. Examples

Consider the frequency stability measurements shown in Figure 1. In the region 10 ms $\leq \tau \leq 1.5$ sec. one has the $\tau^{-1/2}$ followed by τ^{-1} type dependence, which is characteristic of white phase noise. The bandwidth of the oscillator may be determined from the break frequency between the two dependences.

That is

$$f_H = 1/2\pi\tau; \quad \tau = 300 \text{ ms}$$
$$= 0.53 \text{ Hz}$$

Now, using (31) one obtains

$$K_0 = \frac{2\pi \cdot (5\times 10^6)^2 \cdot (1)^2 \cdot (2\times 10^{-10})^2}{3\cdot (0.53)}$$

$$= 3.97 \times 10^{-6}$$

Above 1.5 seconds the τ dependence is τ^0 which is flicker frequency noise. Using (34),

$$K_3 = \frac{(5\times 10^6)^2\ (1.3\times 10^{-11})^2}{4\ \ell n\ 2}$$

$$= 3.52 \times 10^{-7}$$

The intersection frequency between these two noise spectra may be found by using

$$K_0 = \frac{K_3}{f^3}$$

or

$$f = \sqrt[3]{\frac{K_3}{K_0}} = 0.045\ \text{Hz}$$

The spectral density of the phase fluctuations is then:

$$S_\varphi(f) = 3.52\times 10^{-7}/f^3,\ 0 \le f \le 0.045$$

$$= 3.97\times 10^{-6}, \qquad 0.045 \le f \le 0.53$$

$$= 3.97\times 10^{-6}/f^2,\ 0.053 \le f$$

The third term here results from the filtering action of the oscillator output. Actually, the last two terms can be combined by applying a low pass filter to the white phase noise spectrum, i. e.

$$S_\varphi(f) = \frac{3.97\times 10^{-6}}{1+(f/.53)^2},\ 0.045 \le f$$

These results are shown in Figure 3.

Now using (38) and (39)

$$\sigma_{\varphi}^2 (f_L, 0.53) = 3.52 \times 10^{-7} \left[\frac{(0.53)^2 - f_L^2}{(0.53\, f_L)^2}\right]$$

$$+ 2\pi \times 3.47 \times 10^{-6} \left(0.53 - \frac{0.045}{\pi}\right)$$

$$= 3.52 \times 10^{-7} \left[\frac{(0.53)^2 - f_L^2}{(0.53\, f_L)^2}\right] + 1.32 \times 10^{-5}$$

Note that this result is still dependent on what the lower frequency limit is. The ultimate application for which the oscillator is intended usually determines f_L. However, as an example, let $f_L = 0.01$ Hz, then

$$\sigma_{\varphi}^2 (0.01, 0.53) \simeq 3.51 \times 10^{-3}$$

Another example of this type of conversion can be illustrated by measurements made on the Nimbus clock. The solid line in Figure 4 shows the results of the time domain frequency stability measurements made on this oscillator. The τ^{-1} region of these measurements, however, represents the limit of the measuring capabilities of the equipment. Therefore, the frequency spectrum of the phase fluctuations was measured directly. These results are shown as the solid line in Figure 5. The lowest offset frequency measurable was 5 Hz; however, the low frequency information is contained in the τ^0 dependent region of the time domain measurements. This region represents flicker frequency noise. Using (34) and $\upsilon_0 = 3.2$ MHz,

$$K_3 = \frac{(3.2 \times 10^6)^2 \cdot (2.5 \times 10^{-10})^2}{4 \ln 2}$$

$$= 5.37 \times 10^{-7}$$

This noise spectrum is plotted as the slashed line in Figure 5, and the sum of the two spectra is the total phase noise frequency spectrum. Now using the measured phase fluctuation spectrum for the flicker phase noise, one may calculate the time domain frequency stability for this from (3), assuming that the half bandwidth involved is the half bandwidth of the counter ($\sim$ 8KHz).

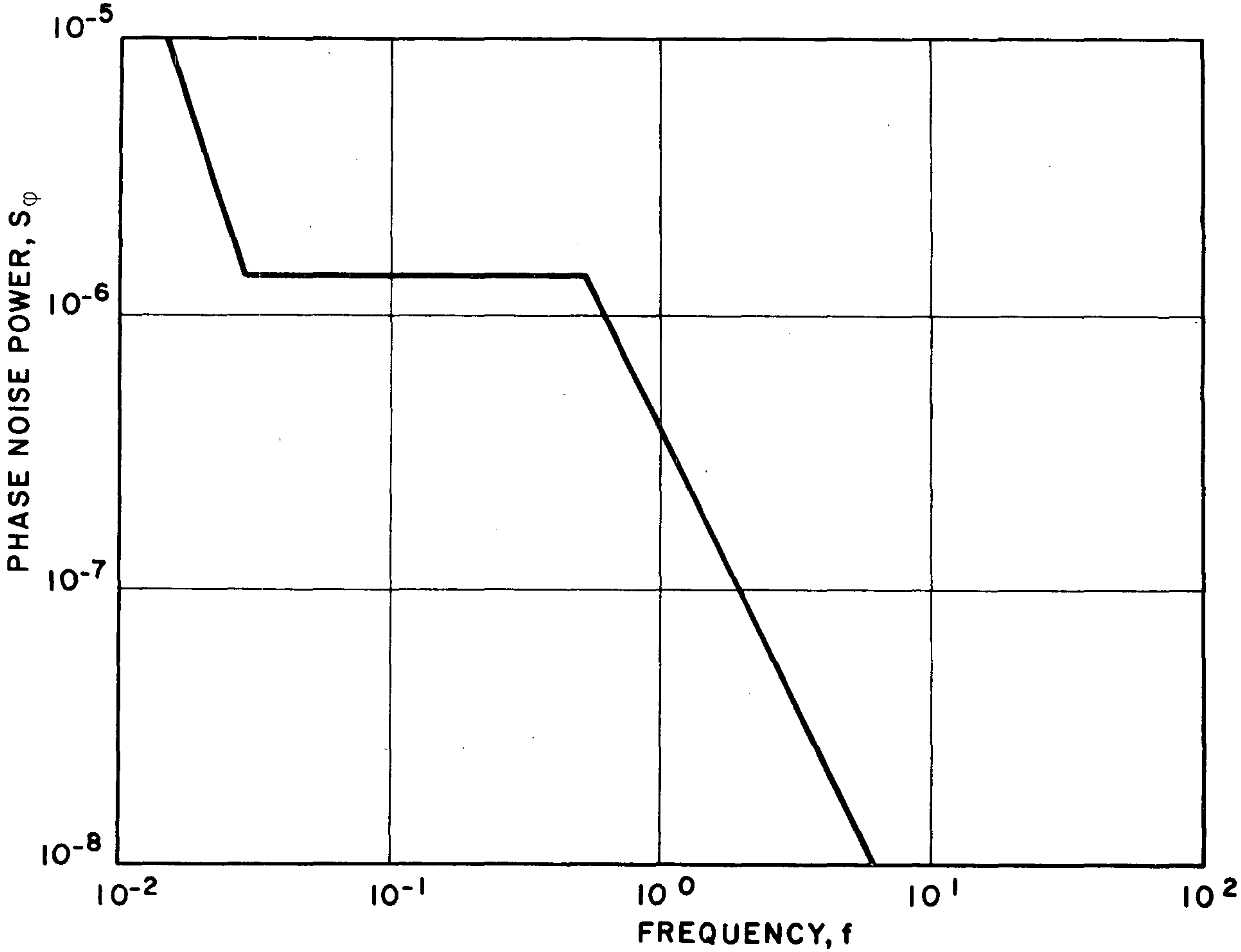

Figure 3. The Spectrum of Phase Noise Fluctuations as Derived from the Time Domain Frequency Stability Measurement Shown in Figure 1

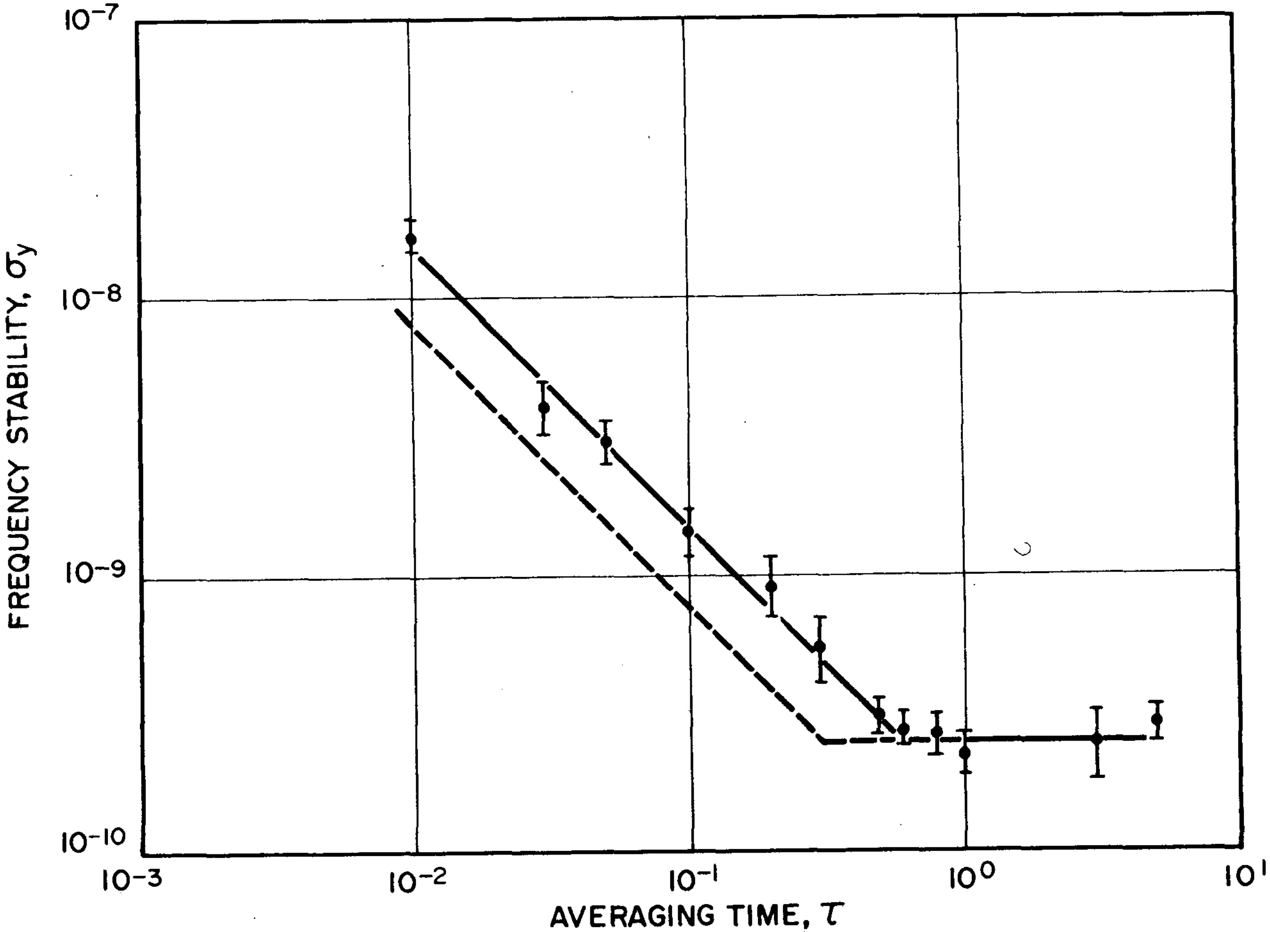

Figure 4. Time Domain Frequency Stability of Nimbus Clock Reference Oscillator (Solid Line-Measured Data, Limited by Measuring Equipment Stability-Dashed Line-Calculated Frequency Stability)

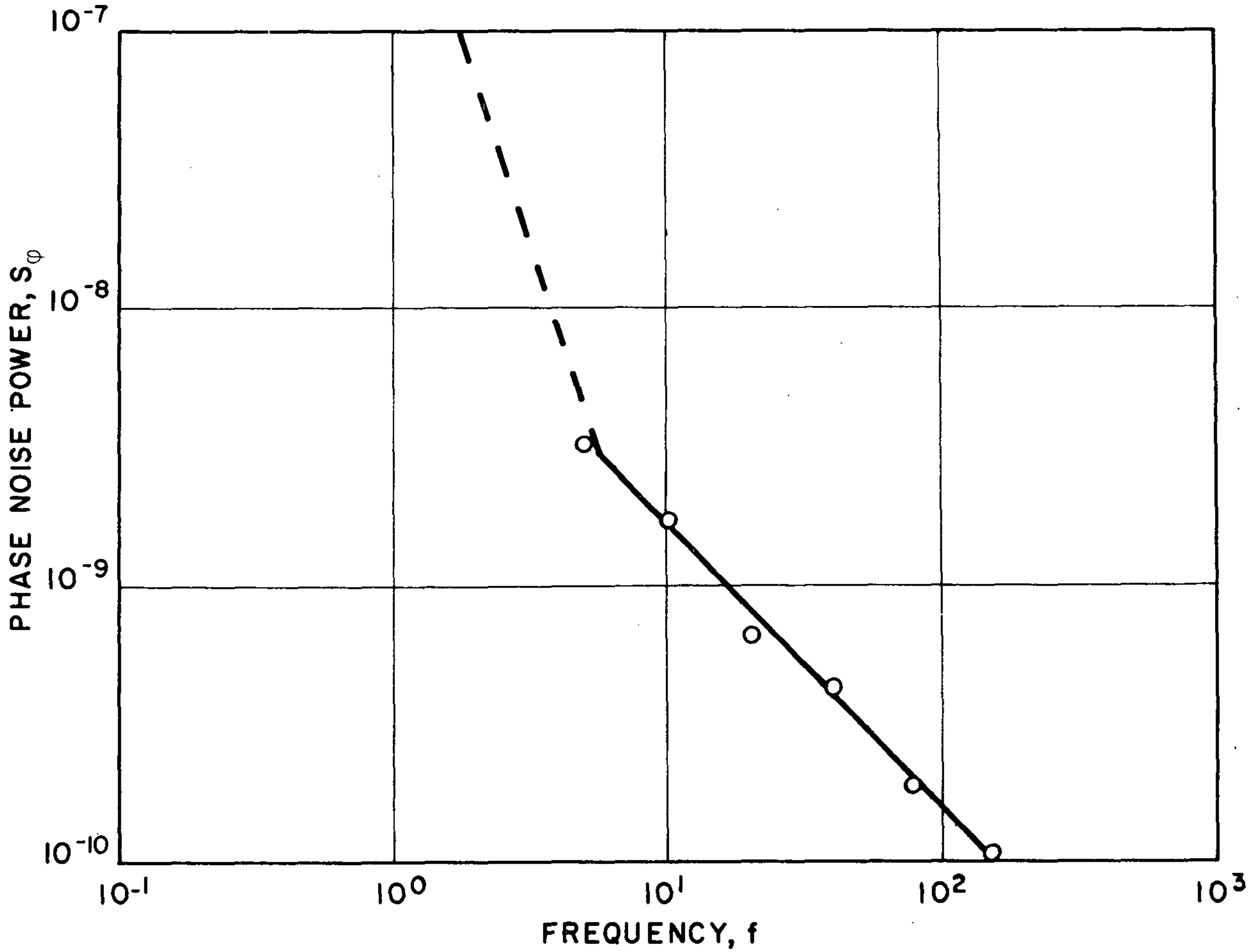

Figure 5. Phase Noise Spectra (Solid Line-Measured Data, Dashed Line-Spectra Calculated from the τ^{0} Portion of Figure 4)

$$\sigma_y(\tau) = \frac{(1.6 \times 10^{-8})^{1/2}}{\tau\,(\pi \times 3.2 \times 10^{6})}$$

At $\tau = 1$ sec.

$$\sigma_y(\tau = 1) = 7.85 \times 10^{-11}$$

and

$$\sigma_y(\tau) = (7.85 \times 10^{-11})/\tau$$

The actual time domain frequency stability is shown as the dashed line in Figure 4.

VI. Conclusions

It has been shown that conversions between frequency stability and phase stability are possible, provided that (1) the exact definitions of frequency stability and phase stability are properly understood and (2) measurements of frequency stability made in the time domain are correctly interpreted. It is important to observe that the practical measure of frequency stability in the time domain depends on: (1) N, the number of samples used to determine the variance of the instantaneous frequency fluctuations; (2) T, the period from the start of one sample to the start of the next; (3) τ, the duration of each sample measurement; (4) m, the number of measured variances used to compute the estimated average variance; and (5) f_H, the system half-bandwidth. The value of these quantities should be included in both the specification and reporting of frequency stability requirements. The exact value of these quantities will often be a function of the system performance requirements; however, in general, the Allen variance ($N = 2$, $T = \tau$) is the preferred measure. The quantities of importance to the value of the phase stability are the lower and upper cutoff frequencies, which are usually determined by system requirements.

When the spectrum of either the instantaneous frequency or phase fluctuations is measured knowledge of both spectra is obtained, since the two are related by $(2\pi f)^2$. The phase fluctuation spectrum may be integrated directly to obtain the phase stability. The frequency domain measure of the frequency stability is the spectrum of the instantaneous frequency fluctuations divided by $2\pi\upsilon_0$, where υ_0 is the nominal oscillator frequency. The time domain measure of the frequency stability can be obtained by integrating the spectrum of instantaneous frequency fluctuations multiplied by the proper weighting function.

Conversion from frequency stability measurements made in the time domain to a value of the phase stability is the most difficult procedure to accomplish. The conversion requires the determination of the frequency dependence of the spectrum of instantaneous phase fluctuations from the time dependence of the frequency stability. Since the relationship between the two is not always unique, some ambiguity may arise. It has been shown that the ambiguity may be reduced by additional time domain measurements and/or knowledge of the type of oscillator system. Once the frequency dependence of the phase fluctuations has been determined, the approximate value of phase fluctuation spectrum can be calculated from the formulas given in the text. The phase stability is then obtained from the phase fluctuation spectrum by integration. This method has been demonstrated with several examples.

The only inaccuracies in the method of converting between frequency and phase stability come about through the sharp intersections between regions of different frequency dependence in the spectra descriptions of the noise. Naturally in reality there is a smooth transition from one region to the next, without the sharp intersections as in Figure 3. It is expected that the resultant error is no more than 5%.

VII. Acknowledgments

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References

1. For example: "System Study for the Random Access Measurement System (RAMS)," A. E. Arndt, J. L. Burgess, and D. L. Reed, NASA Goddard Space Flight Center Report, S-752-70-376, October, 1970.

2. "Characterization of Frequency Stability," J. A. Barnes, et. al., IEEE Trans. Instrumentation and Measurement, vol. IM-20, pp. 105-120, May 1971.

3. "Some Aspects of the Theory and Measurement of Frequency Fluctuations in Frequency Standards," L. S. Cutler and C. L. Searle, Proc. IEEE, vol. 54, pp. 136-154, February, 1966.

4. "A Test Set for the Accurate Measurement of Phase Noise on High-Quality Signal Sources," D. G. Meyer, IEEE Trans. Instrumentation and Measurement, vol. IM-19, pp. 215-227, November, 1970.

5. "The Specification of Oscillator Characteristics from Measurements Made in the Frequency Domain," R. Vessot, L. Mueller, and J. Vanier, Proc. IEEE, vol. 54, pp. 199-207, February, 1966.

6. "Short-Term Frequency Stability: Characterization, Theory, and Measurement," E. J. Baghdady, R. N. Lincoln and B. D. Nelin, Proc. IEEE, vol. 53, pp. 704-722, July, 1965.

7. "The Effects of Noise in Oscillators," E. Hafner, Proc. IEEE, vol. 54, pp. 179-198, February, 1966.