FINAL REPORT

ANALYSIS OF CAVITATION BUBBLE DYNAMICS IN A LIQUID

by

L. L. Fontenot
and
YOUNG CHING LEE

UNIDEV, Inc.
Huntsville, Alabama

October 1971
Contract No. NAS8-26489
UNIDEV Report No. UR-00013

Submitted to

George C. Marshall Space Flight Center
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FOREWORD

This report was prepared by UNIDEV, Inc., Huntsville Alabama, under Contract No. NAS8-26489, for the George C. Marshall Space Flight Center, National Aeronautics & Space Administration; entitled "Analysis of Cavitation Bubble Dynamics in a Liquid". The work was administered under technical direction of the Aero-Astrodynamics Laboratory, with Mr. M. H. Rheinfurth acting as technical monitor.

The study was performed by the Solid and Fluid Mechanics Department, with Dr. L. L. Fontenot as Director, and the Principal Investigator. Mr. Young C. Lee served as Investigator and Computer Programmer.

ABSTRACT

General differential equations governing the dynamics of the cavitation bubbles in a liquid were derived. With the assumption of spherical symmetry the governing equations were simplified. Closed form solutions were obtained for simple cases, and numerical solutions were calculated for complicated ones. The Growth and the collapse of the bubble were analyzed, Oscillations of the bubbles were studied, and the stability of the cavitation bubbles were investigated. The results show that the cavitation bubbles are unstable, and the oscillation is not sinusoidal.

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Section I

INTRODUCTION

It is a physical property of any substance that no matter how dense it is, there always exist numerous sub-microscopic dimension cavities called nucleii. It is from these nucleii that the cavitation bubbles in a liquid begin to grow. Inside these nucleii, there may be dissolved gases (such as air) and vapor of the liquid. In equilibrium, the total pressure inside the nucleus is equal to the local pressure of the liquid at the boundary. When the pressure at the boundary falls below the vapor pressure of the liquid at the local temperature, the liquid will vaporize, and thus the cavitation bubble will grow. As the bubble grows larger and larger, it will become unstable and collapse into small bubbles. The growing velocity and limit of the size of the bubble depend on the properties of the liquid and the parameters of the flow field. If some of the properties of the liquid flow field are fluctuating, the growth of the cavitation bubble may be oscillating.

The cavitation bubble will affect the dynamics of the liquid and cause damage to the pump and propeller operating in the liquid. In this report, the equations governing the dynamics of the cavitation bubble were derived; the growth and collapse of the bubbles were analyzed; the oscillations of the bubbles were studied, and the stability of the cavitation bubbles investigated.

It was found that the general equations governing the dynamics of the cavitation bubble are difficult to solve without simplification. With the assumption of spherical symmetry and other simplifications, the equations were reduced to tractable forms. Closed form solutions were obtained for the simple cases, and numerical solutions were calculated for the complicated ones. Data were presented in both tabulated and graphical forms. The results show that the cavitation bubbles are unstable, the oscillation is not sinusoidal, and the growing velocity is subsonic in general.

Section II

GENERAL FORMULATION A OF CAVITATION BUBBLES

2.1 ARBITRARY INTERFACE

Denote the bubble wall by the closed surface

$$S(r,t) = S(x,y,z,t) = 0 \tag{1}$$

The surface S divides the whole space into two regions, R' and R. The external region R is occupied by a liquid which may have a substantial amount of dissolved gas. The internal region R' is occupied by a mixture of the gas and vapor of the liquid. We denote flow parameters in R by unprimed symbols, and corresponding parameters in R' by primed symbols. Quantities pertaining to the liquid or its vapor will be denoted by the subscript 1, and those pertaining to the gas by subscript 2. Symbols without any subscript refer to total quantities. Thus for example, the total mole number density n is given by

$$n = n_1 + n_2 \tag{2}$$

Denote the molecular mass by m, mass density by ρ, and fluid velocity by V; then we have

$$\rho_1 = m_1 n_1 \quad , \quad \rho_2 = m_2 n_2 \tag{3}$$

$$\rho = \rho_1 + \rho_2 \quad , \quad \rho\vec{V} = \rho_1\vec{V}_1 + \rho_2\vec{V}_2 \tag{4}$$

Similar relations hold for primed quantities.

If the thermal diffusion between the gases is neglected, the governing equations in R' may be written as

$$\frac{\partial \rho_1'}{\partial t} + \nabla \cdot (\rho_1'\vec{V}_1') = 0 \tag{5}$$

$$\frac{\partial \rho_2'}{\partial t} + \nabla \cdot (\rho_2'\vec{V}_2') = 0 \tag{6}$$

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \vec{V}') = 0 \tag{7}$$

$$\rho' \left[\frac{\partial \vec{V}'}{\partial t} + (\vec{V}' \cdot \nabla)\vec{V}' \right] = \nabla \cdot \vec{\tau}' + \rho' \vec{b} \tag{8}$$

$$\rho' \left[\frac{\partial U'}{\partial t} + (\vec{V}' \cdot \nabla)U' \right] + \nabla \cdot \vec{h}' = Tr(\nabla \vec{V}' \cdot \vec{\tau}') + \rho' q' \tag{9}$$

where:

$$\vec{h}' = -\lambda' \nabla T' + \frac{5}{2} kT' [n_1' \vec{V}_1' + n_2' \vec{V}_2' - n' \vec{V}'] \tag{10}$$

$$\vec{V}_1' - \vec{V}_2' = - \frac{n'^2}{n_1' n_2'} D_{12}' [\nabla \frac{n_1'}{n'} + \frac{n_1' n_2' (m_2 - m_1)}{n' \rho'} \quad \frac{\nabla P'}{P'}] \tag{11}$$

$$U' = U'(T', S')$$

In tensorial notations, the component of the stress tensor τ' is given by

$$\tau'^{\,i}_{j} = - p' \delta^{i}_{j} + 2\mu' (\varepsilon_{j}'^{\,i} - \frac{1}{3} \delta^{i}_{j} \varepsilon_{k}'^{\,k}) + \beta' \varepsilon_{k}'^{\,k} \delta^{i}_{j} \tag{13}$$

where:

$$\varepsilon'^{\,i}_{j} = \frac{1}{2}(V'_{i,j} + V'_{j,i}) \tag{14}$$

In the foregoing set of equations, ∇ is the gradient operator, s is the entropy, T the temperature, p the pressure, U the internal energy per unit mass, $\vec{h}$ the heat flux vector, $\vec{b}$ the body force per unit mass, and q the heat generated per unit time and unit volume. The coefficients of thermal conductivity λ, shear viscosity μ, bulk viscosity β, and mutual diffusion D_{12} will in general depend on the temperature and the relative concentration of the gases.

In R, one will make use of the fact that $\rho_2 << \rho_1$, i.e., the density of the dissolved gas is much smaller than the density of the liquid.

Thus one has essentially

$$\vec{V}_2 = \vec{V}_1 - D \left(\frac{\rho}{\rho_2}\right) \nabla \left(\frac{\rho_2}{\rho}\right)$$

$$\rho\vec{V} = \rho\vec{V}_1 - D\rho\nabla\left(\frac{\rho_2}{\rho}\right)$$

and the governing equations become

$$\frac{\partial\rho_1}{\partial t} + \nabla \cdot (\rho_1\vec{V}_1) = 0 \qquad (15)$$

$$\frac{\partial\rho_2}{\partial t} + \nabla \cdot (\rho_2\vec{V}_1) = D\nabla \cdot \left[\rho\nabla \quad \frac{\rho_2}{\rho}\right] \qquad (16)$$

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{V}) = 0 \qquad (17)$$

$$\rho \left[\frac{\partial\vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}\right] = \nabla\vec{\tau} + \rho\vec{b} \qquad (18)$$

$$\rho \left[\frac{\partial U}{\partial t} + (\vec{V} \cdot \nabla)U\right] + \nabla \cdot \vec{h} = \mathrm{Tr}(\nabla\vec{V} \cdot \vec{\tau}) + \rho q \qquad (19)$$

where

$$\vec{h} = -\lambda\nabla T \qquad (20)$$

$$U = U(T,S) \quad \text{(or some equivalent equation of state)} \qquad (21)$$

$$\tau_j{}^i = -\rho\delta_j{}^i + 2\mu\left(\varepsilon_j{}^i - \frac{1}{3}\,\delta_j{}^i\varepsilon_k{}^k\right) + \beta\ \varepsilon_k{}^k\delta_j{}^i \qquad (22)$$

and

$$\varepsilon_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i}) \qquad (23)$$

At the bubble wall S, the flow quantities in R and R' may be connected by applying the conservation laws across S. Let the unit normal at $S(r,t) = 0$ be $\vec{N}$; thus

$$\vec{N} = \nabla S / |\nabla S| \tag{24}$$

and its ith covariant component is

$$N_i = S_{,i} / |\nabla S| \tag{25}$$

Then, on $S(r,t) = 0$, we have

$$\rho_1\left(\vec{V}_1 \cdot \nabla S + \frac{\partial S}{\partial t}\right) = \rho_1'\left(\vec{V}_1' \cdot \nabla S + \frac{\partial S}{\partial t}\right) \tag{26}$$

$$\rho_2\left(\vec{V}_1 \cdot \nabla S + \frac{\partial S}{\partial t}\right) - D\rho\nabla\left(\frac{\rho_2}{\rho}\right) \cdot \nabla S = \rho_2'\left(\vec{V}_2' \cdot \nabla S + \frac{\partial S}{\partial t}\right) \tag{27}$$

$$\rho\left(\vec{V} \cdot \nabla S + \frac{\partial S}{\partial t}\right) = \rho'\left(\vec{V}' \cdot \nabla S + \frac{\partial S}{\partial t}\right) \tag{28}$$

$$\begin{aligned} \rho(V_i)\left(\vec{V} \cdot \nabla S + \frac{\partial S}{\partial t}\right) - \tau_i{}^j S_{,j} &= \rho' V_i'\left(\vec{V}' \cdot \nabla S + \frac{\partial S}{\partial t}\right) \\ &- \tau_i{}'^j S_{,j} - \sigma S_{,i}(\nabla \cdot \vec{N}), \quad i = 1,2,3 \end{aligned} \tag{29}$$

$$\begin{aligned} &\rho\left(U + \frac{V^2}{2}\right)\left(\vec{V} \cdot \nabla S + \frac{\partial S}{\partial t}\right) - \tau^{ij} V_i S_{,j} + \vec{h} \cdot \nabla S \\ &= \rho'\left(U' + \frac{V'^2}{2}\right)\left(\vec{V}' \cdot \nabla S + \frac{\partial S}{\partial t}\right) \\ &- \tau'^{ij} V_i' S_{,j} + \vec{h}' \cdot \nabla S + \sigma \frac{\partial S}{\partial t}(\nabla \cdot \vec{N}) \end{aligned} \tag{30}$$

Also from the local thermodynamical equilibrium, we have

$$T = T' \tag{31}$$

and

$$\rho_2 = \rho_2(p_2',T) \tag{32}$$

or

$$\rho_2 = \alpha p_2' \tag{33}$$

One may remark that σ is the surface tension coefficient, and $\nabla \cdot N = \frac{1}{R_1} + \frac{1}{R_2}$, where R_1 and R_2 are two principal radii of curvature at the point of interest on $S = 0$. An equation of state is needed to relate the partial pressure p'_2 to ρ'_2 and T'.

The foregoing then is the general set of governing equations for the bubble dynamics.

2.2 SPHERICAL BUBBLES

When the bubbles possess spherical symmetry, every physical quantity will depend only on the radial distance from the center, and velocity components other than the radial one will be zero. Thus the governing differential equations will be greatly simplified. The equation for the bubble wall will be

$$r - R(t) = 0 \tag{1}$$

The unit normal on the bubble wall will be

$$\vec{N} = \frac{\vec{r}}{r} \tag{2}$$

Also

$$\nabla \cdot \vec{N} = \frac{2}{r} \tag{3}$$

By use of the general formulations in Section 2.1, it can be shown that the governing equations for the vapor and gas inside the bubble are:

$$\frac{\partial \rho_1'}{\partial t} + \frac{\partial}{\partial r} (\rho_1' V_1') + \frac{2\rho_1' V_1'}{r} = 0 \tag{4}$$

$$\frac{\partial \rho_2'}{\partial t} + \frac{\partial}{\partial r} (\rho_2' V_2') + \frac{2\rho_2' V_2'}{r} = 0 \tag{5}$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial r} (\rho' V') + \frac{2\rho' V'}{r} = 0 \tag{6}$$

$$\begin{aligned} \frac{\partial V'}{\partial t} + V' \frac{\partial V'}{\partial r} = \frac{1}{\rho'} \left(\frac{4\mu'}{3} + \beta'\right) \frac{\partial}{\partial r} \left(\frac{\partial V'}{\partial r} + \frac{2V'}{r}\right) \\ + b - \frac{1}{\rho'} \frac{\partial p'}{\partial r} \end{aligned} \tag{7}$$

$$\begin{aligned} \rho' \frac{\partial U'}{\partial t} + V' \frac{\partial U'}{\partial r} = - p' \left(\frac{\partial V'}{\partial r} + \frac{2V'}{r}\right) \\ + \beta' \left(\frac{\partial V'}{\partial r} + \frac{2V'}{r}\right)^2 + \frac{4\mu'}{3} \left(\frac{\partial V'}{\partial r} - \frac{V'}{r}\right)^2 \\ - \frac{\partial h'}{\partial r} - \frac{2h'}{r} + \rho' q' \end{aligned} \tag{8}$$

$$h' = -\lambda' \frac{\partial T'}{\partial r} + \frac{5}{2} kT' [n_1' V_1' + n_2' V_2' - n' V'] \tag{9}$$

$$V_1' - V_2' = - \frac{n'^2}{n_1' n_2'} D_{12}' \left[\frac{\partial}{\partial r} \left(\frac{n_1'}{n'}\right) + \frac{n_1' n_2' (m_2 - m_1)}{n' \rho' p'} \frac{\partial p'}{\partial r} \right] \tag{10}$$

$$U' = U'(T', S') \tag{11}$$

The governing equations for the liquid and dissolved gas outside the bubble will be:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial r} (\rho_1 V_1) + \frac{2\rho V_1}{r} = 0 \tag{12}$$

$$\frac{\partial \rho_2}{\partial t} + V_1 \frac{\partial \rho_2}{\partial r} + \rho_2 \left(\frac{\partial V_1}{\partial r} + \frac{2V_1}{r}\right) = \frac{D}{r^2} \frac{\partial}{\partial r} \left[\rho r^2 \frac{\partial}{\partial r} \left(\frac{\rho_2}{\rho}\right)\right] \tag{13}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho V) + \frac{2\rho V}{r} = 0 \tag{14}$$

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{1}{\rho} \left(\frac{4\mu}{3} + \beta \right) \frac{\partial}{\partial r} \left(\frac{\partial V}{\partial r} + \frac{2V}{r} \right) + b \tag{15}$$

$$\begin{aligned} \rho \left(\frac{\partial U}{\partial t} + V \frac{\partial U}{\partial r} \right) = - p \left(\frac{\partial V}{\partial r} + \frac{2V}{r} \right) + \beta \left(\frac{\partial V}{\partial r} + \frac{2V}{r} \right)^2 + \frac{4\mu}{3} \left(\frac{\partial V}{\partial r} - \frac{V}{r} \right)^2 \\ + \lambda \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right) + \rho q \end{aligned} \tag{16}$$

$$U = U(T,S) \tag{17}$$

The boundary conditions at the bubble wall r = R (t) will be:

$$\rho_1 (V_1 - \dot{R}) = \rho_1{}' (V_1{}' - \dot{R}) \tag{18}$$

$$\rho_2 (V_1 - \dot{R}) - D\rho \frac{\partial}{\partial r} \left(\frac{\rho_2}{\rho} \right) = \rho_2{}' (V_2{}' - \dot{R}) \tag{19}$$

$$\rho (V - \dot{R}) = \rho' (V' - \dot{R}) \tag{20}$$

$$\begin{aligned} \rho V (V - \dot{R}) + p - \frac{4\mu}{3} \left(\frac{\partial V}{\partial r} - \frac{V}{r} \right) - \beta \left(\frac{\partial V}{\partial r} + \frac{2V}{r} \right) \\ = \rho' V' (V' - \dot{R}) + p' - \frac{4\mu'}{3} \left(\frac{\partial V'}{\partial r} - \frac{V'}{r} \right) \\ - \beta' \left(\frac{\partial V'}{\partial r} + \frac{2V'}{r} \right) - \frac{2\sigma}{R} \end{aligned} \tag{21}$$

$$\begin{aligned} \rho \left(U + \frac{V^2}{2} \right) (V - \dot{R}) + V \left[p - \frac{4\mu}{3} \left(\frac{\partial V}{\partial r} - \frac{V}{r} \right) - \beta \left(\frac{\partial V}{\partial r} + \frac{2V}{r} \right) \right] \\ - \lambda \frac{\partial T}{\partial r} = \rho' \left(U' + \frac{V'^2}{2} \right) (V' - \dot{R}) \\ + V' \left[p' - \frac{4\mu'}{3} \left(\frac{\partial V'}{\partial r} - \frac{V'}{r} \right) - \beta' \left(\frac{\partial V'}{\partial r} + \frac{2V'}{r} \right) \right] \\ + h' - \frac{2\sigma}{R} \dot{R} \end{aligned} \tag{22}$$

$$T = T' \tag{23}$$

$$\rho_2 = \alpha p_2{}' \tag{24}$$

Equations (21) and (22) may be re-written as

$$\begin{aligned} p + \frac{2\sigma}{R} &= p' + \rho'(V' - \dot{R})\ (V' - V) + \frac{4\mu}{3}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) \\ &\quad + \beta\left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right) - \frac{4\mu'}{3}\left(\frac{\partial V'}{\partial r} - \frac{V'}{r}\right) \\ &\quad - \beta'\left(\frac{\partial V'}{\partial r} + \frac{2V'}{r}\right) \end{aligned} \tag{25}$$

$$\begin{aligned} \lambda\frac{\partial T}{\partial r} + h' = \rho'(\dot{R} - V')\Bigg[L &+ \frac{4\mu}{3\rho}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) + \frac{\beta}{\rho}\left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right) \\ &- \frac{4\mu'}{3\rho'}\left(\frac{\partial V'}{\partial r} - \frac{V'}{r}\right) - \frac{\beta'}{\rho'}\left(\frac{\partial V'}{\partial r} + \frac{2V'}{r}\right) \\ &- \frac{1}{2}(\dot{R} - V)^2 + \frac{1}{2}(\dot{R} - V')^2\Bigg] \end{aligned} \tag{26}$$

where the latent heat per unit mass for the condensation of vapor is

$$L = U' + \frac{p'}{\rho} - U - \frac{p}{\rho} \tag{27}$$

2.3 SPHERICAL BUBBLES WITH UNIFORM INTERNAL PRESSURE

Since the diameter of the spherical bubble is small, the internal pressure may be assumed to be uniform. In this case, all physical quantities will be essentially functions of time only. The velocities of the vapor and gas inside the bubble will be

$$V_1{}' = -\frac{r}{3\rho_1{}'}\frac{d\rho_1{}'}{dt}\ ,\quad V_2{}' = -\frac{r}{3\rho_2{}'}\frac{d\rho_2{}'}{dt}\ ,\quad V' = -\frac{r}{3\rho'}\frac{d\rho'}{dt} \tag{28}$$

The heat flux will be given by

$$h' = -r\left(\frac{\rho'}{3}\frac{dU'}{dt} - \frac{p'}{3\rho'}\frac{d\rho'}{dt}\right) \tag{29}$$

In order that the assumption of uniform bubble internal pressure be valid, the velocity of the bubble wall must be smaller than the speed of sound and the various speeds of transport in the liquid. The following assumptions can be made to further simplify the governing equations.

(a) Bulk viscosity coefficients are small

(b) Since the liquid is essentially incompressible and the effect of viscosity is small, the interaction between compressibility and viscosity may be neglected.

(c) $V = V_1$ and $\rho = \rho_1$

(d) At $r = R$, $V = V_1 = \dot{R}$

(e) $\frac{1}{\rho_1}\frac{D\rho_1}{Dt} << \frac{1}{\rho_2}\frac{D\rho_2}{Dt}$ and $\frac{1}{\rho}\frac{D\rho}{Dt} << \frac{1}{\rho_2}\frac{D\rho_2}{Dt}$

(f) $\dot{R}^2 << \frac{p'}{\rho'}$

Based on the above assumptions, it can be shown that the governing equations for the liquid and dissolved gas outside the bubble are

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial r}(\rho V) + \frac{2\rho V}{r} = 0 \qquad (30)$$

$$\frac{\partial\rho_2}{\partial t} + V\frac{\partial\rho_2}{\partial r} = D\left[\frac{\partial^2\rho_2}{\partial r^2} + \frac{2}{r}\frac{\partial\rho_2}{\partial r}\right] \qquad (31)$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (32)$$

$$\rho\left(\frac{\partial U}{\partial t} + V\frac{\partial U}{\partial r}\right) = -p\left(\frac{\partial V}{\partial r} + \frac{2V}{r}\right) + \frac{4\mu}{3}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)^2 + \lambda\left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right) + \rho q \qquad (33)$$

$$U = U(T,S) \qquad (34)$$

At $r = R(t)$, the boundary conditions will be

$$V = \dot{R} \qquad (35)$$

$$D \frac{\partial \rho_2}{\partial r} = \frac{1}{4\pi R^2} \frac{d}{dt}\left(\frac{4}{3}\pi R^3 \rho_2'\right) \tag{36}$$

$$p + \frac{2\sigma}{R} = p' + \frac{4\mu}{3}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) \tag{37}$$

$$\lambda \frac{\partial T}{\partial r} = \frac{1}{4\pi R^2} \frac{d}{dt}\left(\frac{4}{3}\pi R^3 \rho'\right)\left[L + \frac{4\mu}{3\rho}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right)\right] + \frac{\rho' R}{3}\frac{dU'}{dt} + p'\dot{R} \tag{38}$$

$$T = T' \tag{39}$$

and

$$\rho_2 = \alpha p_2' \tag{40}$$

The equations governing the uniform vapor and gas inside the bubble will be unimportant in the solution of the problem.

Section III

GROWTH AND COLLAPSE OF CAVITATION BUBBLES, INERTIA EFFECT DOMINATING

3.1 DIFFERENTIAL EQUATIONS GOVERNING GROWTH AND COLLAPSE OF SPHERICAL BUBBLES, INERTIAL EFFECT DOMINATING

When the inertial effect is the dominating factor for the bubble dynamics, as in the case of rapid collapse of vapor bubbles in a liquid, the thermal and diffusive effects can be neglected in the analysis. In this case, it can be shown that the governing differential equations are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho V) + \frac{2\rho V}{r} = 0 \tag{1}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2}$$

At the bubble wall, the boundary conditions are:

$$V = \dot{R} \text{ at } r = R(t) \tag{3}$$

and

$$p + \frac{2\sigma}{R} = p' + \frac{4\mu}{3}\left(\frac{\partial V}{\partial r} - \frac{V}{r}\right) \tag{4}$$

In general, for most liquids of interest, the pressure will be function of the density only:

$$p = p(\rho) \tag{5}$$

It can be shown that the following solutions satisfy equation (2)

$$V = \frac{R^2\dot{R}}{r^2} \tag{6}$$

$$p = p_\infty(t) + \rho\left(\frac{R^2\ddot{R} + 2R\dot{R}^2}{r} - \frac{R^4\dot{R}^2}{2r^4}\right) \tag{7}$$

where $p_\infty(t)$ is the pressure field at infinity. The equation governing the motion of the bubble wall is determined from the boundary condition, equation (4), to be

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'(R) - p_\infty(t) - \frac{4\mu\dot{R}}{R} - \frac{2\sigma}{R}\right] \tag{8}$$

3.2 SOLUTION OF EQUATION 8, CASE 1

If the right-hand side of equation (8) is not a function of time, the equation can be easily integrated. Let

$$p'(R) - p_\infty(t) - \frac{4\mu\dot{R}}{R} - \frac{2\sigma}{R} = P \tag{9}$$

then equation (8) becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P}{\rho}$$

or

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = \frac{2P}{\rho}R^2\dot{R} = \frac{2P}{3\rho}\frac{d}{dt}(R^3) \tag{10}$$

Integration from $t = t_o$ to $t = t$ yields

$$R^3\dot{R}^2\Big|_{t_o}^{t} = \frac{2P}{3\rho}R^3\Big|_{t_o}^{t}$$

Substitution of the limits and simplification gives

$$\dot{R} = \sqrt{\left(\frac{R_o}{R}\right)^3 \dot{R}_o^{\,2} + \frac{2P}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right]} \tag{11}$$

where $\dot{R}_o = \dot{R}(t_o)$

Equation (11) is the simplest solution of equation (8). It shows that the velocity of the bubble wall is a function of P/ρ and the radius R. The graphs of equation (11) with P/ρ as parameter are shown in Figure 3-1.

3.3 SOLUTION OF EQUATION 8, CASE 2

If P in equation (9) is assumed to be a function of R by the following relation

$$\frac{P}{P_o} = \left(\frac{R_o}{R}\right)^3 \quad \text{or} \quad P = P_o\left(\frac{R_o}{R}\right)^3 \tag{12}$$

where R_o is the radius at $t = t_o$, and P_o some reference pressure. Substitution of equation (12) into equation (10) gives the differential equation of the velocity of the bubble wall as

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P_o}{\rho}\left(\frac{R_o}{R}\right)^3$$

or

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = \frac{2P_oR_o^3}{\rho}\,\frac{\dot{R}}{R} \tag{13}$$

Integration from $t = t_o$ to $t = t$ gives

$$R^3\dot{R}^2\Big|_{t_o}^{t} = \frac{2P_oR_o^3}{\rho}\,\ell n\, R\Big|_{t_o}^{t}$$

Substitution of the limits and simplification gives

$$\dot{R}^2 = \left(\frac{R_o}{R}\right)^3 \dot{R}_o^{\,2} + \frac{2P_o}{\rho}\left(\frac{R_o}{R}\right)^3 \ell n\left(\frac{R}{R_o}\right) \tag{14}$$

Equation (14) shows the velocity of the bubble wall as a function of R_o/R. In equation (14) differentiation of $\dot{R}$ with respect to R/R_o and setting the derivative equal to zero gives the value of R/R_o for the maximum velocity of the bubble wall velocity as:

$$\frac{R}{R_o} = \exp\left(\frac{1}{3} - \frac{R_o^{\ 2}\rho}{2P_o}\right) \tag{15}$$

For R_o = 10 m/sec, P_o/ρ = 600 $(m/sec)^2$, equation (15) gives

$$\frac{R}{R_o} = 1.28$$

3.4 SOLUTION OF EQUATION 8, CASE 3

If the coefficient of shear vicosity μ is small and neglected, equation (8) then becomes

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'(R) - P_\infty(t) - \frac{2\sigma}{R}\right] \tag{17}$$

The pressure inside the bubble may be assumed to be uniform and follow Boyle Law.

$$\frac{p'}{p'_o} = \left(\frac{R_o}{R}\right)^3 \tag{18}$$

where $p'_o = p'(R_o)$.

Also the pressure at infinity p_∞ is assumed to be constant. Substitution of equation (18) into (17) gives

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'_o\left(\frac{R_o}{R}\right)^3 - p_\infty - \frac{2\sigma}{R}\right] \tag{19}$$

or

$$\frac{d}{dt}(R^3\dot{R}^2) = \frac{1}{\rho}\left[\frac{2\ p_o'\ R_o{}^3\ \dot{R}}{R} - 2\ p_\infty\ R^2\dot{R} - 4\sigma R\dot{R}\right]$$

Integration from $t = t_o$ to $t = t$ gives

$$R^3\dot{R}^2\Big|_{t_o}^{t} = \frac{1}{\rho}\left[2p_o'R_o^{\ 3}\ \ln R - 2p_\infty\frac{R^3}{3} - 4\sigma\frac{R^2}{2}\right]_{t_o}^{t}$$

Substitution of the limits and simplification gives

$$\dot{R}^2 = \left(\frac{R_o}{R}\right)^3 \dot{R}_o^{\,2} + \frac{1}{\rho}\left\{2p'_o\left(\frac{R_o}{R}\right)^3 \ell n\left(\frac{R}{R_o}\right) - \frac{2p_\infty}{3}\left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{R}\left[1 - \left(\frac{R_o}{R}\right)^2\right]\right\} \quad (20)$$

3.5 SOLUTION OF EQUATION 8, CASE 4

Here a more general closed form solution will be obtained. In this solution, the liquid pressure (p_∞) and the surface tension (σ) are assumed to be constant, the shear viscosity is assumed to be small and can be neglected, and the gas pressure inside the bubble is assumed to vary as

$$\frac{p'}{p'_o} = \left(\frac{R_o}{R}\right)^n \quad (21a)$$

or

$$p' = p'_o\left(\frac{R_o}{R}\right)^n \quad (21b)$$

where p'_o = gas pressure at $t = t_o$.

R_o = radius of bubble at $t = t_o$.

n = constant.

Substitution of equation (21b) into equation (8) gives

$$\frac{1}{2R^2\dot{R}} \frac{d}{dt}(R^3\dot{R}^2) = \frac{1}{\rho}\left[p'_o\left(\frac{R_o}{R}\right)^n - p_\infty - \frac{2\sigma}{R}\right] \quad (22)$$

Equation (22) may be written as

$$\frac{d}{dt}(R^3\dot{R}^2) = \frac{p'_o}{\rho} 2R_o^{\,n} R^{2-n}\dot{R} - \frac{2p_\infty}{\rho}R^2\dot{R} - \frac{4\sigma}{\rho}R\dot{R}$$

$$= \frac{2p'_o}{\rho} R_o^{\,n} \frac{1}{3-n} \frac{d}{dt}\left(R^{3-n}\right) - \frac{2p_\infty}{3\rho}\frac{d}{dt}(R^3) - \frac{2\sigma}{\rho}\frac{d}{dt}(R^2) \quad (23)$$

Integration from $t = t_o$ to $t = t$ gives

$$R^3\dot{R}^2 \Big|_{t_o}^{t} = \left[\frac{2p'_o}{\rho} R_o^{\,n} \frac{R^{3-n}}{3-n} - \frac{2p_\infty}{3\rho} R^3 - \frac{2\sigma}{\rho} R^2 \right]_{t_o}^{t} \tag{24}$$

$n \neq 3$

Substitution of the limits yields

$$R^3\dot{R}^2 - R_o^3\dot{R}_o^2 = \frac{2p'_o}{\rho} R_o^{\,n}\left(\frac{R^{3-n} - R_o^{3-n}}{3-n}\right) - \frac{2p_\infty}{3\rho}(R^3 - R_o^3) - \frac{2\sigma}{\rho}(R^2 - R_o^2) \tag{25}$$

It is convenient to assume that the bubble begins to grow at $t = t_o$, then

$$\dot{R}_o = 0 \tag{26}$$

Using equation (26) one obtains

$$R^3\dot{R}^2 = \frac{2p'_o}{\rho} R_o^{\,n}\left(\frac{R^{3-n} - R_o^{3-n}}{3-n}\right) - \frac{2p_\infty}{3\rho}(R^3 - R_o^3) - \frac{2\sigma}{\rho}(R^2 - R_o^2) \tag{27}$$

or

$$\dot{R} = \left\{ \frac{2}{3-n} \frac{p'_o}{\rho}\left[\left(\frac{R_o}{R}\right)^n - \left(\frac{R_o}{R}\right)^3\right] - \frac{2p_\infty}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R}\left[1 - \left(\frac{R_o}{R}\right)^2\right] \right\}^{1/2} \tag{28}$$

$n \neq 3$

For most liquids, the surface tension will be small and may be neglected. Then equation (28) reduces to

$$\dot{R} = \sqrt{\frac{2}{3-n}\left(\frac{p'_o}{\rho}\right)\left[\left(\frac{R_o}{R}\right)^n - \left(\frac{R_o}{R}\right)^3\right] - \frac{2}{3}\frac{p_\infty}{\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right]} \tag{29}$$

$n \neq 3$

When n = 1 equation (29) becomes

$$\dot{R} = \sqrt{\left(\frac{p'_o}{\rho}\right)\left[\frac{R_o}{R} - \left(\frac{R_o}{R}\right)^3\right] - \frac{2}{3}\left(\frac{p_\infty}{\rho}\right)\left[1 - \left(\frac{R_o}{R}\right)^3\right]} \tag{30}$$

When n = 1/2 equation (29) becomes

$$\dot{R} = \sqrt{\frac{4}{5}\left(\frac{p'_o}{\rho}\right)\left[\sqrt{\frac{R_o}{R}} - \left(\frac{R_o}{R}\right)^3\right] - \frac{2}{3}\,\frac{p_\infty}{\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right]} \tag{31}$$

3.6 THE TIME OF GROWTH

The time for the spherical bubble to grow from radius (R_o) to radius (R) can be obtained as follows: Equation (29) can be written as

$$dt = \frac{dR}{\sqrt{\frac{2}{3-n}\left(\frac{p'_o}{\rho}\right)\left[\left(\frac{R_o}{R}\right)^n - \left(\frac{R_o}{R}\right)^3\right] - \frac{2}{3}\left(\frac{p_\infty}{\rho}\right)\left[1 - \left(\frac{R_o}{R}\right)^3\right]}} \tag{32}$$

Integration gives

$$t - t_o = \int_{R_o}^{R} \frac{dR}{\sqrt{\frac{2}{3-n}\left(\frac{p'_o}{\rho}\right)\left[\left(\frac{R_o}{R}\right)^n - \left(\frac{R_o}{R}\right)^3\right] - \frac{2}{3}\left(\frac{p_\infty}{\rho}\right)\left[1 - \left(\frac{R_o}{R}\right)^3\right]}} \tag{33}$$

This integration on the right hand side of equation (33) cannot be carried out in closed form and has to be performed by numerical integration.

3.7 NUMERICAL DATA

Computer programs were developed to calculate the growing velocity of the bubble radius for different values of n and other parameters in equation (29). Computer programs were also written to calculate the time of growth as function of the ratio of radius to initial radius in equation (33).

For

$$n = 1$$

$$p_\infty/\rho = 200\ (\mathrm{m/sec})^2$$

$$R_o = 0.02 \text{ inch} = 0.0508 \text{ cm}$$

the growing velocity $\dot{R}$ for $p'_o/p_\infty = 5, 10, 15$ as function of R/R_o are given in Table 3-2 and plotted in Figure 3-2; the growing time versus R/R_o are shown in Table 3-3 and plotted in Figure 3-3.

For

$$n = 1/2$$

$$p_\infty/\rho = 200\ (\mathrm{m/sec})^2$$

$$R_o = 0.02 \text{ inch} = 0.0508 \text{ cm}$$

the growing velocity $\dot{R}$ for $p'_o/p_\infty = 5, 10, 15$ as function of R/R_o are given in Table 3-4 and plotted in Figure 3-4; the growing time versus R/R_o are shown in Table 3-5 and plotted in Figure 3-5.

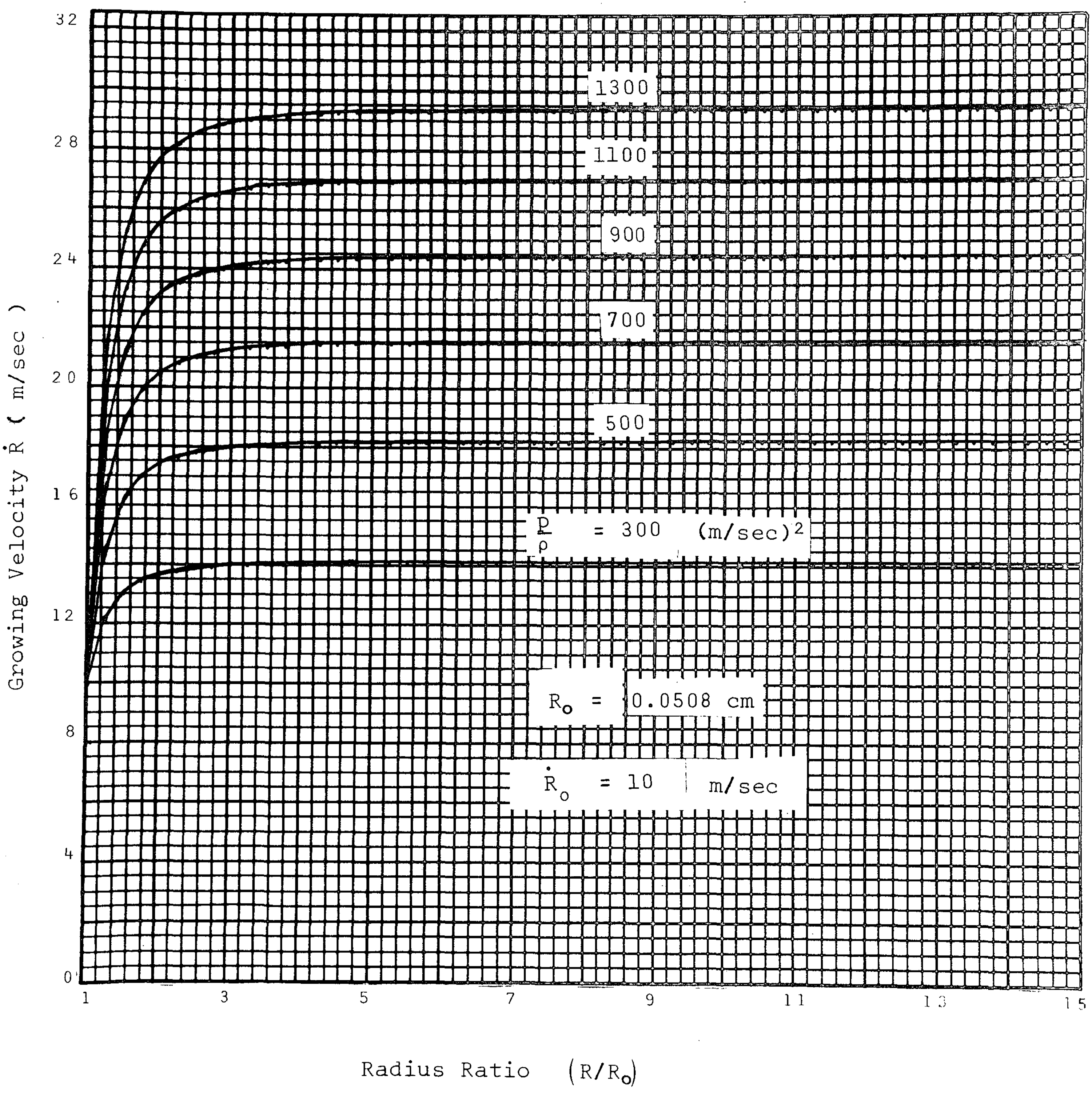

FIG. 3.1 BUBBLE WALL VELOCITY VERSUS RADIUS RATIO , EQUATION (11)

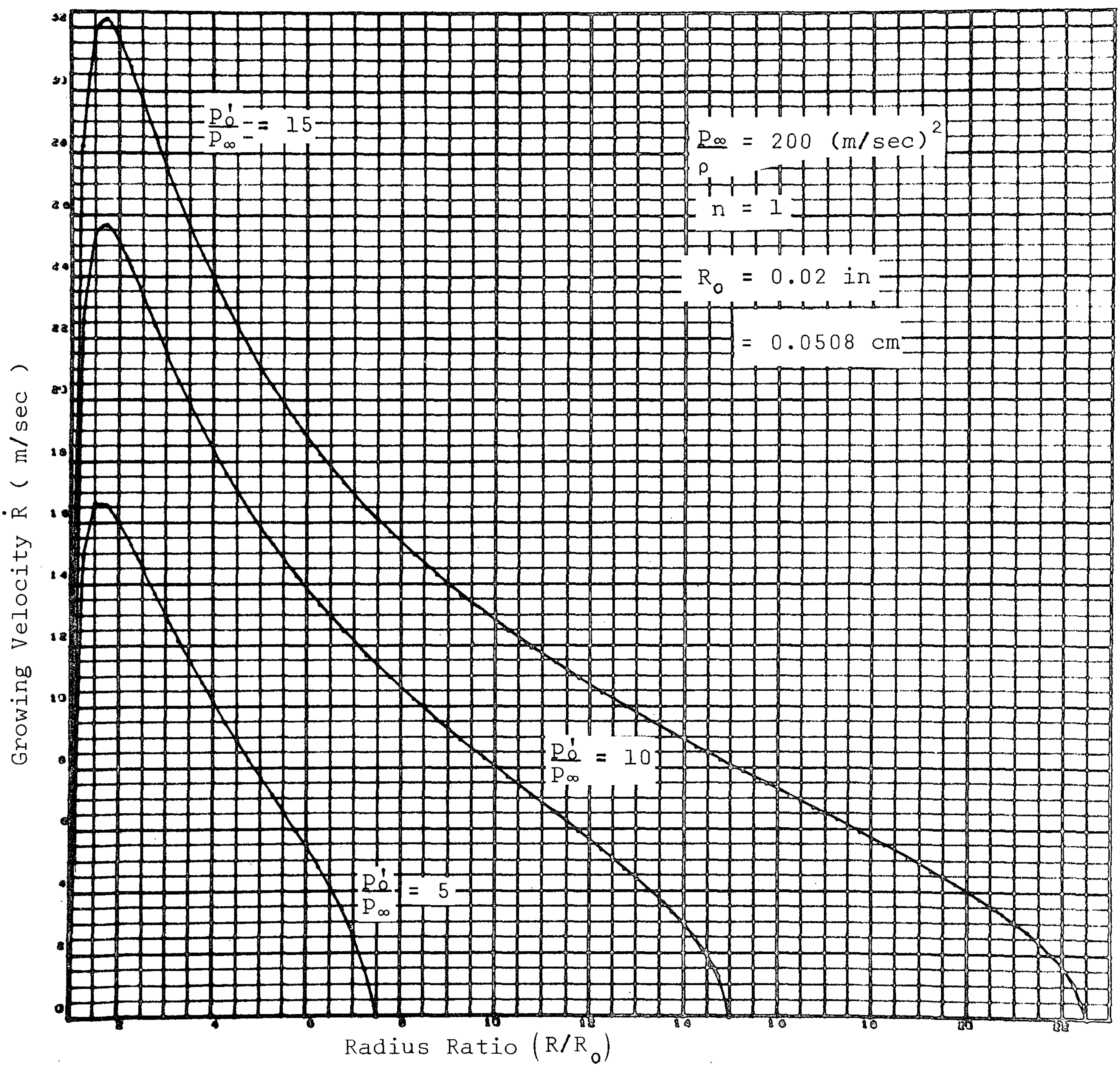

FIG. 3.2 GROWING VELOCITY VERSUS RADIUS RATIO

, EQUATION (29), n = 1

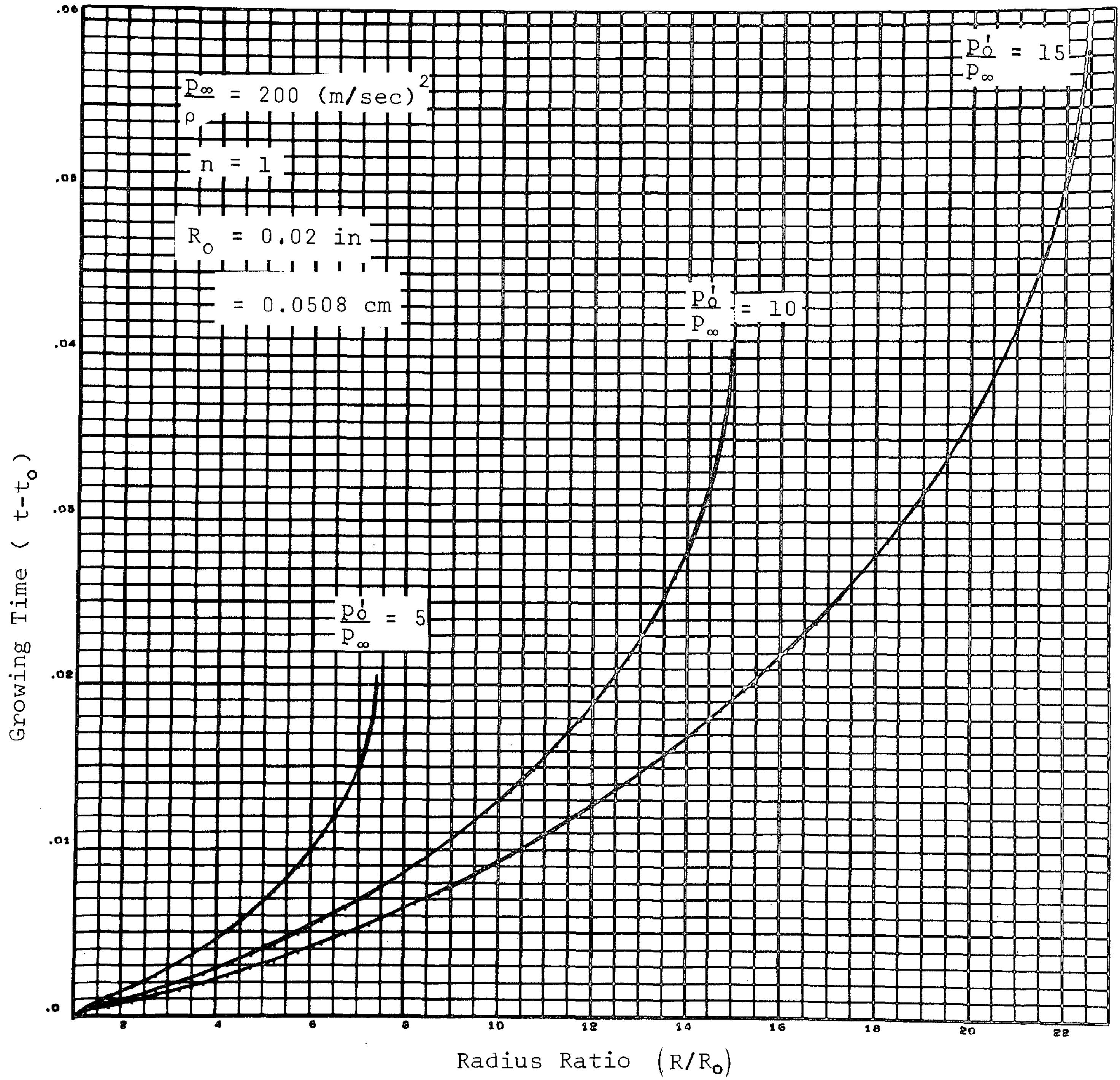

FIG. 3.3 GROWING TIME VERSUS RADIUS RATIO , EQUATION (33), n = 1

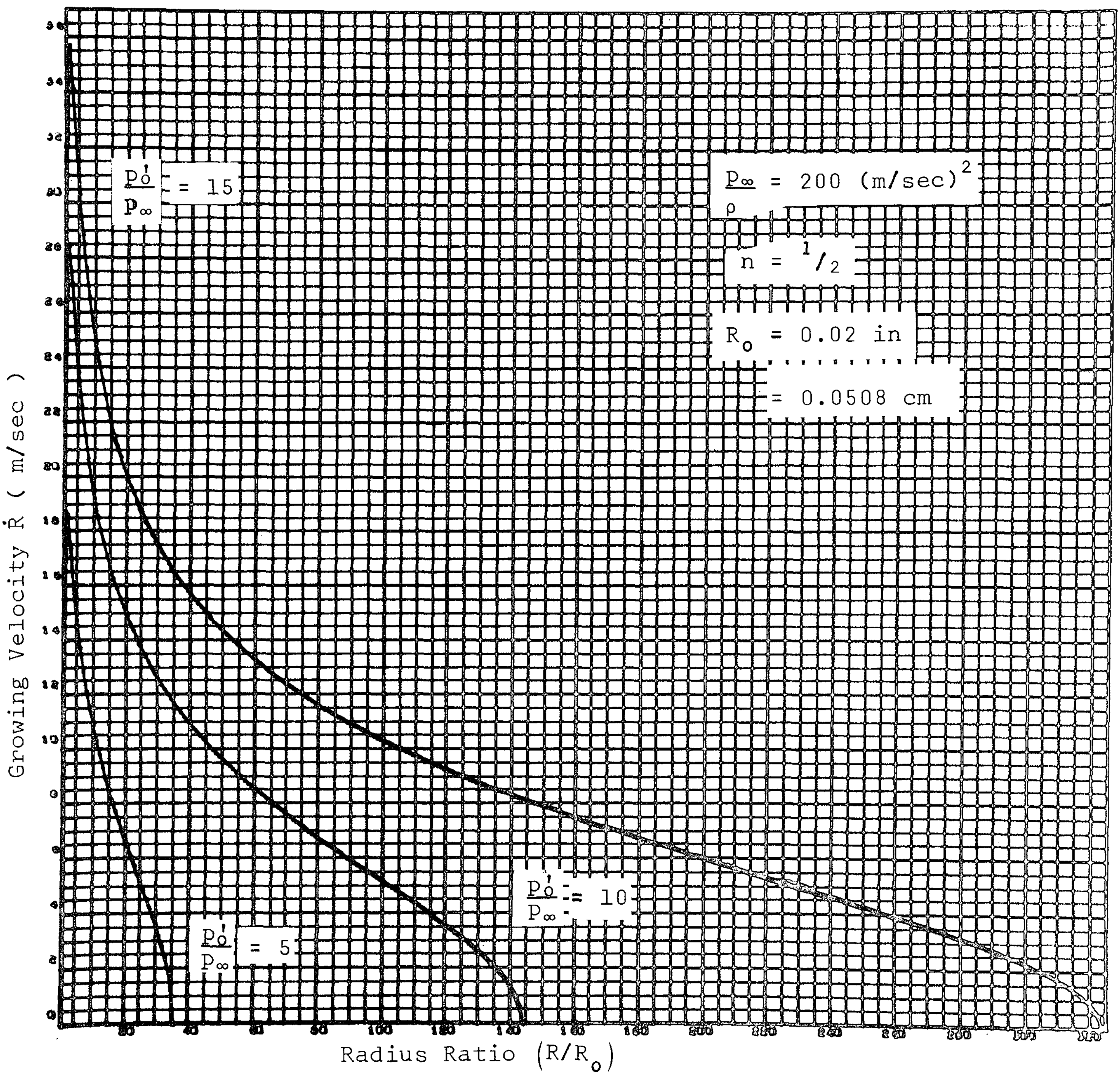

FIG. 3.4 GROWING VELOCITY VERSUS RADIUS RATIO , EQUATION (29), n = $^1/_2$

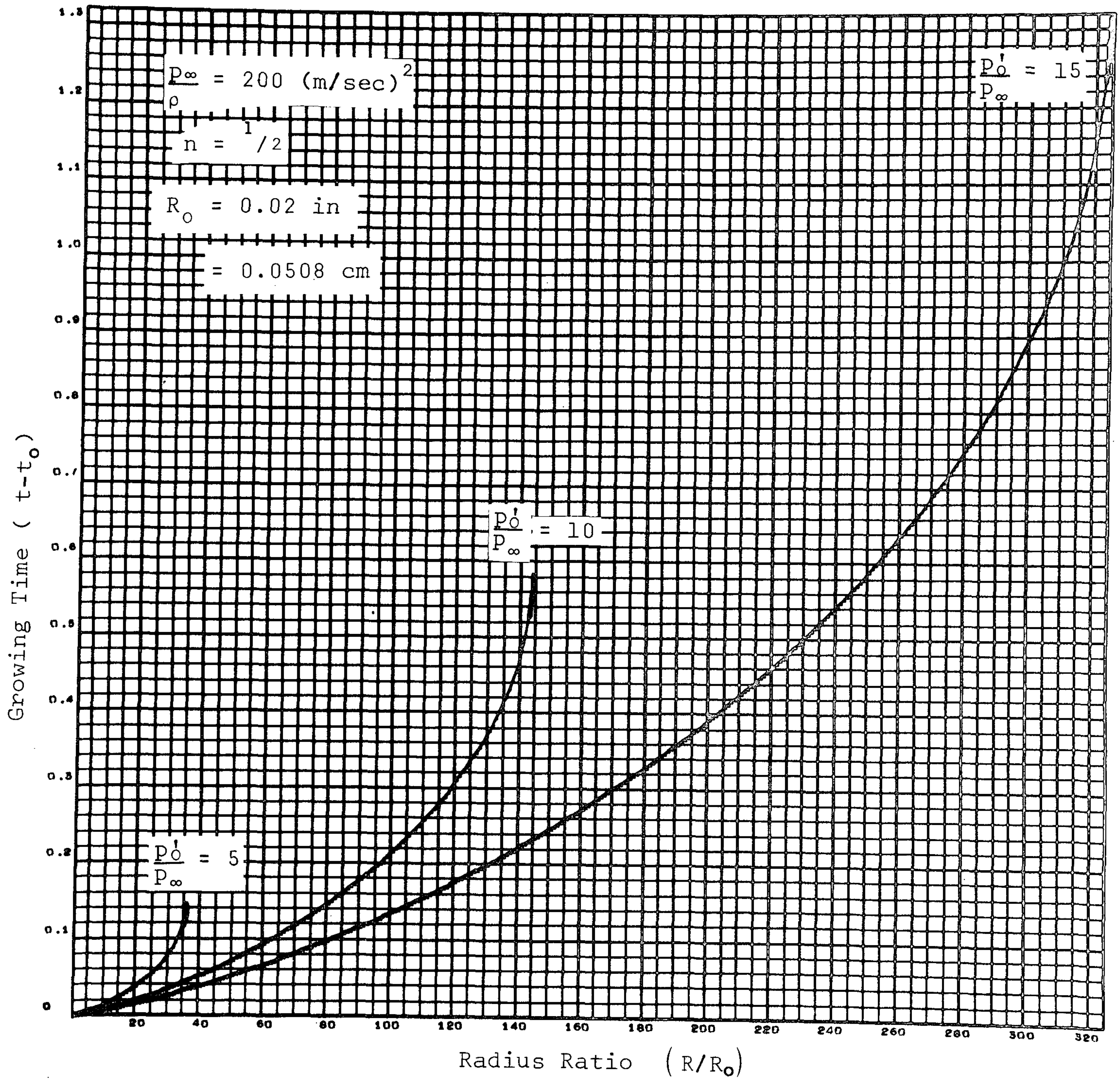

FIG. 3.5 GROWING TIME VERSUS RADIUS RATIO

, EQUATION (33), n = $^1/_2$

TABLE 3.1 GROWING VELOCITY OF BUBBLE WALL VERSUS RADIUS RATIO, EQUATION (11)

σ/ρ Radius Ratio	300(m/sec)^2	500(m/sec)^2	700(m/sec)^2	900(m/sec)^2	1100(m/sec)^2	1300(m/sec)^2
R/R_c	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)
1	10	10	10	10	10	10
1.25	12.2	14.624	16.701	18.55	20.225	21.775
1.5	13.05	16.254	18.92	21.26	23.36	25.29
2	13.7	17.44	20.514	23.184	25.58	27.76
3	14.01	18.02	21.286	24.114	26.64	28.953
4	14.087	18.16	21.47	24.335	26.9	29.235
5	14.114	18.21	21.53	24.413	26.99	29.335
6	14.126	18.23	21.56	24.448	27.026	29.38
7	14.132	18.24	21.58	24.465	27.046	29.4
8	14.135	18.245	21.586	24.475	27.057	29.414
9	14.137	18.248	21.59	24.481	27.064	29.421
10	14.139	18.251	21.594	24.485	27.068	29.426
15	14.141	18.256	21.6	24.492	27.077	29.435

TABLE 3.2 GROWING VELOCITY VERSUS RADIUS RATIO, EQUATION (29), n = 1

$\frac{p_\infty}{\rho} = 200\ (m/sec)^2$, R_o = 0.02 inch = 0.0508 cm					
$\frac{p_o'}{p_\infty} = 5$		$\frac{p_o'}{p_\infty} = 10$		$\frac{p_o'}{p_\infty} = 15$	
R/R	$\dot{R}$ (m/sec)	R/R	$\dot{R}$ (m/sec)	R/R	$\dot{R}$ (m/sec)
1.0	0.00	1.0	0.00	1.0	0.00
1.1	11.16	1.1	16.8	1.1	20.98
1.2	14.09	1.3	23.57	1.2	26.6
1.4	16.28	1.5	25.43	1.4	31.06
1.6	16.74	1.7	25.75	1.6	32.28
1.8	16.54	1.9	25.44	1.7	32.37
2.0	16.07	3.0	21.55	1.9	32.06
3.0	12.96	4.0	18.37	3.0	27.58
3.5	11.5	5.0	15.87	7.0	16.94
4.0	10.16	9.0	9.29	11.0	11.71
5.0	7.73	11.0	6.86	15.0	8.11
6.0	5.41	13.0	4.43	19.0	4.91
7.0	2.65	14.9	0.58	21.0	3.04
7.38	0.1089	14.935	0.1408	22.455	0.1182
7.385	0.00	14.94	0.00	22.46	0.00

TABLE 3.3 GROWING TIME VERSUS RADIUS RATIO, EQUATION (33), n = 1

$\frac{p_\infty}{\rho} = 200(m/sec)^2$, R_o = 0.02 inch = 0.0508 cm					
$\frac{p_o'}{p_\infty} = 5$		$\frac{p_o'}{p_\infty} = 10$		$\frac{p_o'}{p_\infty} = 15$	
R/R_o	$t - t_o$(sec)	R/R_o	$t - t_o$(sec)	R/R_o	$t - t_o$(sec)
1	.00000000	1	.00000000	1	.00000000
2	.14610374-02	2	.96031892-03	2	.76802543-03
3	.28431033-02	3	.18174659-02	4	.22230087-02
4	.45884556-02	4	.28248990-02	6	.41223507-02
5	.68446598-02	5	.39986311-02	8	.64853525-02
6	.99183652-02	6	.53505446-02	10	.93446914-02
7	.14965961-01	7	.68965938-02	12	.12763646-01
7.05	.15356275-01	8	.86595678-02	14	.16850987-01
7.1	.15780662-01	9	.10672592-01	16	.21795669-01
7.15	.16246411-01	10	.12985089-01	17	.24690312-01
7.2	.16767765-01	11	.15674313-01	18	.27958857-01
7.25	.17369846-01	12	.18872646-01	20	.36198124-01
7.3	.18105884-01	13	.22841034-01	21	.41866745-01
7.35	.19144090-01	14	.28267025-01	22	.50423378-01
7.38	.20572564-01	14.935	.40513069-01	22.455	.60710941-01

TABLE 3.4 GROWING VELOCITY VERSUS RADIUS RATIO, EQUATION (29), $n = {}^{1}/_{2}$

$\frac{P_\infty}{\rho} = 200 (m/sec)^2$		$R_o = 0.02$ inch $= 0.0508$ cm			
$\frac{P_o'}{P_\infty} = 5$		$\frac{P_o'}{P_\infty} = 10$		$\frac{P_o'}{P_\infty} = 15$	
R/R_o	$\dot{R}$ (m/sec)	R/R_o	$\dot{R}$ (m/sec)	R/R_o	$\dot{R}$ (m/sec)
1.0	0.00	1.0	0.00	1.0	0.00
1.1	11.34	1.1	17.04	1.1	21.26
1.3	16.27	2.0	28.54	1.5	33.98
1.5	17.95	3.0	27.13	2.0	35.78
1.7	18.56	10.0	19.27	3.0	34.18
1.9	18.7	30.0	12.6	10.0	24.97
3.0	17.43	50.0	9.64	20.0	20.08
4.0	16.01	70.0	7.61	30.0	17.46
10.0	10.91	90.0	5.94	100.0	10.33
15.0	8.55	110.0	4.38	200.0	6.03
25.0	5.16	120.0	3.57	300.0	2.29
30.0	3.56	140.0	1.38	320.0	0.91
35.0	1.37	143.0	0.68	323.0	0.45
35.99	0.06475	143.995	0.04371	323.995	0.03119
35.995	0.00	144.[illegible]	0.00	324.[illegible]	0.00

TABLE 3.5 GROWING TIME VERSUS RADIUS RATIO, EQUATION (33), $n = 1/2$

$\frac{p_\infty}{\rho} = 200 (m/sec)^2$, R_o = 0.02 inch=0.508 cm					
$\frac{p_o'}{p_\infty} = 5$		$\frac{p_o'}{p_\infty} = 10$		$\frac{p_o'}{p_\infty} = 15$	
R/R_o	$t - t_o$ (sec)	R/R_o	$t - t_o$ (sec)	R/R_o	$t - t_o$ (sec)
1	.00000000	1	.00000000	1	.00000000
5	.49730848-02	16	.14724970-01	36	.33107955-01
9	.11158395-01	32	.37706516-01	72	.84802873-01
13	.18883428-01	48	.66972756-01	108	.15064630+00
17	.28257247-01	64	.10268568+00	144	.23099661+00
19	.33647884-01	80	.14582738+00	180	.32806218+00
21	.39583583-01	88	.17075959+00	198	.38415800+00
23	.46150076-01	96	.19843756+00	216	.44643169+00
25	.53470694-01	104	.22945970+00	234	.51622950+00
27	.61730372-01	112	.26475213+00	252	.59563499+00
29	.71226637-01	120	.30587830+00	270	.68816558+00
31	.82493776-01	128	.35588346+00	288	.80067248+00
33	.96697221-01	136	.42260022+00	306	.95077552+00
35	.11783461+00	143	.52899947+00	323	.12341412+01
35.99	.14562532+00	143.9948	.58399516+00	323.9949	.13163100+01

Section IV

GROWTH AND COLLAPSE OF CAVITATING BUBBLES, THERMAL EFFECT DOMINATING

4.1 DIFFERENTIAL EQUATION GOVERNING GROWTH OF SPHERICAL BUBBLES, THERMAL EFFECT DOMINATING

The differential equation governing the growth and collapse of spherical bubbles is (see equation (8) in Section 3.1):

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'(R) - p_\infty(t) - \frac{4\mu\dot{R}}{R} - \frac{2\sigma}{R}\right] \tag{1}$$

where

R = radius of spherical bubble

$\dot{R} = \frac{dR}{dt}$ = growing velocity of the radius

ρ = mass density of the liquid

$p'(R)$ = pressure inside the bubble, is a function of R, and hence a function of time in general

$p_\infty(t)$ = Liquid pressure far from the bubble

μ = coefficient of shear viscosity

σ = surface tension coefficient of the bubble surface.

When the thermal effect is dominating, the effects of viscosity and surface tension may be neglected. Thus one obtains the differential equation governing the growth of the spherical bubble when the thermal effect is dominating, as

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}(p' - p_\infty) \tag{2}$$

4.2 SOLUTION OF EQUATION (2) FOR CONSTANT VAPOR PRESSURE

When the temperature of the liquid is kept constant, the vapor pressure will be independent of time. If the liquid pressure is also kept constant, then equation (2) can be easily solved to yield the growing velocity of the bubble wall as

$$\dot{R} = \sqrt{\frac{2(p'-p_\infty)}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right] + \left(\frac{R_o}{R}\right)^3 \dot{R}_o^{\,2}} \tag{3}$$

where

R_o = Radius of bubble at $t = t_o$

$\dot{R}_o$ = Growing velocity of bubble wall at $t = t_o$

It is convenient to count the time when the bubble begins to grow. Hence

$$t_o = 0$$

$$\dot{R}_o = 0 \tag{4}$$

Thus, equation (2) reduces to

$$\dot{R} = \sqrt{\frac{2}{3}\,\frac{(p'-p_\infty)}{\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right]} \tag{5}$$

The time duration of growth can be obtained by simple integration:

$$t = \int_{R_o}^{R} \frac{dR}{\sqrt{\frac{2}{3}\left(\frac{p'-p_\infty}{\rho}\right)\left[1 - \left(\frac{R_o}{R}\right)^3\right]}} \tag{6}$$

4.3 SOLUTION OF EQUATION (2) FOR SLOWLY VARYING PRESSURE

When the liquid is under heating, the temperature and the vapor pressure will be increasing. In this case, p' will be a function of time, and the solution of equation (2) will not be so straightforward. However, when the heating is slow, equation (2) may be solved by assuming constant p' without introducing much error. Thus all the formulations in Section 4.2 apply to this case.

4.4 NUMERICAL DATA

Computer programs were developed to perform numerical analysis of equations (5) and (6). For

$$\frac{p'-p_\infty}{\rho} = 300,\ 500,\ 700,\ 900,\ 1100,\ 1300\ (\text{m/sec})^2$$

the growing velocity ($\dot{R}$) versus the radius ratio (R/R_o) is plotted in Figure 4-1 and tabulated in Table 4-1 for equation (5): the growing time (t) versus the radius ratio (R/R_o) is plotted in Figure 4-2 and tabulated in Table 4-2 for equation (6).

It is seen that the growing velocity approaches the asymptotic value

$$\sqrt{\frac{2}{3}\left(\frac{p'-p_\infty}{\rho}\right)}$$

very fast as the bubble grows.

Although the bubble growing velocity never decreases, the growth will, however, eventually stop. Because the cavitation bubble is always unstable, the stopping of growth is due to the collapse of the bubble.

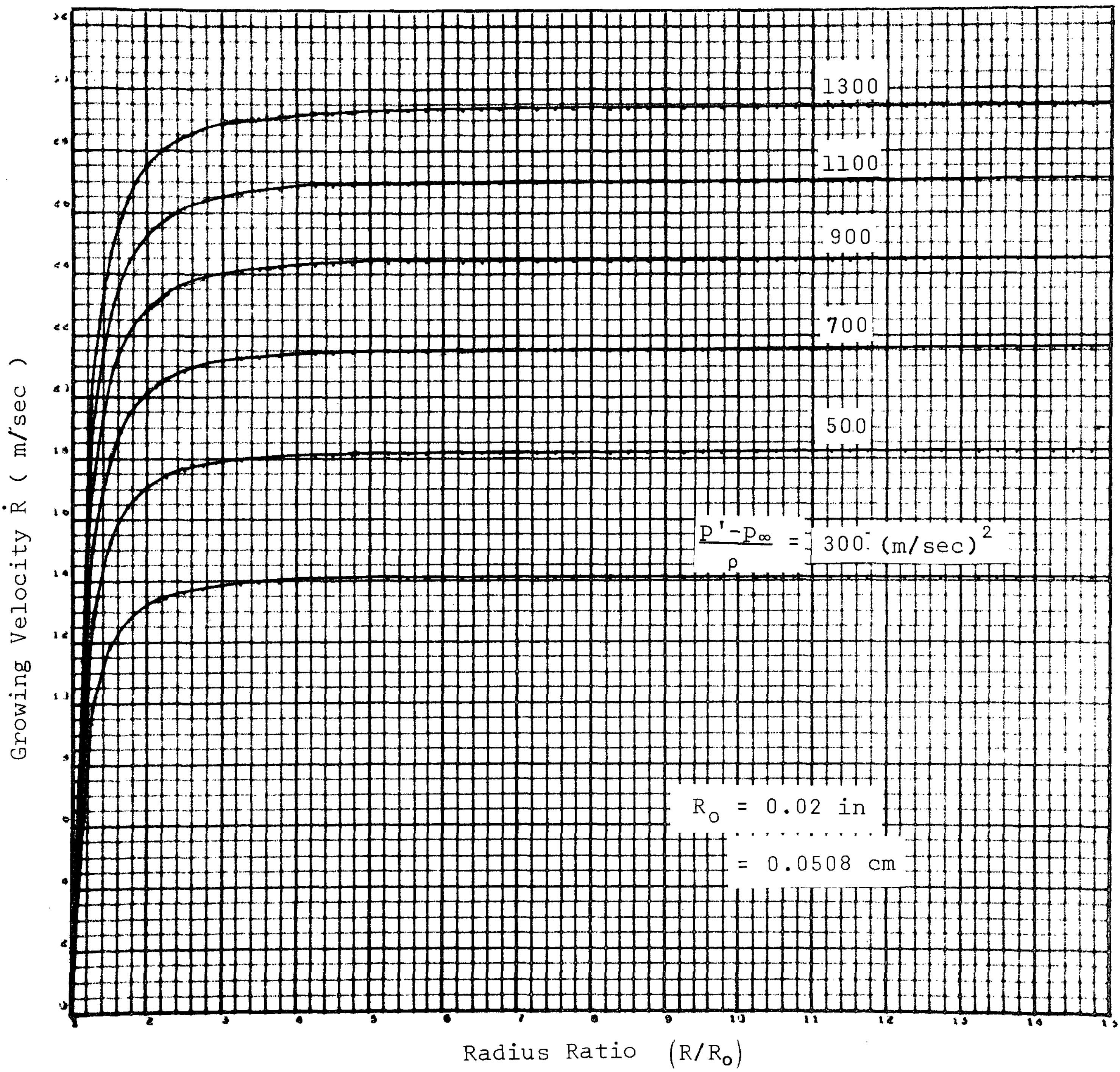

FIG. 4.1 GROWING VELOCITY VERSUS RADIUS RATIO

, EQUATION (5)

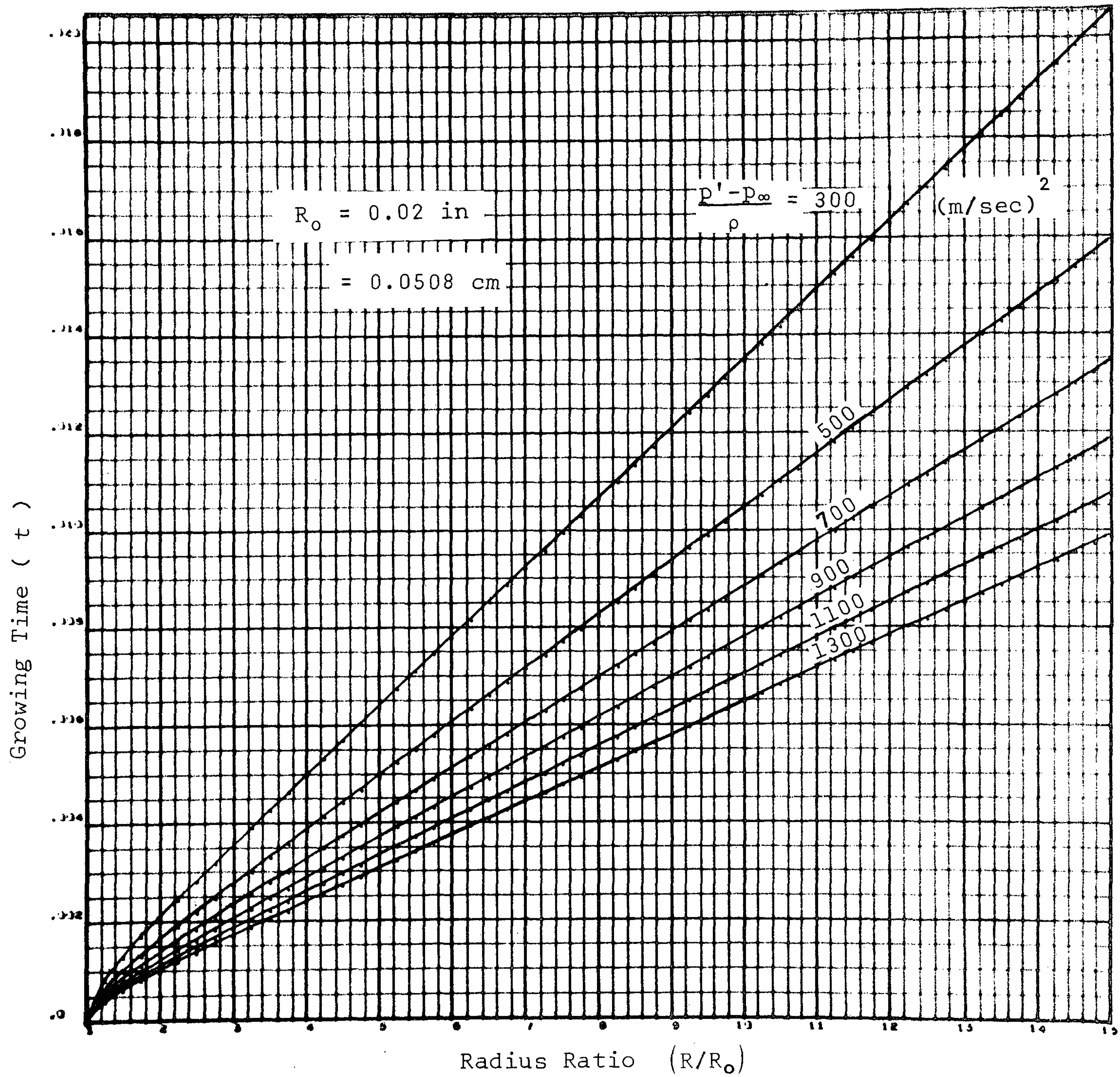

FIG. 4.2 GROWING TIME VERSUS RADIUS RATIO

, EQUATION (6)

TABLE 4.1 GROWING VELOCITY VERSUS RADIUS RATIO, EQUATION (5)

$\frac{p'-p_\infty}{\rho}$ / RADIUS RATIO	$300(m/sec)^2$	$500(m/sec)^2$	$700(m/sec)^2$	$900(m/sec)^2$	$1100(m/sec)^2$	$1300(m/sec)^2$
R/R_o	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)	$\dot{R}$(m/sec)
1	0	0	0	0	0	0
1.25	10.12	13.06	15.46	17.53	19.38	21.07
1.5	11.87	15.32	18.13	20.55	22.72	24.70
2	13.23	17.08	20.21	22.91	25.33	27.54
3	13.88	17.92	21.20	24.04	26.57	28.89
4	14.03	18.11	21.43	24.30	26.87	29.21
5	14.09	18.18	21.52	24.40	26.97	29.32
6	14.11	18.215	21.55	24.44	27.02	29.37
7	14.12	18.23	21.57	24.46	27.04	29.40
8	14.128	18.24	21.581	24.47	27.054	29.41
9	14.132	18.245	21.588	24.48	27.062	29.419
10	14.135	18.248	21.592	24.483	27.067	29.424
15	14.14	18.255	21.599	24.49	27.076	29.43

TABLE 4.2 GROWING TIME VERSUS RADIUS RATIO, EQUATION (6)

$\frac{p'-p_\infty}{\rho}$ / RADIUS RATIO	300 $(m/sec)^2$	500 $(m/sec)^2$	700 $(m/sec)^2$	900 $(m/sec)^2$	1100 $(m/sec)^2$	1300 $(m/sec)^2$
R/R_o	t (sec)	t (sec)	t (sec)	t (sec)	t (sec)	t (sec)
1	.00000000	.00000000	.00000000	.00000000	.00000000	.00000000
2	.20979336-02	.16270816-02	.13789248-02	.12170827-02	.11016646-02	.10140386-02
3	.35646598-02	.27630982-02	.23390341-02	.20638494-02	.18675940-02	.17185911-02
4	.49964936-02	.38721569-02	.32763598-02	.28905191-02	.26153449-02	.24064219-02
5	.64187592-02	.49738390-02	.42074511-02	.37116645-02	.33580988-02	.30896561-02
6	.78373242-02	.60726548-02	.51361199-02	.45306735-02	.40989203-02	.37711128-02
7	.92541549-02	.71701272-02	.60636534-02	.53486811-02	.48388360-02	.44517362-02
8	.10670065-01	.82668862-02	.69905840-02	.61661570-02	.55782708-02	.51319173-02
9	.12085440-01	.93632312-02	.79171647-02	.69833245-02	.63174265-02	.58118417-02
10	.13500484-01	.10459320-01	.88435286-02	.78003007-02	.70564093-02	.64916069-02
15	.20572733-01	.15937846-01	.13473728-01	.11883663-01	.10749987-01	.98892031-02

Section V

OSCILLATION OF CAVITATION BUBBLES

5.1 DIFFERENTIAL EQUATION GOVERNING THE OSCILLATION OF SPHERICAL BUBBLES

The differential equation governing the motion of the bubble wall of a spherical bubble is (see eq. 8 in Section 3.1)

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'(R) - p_\infty(t) - \frac{4\mu\dot{R}}{R} - \frac{2\sigma}{R}\right] \tag{1}$$

where

R = radius of the spherical bubble.

$\dot{R} = \frac{dR}{dt}$ = growing velocity of the radius.

ρ = mass density of the liquid.

$p'(R)$ = pressure inside the bubble, is a function of R.

$P_\infty(t)$ = liquid pressure far from the bubble.

μ = coefficient of shear viscosity.

σ = surface tension coefficient of the bubble surface.

Equation (1) is also the nonlinear differential equation governing the oscillation of the spherical bubble.

5.2 FREE OSCILLATION OF SPHERICAL BUBBLES

Setting the forcing pressure $p_\infty(t)$ equal to a constant, one obtains the differential equation for free oscillation of spherical bubbles as:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p'(R) - p_\infty - \frac{4\mu\dot{R}}{R} - \frac{2\sigma}{R}\right] \tag{2}$$

Since

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{2R^2\dot{R}}\frac{d}{dt}(R^3\dot{R}^2) \tag{3}$$

equation (2) may be written as

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = \frac{2}{3}\left(\frac{p'-p_\infty}{\rho}\right)\frac{d}{dt}\left(R^3\right) - \frac{2\sigma}{\rho}\frac{d}{dt}\left(R^2\right) - \frac{8\mu}{\rho}R\dot{R}^2 \qquad (4)$$

If the viscosity is small and the last term in eq. (4) may be neglected, one obtains

$$\frac{d}{dt}\left(R^3\dot{R}^2\right) = \frac{2}{3}\left(\frac{p'-p_\infty}{\rho}\right)\frac{d}{dt}\left(R^3\right) - \frac{2\sigma}{\rho}\frac{d}{dt}\left(R^2\right) \qquad (5)$$

Integration from $t = t_o$ to $t = t$ yields

$$R^3\dot{R}^2\Big|_{t_o}^{t} = \frac{2}{3\rho}\int_{t_o}^{t} p'\frac{d}{dt}\left(R^3\right)dt - \frac{2p_\infty}{3\rho}R^3\Big|_{t_o}^{t} - \frac{2\sigma}{\rho}R^2\Big|_{t_0}^{t}$$

or

$$R^3\dot{R}^2 - R_o^3\dot{R}_o^2 = \frac{2}{3\rho}\int_{t_o}^{t} p'(R)\frac{d}{dt}\left(R^3\right)dt - \frac{2p_\infty}{3\rho}\left(R^3 - R_o^3\right) - \frac{2\sigma}{\rho}\left(R^2 - R_o^2\right) \qquad (6)$$

where

R_o = radius of bubble at $t = t_o$

$\dot{R}_o = \frac{dR}{dt}$ at $t = t_0$

Let the time $t = t_o$ be the time when the rate of increase of the bubble radius is equal to zero, namely

$$\dot{R}_o = \frac{dR}{dt}\Big|_{t=t_o} = 0 \qquad (7)$$

R_o = maximum radius of bubble during oscillation.

Thus equation (6) reduces to

$$R^3\dot{R}^2 = \frac{2}{3\rho} \int_{t_o}^{t} p'(R) \frac{d}{dt} (R^3) - \frac{2p_\infty}{3\rho} (R^3 - R_o^3) - \frac{2\sigma}{\rho} (R^2 - R_o^2)$$

or

$$\dot{R}^2 = \frac{2}{3\rho R^3} \int_{t_o}^{t} p'(R) \frac{d}{dt} (R^3)\, dt - \frac{2p_\infty}{3\rho} \left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R} \left[1 - \left(\frac{R_o}{R}\right)^2\right] \tag{8}$$

If it is assumed that

$$\frac{p'(R)}{p_o'} = \left(\frac{R_o}{R}\right)^{3\gamma} \qquad \text{or} \qquad p'(R) = p_o' \left(\frac{R_o}{R}\right)^{3\gamma} \tag{9}$$

where

γ = ratio of specific heat of the vapor inside the bubble.

p_o' = pressure inside the bubble when $R = R_o$

Substitution of equation (9) into the integral in equation (8) and evaluating yields

$$\frac{2}{3\rho R^3} \int_{t_o}^{t} p'(R) \frac{d}{dt} (R^3)\, dt = \frac{2p_o'}{(3-3\gamma)\rho} \left[\left(\frac{R_o}{R}\right)^{3\gamma} - \left(\frac{R_o}{R}\right)^3\right] \tag{10}$$

Substitution of equation (10) into equation (8) gives

$$\dot{R}^2 = \frac{2p_o'}{(3-3\gamma)\rho} \left[\left(\frac{R_o}{R}\right)^{3\gamma} - \left(\frac{R_o}{R}\right)^3\right] - \frac{2p_\infty}{3\rho} \left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R} \left[1 - \left(\frac{R_o}{R}\right)^2\right] \tag{11}$$

5.3 VELOCITY VARIATION DURING OSCILLATION

Equation (11) gives the square of the growing velocity as:

$$\dot{R}^2 = \frac{2p_o'}{(3-3\gamma)\rho}\left[\left(\frac{R_o}{R}\right)^{3\gamma} - \left(\frac{R_o}{R}\right)^3\right] - \frac{2p_\infty}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R}\left[1 - \left(\frac{R_o}{R}\right)^2\right] \tag{12}$$

where

R_o = Maximum radius of the bubble during oscillation.

R = Radius of the bubble at any time.

p_o' = Pressure inside the bubble when $R = R_o$

ρ = Mass density of the bubble.

p_∞ = Pressure in the liquid.

σ = Surface tension coefficient of the bubble surface.

γ = Ratio of specific heat of vapor and/or gases inside the bubble.

Note that $\frac{p_o'}{\rho}$, $\frac{p_\infty}{\rho}$ and $\frac{\sigma}{\rho R}$ are non-dimensional and may be designated the minimum vapor pressure, the liquid pressure and the surface tension parameter, respectively. The surface tension parameter changes as the radius grows. For

$$\frac{p_o'}{\rho} = 1 \ (\text{m/sec})^2$$

$$\frac{\sigma}{\rho} = 0.0254 \ (\text{m}^3/\text{sec}^2)$$

$$R_o = 0.02 \text{ in} = 0.0508 \text{ cm}$$

and

$$\frac{p_\infty}{\rho} = 50,\ 100,\ 150 \ (\text{m/sec})^2 \tag{13}$$

the growing velocity as a function of the ratio of radius to maximum radius was analyzed. The results are tabulated in Table 5.1 and plotted in Figure 5-1, with the growing velocity versus the ratio of radius to maximum radius. It is seen that the growing velocity increases very fast and reaches the maximum very quickly as the bubble grows from the minimum radius and then decreases very fast after the maximum velocity. For the majority of the time, the growing velocity is low, as compared to the maximum. At the maximum and the minimum radius, the growing velocity is zero.

As can be seen in Figure 5-1, (or Table 5-1), the maximum growing velocities for the three cases are:

$$\begin{aligned}
\dot{R} &= 990 \text{ m/sec} , \quad \text{for } \frac{p_\infty}{\rho} = 50 \ (\text{m/sec})^2, \text{ at } \frac{R}{R_o} = 0.034 \\
\dot{R} &= 1456 \text{ m/sec} , \quad \text{for } \frac{p_\infty}{\rho} = 100 \ (\text{m/sec})^2, \text{ at } \frac{R}{R_o} = 0.028 \\
\dot{R} &= 1998 \text{ m/sec} , \quad \text{for } \frac{p_\infty}{\rho} = 150 \ (\text{m/sec})^2, \text{ at } \frac{R}{R_o} = 0.024
\end{aligned} \tag{14}$$

5.4 DETERMINATION OF MINIMUM AND MAXIMUM RADII

At the minimum and the maximum radius during oscillation, the growing velocity vanishes. From Table 5.1, when $\dot{R} = 0$ it is seen that minimum radius ratios are

$$\begin{aligned}
\frac{R}{R_o} &= 0.0256 , \quad \text{for } \frac{p_\infty}{\rho} = 50 \ (\text{m/sec})^2 \\
\frac{R}{R_o} &= 0.0213 , \quad \text{for } \frac{p_\infty}{\rho} = 100 \ (\text{m/sec})^2 \\
\frac{R}{R_o} &= 0.0183 , \quad \text{for } \frac{p_\infty}{\rho} = 150 \ (\text{m/sec})^2
\end{aligned} \tag{15}$$

Hence, when the maximum radius is specified, the minimum radius can be determined from the minimum radius ratio. Here, the maximum radius is assumed to be

$$R_o = 0.02 \text{ in} = 0.0508 \text{ cm} \tag{16}$$

Hence the minimum radius will be

$$
\begin{aligned}
R_m &= 0.0256 \times 0.0508 = 0.00130048 \text{ cm}, \quad \text{for } \frac{p_\infty}{\rho} = 50 \ (\text{m/sec})^2 \\
R_m &= 0.0213 \times 0.0508 = 0.00108204 \text{ cm}, \quad \text{for } \frac{p_\infty}{\rho} = 100 \ (\text{m/sec})^2 \\
R_m &= 0.0183 \times 0.0508 = 0.00092964 \text{ cm}, \quad \text{for } \frac{p_\infty}{\rho} = 150 \ (\text{m/sec})^2
\end{aligned}
\tag{17}
$$

5.5 DETERMINATION OF GROWING TIME FROM MAXIMUM TO MINIMUM RADIUS

To determine the growing time, one writes equation (12) as

$$\dot{R} = F$$

or

$$dt = \frac{dR}{F} \tag{18}$$

where

$$F = F\left(\frac{R_o}{R}\right) = \left\{ \frac{2p_o'}{(3-3\gamma)\rho} \left[\left(\frac{R_o}{R}\right)^{3\gamma} - \left(\frac{R_o}{R}\right)^3\right] - \frac{2p_\infty}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R}\left[1 - \left(\frac{R_o}{R}\right)^2\right] \right\}^{1/2} \tag{19}$$

Integration of equation (19) from the minimum radius (R_m) to R = R yield the growing time

$$t = \int_{R_m}^{R} \frac{dR}{F} \tag{20}$$

The integration in equation (20) cannot be carried out in closed form and has to be integrated by numerical method. Results of the numerical integration of equation (20) for the three cases specified in equation (13) are tabulated in Table 5-2 and plotted in Figure 5-2 with growing time versus the ratio of radius to maximum radius. It is seen that the majority of the time is spent for $\frac{R}{R_o} = 0.5$ through 1, since the growing velocity for this later part of growing is relatively small.

5.6 DETERMINATION OF PERIOD AND FREQUENCY OF OSCILLATION

The period of oscillation is equal to twice the growing time from the minimum to the maximum radius. It is seen in Table 5-2 that the growing times from the minimum to the maximum radius are

$$\begin{aligned} t &= 0.001491 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} = 50 \ (\text{m/sec})^2 \\ t &= 0.001299 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} = 100 \ (\text{m/sec})^2 \\ t &= 0.001166 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} = 150 \ (\text{m/sec})^2 \end{aligned} \tag{21}$$

Hence the periods of oscillation are

$$\begin{aligned} T &= 2 \times 0.001491 = 0.002982 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} \ 50 \ (\text{m/sec})^2 \\ T &= 2 \times 0.001299 = 0.002598 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} \ 100 \ (\text{m/sec})^2 \\ T &= 2 \times 0.001166 = 0.002332 \text{ sec} \quad , \text{ for } \frac{p_\infty}{\rho} \ 150 \ (\text{m/sec})^2 \end{aligned} \tag{22}$$

The frequency of oscillation is equal to the reciprocal of the period:

$$f = \frac{1}{T} \tag{23}$$

Hence the frequencies of oscillation are

$$\begin{aligned} f &= \frac{1}{0.002982} = 335.35 \text{ cps} \quad , \text{ for } \frac{p_\infty}{\rho} = 50 \ (\text{m/sec})^2 \\ f &= \frac{1}{0.002598} = 384.91 \text{ cps} \quad , \text{ for } \frac{p_\infty}{\rho} = 100 \ (\text{m/sec})^2 \\ f &= \frac{1}{0.002332} = 428.82 \text{ cps} \quad , \text{ for } \frac{p_\infty}{\rho} = 150 \ (\text{m/sec})^2 \end{aligned} \tag{24}$$

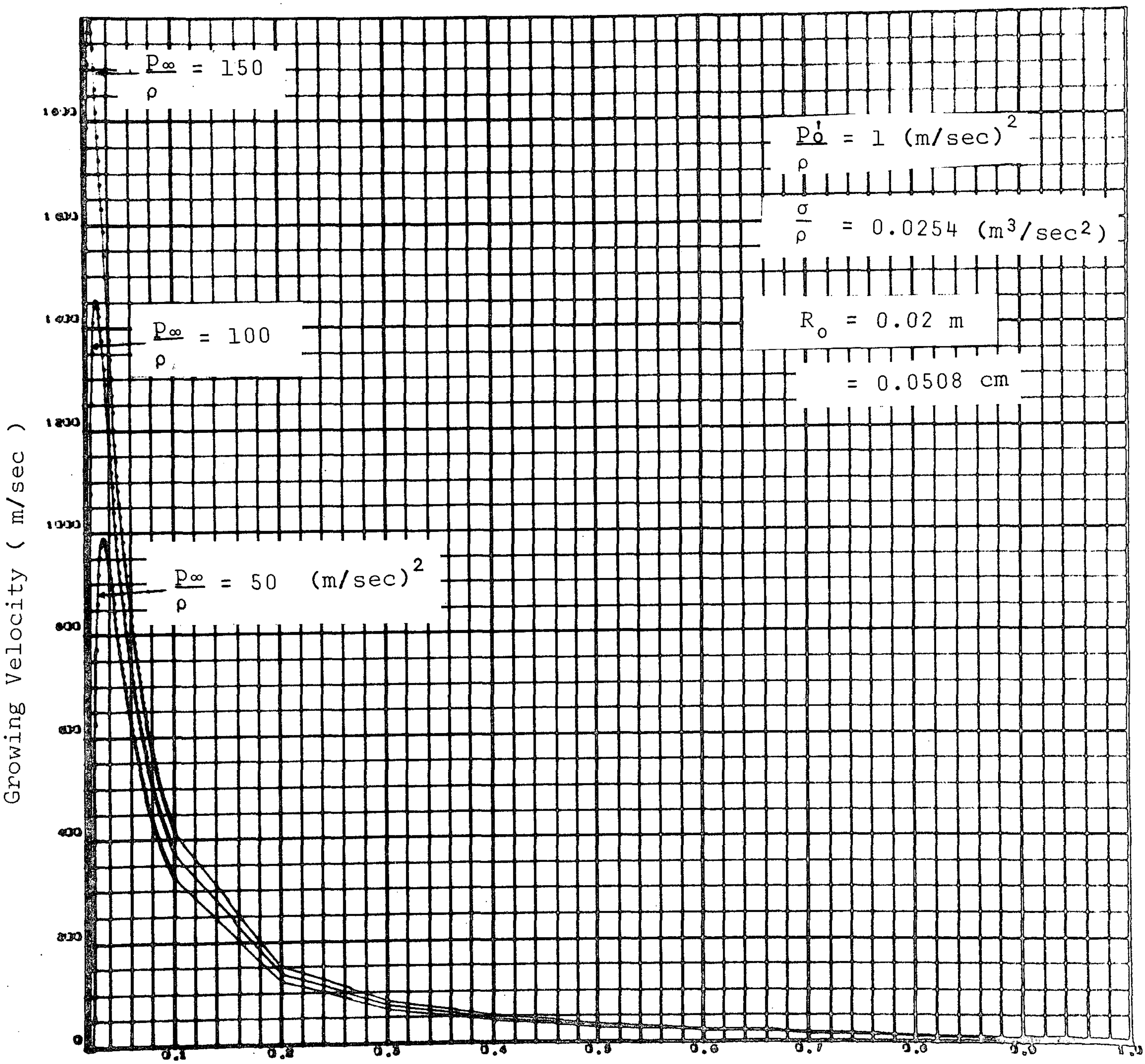

FIG. 5.1 GROWING VELOCITY VERSUS RADIUS RATIO, EQUATION (12)

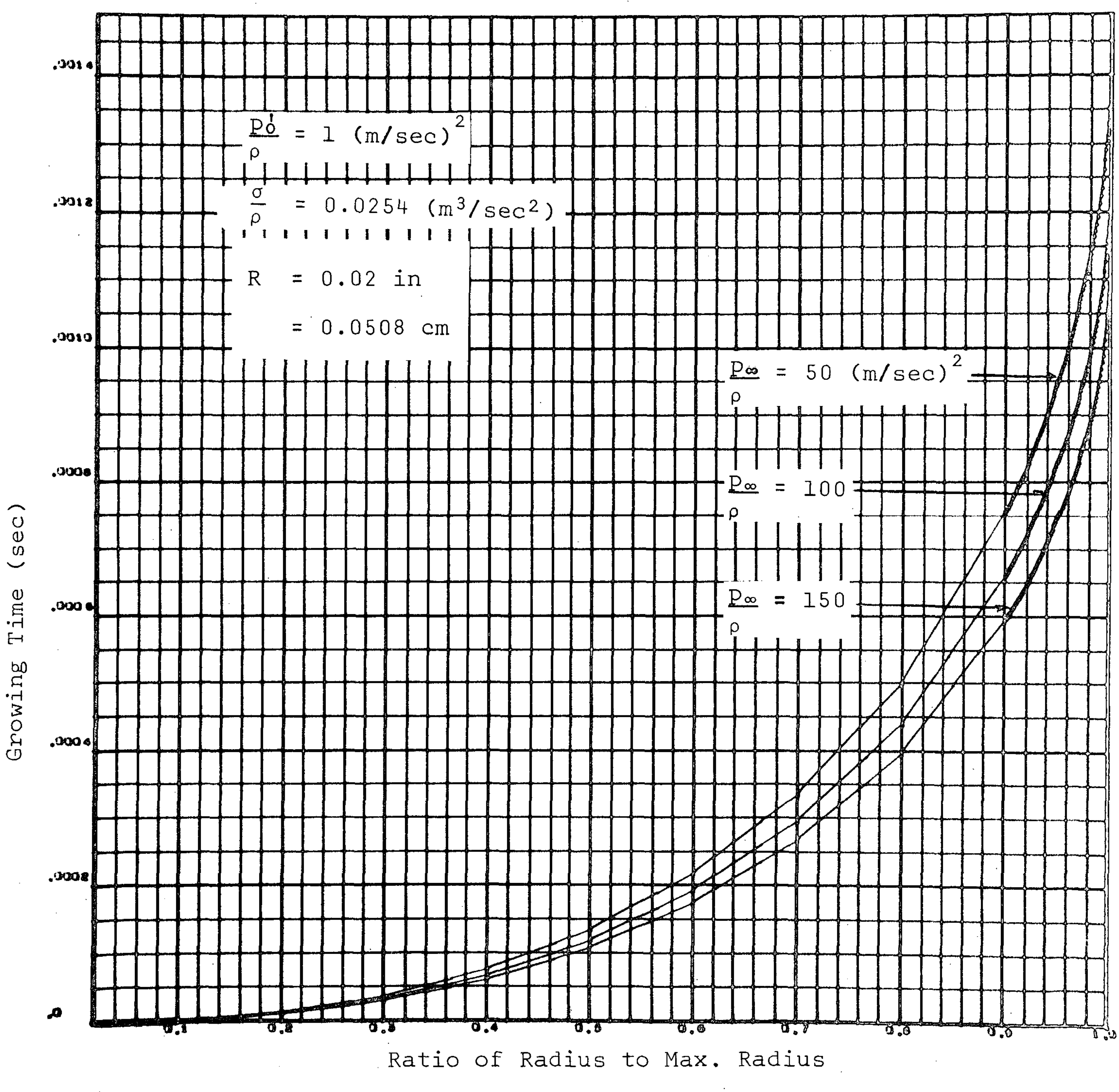

FIG. 5.2 GROWING TIME VERSUS RADIUS RATIO, EQUATION (20)

TABLE 5.1 GROWING VELOCITY VERSUS RADIUS RATIO, EQUATION (12)

$\frac{p_o'}{\rho} = 1\ (m/sec)^2$, $\frac{\sigma}{\rho} = 0.0254 (m^3/sec^2)$, $R_o = 0.0508$ cm					
$\frac{p_\infty}{\rho} = 50 (m/sec)^2$		$\frac{p_\infty}{\rho} = 100 (m/sec)^2$		$\frac{p_\infty}{\rho} = 150 (m/sec)^2$	
R/R_o	$\dot{R}$ (m/sec)	R/R_o	$\dot{R}$ (m/sec)	R/R_o	$\dot{R}$ (m/sec)
.0256	.00000000	.0213	.00000000	.0183	.00000000
.0257	.59016947+02	.0214	.16607001+03	.0184	.19153883+03
.03	.92069067+03	.027	.14457208+04	.023	.19763177+04
.033	.98651910+03	.028	.14563983+04	.024	.19980717+04
.034	.98951041+03	.029	.14541866+04	.025	.19949377+04
.035	.98705723+03	.04	.11782545+04	.04	.13816960+04
.06	.63052161+03	.06	.74286283+03	.06	.84031713+03
.08	.44153809+03	.08	.50992818+03	.08	.57017332+03
.1	.32795098+03	.1	.37530237+03	.1	.41731510+03
.201	.12118630+03	.201	.13695829+03	.201	.15109278+03
.302	.65439157+02	.302	.73886077+02	.302	.81461780+02
.403	.41296041+02	.403	.46704581+02	.403	.51548742+02
.504	.28162379+02	.504	.31939850+02	.504	.35315555+02
.605	.19931228+02	.605	.22681422+02	.605	.25132449+02
.706	.14193929+02	.706	.16212936+02	.706	.18006971+02
.807	.97733811+01	.807	.11207601+02	.807	.12478047+02
.908	.58624441+01	.908	.67499404+01	.908	.75336008+01
1	.00000000	1	.00000000	1	.00000000

TABLE 5.2 GROWING TIME VERSUS RADIUS RATIO, EQUATION (20)

$\frac{p_o'}{\rho} = 1(m/sec)^2$, $\frac{\sigma}{\rho} = 0.0254(m^3/sec^2)$, $R_o = 0.0508$ cm					
$\frac{p_\infty}{\rho} = 50(m/sec)^2$		$\frac{p_\infty}{\rho} = 100(m/sec)^2$		$\frac{p_\infty}{\rho} = 150(m/sec)^2$	
R/R_o	t (sec)	R/R_o	t (sec)	R/R_o	t (sec)
.0256	.00000000	.0213	.00000000	.0183	.00000000
.0257	.33888829-08	.0214	.44848466-08	.0184	.10441826-08
.03	.16035638-06	.027	.11608848-06	.023	.70865008-07
.033	.22270341-06	.028	.12986035-06	.024	.80917655-07
.034	.24293612-06	.029	.14359555-06	.025	.90927471-07
.035	.26316590-06	.04	.30949377-06	.04	.26928377-06
.06	.89061124-06	.06	.74348303-06	.06	.64818597-06
.08	.16567892-05	.08	.14016946-05	.08	.12341341-05
.1	.27164283-05	.1	.23240837-05	.1	.20617139-05
.201	.13832176-04	.201	.12128094-04	.201	.10930404-04
.302	.37323278-04	.302	.32931417-04	.302	.29797800-04
.403	.76898401-04	.403	.67954348-04	.403	.61547466-04
.504	.13680060-03	.504	.12084343-03	.504	.10942231-03
.605	.22270192-03	.605	.19645095-03	.605	.17772590-03
.706	.34332052-03	.706	.30229993-03	.706	.27313415-03
.807	.51501863-03	.807	.45230314-03	.807	.40801848-03
.908	.77959753-03	.908	.68251800-03	.908	.61451709-03
1	.14910834-02	1	.12988718-02	1	.11659031-02

Section VI

STABILITY OF CAVITATION BUBBLES

6.1 DIFFERENTIAL EQUATION GOVERNING BUBBLE STABILITY

If some infinitesimally small deviation from the spherical symmetry is given to the spherical bubble, one would like to know whether this perturbation will grow or diminish. The stability equations are thus the linearized forms of the equations that govern the dynamics of the spherical bubble. To analyze the stability problem, the perturbation is expressed in terms of spherical harmonics. Thus the equation of the bubble wall is:

$$r = R + \sum_n a_n Y_n \qquad (1)$$

where

$R = R(t)$ = the unperturbed radius of the bubble.

Y_n = spherical harnomic of degree n

$a_n = a_n(t)$ = the amplitude of perturbation

and

$$|a_n| << R \qquad (2)$$

It can be shown that the stability equation for a_n is

$$\frac{d^2 a_n}{dt^2} + 3\frac{\dot{R}}{R}\frac{da_n}{dt} - A_n a_n = 0 \qquad (3)$$

where

$$A_n = \frac{[n(n-1)\rho - (n+1)(n+2)\rho']\ddot{R} - (n-1)n(n+1)(n+2)\sigma/R^2}{R[n\rho + (n+1)\rho']} \qquad (4)$$

It is noted that the mode with n = 1 corresponds to a translation of the bubble as a whole, and only the modes with $n \geq 2$ are relevant to the distortion of the spherical bubble. The density of the vapor (ρ'), being much smaller than the density of the liquid (ρ), may be neglected, and the expression for A_n reduces

$$A_n = \frac{(n-1)\ddot{R}}{R} - (n-1)(n+1)(n+2)\frac{\sigma}{\rho R^3} \tag{5}$$

By the substitution of

$$a_n = b_n \exp\left[-\frac{3}{2}\int_{t_o}^{t} \frac{\dot{R}}{R}\,dt\right] = \left(\frac{R_o}{R}\right)^{3/2} b_n \tag{6}$$

where R_o = the value of R at $t = t_o$.

Equation (3) can be transformed to

$$\frac{d^2 b_n}{dt^2} + B_n b_n = 0 \tag{7}$$

where

$$B_n = (n-1)(n+1)(n+2)\frac{\sigma}{\rho R^3} - \frac{(n+1/2)\ddot{R}}{R} - \frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2 \tag{8}$$

6.2 CRITERIA FOR STABILITY OR INSTABILITY

It can easily be shown that the criteria of stability of differential equations in the type of equation (7) are

$$\begin{aligned} &B_n \geq 0 \quad \text{stable} \\ &B_n < 0 \quad \text{unstable} \end{aligned} \tag{9}$$

Thus these are the criteria of the stability and unstability of the spherical cavitation bubble. Here, instability means that the amplitude of disturbance is increasing expotentially, while stability means that the amplitude of disturbance is decreasing expotentially.

6.3 DISCUSSION OF THE CRITERIA

(a) For $n \geq 2$, the first term in the expression of B_n in equation (8):

$$(n-1)(n+1)(n+2)\,\frac{\sigma}{\rho R^3}$$

is always positive. Hence the effect of the surface tension (σ) is always stabilizing, and this affirms common physical reasoning that the surface tension is to prevent the bubble from collapsing.

(b) When the bubble is growing:

$$\dot{R} > 0$$

and when the growing velocity increases with time

$$\frac{d}{dt}(\dot{R}) = \ddot{R} > 0$$

the criteria of stability and unstability will be

$$\text{stable:} \quad (n-1)(n+1)(n+2)\,\frac{\sigma}{\rho R^3} \geq (n+\tfrac{1}{2})\,\frac{\ddot{R}}{R} + \frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2 \tag{10}$$

$$\text{unstable:} \quad (n-1)(n+1)(n+2)\,\frac{\sigma}{\rho R^3} < (n+\tfrac{1}{2})\,\frac{\ddot{R}}{R} + \frac{3}{4}\left(\frac{\dot{R}}{R}\right)^2 \tag{11}$$

Since the surface tension of the vapor-liquid interface is small for many liquids; when the bubble is growing, and the growing velocity increases rapidly with time, as in the case of boiling, the situation will be unstable.

6.4 REGIONS OF STABILITY AND INSTABILITY

For the convenience of parameter study, equations (10) and (11) are reduced to non-dimensional forms:

Stable:

$$\frac{\sigma\tau^2}{\rho R_o^3} \geq F$$

2

Unstable:

$$\frac{\sigma\tau^2}{\rho R_o^3} < F$$

where

$$F = F(R,\dot{R},\ddot{R}) = \frac{1}{(n-1)(n+1)(n+2)}\left[\left(n+\frac{1}{2}\right)\frac{\ddot{R}\tau^2}{R} + \frac{3}{4}\left(\frac{\dot{R}\tau}{R}\right)^2\right]\left(\frac{R}{R_o}\right)^3$$

R_o = radius of bubble at some specific time (for example initial time)

τ = some characteristic time of the bubble (for example the period of oscillation)

Thus the regions of stability and instability can be plotted with surface tension parameter $\frac{\sigma\tau^2}{\rho R_o^3}$ versus velocity parameter $\frac{\dot{R}\tau}{R}$, acceleration parameter $\frac{\ddot{R}\tau^2}{R}$ or radius ratio $\frac{R}{R_o}$.

with acceleration parameter $\frac{\ddot{R}\tau^2}{R} = 2$

velocity parameter $\frac{\dot{R}\tau}{R} = 10$

degree of harmonic n = 2, 3, 4

value of function F versus $\frac{R}{R_o}$ are given in Table 6.1; the curve of function F versus $\frac{R}{R_o}$ and the region of stability and instability are shown in Figure 6.1.

6.5 BUBBLE SIZE AT COLLAPSE

Equation (28) in Section 3 gives the square of the growing velocity as

$$\dot{R}^2 = \frac{2p'_o}{(3-\gamma)\rho}\left[\left(\frac{R_o}{R}\right)^\gamma - \left(\frac{R_o}{R}\right)^3\right] - \frac{2p_\infty}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^3\right] - \frac{2\sigma}{\rho R}\left[1 - \left(\frac{R_o}{R}\right)^2\right] \quad (12)$$

where

p'_o = pressure inside the bubble

p_∞ = pressure in the liquid

R_o = radius of bubble at initial time

γ = ratio of specific heat of vapor gas inside the bubble

It can be shown that when the right hand side of equation (12) becomes negative, the bubble will collapse. Equating the right hand side of equation (12) to zero, one obtains the expression relating the bubble size at the time of collapse to other parameters as

$$\frac{2p'_o}{(3-\gamma)\rho}\left[\left(\frac{R_o}{R}\right)^{\gamma} - \left(\frac{R_o}{R}\right)^{3}\right] = \frac{2p_\infty}{3\rho}\left[1 - \left(\frac{R_o}{R}\right)^{3}\right] + \frac{2\sigma}{\rho R}\left[1 - \left(\frac{R_o}{R}\right)^{2}\right] \tag{13}$$

Equation (13) can be written in non-dimensional form as

$$\frac{p'_o}{p_\infty} = \frac{(3-\gamma)}{3}\,\frac{\left[\left(\frac{R}{R_o}\right)^{3} - 1\right]\left(\frac{R}{R_o}\right)^{\gamma}}{\left(\frac{R}{R_o}\right)^{3} - \left(\frac{R}{R_o}\right)^{\gamma}}$$

$$+ (3-\gamma)\,\frac{\left[\left(\frac{R}{R_o}\right)^{2} - 1\right]\left(\frac{R}{R_o}\right)^{1+\gamma}}{\left(\frac{R}{R_o}\right)^{3} - \left(\frac{R}{R_o}\right)^{\gamma}}\left(\frac{\sigma}{p_\infty R}\right) \tag{14}$$

The surface tension term in equation (14) is small and may be neglected. By doing so, one obtains the pressure ratio versus the bubble size at collapse as

$$\frac{p'_o}{p_\infty} = \frac{(3-\gamma)}{3}\,\frac{\left[\left(\frac{R}{R_o}\right)^{3} - 1\right]\left(\frac{R}{R_o}\right)^{\gamma}}{\left(\frac{R}{R_o}\right)^{3} - \left(\frac{R}{R_o}\right)^{\gamma}} \tag{15}$$

For specific heat ratio

$\gamma = 1.4,\ 1$

the curve of the pressure ratio $\frac{p'_o}{p_\infty}$ versus the bubble radius ratio $\frac{R}{R_o}$ at collapse is shown in Table 6.2 and Figure 6.2.

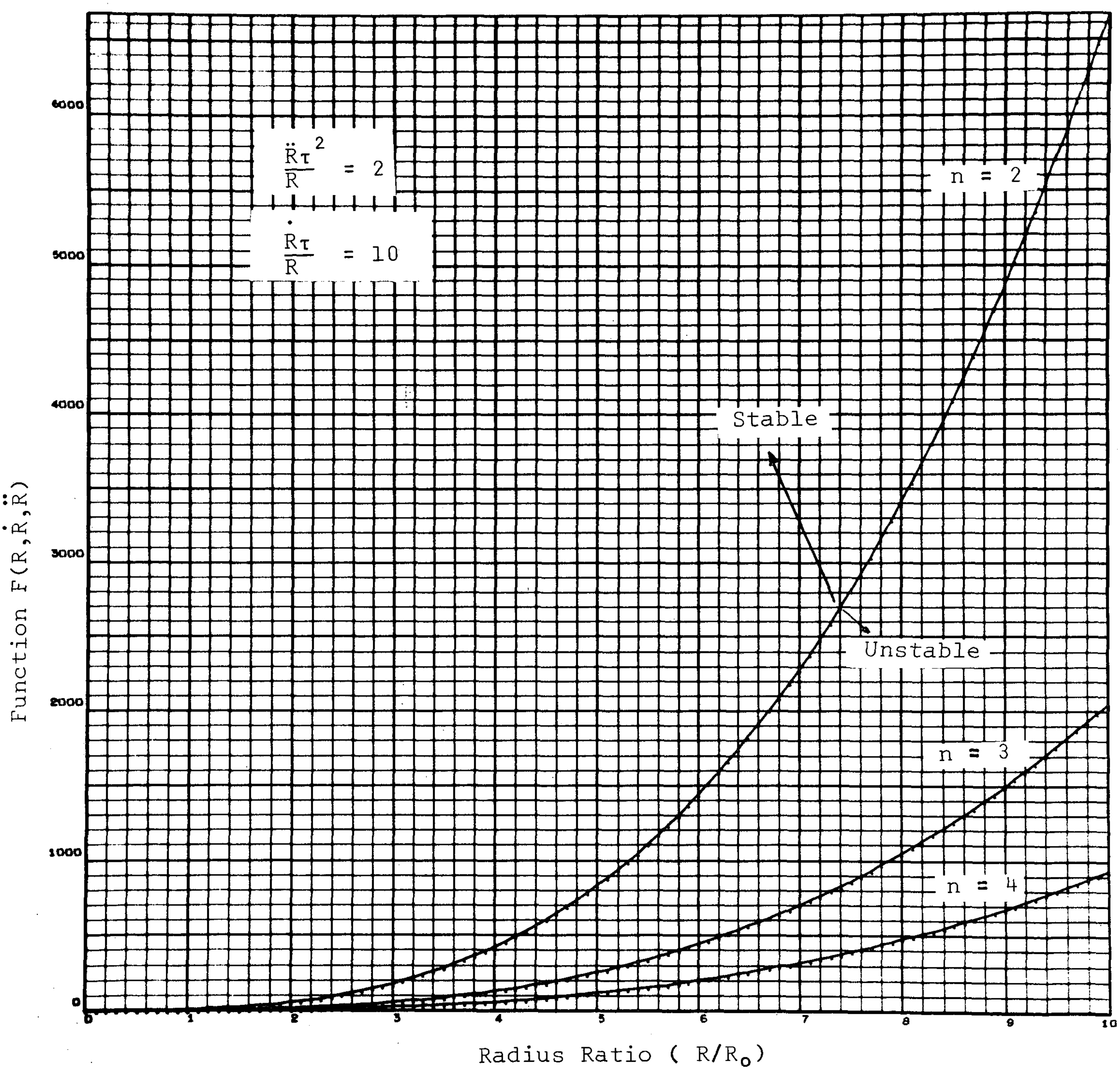

FIG. 6.1 REGIONS OF STABILITY AND INSTABILITY

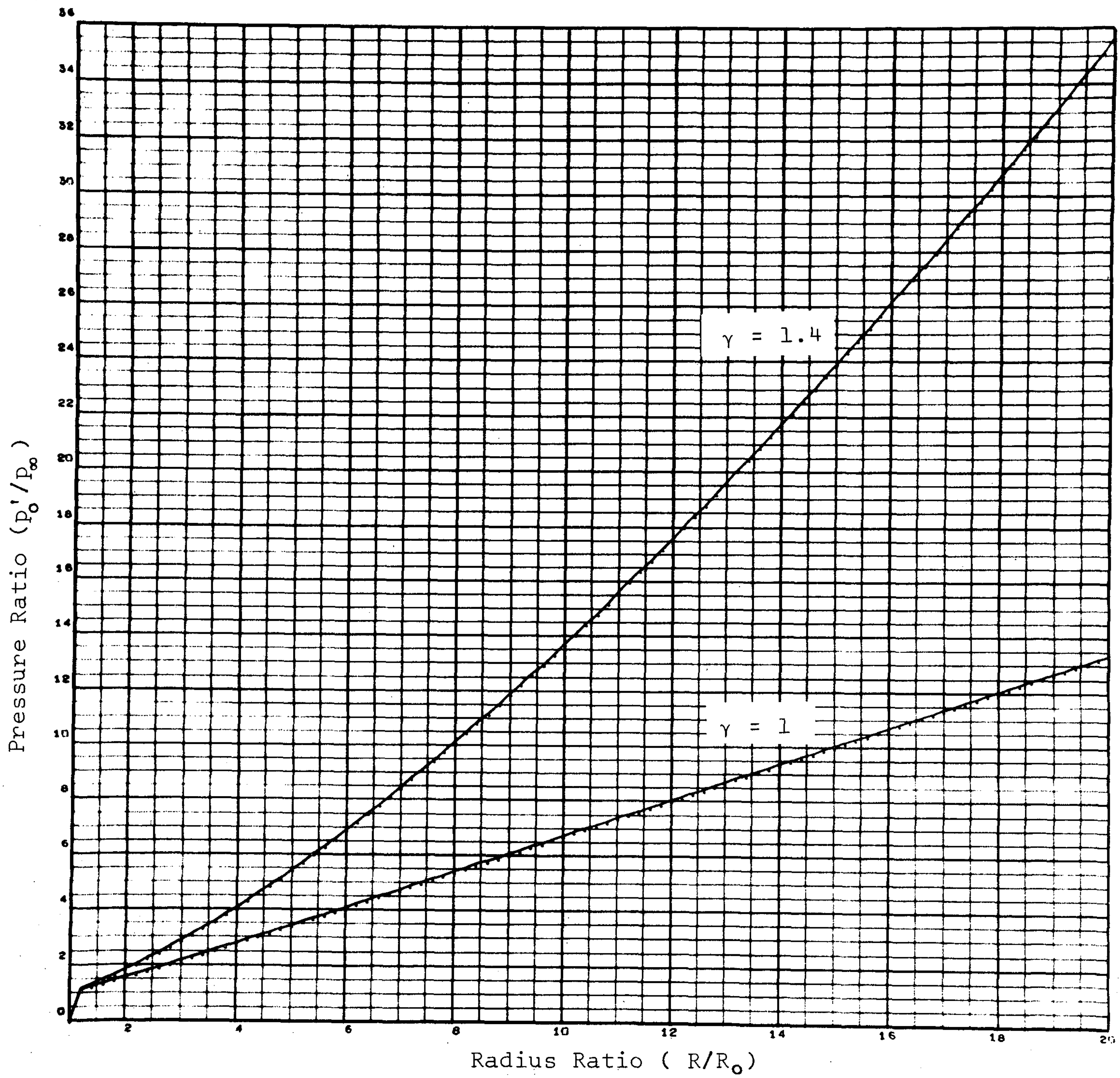

FIG.6.2 PRESSURE RATIO $\left(\frac{P_o'}{P_\infty}\right)$ VERSUS BUBBLE RADIUS RATIO $\left(\frac{R}{R_o}\right)$ AT COLLAPSE

TABLE 6.1 REGIONS OF STABILITY AND INSTABILITY

$$\frac{\ddot{R}\tau^2}{R} = 2 \quad , \quad \frac{\dot{R}\tau}{R} = 10$$

$\frac{R}{R_o}$	n = 2 $F(R,\dot{R},\ddot{R})$	n = 3 $F(R,\dot{R},\ddot{R})$	n = 4 $F(R,\dot{R},\ddot{R})$
0	00000000	00000000	00000000
1	.66666667+01	.20500000+01	.93333333+00
2	.53333333+02	.16400000+02	.74666667+01
3	.18000000+03	.55350000+02	.25200000+02
4	.42666667+03	.13120000+03	.59733333+02
5	.83333333+03	.25625000+03	.11666667+03
6	.14400000+04	.44280000+03	.20160000+03
7	.22866667+04	.70315000+03	.32013333+03
8	.34133333+04	.10496000+04	.47786667+03
9	.48600000+04	.14944500+04	.68040000+03
10	.66666667+04	.20500000+04	.93333333+03

TABLE 6.2 PRESSURE RATIO (p_o'/p_∞) VERSUS BUBBLE RADIUS RATIO(R/R_o) AT COLLAPSE

Radius Ratio R/R_o	$\gamma = 1$ $\frac{p_o'}{p_\infty}$	$\gamma = 1.4$ $\frac{p_o'}{p_\infty}$	Radius Ratio R/R_o	$\gamma = 1$ $\frac{p_o'}{p_\infty}$	$\gamma = 1.4$ $\frac{p_o'}{p_\infty}$
1	.00000000	.00000000	11	.73888889+01	.15632000+02
2	.15555556+01	.18377832+01	12	.80512821+01	.17610667+02
3	.21666667+01	.28891200+01	13	.87142857+01	.19653333+02
4	.28000000+01	.41027200+01	14	.93777778+01	.21760000+02
5	.34444444+01	.54508613+01	15	.10041667+02	.23940736+02
6	.40952381+01	.69152000+01	16	.10705882+02	.26133333+02
7	.47500000+01	.84837333+01	17	.11370370+02	.28426667+02
8	.54074074+01	.10147200+02	18	.12035088+02	.30773333+02
9	.60666667+01	.11893333+02	19	.12700000+02	.33173333+02
10	.67272727+01	.13728165+02	20	.13365079+02	.35645067+02

Section VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 CONCLUSIONS

The general differential equations governing the dynamics of the cavitation bubbles are complicated and difficult to solve. However, with the assumption of spherical symmetry and some further simplification, the equations were reduced to tractable forms. In this report, the growth and collapse of cavitation bubbles were studied, the oscillations of the bubbles were analyzed, and the stability criteria investigated. It was found that the cavitation bubbles are unstable in general; that the frequency of oscillation is high, and the oscillation not sinusoidal; that there is much variation in the growing velocity, and that the growing velocity is subsonic in general.

7.2 RECOMMENDATIONS

It is considered that the theory and results on cavitation bubbles developed in this study can be advantageously applied to investigate the effect of cavitation bubbles on the dynamics of propellant feed lines in the space rocket engines. Since the cavitation compliance of the turbopump is important in the analyses of the structural and propulsion systems coupling longitudinal stability (POGO), the following are recommended for future study:

(a) Establish a mathematical model and derive the dynamic equations of the propellant feedline system of aerospace rocket engines with the effect of cavitation bubbles included.

(b) Solve the dynamic equations and determine the dynamic characteristics of the propellant feedline system.

(c) Investigate the effect of cavitation bubbles on the dynamics of the feedline system and on the POGO problem.

Section VIII

APPENDICES

8.1 APPENDIX A. NOMENCLATURE

Notation	Description
$a_n = a_n(t)$	Amplitude of perturbation of bubble radius in nth harmonic
b	Body force per unit mass
b_n	$\left(\frac{R_o}{R}\right)^{-3/2} a_n$
D_{12}	coefficient of mass diffusion
f	Frequencies of Oscillation
h	Heat flux Vector
k	Boltzmann constant
m	Molecular mass
N	Unit normal at the bubble wall
n	Number density
P_o	Some reference pressure
p	Pressure
$p'(R)$	Pressure inside the bubble
p'_o	Pressure inside the bubble when $R = R_o$
$p_\infty(t)$	Pressure field at infinity
q	Heat generated per unit time and unit volume
r	Eulerian spatial radius
R	Radius of bubble
$R_o = R(t_o)$	Radius of bubble at $t = t_o$
$\dot{R} = \frac{dR}{dt}$	Growing velocity
$\dot{R}_o$	Growing velocity at $t = t_o$

Notation	Description
$\ddot{R} = \frac{d^2R}{dt^2}$	Growing acceleration
R_m	Minimum radius of bubble
s	Entropy per unit mass
T	Temperature
t	Growing time
U	Internal energy per unit mass
V	Growing velocity
Y_n	Spherical harmonic of degree n
β	Bulk viscosity
γ	Ratio of specific heat of the vapor inside the bubble
λ	Coefficients of thermal conductivity
μ	Shear viscosity
ρ	Mass density
σ	Surface tension coefficient
τ'	Component of the tensor
∇	Gradient operator