TECHNIQUES OF ORBITAL DECAY AND LONG-TERM EPHEMERIS PREDICTION FOR SATELLITES IN EARTH ORBIT

Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

AERO-ASTRODYNAMICS LABORATORY

HUNTSVILLE, ALABAMA

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COMPUTER SCIENCES CORPORATION

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8300 South Whitesburg Drive

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Major Offices and Facilities Throughout the World

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November 1971

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FOREWORD

This report, which supersedes the interim report dated May 1971, presents the results of work performed by Computer Sciences Corporation's Aerospace Systems Center while under contract to the Aero-Astrodynamics Laboratory of the George C. Marshall Space Flight Center, Contract NAS8-26113.

The authors are grateful to Messrs. L. D. Mullins and B. S. Perrine (MSFC-S&E-AERO-MMD) for their technical assistance and to Messrs. W. J. Elkins and M. M. Hansing (CSC) for their programming support.

SUMMARY

The methods of both special and general perturbation theory are employed in solving the equations of motion for a satellite subjected to the perturbational effects of earth oblateness and atmospheric drag. In the special perturbation method, Cowell and variation-of-parameters formulations of the motion equations are implemented and numerically integrated by means of a MARVES (Marshall Vehicle Engineering Simulation System) computer program. Variations in the orbital elements due to drag are computed using the 1970 Jacchia atmospheric density model, which includes the effects of semiannual variations, diurnal bulge, solar activity, and geomagnetic activity. In the general perturbation method, two-variable asymptotic series and the automated manipulation capabilities of FORMAC (Formula Manipulation Compiler) are used to obtain analytical solutions to the variation-of-parameters equations. Solutions are obtained when considering the effect of oblateness only (J_2 and J_3) and the combined effects of oblateness and drag. These solutions are then numerically evaluated by means of a FORTRAN program in which an updating scheme is used to maintain accurate epoch values of the elements. The atmospheric density function is approximated by a Fourier series in true anomaly, and the 1970 Jacchia model is used to periodically update the Fourier coefficients. The accuracy of both methods is demonstrated by comparing computed orbital elements to actual elements (or elements computed by standard MSFC programs) over time spans of up to 8 days for the special perturbation method and up to 356 days for the general perturbation method.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

NOMENCLATURE

Mathematical Symbols

a	semimajor axis
a, b, c	a set of constants arising in the solution of the ordinary differential equations having $\tilde{t}$ as the independent variable (explicitly defined in Appendix F)
a_i, b_i	Fourier coefficients appearing in Fourier series approximation to atmospheric density function
A	a constant arising in the solution of the ordinary differential equations having $\tilde{t}$ as the independent variable (explicitly defined in Appendix E)
(A/m)	satellite area/mass ratio
b	angle measured normal to orbital plane in direction of normal perturbative acceleration
B	orbital element defined as $a^{-1/2}$
$C_1, C_2, \ldots$	a set of constants arising in the solution of the ordinary differential equations having $\tilde{t}$ as the independent variable (explicitly defined in Appendix E)
$C(\tilde{t})$	integration "constant" associated with asymptotic series solution development (see Paragraph 4.3.1)
C_D	aerodynamic drag coefficient
D	drag force magnitude per unit mass
D_2	a constant arising in the solution of the ordinary differential equations having $\tilde{t}$ as the independent variable (explicitly defined in Appendix F)
e	eccentricity
h	specific angular momentum

i	inclination relative to earth equatorial plane
J_2, J_3, J_4	coefficients of second, third and fourth harmonics, respectively, of earth gravitational potential
K^*	constant arising in the formulation of the differential equations of motion for tangential atmospheric drag (explicitly defined in Paragraph 2.5.2)
M	mean anomaly
n	mean motion defined as $(\mu/a^3)^{1/2}$
p	semilatus parameter defined as $a(1-e^2)$
$\bar{r}$	geocentric radius vector
$\ddot{\bar{r}}$	perturbative acceleration vector
r_e	equatorial radius of earth
R	perturbative gravitational potential function
t	time
$\bar{t}$	fast time variable defined as $t(1+\alpha_2'\epsilon^2)$
$\tilde{t}$	slow time variable defined as ϵt
u	argument of latitude $(\nu+\omega)$
$\bar{v}$	inertial velocity vector
$\bar{v}_R$	relative velocity vector
α	a constant arising in the solution of the ordinary differential equations having $\tilde{t}$ as the independent variable (explicitly defined in Appendix E)

NOMENCLATURE (Continued)

$\alpha_i, \beta_i, \gamma_i, \delta_i$	a set of constants used to represent different linear combinations of the Fourier coefficients $\underline{a}_i$ and $\underline{b}_i$ (explicitly defined in Appendix F)
α'	right ascension of satellite subpoint
α_f	flight path angle (positive above local horizontal)
β'	angle between local latitude and orbital planes
δ	declination of satellite subpoint
ϵ	small perturbative parameter defined as $(3/2)J_2$
η	transformation parameter defined as e $\sin\omega$
λ	angle between local longitude and orbital planes
λ_1, λ_2	constants appearing in the $\underline{\xi}$ and $\underline{\eta}$ solutions for oblateness/drag (explicitly defined in Appendix F)
μ	earth gravitational constant
ν	true anomaly
ξ	transformation parameter defined as e $\cos\omega$
ρ	atmospheric density
ϕ	angle between radius and velocity vectors
ω	argument of perigee
ω_e	magnitude of earth rotational velocity vector
Ω	right ascension of ascending node

NOTE: The subscript "0" is used to denote the epoch (or reference) value of an element or element function (i.e., e_0, $e_0^{(0)}$, $e_0^{(1)}$).

NOMENCLATURE (Continued)

Element Types and Variations

long-periodic variation - A variation periodic with respect to $\underline{\omega}$ or multiples of $\underline{\omega}$; for example, $\sin\omega$.

mean orbital elements - The osculating elements with the short periodic variations removed.

osculating orbital elements - The instantaneous elements defining the continually changing elliptical orbit.

secular variation - A steady nonoscillatory variation from the epoch value, i.e., a variation directly proportional to the independent variable; for example, Ct.

short-periodic variation - A variation periodic with respect to linear combinations of $\underline{\nu}$ and $\underline{\omega}$; for example, $\cos(\nu+\omega)$.

SECTION 1 - INTRODUCTION

The purpose of this study is to develop techniques of orbital decay and long-term ephemeris prediction for satellites in elliptical earth-orbit. These techniques are to be accurate and flexible, and are to lead themselves to rapid computation. In order to meet current needs, emphasis is to be placed on the development of ephemeris prediction techniques for low-eccentricity, near-earth orbits when considering the perturbational effects of earth oblateness and atmospheric drag.

Classically, two general methods of attack are available for solving this problem. These methods are known as the methods of special and general perturbations, respectively. In both methods, the equations of motion may be formulated either as three second-order differential equations (for the perturbative accelerations) or as six first-order differential equations (for some set of fundamental orbital elements). The two methods differ in that special perturbation formulations (such as Cowell's, Encke's, variation-of-parameters, etc.) employ various numerical integration procedures (such as Runge-Kutta, Fehlberg, Shanks, etc.) to obtain the solution, while general perturbation techniques (such as variation-of-parameters, variation-of-coordinates, etc.) generally employ series expansions (such as Taylor's, multivariable asymptotic, etc.) combined with analytical integration to achieve the desired solution. In choosing one method or the other, one must keep in mind both the nature of the orbit under consideration and the nature of the solutions desired.

The main advantages of the special perturbation method lie in simplicity of formulation, applicability to any type of orbit in any perturbing force field, and compactness of storage requirements for program solution. This method is ideally suited for calculating orbits of limited duration. The main disadvantages inherent in this method are the inducement of errors (truncation and round-off) due to the numerical nature of the process, the resulting lack of application to orbits of long duration, and the extensive computation time required for solution.

The primary advantages of the general perturbation method lie in its applicability to orbits of long duration, its relatively rapid computer solution time, and its ability to provide a clearer geometric conception of the effects of the various perturbations. On the other hand, in applying this method one is faced with much analytical labor in formulating the equations to include various perturbations and in obtaining the solutions to these equations.

To achieve extended applicability in attacking the problem at hand, it was decided to employ formulations of both methods. In the special perturbation method both the Cowell and the variation-of-parameters formulations are employed, while the general perturbation method consists of the variation-of-parameters formulation using two-variable asymptotic series expansions. To alleviate the analytical labor required, the automated manipulation capabilities of the FORMAC (Formula Manipulation Compiler) language are utilized.

SECTION 2 - DERIVATION OF THE VARIATION-OF-PARAMETERS DIFFERENTIAL EQUATIONS OF MOTION (LAGRANGE'S PLANETARY EQUATIONS)

The purpose of this section is to derive, by the method of perturbative differentiation, the differential equations of motion for a selected set of orbital elements (or parameters) when considering the perturbational effects of earth oblateness and tangential atmospheric drag. Since perturbative forces are additive, the differential equations for each perturbational effect can be formulated separately. This set of differential equations will then be solved numerically by the methods of special perturbation theory and analytically by the methods of general perturbation theory.

2.1 SELECTED ORBITAL ELEMENT SET

The orbital element set selected for consideration is

$$(B, e, i, \Omega, \omega, M(\text{or } \nu); t)$$

where $B = a^{-1/2}$ (defined for mathematical simplification)

a = semimajor axis

e = eccentricity

i = inclination relative to earth equatorial plane

Ω = right ascension of ascending node

ω = argument of perigee

M = mean anomaly

ν = true anomaly

t = time (independent variable)

Although only one anomaly angle is needed in the element set, it is advantageous to consider both $\underline{M}$ and $\underline{\nu}$. The differential equation for $\underline{M}$ is more amenable to asymptotic series solution; on the other hand, it is mathematically easier to derive the differential equations for all elements in terms of $\underline{\nu}$. Consequently, the differential equation for M is derived and solved; ν is then obtained by a Fourier-Bessel expansion involving M and e.

2.2 PERTURBATIVE DIFFERENTIATION

In the theory of perturbative differentiation, the variation (time-derivative) of any element f is considered as the sum of two parts; i. e.,

$$\frac{df}{dt} = \dot{f} + \grave{f}$$

where $\dot{f}$ (f-dot) is the Keplerian variation that remains if all disturbing forces are suddenly removed and $\grave{f}$ (f-grave) is the perturbative variation caused by the disturbing forces. There are three types of variations which arise in the theory; namely,

Type 1: $\frac{df}{dt} = \dot{f}$, where $\grave{f} = 0$

Type 2: $\frac{df}{dt} = \grave{f}$, where $\dot{f} = 0$

Type 3: $\frac{df}{dt} = \dot{f} + \grave{f}$, where both parts exist

Since the velocity associated with the osculating orbit at the point of tangency is the same as the actual velocity of the perturbed satellite, the components of $\frac{d\bar{r}}{dt}$ in an inertial coordinate system are of the first type. Variations of the second type arise for elements that would be constant in Keplerian motion, such as a, e, i, Ω, and ω. Elements referred to a perturbed reference direction, such as M and ν, are of the third type.

It follows, then, that the basic differential equations of motion for the selected elements are

$$\frac{dB}{dt} = \grave{B} \tag{2-1}$$

$$\frac{de}{dt} = \grave{e} \tag{2-2}$$

$$\frac{di}{dt} = i' \tag{2-3}$$

$$\frac{d\Omega}{dt} = \Omega' \tag{2-4}$$

$$\frac{d\omega}{dt} = \omega' \tag{2-5}$$

$$\frac{dM}{dt} = \dot{M} + M' = n + M' \tag{2-6}$$

The next step is to obtain the perturbative variations indicated above.

Two techniques are commonly used to obtain the perturbative variation $\underline{f}'$ of an element $\underline{f}$. The first technique consists of developing the total variation of the element and then removing the Keplerian part; i.e.,

$$f' = \frac{df}{dt} - \dot{f}$$

The second technique consists of using perturbative differentiation, which involves taking the grave-derivative of a given expression in which only the variations due to the disturbing forces are considered. The second technique is used here to obtain the perturbative variations of the elements. (For a further discussion of perturbative differentiation, see References 1 and 2; particularly p. 21 of Reference 1.)

2.3 PERTURBATIVE VARIATION EQUATIONS

It is necessary to obtain the perturbative variations $\underline{B}'$, $\underline{e}'$, $\underline{i}'$, $\underline{\Omega}'$, $\underline{\omega}'$ and $\underline{M}'$ in terms of the orbital elements and the perturbative acceleration vector $\dot{\bar{r}}'$, resolved as follows (Reference 2, p. 284):

$$\dot{\bar{r}}' = \dot{r}'\,\bar{U} + r\dot{\nu}'\bar{V} + r\dot{b}'\bar{W}$$

where $\bar{U}$ = unit vector in direction of increasing $\bar{r}$ (radial)

$\bar{V}$ = unit vector perpendicular to $\bar{r}$ in orbital plane (transverse)

$\bar{W}$ = unit vector perpendicular to orbital plane (orthogonal)

As will be seen in the next section, the perturbative acceleration components $\underline{\dot{r}'}$, $\underline{r\dot{\nu}'}$ and $\underline{r\dot{b}'}$ can also be obtained in terms of the orbital elements via the disturbing function $\underline{R}$.

Although derivation of the perturbative variation equations by the method of perturbative differentiation is straightforward, it is mathematically tedious; consequently, the procedure is presented in Appendix A. The results are (also see Reference 1, p. 22, and Reference 2, pp. 247 and 284):

$$B' = \frac{1}{Bp}\left[-\frac{r\,\dot{r}'}{\sqrt{\mu p}}\left(e\,\frac{p}{r}\sin\nu\right) - \frac{r^2\dot{\nu}'}{\sqrt{\mu p}}\left(\frac{p}{r}\right)^2\right] \tag{2-7}$$

$$e' = \frac{r\,\dot{r}'}{\sqrt{\mu p}}\left(\frac{p}{r}\sin\nu\right) + \frac{r^2\dot{\nu}'}{\sqrt{\mu p}}\left[\left(\frac{p}{r}+1\right)\cos\nu + e\right] \tag{2-8}$$

$$i' = \frac{r^2\dot{b}'\cos u}{\sqrt{\mu p}} \tag{2-9}$$

$$\Omega' = \frac{r^2\dot{b}'}{\sqrt{\mu p}}\,\frac{\sin u}{\sin i} \tag{2-10}$$

$$\omega' = -\Omega'\cos i - \frac{r\,\dot{r}'}{\sqrt{\mu p}\,e}\left(\frac{p}{r}\cos\nu\right) + \frac{r^2\dot{\nu}'}{\sqrt{\mu p}\,e}\left(\frac{p}{r}+1\right)\sin\nu \tag{2-11}$$

$$M' = -\left(1-e^2\right)^{1/2}\left[\omega' + \Omega'\cos i + \frac{2r\,\dot{r}'}{\sqrt{\mu p}}\right] \tag{2-12}$$

2.4 PERTURBATIVE ACCELERATION COMPONENTS

2.4.1 Earth Oblateness

The perturbative acceleration vector $\dot{\bar{r}}'$ due to an axially symmetric oblate earth can be written as the gradient of the perturbative potential function $\underline{R}$ (per unit mass), which becomes, in spherical coordinates,

$$\dot{\bar{r}}' = \frac{\partial R}{\partial r}\,\bar{i} + \frac{1}{r\cos\delta}\cancelto{0}{\frac{\partial R}{\partial \alpha'}}\,\bar{j} + \frac{1}{r}\frac{\partial R}{\partial \delta}\,\bar{k}$$

where (see Figure 2-1)

$\bar{i}$ = unit vector in direction of increasing $\bar{r}$

$\bar{j}$ = unit vector in direction of increasing α'

$\bar{k}$ = unit vector in direction of increasing δ

and (Reference 3, p. 49)

$$R = -\frac{\mu}{r}\left[J_2\left(\frac{r_e}{r}\right)^2\left(\frac{3}{2}\sin^2\delta - \frac{1}{2}\right) + J_3\left(\frac{r_e}{r}\right)^3\left(\frac{5}{2}\sin^2\delta - \frac{3}{2}\right)\sin\delta + J_4\left(\frac{r_e}{r}\right)^4\left(\frac{35}{8}\sin^4\delta - \frac{30}{8}\sin^2\delta + \frac{3}{8}\right)\right] \tag{2-13}$$

(NOTE: J_3 and J_4 are negative numbers.)

As previously mentioned, the general expression for the perturbative acceleration vector can be written as

$$\ddot{\bar{r}}' = \ddot{r}'\,\bar{U} + r\dot{\nu}'\,\bar{V} + r\dot{b}'\,\bar{W}$$

where $\bar{U}$ = unit vector in direction of increasing $\bar{r}$ (radial)

$\bar{V}$ = unit vector perpendicular to $\bar{r}$ in orbital plane (transverse)

$\bar{W}$ = unit vector perpendicular to orbital plane (orthogonal)

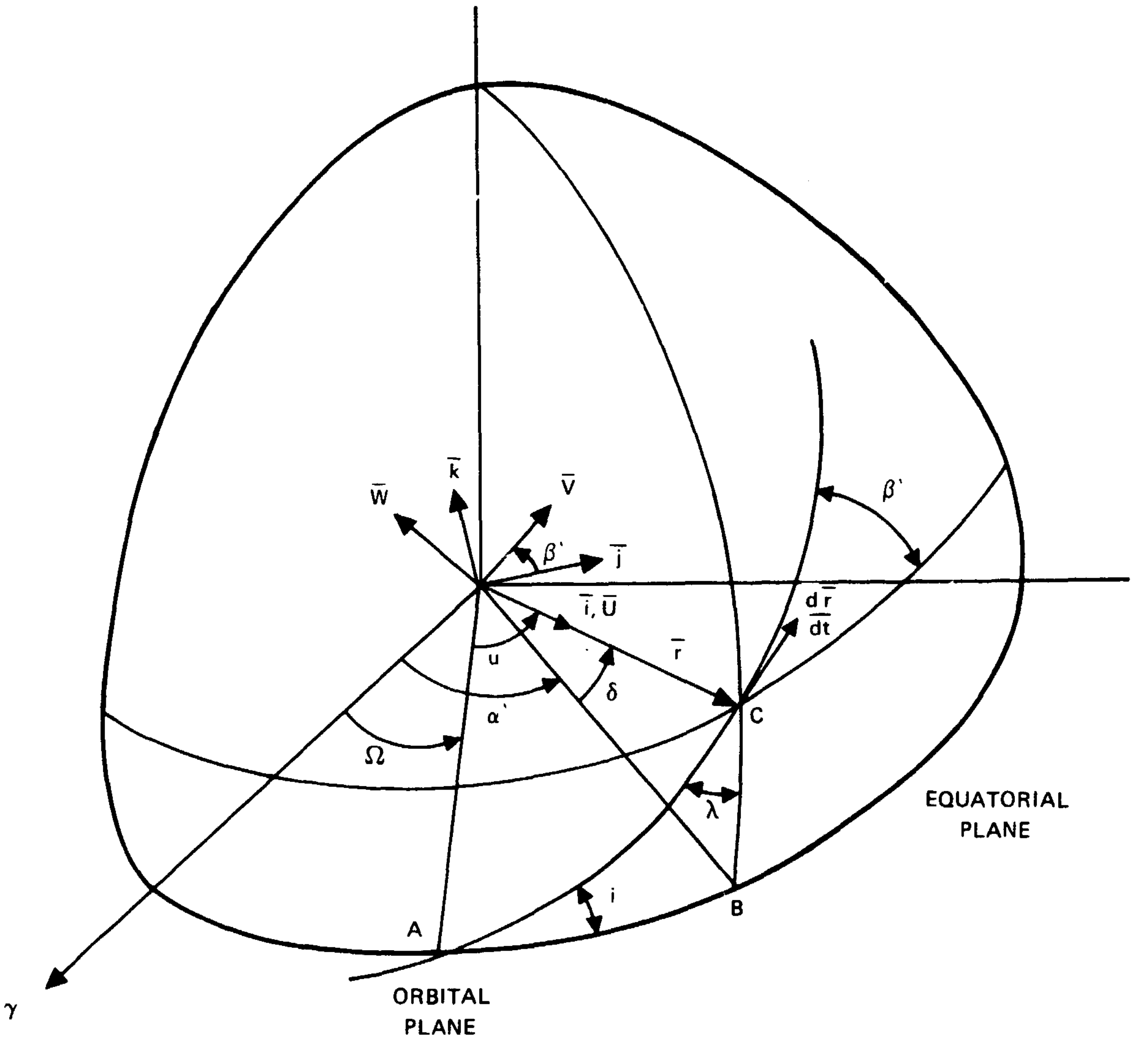


Figure 2-1. Intersection of Equatorial and Orbital Planes

The transformation between the $(\bar{i}, \bar{j}, \bar{k})$ system and the $(\bar{U}, \bar{V}, \bar{W})$ system is obtained via a right-hand rotation about the i-axis through an angle $\underline{\beta}'$, as seen from Figure 2-1. Thus

$$\begin{pmatrix} \dot{r}' \\ \\ r\dot{\nu}' \\ \\ r\dot{b}' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \\ 0 & \cos\beta' & \sin\beta' \\ \\ 0 & -\sin\beta' & \cos\beta' \end{pmatrix} \begin{pmatrix} \frac{\partial R}{\partial r} \\ \\ 0 \\ \\ \frac{1}{r}\frac{\partial R}{\partial \delta} \end{pmatrix}$$

resulting in the scalar equations

$$\dot{r}' = \frac{\partial R}{\partial r}$$

$$r\dot{\nu}' = \frac{\sin\beta'}{r}\frac{\partial R}{\partial \delta}$$

$$r\dot{b}' = \frac{\cos\beta'}{r}\frac{\partial R}{\partial \delta}$$

Performing the indicated differentiation on Equation (2-13) yields

$$\dot{r}' = -\frac{3}{2}\mu J_2 r_e^2\left(\frac{1}{r^4}\right)(1-3\sin^2\delta) - 2\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(3-5\sin^2\delta)\sin\delta + \frac{5}{8}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)(35\sin^4\delta - 30\sin^2\delta + 3) \qquad (2\text{-}14)$$

$$r\dot{\nu}' = \sin\beta'\left[-3\mu J_2 r_e^2\left(\frac{1}{r^4}\right)\sin\delta\cos\delta + \frac{3}{2}\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(1-5\sin^2\delta)\cos\delta - \frac{5}{2}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)(7\sin^2\delta - 3)\sin\delta\cos\delta\right] \qquad (2\text{-}15)$$

$$r\dot{b}' = \cos\beta'\left[-3\mu J_2 r_e^2\left(\frac{1}{r^4}\right)\sin\delta\cos\delta + \frac{3}{2}\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(1-5\sin^2\delta)\cos\delta - \frac{5}{2}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)(7\sin^2\delta - 3)\sin\delta\cos\delta\right] \qquad (2\text{-}16)$$

It is now necessary to express $\underline{\beta}'$ and $\underline{\delta}$ in terms of the orbital elements. Referring to Figure 2-1, from the spherical triangle ABC

$$\sin\delta = \sin i \sin u \qquad (2\text{-}17)$$

$$\cos\delta = \sqrt{1-\sin^2 i \sin^2 u} \qquad (2\text{-}18)$$

Also, from the same triangle

$$\cos u = \operatorname{ctn} \lambda \ \operatorname{ctn} i$$

or

$$\tan \lambda = \frac{\operatorname{ctn} i}{\cos u}$$

But

$$1 + \tan^2\lambda = \frac{1}{\cos^2 \lambda}$$

thus

$$\cos \lambda = \frac{1}{\sqrt{1 + \tan^2 \lambda}} = \frac{1}{\sqrt{1 + \left(\frac{\operatorname{ctn} i}{\cos u}\right)^2}} = \frac{\cos u}{\sqrt{\operatorname{ctn}^2 i + \cos^2 u}}$$

Since $\beta' = 90^\circ - \lambda$ (i.e., latitude and longitude lines are perpendicular), then

$$\sin \beta' = \sin (90^\circ - \lambda) = \cos \lambda$$

or

$$\sin \beta' = \frac{\cos u}{\sqrt{\operatorname{ctn}^2 i + \cos^2 u}}$$

and

$$\cos \beta' = \frac{\operatorname{ctn} i}{\sqrt{\operatorname{ctn}^2 i + \cos^2 u}}$$

However,

$$\sqrt{\operatorname{ctn}^2 i + \cos^2 u} = \frac{1}{\sin i} \sqrt{\cos^2 i + \cos^2 u \sin^2 i} = \frac{1}{\sin i} \sqrt{1 - \sin^2 i \sin^2 u}$$

hence

$$\sin \beta' = \frac{\cos u \sin i}{\sqrt{1 - \sin^2 i \sin^2 u}} \tag{2-19}$$

$$\cos \beta' = \frac{\cos i}{\sqrt{1 - \sin^2 i \sin^2 u}} \tag{2-20}$$

Substituting Equations (2-17) through (2-20) into Equations (2-14) through (2-16) yields, after simplification, the desired results.

$$\dot{r}' = -\frac{3}{2}\mu J_2 r_e^2\left(\frac{1}{r^4}\right)(1-3\sin^2 i \sin^2 u) - 2\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(3-5\sin^2 i \sin^2 u) \sin i \sin u$$
$$+\frac{5}{8}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)(35\sin^4 i \sin^4 u - 30\sin^2 i \sin^2 u + 3) \quad (2\text{-}21)$$

$$r\dot{\nu}' = -\frac{3}{2}\mu J_2 r_e^2\left(\frac{1}{r^4}\right)\sin^2 i \sin 2u + \frac{3}{2}\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(1-5\sin^2 i \sin^2 u)\cos u \sin i$$
$$-\frac{5}{4}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)\sin^2 i \sin 2u\,(7\sin^2 i \sin^2 u - 3) \quad (2\text{-}22)$$

$$r\dot{b}' = -3\mu J_2 r_e^2\left(\frac{1}{r^4}\right)\sin i \cos i \sin u + \frac{3}{2}\mu J_3 r_e^3\left(\frac{1}{r^5}\right)(1-5\sin^2 i \sin^2 u)\cos i$$
$$-\frac{5}{4}\mu J_4 r_e^4\left(\frac{1}{r^6}\right)\sin 2i \sin u\,(7\sin^2 i \sin^2 u - 3) \quad (2\text{-}23)$$

After converting to units of earth-radii and performing trigonometric-identity manipulations, it can be shown that Equations (2-21) through (2-23) agree with Reference 2, p.288, and Reference 4, p. 193.

2.4.2 Tangential Atmospheric Drag

The perturbative acceleration vector $\ddot{\bar{r}}'$ due to a tangential atmospheric drag force can be written as

$$\ddot{\bar{r}}' = -D\bar{T} + O\bar{N} + O\bar{W} = -D\bar{T}$$

where (see Figure 2-2)

$\bar{T}$ = unit vector along orbit tangent in direction of motion (tangential)

$\bar{N}$ = unit vector perpendicular to orbit tangent (normal)

$\bar{W}$ = unit vector perpendicular to orbital plane (orthogonal)

and

$$D = \frac{1}{2}\left(\frac{A}{m}\right) C_D \rho v_R^2$$

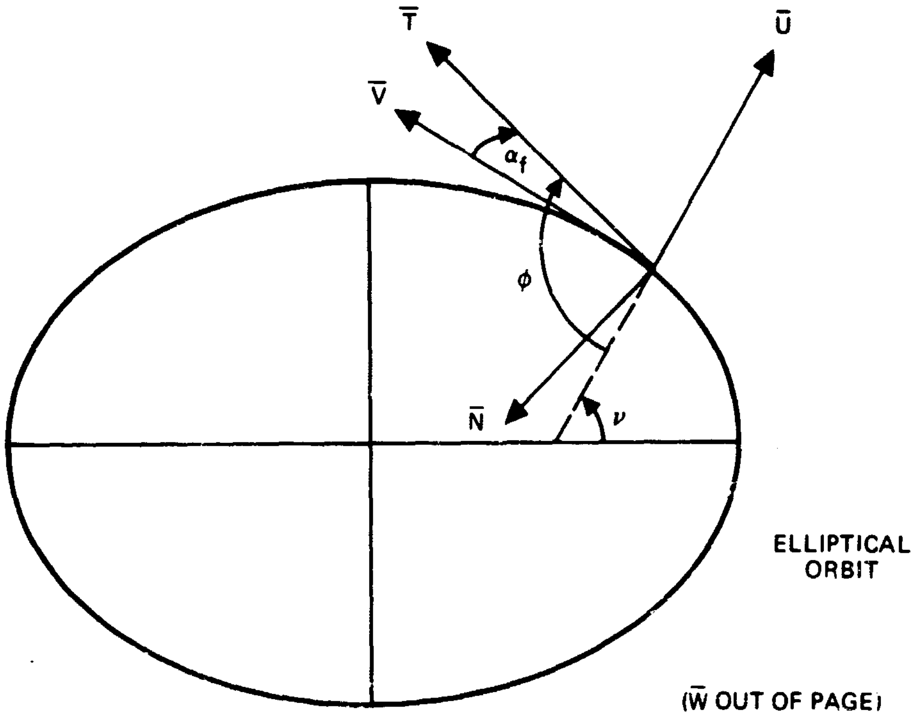


Figure 2-2. Cross-Sectional View of Elliptical Orbit

The relative velocity $\underline{v}_R$ can be approximated in terms of the inertial velocity $\underline{v}$ by (Reference 5, p. 165)

$$v_R = v \left(1 - \frac{\omega_e \cos i}{n}\right)$$

Since the inertial velocity of a satellite in an elliptical orbit is given by (Reference 6, p. 80)

$$v^2 = \frac{\mu}{p} \left(1 + e^2 + 2e \cos \nu\right) = \frac{\mu B^2}{(1-e^2)} \left(1 + e^2 + 2e \cos \nu\right)$$

the drag force magnitude per unit mass can be written as

$$D = \frac{1}{2}\left(\frac{A}{m}\right) C_D \, \rho \, \frac{\mu B^2}{(1-e^2)} \left(1 + e^2 + 2e \cos \nu\right) \left(1 - \frac{\omega_e \cos i}{n}\right)^2 \tag{2-24}$$

As discussed in the previous section, the general expression for the perturbative acceleration vector is

$$\dot{\bar{r}}' = \dot{r}' \, \bar{U} + r\dot{\nu}' \, \bar{V} + r\dot{b}' \, \bar{W}$$

From Figure 2-2, it can be seen that the transformation matrix relating the $(\bar{U}, \bar{V}, \bar{W})$ system to the $(\bar{T}, \bar{N}, \bar{W})$ system is obtained via a right-hand rotation about the $\bar{W}$-axis through an angle $(180^\circ + \phi)$, i.e.,

$$[T] = \begin{pmatrix} \cos(180^\circ + \phi) & \sin(180^\circ + \phi) & 0 \\ -\sin(180^\circ + \phi) & \cos(180^\circ + \phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\phi & -\sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} \dot{r}' \\ r\dot{\nu}' \\ r\dot{b}' \end{pmatrix} = \begin{pmatrix} -\cos\phi & -\sin\phi & 0 \\ \sin\phi & -\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -D \\ 0 \\ 0 \end{pmatrix}$$

resulting in the scalar equations

$$\dot{r}' = D \cos\phi \tag{2-25}$$

$$r\dot{\nu}' = -D \sin\phi \tag{2-26}$$

$$r\dot{b}' = 0 \tag{2-27}$$

The angle $\underline{\phi}$ is related to the orbital elements by (Reference 6, p. 83)

$$\sin\phi = \frac{1 + e\cos\nu}{(1 + e^2 + 2e\cos\nu)^{1/2}} \tag{2-28}$$

$$\cos\phi = \frac{-e\sin\nu}{(1 + e^2 + 2e\cos\nu)^{1/2}} \tag{2-29}$$

Substituting Equation (2-24) and Equations (2-28) and (2-29) into Equations (2-25) through (2-27) yields the desired results

$$\dot{r}' = -\frac{1}{2}\left(\frac{A}{m}\right) C_D\, \rho\, \frac{\mu B^2 e \sin\nu}{(1-e^2)} \left(1 + e^2 + 2e\cos\nu\right)^{1/2} \left(1 - \frac{\omega_e \cos i}{n}\right)^2 \tag{2-30}$$

$$r\dot{\nu}' = -\frac{1}{2}\left(\frac{A}{m}\right) C_D\, \rho\, \frac{\mu B^2 (1 + e\cos\nu)}{(1-e^2)} \left(1 + e^2 + 2e\cos\nu\right)^{1/2} \left(1 - \frac{\omega_e \cos i}{n}\right)^2 \tag{2-31}$$

$$r\dot{b}' = 0 \tag{2-32}$$

2.5 DIFFERENTIAL EQUATIONS IN FINAL FORM

2.5.1 Earth Oblateness

Expressing the earth oblateness differential equations in final form requires substituting Equations (2-21) through (2-23) into Equations (2-7) through (2-12), simplifying, and then substituting the corresponding results into Equations (2-1) through (2-6). To illustrate this procedure, the final form of the differential equation for the element $\underline{i}$ will be derived. The equations for all other elements can be obtained in a similar manner.

Substituting Equation (2-23) into Equation (2-9) yields

$$i' = \frac{\cos u}{\sqrt{\mu p}} \Big[-3\mu J_2 r_e^2 \left(\frac{1}{r^3}\right) \sin i \cos i \sin u + \frac{3}{2}\mu J_3 r_e^3 \left(\frac{1}{r^4}\right) (1 - 5\sin^2 i \sin^2 u) \cos i$$
$$- \frac{5}{4}\mu J_4 r_e^4 \left(\frac{1}{r^5}\right) \sin 2i \sin u \,(7\sin^2 i \sin^2 u - 3)\Big]$$

Substituting

$$r = \frac{a(1-e^2)}{1 + e\cos\nu} = \frac{p}{1 + e\cos\nu}$$

and performing trigonometric-identity manipulations yields

$$\begin{aligned} i' = &-\frac{3\sqrt{\mu}\, J_2\, r_e^{\,2}\,(1 + e\cos\nu)^3}{4\,p^{7/2}} \sin 2\,i \sin 2\,u \\ &+ \frac{3\sqrt{\mu}\, J_3\, r_e^{\,3}\,(1 + e\cos\nu)^4}{2\,p^{9/2}} \cos i \cos u\,(1-5\sin^2 i \sin^2 u) \\ &- \frac{5\sqrt{\mu}\, J_4\, r_e^{\,4}\,(1 + e\cos\nu)^5}{8\,p^{11/2}} \sin 2\,i \sin 2\,u\,(7\sin^2 i \sin^2 u - 3) \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} i' = &-\frac{\left(\frac{3}{2} J_2\right)\sqrt{\mu}\, r_e^{\,2}\left(1 + e\cos\nu\right)^3}{2\,p^{7/2}} \sin 2\,i \sin 2\,u \\ &+ \frac{\left(\frac{3}{2} J_2\right)\left(\frac{J_3}{J_2}\right)\sqrt{\mu}\, r_e^{\,3}\,(1 + e\cos\nu)^4}{p^{9/2}} \cos i \cos u\,(1-5\sin^2 i \sin^2 u) \\ &- \frac{5\left(\frac{3}{2} J_2\right)\left(\frac{J_4}{J_2}\right)\sqrt{\mu}\, r_e^{\,4}\,(1 + e\cos\nu)^5}{12\,p^{11/2}} \sin 2\,i \sin 2\,u\,(7\sin^2 i \sin^2 u - 3) \end{aligned}$$

Defining

$$\epsilon = \frac{3}{2} J_2 \qquad \text{(a small parameter)} \tag{2-33}$$

This equation becomes

$$\begin{aligned} i' = &-\epsilon\,\frac{\sqrt{\mu}\, r_e^{\,2}\,(1 + e\cos\nu)^3}{2\,p^{7/2}} \sin 2\,i \sin 2\,u \\ &+ \epsilon\,\frac{\left(\frac{J_3}{J_2}\right)\sqrt{\mu}\, r_e^{\,3}\,(1 + e\cos\nu)^4}{p^{9/2}} \cos i \cos u\,(1-5\sin^2 i \sin^2 u) \\ &- \epsilon\,\frac{5\left(\frac{J_4}{J_2}\right)\sqrt{\mu}\, r_e^{\,4}\,(1 + e\cos\nu)^5}{12\,p^{11/2}} \sin 2\,i \sin 2\,u\,(7\sin^2 i \sin^2 u - 3) \end{aligned}$$

Finally, substituting this equation into Equation (2-3), noting that

$$\frac{1}{p} = \frac{1}{a(1-e^2)} = \frac{B^2}{(1-e^2)}$$

yields the desired result

$$\frac{di}{dt} = -\epsilon \frac{\sqrt{\mu}\, B^7 r_e^2 (1 + e \cos \nu)^3}{2(1-e^2)^{7/2}} \sin 2 i \sin 2 u$$

$$+ \epsilon \frac{\left(\frac{J_3}{J_2}\right)\sqrt{\mu}\, B^9 r_e^3 (1 + e \cos \nu)^4}{(1-e^2)^{9/2}} \cos i \cos u \,(1-5 \sin^2 i \sin^2 u)$$

$$- \epsilon \frac{5\left(\frac{J_4}{J_2}\right) \sqrt{\mu}\, B^{11} r_e^4 (1 + e \cos \nu)^5}{12\,(1-e^2)^{11/2}} \sin 2 i \sin 2 u \,(7 \sin^2 i \sin^2 u - 3)$$

(NOTE: An optional formulation of this equation would be one in which the higher earth harmonics (J_3 and J_4 are treated as "higher-order" perturbing terms; i.e., as ϵ^2 terms. As will be shown in the solution procedure of Section 4, however, treating these harmonics as ϵ-order terms yields mean orbital elements which include long-periodic as well as secular variations.)

As previously mentioned, the equations for the other elements can be obtained in a similar manner. The complete set of differential equations when considering earth oblateness is presented on the following pages.

$$\frac{dB}{dt} = \epsilon \frac{\sqrt{\mu}\, B^8 r_e^2 (1 + e \cos \nu)^4}{(1-e^2)^{9/2}} \left[e \sin \nu \,(1 - 3 \sin^2 i \sin^2 u) + (1 + e \cos \nu) \sin^2 i \sin 2 u \right] \tag{2-34}$$

$$+ \frac{4}{3} \epsilon \frac{\sqrt{\mu}\, B^{10} \left(\frac{J_3}{J_2}\right) r_e^3 (1 + e \cos \nu)^5}{(1-e^2)^{11/2}} \left[e \sin \nu \sin i \sin u \,(3 - 5 \sin^2 i \sin^2 u) - \frac{3}{4} (1 + e \cos \nu)\,(1 - 5 \sin^2 i \sin^2 u) \cos u \sin i \right]$$

$$- \frac{5}{12} \epsilon \frac{\sqrt{\mu}\, B^{12} \left(\frac{J_4}{J_2}\right) r_e^4 (1 + e \cos \nu)^6}{(1-e^2)^{13/2}} \left[e \sin \nu \,(35 \sin^4 i \sin^4 u - 30 \sin^2 i \sin^2 u - 3) - 2 (1 + e \cos \nu) \sin^2 i \sin 2 u \,(7 \sin^2 i \sin^2 u - 3) \right]$$

$$\frac{de}{dt} = -\epsilon \frac{\sqrt{\mu}\, B^7 r_e^2 (1 + e \cos \nu)^3}{(1-e^2)^{7/2}} \left[\sin \nu \,(1 + e \cos \nu)\,(1 - 3 \sin^2 i \sin^2 u) + \sin^2 i \sin 2u \,(2 \cos \nu + e \cos^2 \nu + e) \right] \tag{2-35}$$

$$- \frac{4}{3} \epsilon \frac{\sqrt{\mu}\, B^9 \left(\frac{J_3}{J_2}\right) r_e^3 (1 + e \cos \nu)^4}{(1-e^2)^{9/2}} \left[\sin i \sin u \sin \nu \,(1 + e \cos \nu)\,(3 - 5 \sin^2 i \sin^2 u) - \frac{3}{4} (1 - 5 \sin^2 i \sin^2 u)\,(2 \cos \nu + e \cos^2 \nu + e) \cos u \sin i \right]$$

$$+ \frac{5}{12} \epsilon \frac{\sqrt{\mu}\, B^{11} \left(\frac{J_4}{J_2}\right) r_e^4 (1 + e \cos \nu)^5}{(1-e^2)^{11/2}} \left[\sin \nu \,(1 + e \cos \nu)\,(35 \sin^4 i \sin^4 u - 30 \sin^2 i \sin^2 u + 3) - 2 \sin^2 i \sin 2 u \,(7 \sin^2 i \sin^2 u - 3)\,(2 \cos \nu + e \cos^2 \nu + e) \right]$$

$$\frac{di}{dt} = -\epsilon \frac{\sqrt{\mu}\, B^7 r_e^{\,2} (1+e\cos\nu)^3}{2\,(1-e^2)^{7/2}} \sin 2i \sin 2u \tag{2-36}$$

$$+\ \epsilon \frac{\sqrt{\mu}\, B^9 \left(\frac{J_3}{J_2}\right) r_e^{\,3} (1+e\cos\nu)^4}{(1-e^2)^{9/2}} \cos i \cos u \,(1-5\sin^2 i \sin^2 u)$$

$$-\frac{5}{12}\epsilon \frac{\sqrt{\mu}\, B^{11} \left(\frac{J_4}{J_2}\right) r_e^{\,4} (1+e\cos\nu)^5}{(1-e^2)^{11/2}} \sin 2i \sin 2u \,(7\sin^2 i \sin^2 u - 3)$$

$$\frac{d\Omega}{dt} = -2\epsilon \frac{\sqrt{\mu}\, B^7 r_e^{\,2} (1+e\cos\nu)^3}{(1-e^2)^{7/2}} \sin^2 u \cos i \tag{2-37}$$

$$+\ \epsilon \frac{\sqrt{\mu}\, B^9 \left(\frac{J_3}{J_2}\right) r_e^{\,3} (1+e\cos\nu)^4}{(1-e^2)^{9/2}} \operatorname{ctn} i \sin u \,(1-5\sin^2 i \sin^2 u)$$

$$-\frac{5}{3}\epsilon \frac{\sqrt{\mu}\, B^{11} \left(\frac{J_4}{J_2}\right) r_e^{\,4} (1+e\cos\nu)^5}{(1-e^2)^{11/2}} \cos i \sin^2 u \,(7\sin^2 i \sin^2 u - 3)$$

$$\frac{d\omega}{dt} = \epsilon \frac{\sqrt{\mu}\, B^7 r_e^{2} (1+e \cos \nu)^3}{e\,(1-e^2)^{7/2}} \left[2\, e \sin^2 u \cos^2 i + \cos \nu\, (1+e \cos \nu)\, (1-3 \sin^2 i \sin^2 u) - \sin \nu \sin^2 i \sin 2u\, (2+e \cos \nu)\right] \tag{2-38}$$

$$+ \epsilon \frac{\sqrt{\mu}\, B^9 \left(\frac{J_3}{J_2}\right) r_e^{3} (1+e \cos \nu)^4}{e\,(1-e^2)^{9/2}} \left[(1-5 \sin^2 i \sin^2 u)\, (-e \operatorname{ctn} i \cos i \sin u + \sin \nu \cos u \sin i \left\{2+e \cos \nu \right\}) + \frac{4}{3}\, (1+e \cos \nu) \right.$$

$$\left. (3-5 \sin^2 i \sin^2 u) \cos \nu \sin i \sin u\right]$$

$$+ \frac{5}{3} \epsilon \frac{\sqrt{\mu}\, B^{11} \left(\frac{J_4}{J_2}\right) r_e^{4} (1+e \cos \nu)^5}{e\,(1-e^2)^{11/2}} \left[(7 \sin^2 i \sin^2 u - 3)\, (e \cos^2 i \sin^2 u - \frac{1}{2} \sin^2 i \sin 2\, u \sin \nu \left\{2+e \cos \nu \right\}) - \frac{1}{4} (1+e \cos \nu) \right.$$

$$\left. (35 \sin^4 i \sin^4 u - 30 \sin^2 i \sin^2 u + 3) \cos \nu\right]$$

$$\frac{dM}{dt} = n - \epsilon \frac{\sqrt{\mu}\, B^7 r_e^{2} (1+e \cos \nu)^3}{e\,(1-e^2)^3} \left[(1-3 \sin^2 i \sin^2 u)\, (-2e + \cos \nu \left\{1+e \cos \nu \right\}) - \sin \nu \sin^2 i \sin 2\, u\, (2+e \cos \nu)\right] \tag{2-39}$$

$$- \frac{4}{3} \epsilon \frac{\sqrt{\mu}\, B^9 \left(\frac{J_3}{J_2}\right) r_e^{3} (1+e \cos \nu)^4}{e\,(1-e^2)^4} \left[\sin i \sin u\, (3-5 \sin^2 i \sin^2 u)\, (-2e + \cos \nu \left\{1+e \cos \nu \right\}) + \frac{3}{4} \sin \nu \cos u \sin i\, (2+e \cos \nu)\, (1-5 \sin^2 i \sin^2 u)\right]$$

$$+ \frac{5}{12} \epsilon \frac{\sqrt{\mu}\, B^{11} \left(\frac{J_4}{J_2}\right) r_e^{4} (1+e \cos \nu)^5}{e\,(1-e^2)^5} \left[(35 \sin^4 i \sin^4 u - 30 \sin^2 i \sin^2 u + 3)\, (-2e + \cos \nu \left\{1+e \cos \nu \right\}) + 2 \sin^2 i \sin 2\, u\, (7 \sin^2 i \sin^2 u - 3) \right.$$

$$\left. \sin \nu\, (2+e \cos \nu)\right]$$

2.5.2 Tangential Atmospheric Drag

Expressing the drag differential equations in final form requires substituting Equations (2-30) through (2-32) into Equations (2-7) through (2-12), simplifying, and then substituting the corresponding results into Equations (2-1) through (2-6). To illustrate this procedure, the final form of the differential equation for the element $\underline{e}$ will be derived. The equations for all other elements can be obtained in a similar manner.

Substituting Equations (2-25) and (2-26) into Equation (2-8) yields

$$e' = \frac{1}{\sqrt{\mu p}}\left[rD\cos\phi\left(\frac{p}{r}\sin\nu\right) - rD\sin\phi\left\{\left(\frac{p}{r}+1\right)\cos\nu + e\right\}\right]$$

Substituting

$$r = \frac{p}{1+e\cos\nu} \qquad \frac{p}{r} = 1+e\cos\nu$$

the equation becomes

$$e' = \frac{D}{\sqrt{\mu p}}\left[\frac{p\cos\phi}{1+e\cos\nu}(1+e\cos\nu)\sin\nu - \frac{p\sin\phi}{1+e\cos\nu}(2+e\cos\nu)\cos\nu - \frac{p\,e\sin\phi}{1+e\cos\nu}\right]$$

$$= \frac{D\sqrt{p}}{\sqrt{\mu}(1+e\cos\nu)}\left[\cos\phi\sin\nu(1+e\cos\nu) - \sin\phi\cos\nu(2+e\cos\nu) - e\sin\varphi\right]$$

Substituting Equations (2-28) and (2-29) yields

$$e' = \frac{D\sqrt{p}}{\sqrt{\mu}(1+e\cos\nu)(1+e^2+2e\cos\nu)^{1/2}}\left[-e\sin^2\nu(1+e\cos\nu) - \cos\nu(1+e\cos\nu)(2+e\cos\nu) - e(1+e\cos\nu)\right]$$

$$= -\frac{2D\sqrt{p}}{\sqrt{\mu}(1+e^2+2e\cos\nu)^{1/2}}(e+\cos\nu)$$

Substituting Equation (2-24) and noting that

$$p = a(1-e^2) = \frac{(1-e^2)}{B^2}$$

results in

$$e' = -\frac{2\sqrt{1-e^2}\,(e+\cos\nu)}{B\sqrt{\mu}(1+e^2+2e\cos\nu)^{1/2}}\left[\frac{1}{2}\left(\frac{A}{m}\right)C_D\,\rho\frac{\mu B^2}{(1-e^2)}(1+e^2+2e\cos\nu)\right]\left(1-\frac{\omega_e\cos i}{n}\right)^2$$

$$= -\left(\frac{A}{m}\right)\frac{C_D\,\rho\sqrt{\mu}\,B(e+\cos\nu)(1+e^2+2e\cos\nu)^{1/2}}{(1-e^2)^{1/2}}\left(1-\frac{\omega_e\cos i}{n}\right)^2$$

Since

$$n = \sqrt{\mu}\, B^3$$

the equation can be written as

$$e' = -\left(\frac{A}{m}\right) \frac{C_D\, \rho\, \mu\, B^4\, (e+\cos\nu)\, (1+e^2+2e\cos\nu)^{1/2}}{n\, (1-e^2)^{1/2}} \left(1 - \frac{\omega_e \cos i}{n}\right)^2$$

In terms of the small parameter ϵ as defined by Equation (2-33), this becomes

$$e' = -\epsilon \left(\frac{A}{m}\right)\left(\frac{1}{J_2}\right) \frac{2\, C_D\, \rho\, \mu\, B^4\, (e+\cos\nu)\, (1+e^2+2e\cos\nu)^{1/2}}{3n\, (1-e^2)^{1/2}} \left(1 - \frac{\omega_e \cos i}{n}\right)^2$$

Finally, substituting this equation into Equation (2-2), and defining

$$K^* = \frac{2}{3}\left(\frac{A}{m}\right)\left(\frac{1}{J_2}\right) C_D\, \mu \left(1 - \frac{\omega_e \cos i}{n}\right)^2 \tag{2-40}$$

yields the desired result

$$\frac{de}{dt} = -\epsilon\, \frac{K^*\, \rho\, B^4\, (e+\cos\nu)\, (1+e^2+2e\cos\nu)^{1/2}}{n\, (1-e^2)^{1/2}}$$

As previously mentioned, the equations for the other elements are obtained in a similar manner. The complete set of differential equations when considering tangential atmospheric drag is presented on the following page.

Differential Equations of Motion When
Considering Tangential Atmospheric Drag

$$\frac{dB}{dt} = \epsilon \frac{K^* \rho B^5 (1+e^2+2e\cos\nu)^{3/2}}{2n(1-e^2)^{3/2}} \tag{2-41}$$

$$\frac{de}{dt} = -\epsilon \frac{K^* \rho B^4 (e+\cos\nu)(1+e^2+2e\cos\nu)^{1/2}}{n(1-e^2)^{1/2}} \tag{2-42}$$

$$\frac{di}{dt} = 0 \qquad (\text{since } r\,\dot{b}' = 0) \tag{2-43}$$

$$\frac{d\Omega}{dt} = 0 \qquad (\text{since } r\,\dot{b}' = 0) \tag{2-44}$$

$$\frac{d\omega}{dt} = -\epsilon \frac{K^* \rho B^4 \sin\nu (1+e^2+2e\cos\nu)^{1/2}}{ne(1-e^2)^{1/2}} \tag{2-45}$$

$$\frac{dM}{dt} = \epsilon \frac{K^* \rho B^4 \sin\nu (1+e^2+2e\cos\nu)^{1/2}}{n}\left(\frac{1}{e} + \frac{e}{1+e\cos\nu}\right) \tag{2-46}$$

SECTION 3 - SPECIAL PERTURBATION METHOD OF SOLUTION

In Section 2, the differential equations of motion for a selected set of orbital elements were derived when considering the perturbational effects of earth oblateness and tangential atmospheric drag. The purpose of this section is to discuss the procedure by which these equations, along with a Cowell formulation of the motion equations, are numerically solved. Included is a synopsis of the MARVES computer program which has been implemented to perform this so-called special perturbation method of solution.

3.1 GENERAL

The term "special perturbations" refers to a technique for the prediction of an orbit by numerical integration, so as to include the effects of various perturbative forces that cause the trajectory to deviate from some reference orbit (Reference 2, pp. 227-228). The basic procedure is the generation of the next step or increment of the state variables representing the orbiting body when having a complete knowledge of the preceding variables (Reference 7, pp. 220-221). Specifically, one begins with some epoch state and integrates, numerically, a set of three second-order or six first-order differential equations of motion.

The variation-of-parameters formulation involves the integration of six first-order equations (often referred to as the Lagrange planetary equations) which are functions of the selected orbital elements. As is evident in the literature (Reference 2, p. 243; Reference 8, pp. 235-236), there is no "best" set of fundamental elements to employ, and the choice is dictated by the application in mind. In the Cowell formulation, three second-order motion equations for the perturbative rectangular accelerations are integrated to obtain the current state variables (position and velocity).

3.2 VARIATION-OF-PARAMETERS FORMULATION

In the variation-of-parameters formulation, six first-order element rate equations are numerically integrated; these equations reflect perturbations due to earth oblateness (second, third, and fourth harmonics) and atmospheric drag (using a 1970

Jacchia atmospheric density model). The oblateness equations are identically those presented in Paragraph 2.5.1, whereas the drag equations differ from those presented in Paragraph 2.5.2 in that a three-dimensional rather than tangential drag force is considered. (NOTE: To readily obtain analytical solutions to the equations, it is necessary to assume a tangential atmospheric drag; however, this assumption is not required when numerically integrating the equations.)

3.3 COWELL FORMULATION

In the Cowell formulation, the equations of motion are expressed in rectangular form and integrated twice to obtain the velocity and position. These equations have the standard form:

$$\frac{d^2x}{dt^2} = \ddot{\underline{x}} + \dot{x}^{\backslash} \qquad (x \rightarrow y, z)$$

where $\ddot{\underline{x}}$ represents the central force term, and $\dot{\underline{x}}^{\backslash}$, the perturbative term, represents the accumulated effects of all perturbations acting. The perturbations included in this formulation also consist of earth oblateness (second, third, and fourth harmonics) and atmospheric drag (using a 1970 Jacchia atmospheric density model).

3.4 SYNOPSIS OF MARVES COMPUTER PROGRAM FOR NUMERICALLY INTEGRATING THE EQUATIONS OF MOTION

A MARVES/FORTRAN double-precision special perturbations program has been developed for the UNIVAC 1108 and is currently available through the MSFC Computation Laboratory. This program provides, on user option, either the Cowell or variation-of-parameters formulations.

The program is modular in design, with FORTRAN subprograms selectively linked and controlled by two MARVES driver programs. This configuration allows user selection from a library of simulation routines and high precision numerical integration schemes currently operational and available to MARVES users (Reference 9). These integration schemes include a variety of single and multistep methods with provisions for optimum step-size prediction based on the resultant truncation error.

When selecting the "best" method for solving the set of differential equations for a particular orbit, many factors must be considered, such as: required accuracy, number of integration interrupts, frequency of computer printout, and integration step-size limitations. Reference 9 contains a thorough discussion of the methods currently available in MARVES, along with some generalizations that can be made about method selection.

Many other desired features are incorporated into the program, such as critical time events, the nearest Besselian year coordinate transformation, and the 1970 Jacchia atmospheric density model. Also included is a solar-ephemeris computation routine that eliminates the need for read/interpolation of Jet Propulsion Laboratory (JPL) ephemeris tapes.

A complete description of this MARVES program (referred to as the SPERTB program) is given in Reference 10.

SECTION 4 - GENERAL PERTURBATION METHOD OF SOLUTION

In Section 2, the differential equations of motion for a selected set of orbital elements were derived when considering the perturbational effects of earth oblateness and tangential atmospheric drag. This section analytically solves these equations by the method of two-variable asymptotic series. (To date, complete solutions have been obtained for both oblateness ($\underline{J}_2$ and $\underline{J}_3$) and oblateness/drag combined.) Included are synopses of the FORMAC computer program used in obtaining the analytical solutions and the FORTRAN program used in numerically evaluating these solutions.

4.1 GENERAL

In the method classically known as general perturbations, six first-order equations of motion can be formulated as functions of some fundamental set of orbital elements. The perturbation effects are expressed analytically, and the element solutions are generally obtained by analytical integration of series expansions in one form or another. These solutions are explicit functions of time, constants of the problem and constants of integration. They define the vehicle state at any instant in time, as the epoch state conditions make the problem completely determinant.

The primary difficulty in the general perturbation method has always been the overwhelming amount of analytical labor required to obtain the solutions. However, the state of the art in computer technology is such that automated manipulation languages, i.e., languages for doing symbolic as opposed to strictly numerical mathematics, are now generally available. Consequently, many of these burdensome analytical tasks, such as series manipulations, function expansion, differentiation and integration, can now be alleviated.

The language selected for use in this development is FORMAC (FORMULA MANIPULATION COMPILER). This language, currently available through the MSFC Computation Laboratory, was developed by IBM, and contains a wide range of analytical capabilities (Reference 11). Consequently, it has proven itself a valuable tool for the application at hand.

4.2 BASIC THEORY OF THE TWO-VARIABLE ASYMPTOTIC SERIES EXPANSION METHOD

As was indicated earlier, general perturbation techniques employ series expansions for assumed element solutions. These expansions result in correspondingly expanded differential equations which are then analytically integrated. The type expansions employed in this study are classically known as asymptotic series expansions. It is the purpose of this section to provide an outline of the theoretical basis for such expansions, illustrating those concepts required in the particular application at hand.

The discussion begins with some basic definitions and nomenclature.

Definition 1

Let $f(t, \epsilon)$ and $g(\epsilon)$ be real-valued functions, where ϵ is a small positive parameter and t ranges continuously over some set S of nonnegative reals. Then, a measure of the relative magnitudes of $f(t, \epsilon)$ and $g(\epsilon)$ may be obtained if a real (finite) K exists such that:

$$\lim_{\epsilon \to 0} \left| \frac{f(t, \epsilon)}{g(\epsilon)} \right| \le K$$

for all t in S. Symbolically, the existence of this limit is denoted by writing:

$$f(t, \epsilon) = O(g(\epsilon))$$

which reads "$f(t, \epsilon)$ is of the order of $g(\epsilon)$." The existence of the limit for all t in S makes this relation uniform in that K can be chosen independently of t. The function $g(\epsilon)$ is called the gauge function, and when $K = 1$, $f(t, \epsilon)$ is said to be asymptotically equal to $g(\epsilon)$. If t is a function of several real variables, the relation is said to be multivariable (Reference 12, pp. 180-185; Reference 13, pp. 1-3; and Reference 1, pp. 16-17).

For purposes of clarification, consider the following example:

Let $\qquad t > 0,\ 0 < \epsilon << 1,\ f(t, \epsilon) = \epsilon^2 \sin t$ and $g(\epsilon) = \epsilon$

Then clearly, $$f(t, \epsilon) = O(g(\epsilon))$$

since
$$\lim_{\epsilon \to 0} \left| \frac{\epsilon^2 \sin t}{\epsilon} \right| \leq \epsilon$$

for all $t > 0$; i.e., uniformly in t as $\epsilon \to 0$.

Another symbol, which is often used to measure the relative magnitude of two functions, bears a simple relationship to the order symbol of Definition 1. If $f(t, \epsilon)$, $g(\epsilon)$ and ϵ are as previously defined, then this alternate measurement is obtained when:

$$\lim_{\epsilon \to 0} \left| \frac{f(t, \epsilon)}{g(\epsilon)} \right| = 0$$

for all t in S. Symbolically:

$$f(t, \epsilon) = o(g(\epsilon))$$

and is read "$f(t, \epsilon)$ is small o of $g(\epsilon)$." (When both symbols are employed, $f(t, \epsilon) = O(g(\epsilon))$ is often read "$f(t, \epsilon)$ is large O of $g(\epsilon)$.")

The symbol small o, though not employed herein, is related to the large O of Definition 1 by:

$$o(O(g(\epsilon))) = o(g(\epsilon))$$

Definition 2

Let $g_i(\epsilon)$, $i = 0, 1, 2, \ldots$, be a sequence of real-valued functions of the small (positive) parameter ϵ. Then, this sequence is called an asymptotic sequence for $\epsilon \to 0$ if, for each i (Reference 12, pp. 182-183; Reference 13, pp. 2-3; and Reference 1, p. 17):

$$\lim \frac{g_{i+1}(\epsilon)}{g_i(\epsilon)} = 0$$

Such a sequence is illustrated in the following example:

Let the sequence $g_i(\epsilon)$, $i = 0, 1, 2, \ldots$, be defined by

$$g_i(\epsilon) = \epsilon^{\lambda_i}, \quad \lambda_{i+1} > \lambda_i > 0$$

for all $\underline{i}$. Then this sequence is an asymptotic sequence, since

$$\lim_{\epsilon \to 0} \frac{g_{i+1}(\epsilon)}{g_i(\epsilon)} = \lim_{\epsilon \to 0} \frac{\epsilon^{\lambda_{i+1}}}{\epsilon^{\lambda_i}} = 0$$

for each $\underline{i}$.

Definition 3

Let $g_i(\epsilon)$ and $f^{(i)}(t)$ be real-valued functions of the small parameter $\underline{\epsilon}$ and the real nonnegative variable $\underline{t}$, respectively. Then, the sequence of partial sums:

$$\sum_{i=0}^{N} g_i(\epsilon) f^{(i)}(t)$$

is called an asymptotic expansion to $\underline{N}$ terms of a function $x(t, \epsilon)$ as $\epsilon \to 0$ when:

$$X(t, \epsilon) = \sum_{i=0}^{N} g_i(\epsilon) f^{(i)}(t) + O\left(g_{i+1}(\epsilon)\right)$$

as $\epsilon \to 0$. The asymptotic expansion is said to be uniformly valid when it holds for all $\underline{t}$ in some set $\underline{S}$ of nonnegative reals, i.e., when $O(g_{i+1}(\epsilon))$ is uniform in $\underline{t}$. If $\underline{t}$ is expressed, at least formally, as a function of several variables, then the expansion is said to be a multivariable asymptotic expansion. Such an expansion would have the form:

$$X(t, \epsilon) = \sum_{i=0}^{N} g_i(\epsilon) f^{(i)}(\bar{t}, \tilde{t}, \ldots) + O\left(g_{i+1}(\epsilon)\right)$$

as $\epsilon \to 0$. For purposes of preserving the uniform validity of the expansion (Reference 13, pp. 79-82, and Reference 1, p. 17), the variables $\bar{t}, \tilde{t}, \ldots$ are taken as functions of $\underline{\epsilon}$ multiplied linearly by $\underline{t}$. Here, $\underline{\bar{t}}$ is termed the fast variable while $\underline{\tilde{t}}$ is termed the slow variable (Reference 1, pp. 16-17). The first, second, and third approximations to $x(t, \epsilon)$ are given as:

$$g_0(\epsilon)f^{(0)}(t)$$

$$g_0(\epsilon)f^{(0)}(t) + g_1(\epsilon)f^{(1)}(t)$$

$$g_0(\epsilon)f^{(0)}(t) + g_1(\epsilon)f^{(1)}(t) + g_2(\epsilon)f^{(2)}(t)$$

To fix these ideas, take $t > 0$ and $x(t, \epsilon) = e^{\epsilon t}$, where ϵ is a small positive parameter. Let $g_i(\epsilon) = \epsilon^i/i!$ and $f^{(i)}(t) = t^i$, $i = 0, 1, 2, \ldots$. Then

$$\lim_{\epsilon \to 0} \frac{g_{i+1}(\epsilon)}{g_i(\epsilon)} = \lim_{\epsilon \to 0} \frac{\epsilon}{i+1} = 0$$

so that $g_i(\epsilon)$, $i = 0, 1, 2, \ldots$ is an asymptotic sequence. The sequence of partial sums:

$$\sum_{i=0}^{N} \frac{\epsilon^i t^i}{i!}$$

is an asymptotic expansion to N terms of $x(t, \epsilon) = e^{\epsilon t}$.

Note that in this example the asymptotic expansion was convergent. However, there is to be no convergence requirement imposed on such expansions, and some expansions may converge for some range of ϵ, or may diverge for all ϵ. The practical applicability of the method is not determined by convergence of the series when $i \to \infty$, but by its asymptotic properties for a fixed value if i when $\epsilon \to 0$ (Reference 14, pp. 40-41).

Hence, an important characteristic of asymptotic expansions is that the error made in approximating the given function by such an expansion is of the order of the first neglected term (Reference 1, p. 17). For this reason, it is important that one make a wise choice for the small parameter ϵ when using this method.

Consider $x(t, \epsilon)$ a function that is to be approximated by a two-variable asymptotic series expansion. Then, $\underline{t}$ will be functionally related to two variables, say $\underline{\bar{t}}$ and $\underline{\tilde{t}}$, in a linear fashion through $\underline{\epsilon}$. Here again, $\underline{\bar{t}}$ will be termed a fast variable and $\underline{\tilde{t}}$ a slow variable. Further, suppose that $x(t, \epsilon)$ represents the solution to a differential equation whose independent variable is $\underline{t}$. To apply the technique, the differential equation must be expressed as a function of both $\underline{t}$ (at least implicitly) and $\underline{\epsilon}$.

Thus, the initial value problem for an ordinary differential equation is converted, through use of a two-variable expansion, to one involving partial differential equations in $\underline{\bar{t}}$ and $\underline{\tilde{t}}$. The two-variable asymptotic solution of the transformed problem will then involve certain undetermined functions which are defined by postulating that the problem possess a consistent asymptotic solution which is uniformly valid (at least to values of $\underline{t}$ of the order of the reciprocal of the small parameter).

There are two concepts that aid in arriving at uniformly valid solutions, as opposed to those which are initially valid (i. e., valid over some initial portions of their ranges). These are called the first and second uniformity conditions, respectively (Reference 1, p. 18).

The <u>first uniformity condition</u> states that a multivariable asymptotic solution to a small parameter dependent differential equation cannot contain secular terms in the fast variable $\underline{\bar{t}}$ (i. e., terms proportional to $\underline{\bar{t}}$), if the solution when $\epsilon = 0$ does not contain such terms. In short, if the solution to the differential equation when $\epsilon = 0$ is bounded in the fast variable, the solution procedure cannot unbound the solution when $\epsilon \neq 0$. Note that this condition is applicable only if the $\epsilon = 0$ solution is initially bounded (Reference 1, p. 18).

The <u>second uniformity condition</u> is a result of the uniform validity requirement, and this condition states that:

$$\lim_{\epsilon \to 0} \frac{g_{i+1}(\epsilon)\, f^{(i+1)}(\bar{t}, \tilde{t})}{g_i(\epsilon)\, f^{(i)}(\bar{t}, \tilde{t})} = 0$$

for each $\underline{i}$ and all $\underline{t}$ of some set $\underline{S}$ of nonnegative reals. Simply stated, the ratio:

$$\frac{f^{(i+1)}(\bar{t}, \tilde{t})}{f^{(i)}(\bar{t}, \tilde{t})}$$

cannot contain terms secular in the slow variable $\tilde{t}$. This condition may be employed to eliminate nonuniform results even when the first uniformity condition cannot be applied (Reference 1, p. 18 and Reference 15, pp. 206-224).

In Paragraph 4.3, the two-variable asymptotic series expansion method will be employed in obtaining solutions to the variation-of-parameters equations derived in Section 2. Thus, the function $x(t, \epsilon)$ to be approximated by these two-variable expansions will represent some osculating element; ϵ will be a small parameter arising through the perturbational effects, and $\bar{t}$, $\tilde{t}$ will be two time-scale variables associated with the time t.

4.3 APPLICATION OF THE TWO-VARIABLE ASYMPTOTIC SERIES METHOD TO THE DIFFERENTIAL EQUATIONS OF MOTION

This method assumes that the solutions to the equations of motion can be expressed as asymptotic series in two variables ($\bar{t}$ and $\tilde{t}$), i. e.,

$$B = B^{(0)}(\bar{t}, \tilde{t}) + \epsilon B^{(1)}(\bar{t}, \tilde{t}) + \epsilon^2 B^{(2)}(\bar{t}, \tilde{t}) + \ldots$$

$$e = e^{(0)}(\bar{t}, \tilde{t}) + \epsilon e^{(1)}(\bar{t}, \tilde{t}) + \epsilon^2 e^{(2)}(\bar{t}, \tilde{t}) + \ldots, \text{ etc.}$$

where $B^{(0)}$, $B^{(1)}$, $B^{(2)}, \ldots$, $e^{(0)}$, $e^{(1)}, \ldots$ are functions of time (i. e., solutions) as yet to be determined and:

$$\bar{t} = t\,(1+\alpha_2' \epsilon^2) \qquad \text{(fast variable)} \qquad (4\text{-}1)$$

$$\tilde{t} = \epsilon t \qquad \text{(slow variable)} \qquad (4\text{-}2)$$

with α_2' being an undetermined constant.

In the asymptotic series expansion for a given element, the first term is referred to herein as the first approximation to the total solution, and the sum of the first and second terms is referred to as the second approximation. For the element

$\underset{\smile}{e}$, as an example:

$$e = e^{(0)} \qquad \text{(first approximation)}$$

$$e = \underbrace{e^{(0)}}_{\text{super-zero solution}} + \epsilon \underbrace{e^{(1)}}_{\text{super-one solution}} \qquad \text{(second approximation)}$$

These approximations will now be derived for the set of elements (B, e, i, Ω, ω and M). First approximations will be obtained when considering both oblateness and oblateness/drag. Second approximations will be obtained in terms of the super-one solutions due to oblateness only, as it will be shown that the super-one solutions due to drag are negligible. The general procedure for obtaining the third approximations will be outlined.

4.3.1 Obtaining the First Approximations to the Solutions

The desired first (and second) approximations are obtained by solving the variation-of-parameters equations (oblateness only or oblateness/drag) when considering only terms of the order of $\underline{\epsilon}$ (i.e., neglecting terms $\geq \epsilon^2$). Since these equations are highly coupled, their solutions must be obtained simultaneously (at least in theory). However, by making reasonable assumptions, the solutions for each element can be obtained separately up to a point - this point being the formulation of a set of first-order ordinary differential equations having $\underline{\tilde{t}}$ as the independent variable. To illustrate the procedure leading to this point, the equation for a representative element will be considered in detail.

4.3.1.1 Oblateness Only

The element $\underline{i}$ is taken as the representative element, so it is necessary to expand each element appearing in Equation (2-36) to the first-order of $\underline{\epsilon}$. From Appendix B:

$$\cos i = \cos i^{(0)} + \epsilon\left[-i^{(1)} \sin i^{(0)}\right] + \epsilon^2\left[\quad\right] + \ldots$$

$$\sin^2 u = \sin^2 u^{(0)} + \epsilon\left[u^{(1)} \sin 2u^{(0)}\right] + \epsilon^2\left[\quad\right] + \ldots$$

$$\left(1 + e\cos\nu\right) = \left(1 + e^{(0)}\cos\nu^{(0)}\right) + \epsilon\left[\quad\right] + \epsilon^2\left[\quad\right] + \ldots, \text{ etc.}$$

Therefore, to the first order of ϵ, Equation (2-36) becomes, when considering only J_2 and J_3 (the solution procedure has not yet been extended to higher harmonics):

$$\frac{di}{dt} = -\epsilon \frac{\sqrt{\mu}\, r_e^{\,2}\, B^{(0)^7}\left(1 + e^{(0)}\cos\nu^{(0)}\right)^3}{2\left(1-e^{(0)^2}\right)^{7/2}} \sin 2i^{(0)} \sin 2u^{(0)} \tag{4-3}$$

$$+\ \epsilon \frac{\sqrt{\mu}\, r_e^{\,3}\, B^{(0)^9}\left(\frac{J3}{J2}\right)\left(1 + e^{(0)}\cos\nu^{(0)}\right)^4}{\left(1-e^{(0)^2}\right)^{9/2}} \cos i^{(0)} \cos u^{(0)} \left(1-5\sin^2 i^{(0)} \sin^2 u^{(0)}\right)$$

The solution method begins by assuming that Equation (4-3) has the asymptotic series solution:

$$i\,(\bar{t}, \tilde{t}) = i^{(0)}\,(\bar{t}, \tilde{t}) + \epsilon i^{(1)}\,(\bar{t}, \tilde{t}) + \ldots \tag{4-4}$$

where

$$\bar{t} = t\left(1 + \alpha_2' \epsilon^2\right) \tag{4-5}$$

$$\tilde{t} = \epsilon t \tag{4-6}$$

Differentiating Equation (4-4) with respect to time yields:

$$\frac{di}{dt} = \frac{\partial i}{\partial \bar{t}}\frac{d\bar{t}}{dt} + \frac{\partial i}{\partial \tilde{t}}\frac{d\tilde{t}}{dt}$$

$$= \left(\frac{\partial i^{(0)}}{\partial \bar{t}} + \epsilon\frac{\partial i^{(1)}}{\partial \bar{t}} + \ldots\right)\frac{d\bar{t}}{dt} + \left(\frac{\partial i^{(0)}}{\partial \tilde{t}} + \epsilon\frac{\partial i^{(1)}}{\partial \tilde{t}} + \ldots\right)\frac{d\tilde{t}}{dt}$$

which becomes upon differentiating Equations (4-5) and (4-6):

$$\frac{di}{dt} = \left(\frac{\partial i^{(0)}}{\partial \bar{t}} + \epsilon\frac{\partial i^{(1)}}{\partial \bar{t}} + \ldots\right)\left(1 + \alpha_2' \epsilon^2\right) + \left(\frac{\partial i^{(0)}}{\partial \tilde{t}} + \epsilon\frac{\partial i^{(1)}}{\partial \tilde{t}} + \ldots\right)(\epsilon)$$

Rearranging in ascending powers of ϵ yields:

$$\frac{di}{dt} = \frac{\partial i^{(0)}}{\partial \bar{t}} + \epsilon \left[\frac{\partial i^{(1)}}{\partial \bar{t}} + \frac{\partial i^{(0)}}{\partial \tilde{t}} \right] + \epsilon^2 \left[\quad \right] + \ldots \tag{4-7}$$

Equating coefficients of like powers of ϵ from Equations (4-3) and (4-7) results in the following partial differential equations:

$$\frac{\partial i^{(0)}}{\partial \bar{t}} = 0 \tag{4-8}$$

$$\frac{\partial i^{(1)}}{\partial \bar{t}} + \frac{\partial i^{(0)}}{\partial \tilde{t}} = - \frac{\sqrt{\mu}\, r_e^{\,2}\, B^{(0)^7} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^3}{2 \left(1 - e^{(0)^2}\right)^{7/2}} \sin 2 i^{(0)} \sin 2 u^{(0)}$$

$$+ \frac{\sqrt{\mu} \left(\frac{J3}{J2}\right) r_e^{\,3} B^{(0)^9} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^4}{\left(1 - e^{(0)^2}\right)^{9/2}} \cos i^{(0)} \cos u^{(0)} \left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) \tag{4-9}$$

The problem has now reduced to solving these partial differential equations. Equation (4-8) implies that $i^{(0)}$ is either constant or a function of $\tilde{t}$ only. Consequently,

$$\frac{\partial i^{(0)}}{\partial \tilde{t}} = \frac{d\, i^{(0)}}{d \tilde{t}} = \text{function of } \tilde{t} \text{ or constant} \tag{4-10}$$

In light of Equations (4-8) and (4-10), Equation (4-9) can be reduced to an ordinary differential equation if the constant of integration is considered a function of $\tilde{t}$.

The resulting solution in "integral form" is:

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\bar{t} - \int \frac{\sqrt{\mu}\, r_e^{\,2} B^{(0)^7} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^3}{2\left(1 - e^{(0)^2}\right)^{7/2}} \sin 2\, i^{(0)} \sin 2\, u^{(0)}\, d\bar{t} \tag{4-11}$$

$$+\int \frac{\sqrt{\mu}\, r_e^{\,3} B^{(0)^9} \left(\frac{J3}{J2}\right) \left(1 + e^{(0)} \cos \nu^{(0)}\right)^4}{\left(1 - e^{(0)^2}\right)^{9/2}} \cos i^{(0)} \cos u^{(0)} \left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) d\bar{t}$$

$$+ C(\tilde{t})$$

where $C(\tilde{t})$ is the integration constant. Before proceeding to solve the above integrals, it is desirable to transform the variable of integration from $\bar{t}$ to $\nu^{(0)}$. This transformation is taken to be the standard Keplerian transformation (Reference 6, p. 221).

$$\frac{d\nu}{dt} = \frac{\sqrt{\mu}(1 + e \cos \nu)^2}{p^{3/2}} = \frac{\sqrt{\mu} B^3 (1 + e \cos \nu)^2}{(1 - e^2)^{3/2}}$$

Thus,

$$d\bar{t} = \frac{\left(1 - e^{(0)^2}\right)^{3/2}}{\sqrt{\mu} B^{(0)^3} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^2} d\nu^{(0)} \tag{4-12}$$

Substituting Equation (4-12) into Equation (4-11) and simplifying yields

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\bar{t} - \frac{r_e^{\,2}}{2} \int \frac{B^{(0)^4}}{\left(1 - e^{(0)^2}\right)^2} \left(1 + e^{(0)} \cos \nu^{(0)}\right) \sin 2\, i^{(0)} \sin 2u^{(0)} d\nu^{(0)}$$

$$+ r_e^{\,3} \left(\frac{J3}{J2}\right) \int \frac{B^{(0)^6}}{\left(1 - e^{(0)^2}\right)^3} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^2 \cos i^{(0)} \cos u^{(0)} \tag{4-13}$$

$$\left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) d\nu^{(0)} + C(\tilde{t})$$

In order to perform the indicated integration, it is necessary to know the dependence of the element functions $B^{(0)}$, $e^{(0)}$, $i^{(0)}$ and $\omega^{(0)}$ (recall that $u^{(0)} = \omega^{(0)} + \nu^{(0)}$) upon $\nu^{(0)}$. However, these element functions are not yet known - in fact, the determination of these functions is the goal of the present development. Therefore, in order to proceed with the solution development, it is necessary to make a simplifying assumption based on the knowledge that the elements $\underline{B}$, $\underline{e}$, $\underline{i}$ and $\underline{\omega}$ vary slowly with time as compared to the element $\underline{\nu}$. <u>Specifically, it will be assumed that with respect to a $d\nu^{(0)}$ integration, the element functions $B^{(0)}$, $e^{(0)}$, $i^{(0)}$ and $\omega^{(0)}$ are constant.</u> The effect of this assumption on the accuracy of the resultant solutions can be minimized by periodically rectifying the orbit and updating the epoch values of the elements. (As discussed in Paragraph 4.5, an "updating procedure" is used when numerically evaluating the solution equations.)

In "partial consideration" of this assumption, Equation (4-13) can be written as:

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\,\bar{t} + \frac{B^{(0)^4}\, re^2}{2\left(1 - e^{(0)^2}\right)^2} \int -\left(1 + e^{(0)} \cos \nu^{(0)}\right) \sin 2\, i^{(0)} \sin 2\, u^{(0)}\, d\nu^{(0)} \tag{4-14}$$

$$+ \frac{B^{(0)^6}\left(\frac{J3}{J2}\right) re^3}{\left(1 - e^{(0)^2}\right)^3} \int \left(1 + e^{(0)} \cos \nu^{(0)}\right)^2 \cos i^{(0)} \cos u^{(0)} \left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) d\nu^{(0)}$$

$$+ C\,(\tilde{t})$$

or in notational form as:

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\,\bar{t} + K_2(i)\, I_2(i) + K_3(i)\, I_3(i) + C\,(\tilde{t}) \tag{4-15}$$

where

$$K_2(i) = \frac{B^{(0)^4} r_e^2}{2(1 - e^{(0)^2})^2} \tag{4-16}$$

$$K_3(i) = \frac{B^{(0)^6} r_e^3 \left(\frac{J3}{J2}\right)}{(1 - e^{(0)^2})^3} \tag{4-17}$$

and

$$I_2(i) = \int - (1 + e^{(0)} \cos \nu^{(0)}) \sin 2\, i^{(0)} \sin 2\, u^{(0)} d\nu^{(0)} \tag{4-18}$$

$$I_3(i) = \int (1 + e^{(0)} \cos \nu^{(0)})^2 \cos i^{(0)} \cos u^{(0)} (1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)})\, d\nu^{(0)} \tag{4-19}$$

(NOTE: In the above notation, the subscript on K and I indicates the earth-harmonic under consideration; the parenthetical (i) indicates the element i.)

Since $e^{(0)}$, $i^{(0)}$, and $\omega^{(0)}$ are considered constant by the previously-stated assumption, inspection of the integrals given by Equations (4-18) and (4-19) reveals that each integral can be expanded to a series of single-term integrals of the general form:

$$f(e^{(0)}, i^{(0)}, \omega^{(0)}) \int \sin^P \nu^{(0)} \cos^Q \nu^{(0)} d\nu^{(0)} \qquad (P, Q = 0, 1, 2, \ldots)$$

which is directly integrable by "textbook" formulas. Unfortunately, such an expansion procedure results in many single-term integrals; to solve these by hand for each of the six elements would be an overwhelmingly laborious task. However, by utilizing the automated techniques of the FORMAC language, a computer program was written for the IBM 7094 to expand expressions similar to Equations (4-18) and (4-19) and then "solve" the single-term integrals by an identification and substitution procedure. Basically, the program identifies through an iteration process, the values of the exponents P and Q occurring in each single-term integrand; it then substitutes the precoded solution for that particular integral.

In general, the integrated solutions to Equations (4-18) and (4-19) will consist of terms secular in the independent variable $\nu^{(0)}$ and terms non-secular in $\nu^{(0)}$; i.e.,

$$I_2(i) = S_2(i)\,\nu^{(0)} + N_2(i) \tag{4-20}$$

$$I_3(i) = S_3(i)\,\nu^{(0)} + N_3(i) \tag{4-21}$$

where S denotes the secular terms and N the non-secular terms. The FORMAC program prints the answer arrays $\underline{I}_2$, $\underline{I}_3$, and $\underline{S}_2$, $\underline{S}_3$ for each element; since these arrays are very lengthy, they are presented in Appendix C.

In view of Equations (4-20) and (4-21), Equation (4-15) becomes

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\,\bar{t} + K_2(i)\left[S_2(i)\nu^{(0)} + N_2(i)\right] + K_3(i)\left[S_3(i)\nu^{(0)} + N_3(i)\right] + C\,(\tilde{t}) \tag{4-22}$$

As shown in Appendix D, the element function $\nu^{(0)}$ is secularly related to the fast time-variable $\bar{t}$ by

$$\nu_s^{(0)} = n^{(0)}\bar{t} \tag{4-23}$$

Hence, the resolution of $\nu^{(0)}$ into secular and non-secular parts yields

$$\nu^{(0)} = \nu_s^{(0)} + \nu_N^{(0)} = n^{(0)}\,\bar{t} + \nu_N^{(0)} \tag{4-24}$$

where $\nu_N^{(0)}$ is the non-secular part of $\nu^{(0)}$ yet to be determined.

Substituting Equation (4-24) into (4-22) yields

$$i^{(1)} = -\frac{di^{(0)}}{d\tilde{t}}\bar{t} + K_2(i)\,S_2(i)n^{(0)}\bar{t} + K_2(i)\left[S_2(i)\,\nu_N^{(0)} + N_2(i)\right]$$

$$+ K_3(i)\,S_3(i)n^{(0)}\bar{t} + K_3(i)\left[S_3(i)\,\nu_N^{(0)} + N_3(i)\right] + C\,(\tilde{t})$$

which becomes after rearranging

$$i^{(1)} = \left[-\frac{di^{(0)}}{d\tilde{t}} + K_2(i)\,S_2(i)\,n^{(0)} + K_3(i)\,S_3(i)\,n^{(0)}\right]\bar{t} \tag{4-25}$$

$$+ K_2(i)\left[S_2(i)\,\nu_N^{(0)} + N_2(i)\right] + K_3(i)\left[S_3(i)\,\nu_N^{(0)} + N_3(i)\right] + C\,(\tilde{t})$$

At this point, the first uniformity condition (see Paragraph 4.2) can be imposed. Essentially, this condition requires that any approximate solution to the element $\underline{i}$ not contain a secular term in the fast variable $\underline{\bar{t}}$ since the solution to the differential equation for $\underline{i}$ did not contain a secular term when $\epsilon = 0$. In order for this condition to be satisfied, it must be that

$$\left[-\frac{di^{(0)}}{d\tilde{t}} + K_2(i)\,S_2(i)n^{(0)} + K_3(i)\,S_3(i)n^{(0)}\right] = 0 \tag{4-26}$$

In view of Equation (4-26), Equation (4-25) becomes merely

$$i^{(1)} = K_2(i)\left[S_2(i)\,\nu_N^{(0)} + N_2(i)\right] + K_3(i)\left[S_3(i)\,\nu_N^{(0)} + N_3(i)\right] + C\,(\tilde{t}) \tag{4-27}$$

Now, as is evident in Appendix D, one method for obtaining the non-secular part of $\nu^{(0)}$ (i.e., $\nu_N^{(0)}$) would be to evaluate the indicated Fourier series This is not necessary, however, since Equation (4-24) can be rearranged as

$$\nu_N^{(0)} = \nu^{(0)} - n^{(0)} \bar{t}$$

and $\nu^{(0)}$ and $n^{(0)}\bar{t}$ are known. Thus, Equation (4-27) can be written as

$$i^{(1)} = K_2(i)\left[S_2(i)\nu^{(0)} + N_2(i)\right] - K_2(i)\, S_2(i)n^{(0)}\bar{t} + K_3(i)\left[S_3(i)\nu^{(0)} + N_3(i)\right]$$

$$-K_3(i)\, S_3(i)\, n^{(0)}\bar{t} + C\,(\tilde{t})$$

which becomes by Equations (4-20) and (4-21)

$$i^{(1)} = K_2(i)\left[I_2(i) - S_2(i)n^{(0)}\bar{t}\right] + K_3(i)\left[I_3(i) - S_3(i)n^{(0)}\bar{t}\right] + C\,(\tilde{t}) \qquad (4\text{-}28)$$

where $K_2(i)$ and $K_3(i)$ are given by Equations (4-16) and (4-17), and $I_2(i)$, $I_3(i)$, $S_2(i)$, and $S_3(i)$ are obtained from the FORMAC program (see Appendix C). It should be noted that although the appearance of $\bar{t}$ in Equation (4-28) suggests secularity, this secularity is "cancelled" by that appearing in $I_2(i)$ and $I_3(i)$. Consequently, $i^{(1)}$ is non secular in $\bar{t}$, thereby satisfying the first uniformity condition.

Returning to Equation (4-26), it follows that

$$\frac{di^{(0)}}{d\tilde{t}} = K_2(i)\, S_2(i)n^{(0)} + K_3(i)\, S_3(i)n^{(0)} \qquad (4\text{-}29)$$

From the FORMAC results presented in Appendix C

$$S_2(i) = 0 \qquad (4\text{-}30)$$

$$S_3(i) = e^{(0)} \cos i^{(0)} \cos \omega^{(0)} \left(1 - \frac{5}{4} \sin^2 i^{(0)}\right) \qquad (4\text{-}31)$$

Substituting Equations (4-16), (4-17), (4-30) and (4-31) into Equation (4-29) yields

$$\frac{di^{(0)}}{d\tilde{t}} = \frac{B^{(0)^6}\left(\frac{J3}{J2}\right) r_e^{\,3} n^{(0)} e^{(0)} \cos i^{(0)} \cos \omega^{(0)}}{(1 - e^{(0)^2})^3} \left(1 - \frac{5}{4}\sin^2 i^{(0)}\right)$$

This is the first-order ordinary differential equation having $\tilde{t}$ as the independent variable which was referred to at the beginning of this section. Using a procedure identical to that just illustrated for the element i, the corresponding equations for the remaining elements can be formulated. The set of equations for all elements is presented below, along with the (approximate) solutions to the equations as derived in Appendix E. These solutions were obtained by a method set forth in Reference 16, whereby e and ω are considered to vary simultaneously and terms of the order of e^2 (or smaller) are ignored. The constants (A, α, C_1, C_2, ..., C_8) appearing in the solutions are defined in Appendix E.

Element e

$$\frac{de^{(0)}}{d\tilde{t}} = \frac{B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^{\,3} n^{(0)} \cos \omega^{(0)} \sin i^{(0)}}{(1-e^{(0)2})^2}\left(\frac{5}{4}\sin^2 i^{(0)} - 1\right) \tag{4-32}$$

$\Rightarrow$

$$e^{(0)} = \left[A^2 + 2\frac{C_1}{C_2} A \sin(C_2 \tilde{t} + \alpha) + \left(\frac{C_1}{C_2}\right)^2\right]^{1/2} \tag{4-33}$$

Element ω

$$\frac{d\omega^{(0)}}{d\tilde{t}} = \frac{B^{(0)^4} r_e^{\,2} n^{(0)}}{(1-e^{(0)2})^2}\left(2 - \frac{5}{2}\sin^2 i^{(0)}\right) + \frac{B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^{\,3} n^{(0)}}{(1-e^{(0)2})^3}\left[\frac{35}{4} e^{(0)} \sin i^{(0)} \sin \omega^{(0)} \cos^2 i^{(0)}\right.$$

$$\left. - e^{(0)} \csc i^{(0)} \sin \omega^{(0)} + \frac{1}{e^{(0)}} \sin i^{(0)} \sin \omega^{(0)} \left(1 - \frac{5}{4}\sin^2 i^{(0)}\right)\right] \tag{4-34}$$

$\Rightarrow$

$$\omega^{(0)} = \tan^{-1}\left[\frac{A \sin(C_2 \tilde{t} + \alpha) + \frac{C_1}{C_2}}{A \cos(C_2 \tilde{t} + \alpha)}\right] \tag{4-35}$$

Element i

$$\frac{di^{(0)}}{d\tilde{t}} = \frac{B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^{\,3} n^{(0)} e^{(0)} \cos i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^3}\left(1 - \frac{5}{4}\sin^2 i^{(0)}\right) \tag{4-36}$$

$$\Rightarrow$$

$$i^{(0)} = i_0^{(0)} + \frac{C_3}{C_2}\left(e^{(0)} \sin\omega^{(0)} - e_0^{(0)} \sin\omega_0^{(0)}\right) \tag{4-37}$$

Element Ω

$$\frac{d\Omega^{(0)}}{d\tilde{t}} = -\frac{B^{(0)^4} r_e^2 n^{(0)} \cos i^{(0)}}{(1-e^{(0)2})^2} + \frac{B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^3 n^{(0)} e^{(0)} \sin\omega^{(0)}}{(1-e^{(0)2})^3} \left(\operatorname{ctn} i^{(0)} - \frac{15}{8} \sin 2\, i^{(0)}\right) \tag{4-38}$$

$$\Rightarrow$$

$$\Omega^{(0)} = \Omega_0^{(0)} - \frac{C_5}{C_2}\left(e^{(0)} \cos\omega^{(0)} - e_0^{(0)} \cos\omega_0^{(0)}\right) + \left(C_5 \frac{C_1}{C_2} - C_4\right)\left(\tilde{t} - \tilde{t}_0\right) \tag{4-39}$$

Element B

$$\frac{dB^{(0)}}{d\tilde{t}} = 0 \tag{4-40}$$

$$\Rightarrow$$

$$B^{(0)} = B_0^{(0)} \tag{4-41}$$

NOTE: Since

$$n = \sqrt{\mu}\, B^3$$

it follows that

$$n^{(0)} = \sqrt{\mu}\, B^{(0)^3} = \sqrt{\mu}\, B_0^{(0)^3} = n_0^{(0)} \tag{4-42}$$

Element M

$$\frac{dM^{(0)}}{d\tilde{t}} = \frac{B^{(0)^4} r_e^2 n^{(0)}}{(1-e^{(0)2})^{3/2}} \left(1 - \frac{3}{2} \sin^2 i^{(0)}\right) - \frac{4B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^3 n^{(0)}}{3e^{(0)}(1-e^{(0)2})^{5/2}} \left[-3e^{(0)^2} \sin i^{(0)} \sin\omega^{(0)} \left(1 - \frac{5}{4} \sin^2 i^{(0)}\right)\right.$$

$$\left. + \frac{3}{4} \sin i^{(0)} \sin\omega^{(0)} \left(1 - \frac{5}{4} \sin^2 i^{(0)}\right)\right] \tag{4-43}$$

$$\Rightarrow \quad M^{(0)} = \left[n^{(0)}(t - t_o)\right] + M_o^{(0)} + \left(C_6 + \frac{C_1 C_8}{C_2} - 4\frac{C_1 C_7}{C_2}\right)(\hat{t} - \tilde{t}_o)$$

$$- \frac{(C_8 - 4C_7)}{C_2}\left(e^{(0)} \cos \omega^{(0)} - e_o^{(0)} \cos \omega_o^{(0)}\right) \tag{4-44}$$

NOTE: The first term on the right-hand side of Equation (4-44) is the Keplerian change in $\underline{M}$ that takes place during the time interval $(t-t_0)$. When applying the asymptotic series solution method to Equation (2-39) to obtain Equation (4-43), the Keplerian variation is ignored since this variation can be solved in a straightforward manner from the Keplerian equation

$$\frac{dM}{dt} = n$$

Consequently, the Keplerian change must be added to the solution of Equation (4-43).

Element ν

As mentioned in Paragraph 2.1, the element $\underline{\nu}$ is obtained by a Fourier-Bessel expansion involving $\underline{M}$ and $\underline{e}$. To the order of $\underline{e^2}$, this expansion is (Reference 6, p. 89)

$$\nu^{(0)} = M^{(0)} + 2e^{(0)} \sin M^{(0)} + \frac{5}{4} {e^{(0)}}^2 \sin 2M^{(0)} \tag{4-45}$$

4.3.1.2 Oblateness and Drag

The element $\underline{e}$ is taken as the representative element, so it is first necessary to form the composite differential equation for $\underline{e}$ when considering oblateness and drag. Since perturbative forces are additive, this is done by merely adding Equation (2-35), $\underline{J_2}$ and $\underline{J_3}$ terms only, to Equation (2-42). It is then necessary to expand each element in the composite equation to the first order of $\underline{\epsilon}$. Using the expansions presented in Appendix B results in

$$
\begin{aligned}
\frac{de}{dt} = &-\epsilon \frac{\sqrt{\mu}\, B^{(0)^7} r_e^{\,2} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^3}{\left(1-e^{(0)2}\right)^{7/2}} \Big[\sin \nu^{(0)} \left(1 + e^{(0)} \cos \nu^{(0)}\right) \left(1 - 3 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) \\
&+ \sin^2 i^{(0)} \sin 2 u^{(0)} \left(2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)}\right)\Big] \\
&- \frac{4}{3}\, \epsilon \frac{\sqrt{\mu}\, B^{(0)^9} \left(\frac{J_3}{J_2}\right) r_e^{\,3} \left(1 + e^{(0)} \cos \nu^{(0)}\right)^4}{\left(1-e^{(0)2}\right)^{9/2}} \Big[\sin i^{(0)} \sin u^{(0)} \sin \nu^{(0)} \left(1 + e^{(0)} \cos \nu^{(0)}\right)\left(3 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) \\
&- \frac{3}{4}\left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right)\left(2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)}\right)\left(\cos u^{(0)} \sin i^{(0)}\right)\Big] \\
&- \epsilon \frac{K^* \rho\, B^{(0)^4} \left(e^{(0)} + \cos \nu^{(0)}\right)\left(1 + e^{(0)^2} + 2e^{(0)} \cos \nu^{(0)}\right)^{1/2}}{n^{(0)}\left(1-e^{(0)2}\right)^{1/2}}
\end{aligned}
\tag{4-46}
$$

The solution method begins by assuming that Equation (4-46) has the asymptotic series solution

$$
e\,(\bar{t}, \tilde{t}) = e^{(0)}\,(\bar{t}, \tilde{t}) + \epsilon e^{(1)}\,(\bar{t}, \tilde{t}) + \ldots
$$

Following the procedure outlined in the previous section for the element $\underline{i}$, the solution in "integral form" is obtained (corresponding to Equation (4-14):

$$
\begin{aligned}
e^{(1)} = &-\frac{de^{(0)}}{d\tilde{t}}\,\bar{t} + \frac{B^{(0)^4} r_e^{\,2}}{\left(1-e^{(0)2}\right)^2} \int \left(1 + e^{(0)} \cos \nu^{(0)}\right) \Big[\sin \nu^{(0)} \left(1 + e^{(0)} \cos \nu^{(0)}\right)\left(1 - 3 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) \\
&+ \sin^2 i^{(0)} \sin 2 u^{(0)} \left(2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)}\right)\Big]\, d\nu^{(0)} \\
&+ \frac{4 B^{(0)^6} \left(\frac{J_3}{J_2}\right) r_e^{\,3}}{3\left(1-e^{(0)2}\right)^3} \int -\left(1 + e^{(0)} \cos \nu^{(0)}\right)^2 \Big[\sin i^{(0)} \sin u^{(0)} \sin \nu^{(0)} \left(1 + e^{(0)} \cos \nu^{(0)}\right) \\
&\left(3 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right) \\
&- \frac{3}{4}\left(1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)}\right)\left(2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)}\right) \cos u^{(0)} \sin i^{(0)}\Big]\, d\nu^{(0)} \\
&+ \frac{K^* \left(1-e^{(0)2}\right)}{\mu\, B^{(0)2}} \int \frac{\rho \left(e^{(0)} + \cos \nu^{(0)}\right)\left(1 + e^{(0)2} + 2e^{(0)} \cos \nu^{(0)}\right)^{1/2}}{\left(1 + e^{(0)} \cos \nu^{(0)}\right)^2}\, d\nu^{(0)} + c(\tilde{t})
\end{aligned}
\tag{4-47}
$$

or in notational form as

$$e^{(1)} = -\frac{d\dot{e}^{(0)}}{d\tilde{t}}\ \bar{t} + K_2(e)\ I_2(e) + K_3(e)\ I_3(e) + K_D(e)\ I_D(e) + C\ (\tilde{t}) \tag{4-48}$$

where

$$K_2(e) = \frac{B^{(0)^4}\ r_e^2}{(1-e^{(0)2})\,2} \tag{4-49}$$

$$K_3(e) = \frac{4\ B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^3}{3\ (1-e^{(0)2})\,3} \tag{4-50}$$

$$K_D(e) = \frac{K*(1-e^{(0)2})}{\mu B^{(0)2}} \tag{4-51}$$

and

$$I_2(e) = \int -(1 + e^{(0)} \cos \nu^{(0)}) \Big[\sin \nu^{(0)} (1 + e^{(0)} \cos \nu^{(0)}) (1 - 3 \sin^2 i^{(0)} \sin^2 u^{(0)})$$
$$+ \sin^2 i^{(0)} \sin 2u^{(0)}\ (2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)})\Big]\, d\nu^{(0)} \tag{4-52}$$

$$I_3(e) = \int -(1 + e^{(0)} \cos \nu^{(0)})\ 2 \Big[\sin i^{(0)} \sin u^{(0)} \sin \nu^{(0)} (1 + e^{(0)} \cos \nu^{(0)}) (3 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)})$$
$$- \frac{3}{4} (1 - 5 \sin^2 i^{(0)} \sin^2 u^{(0)})\ (2 \cos \nu^{(0)} + e^{(0)} \cos^2 \nu^{(0)} + e^{(0)})\ \cos u^{(0)} \sin i^{(0)}\Big]\, d\nu^{(0)} \tag{4-53}$$

$$I_D(e) = \int - \frac{\rho\ (e^{(0)} + \cos \nu^{(0)})\ (1 + e^{(0)2} + 2e^{(0)} \cos \nu^{(0)})\ 1/2}{(1 + e^{(0)} \cos \nu^{(0)})^2}\ d\nu^{(0)} \tag{4-54}$$

(NOTE: In the above notation, the numerical subscript on K and I indicates the earth-harmonic under consideration, the subscript D indicates drag, and the parenthetical (e) indicates the element e.)

Since $e^{(0)}$, $i^{(0)}$, and $\omega^{(0)}$ (recall that $u^{(0)} = \omega^{(0)} + \nu^{(0)}$) are considered constant by the previously stated assumption, inspection of the integrals given by Equations (4-52) and (4-53) reveals that each integral can be expanded to a series of single-term integrals of the general form

$$f(e^{(0)},\ i^{(0)},\ \omega^{(0)}) \int \sin^P \nu^{(0)} \cos^Q \nu^{(0)} d\nu^{(0)} \qquad (P,\ Q = 0,\ 1,\ 2, \ldots\ldots)$$

which is directly integrable by "textbook" formulas and, therefore, the FORMAC program. However, in order to readily integrate Equation (4-54), it is convenient to employ the binomial series approximations

$$(1 + e^{(0)^2} + 2e^{(0)} \cos \nu^{(0)})^{1/2} = 1 + e^{(0)} \cos \nu^{(0)} + 1/2\ e^{(0)^2} \sin^2 \nu^{(0)} + \ldots \quad (4\text{-}55)$$

$$(1 + e^{(0)} \cos \nu^{(0)})^{-2} = 1 - 2e^{(0)} \cos \nu^{(0)} + 3e^{(0)^2} \cos^2 \nu^{(0)} + \ldots . \quad (4\text{-}56)$$

It is also necessary to know the functional dependency of atmospheric density ρ upon true anomaly $\nu^{(0)}$. In the past, this dependency has been established by using very simple models of atmospheric density, such as an exponential model or a power-law model. Though convenient to work with, these types of models do not provide realistic simulations of the actual environment since they are structured to represent the variation of density with altitude only. Density actually varies with solar and geomagnetic activity, time of year and position relative to the sub-solar point (diurnal bulge), as well as with altitude.

Realistic simulations of long-term satellite motion must include these additional variations in the density model. For example, using a simple density model (the 1959 ARDC) to compute the lifetime of Satellite 1961ϵ results in a lifetime of 179.1 days. The actual lifetime was 525.5 days - an error of 66%! On the other hand, using a realistic model (the 1970 Jacchia) produced a lifetime of 537.9 days; an error of only 2.4%.

The difficulty with using a realistic density model is in expressing density as a function of true anomaly. An examination of the 1970 Jacchia model shows how complex a realistic model is and, consequently, how difficult it would be to implement directly into a general perturbation technique. Yet, in order for the general perturbation technique to be as accurate as numerical solutions, it is desirable to use the 1970 Jacchia model.

A rather unique approach to the use of a realistic density model is taken in this study. Specifically, the variation of $\underline{\rho}$ with $\nu^{(0)}$ is approximated by the Fourier series

$$\rho = 1/2\ a_0 + \sum_{k=1}^{4} [a_k \cos k\nu^{(0)} + b_k \sin k\nu^{(0)}] \tag{4-57}$$

where $\underline{a}_0$, $\underline{a}_k$, and $\underline{b}_k$ are Fourier coefficients determined in the following manner: A table of density values is computed for intervals of true anomaly around one orbital revolution by numerically evaluating the 1970 Jacchia model. Integrals associated with determination of the Fourier coefficients are then computed by the Trapezoidal Rule. (It was found that the Fourier series using coefficients through $\underline{a}_4$ and $\underline{b}_4$ give an excellent approximation to the functional dependency of density upon true anomaly.)

Because of the dynamic nature of the density function, the series approximation will not hold for long periods of time. (In fact, this is one area in which further study is recommended - see Section 6.) The length of time depends somewhat upon the amount of resolution in the density input data (solar flux, geomagnetic index, etc.) and upon the orbital conditions. For instance, if daily values of solar flux and heating parameters are used, the series would need to be evaluated at least daily. If the orbit is in a state of rapid decay, the series could require more frequent evaluation. As discussed in Paragraph 4.5, the Fourier coefficients are updated at required intervals when numerically evaluating the solution equations.

Returning to the solution procedure and substituting Equations (4-55) through (4-57) into Equation (4-54) yields

$$I_D(e) = \int - \Big[(1/2\, a_0 + a_1 \cos \nu^{(0)} + \ldots + b_4 \sin 4\, \nu^{(0)}) (e^{(0)} + \cos \nu^{(0)}) \qquad (4\text{-}58)$$

$$(1 + e^{(0)} \cos \nu^{(0)} + 1/2 e^{(0)^2} \sin^2 \nu^{(0)}) (1 - 2e^{(0)} \cos \nu^{(0)} + 3e^{(0)^2} \cos^2 \nu^{(0)}) \Big] d\nu^{(0)}$$

The FORMAC program is utilized to expand Equations (4-52), (4-53) and (4-58) and then "solve" the single-term integrals. In general, the integrated solutions will consist of terms secular in the independent variable $\nu^{(0)}$ and terms non-secular in $\nu^{(0)}$; i.e.,

$$I_2(e) = S_2(e)\nu^{(0)} + N_2(e) \qquad (4\text{-}59)$$

$$I_3(e) = S_3(e)\nu^{(0)} + N_3(e) \qquad (4\text{-}60)$$

$$I_D(e) = S_D(e)\nu^{(0)} + N_D(e) \qquad (4\text{-}61)$$

where S denotes the secular terms and N the non-secular terms. The FORMAC program prints the answer arrays $\underline{I}_2$, $\underline{I}_3$, $\underline{I}_D$ and $\underline{S}_2$, $\underline{S}_3$, $\underline{S}_D$ for each element; since these arrays are very lengthy, they are presented in Appendix C.

In view of Equations (4-59) through (4-61), Equation (4-48) becomes

$$e^{(1)} = -\frac{de^{(0)}}{d\tilde{t}}\, \bar{t} + K_2(e) \left[S_2(e)\nu^{(0)} + N_2(e) \right] + K_3(e) \left[S_3(e)\nu^{(0)} + N_3(e) \right]$$

$$+ K_D(e) \left[S_D(e)\nu^{(0)} + N_D(e) \right] + C(\tilde{t}) \qquad (4\text{-}62)$$

As shown in Appendix D, the element function $\nu^{(0)}$ is secularly related to the fast time-variable $\bar{t}$ by:

$$\nu_s^{(0)} = n^{(0)} \bar{t} \tag{4-23}$$

Hence, the resolution of $\nu^{(0)}$ into secular and nonsecular parts yields:

$$\nu^{(0)} = \nu_s^{(0)} + \nu_N^{(0)} = n^{(0)} \bar{t} + \nu_N^{(0)} \tag{4-24}$$

where $\nu_N^{(0)}$ is the nonsecular part of $\nu^{(0)}$ yet to be determined.

Substituting Equation (4-24) into Equation (4-62) and rearranging yields:

$$\begin{aligned} e^{(1)} = &\left[-\frac{de^{(0)}}{d\bar{t}} + K_2(e) S_2(e) n^{(0)} + K_3(e) S_3(e) n^{(0)} + K_D(e) S_D(e) n^{(0)} \right] \bar{t} \\ &+ K_2(e) \left[S_2(e) \nu_N^{(0)} + N_2(e) \right] + K_3(e) \left[S_3(e) \nu_N^{(0)} + N_3(e) \right] \\ &+ K_D(e) \left[S_D(e) \nu_N^{(0)} + N_D(e) \right] + C(\bar{t}) \end{aligned} \tag{4-63}$$

At this point, the first uniformity condition (see Paragraph 4.2) can be imposed. Essentially, this condition requires that any approximate solution to the element $\underline{e}$ not contain a secular term in the fast variable $\bar{t}$, since the solution to the differential equation for $\underline{e}$ did not contain a secular term when $\epsilon = 0$. In order for this condition to be satisfied, it must be that:

$$\left[-\frac{de^{(0)}}{d\bar{t}} + K_2(e) S_2(e) n^{(0)} + K_3(e) S_3(e) n^{(0)} + K_D(e) S_D(e) n^{(0)} \right] = 0 \tag{4-64}$$

In view of Equation (4-64), Equation (4-63) merely becomes:

$$\begin{aligned} e^{(1)} = &K_2(e) \left[S_2(e) \nu_N^{(0)} + N_2(e) \right] + K_3(e) \left[S_3(e) \nu_N^{(0)} + N_3(e) \right] \\ &+ K_D(e) \left[S_D(e) \nu_N^{(0)} + N_D(e) \right] + C(\bar{t}) \end{aligned} \tag{4-65}$$

Now, as is evident in Appendix D, one method for obtaining the nonsecular part of $\nu^{(0)}$ (i.e., $\nu_N^{(0)}$) would be to evaluate the indicated Fourier series. This is not necessary, however, since Equation (4-24) can be rearranged as:

$$\nu_N^{(0)} = \nu^{(0)} - n^{(0)} \bar{t}$$

and $\nu^{(0)}$ and $n^{(0)} \bar{t}$ are known. Thus, when considering Equations (4-59) through (4-61), Equation (4-65) can be written as:

$$e^{(1)} = K_2(e) \left[I_2(e) - S_2(e)\, n^{(0)} \bar{t} \right] + K_3(e) \left[I_3(e) - S_3(e)\, n^{(0)} \bar{t} \right] + K_D(e) \left[I_D(e) - S_D(e)\, n^{(0)} \bar{t} \right] + C(\bar{t}) \tag{4-66}$$

It should be noted that, although the appearance of $\bar{t}$ in Equation (4-66) suggests secularity, this secularity is cancelled by that appearing in $I_2(e)$, $I_3(e)$, and $I_D(e)$. Consequently, $e^{(1)}$ is nonsecular in $\bar{t}$, thereby satisfying the first uniformity condition.

Returning to Equation (4-64), it follows that:

$$\frac{de^{(0)}}{d\bar{t}} = K_2(e)\, S_2(e)\, n^{(0)} + K_3(e)\, S_3(e)\, n^{(0)} + K_D(e)\, S_D(e)\, n^{(0)} \tag{4-67}$$

From the FORMAC results presented in Appendix C:

$$S_2(e) = 0 \tag{4-68}$$

$$S_3(e) = \frac{3}{4} \sin i^{(0)} \cos \omega^{(0)} (1 - e^{(0)2}) (\frac{5}{4} \sin^2 i^{(0)} - 1) \tag{4-69}$$

$$S_D(e) = (-\frac{1}{2} a_1 + \frac{3}{2} b_3) + (-\frac{1}{4} a_0 + \frac{1}{4} a_2)\, e^{(0)} \tag{4-70}$$

(Recall that $\underline{a}_0$, $\underline{a}_1$, $\underline{a}_2$, and $\underline{b}_3$ are the Fourier coefficients appearing in the density function approximation.)

Substituting Equations (4-49) through (4-51) and Equations (4-68) through (4-70) into Equation (4-67) yields:

$$\frac{de^{(0)}}{d\tilde{t}} = \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}\, n^{(0)} \sin i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^2} \left(\frac{5}{4}\sin^2 i^{(0)} - 1\right)$$

$$+ \frac{n^{(0)} K^* (1-e^{(0)2})}{\mu B^{(0)2}} \left[\left(-\frac{1}{2}a_1 + \frac{3}{2}b_3\right) + \left(-\frac{1}{4}a_0 + \frac{1}{4}a_2\right) e^{(0)}\right]$$

This is the first-order ordinary differential equation having $\tilde{t}$ as the independent variable which was referred to at the beginning of this section. Using a procedure identical to that just illustrated for the element e, the corresponding equations for the remaining elements can be formulated. The set of equations for all elements is presented below, along with the (approximate) solutions to the equations as derived in Appendix F. The constants (a, b, c, λ_1, λ_2, C_1, C_2, ..., C_8, D_2, β_0, β_1, δ_{-1}, δ_0 and δ_1) appearing in the solutions are defined in Appendix F.

Element e

$$\frac{de^{(0)}}{d\tilde{t}} = \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}\, n^{(0)} \sin i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^2} \left(\frac{5}{4}\sin^2 i^{(0)} - 1\right) \tag{4-71}$$

$$+ \frac{n^{(0)} K^* (1-e^{(0)2})}{\mu B^{(0)2}} \left[\left(-\frac{1}{2}a_1 + \frac{3}{2}b_3\right) + \left(-\frac{1}{4}a_0 + \frac{1}{4}a_2\right) e^{(0)}\right]$$

$\Rightarrow$

$$e^{(0)} = (\xi^2 + \eta^2)^{1/2}$$

(4-72)

where

$$\xi = e_{xp}^{\,b\tilde{t}} \left[\lambda_1 \cos c\,\tilde{t} + \lambda_2 \sin c\,\tilde{t}\right] - \frac{ab}{b^2 + c^2}$$

$$\eta = e_{xp}^{\,b\tilde{t}} \left[-\lambda_1 \sin c\,\tilde{t} + \lambda_2 \cos c\,\tilde{t}\right] - \frac{ac}{b^2 + c^2}$$

Element ω

$$\frac{d\omega^{(0)}}{d\bar{t}} = \frac{B^{(0)4}\, r_e^{\,2}\, n^{(0)}}{(1-e^{(0)2})^2}\left(2-\frac{5}{2}\sin^2 i^{(0)}\right) + \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}\, n^{(0)}}{(1-e^{(0)2})^3}\left[\frac{35}{4}\, e^{(0)} \sin i^{(0)} \sin \omega^{(0)} \cos^2 i^{(0)}\right.$$

$$\left. - e^{(0)} \csc i^{(0)} \sin \omega^{(0)} + \frac{1}{e^{(0)}} \sin i^{(0)} \sin \omega^{(0)} \left(1-\frac{5}{4}\sin^2 i^{(0)}\right)\right] \tag{4-73}$$

$$+ \frac{n^{(0)} K^* (1-e^{(0)2})}{B^{(0)2}\, e^{(0)}\, \mu}\left[-\frac{1}{2} b_1 - \frac{3}{2} b_3 + \frac{1}{4} b_2\, e^{(0)}\right]$$

$\Rightarrow$

$$\omega^{(0)} = \tan^{-1}\left(\frac{\eta}{\xi}\right) \tag{4-74}$$

where ξ and η are given on the previous page.

Element i

$$\frac{d i^{(0)}}{d\bar{t}} = \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}\, n^{(0)}\, e^{(0)} \cos i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^3}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right) \tag{4-75}$$

$\Rightarrow$

$$i^{(0)} = i_0^{(0)} + \frac{C_3}{C_2}\left(e^{(0)} \sin \omega^{(0)} - e_0^{(0)} \sin \omega_0^{(0)}\right) \tag{4-76}$$

Element Ω

$$\frac{d\Omega^{(0)}}{d\bar{t}} = -\frac{B^{(0)4}\, r_e^{\,2}\, n^{(0)} \cos i^{(0)}}{(1-e^{(0)2})^2} + \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}\, n^{(0)}\, e^{(0)} \sin \omega^{(0)}}{(1-e^{(0)2})^3}\left(\operatorname{ctn} i^{(0)} - \frac{15}{8}\sin 2\, i^{(0)}\right) \tag{4-77}$$

$\Rightarrow$

$$\Omega^{(0)} = \Omega_0^{(0)} - \frac{C_5}{C_2}\left(e^{(0)} \cos \omega^{(0)} - e_0^{(0)} \cos \omega_0^{(0)}\right) + \left(C_5 \frac{C_1}{C_2} - C_4\right)(\bar{t}-\bar{t}_0) \tag{4-78}$$

Element B

$$\frac{dB^{(0)}}{d\bar{t}} = \frac{n^{(0)} K^*}{2B^{(0)} \mu} \left[\frac{1}{2} a_0 + \left(\frac{1}{2} a_1 - \frac{3}{2} b_3\right) e^{(0)}\right] \tag{4-79}$$

$\Rightarrow$

$$B^{(0)} = \frac{B_0^{(0)} D_2}{2} (\beta_0 + \beta_1 e_0^{(0)}) (\bar{t} - \bar{t}_0) + B_0^{(0)} \tag{4-80}$$

NOTE: Since

$$n = \sqrt{\mu} B^3$$

it follows that

$$n^{(0)} = \sqrt{\mu} B^{(0)^3} \tag{4-81}$$

Element M

$$\begin{aligned}\frac{dM^{(0)}}{d\bar{t}} = {} & \frac{B^{(0)^4} r_e^{\,2} n^{(0)}}{(1 - e^{(0)2})^{3/2}} \left(1 - \frac{3}{2} \sin^2 i^{(0)}\right) - \frac{4B^{(0)^6} \left(\frac{J_3}{J_2}\right) r_e^{\,3} n^{(0)}}{3e^{(0)} (1 - e^{(0)2})^{5/2}} \Big[- 3e^{(0)^2} \sin i^{(0)} \sin \omega^{(0)} (1 - \frac{5}{4} \sin^2 i^{(0)}) \\ & + \frac{3}{4} \sin i^{(0)} \sin \omega^{(0)} \left(1 - \frac{5}{4} \sin^2 i^{(0)}\right)\Big] + \frac{n^{(0)} K^* (1 - e^{(0)2})^{3/2}}{B^{(0)2} \mu} \left[\left(\frac{1}{2} b_1 + \frac{3}{2} b_3\right) \frac{1}{e^{(0)}} + \left(\frac{1}{2} b_1 + \frac{3}{4} b_2 + \frac{3}{2} b_3\right) \right. \\ & \left. + \left(\frac{9}{16} b_1 - \frac{1}{2} b_2 + \frac{27}{16} b_3\right) e^{(0)} \right]\end{aligned} \tag{4-82}$$

$\Rightarrow$

$$\begin{aligned} M^{(0)} = {} & \left[n^{(0)} (t - t_0)\right] + M_0^{(0)} + \left[C_6 + (4C_7 - C_8) \frac{ac}{b^2 + c^2} + D_2 \left(\frac{\delta_{-1}}{e_0^{(0)}} + \delta_0 + \delta_1 e_0^{(0)}\right)\right] [\bar{t} - \bar{t}_0] \\ & - (4 C_7 - C_8) \left[\frac{e_{xp}^{b\bar{t}}}{b^2 + c^2} \left\{ (\lambda_2 c - \lambda_1 b) \sin c \bar{t} + (\lambda_2 b + \lambda_1 c) \cos c \bar{t} \right\} \right. \\ & \left. - \frac{e_{xp}^{b \bar{t}_0}}{b^2 + c^2} \left\{ (\lambda_2 c - \lambda_1 b) \sin c \bar{t}_0 + (\lambda_2 b + \lambda_1 c) \cos c \bar{t} \right\} \right] \end{aligned} \tag{4-83}$$

NOTE: The first term on the right-hand side of Equation (4-83) is the Keplerian change in $\underline{M}$ that takes place during the time interval $(t - t_0)$.

Element ν

As mentioned in Paragraph 2.1, the element $\underline{\nu}$ is obtained by a Fourier-Bessel expansion involving $\underline{M}$ and $\underline{e}$. To the order of $\underline{e}^2$, the expansion is (Reference 6, p. 39):

$$\nu^{(0)} = M^{(0)} + 2\, e^{(0)} \sin M^{(0)} + \frac{5}{4}\, {e^{(0)}}^2 \sin 2\, M^{(0)} \tag{4-84}$$

4.3.2. Obtaining the Second Approximations to the Solutions

It so happens that the second approximations are very nearly obtained during the process of deriving the first approximations, since the super-one solutions are merely functions of the super-zero solutions and an integration constant. The procedure for completing the derivation of the second approximations will now be illustrated. As mentioned at the beginning of Paragraph 4.3, and as will be more thoroughly discussed at the end of this section, drag need not be considered since the super-one solutions due to drag are negligible.

The element $\underline{i}$ will again be considered in detail as a representative element. Recall Equation (4-28):

$$i^{(1)} = K_2\,(i) \left[I_2\,(i) - S_2\,(i)\, n^{(0)}\, \bar{t} \right] + K_3\,(i) \left[I_3\,(i) - S_3\,(i)\, n^{(0)}\, \bar{t} \right] + C\,(\tilde{t}) \tag{4-28}$$

where $K_2(i)$ and $K_3(i)$ are given by Equations (4-16) and (4-17), $n^{(0)}$ is given by Equation (4-81), and $I_2(i)$, $I_3(i)$, and $S_3(i)$ are obtained from the FORMAC program (see Appendix C). Hence, once the first approximations are known, $i^{(1)}$ can be computed from Equation (4-28) after the constant of integration $C\,(\tilde{t})$ has been determined. In theory, a second application of the first uniformity condition (see Paragraph 4.2) would provide a means of determining $C(\tilde{t})$; unfortunately, this requires at least a partial formulation of the third approximation (see Paragraph 4.3.3).

To readily proceed with the solution development for $i^{(1)}$, it is convenient to make a simplifying assumption based on the supposition that $C(\tilde{t})$, being a function of the slow time-variable $\tilde{t}$, is slowly varying itself. Specifically, it will be assumed that $C(\tilde{t})$ is a constant, C. The effect of this assumption can be minimized by periodically rectifying the orbit and updating the epoch value of this constant. (As discussed in Paragraph 4.5, an updating procedure is used when numerically evaluating the solution equations and, as shown in the plots of Appendix H, the $C(\tilde{t})$ for each element remains sufficiently constant over time intervals which are not extreme. Furthermore, as discussed in Appendix G, these plots well describe the functional form of the $C(\tilde{t})$'s obtainable when considering the third approximation.)

In considering this assumption, Equation (4-28) can be written as:

$$i^{(1)} = K_2(i)\left[I_2(i) - S_2(i)\, n^{(0)}\bar{t}\right] + K_3(i)\left[I_3(i) - S_3(i)\, n^{(0)}\bar{t}\right] + C \qquad (4\text{-}85)$$

The constant C can now be evaluated from initial (or epoch) conditions. From Equation (4-14), it can be seen that C is the constant associated with a $d\nu^{(0)}$ integration in which all other element functions are considered constant. So, at epoch time t_0, Equation (4-85) becomes:

$$i_0^{(1)} = \left[K_2(i)\left\{I_2(i) - S_2(i)\, n^{(0)}\bar{t}\right\} + K_3(i)\left\{I_3(i) - S_3(i)\, n^{(0)}\bar{t}\right\}\right]_{\bar{t}_0} + C \qquad (4\text{-}86)$$

where $[\;]_{\bar{t}_0}$

indicates that the functions of $\nu^{(0)}$ within the bracket are to be evaluated using the epoch value $\nu_0^{(0)}$. Functions of the other elements (such as $\sin\omega^{(0)}$) are evaluated using current values. For example:

$$[\sin\omega^{(0)}\cos\nu^{(0)}]_{\bar{t}_0} = \sin\omega^{(0)}\cos\nu_0^{(0)}$$

(The FORTRAN program discussed in Paragraph 4.5 employs a procedure for updating this epoch value.)

In view of Equation (4-86), Equation (4-85) becomes:

$$i^{(1)} = i_0^{(1)} + K_2(i) \left| I_2(i) - S_2(i)\, n^{(0)} \bar{t} \right| + K_3(i) \left| I_3(i) - S_3(i)\, n^{(0)} \bar{t} \right| - \left| K_2(i) \left\{ I_2(i) - S_2(i)\, n^{(0)} \bar{t} \right\} + K_3(i) \left\{ I_3(i) - S_3(i)\, n^{(0)} \bar{t} \right\} \right|_{\bar{t}_0} \qquad (4\text{-}87)$$

Since $\underline{K}_2$ and $\underline{K}_3$ are not functions of $\nu_0^{(0)}$, Equation (4-87) can be written as:

$$i^{(1)} = i_0^{(1)} + K_2(i) \left| I_2(i) - S_2(i)\, n^{(0)} \bar{t} \right| + K_3(i) \left| I_3(i) - S_3(i)\, n^{(0)} \bar{t} \right| - K_2(i) \left| I_2(i) - S_2(i)\, n^{(0)} \bar{t} \right|_{\bar{t}_0} - K_3(i) \left| I_3(i) - S_3(i)\, n^{(0)} \bar{t} \right|_{\bar{t}_0}$$

or more concisely as:

$$i^{(1)} = i_0^{(1)} + K_2(i) \left| I_2(i) - S_2(i)\, n^{(0)} \bar{t} \right|_{\bar{t}_0}^{\bar{t}} + K_3(i) \left| I_3(i) - S_3(i)\, n^{(0)} \bar{t} \right|_{\bar{t}_0}^{\bar{t}} \qquad (4\text{-}88)$$

where $K_2(i)$ and $K_3(i)$ are given by Equations (4-16) and (4-17), $n^{(0)}$ is given by Equation (4-81), and $I_2(i)$, $I_3(i)$, $S_2(i)$ and $S_3(i)$ are obtained from the FORMAC printout (see Appendix C).

To this point, the solution for $i^{(1)}$ has been considered in notational form. For a more revealing look into the actual solution, it is necessary to substitute Equations (4-16), (4-17), and (4-81) and the FORMAC results $I_2(i)$, $I_3(i)$, $S_2(i)$, and $S_3(i)$ into Equation (4-88). The solution resulting from these substitutions is presented in Equation (4-89).

Using a procedure identical to that just illustrated for the element $\underline{i}$, the corresponding equations for the remaining elements can be obtained. The equations for all six elements are summarized in notational form following Equation (4-89).

$i^{(1)}$ Solution

$$
\begin{aligned}
i^{(1)} = i_0^{(1)} + \frac{B^{(0)4}\, r_e^2}{2\,(1-e^{(0)2})^2} \Big[& -2\, e^{(0)} \sin 2i^{(0)} \sin \nu^{(0)} \sin \omega^{(0)} \cos \omega^{(0)} + \frac{4}{3} e^{(0)} \sin 2i^{(0)} \sin^2 \nu^{(0)} \sin^2 \omega^{(0)} \cos \nu^{(0)} - \frac{2}{3} e^{(0)} \sin 2\, i^{(0)} \sin^2 \nu^{(0)} \cos \nu^{(0)} \\
& + \frac{4}{3} e^{(0)} \sin 2\, i^{(0)} \sin^3 \nu^{(0)} \sin \omega^{(0)} \cos \omega^{(0)} - \frac{4}{3} e^{(0)} \sin 2\, i^{(0)} \sin^2 \omega^{(0)} \cos \nu^{(0)} + \frac{2}{3} e^{(0)} \sin 2\, i^{(0)} \cos \nu^{(0)} - 2 \sin 2\, i^{(0)} \sin \nu^{(0)} \sin \omega^{(0)} \cos \nu^{(0)} \\
& \cos \omega^{(0)} - \sin 2\, i^{(0)} \sin^2 \nu^{(0)} + 2 \sin 2\, i^{(0)} \sin^2 \nu^{(0)} \sin^2 \omega^{(0)} + 2 e^{(0)} \sin 2\, i^{(0)} \sin \nu_0^{(0)} \sin \omega^{(0)} \cos \omega^{(0)} - \frac{4}{3} e^{(0)} \sin 2\, i^{(0)} \sin^2 \nu_0^{(0)} \sin^2 \omega^{(0)} \\
& \cos \nu_0^{(0)} + \frac{2}{3} e^{(0)} \sin 2\, i^{(0)} \sin^2 \nu_0^{(0)} \cos \nu_0^{(0)} - \frac{4}{3} e^{(0)} \sin 2\, i^{(0)} \sin^3 \nu_0^{(0)} \sin \omega^{(0)} \cos \omega^{(0)} + \frac{4}{3} e^{(0)} \sin 2\, i^{(0)} \sin^2 \omega^{(0)} \cos \nu_0^{(0)} - \frac{2}{3} e^{(0)} \sin 2\, i^{(0)} \\
& \cos \nu_0^{(0)} + 2 \sin 2\, i^{(0)} \sin \nu_0^{(0)} \sin \omega^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)} + \sin 2\, i^{(0)} \sin^2 \nu_0^{(0)} - 2 \sin 2\, i^{(0)} \sin^2 \nu_0^{(0)} \sin^2 \omega^{(0)} \Big] \\
+ \frac{B^{(0)6} \left(\frac{J_3}{J_2}\right) r_e^3}{(1-e^{(0)2})^3} \Big[& -5 \sin^2 i^{(0)} \sin \nu^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \omega^{(0)} - 5 \sin^2 i^{(0)} \sin^2 \nu^{(0)} \sin \omega^{(0)} \cos i^{(0)} \cos \nu^{(0)} + \frac{20}{3} \sin^2 i^{(0)} \sin^2 \nu^{(0)} \sin^3 \omega^{(0)} \\
& \cos i^{(0)} \cos \nu^{(0)} + \frac{20}{3} \sin^2 i^{(0)} \sin^3 \nu^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \omega^{(0)} - \frac{5}{3} \sin^2 i^{(0)} \sin^3 \nu^{(0)} \cos i^{(0)} \cos \omega^{(0)} - \frac{5}{3} \sin^2 i^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} \cos \nu^{(0)} \\
& + \sin \nu^{(0)} \cos i^{(0)} \cos \omega^{(0)} + \sin \omega^{(0)} \cos i^{(0)} \cos \nu^{(0)} - \frac{5}{4} e^{(0)} \nu^{(0)} \sin^2 i^{(0)} \cos i^{(0)} \cos \omega^{(0)} + e^{(0)} \nu^{(0)} \cos i^{(0)} \cos \omega^{(0)} - 10\, e^{(0)} \sin^2 i^{(0)} \sin \nu^{(0)} \\
& \sin^2 \omega^{(0)} \cos i^{(0)} \cos \nu^{(0)} \cos \omega^{(0)} + \frac{5}{4} e^{(0)} \sin^2 i^{(0)} \sin \nu^{(0)} \cos i^{(0)} \cos \nu^{(0)} \cos \omega^{(0)} - 10\, e^{(0)} \sin^2 i^{(0)} \sin^2 \nu^{(0)} \sin \omega^{(0)} \cos i^{(0)} \\
& + 15\, e^{(0)} \sin^2 i^{(0)} \sin^2 \nu^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} + 10\, e^{(0)} \sin^2 i^{(0)} \sin^3 \nu^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \nu^{(0)} \cos \omega^{(0)} - \frac{5}{2} e^{(0)} \sin^2 i^{(0)} \sin^3 \nu^{(0)} \\
& \cos i^{(0)} \cos \nu^{(0)} \cos \omega^{(0)} + \frac{15}{2} e^{(0)} \sin^2 i^{(0)} \sin^4 \nu^{(0)} \sin \omega^{(0)} \cos i^{(0)} + 10\, e^{(0)} \sin^2 i^{(0)} \sin^4 \nu^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} + e^{(0)} \sin \nu^{(0)} \\
& \cos i^{(0)} \cos \nu^{(0)} \cos \omega^{(0)} - e^{(0)} \sin^2 \nu^{(0)} \sin \omega^{(0)} \cos i^{(0)}
\end{aligned}
\tag{4-89}
$$

$$+ 5 \sin^2 i^{(0)} \sin \nu_0^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \omega^{(0)} + 5 \sin^2 i^{(0)} \sin^2 \nu_0^{(0)} \sin \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} - \frac{20}{3} \sin^2 i^{(0)} \sin^2 \nu_0^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)}$$

$$- \frac{20}{3} \sin^2 i^{(0)} \sin^3 \nu_0^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \omega^{(0)} + \frac{5}{3} \sin^2 i^{(0)} \sin^3 \nu_0^{(0)} \cos i^{(0)} \cos \omega^{(0)} + \frac{5}{3} \sin^2 i^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} - \sin \nu_0^{(0)} \cos i^{(0)} \cos \omega^{(0)}$$

$$- \sin \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} + \frac{5}{4} e^{(0)} \nu_0^{(0)} \sin^2 i^{(0)} \cos i^{(0)} \cos \omega^{(0)} - e^{(0)} \nu_0^{(0)} \cos i^{(0)} \cos \omega^{(0)} + 10\, e^{(0)} \sin^2 i^{(0)} \sin \nu_0^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)}$$

$$- \frac{5}{4} e^{(0)} \sin^2 i^{(0)} \sin \nu_0^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)} + 10\, e^{(0)} \sin^2 i^{(0)} \sin^2 \nu_0^{(0)} \sin \omega^{(0)} \cos i^{(0)} - 15\, e^{(0)} \sin^2 i^{(0)} \sin^2 \nu_0^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)}$$

$$- 10\, e^{(0)} \sin^2 i^{(0)} \sin^3 \nu_0^{(0)} \sin^2 \omega^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)} + \frac{5}{2} e^{(0)} \sin^2 i^{(0)} \sin^3 \nu_0^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)} - \frac{15}{2} e^{(0)} \sin^2 i^{(0)} \sin^4 \nu_0^{(0)} \sin \omega^{(0)}$$

$$\cos i^{(0)} - 10\, e^{(0)} \sin^2 i^{(0)} \sin^4 \nu_0^{(0)} \sin^3 \omega^{(0)} \cos i^{(0)} - e^{(0)} \sin \nu_0^{(0)} \cos i^{(0)} \cos \nu_0^{(0)} \cos \omega^{(0)} + e^{(0)} \sin^2 \nu_0^{(0)} \sin \omega^{(0)} \cos i^{(0)}$$

$$- n^{(0)} \bar{t} \left(- \frac{5}{4} e^{(0)} \sin^2 i^{(0)} \cos i^{(0)} \cos \omega^{(0)} + e^{(0)} \cos i^{(0)} \cos \omega^{(0)}\right) + n^{(0)} \bar{t}_0 \left(- \frac{5}{4} e^{(0)} \sin^2 i^{(0)} \cos i^{(0)} \cos \omega^{(0)} + e^{(0)} \cos i^{(0)} \cos \omega^{(0)}\right) \Bigg|$$

Element e

$$e^{(1)} = e_0^{(1)} + K_2(e)\,[I_2(e) - S_2(e)\,n^{(0)}\,\bar{t}\,]_{\bar{t}_0}^{\bar{t}} + K_3(e)\,[I_3(e) - S_3(e)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} \tag{4-90}$$

$$K_2(e) = \frac{B^{(0)4}\,r_e^{\,2}}{(1-e^{(0)2})^2} \tag{4-91}$$

$$K_3(e) = \frac{4B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{3\,(1-e^{(0)2})^3} \tag{4-92}$$

Element ω

$$\omega^{(1)} = \omega_0^{(1)} + K_2(\omega)\,[I_2(\omega) - S_2(\omega)\,n^{(0)}\,\bar{t}\,]_{\bar{t}_0}^{\bar{t}} + K_3(\omega)\,[I_3(\omega) - S_3(\omega)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} \tag{4-93}$$

$$K_2(\omega) = \frac{B^{(0)4}\,r_e^{\,2}}{e^{(0)}\,(1-e^{(0)2})^2} \tag{4-94}$$

$$K_3(\omega) = -\frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{e^{(0)}\,(1-e^{(0)2})^3} \tag{4-95}$$

Element i

$$i^{(1)} = i_0^{(1)} + K_2(i)\,[I_2(i) - S_2(i)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} + K_3(i)\,[I_3(i) - S_3(i)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} \tag{4-96}$$

$$K_2(i) = \frac{B^{(0)4}\,r_e^{\,2}}{2\,(1-e^{(0)2})^2} \tag{4-97}$$

$$K_3(i) = \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{(1-e^{(0)2})^3} \tag{4-98}$$

Element Ω

$$\Omega^{(1)} = \Omega_0^{(1)} + K_2(\Omega)\,[I_2(\Omega) - S_2(\Omega)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} + K_3(\Omega)\,[I_3(\Omega) - S_3(\Omega)\,n^{(0)}\,\bar{t}]_{\bar{t}_0}^{\bar{t}} \tag{4-99}$$

$$K_2(\Omega) = \frac{2B^{(0)4}\,r_e^{\,2}}{(1-e^{(0)2})^2} \tag{4-100}$$

$$K_3(\Omega) = \frac{B^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{(1-e^{(0)2})^3} \tag{4-101}$$

Element B

$$B^{(1)} = B_0^{(1)} + K_2(B)\left[I_2(B) - S_2(B)\, n^{(0)}\, \bar{t}\right]_{\bar{t}_0}^{\bar{t}} + K_3(B)\left[I_3(B) - S_3(B)\, n^{(0)}\, \bar{t}\right]_{\bar{t}_0}^{\bar{t}} \qquad (4\text{-}102)$$

$$K_2(B) = \frac{B^{(0)^5}\, r_e^{\,2}}{(1-e^{(0)2})^3} \qquad (4\text{-}103)$$

$$K_3(B) = \frac{4B^{(0)^7}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{3\,(1-e^{(0)2})^4} \qquad (4\text{-}104)$$

Element M

$$M^{(1)} = M_0^{(1)} + K_2(M)\left[I_2(M) - S_2(M)\, n^{(0)}\, \bar{t}\right]_{\bar{t}_0}^{\bar{t}} + K_3(M)\left[I_3(M) - S_3(M)\, n^{(0)}\, \bar{t}\right]_{\bar{t}_0}^{\bar{t}} \qquad (4\text{-}105)$$

$$K_2(M) = \frac{B^{(0)^4}\, r_e^{\,2}}{e^{(0)}\,(1-e^{(0)2})^{3/2}} \qquad (4\text{-}106)$$

$$K_3(M) = -\,\frac{4B^{(0)^6}\left(\frac{J_3}{J_2}\right) r_e^{\,3}}{3e^{(0)}\,(1-e^{(0)2})^{5/2}} \qquad (4\text{-}107)$$

As discussed in the beginning of Paragraph 4.3, the second approximations to the solutions are then formulated as

$$e = e^{(0)} + \epsilon\, e^{(1)} \qquad (4\text{-}108)$$

$$\omega = \omega^{(0)} + \epsilon\, \omega^{(1)} \qquad (4\text{-}109)$$

$$i = i^{(0)} + \epsilon\, i^{(1)} \qquad (4\text{-}110)$$

$$\Omega = \Omega^{(0)} + \epsilon\, \Omega^{(1)} \qquad (4\text{-}111)$$

$$B = B^{(0)} + \epsilon\, B^{(1)} \qquad (4\text{-}112)$$

$$M = M^{(0)} + \epsilon\, M^{(1)} \qquad (4\text{-}113)$$

The second approximation for true anomaly $\underline{\nu}$ is obtained by a Fourier-Bessel expansion involving $\underline{M}$ and $\underline{e}$. To the order of $\underline{e}^2$, this expansion is (Reference 6, p. 89)

$$\nu = M + 2\, e \sin M + \frac{5}{4} e^2 \sin 2\, M \qquad (4\text{-}114)$$

where $\underline{e}$ and $\underline{M}$ are given by Equations (4-108) and (4-113), respectively.

As mentioned at the beginning of this section, the super-one solutions due to drag are negligible. This fact is illustrated by the following consideration. The second approximation to the total solution has the form:

$$E = E^{(0)} + \epsilon\, E^{(1)}$$

where $\underline{E}$ represents any orbital element in the set (B, e, i, Ω, ω, and M). The $\epsilon E^{(1)}$ terms are short-periodic (see Paragraph 4.3.4) and are composed of integrals of the form:

$$E^{(1)} = K_2\,(E) \int [\text{Periodic}]\; d\,\nu^{(0)} + K_3\,(E) \int [\text{Periodic}]\; d\,\nu^{(0)} + K_D\,(E) \int [\text{Periodic}]\; d\,\nu^{(0)}$$

where $K_2(E)$ is the constant associated with $\underline{J}_2$ effects, $K_3(E)$ the constant associated with $\underline{J}_3$ effects, and $K_D(E)$ the constant associated with drag effects. Since the above integrands are composed of trigonometric functions which do not yield overall solutions secular in $\nu^{(0)}$, the integrated terms will be trigonometric functions having amplitudes proportional to the respective constant $K_2(E)$, $K_3(E)$ or $K_D(E)$. An order-of-magnitude analysis has revealed that $K_D(E)$ is considerably smaller than $K_2(E)$ and $K_3(E)$ for each element. Specifically, for a low-eccentricity orbit (e = 0.0055) and a $C_D(A/m)$ of 0.02 m^2/kg, the relative magnitudes of these constants were found to be approximately:

E	$\epsilon K_2(E)$	$\epsilon K_3(E)$	$\epsilon K_D(E)$
B	1×10^{-5}	2×10^{-8}	0.8×10^{-16}
e	1×10^{-3}	1×10^{-6}	0.9×10^{-14}
ω	8.8 deg	0.009 deg	0.9×10^{-10}
M	9.0 deg	0.009 deg	0.9×10^{-10}

Because $\epsilon K_D(E)$ is 10 to 11 orders of magnitude less than $\epsilon K_2(E)$ and 8 to 9 orders of magnitude less than $\epsilon K_3(E)$, it would appear that drag effects can justifiably be neglected in deriving the super-one solutions. To verify this, a computer run was made for the elements $\underline{B}$, $\underline{e}$, and $\underline{\omega}$ in which even the super-one solutions due to $\underline{J_3}$ were neglected. As expected, there was very little difference in the super-one solutions with and without the effects of $\underline{J_3}$. Consequently, there would be even less difference in the super-one solutions with and without the effects of drag.

4.3.3 Procedure for Obtaining the Third Approximations to the Solutions

The third approximation of the solution for any element $\underline{E}$ has the form:

$$E = E^{(0)} + \epsilon E^{(1)} + \epsilon^2 E^{(2)}$$

Procedures for obtaining the $E^{(2)}$ solutions have been established when considering oblateness only, but as yet have not been executed to the point of completely determining the third approximations. These procedures are outlined in this section. Included within their development are the steps necessary to obtain expressions for $C(\tilde{t})$; a detailed discussion of these steps is given in Appendix G.

The procedure begins by further expanding the basic differential equations of satellite motion (Equations (2-34) through (2-39)) to the order of ϵ^2. These equations will have the form:

$$\frac{dE}{dt} = \epsilon\, f\,(E^{(0)}) + \epsilon^2\, g\,(E^{(0)}, E^{(1)})$$

Specifically, for the element i:

$$\frac{di}{dt} = \epsilon\, f_i\,(B^{(0)}, e^{(0)}, i^{(0)}, \omega^{(0)}, \nu^{(0)}) + \epsilon^2\, g_i\,(B^{(0)}, B^{(1)}, e^{(0)}, e^{(1)}, i^{(0)}, i^{(1)}, \omega^{(0)}, \omega^{(1)}, \nu^{(0)}, \nu^{(1)})$$

(4-115)

which is a functional extension of Equation (4-3). The asymptotic series solution for i has the form:

$$i\,(t) = i^{(0)}\,(\tilde{t}) + \epsilon\, i^{(1)}\,(\bar{t}, \tilde{t}) + \epsilon^2\, i^{(2)}\,(\bar{t}, \tilde{t}) \tag{4-116}$$

where

$$\bar{t} = t\,(1 + \alpha_2'\,\epsilon^2)$$

$$\tilde{t} = \epsilon\, t$$

Differentiating Equation (4-116) with respect to time yields

$$\frac{di}{dt} = \frac{\partial\, i^{(0)}}{\partial\, \bar{t}} + \epsilon \left[\frac{\partial\, i^{(1)}}{\partial\, \bar{t}} + \frac{\partial\, i^{(0)}}{\partial\, \tilde{t}} \right] + \epsilon^2 \left[\frac{\partial\, i^{(2)}}{\partial\, \bar{t}} + \alpha_2' \frac{\partial\, i^{(0)}}{\partial\, \bar{t}} + \frac{\partial\, i^{(1)}}{\partial\, \tilde{t}} \right] \tag{4-117}$$

which is an extension of Equation (4-7). Equating coefficients of like powers of ϵ from Equations (4-115) and (4-117) results in three partial differential equations to be solved (two of which are the same as before):

$$\frac{\partial\, i^{(0)}}{\partial\, \bar{t}} = 0 \tag{4-8}$$

$$\frac{\partial i^{(1)}}{\partial \bar{t}} + \frac{\partial i^{(0)}}{\partial \tilde{t}} = f_i \tag{4-9}$$

$$\frac{\partial i^{(2)}}{\partial \bar{t}} + \alpha'_2 \frac{\partial i^{(0)}}{\partial \bar{t}} + \frac{\partial i^{(1)}}{\partial \tilde{t}} = g_i \tag{4-118}$$

Solving Equations (4-8) and (4-9) produced second approximations in which it was necessary to assume the integration constant $C(\tilde{t})$ to be a true constant (see Paragraph 4.3.2). Solution of Equation (4-118) will permit the functional determination of $C(\tilde{t})$, as well as a partial determination of $i^{(2)}$. (Thus, the process of obtaining the higher-order solution $E^{(2)}$ serves to complete the $E^{(1)}$ solution.)

Equation (4-118) will now be considered in more detail. Equation (4-8) implies that $i^{(0)}$ is a function of $\underline{\tilde{t}}$ only, thus:

$$\frac{\partial i^{(2)}}{\partial \bar{t}} + \frac{\partial i^{(1)}}{\partial \tilde{t}} = g_i \tag{4-119}$$

The solution for $i^{(1)}$ is composed of terms nonsecular in $\underline{\bar{t}}$ and a function which depends only upon $\underline{\tilde{t}}$, i.e., from Equation (4-28):

$$i^{(1)} = f_N\left(E^{(0)}\right) + C_i(\tilde{t})$$

where $$f_N = K_2(i)\left[I_2(i) - S_2(i)\, n^{(0)}\, \bar{t}\right] + K_3(i)\left[I_3(i) - S_3(i)\, n^{(0)}\, \bar{t}\right]$$

Equation (4-119) can, therefore, be written as

$$\frac{\partial i^{(2)}}{\partial \bar{t}} + \frac{\partial f_N}{\partial \tilde{t}} + \frac{d\, C_i(\tilde{t})}{d\, \tilde{t}} = g_i \tag{4-120}$$

The solution of Equation (4-120) can be obtained by the same procedure used to solve Equation (4-9). Thus:

$$i^{(2)} = -\int \frac{\partial f_N}{\partial \bar{t}}\, d\,\bar{t} - \frac{d\,C_i(\tilde{t})}{d\,\tilde{t}}\,\bar{t} + \int g_i\, d\,\bar{t} + C'_i(\tilde{t}) \tag{4-121}$$

where $C'_i\,(\tilde{t})$ is a constant of the $\bar{t}$ integration associated with the $i^{(2)}$ solution. At this point, it is necessary to again apply the first uniformity condition. Terms secular in $\bar{t}$ are collected in Equation (4-121) and set equal to zero, yielding:

$$\frac{d\,C_i(\tilde{t})}{d\,\tilde{t}}\,\bar{t} = \int_{\text{sec.}} g_i\, d\,\bar{t} - \int_{\text{sec.}} \frac{\partial f_N}{\partial \bar{t}}\, d\,\bar{t} \tag{4-122}$$

This equation allows the functional determination of $C(\tilde{t})$, and its solution is discussed in Appendix G.

The $i^{(2)}$ solution thus becomes only a function of terms nonsecular in $\bar{t}$ and the integration constant, i.e.,

$$i^{(2)} = -\int_{\text{N.S.}} \frac{\partial f_N}{\partial \bar{t}}\, d\,\bar{t} + \int_{\text{N.S.}} g_i\, d\,\bar{t} + C'_i(\tilde{t}) \tag{4-123}$$

These nonsecular terms can be evaluated by the same technique used to arrive at Equation (4-28). Again, the integration constant, $C'_i\,(\tilde{t})$, must be assumed truly constant or determined from the $E^{(3)}$ solution.

The procedure for computing the complete $E^{(2)}$ solutions, as outlined above, is relatively straightforward; however, it involves many long expressions and taxes even the capabilities of FORMAC. It is uncertain at this point what benefits would be derived, in relation to the work involved, by obtaining the complete $E^{(2)}$ solutions. As will be discussed in Paragraph 4.3.4, element solutions are composed of short-periodic, long-periodic and secular components. Secular and long-periodic terms are the most important in long-term ephemeris prediction, and these terms are presently contained entirely within the $E^{(0)}$ solutions. The short-periodic terms, on the other hand, are contained entirely within the $E^{(1)}$ solutions. It is anticipated that the $C(\tilde{t})$s will add $\underline{\tilde{t}}$ secularity to the $E^{(1)}$ solutions; however, the $E^{(2)}$ solutions will be purely periodic until the $E^{(3)}$ solutions are evaluated, at least to the point of determining $C'(\tilde{t})$. Thus, the $E^{(2)}$ terms will, most likely, only add terms on the order of $\underline{\varepsilon}^2$ to the periodic solutions. Since these indications have not been completely verified, it is recommended that the nature of the $E^{(2)}$ solutions be further investigated before undertaking the laborious procedure of completely solving for them.

Undoubtedly, carrying the $E^{(2)}$ solution procedure to the point of determining $C(\tilde{t})$ for each element would yield certain benefits. For instance, having a functional expression for each $C(\tilde{t})$ would eliminate one requirement for periodically updating the epoch values of the elements and associated parameters. (Since each $C(\tilde{t})$ is presently assumed constant, it is one of the parameters that must be periodically updated.) A second benefit would result from the fact that second-order $\underline{J}_2$ and $\underline{J}_4$ secular effects could most likely be represented in the overall solutions via $C(\tilde{t})$. (The importance of these effects is discussed in Section 5.)

4.3.4 Physical Interpretation of the Solution Components

The analytical investigation of perturbational effects on a satellite shows that (Reference 3, pp. 361-362):

1. Certain elements experience secular variations from their epoch values, as well as periodic variations about these epoch values
2. Other elements have only periodic variations.

Earth oblateness, for example, causes secular variations in the elements $\underline{\Omega}$, $\underline{\omega}$ and $\underline{M}$, and very small periodic variations in all the elements. Similarly, atmospheric drag causes secular variations in $\underline{B}$ (or $\underline{a}$), $\underline{e}$ and $\underline{M}$, and very small periodic variations in all the elements.

Among the periodic variations, a distinction is made between long-periodic variations (periodic with respect to $\underline{\omega}$ or multiples of $\underline{\omega}$) and short-periodic variations (periodic with respect to linear combinations of $\underline{\nu}$ and $\underline{\omega}$). To visualize these effects, consider Figure 4-1. The superposition of all variations depicted in Figure 4-1 yields what is referred to as the osculating element. Consequently, the set of osculating (or instantaneous) elements defines the continually changing elliptical orbit.

Inspection of the solution equations for $e^{(0)}$, $\omega^{(0)}$, $i^{(0)}$, $\Omega^{(0)}$, $B^{(0)}$ and $M^{(0)}$ (disregarding, for the purpose of this discussion, the Keplerian variation in $M^{(0)}$) reveals that these solutions are secular with respect to $\underline{\tilde{t}}$ and/or periodic with respect to $\underline{\omega}$. Consequently, these solution components represent a superposition of the secular and long-periodic variations depicted in Figure 4-1, and, as such, represent the mean elements. (A mean element is normally defined as the osculating element minus the short-periodic variation; however, as discussed in Appendix I, there are other convenient definitions for a mean element.)

Inspection of the solution equations for $e^{(1)}$, $\omega^{(1)}$, $i^{(1)}$, $\Omega^{(1)}$, $B^{(1)}$, and $M^{(1)}$ reveals that these solutions are short-periodic. Consequently, these solution components represent short-periodic variations of the elements about their mean values.

In summary, letting $\underline{E}$ denote any element in the set (B, e, i, Ω, ω and M), the physical interpretation of the solution components is as follows:

$$E \qquad E^{(0)} \qquad + \qquad \varepsilon E^{(1)}$$

short-periodic variation about the mean

mean value of the element

osculating value of the element

(NOTE: Since the $E^{(2)}$ solution components have not been derived, they are not depicted above. As discussed in Paragraph 4.3.3, it is anticipated that the $E^{(2)}$ component will add secularity (with respect to the slow time-variable $\underline{\tilde{t}}$) to the $E^{(1)}$ component through the constant of integration $C(\tilde{t})$.)

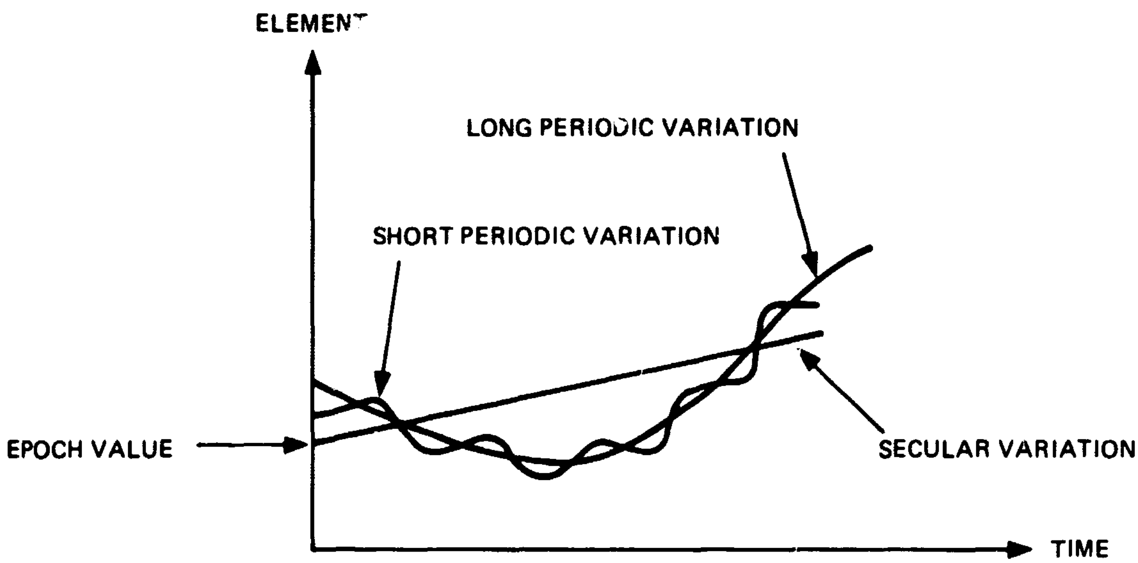


Figure 4-1. Typical Orbital Element Variations

4.4 SYNOPSIS OF THE FORMAC PROGRAM USED IN OBTAINING THE ASYMPTOTIC SERIES SOLUTIONS

As was indicated in Paragraph 4.1, general perturbation methods require a great amount of analytical labor in formulating and integrating the equations of motion. Equations such as (4-18) and (4-58), for example, involve a large number of integrals of the form:

$$\int \sin^P x \cos^Q x \, dx \qquad (P, Q = 0, 1, 2, \ldots)$$

While these integrals are basic, each generally requires several tedious recursions in its analytical evaluation.

In addition, more complex integral forms may arise, such as

$$K \int \frac{\sin^P x \cos^Q x}{(1 + e \cos x)^N} \, dx \quad (N = 1, 2, \ldots)$$

To alleviate the analytical labor required in performing numerous evaluations of both integral forms, an IBM 7094 FORMAC program (IDIGTE) was developed which provides the required expansion and integration capabilities. This program consists of a FORMAC driver and a set of subroutines which effect the required integrations. The driver performs all required manipulations of each input integrand, determines the integration parameters $\underline{P}$, $\underline{Q}$, $\underline{N}$ and the "constant" $\underline{K}$ and then transmits these quantities to the driver routine of the integration package (the set of routines which perform the required integrations). The integration package driver then identifies the integrand involved, makes any necessary variable transformations, and calls upon the proper subroutine to carry out the integration.

The complete solution of an integrand usually requires solving several sub-integrals (special cases), and each integration package subroutine is designed to integrate a given type of subintegral. Basically, each integration is carried out by substituting the prederived and precoded solution for that particular integral (i.e., the integral determined by the values of $\underline{P}$, $\underline{Q}$ and $\underline{N}$). These "integrated" results are then transmitted back to the integration package driver (where inverse transforms are performed, if necessary), and the results passed on to the FORMAC driver for simplification and output.

A detailed description of this program is provided in Reference 17.

4.5 SYNOPSIS OF THE GENERAL PERTURBATION FORTRAN PROGRAM FOR NUMERICALLY EVALUATING THE ASYMPTOTIC SERIES SOLUTIONS

A FORTRAN double-precision computer program (GENPUR) has been developed for the UNIVAC 1108 to numerically evaluate the analytical solutions derived in Paragraph 4.3. Currently, the program reflects only those perturbations due to earth oblateness ($\underline{J}_2$ and $\underline{J}_3$) and tangential atmospheric drag, but it is structured to readily accommodate additional perturbations, such as higher-order harmonics, low-level thrusting, solar radiation, etc. At user option, asymptotic series solutions, through the second approximation, can be evaluated when considering either earth oblateness or the combined effects of oblateness and drag.

As indicated in Paragraph 4.3, certain assumptions were made in order to effect the integrations involved and to evaluate the corresponding integration constants. These assumptions appear to be physically reasonable, if they are considered to hold over time intervals which are not extreme. With this in mind, the program is structured to make use of an updating scheme, whereby the solutions evaluated over a given time interval (Δt) are expressed in terms of constants and epoch values of the elements (both osculating and mean) computed at the beginning of that time interval. These solutions are then used to recompute the constants and epoch values prior to the solution evaluation over the next time interval. Included in this scheme is a procedure for updating the Fourier coefficients appearing in the series approximation to the atmospheric density function (see Paragraph 4.3.1.2). At the beginning of each time interval, these coefficients are evaluated by using the 1970 Jacchia atmospheric density model.

Even though the asymptotic solutions do not yet include $\underline{J}_4$ and second-order $\underline{J}_2$ perturbations, estimates of these effects for the elements $\underline{\omega}$, $\underline{\Omega}$ and $\underline{M}$ were temporarily implemented using Brouwer's solutions (see Section 5) to make meaningful comparisons with actual satellite data.

A complete description of this program (referred to as the GENPUR program) is provided in Reference 18.

SECTION 5 - COMPARISON OF RESULTS FROM THE GENERAL PERTURBATION PROGRAM

The asymptotic series solutions through the second approximations have been implemented into a computer program (GENPUR). Output of the GENPUR program consists of mean elements ($E^{(0)}$) and osculating elements ($E^{(0)} + \epsilon E^{(1)}$). (As discussed in Paragraph 4.3.4 and shown in Appendix H, the mean elements contain the very important long-periodic and secular effects, while the osculating elements result from adding short-periodic effects to the mean elements.) The purpose of this section is to thoroughly discuss and compare the results obtainable from GENPUR.

The comparison of GENPUR results is conducted in two parts. First, the validity of the mean element solutions over long time periods is established by a comparison with mean elements derived from Smithsonian tracking data, along with corresponding solutions from the MSFC Orbit Lifetime Program. Numerical integration programs such as COWELL and ENCKE would have been ideal for this comparison, but they are restricted in their application to relatively short time intervals (20 days). On the other hand, use of elements derived from tracking data provides the opportunity of observing the actual behavior of an orbit, since there are usually small forces in an actual environment that are never modeled. In the second part of the comparison, osculating element solutions from GENPUR are compared to the results of two numerical integration programs, COWELL and SPERTB. These solutions are analyzed for only 8 days, since they are merely short periodic additions to the mean elements.

5.1 COMPARISON WITH SMITHSONIAN TRACKING DATA

The Baker-Nunn system operated by the Smithsonian Astrophysical Observatory (SAO) is a source of very accurate satellite tracking data. The purpose of this section is to present a detailed comparison of GENPUR results with SAO tracking data for three satellites, namely: Explorer 7, Explorer 1 and SA-5.

The Baker-Nunn camera is an instrument with very high accuracy individual measurements. The timing accuracy of observations is approximately 0.001 second (corresponding to an in-track error of 10 meters for a satellite at a 1000-km altitude). Average positions are accurate to within 3 to 4 seconds of arc. The camera takes a time exposure of a satellite which is in sunlight, while the camera is in darkness. The exposure is interrupted by a rapid operation of the shutter so that the photograph appears as a dashed streak of light. The time of the middle interruption is recorded with an atomic clock. Appearing on the photograph with the dashed streak (which is the satellite) will be point sources of light, which are known stars. The locations (right ascensions and declinations) of these stars are accurately predetermined so that the photograph provides a recorded history of where the satellite was in relation to known references. The processing of these pictures is done with extreme care, requiring as long as several weeks to get the final results.

A series of these measurements are then analyzed by the SAO Differential Orbit Improvement program (DOI). The DOI program determines, through a least-squares procedure, the set of orbit elements that most accurately represents the satellite motion during the period of observation. These elements are published for some satellites in SAO Special Reports. An example is given in Figure 5-1 (taken from Reference 19) which shows the elements for the initial history of Satellite 1964-5A (SA-5). The elements are given in 1-day increments of Modified Julian Date (MJD); however, they are not exactly in the form desired. The analysis of this report uses semimajor axis $\underline{a}$, eccentricity $\underline{e}$, inclination $\underline{i}$, right ascension of ascending node $\underline{\Omega}$, argument of perigee $\underline{\omega}$, and mean anomaly $\underline{M}$. Columns 2, 3, 4, and 6 of Figure 5-1 give $\underline{\omega}$, $\underline{\Omega}$, and $\underline{i}$ in degrees, and $\underline{M}$ in revolutions. Column 5 presents the history of eccentricity, and Column 9 the history of perigee radius in megameters. Semimajor axis is obtained from

$$a = \frac{r_p}{1 - e}$$

Anomalistic mean motion (in revolutions per day) and its first derivative are given in Columns 7 and 8. Information pertaining to the number of observations on which each

T (MJD)	ω	Ω	i	e	M	n	n′/2	q	N	D	σ
38426.0	150.21 4	161.797 5	31.4561 9	.03580 1	.0961 1	15.193621 2	.336E-3 2	6.637593	59	4	.91
38427.0	159.89 4	155.259 7	31.457 1	.03581 1	.2918 1	15.194299 3	.335E-3 2	6.637308	40	4	.90
38428.0	170.10 4	148.723 7	31.457 1	.03564 1	.4867 1	15.194956 3	.329E-3 3	6.638297	44	4	1.04
38429.0	180.23 3	142.187 4	31.458 8	.03550 8	.68250 8	15.195589 2	.306E-3 1	6.639080	34	4	.56
38430.0	190.38 3	135.654 6	31.460 1	.03536 9	.87875 9	15.196172 2	.275E-3 4	6.639850	23	4	.59
38431.0	200.61 1	129.12 2	31.460 5	.03531 3	.0753 3	15.196828 9	.325E-3 6	6.640029	20	4	1.66
38432.0	210.60 8	122.60 1	31.460 2	.03516 2	.2732 2	15.197533 6	.360E-3 7	6.640865	17	4	.92
38433.0	220.97 7	116.05 1	31.461 2	.03503 1	.4707 2	15.198241 4	.347E-3 3	6.641545	16	4	.73
38434.0	230.90 3	109.513 5	31.460 1	.03493 2	.67023 9	15.198943 4	.342E-3 4	6.642017	29	4	.95
38435.0	241.28 2	102.980 3	31.460 1	.03489 1	.86907 6	15.199648 3	.362E-3 2	6.642050	31	4	.84
38436.0	251.58 3	96.443 5	31.459 2	.03485 2	.06881 9	15.200309 7	.334E-3 6	6.642143	32	4	1.34
38437.0	261.89 1	89.904 2	31.459 1	.03478 1	.26917 4	15.200922 2	.288E-3 2	6.642470	37	4	.64
38438.0	272.11 2	83.362 3	31.459 1	.03473 2	.47027 5	15.201555 2	.325E-3 3	6.642649	35	4	.83
38439.0	282.30 1	76.827 2	31.4598 7	.03472 1	.67210 3	15.202240 2	.378E-3 2	6.642519	34	4	.44
38440.0	292.61 1	70.288 2	31.4596 7	.03472 1	.87433 3	15.202988 2	.372E-3 2	6.642280	37	4	.45
38441.0	302.89 1	63.744 2	31.4593 7	.034733 8	.07734 3	15.203690 1	.347E-3 1	6.641988	42	4	.49
38442.0	313.055 9	57.200 2	31.4590 5	.034772 7	.28125 2	15.204350 1	.317E-3 1	6.641526	40	4	.37
38443.0	323.32 1	50.659 2	31.4585 7	.034843 6	.48567 3	15.204985 2	.315E-3 2	6.640854	37	4	.39
38444.0	333.55 1	44.113 2	31.4585 7	.034879 6	.69073 3	15.205704 1	.356E-3 1	6.640392	40	4	.42
38445.0	343.70 1	37.571 2	31.4583 8	.034926 6	.89670 4	15.206420 2	.360E-3 2	6.639860	40	4	.43
38446.0	353.86 2	31.025 3	31.4586 9	.034987 7	.10334 5	15.207168 2	.379E-3 2	6.639226	43	4	.52
38447.0	4.01 2	24.481 3	31.4581 8	.035083 7	.31076 4	15.207928 2	.387E-3 2	6.638344	52	4	.57
38448.0	14.06 1	17.935 2	31.4580 9	.035177 8	.51921 4	15.208710 2	.392E-3 1	6.637470	53	4	.61
38449.0	24.17 1	11.390 3	31.458 1	.03512 1	.72819 4	15.209528 2	.451E-3 2	6.637590	59	4	.83
38450.0	34.182 9	4.834 2	31.457 8	.035259 7	.93832 2	15.210449 1	.478E-3 1	6.636399	60	4	.56
38451.0	44.16 1	358.293 2	31.4573 9	.035335 9	.14949 3	15.211448 2	.508E-3 1	6.635586	67	4	.72
38452.0	54.13 1	351.746 3	31.457 1	.03538 1	.36166 3	15.212473 2	.519E-3 2	6.634965	73	4	.85
38453.0	64.18 1	345.197 3	31.459 1	.03534 1	.57464 4	15.213436 2	.462E-3 2	6.634985	63	4	1.03
38454.0	74.15 2	338.651 4	31.458 1	.03535 1	.78874 4	15.214332 2	.424E-3 2	6.634611	58	4	1.13

SAO mean elements -- Satellite 1964 5A

1-29 February 1964

Figure 5-1. Example of Orbit Elements from SAO Reports set of elements is based and accuracy of these observations is given in Columns 10, 11 and 12.

The first approach taken in presenting the comparison of computed orbit elements to SAO elements was to simply plot the histories of SAO elements with a solid line and the computed elements with points. In this manner the actual behavior of the element could be observed, as well as how closely the mathematical simulation duplicated it. However, the angular elements $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$ revolve through several hundred degrees (and in the case of $\underline{M}$, several thousand revolutions). Thus, a small deviation of the computed from SAO would be unnoticed, so a second method of presentation is used - a plot of the difference of computed minus SAO. These are much more revealing for the three angles $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$, and are the only ones presented for them.

Also, the computed results of the MSFC Orbit Lifetime Program (Reference 20) are shown in the comparisons. This program is indicative of the current state-of-the-art in long-term ephemeris prediction, and, as such, provides a standard basis for evaluating the GENPUR results.

As discussed in Appendix I, there are various ways of defining a mean element. The GENPUR definition is essentially osculating minus short-periodic, whereas the Orbit Lifetime Program and the SAO DOI program use Kozai's mean elements. The essential difference is that in defining mean $\underline{a}$, Kozai subtracts an additional term (see Appendix I). In the following comparisons, this term is added back to the Lifetime Program solution for $\underline{a}$ and to the SAO definition of $\underline{a}$ so that all are equivalent.

The GENPUR program is in a developmental state; consequently, it presently lacks, among other things, representation of second-order $\underline{J}_2$ and $\underline{J}_4$ effects. These effects are very important for the elements $\underline{\Omega}$ and $\underline{\omega}$, and have a slight effect on $\underline{M}$. To illustrate their importance, orbits of two satellites were simulated by GENPUR, with and without an approximate solution for these effects. The approximate solutions were obtained from Brouwer's theory (Reference 21) and are in the form of corrections added to the GENPUR mean elements at each time point. These corrections are $\left(\delta \equiv \sqrt{1 - e^{(0)2}}\right)$:

$$\Delta\Omega = n^{(0)}\bar{t}\left\{\frac{3}{32} J_2^{\,2}\left(\frac{r_e}{p}\right)^4 \left[(-5 + 12\delta + 9\delta^2)\cos i^{(0)} - (35 + 36\delta + 5\delta^2)\cos^3 i^{(0)}\right]\right.$$
$$\left. - \frac{15}{32} J_4\left(\frac{r_e}{p}\right)^4 (5 - 3\delta^2)\cos i^{(0)} \left(3 - 7\cos^2 i^{(0)}\right)\right\} \tag{5-1}$$

$$\Delta\omega = n^{(0)}\bar{t}\left\{\frac{3}{128} J_2^{\,2}\left(\frac{r_e}{p}\right)^4 \left[-35 + 24\delta + 25\delta^2 + (90 - 192\delta - 126\delta^2)\cos^2 i^{(0)}\right.\right.$$
$$\left. + (385 + 360\delta + 45\delta^2)\cos^4 i^{(0)}\right] \tag{5-2}$$
$$\left. - \frac{15}{128} J_4\left(\frac{r_e}{p}\right)^4 \left[21 - 9\delta^2 + (-270 + 126\delta^2)\cos^2 i^{(0)} + (385 - 189\delta^2)\cos^4 i^{(0)}\right]\right\}$$

$$\Delta M = n^{(0)}\bar{t}\left\{\frac{3}{128} J_2^{\,2}\left(\frac{r_e}{p}\right)^4 \left[-15 + 16\delta + 25\delta^2 + (30 - 96\delta - 90\delta^2)\cos^2 i^{(0)}\right.\right.$$
$$\left. + (105 + 144\delta + 25\delta^2)\cos^4 i^{(0)}\right] \tag{5-3}$$
$$\left. - \frac{45}{128} J_4\left(\frac{r_e}{p}\right)^4 e^{(0)2} \left[3 - 30\cos^2 i^{(0)} + 35\cos^4 i^{(0)}\right]\right\}$$

Figures 5-2 through 5-5 show the GENPUR errors in Ω and ω for the Explorer 7 and Explorer 1 satellites with and without these approximations. (There was little noticeable difference in M.) For Explorer 7, the approximations unfortunately increase the error in Ω from 0.3° to -0.7°. However, they decrease the error in ω from a secular 1.1° to a random $\pm$0.2°. For Explorer 1, the effects are much more drastic. The approximations reduce the error in Ω from 6.5° to 0.45°, and in ω from -8.3° to -0.2°. Because these second-order J_2 and J_4 effects are so important, the Brouwer approximations given above presently remain in the GENPUR program and will be included in all subsequent comparisons.

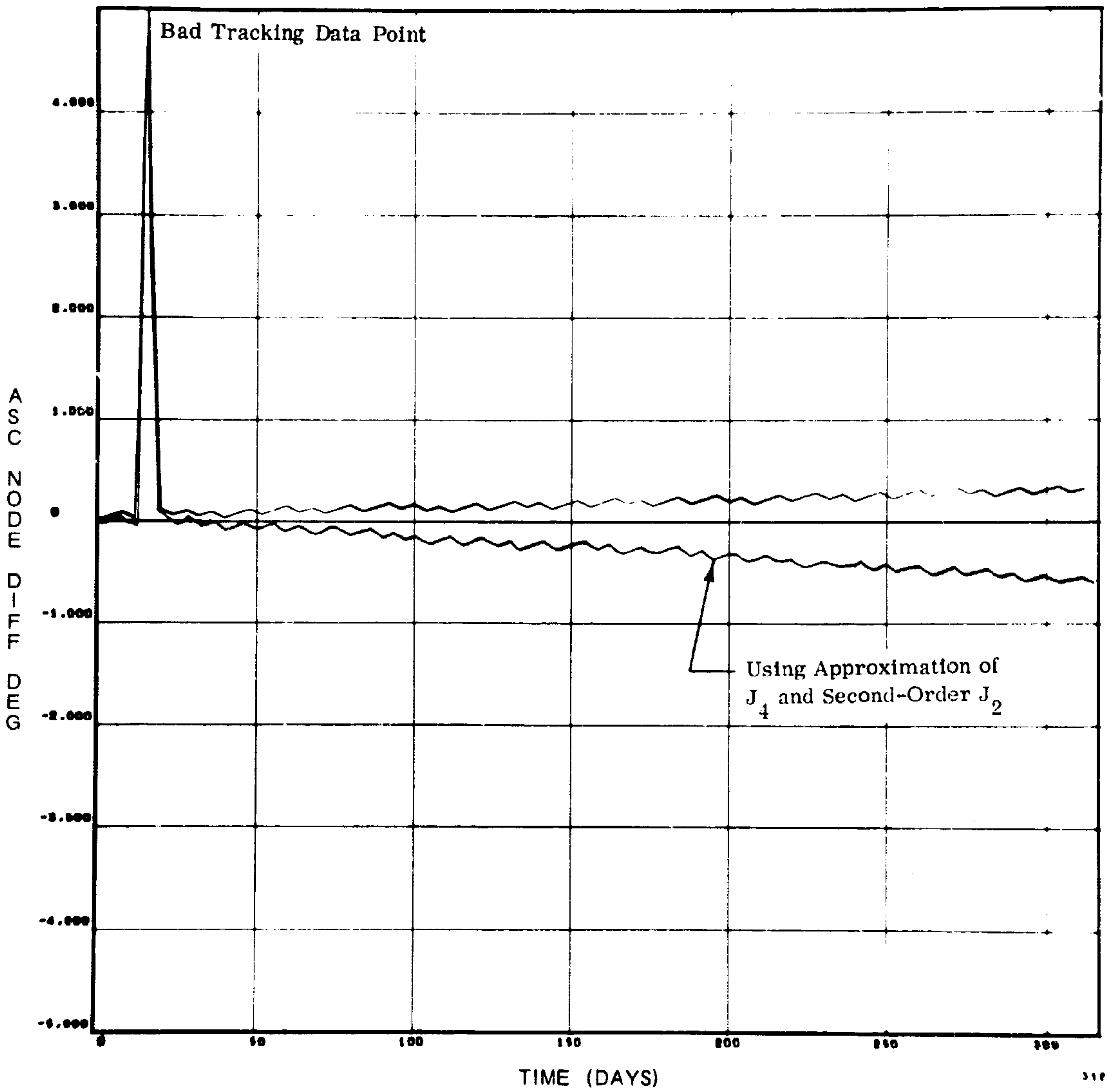


Figure 5-2. GENPUR Error in Ascending Node for Explorer 7

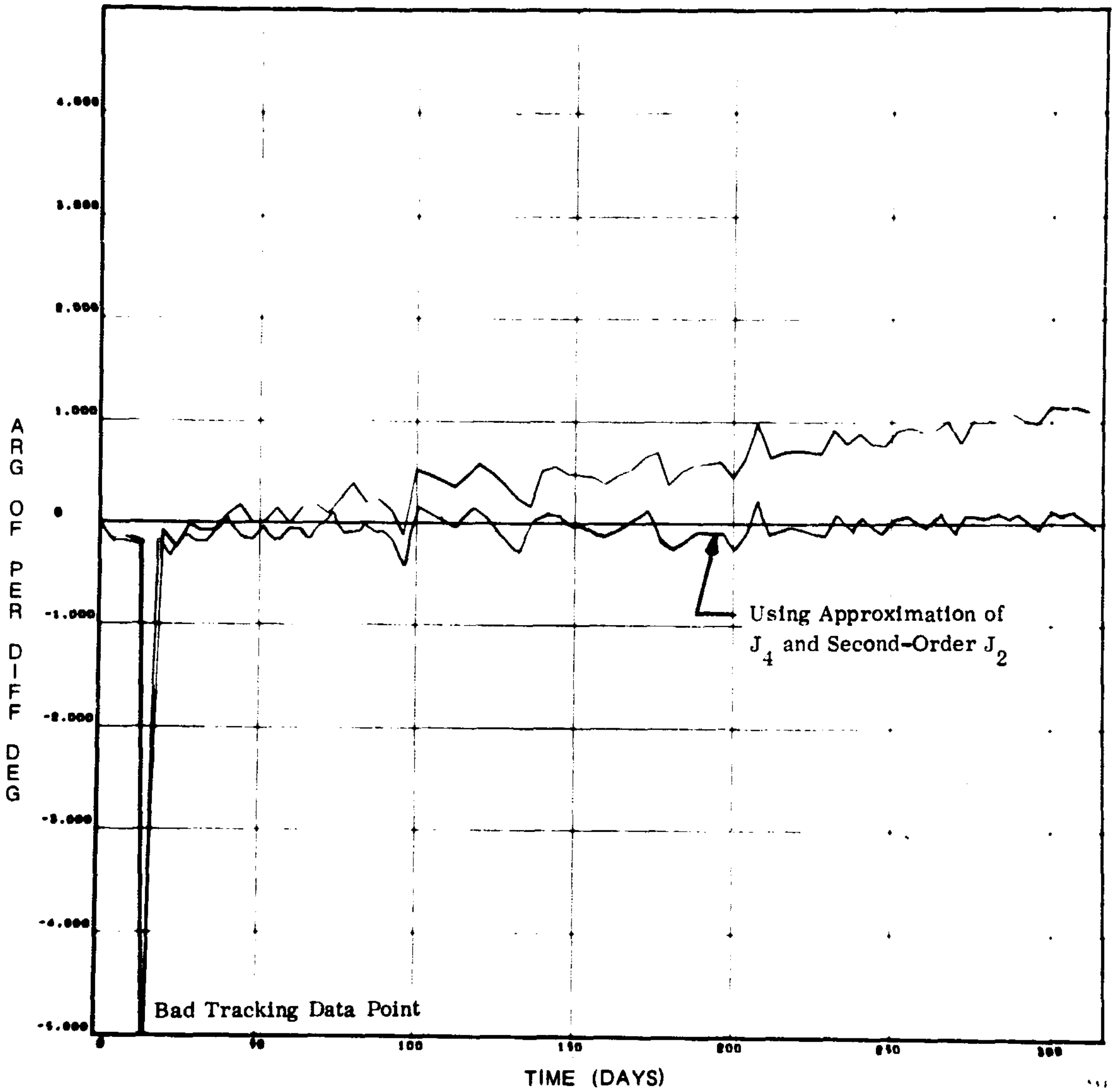


Figure 5-3. GENPUR Error in Argument of Perigee for Explorer 7

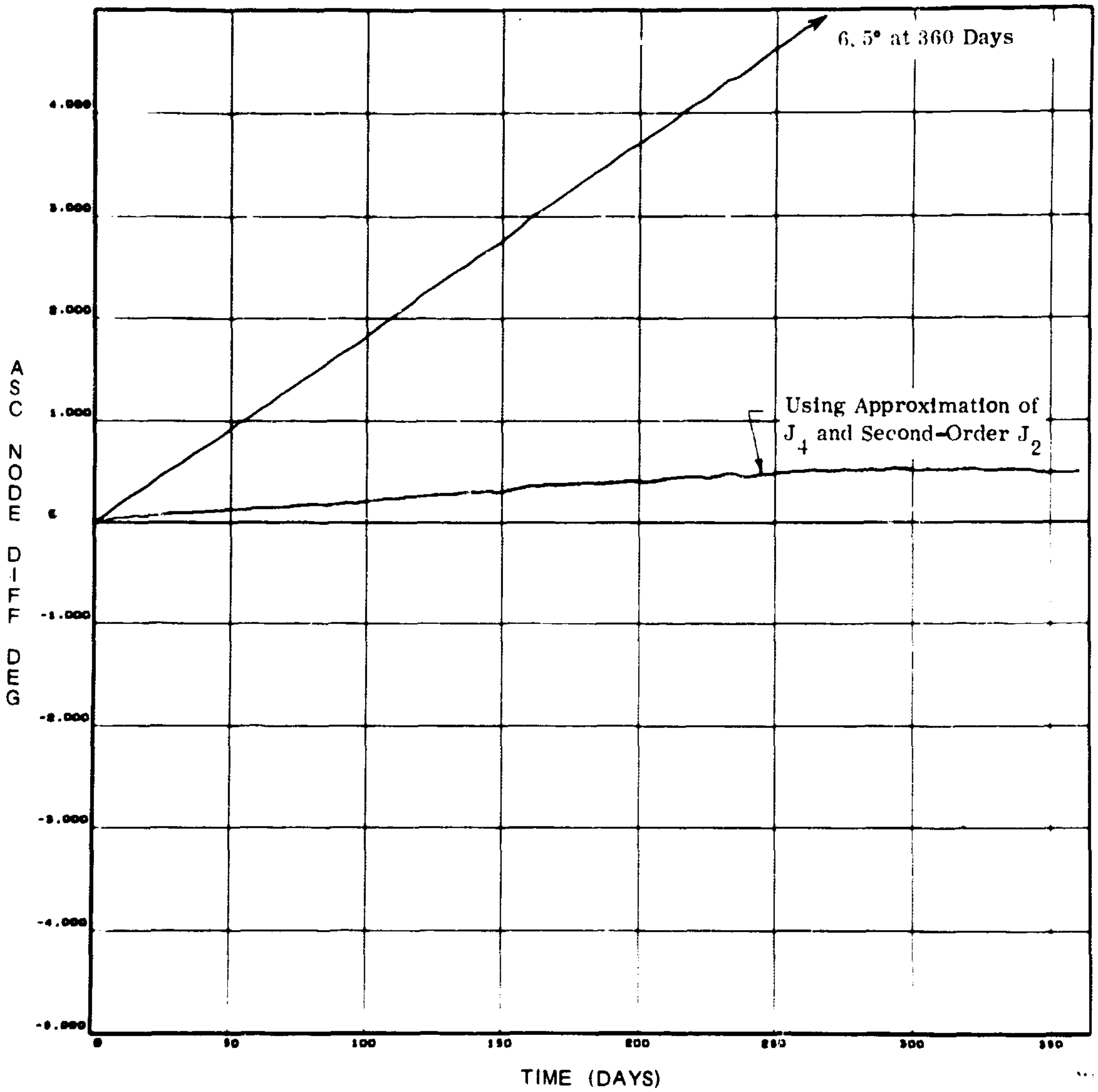


Figure 5-4. GENPUR Error in Ascending Node for Explorer 1

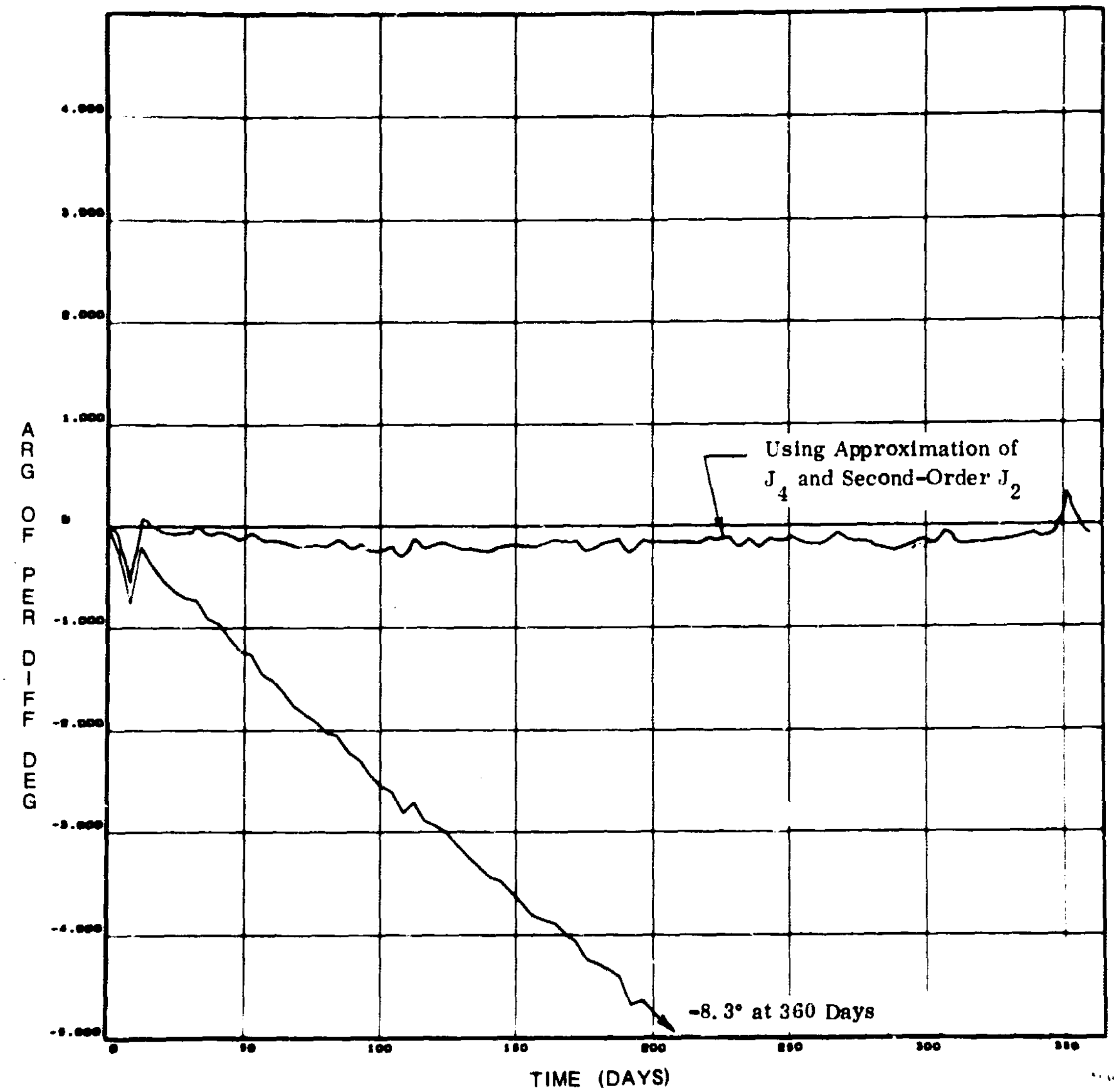


Figure 5-5. GENPUR Error in Argument of Perigee for Explorer 1

5.1.1 Explorer 7 Comparison

A 344-day history of the orbit elements for Explorer 7 beginning on 31 March 1962 was computed by the GENPUR and Orbit Lifetime Programs. Initial mean elements and ballistic coefficients for each program are given below in Table 5-1. The m/C_DA values were adjusted in each program to yield the best overall simulation of the decay; in this case, the resultant values were the same (40 kg/m^2). (Note the difference of 1.1 km in initial semimajor axis due to the definition of Kozai's mean elements used by the Lifetime Program.)

Table 5-1. Initial Conditions for Explorer 7

MEAN ELEMENTS	GENPUR	ORBIT LIFETIME
a (km)	7193.0	7191.9
e	0.03545	0.03545
i (degrees)	50.305	50.305
Ω (degrees)	344.40	344.40
ω (degrees)	232.44	232.44
M (degrees)	179.46	179.46
m/C_DA (kg/m^2)	40.0	40.0

The histories of semimajor axis and eccentricity are shown in Figure 5-6. The solid line is a connection of each SAO element point (given at 4-day intervals). The asterisks represent simulation results from the GENPUR program, and the circles are results from the MSFC Orbit Lifetime Program. Both simulations are nearly coincident for $\underline{a}$ and $\underline{e}$, and both show extremely good agreement with the SAO elements. (Recall that the output of the Lifetime Program and the SAO values of semimajor axis have been adjusted to remove the Kozai correction.) Semimajor axis decays only slightly (0.5 km) during this interval, so that the orbit is essentially free of significant drag effects. The long-period variation in eccentricity due to $\underline{J_3}$ is very evident, having a period of approximately 110 days and an amplitude of 0.0008. Note also the relatively rough nature of the tracking data, especially for $\underline{a}$. There seems to be bad tracking

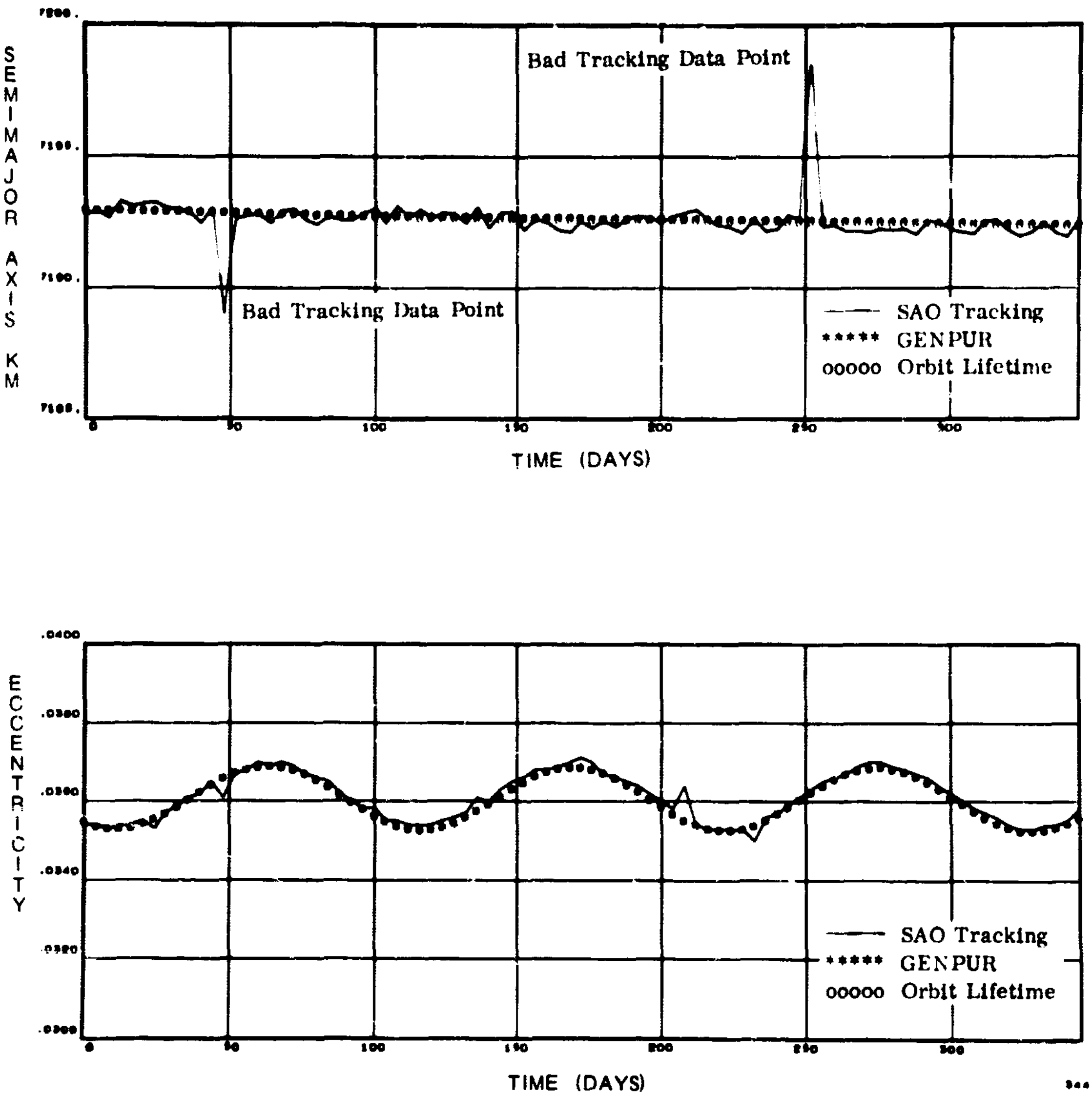
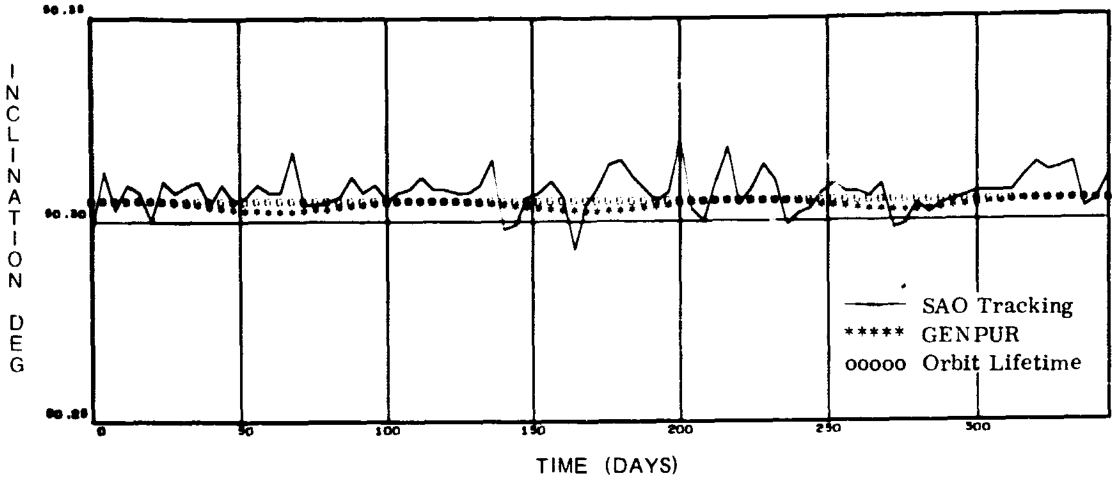


Figure 5-6. Semimajor Axis and Eccentricity for Explorer 7



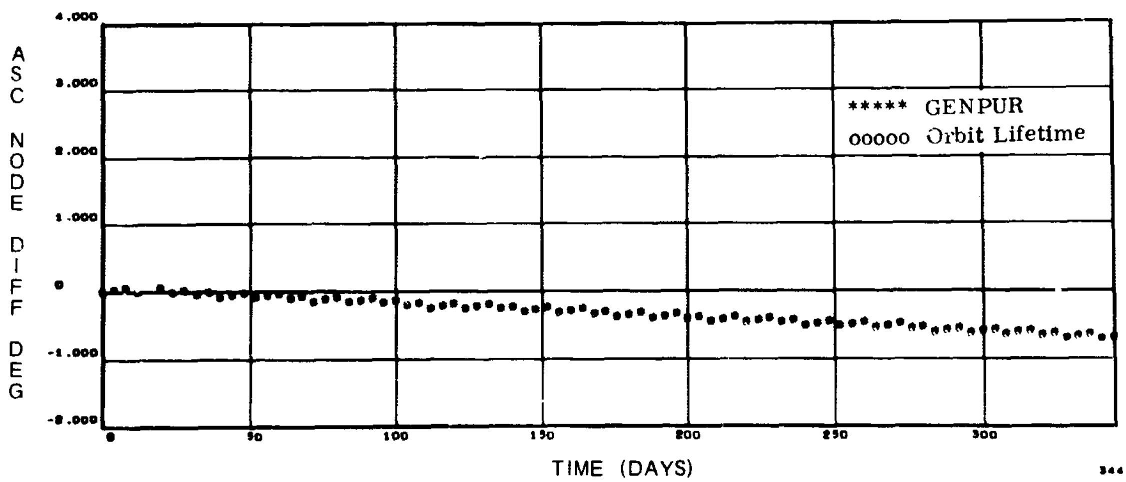
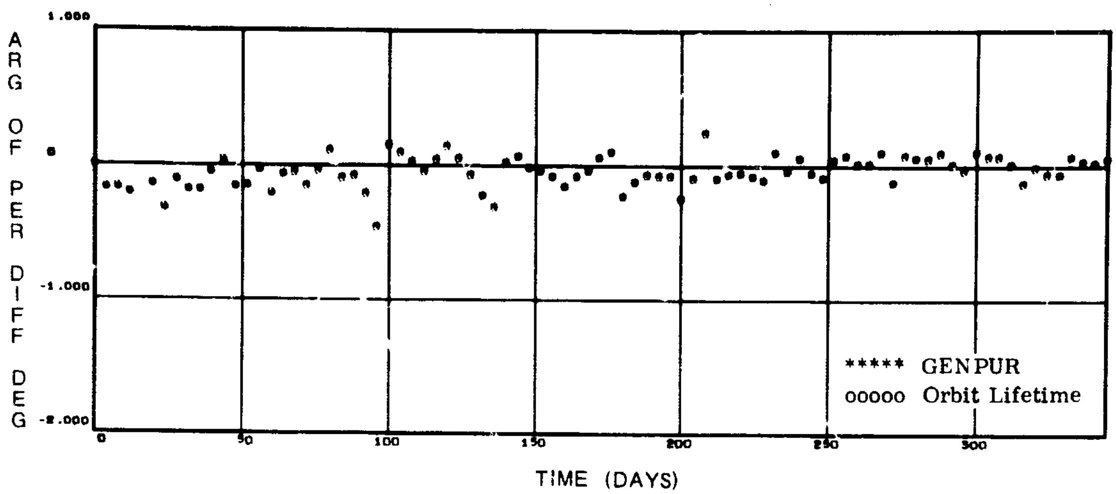


Figure 5-7. Inclination and Errors in Ascending Node for Explorer 7



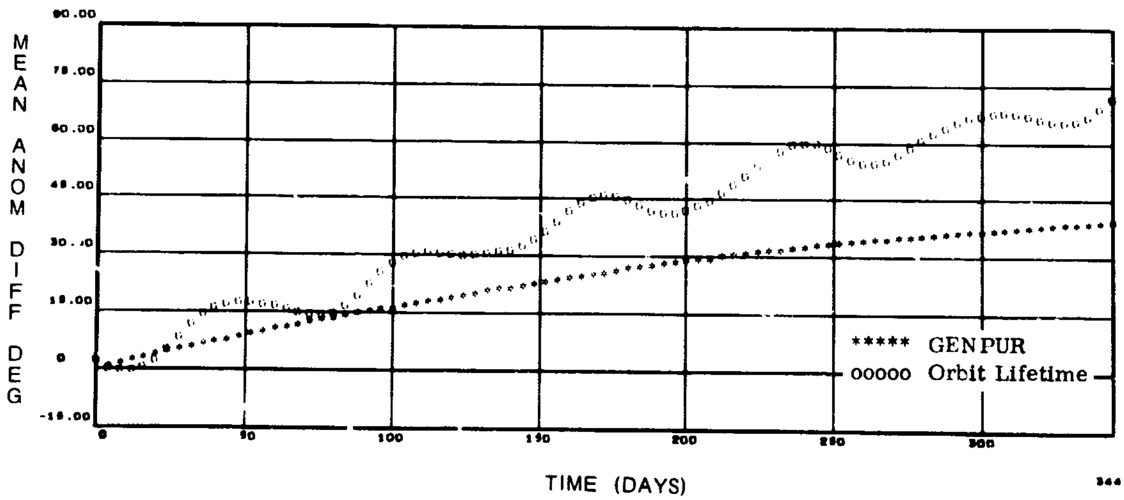


Figure 5-8. Errors in Argument of Perigee and Mean Anomaly for Explorer 7

data points in the values of $\underline{a}$ at 48 and 252 days. (These characteristics are evident for each element and each satellite.)

Figure 5-7 shows two different types of plots. The top figure presents SAO tracking values of inclination, along with individual values of inclination from the GENPUR and Orbit Lifetime Programs. Tracking values of inclination are fairly rough, at least on the scale being used. The Lifetime Program holds inclination constant at the initial value (50.305°). The GENPUR program simulates the secular change in inclination due to drag, not evident in this figure, and the periodic change due to $\underline{J}_3$, which can be seen. (The advantage of having inclination vary is not apparent for Explorer 7, but will be for Explorer 1.)

The bottom half of Figure 5-7 shows the error in ascending node produced by each program, i.e., computed value minus SAO value. Again the results of each program are nearly identical and both show fair agreement with the SAO elements. There is a secular buildup of error in ascending node to -0.7° for each program. (Recall that the GENPUR program uses Brouwer's equations to approximate the $\underline{J}_4$ and second-order $\underline{J}_2$ effects in $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$.)

In-track position of a satellite is primarily a function of argument of perigee (ω) and mean anomaly (M). The mean anomaly typically undergoes 5000 revolutions in 340 days. It is extremely sensitive to small changes in semimajor axis. For example, an error of only 0.4 km in semimajor axis can result in an error of 170° in mean anomaly after 340 days. Mean anomaly is very sensitive to gravity and drag perturbations; thus, it provides a significant measurement of the accuracy of a simulation. Figure 5-8 shows the errors of both programs in $\underline{\omega}$ and $\underline{M}$. Both have almost identical simulations of $\underline{\omega}$ with no apparent secular error, but only a random error of $\pm 0.2°$. These differences may, in fact, be due to limitations on the accuracy of the tracking data, rather than inaccuracy in the simulations. The simulations of mean anomaly are somewhat different. The Orbit Lifetime Program shows a periodic and secular error buildup of nearly 75°. The GENPUR program, on the other hand, exhibits only a secular error buildup of 40°.

5.1.2 Explorer 1 Comparison

A 356-day history of the orbit elements for Explorer 1 beginning on 2 January 1964 is shown in Figures 5-9 through 5-11. Elements derived from SAO tracking data are shown along with computed solutions from the GENPUR and Orbit Lifetime programs. Initial mean elements and ballistic coefficients for each program are given below in Table 5-2. Again, the $m/C_D A$ values were adjusted in each program to yield the best overall simulation of the decay; in this case, the resultant values were different. It is thought, that the reason is due to the fact that short-periodic perturbations in altitude were not considered when determining the Fourier coefficients of the GENPUR program. (The difference between the two definitions of $\underline{a}$ amounts to 5.0 km for this orbit.)

Table 5-2. Initial Conditions for Explorer 1

MEAN ELEMENTS	GENPUR	ORBIT LIFETIME
a (km)	7368.14	7363.14
e	0.08747	0.08747
i (degrees)	33.198	33.198
Ω (degrees)	34.01	34.01
ω (degrees)	151.27	151.27
M (degrees)	50.112	50.112
$m/C_D A (kg/m^2)$	22.28	25.0

The histories of semimajor axis and eccentricity are shown in Figure 5-9. The same plotting symbols as before are used, so that the straight line is a connection of SAO elements, asterisks represent GENPUR results and circles are Orbit Lifetime results. Both simulations are nearly coincident for $\underline{a}$ and show reasonably good agreement with the SAO elements. The simulations are initially about 0.8 km higher than actual and then fall about 0.6 km below actual after 350 days. The reason for this behavior is due to omission of daily values of solar flux $F_{10.7}$ and geomagnetic index A_p in the density model. (The Lifetime Program, when using the daily values, showed nearly

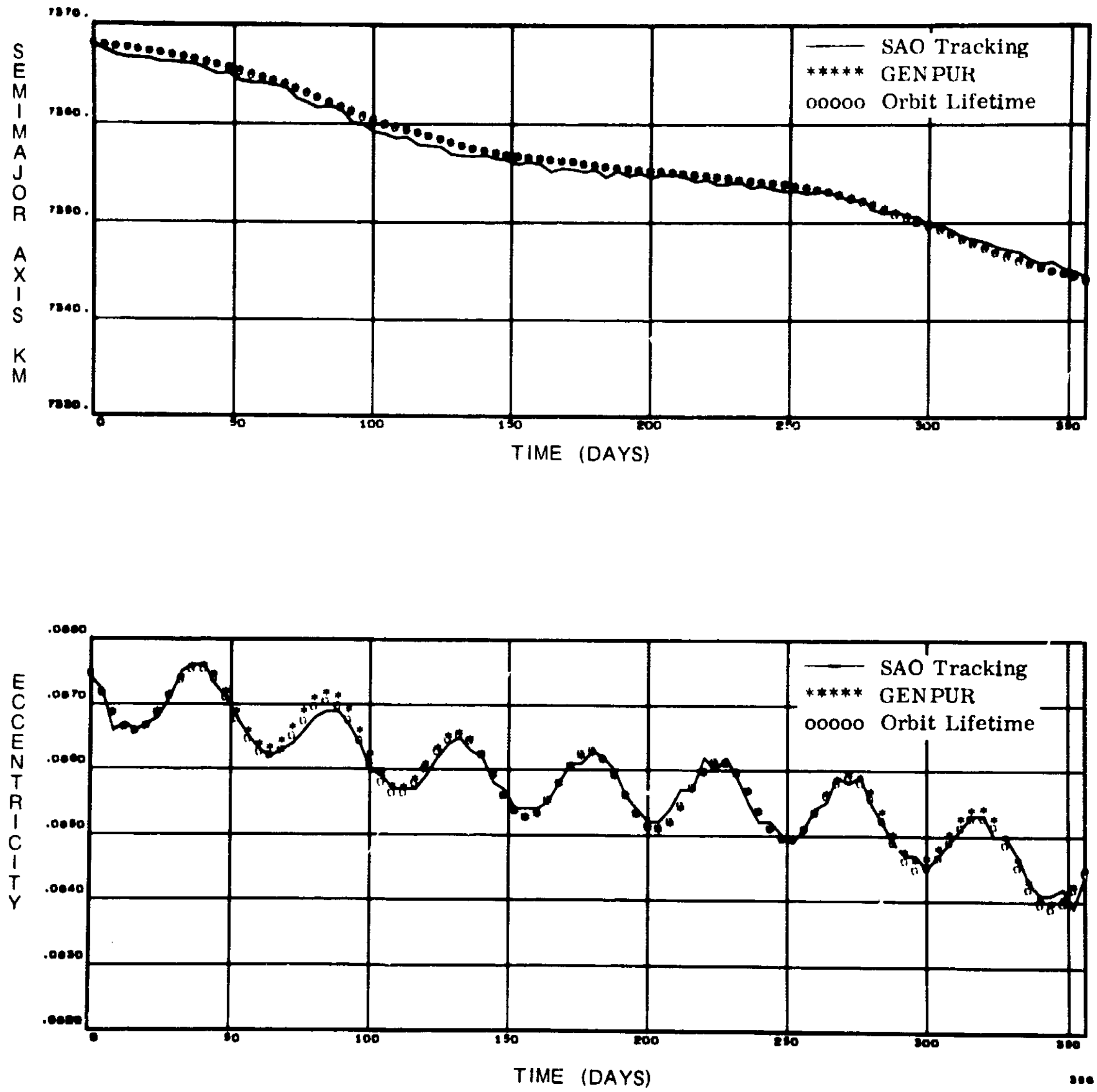


Figure 5-9. Semimajor Axis and Eccentricity for Explorer 1

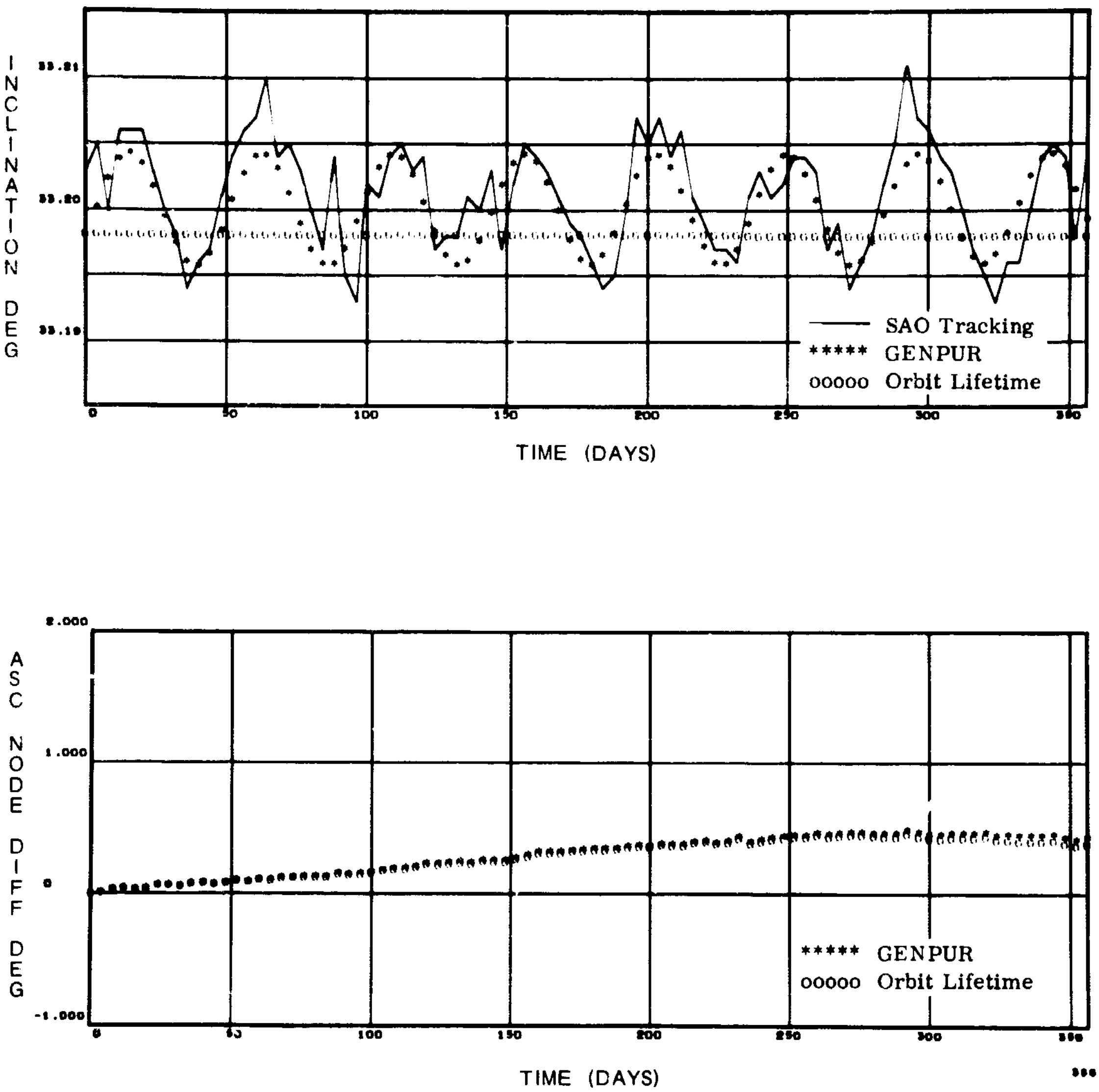
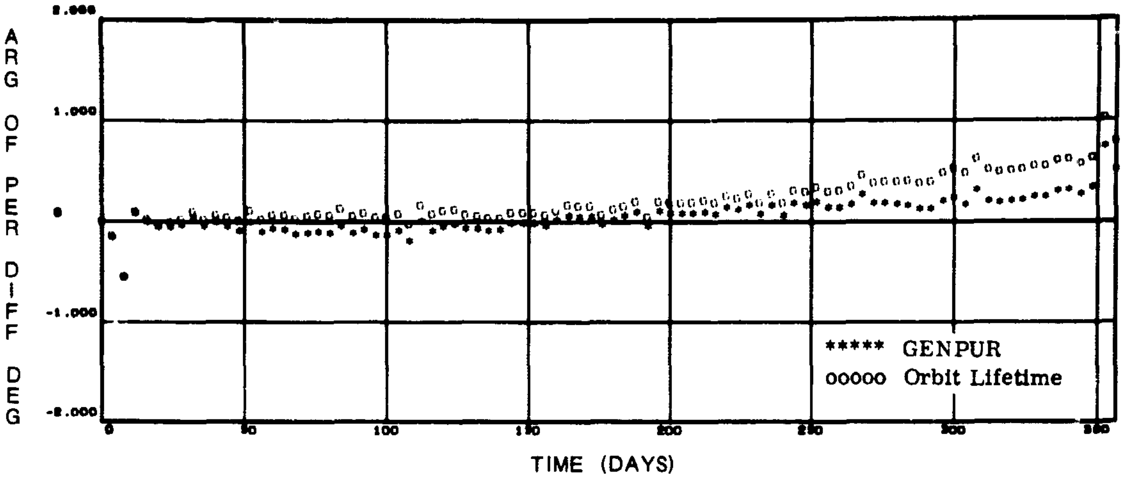


Figure 5-10. Inclination and Errors in Ascending Node for Explorer 1



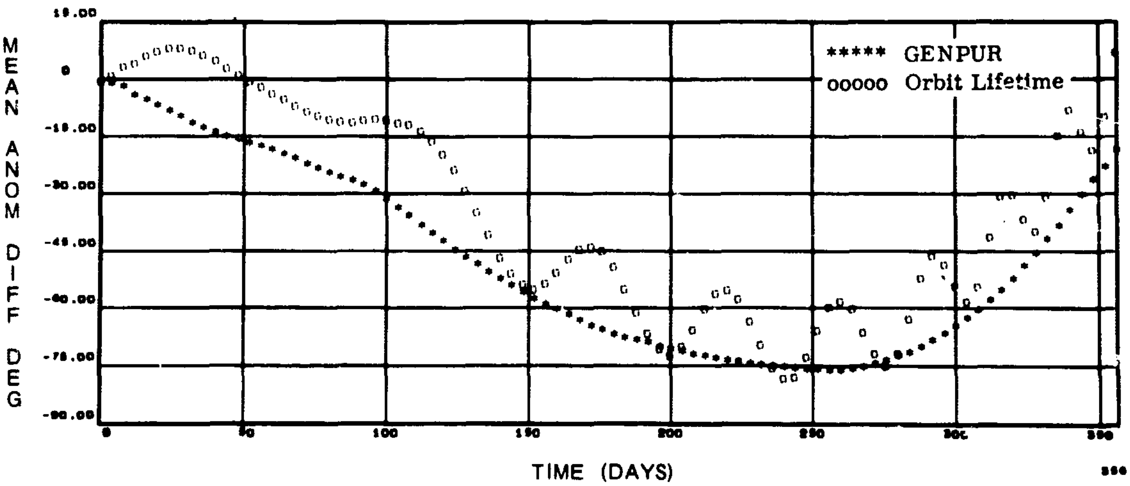


Figure 5-11. Errors in Argument of Perigee and Mean Anomaly for Explorer 1

perfect agreement.) As yet, input of daily $F_{10.7}$ and A_p values is not available for GENPUR; hence, both programs were run in a simulated preflight condition using only mean values of $F_{10.7}$ and regression values of A_p. Note that this orbit is affected considerably more by drag due to the lower perigee than was the orbit of Explorer 7. Semimajor axis decayed 24 km, rather than the 0.5 km for Explorer 7.

The lower half of Figure 5-9 depicts eccentricity. Both simulations agree well with SAO elements, but they are not coincident. Long-period effects of $\underline{J}_3$ are again clearly evident with a period of 48 days and an amplitude of 0.006. A secular decrease in the magnitude of $\underline{e}$ due to drag is also noticeable.

The upper half of Figure 5-10 shows the computed simulations and SAO values of inclination. The long-period variation due to $\underline{J}_3$ is clear, and is reasonably well simulated by GENPUR. In the Lifetime Program, however, inclination is held constant. Therefore, the GENPUR program shows a significant advantage over the Lifetime Program in simulating inclination.

The lower half of Figure 5-10 shows the errors of GENPUR and Orbit Lifetime in simulating ascending node. Both programs show very similar results, having maximum errors of 0.45°.

Errors in the critical in-track angles $\underline{\omega}$ and $\underline{M}$ are shown in Figure 5-11. GENPUR results are better than the Lifetime Program for $\underline{\omega}$. GENPUR errors grow to a maximum of only 0.3° whereas Orbit Lifetime errors in $\underline{\omega}$ grow to 0.6°. Errors in mean anomaly for GENPUR are smoothly varying with a maximum of -75°. Maximum error in the Lifetime Program is also -75°, but note the peculiar periodic nature that it exhibits (which was also evident in Explorer 7). The error in mean anomaly from the GENPUR program is easily explained in terms of the error in semimajor axis. Simulations of mean anomaly are very dependent upon an accurate value of $\underline{a}$. (Recall that there were small errors in the simulations of $\underline{a}$ for Explorer 1, Figure 5-9.) Initially, computed $\underline{a}$ was too large, which means theoretically that the orbital mean motion (equal to $(\mu/a^3)^{1/2}$) would be too slow and mean anomaly would not change as rapidly as it should. Figure 5-11 shows that this is, in fact, what actually happened. The GENPUR value of mean anomaly initially falls below actual. Then, as the computed a becomes close to the actual a at 260 days and falls below actual at 275 days, the error in mean anomaly levels off at the maximum -75° and returns to only -18°. Therefore, had the GENPUR simulation of a been better, the error in mean anomaly would have been much less.

5.1.3 SA-5 Comparison

A 334-day history of the orbit elements for the SA-5 satellite beginning on 1 February 1964 is shown in Figures 5-12 through 5-14. Elements derived from SAO tracking data, at 2-day intervals rather than the 4-day intervals of the previous satellites, are shown along with computed solutions from the GENPUR and Orbit Lifetime Programs. Initial mean elements and ballistic coefficients for each program are given in Table 5-3. (The difference between the two definitions of a amounts to 5.64 km for this orbit.)

Table 5-3. Initial Conditions for SA-5

MEAN ELEMENTS	GENPUR	ORBIT LIFETIME
a (km)	6889.68	6884.04
e	0.0358	0.0358
i (degrees)	31.4561	31.4561
Ω (degrees)	161.797	161.797
ω (degrees)	150.01	150.01
M (degrees)	34.56	34.56
m/C_DA (kg/m^2)	88.22	106.0

The histories of semimajor axis and eccentricity are shown in Figure 5-12. Plotting symbols and notation are the same as before. Both simulations are nearly coincident for the element a, but neither agrees very well with the SAO values. The simulations agree reasonably well for the first 60 days, but rise above actual by 2 km at 100 days and then fall below actual by -3 km at 334 days. This error was encountered

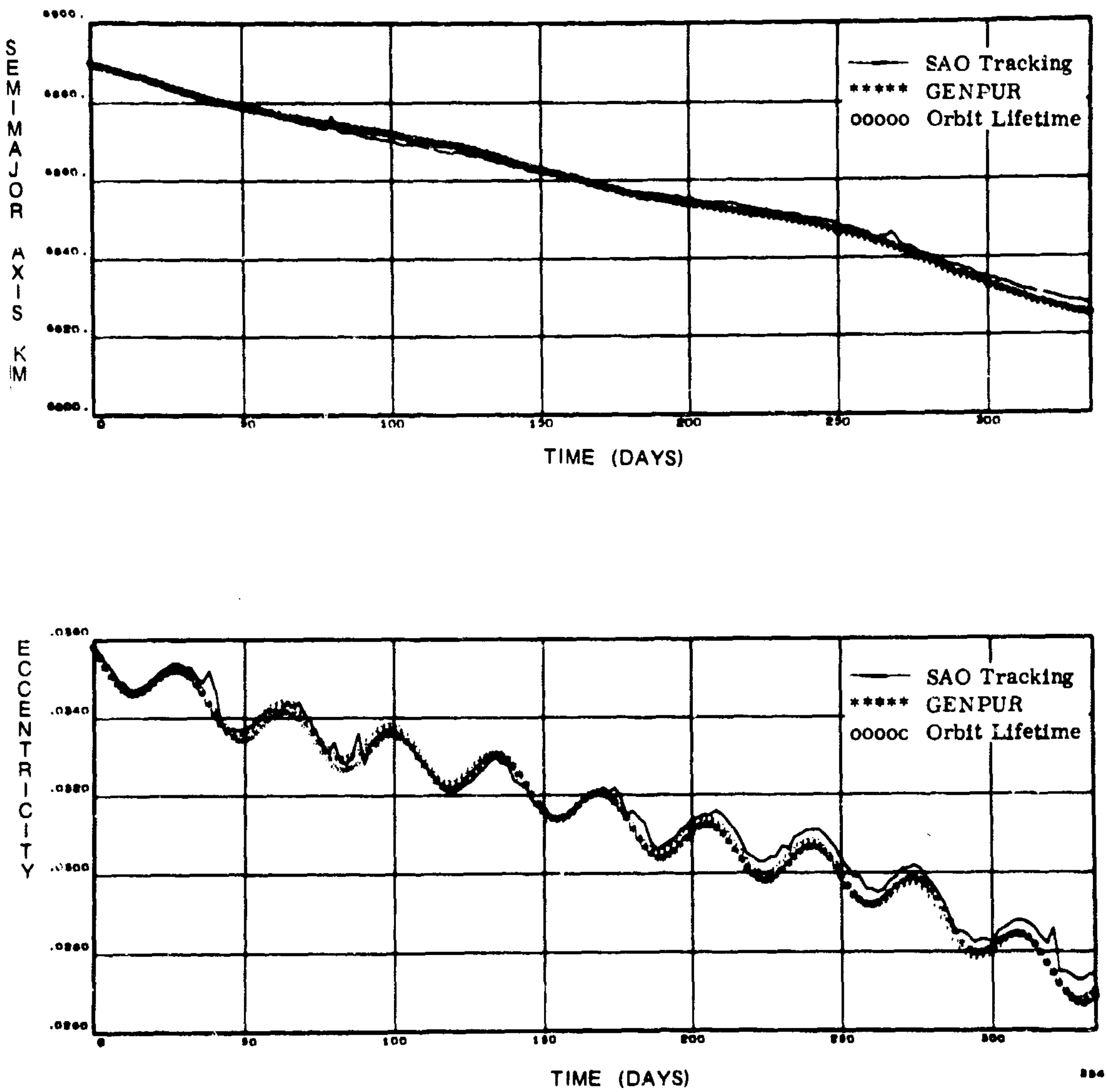


Figure 5-12. Semimajor Axis and Eccentricity for SA-5

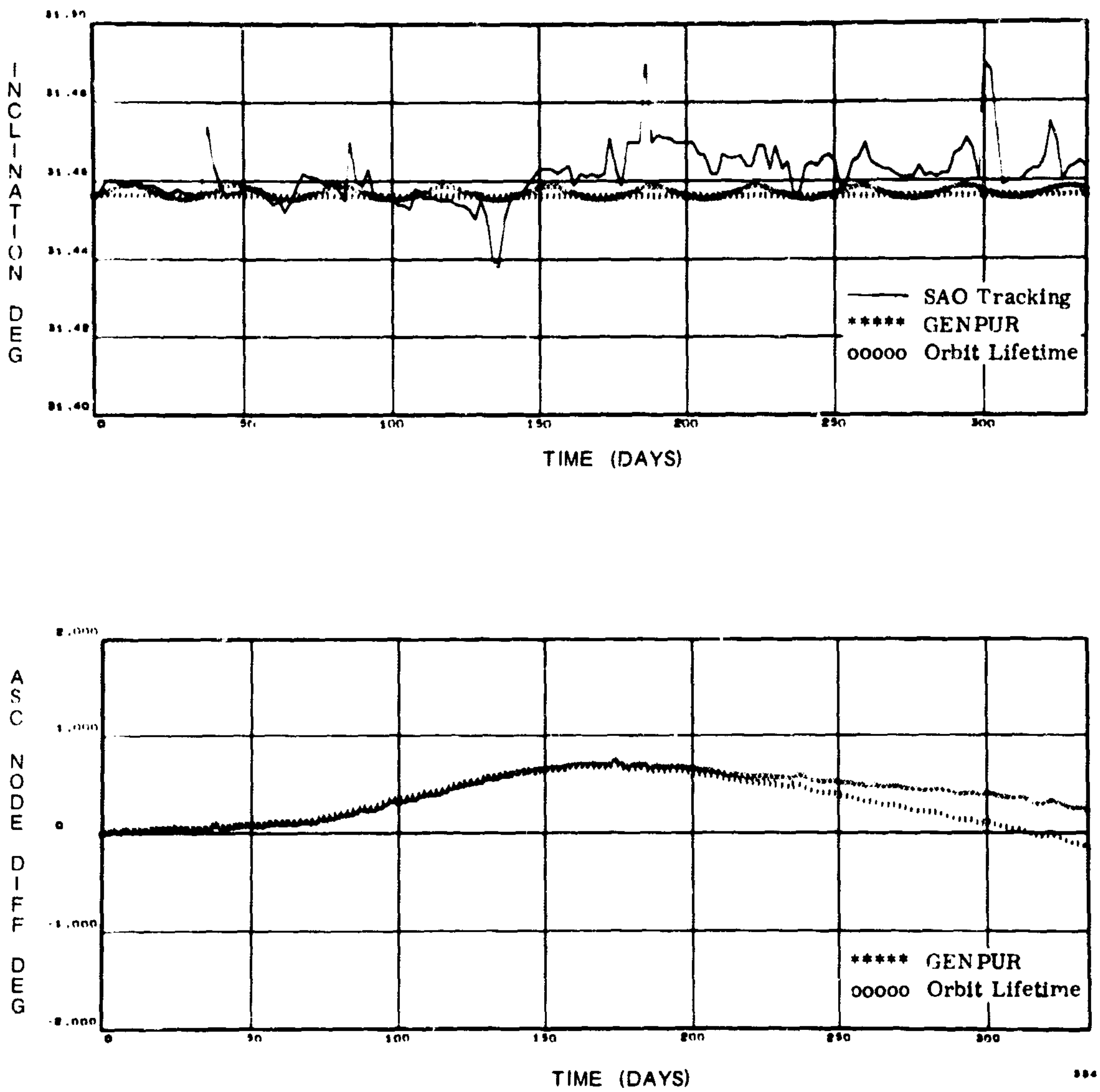


Figure 5-13. Inclination and Errors in Ascending Node for SA-5

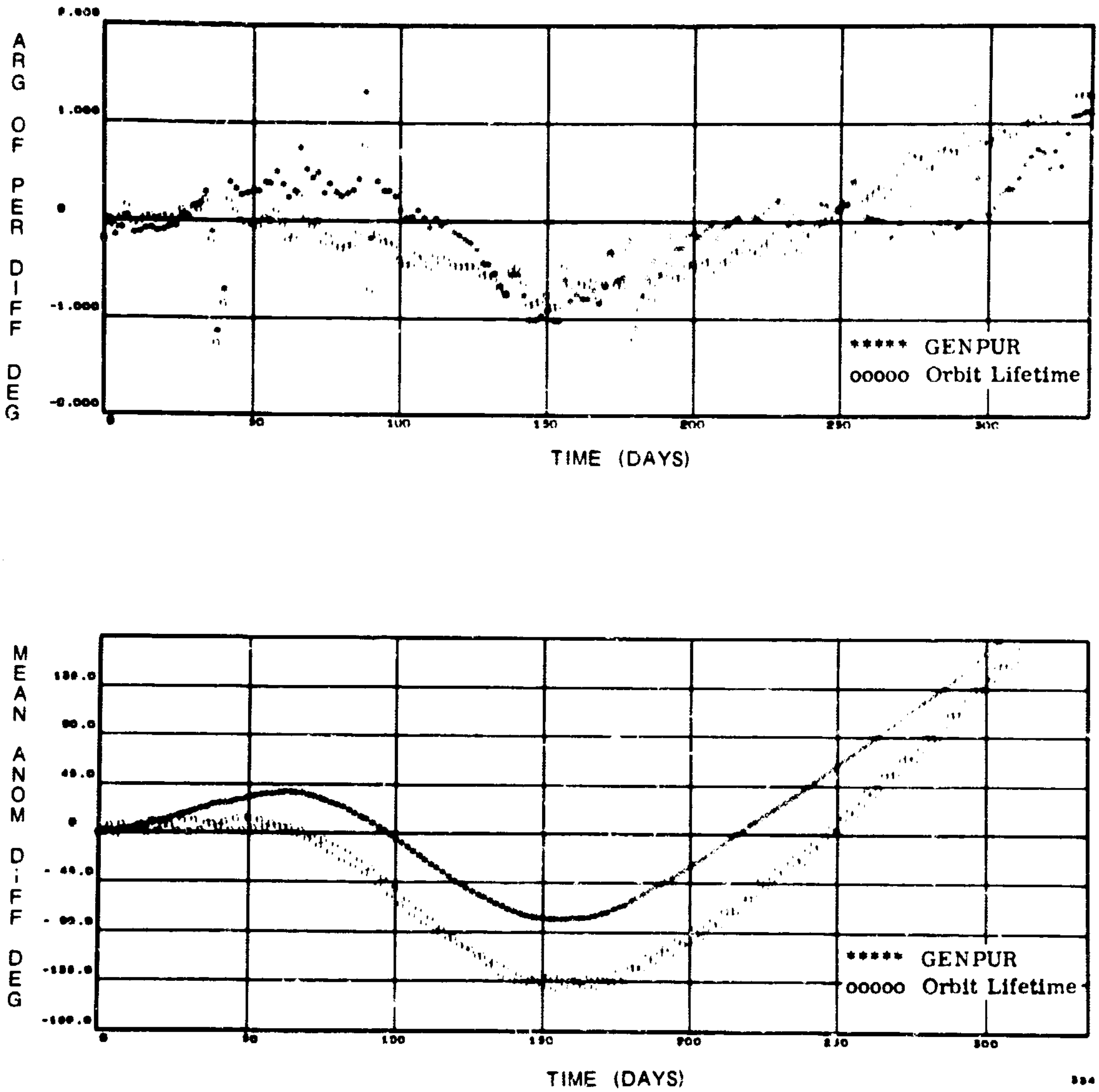


Figure 5-14. Errors in Argument of Perigee and Mean Anomaly for SA-5

in a previous study and is believed to be due to an inaccuracy of the 1970 Jacchia density model at lower altitudes. Since the semimajor axis is not well simulated, it is anticipated at this point that there will be relatively large errors in the simulations of $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$. Of the three satellites investigated, drag had the most significant effect on SA-5. The initial perigee altitude was only 270 km (versus 340 km for Explorer 1 and 560 km for Explorer 7) so that the atmospheric density at perigee was significantly greater than for the other satellites. In 334 days, the semimajor axis of SA-5 decayed by 64 km versus 24 km for Explorer 1 and 0.5 km for Explorer 7.

The lower half of Figure 5-12 depicts eccentricity. The two simulations are nearly coincident, but again do not agree well with SAO values. The reason is the same as for the discrepancy in $\underline{a}$. Long-period effects in $\underline{e}$ are clearly evident with a period of 36 days and an amplitude of 0.0008. A secular decrease in the magnitude of $\underline{e}$ due to drag is also noticeable.

The upper half of Figure 5-13 shows computed simulations and SAO values of inclination. Long-period variations are not evident in the SAO values. In fact, the random fluctuations in the SAO data are larger than the amplitude of the long-periodicity, implying that the resolution of the SAO data was not accurate enough to show the long-periodicity. The SAO elements also show a very interesting phenomena at 150 days, where the average value of inclination seems to change from 31.456° to 31.465°. It is hard to imagine what physical force could cause this change other than a powered plane-change maneuver; however, no such maneuver was performed by SA-5. It can be concluded that the GENPUR simulation of inclination for SA-5 is as accurate as the SAO elements.

The lower half of Figure 5-13 shows the errors of GENPUR and Orbit Lifetime Programs in simulating $\underline{\Omega}$. The two programs agree with each other for the first 210 days, but then the errors diverge. The error of the Lifetime Program decreases more rapidly than does that of the GENPUR program. The reason is that, at this point, the Lifetime Program simulation of $\underline{a}$ falls slightly below that of GENPUR. Neither program shows particularly good agreement with SAO values, both having a maximum error of nearly 0.7°. This was as expected, since $\underline{a}$ was not well simulated.

Errors in the critical in-track angles $\underline{\omega}$ and $\underline{M}$ are shown in Figure 5-14. The two programs are not coincident in simulating $\underline{\omega}$, but it would be hard to say which is better. Both show errors ranging from -1.0° to +1.2°. Again the trouble is due to a poor simulation of $\underline{a}$.

Errors in mean anomaly are very dependent upon the simulation of $\underline{a}$. Thus, the poor simulations of $\underline{a}$ by both programs are very evident in their large errors in $\underline{M}$. The GENPUR error in $\underline{M}$ ranges from -80° to more than +180°. The Lifetime Program error ranges from -135° to more than +180°. Simulations of the elements of SA-5 clearly demonstrate the importance in orbit ephemeris prediction of having a good simulation of semimajor axis (which depends upon the use of an accurate density model).

5.1.4 Summary of Tracking Data Comparisons

More than 300 orbit days for Explorer 7, Explorer 1 and SA-5 have been simulated by the GENPUR and Orbit Lifetime Programs. A summary of the errors of the simulations for each orbit element and each satellite is shown in Table 5-4. (Recall that the GENPUR program does not yet contain asymptotic expansion solutions for second-order secular effects of $\underline{J}_2$ and $\underline{J}_4$, but uses Brouwer's equations.) A $\pm$ sign indicates that the error was more or less random, and is the type desirable for all the errors. A single number means that the error steadily increased to the value given, whereas two numbers indicate that the error grew to the first number and then reversed direction and attained the level of the second number. For example, the error by GENPUR in $\underline{a}$ for Explorer 1 first grew to 0.8 km and then reversed direction to -0.6 km.

In general, the errors of both programs are nearly equal with one or two exceptions. Having the $\underline{J}_3$ effects on inclination included in GENPUR results in only a -0.003° error rather than the -0.008° to 0.004° error of the Lifetime Program (for Explorer 1). The argument of perigee for Explorer 1 was better simulated by the GENPUR program, as was the mean anomaly for Explorer 7.

Maximum Error During the Full Simulation in:		EXPLORER 7		EXPLORER 1		SA-5	
		GENPUR	Lifetime	GENPUR	Lifetime	GENPUR	Lifetime
a	(km)	±0.3	±0.3	0.8, -0.6	0.8, -0.6	2.0, -3.0	2.0, -3.0
e		-0.0002	-0.0002	±0.0003	±0.0002	-0.0006	-0.0006
i	(deg)	±0.01	±0.01	-0.003	-0.003, 0.004	-0.01	-0.01
Ω	(deg)	-0.7	-0.7	0.45	0.45	0.7	0.7
ω	(deg)	±0.2	±0.2	0.3	0.6	-1.0, 1.2	-1.0, 1.2
M	(deg)	40	70	-75	-75	-80, 180	-135, 180

Table 5-4. Summary of Errors by GENPUR and Lifetime Programs in Simulating Orbits of Three Satellites

Note: The GENPUR simulations contained Brouwer's approximations of second-order J_2 and J_4 effects.

5.2 COMPARISON WITH NUMERICAL INTEGRATION PROGRAMS

Once an accurate history of mean elements is available, the osculating elements can be obtained by adding short-periodic terms. These short-periodic terms are primarily functions of the mean elements. It is not necessary to verify osculating elements for long time intervals, providing the mean elements are good. (If the osculating elements are good for short periods of time, they will be good throughout any given interval providing the mean elements remain satisfactory.)

SAO tracking data do not contain osculating elements. Therefore, a different method of comparison was necessary to verify osculating element solutions. The MSFC COWELL and SPERTB numerical integration programs use osculating elements exclusively; therefore, Table 5-5 shows a comparison of GENPUR osculating elements to those from the COWELL and SPERTB programs. Initial conditions are the initial SAO elements for Explorer 7. A period of 8 days was simulated, so that the GENPUR mean elements experienced little error.

In Table 5-5, the osculating element solutions are shown at the end of 1 day and 8 days. Four simulations were run, namely: the COWELL program, the SPERTB program with and without $\underline{J}_4$ effects, and the GENPUR program without $\underline{J}_4$ (and second-order $\underline{J}_2$) effects. The GENPUR results are within the differences between the COWELL and SPERTB programs; thus, the GENPUR osculating element solutions are excellent.

Osculating Elements	AFTER 1 DAY				AFTER 8 DAYS			
	COWELL	SPERTB	SPERTB (w/o J_4)	GENPUR (w/o J_4 & J_2^2)	COWELL	SPERTB	SPERTB (w/o J_4)	GENPUR (w/o J_4 & J_2^2)
e	0.035699	0.035698	0.035698	0.035679	0.035456	0.035460	0.035453	0.035432
ω (deg)	235.734	235.733	235.733	235.713	260.436	260.352	260.428	260.353
i (deg)	50.299	50.299	50.299	50.299	50.304	50.304	50.304	50.305
Ω (deg)	340.188	340.188	340.188	340.192	310.865	310.869	310.865	310.900
a (km)	7192.741	7192.735	7192.737	7192.812	7194.044	7193.997	7193.980	7194.032
M (deg)	263.862	263.861	263.866	263.827	133.290	133.450	133.459	133.377

Table 5-5. Short Term Comparisons of Osculating Elements

SECTION 6 - CONCLUSIONS AND RECOMMENDATIONS

The basic objective of this research project has been to develop, through applied research in general perturbation theory, perturbation techniques that provide an accurate and rapid long-term ephemeris prediction capability for satellites in earth orbit. The approach taken was to use two-variable asymptotic series in obtaining approximate solutions to the Lagrange planetary equations of orbit motion. This technique constitutes a relatively new approach to the ephemeris prediction problem and, while it is not yet on a rigorous mathematical basis, offers several potential advantages (as discussed in Paragraph 4.2). In this study, it was found that two-variable asymptotic series can be successfully applied to the problem of artificial satellite motion under the combined influence of gravity and drag. The first and second approximations of element solutions derived by asymptotic series agree in form to those derived in other established theories.

Of the potential advantages which the asymptotic series method offers, two were found to be of significant aid thus far. Since the method employs two time scales, the solutions obtained tend to group naturally by physical effects, i.e., they group into secular, long-periodic and short-periodic components. Therefore, it is not necessary to use a procedure such as Kozai's in which the disturbing function is resolved into secular, long-periodic, and short-periodic parts. Second, the error involved in a given series approximation is of the order of the first neglected term. Consequently, the asymptotic solutions are naturally structured to include the dominating effects of each perturbation in the initial approximation. Furthermore, a control of the expected error is provided by selection of the expansion parameter $\underline{\epsilon}$.

Currently, the asymptotic solutions have been obtained through the second approximations when considering earth oblateness and tangential atmospheric drag. In these approximations, it was found that the $E^{(0)}$ solutions contain the very important secular and long-periodic effects, while the $E^{(1)}$ solutions contain the short-periodic effects. The $E^{(0)}$ solutions were derived by first obtaining simultaneous solutions to the differential equations for the elements $e^{(0)}$ and $\omega^{(0)}$; these solutions were then used to obtain solutions for the remaining elements. These solutions were carried only through the first power of eccentricity. Because of the importance of the $E^{(0)}$ components in the total solution, it is recommended that they be investigated further. Specifically, extension of the simultaneous solutions to include more elements and retention of higher orders of eccentricity are recommended.

As indicated in Paragraph 4.3.2, functional forms of the integration "constants" $C(\tilde{t})$ have not been analytically determined, and are currently evaluated by use of an updating procedure. If the functional form of each $C(\tilde{t})$ was available, one requirement for using the update procedure would be eliminated; furthermore, the second-order secular effects of oblateness perturbations would be contained in these functions (see Paragraph 4.3.3). Analytic determination of these "constants" requires partial development of the $E^{(2)}$ solutions. It is anticipated that the $E^{(2)}$ solutions, themselves, will be purely periodic except for their integration "constant", $C'(\tilde{t})$. Therefore, it is recommended that the $E^{(2)}$ solutions be investigated, at least to the point of determining the functional forms of $C(\tilde{t})$.

During the study, it was found that some form of automated manipulation capability is absolutely essential to the accurate and timely solutions of the equations involved. Many operations on very lengthy expressions are required, such as expansions, integrations, substitutions, simplifications, etc. Furthermore, an automated method for uniform presentation of results is highly desirable. Therefore, the FORMAC language was used to write a computer program that performs these operations and presents the results in a convenient manner. As a result, a great deal of experience was gained in the use of FORMAC; and limitations of the language, such as lack of identity recognition, core storage requirement, problems in subroutine communication, etc., were encountered. (A thorough discussion of these problems is given in Reference 17.) The necessity of this automated manipulation capability in providing accurate and timely analytical results cannot be overemphasized, and the development of the FORMAC program is considered to be a major accomplishment of the project.

The second approximation solutions of orbital motion using two-variable asymptotic expansions have been implemented into a UNIVAC 1108 computer program (GENPUR). A comparative study of the results obtained using this program showed it to be very accurate, especially when Brouwer's approximations (Reference 21) of the second-order $\underline{J}_2$ and $\underline{J}_4$ effects are used. (As yet, these effects have not been determined by the methods of asymptotic expansion.) For example, errors in the solutions for the short-period effects in mean anomaly for Explorer 7 were less than 0.09 degree during 8 days, which was less than the difference between the standard COWELL and SPERTB (Reference 10) numerical integration programs. Furthermore, errors in the long-term solutions (i.e., mean element solutions) were generally less than or equal to the errors of the MSFC Lifetime Program (Reference 20). (These errors in the long-term solutions may possibly be reduced when the approximations of $\underline{J}_4$ and $\underline{J}_2^2$ effects are replaced by the asymptotic series solutions.)

The run time required for an ephemeris prediction program is always of utmost importance. The GENPUR program is extremely fast and has the potential of being even faster. For example, the run time required for the simulation of the Explorer 7 satellite over a 360-day period was 94.5 seconds when using an update interval of 24 hours. Increasing this interval to 96 hours resulted in no noticeable loss of accuracy, and the run time was reduced to only 23.4 seconds. In comparison, the run time required for the same orbit using the MSFC Lifetime Program (with a 2-day step) was 148.9 seconds.

Even in its present developmental state, the GENPUR program has clearly demonstrated the soundness of the approach taken herein to compute long-term satellite ephemeris. Before being placed in a production status, however, there are certain additions to the program which should be made. Errors in the element solutions for $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$ could possibly be reduced by an accurate representation of $\underline{J}_2^2$ and $\underline{J}_4$ effects. Even with the approximations now being used, the GENPUR error is comparable to, or less than, that of the Lifetime Program. Since two-variable asymptotic series represents a different approach to ephemeris prediction, it is quite possible that this method could result in more accuracy than existing solution methods.

Another addition recommended for GENPUR, which could result in much faster run times than even the 23.4 seconds mentioned previously, is the use of analytical expressions for the Fourier coefficients. One innovative feature of the GENPUR technique that contributes to its speed has been the use of Fourier series expansions to represent drag effects. The Fourier coefficients are presently determined by use of the 1970 Jacchia density model at frequent intervals. If the variations of these coefficients for periods of 20 or 30 days could be established analytically, a run time of only 6 seconds would be a possibility. Furthermore, the successful development of such a model would represent a significant advancement in the state-of-the-art of satellite ephemeris prediction.

An increase in the flexibility of the GENPUR program is also recommended. A wide variety of input coordinate systems, as provided in the Lifetime Program, would be advantageous. The satellite physical characteristics (mass, drag coefficient, and area) must now be held constant in the program. Providing input options for these items which allow variations with time and/or orbital position would be extremely useful in orbit analyses. Also, it would be desirable to have an input option for daily values of solar flux and heating parameters. This flexibility could be easily achieved within GENPUR by incorporating many of the corresponding routines of the Lifetime Program.

Once the GENPUR program has been extended as recommended above, it will represent an even more valuable tool for conducting astrodynamic investigations. For example, King-Hele has stated that the upper atmosphere rotates at a faster rate than the earth, but other investigators have failed to confirm this finding. By using the GENPUR program to study the long-term evolution of inclination for various orbits, an independent estimate of upper atmosphere rotation could be made. Another problem which has received little attention is the exact nature of the final decay of eccentricity.

It is well-known that an eccentric orbit becomes nearly circular before its ultimate decay, but whether it becomes zero or reaches a limiting value is uncertain. Furthermore, because of its extremely fast run time, GENPUR is ideal for parametric studies to identify characteristics of various classes of orbits to aid in mission planning activities.

In summary, this study has demonstrated the successful application of two-variable asymptotic expansions and the automated manipulation capabilities of FORMAC to the satellite motion problem. The resulting GENPUR computer program, although in a developmental state, has clearly exhibited the potential of being more accurate and much faster than any existing long-term ephemeris prediction program.

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APPENDIX A - DERIVATION OF THE PERTURBATIVE VARIATION EQUATIONS

To illustrate the procedure for obtaining Equations (2-7) through (2-12) by the method of perturbative differentiation, the equation for $\grave{e}$ will be derived.

The polar equation for an ellipse is

$$r = \frac{h^2/\mu}{(1+e\cos\nu)} = \frac{\rho}{(1+e\cos\nu)} \tag{A-1}$$

where the specific angular momentum is given by

$$h = r^2\dot{\nu} \tag{A-2}$$

Taking the dot-derivative of Equation (A-1) results in

$$\dot{r} = \frac{h^2 e\dot{\nu}\sin\nu}{\mu(1+e\cos\nu)^2}$$

or, after substituting Equations (A-1) and (A-2)

$$\dot{r} = \frac{\mu e\sin\nu}{h} \tag{A-3}$$

Substituting Equation (A-2) into Equation (A-1) yields

$$e\cos\nu = \frac{r^3\dot{\nu}^2}{\mu} - 1$$

It is now necessary to take the grave-derivative of this expression, remembering that $\grave{r} = 0$ (see Paragraph 2.2). Thus,

$$\grave{e}\cos\nu - e\grave{\nu}\sin\nu = \frac{2r^3\dot{\nu}\grave{\dot{\nu}}}{\mu} = \frac{2r^2\grave{\dot{\nu}}}{\sqrt{\mu\rho}}\left(\frac{\rho}{r}\right) \tag{A-4}$$

Similarly, substituting Equation (A-2) into Equation (A-3) yields

$$e\sin\nu = \frac{r^2\dot{r}\dot{\nu}}{\mu}$$

which becomes, after taking the grave-derivative

$$e' \sin\nu + e\nu' \cos\nu = \frac{r^2}{\mu}\left(\dot{r}\dot{\nu}' + \dot{r}'\dot{\nu}\right) = \frac{r^2\dot{\nu}' e \sin\nu}{\sqrt{\mu p}} + \frac{\dot{r}\dot{r}'}{\sqrt{\mu p}}\left(\frac{p}{r}\right) \qquad \text{(A-5)}$$

Multiplying Equation (A-4) by cos ν and Equation (A-5) by sin ν and then adding the results yields

$$e' = \frac{2r^2\dot{\nu}'}{\sqrt{\mu p}}\left(\frac{p}{r}\right)\cos\nu + \frac{r^2\dot{\nu}' e \sin^2\nu}{\sqrt{\mu p}} + \frac{r\dot{r}'}{\sqrt{\mu p}}\left(\frac{p}{r}\right)\sin\nu$$

$$= \frac{r\dot{r}'}{\sqrt{\mu p}}\left(\frac{p}{r}\right)\sin\nu + \frac{r^2\dot{\nu}'}{\sqrt{\mu p}}\left[\cos\nu\left(2\frac{p}{r} - e\cos\nu\right) + e\right]$$

Since

$$e\cos\nu = \frac{p}{r} - 1$$

this becomes

$$e' = \frac{r\dot{r}'}{\sqrt{\mu p}}\left(\frac{p}{r}\sin\nu\right) + \frac{r^2\dot{\nu}'}{\sqrt{\mu p}}\left[\left(\frac{p}{r}+1\right)\cos\nu + e\right] \qquad \text{(2-8)}$$

(NOTE: Equation (2-8) agrees with Reference 2, p. 247)

The perturbative variation equations for the remaining elements can be derived in a similar manner.

APPENDIX B - ASYMPTOTIC EXPANSION OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

To illustrate the procedure for asymptotically expanding trigonometric and exponential functions, the functions

$$\cos x \qquad \sin^2 x \qquad (1-e^2)^{-7/2}$$

$$\sin x \qquad (1+e\cos x)^3$$

will be considered, where $\underline{x}$ is used to denote any angle element.

Begin by assuming the asymptotic series

$$x = x^{(0)} + \underbrace{\epsilon x^{(1)} + \epsilon^2 x^{(2)} + \ldots}_{b} = x^{(0)} + b$$

<u>cos x</u>

$$\cos x = \cos(x^{(0)} + b) = \cos x^{(0)} \cos b - \sin x^{(0)} \sin b$$

But

$$\cos b = 1 - \frac{b^2}{2} + \ldots = 1 - \frac{1}{2}(\epsilon x^{(1)} + \epsilon^2 x^{(2)})^2 = 1 - \frac{1}{2}\epsilon^2 x^{(1)2} + \ldots$$

$$\sin b = b - \frac{b^3}{6} + \ldots = \epsilon x^{(1)} + \epsilon^2 x^{(2)} + \ldots$$

Thus,

$$\cos x = \cos x^{(0)}(1 - \tfrac{1}{2}\epsilon^2 x^{(1)2}) - \sin x^{(0)}(\epsilon x^{(1)} + \epsilon^2 x^{(2)})$$

$$= \cos x^{(0)} + \epsilon[-x^{(1)} \sin x^{(0)}] + \epsilon^2[\quad] + \ldots$$

<u>sin x</u>

$$\sin x = \sin(x^{(0)} + b) = \sin x^{(0)} \cos b + \cos x^{(0)} \sin b$$

$$= \sin x^{(0)}(1 - \tfrac{1}{2}\epsilon^2 x^{(1)2}) + \cos x^{(0)}(\epsilon x^{(1)} + \epsilon^2 x^{(2)})$$

$$= \sin x^{(0)} + \epsilon[x^{(1)} \cos x^{(0)}] + \epsilon^2[\quad] + \ldots$$

$\underline{\sin^2 x}$

Note that sin x has the form

$$\sin x = a_1 + a_2\epsilon + a_3\epsilon^2$$

$\Longrightarrow$

$$\sin^2 x = [(a_1+a_2\epsilon)+a_3\epsilon^2]^2 = (a_1+a_2\epsilon)^2 + 2(a_1+a_2\epsilon)a_3\epsilon^2 + a_3^2\epsilon^4$$

$$= a_1^2 + \epsilon[2a_1a_2] + \epsilon^2[\quad] + \ldots$$

Thus,

$$\sin^2 x = \sin^2 x^{(0)} + \epsilon[2x^{(1)}\sin x^{(0)}\cos x^{(0)}] + \epsilon^2[\quad] + \ldots$$

$$= \sin^2 x^{(0)} + \epsilon[x^{(1)}\sin 2x^{(0)}] + \epsilon^2[\quad] + \ldots$$

$\underline{(1 + e\cos x)^3}$

From the previous expansion for cos x,

$$(1+e\cos x) = 1 + (e^{(0)} + \epsilon e^{(1)} + \ldots)(\cos x^{(0)} + \epsilon[-x^{(1)}\sin x^{(0)}])$$

$$= 1 + e^{(0)}\cos x^{(0)} + \epsilon e^{(0)}(-x^{(1)}\sin x^{(0)}) + \epsilon e^{(1)}\cos x^{(0)} + \ldots$$

$$= [1 + e^{(0)}\cos x^{(0)}] + \epsilon[e^{(1)}\cos x^{(0)} - e^{(0)}x^{(1)}\sin x^{(0)}] + \epsilon^2[\quad] + \ldots$$

Note that (1 + e cos x) has the form

$$(1 + e\cos x) = a_1 + a_2\epsilon + a_3\epsilon^2$$

$\Longrightarrow$

$$(1+e\cos x)^3 = [(a_1+a_2\epsilon)+a_3\epsilon^2]^3 = (a_1+a_2\epsilon)^3 + 3(a_1+a_2\epsilon)^2 a_3\epsilon^2 + \ldots$$

$$= a_1^3 + \epsilon[3a_1^2 a_2] + \epsilon^2[\quad] + \ldots$$

Thus,

$$(1+e\cos x)^3 = \left[1+e^{(0)}\cos x^{(0)}\right]^3 + \epsilon\left[3(1+e^{(0)}\cos x^{(0)})^2(e^{(1)}\cos x^{(0)} - e^{(0)}x^{(1)}\sin x^{(0)})\right]$$

$$+\epsilon^2[\quad] + \ldots$$

<u>$(1-e^2)^{-7/2}$</u>

$$(1-e^2) = 1-(e^{(0)}+\epsilon e^{(1)}+\epsilon^2 e^{(2)}+\ldots)^2 = 1-\left[(e^{(0)}+\epsilon e^{(1)})+\epsilon^2 e^{(2)}\right]^2$$

$$= 1-(e^{(0)2}+2e^{(0)}e^{(1)}\epsilon+\epsilon^2 e^{(1)2}) - 2(e^{(0)}+\epsilon e^{(1)})\epsilon^2 e^{(2)} - \epsilon^4 e^{(2)2}$$

$$= \left[1-e^{(0)2}\right] + \epsilon\left[-2e^{(0)}e^{(1)}\right] + \epsilon^2\left[-e^{(1)2} - 2e^{(0)}e^{(2)}\right] + \ldots$$

Then,

$$(1-e^2)^{-7/2} = \left[\{1-e^{(0)2}\} + \{\epsilon(-2e^{(0)}e^{(1)}) + \epsilon^2(-e^{(1)2} - 2e^{(0)}e^{(2)})\}\right]^{-7/2}$$

$$= (1-e^{(0)2})^{-7/2} + (-\tfrac{7}{2})(1-e^{(0)2})^{-9/2}\left(\epsilon\left[-2e^{(0)}e^{(1)}\right] + \epsilon^2\left[-e^{(1)2} - 2e^{(0)}e^{(2)}\right]\right)$$

$$= \left[1-e^{(0)2}\right]^{-7/2} + \epsilon\left[7(1-e^{(0)2})^{-9/2}e^{(0)}e^{(1)}\right] + \epsilon^2[\quad] + \ldots$$

APPENDIX C - PRESENTATION OF THE INTEGRATION RESULTS OBTAINED FROM THE FORMAC PROGRAM WHEN CONSIDERING EARTH OBLATENESS (J_2 AND J_3) AND DRAG

As mentioned in Paragraph 4.3.1, a FORMAC computer program is used to solve integrals of the form given by Equations (4-18), (4-19), (4-52), (4-53), and (4-58) for each element. The program prints the total integrated results $\underline{I}_2$, $\underline{I}_3$, and $\underline{I}_D$ for each element, as well as the secular (with respect to $\nu^{(0)}$) parts $\underline{S}_2$, $\underline{S}_3$, and $\underline{S}_D$ (see Equations (4-20), (4-21), (4-59), (4-60), and (4-61)). These answer arrays are presented on the following pages. As discussed in Paragraph 4.3.2, it is not necessary to consider drag when formulating the second approximations to the solutions since the super-one solutions due to drag are negligible. Consequently, $\underline{I}_D$ for each element is not required and, therefore, is not presented.

To maintain consistency with the assumptions made in solving the set of ordinary differential equations having $\underline{\tilde{t}}$ as the independent variable (see Appendices E and F), the arrays generally include only terms through the order of $\underline{e}$ (i.e., terms on the order of $\underline{e}^2$, or smaller, are ignored).

Recall that the numerical subscript on $\underline{I}$ and $\underline{S}$ indicates the earth harmonic under consideration, the subscript $\underline{D}$ indicates drag, and the parenthetical (x) indicates the element $\underline{x}$. For example,

$I_2(\Omega)$ = Total integrated result for the element $\underline{\Omega}$ when considering the second harmonic (J_2)

$S_3(e)$ = Secular part of the total integrated result for the element $\underline{e}$ when considering the third harmonic (J_3)

$S_D(e)$ = Secular part of the total integrated result for the element $\underline{e}$ when considering drag

Also, recall that in the $\underline{S}_D$ array for each element, $\underline{a}_K$ and $\underline{b}_K$ are the Fourier coefficients appearing in the atmospheric density function approximation (see Equation (4-57).

Element B

$$
\begin{aligned}
I_2(B) = e^{(0)}\Big[& 6\sin^2 i^{(0)} \sin\nu^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} - 6\sin^2 i^{(0)} \sin^2\nu^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} + 3\sin^2 i^{(0)} \sin^2\nu^{(0)} \cos\nu^{(0)} \\
& - 6\sin^2 i^{(0)} \sin^3\nu^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} + 3\sin^2 i^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} - \cos\nu^{(0)} \Big] + 2\sin^2 i^{(0)} \sin\nu^{(0)} \sin\omega^{(0)} \cos\nu^{(0)} \cos\omega^{(0)} \\
& + \sin^2 i^{(0)} \sin^2\nu^{(0)} - 2\sin^2 i^{(0)} \sin^2\nu^{(0)} \sin^2\omega^{(0)}
\end{aligned}
$$

$$S_2(B) = 0$$

$$
\begin{aligned}
I_3(B) = e^{(0)}\Big[& -3\sin i^{(0)} \sin\nu^{(0)} \cos\nu^{(0)} \cos\omega^{(0)} + 3\sin i^{(0)} \sin^2\nu^{(0)} \sin\omega^{(0)} + 15\sin^3 i^{(0)} \sin\nu^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} \cos\omega^{(0)} \\
& + 15\sin^3 i^{(0)} \sin^2\nu^{(0)} \sin\omega^{(0)} - 25\sin^3 i^{(0)} \sin^2\nu^{(0)} \sin^3\omega^{(0)} - 20\sin^3 i^{(0)} \sin^3\nu^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} \cos\omega^{(0)} \\
& + 5\sin^3 i^{(0)} \sin^3\nu^{(0)} \cos\nu^{(0)} \cos\omega^{(0)} - 15\sin^3 i^{(0)} \sin^4\nu^{(0)} \sin\omega^{(0)} + 20\sin^3 i^{(0)} \sin^4\nu^{(0)} \sin^3\omega^{(0)} \Big] \\
& - \tfrac{3}{4}\sin i^{(0)} \sin\nu^{(0)} \cos\omega^{(0)} - \tfrac{3}{4}\sin i^{(0)} \sin\omega^{(0)} \cos\nu^{(0)} + \tfrac{15}{4}\sin^3 i^{(0)} \sin\nu^{(0)} \sin^2\omega^{(0)} \cos\omega^{(0)} \\
& + \tfrac{15}{4}\sin^3 i^{(0)} \sin^2\nu^{(0)} \sin\omega^{(0)} \cos\nu^{(0)} - 5\sin^3 i^{(0)} \sin^2\nu^{(0)} \sin^3\omega^{(0)} \cos\nu^{(0)} - 5\sin^3 i^{(0)} \sin^3\nu^{(0)} \sin^2\omega^{(0)} \cos\omega^{(0)} \\
& + \tfrac{5}{4}\sin^3 i^{(0)} \sin^3\nu^{(0)} \cos\omega^{(0)} + \tfrac{5}{4}\sin^3 i^{(0)} \sin^3\omega^{(0)} \cos\nu^{(0)}
\end{aligned}
$$

$$S_3(B) = 0$$

$$S_D(B) = \tfrac{1}{2}a_0 + \left(\tfrac{1}{2}a_1 - \tfrac{3}{2}b_3\right)e^{(0)}$$

Element e

$$I_2(e) = e^{(0)}\Big[-8\sin^2 i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 4\sin^2 i^{(0)}\sin^2\nu^{(0)} + 11\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}$$

$$+ 6\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 6\sin^2 i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)} + 3\sin^2 i^{(0)}\sin^4\nu^{(0)} - \sin^2\nu^{(0)}\Big]$$

$$- 4\sin^2 i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} + \frac{14}{3}\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} - \frac{7}{3}\sin^2 i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}$$

$$+ \frac{14}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} - \frac{5}{3}\sin^2 i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} - \frac{2}{3}\sin^2 i^{(0)}\cos\nu^{(0)} + \cos\nu^{(0)}$$

$$S_2(e) = 0$$

$$I_3(e) = e^{(0)}\Big[\frac{9}{2}\sin i^{(0)}\sin\nu^{(0)}\cos\omega^{(0)} - \frac{17}{4}\sin i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} - \frac{17}{4}\sin i^{(0)}\sin^3\nu^{(0)}\cos\omega^{(0)}$$

$$+ 5\sin i^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} - \frac{45}{2}\sin^3 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)} - 23\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}$$

$$+ \frac{151}{4}\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)} + \frac{205}{4}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)} - \frac{15}{2}\sin^3 i^{(0)}\sin^3\nu^{(0)}\cos\omega^{(0)}$$

$$+ \frac{81}{4}\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} - 27\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)} - 27\sin^3 i^{(0)}\sin^5\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)}$$

$$+ \frac{27}{4}\sin^3 i^{(0)}\sin^5\nu^{(0)}\cos\omega^{(0)} - \sin^3 i^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} - 7\sin^3 i^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)}\Big]$$

$$- \frac{3}{4}\nu^{(0)}\sin i^{(0)}\cos\omega^{(0)} + \frac{15}{16}\nu^{(0)}\sin^3 i^{(0)}\cos\omega^{(0)} + \frac{9}{4}\sin i^{(0)}\sin\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \frac{9}{4}\sin i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}$$

$$- \frac{15}{2}\sin^3 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \frac{15}{16}\sin^3 i^{(0)}\sin\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \frac{15}{2}\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}$$

$$+ \frac{55}{4}\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^3\omega^{(0)} + \frac{25}{2}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \frac{25}{8}\sin^3 i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$

$$+ \frac{75}{8}\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin\omega^{(0)} - \frac{25}{2}\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^3\omega^{(0)}$$

$$S_3(e) = \left(-\frac{3}{4}\sin i^{(0)}\cos\omega^{(0)} + \frac{15}{16}\sin^3 i^{(0)}\cos\omega^{(0)}\right)\left(1 - e^{(0)2}\right)$$

$$S_p(e) = \left(-\frac{1}{2}a_1 + \frac{3}{2}b_3\right) + \left(-\frac{1}{4}a_0 + \frac{1}{4}a_2\right)e^{(0)}$$

Element i

$$
\begin{aligned}
I_2(i) = e^{(0)}\Big[&-2\sin 2i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} + \tfrac{4}{3}\sin 2i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} \\
&- \tfrac{2}{3}\sin 2i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)} + \tfrac{4}{3}\sin 2i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} - \tfrac{4}{3}\sin 2i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} \\
&+ \tfrac{2}{3}\sin 2i^{(0)}\cos\nu^{(0)}\Big] - 2\sin 2i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \sin 2i^{(0)}\sin^2\nu^{(0)} \\
&+ 2\sin 2i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}
\end{aligned}
$$

$$S_2(i) = 0$$

$$
\begin{aligned}
I_3(i) = e^{(0)}\Big[&-\tfrac{5}{4}\nu^{(0)}\sin^2 i^{(0)}\cos i^{(0)}\cos\omega^{(0)} + \nu^{(0)}\cos i^{(0)}\cos\omega^{(0)} \\
&- 10\sin^2 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \tfrac{5}{4}\sin^2 i^{(0)}\sin\nu^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} \\
&- 10\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)} + 15\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)} \\
&+ 10\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \tfrac{5}{2}\sin^2 i^{(0)}\sin^3\nu^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} \\
&+ \tfrac{15}{2}\sin^2 i^{(0)}\sin^4\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)} - 10\sin^2 i^{(0)}\sin^4\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)} + \sin\text{[illegible]}^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} \\
&- \sin^2\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\Big] - 5\sin^2 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\omega^{(0)} \\
&- 5\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} + \tfrac{20}{3}\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} \\
&+ \tfrac{20}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\omega^{(0)} - \tfrac{5}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)}\cos i^{(0)}\cos\omega^{(0)} \\
&- \tfrac{5}{3}\sin^2 i^{(0)}\sin^3\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} + \sin\nu^{(0)}\cos i^{(0)}\cos\omega^{(0)} + \sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}
\end{aligned}
$$

$$S_3(i) = -\tfrac{5}{4}e^{(0)}\sin^2 i^{(0)}\cos i^{(0)}\cos\omega^{(0)} + e^{(0)}\cos i^{(0)}\cos\omega^{(0)}$$

$$S_D(i) = 0$$

Element Ω

$$
\begin{aligned}
I_2(\Omega) = e^{(0)}\Big[& -\sin\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)} - \tfrac{2}{3}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \tfrac{2}{3}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)} \\
& - \tfrac{1}{3}\sin^3\nu^{(0)}\cos i^{(0)} + \tfrac{2}{3}\sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}\Big] - \tfrac{1}{2}\nu^{(0)}\cos i^{(0)} - \sin\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} \\
& + \tfrac{1}{2}\sin\nu^{(0)}\cos i^{(0)}\cos\nu^{(0)} - \sin^2\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\cos\omega^{(0)}
\end{aligned}
$$

$$S_2(\Omega) = -\tfrac{1}{2}\cos i^{(0)}$$

$$
\begin{aligned}
I_3(\Omega) = e^{(0)}\Big[& -\tfrac{15}{4}\nu^{(0)}\sin i^{(0)}\sin\omega^{(0)}\cos i^{(0)} + \nu^{(0)}\operatorname{ctn} i^{(0)}\sin\omega^{(0)} + \tfrac{15}{4}\sin i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} \\
& - 10\sin i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} - 15\sin i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\omega^{(0)} \\
& - \tfrac{15}{2}\sin i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} + 10\sin i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)} \\
& + 10\sin i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\omega^{(0)} - \tfrac{5}{2}\sin i^{(0)}\sin^4\nu^{(0)}\cos i^{(0)}\cos\omega^{(0)} + \operatorname{ctn} i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} \\
& + \operatorname{ctn} i^{(0)}\sin^2\nu^{(0)}\cos\omega^{(0)}\Big] - 5\sin i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)} - \tfrac{20}{3}\sin i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} \\
& + \tfrac{5}{3}\sin i^{(0)}\sin^2\nu^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 5\sin i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos i^{(0)} \\
& + \tfrac{20}{3}\sin i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)}\cos i^{(0)} + \tfrac{5}{3}\sin i^{(0)}\cos i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} \\
& + \tfrac{10}{3}\sin i^{(0)}\cos i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \operatorname{ctn} i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)} - \operatorname{ctn} i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}
\end{aligned}
$$

$$S_3(\Omega) = -\tfrac{15}{4}e^{(0)}\sin i^{(0)}\sin\omega^{(0)}\cos i^{(0)} + e^{(0)}\operatorname{ctn} i^{(0)}\sin\omega^{(0)}$$

$$S_p(\Omega) = 0$$

<u>**Element ω**</u>

$$I_2(\omega) = e^{(0)}\Big[-\tfrac{5}{2}\nu^{(0)}\sin^2 i^{(0)} + 2\nu^{(0)} - 8\sin^2 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} + \tfrac{5}{2}\sin^2 i^{(0)}\sin\nu^{(0)}\cos\nu^{(0)}$$
$$-11\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} + 6\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} - 3\sin^2 i^{(0)}\sin^3\nu^{(0)}\cos\nu^{(0)}$$
$$+6\sin^2 i^{(0)}\sin^4\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)} + 2\sin\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} + 2\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)}\Big]$$
$$-3\sin^2 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)} - \tfrac{14}{3}\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \tfrac{14}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}$$
$$-\tfrac{7}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)} + \tfrac{7}{3}\sin^2 i^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \sin\nu^{(0)}$$

$$S_2(\omega) = 2e^{(0)} - \tfrac{5}{2}e^{(0)}\sin^2 i^{(0)}$$

$$I_3(\omega) = e^{(0)}\Big[13\sin i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)} + 5\sin i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)} - \tfrac{2}{3}\sin i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}$$
$$+\tfrac{20}{3}\sin i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + 4\sin i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$-\tfrac{20}{3}\sin i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)} - \tfrac{5}{3}\sin i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 10\sin i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$-\csc i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)} + \csc i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 25\sin^3 i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)}$$
$$-47\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \tfrac{14}{3}\sin^3 i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$-\tfrac{125}{3}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)} + 65\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)} - 9\sin^3 i^{(0)}\sin^4\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$+36\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + 27\sin^3 i^{(0)}\sin^5\nu^{(0)}\sin\omega^{(0)}$$
$$-36\sin^3 i^{(0)}\sin^5\nu^{(0)}\sin^3\omega^{(0)} + 6\sin^3 i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} + \tfrac{28}{3}\sin^3 i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}\Big]$$
$$+\nu^{(0)}\sin i^{(0)}\sin\omega^{(0)} - \tfrac{5}{4}\nu^{(0)}\sin^3 i^{(0)}\sin\omega^{(0)} + 3\sin i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} + 3\sin i^{(0)}\sin^2\nu^{(0)}\cos\omega^{(0)}$$
$$+\tfrac{5}{4}\sin^3 i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} - \tfrac{20}{3}\sin^3 i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)} - 15\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)}$$
$$-\tfrac{25}{2}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} + \tfrac{50}{3}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)} - \tfrac{35}{6}\sin^3 i^{(0)}\sin^4\nu^{(0)}\cos\omega^{(0)}$$
$$+\tfrac{50}{3}\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)}$$

$$S_3(\omega) = \sin i^{(0)}\sin\omega^{(0)} - \tfrac{5}{4}\sin^3 i^{(0)}\sin\omega^{(0)}$$

$$S_p(\omega) = -\tfrac{1}{2}b_1 - \tfrac{3}{2}b_3 + \tfrac{1}{4}b_2 e^{(0)}$$

Element M

$$I_2(M) = e^{(0)}\Big[\nu^{(0)} - \tfrac{3}{2}\nu^{(0)}\sin^2 i^{(0)} + \tfrac{3}{2}\sin^2 i^{(0)}\sin\nu^{(0)}\cos\nu^{(0)} + 3\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)}$$
$$- 6\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)} + 3\sin^2 i^{(0)}\sin^3\nu^{(0)}\cos\nu^{(0)} - 6\sin^2 i^{(0)}\sin^4\nu^{(0)}\sin\omega^{(0)}\cos\omega^{(0)}$$
$$- \sin\nu^{(0)}\cos\nu^{(0)}\Big] + 3\sin^2 i^{(0)}\sin\nu^{(0)}\sin^2\omega^{(0)} + \tfrac{14}{3}\sin^2 i^{(0)}\sin^2\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$- \tfrac{14}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)}\sin^2\omega^{(0)} + \tfrac{7}{3}\sin^2 i^{(0)}\sin^3\nu^{(0)} - \tfrac{2}{3}\sin^2 i^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \sin\nu^{(0)}$$

$$S_3(M) = e^{(0)} - \tfrac{3}{2}e^{(0)}\sin^2 i^{(0)}$$

$$I_3(M) = e^{(0)}\Big[3\sin i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)} + \tfrac{17}{4}\sin i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \tfrac{17}{4}\sin i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}$$
$$+ \tfrac{7}{4}\sin i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - 5\sin^3 i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)} - \tfrac{203}{12}\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)}\cos\nu^{(0)}$$
$$- \tfrac{13}{12}\sin^3 i^{(0)}\sin^2\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \tfrac{35}{2}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)} + \tfrac{365}{12}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)}$$
$$+ 27\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)} - \tfrac{27}{4}\sin^3 i^{(0)}\sin^4\nu^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$+ \tfrac{81}{4}\sin^3 i^{(0)}\sin^5\nu^{(0)}\sin\omega^{(0)} - 27\sin^3 i^{(0)}\sin^5\nu^{(0)}\sin^3\omega^{(0)} - \tfrac{1}{12}\sin^3 i^{(0)}\sin^2\omega^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}$$
$$- \tfrac{13}{6}\sin^3 i^{(0)}\cos\nu^{(0)}\cos\omega^{(0)}\Big] + \tfrac{3}{4}\nu^{(0)}\sin i^{(0)}\sin\omega^{(0)} - \tfrac{15}{16}\nu^{(0)}\sin^3 i^{(0)}\sin\omega^{(0)}$$
$$+ \tfrac{9}{4}\sin i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} + \tfrac{9}{4}\sin i^{(0)}\sin^2\nu^{(0)}\cos\omega^{(0)} + \tfrac{15}{16}\sin^3 i^{(0)}\sin\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)}$$
$$- 5\sin^3 i^{(0)}\sin\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)} - \tfrac{45}{4}\sin^3 i^{(0)}\sin^2\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)}$$
$$- \tfrac{75}{8}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin\omega^{(0)}\cos\nu^{(0)} + \tfrac{25}{2}\sin^3 i^{(0)}\sin^3\nu^{(0)}\sin^3\omega^{(0)}\cos\nu^{(0)}$$
$$+ \tfrac{25}{2}\sin^3 i^{(0)}\sin^4\nu^{(0)}\sin^2\omega^{(0)}\cos\omega^{(0)} - \tfrac{25}{8}\sin^3 i^{(0)}\sin^4\nu^{(0)}\cos\omega^{(0)}$$

$$S_3(M) = \tfrac{3}{4}\sin i^{(0)}\sin\omega^{(0)} - \tfrac{15}{16}\sin^3 i^{(0)}\sin\omega^{(0)}$$

$$S_D(M) = \left(\tfrac{1}{2}b_1 + \tfrac{3}{2}b_3\right)\frac{1}{e^{(0)}} + \left(\tfrac{1}{2}b_1 + \tfrac{3}{4}b_2 + \tfrac{3}{2}b_3\right) + \left(\tfrac{9}{16}b_1 - \tfrac{1}{2}b_2 + \tfrac{27}{16}b_3\right)e^{(0)}$$

APPENDIX D - DERIVATION OF THE SECULAR PART OF $\nu^{(0)}$ WITH RESPCT TO $\bar{t}$

From Equation (4-12)

$$\frac{d\nu^{(0)}}{d\bar{t}} = \frac{\sqrt{\mu}\, B^{(0)3}(1+e^{(0)}\cos\nu^{(0)})^2}{(1-e^{(0)2})^{3/2}}$$

or

$$\frac{d\nu^{(0)}}{(1+e^{(0)}\cos\nu^{(0)})^2} = \frac{\sqrt{\mu}\, B^{(0)3}\, d\bar{t}}{(1-e^{(0)2})^{3/2}}$$

Since $e^{(0)}$ and $B^{(0)}$ are considered constant with respect to a fast time-variable integration, then

$$\int\frac{d\nu^{(0)}}{(1+e^{(0)}\cos\nu^{(0)})^2} = \frac{\sqrt{\mu}\, B^{(0)3}}{(1-e^{(0)2})^{3/2}}\bar{t} = \frac{n^{(0)}\bar{t}}{(1-e^{(0)2})^{3/2}} \qquad \text{(D-1)}$$

It can be shown that (Reference 1, p. 201)

$$\int\frac{d\nu^{(0)}}{(1+e^{(0)}\cos\nu^{(0)})^2} = \frac{E^{(0)}-e^{(0)}\sin E^{(0)}}{(1-e^{(0)2})^{3/2}} - C \qquad \text{(D-2)}$$

where $\underline{E}$ is the eccentric anomaly and $\underline{C}$ is the integration constant. Equating Equations (D-1) and (D-2) yields

$$E^{(0)} = \underbrace{n^{(0)}\bar{t}}_{\text{secular}} + \underbrace{e^{(0)}\sin E^{(0)} + C(1-e^{(0)2})^{3/2}}_{\text{nonsecular}} \qquad \text{(D-3)}$$

From Reference 1, p. 209 (or Reference 6, p. 89)

$$\nu^{(0)} = E^{(0)} + \underbrace{2\sum_{j=1}^{\infty}\frac{A^j}{j}\sin jE^{(0)}}_{\text{periodic Fourier series}} \qquad \left[A = \frac{1-\sqrt{1-e^{(0)2}}}{e^{(0)}}\right] \qquad \text{(D-4)}$$

Substituting Equation (D-3) into Equation (D-4) yields

$$\nu^{(0)} = \underbrace{m^{(0)}\bar{t}}_{\text{secular}} + \underbrace{e^{(0)}\sin E^{(0)} + c\left(1-e^{(0)2}\right)^{3/2} + 2\sum_{j=1}^{\infty}\frac{A^j}{j}\sin jE^{(0)}}_{\text{nonsecular}} \qquad \text{(D-5)}$$

The general resolution of $\nu^{(0)}$ into secular and nonsecular parts can be indicated as

$$\nu^{(0)} = \nu_S^{(0)} + \nu_N^{(0)} \qquad \text{(D-6)}$$

Equating Equations (D-5) and (D-6) results in

$$\nu_S^{(0)} = m^{(0)}\bar{t} \qquad \text{(4-23)}$$

Inspection of Equations (D-3) and (4-23) shows that $\nu^{(0)}$ and $E^{(0)}$ have the same secular part.

APPENDIX E - SOLUTION OF THE SET OF ORDINARY DIFFERENTIAL EQUATIONS HAVING $\tilde{\tau}$ AS THE INDEPENDENT VARIABLE WHEN CONSIDERING EARTH OBLATENESS ONLY

E.1 INTRODUCTION

The following set of ordinary differential equations to the order of $e^{(0)}$, as derived in Paragraph 4.3.1.1, will now be solved:

$$\frac{de^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} \sin i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^2} \left(\frac{5}{4}\sin^2 i^{(0)} - 1\right) \tag{4-32}$$

$$\frac{d\omega^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)4} r_e^2 n^{(0)}}{(1-e^{(0)2})^2}\left(2-\frac{5}{2}\sin^2 i^{(0)}\right) + \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)}}{(1-e^{(0)2})^3}\left[\frac{35}{4} e^{(0)} \sin i^{(0)} \sin\omega^{(0)} \cos^2 i^{(0)} - e^{(0)} \csc i^{(0)} \sin\omega^{(0)} + \frac{1}{e^{(0)}} \sin i^{(0)} \sin\omega^{(0)}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right)\right] \tag{4-34}$$

$$\frac{di^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} e^{(0)} \cos i^{(0)} \cos\omega^{(0)}}{(1-e^{(0)2})^3}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right) \tag{4-36}$$

$$\frac{d\Omega^{(0)}}{d\tilde{\tau}} = -\frac{\beta^{(0)4} r_e^2 n^{(0)} \cos i^{(0)}}{(1-e^{(0)2})^2} + \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} e^{(0)} \sin\omega^{(0)}}{(1-e^{(0)2})^3}\left(\operatorname{ctn} i^{(0)} - \frac{15}{8}\sin 2i^{(0)}\right) \tag{4-38}$$

$$\frac{dM^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)4} r_e^2 n^{(0)}}{(1-e^{(0)2})^{3/2}}\left(1-\frac{3}{2}\sin^2 i^{(0)}\right) + \frac{4\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)}}{3e^{(0)}(1-e^{(0)2})^{5/2}}\left[3e^{(0)2} \sin i^{(0)} \sin\omega^{(0)}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right) - \frac{3}{4}\sin i^{(0)} \sin\omega^{(0)}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right)\right] \tag{4-43}$$

The method of solution is based on one set forth in Reference 16, whereby $e^{(0)}$ and $\omega^{(0)}$ are considered to vary simultaneously and the solutions for the remaining elements are obtained as functions of $e^{(0)}$ and $\omega^{(0)}$. In consideration of this, the following (assumed) constants are defined:

$$C_1 = \frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^3\, m_0^{(0)} \sin i_0^{(0)}}{(1-e_0^{(0)2})^2}\left(\frac{5}{4}\sin^2 i_0^{(0)} - 1\right) \tag{E-1}$$

$$C_2 = \frac{\beta_0^{(0)4}\, m_0^{(0)}\, r_e^2\left(2-\frac{5}{2}\sin^2 i_0^{(0)}\right)}{(1-e_0^{(0)2})^2} \tag{E-2}$$

$$C_2' = \frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) m_0^{(0)}\, r_e^3}{(1-e_0^{(0)2})^3}\left(\frac{35}{4}\sin i_0^{(0)}\cos^2 i_0^{(0)} - \csc i_0^{(0)}\right) \tag{E-3}$$

$$C_2'' = \frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) m_0^{(0)}\, r_e^3 \sin i_0^{(0)}}{(1-e_0^{(0)2})^3}\left(1-\frac{5}{4}\sin^2 i_0^{(0)}\right) = -\frac{C_1}{(1-e_0^{(0)2})} \tag{E-4}$$

$$C_3 = \frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^3\, m_0^{(0)} \cos i_0^{(0)}}{(1-e_0^{(0)2})^3}\left(1-\frac{5}{4}\sin^2 i_0^{(0)}\right) \tag{E-5}$$

$$C_4 = \frac{\beta_0^{(0)4}\, m_0^{(0)}\, r_e^2 \cos i_0^{(0)}}{(1-e_0^{(0)2})^2} \tag{E-6}$$

$$C_5 = \frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^3\, m_0^{(0)}}{(1-e_0^{(0)2})^3}\left(\operatorname{ctn} i_0^{(0)} - \frac{15}{8}\sin 2 i_0^{(0)}\right) \tag{E-7}$$

$$C_6 = \frac{\beta_0^{(0)4}\, r_e^2\, m_0^{(0)}}{(1-e_0^{(0)2})^{3/2}}\left(1-\frac{3}{2}\sin^2 i_0^{(0)}\right) \tag{E-8}$$

$$C_7 = -\frac{\beta_0^{(0)6}\left(\frac{J_3}{J_2}\right) r_e^3\, m_0^{(0)} \sin i_0^{(0)}}{(1-e_0^{(0)2})^{5/2}}\left(1-\frac{5}{4}\sin^2 i_0^{(0)}\right) \tag{E-9}$$

Note that these constants are expressed in terms of the epoch values of the element functions, i.e., $B_0^{(0)}$, $i_0^{(0)}$, $e_0^{(0)}$, and $n_0^{(0)}$. The effect of this assumption on the accuracy of the resultant solutions can be minimized by periodically rectifying the orbit and updating the epoch values of the elements. (As discussed in Paragraph 4.5, an "updating procedure" is used when numerically evaluating the solution equations.)

In terms of these constants, the differential equations become

$$\frac{de^{(0)}}{d\tau} = C_1 \cos \omega^{(0)} \tag{E-10}$$

$$\frac{d\omega^{(0)}}{d\tau} = C_2 + C_2' e^{(0)} \sin \omega^{(0)} + C_2'' \frac{\sin \omega^{(0)}}{e^{(0)}} \tag{E-11}$$

$$\frac{di^{(0)}}{d\tau} = C_3 e^{(0)} \cos \omega^{(0)} \tag{E-12}$$

$$\frac{d\Omega^{(0)}}{d\tau} = -C_4 + C_5 e^{(0)} \sin \omega^{(0)} \tag{E-13}$$

$$\frac{dM^{(0)}}{d\tau} = C_6 - 4C_7 e^{(0)} \sin \omega^{(0)} + C_7 \frac{\sin \omega^{(0)}}{e^{(0)}} \tag{E-14}$$

Equations (E-10) and (E-11) will now be solved simultaneously. The solutions to Equations (E-12) through (E-14) will then be obtained as functions of $e^{(0)}$ and $\omega^{(0)}$.

E.2 SIMULTANEOUS SOLUTION FOR $e^{(0)}$ AND $\omega^{(0)}$

To effect the simultaneous solution for $e^{(0)}$ and $\omega^{(0)}$, the following transformation parameters are used

$$\xi = e^{(0)} \cos \omega^{(0)} \tag{E-15}$$

$$\eta = e^{(0)} \sin \omega^{(0)} \tag{E-16}$$

Differentiating with respect to $\tilde{t}$ yields (where the "dot" indicates $\tilde{t}$ differentiation)

$$\dot{\xi} = \dot{e}^{(0)} \cos \omega^{(0)} - e^{(0)} \dot{\omega}^{(0)} \sin \omega^{(0)} \tag{E-17}$$

$$\dot{\eta} = \dot{e}^{(0)} \sin \omega^{(0)} + e^{(0)} \dot{\omega}^{(0)} \cos \omega^{(0)} \tag{E-18}$$

Substituting Equations (E-10), (E-11), (E-15), and (E-16) into Equations (E-17) and (E-18) yields after simplification

$$\dot{\xi} = -c_2 \eta - c_2' \eta^2 + \frac{c_1}{\xi^2+\eta^2}\left[\frac{\xi^2 - \xi^4 - \xi^2\eta^2 + \eta^2}{1-(\xi^2+\eta^2)}\right]$$

$$\dot{\eta} = c_2 \xi + c_2' \xi \eta - c_1 \frac{\xi \eta}{1-(\xi^2+\eta^2)}$$

To the order of e, i.e., ignoring terms of the order e^2 (or smaller), these equations become merely

$$\dot{\xi} = -c_2 \eta + c_1 \tag{E-19}$$

$$\dot{\eta} = c_2 \xi \tag{E-20}$$

(NOTE: These equations agree in form with Equation (14) of Reference 16.)

In terms of the operator $D = \frac{d}{d\tilde{t}}$, Equations (E-19) and (E-20) can be expressed as

$$\frac{1}{c_2} D(\xi) = \frac{c_1}{c_2} - \eta \tag{E-21}$$

$$\frac{1}{C_2}D(\eta) = \xi \tag{E-22}$$

To solve these equations simultaneously (Reference 22, pp. 198-200), ξ will first be eliminated. Substituting Equation (E-22) into Equation (E-21) yields

$$\frac{C_1}{C_2} - \eta - \frac{1}{C_2}D\left(\frac{1}{C_2}D\eta\right) = 0$$

$$C_1C_2 - C_2^2\eta - D^2(\eta) = 0$$

or

$$(D^2 + C_2^2)\eta = C_1C_2$$

As can be seen, this is a linear nonhomogeneous equation in η. It has the standard solution

$$\eta = K_1 \cos C_2 Z + K_2 \sin C_2 Z + \frac{C_1}{C_2} \tag{E-23}$$

where K_1 and K_2 are undetermined constants.

To obtain the solution for ξ, η will now be eliminated from Equations (E-21) and (E-22). From Equation (E-21)

$$\eta = \frac{C_1}{C_2} - \frac{1}{C_2}D(\xi)$$

implying

$$D(\eta) = D\left(\frac{C_1}{C_2}\right) - D\left(\frac{1}{C_2}D\xi\right) = -\frac{1}{C_2}D^2(\xi) \tag{E-24}$$

Equating Equations (E-22) and (E-24) yields

$$\xi C_2 = -\frac{1}{C_2}D^2(\xi)$$

or

$$(D^2 + C_2^2)\xi = 0$$

As can be seen, this is a linear homogeneous equation in ξ. It has the standard solution

$$\xi = K_3 \cos C_2\tilde{t} + K_4 \sin C_2\tilde{t} \qquad \text{(E-25)}$$

where K_3 and K_4 are undetermined constants.

Now, the constants K_1, K_2, K_3, and K_4 have to be "adjusted" so as to make Equations (E-23) and (E-25) satisfy the original equations. This can be done by substituting Equations (E-23) and (E-25) into Equation (E-20) and seeking relations between these constants. Performing this substitution yields

$$\frac{d}{d\tilde{t}}\left(K_1 \cos C_2\tilde{t} + K_2 \sin C_2\tilde{t} + \frac{C_1}{C_2}\right) - C_2\left(K_3 \cos C_2\tilde{t} + K_4 \sin C_2\tilde{t}\right) = 0$$

which implies

$$-K_1C_2 \sin C_2\tilde{t} + K_2C_2 \cos C_2\tilde{t} - C_2K_3 \cos C_2\tilde{t} - C_2K_4 \sin C_2\tilde{t} = 0$$

This equation is true only if

$$K_3 = K_2$$

and

$$K_4 = -K_1$$

Furthermore, it is convenient to express K_1 and K_2 as

$$K_1 = A \sin\alpha$$
$$K_2 = A \cos\alpha$$

where A (a positive number) and α are constants yet to be determined. Thus, Equations (E-23) and (E-25) become

$$\xi = A \cos(C_2\tilde{t} + \alpha) \qquad \text{(E-26)}$$

$$\eta = A \sin(C_2\tilde{t} + \alpha) + \frac{C_1}{C_2} \qquad \text{(E-27)}$$

Since, from Equations (E-15) and (E-16)

$$e^{(0)} = \sqrt{\xi^2 + \eta^2}$$

$$\omega^{(0)} = \tan^{-1}\left(\frac{\eta}{\xi}\right)$$

it follows that the desired solutions are

$$e^{(0)} = \left[A^2 + 2\frac{C_1}{C_2} A \sin(C_2\tilde{t} + \alpha) + \left(\frac{C_1}{C_2}\right)^2\right]^{1/2} \tag{4-33}$$

$$\omega^{(0)} = \tan^{-1}\left[\frac{A \sin(C_2\tilde{t} + \alpha) + C_1/C_2}{A \cos(C_2\tilde{t} + \alpha)}\right] \tag{4-35}$$

It now remains to determine the constants $\underline{A}$ and $\underline{\alpha}$. This can be done by evaluating Equations (E-26) and (E-27) at the epoch time $\underline{t}_0$, resulting in

$$\xi_0 = A \cos(C_2\tilde{t}_0 + \alpha) \tag{E-28}$$

$$\eta_0 = A \sin(C_2\tilde{t}_0 + \alpha) + \frac{C_1}{C_2} \tag{E-29}$$

Multiplying the first equation by $\sin(C_2\tilde{t}_0 + \alpha)$, the second by $-\cos(C_2\tilde{t}_0 + \alpha)$, and then adding yields

$$\xi_0 \sin(C_2\tilde{t}_0 + \alpha) - \eta_0 \cos(C_2\tilde{t}_0 + \alpha) + \frac{C_1}{C_2}\cos(C_2\tilde{t}_0 + \alpha) = 0$$

or merely

$$\tan(C_2\tilde{t}_0 + \alpha) = \frac{1}{\xi_0}\left(\eta_0 - \frac{C_1}{C_2}\right) \tag{E-30}$$

Define

$$D^* = \frac{1}{\xi_0}\left(\eta_0 - \frac{C_1}{C_2}\right) \tag{E-31}$$

which becomes by Equations (E-15) and (E-16) evaluated at $\underline{t}_0$

$$D^* = \left(\tan\omega_0^{(0)} - \frac{C_1}{C_2 e_0^{(0)} \cos\omega_0^{(0)}}\right) \tag{E-32}$$

Then Equation (E-30) can be written as

$$\tan(C_2\tilde{t}_0+\alpha) = \mathcal{D}^*$$

$$\alpha = \tan^{-1}(\mathcal{D}^*) - C_2\tilde{t}_0 = \tan^{-1}\left[\frac{e_0^{(0)}\sin\omega_0^{(0)} - C_1/C_2}{e_0^{(0)}\cos\omega_0^{(0)}}\right] - C_2\tilde{t}_0 \tag{E-33}$$

Multiplying Equation (E-28) by cos $(C_2\tilde{t}_0 + \alpha)$, Equation (E-29) by sin $(C_2\tilde{t}_0 + \alpha)$, and then adding yields

$$\xi_0\cos(C_2\tilde{t}_0+\alpha) + \eta_0\sin(C_2\tilde{t}_0+\alpha) = A + \frac{C_1}{C_2}\sin(C_2\tilde{t}_0+\alpha)$$

$$A = \xi_0\cos(C_2\tilde{t}_0+\alpha) + \eta_0\sin(C_2\tilde{t}_0+\alpha) - \frac{C_1}{C_2}\sin(C_2\tilde{t}_0+\alpha)$$

$$A = \frac{\xi_0^2}{A} + \eta_0\left(\frac{\eta_0 - C_1/C_2}{A}\right) - \frac{C_1}{C_2}\left(\frac{\eta_0 - C_1/C_2}{A}\right)$$

$$A^2 = \xi_0^2 + \eta_0^2 - 2\eta_0\frac{C_1}{C_2} + \left(\frac{C_1}{C_2}\right)^2$$

$$A = \left[e_0^{(0)2} + \left(\frac{C_1}{C_2}\right)^2 - 2\frac{C_1}{C_2}e_0^{(0)}\sin\omega_0^{(0)}\right]^{1/2} \tag{E-34}$$

E.3 SOLUTION FOR $i^{(0)}$

By Equations (E-15) and (E-26), Equation (E-12) becomes

$$\frac{di^{(0)}}{d\tilde{t}} = C_3 A\cos(C_2\tilde{t}+\alpha)$$

which yields upon integration

$$i^{(0)} = \frac{C_3}{C_2}A\sin(C_2\tilde{t}+\alpha) + K$$

where $\underline{K}$ is the integration constant. Substituting Equation (E-27) yields

$$i^{(0)} = \frac{C_3}{C_2}\left(\eta - \frac{C_1}{C_2}\right) + K = \frac{C_3}{C_2}\left(e^{(0)}\sin\omega^{(0)} - \frac{C_1}{C_2}\right) + K$$

The constant $\underline{K}$ can be determined by evaluating this equation at the epoch time $\underline{t}_0$, resulting in

$$K = i_0^{(0)} - \frac{c_3}{c_2}\left(e_0^{(0)} \sin \omega_0^{(0)} - \frac{c_1}{c_2}\right)$$

Hence, the desired solution is

$$i^{(0)} = i_0^{(0)} + \frac{c_3}{c_2}\left(e^{(0)} \sin \omega^{(0)} - e_0^{(0)} \sin \omega_0^{(0)}\right) \tag{4-37}$$

E.4 SOLUTION FOR $\Omega^{(0)}$

By Equations (E-16) and (E-27), Equation (E-13) becomes

$$\frac{d\Omega^{(0)}}{d\tau} = -c_4 + c_5 A \sin(c_2\tau + \alpha) + c_5 \frac{c_1}{c_2}$$

which yields upon integration

$$\Omega^{(0)} = \left(c_5 \frac{c_1}{c_2} - c_4\right)\tau - \frac{c_5}{c_2} A \cos(c_2\tau + \alpha) + K$$

where $\underline{K}$ is the integration constant. Substituting Equation (E-26) yields

$$\Omega^{(0)} = \left(c_5 \frac{c_1}{c_2} - c_4\right)\tau - \frac{c_5}{c_2}\xi + K$$

$$= \left(c_5 \frac{c_1}{c_2} - c_4\right)\tau - \frac{c_5}{c_2} e^{(0)} \cos \omega^{(0)} + K$$

The constant $\underline{K}$ can be determined by evaluating this equation at the epoch time $\underline{t}_0$, resulting in

$$K = \Omega_0^{(0)} - \left(c_5 \frac{c_1}{c_2} - c_4\right)\tau_0 + \frac{c_5}{c_2} e_0^{(0)} \cos \omega_0^{(0)}$$

Hence, the desired solution is

$$\Omega^{(0)} = \Omega_0^{(0)} - \frac{c_5}{c_2}\left(e^{(0)} \cos \omega^{(0)} - e_0^{(0)} \cos \omega_0^{(0)}\right) + \left(c_5 \frac{c_1}{c_2} - c_4\right)(\tau - \tau_0) \tag{4-39}$$

E.5 SOLUTION FOR $M^{(0)}$

From Equation (E-14)

$$\frac{dm^{(0)}}{d\tau} = c_6 - 4c_7 e^{(0)} \sin \omega^{(0)} + \frac{c_7 e^{(0)} \sin \omega^{(0)}}{e^{(0)2}}$$

This equation can be linearized by approximating $e^{(0)}$ by its epoch value $e_0^{(0)}$ (referred to as "backlining" $e^{(0)}$), resulting in

$$\frac{dm^{(0)}}{dZ} = C_6 + (C_8 - 4C_7)e^{(0)} \sin \omega^{(0)} \tag{E-35}$$

where

$$C_8 = \frac{C_7}{e_0^{(0)2}} \tag{E-36}$$

Then, from Equation (E-16)

$$\frac{dm^{(0)}}{dZ} = C_6 + (C_8 - 4C_7)\eta$$

so that

$$M^{(0)} = C_6 + (C_8 - 4C_7)\int \eta \, dZ + K$$

where $\underline{K}$ is the integration constant.

From Equation (E-27)

$$M^{(0)} = C_6 + (C_8 - 4C_7)\int \left[A \sin(C_2 Z + \alpha) + \frac{C_1}{C_2} \right] dZ + K$$

resulting in the solution

$$M^{(0)} = \left(C_6 + \frac{C_1 C_8}{C_2} - \frac{4C_1C_7}{C_2}\right) Z - \frac{(C_8 - 4C_7)}{C_2} e^{(0)} \cos \omega^{(0)} + K$$

Evaluating $\underline{K}$ at epoch time $\underline{t}_0$ yields

$$M^{(0)} = M_0^{(0)} + \left(C_6 + \frac{C_1 C_8}{C_2} - \frac{4 C_1 C_7}{C_2}\right)(\tau - \tau_0)$$
$$- \frac{(C_8 - 4 C_7)}{C_2}\left(e^{(0)} \cos \omega^{(0)} - e_0^{(0)} \cos \omega_0^{(0)}\right) \qquad (4\text{-}44)$$

As mentioned in Paragraph 4.3.1.1, the Keplerian change in $\underline{M}$ taking place during the time interval $(t - t_0)$ must be added to the above equation.

APPENDIX F - COMPOSITE SOLUTION OF THE SET OF ORDINARY DIFFERENTIAL EQUATIONS HAVING $\tilde{\tau}$ AS THE INDEPENDENT VARIABLE WHEN CONSIDERING EARTH OBLATENESS AND ATMOSPHERIC DRAG

F.1 INTRODUCTION

The following set of ordinary differential equations to the order of $e^{(0)}$, representing the combined effects of earth oblateness and atmospheric drag (Paragraph 4.3.1.2), will now be solved:

$$\frac{de^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} \sin i^{(0)} \cos \omega^{(0)}}{(1-e^{(0)2})^2}\left(\frac{5}{4}\sin^2 i^{(0)} - 1\right) + \frac{n^{(0)} K^* (1-e^{(0)2})}{\mu \beta^{(0)2}}\left[\left(-\frac{1}{2}a_1 + \frac{3}{2}b_3\right) + \left(-\frac{1}{4}a_0 + \frac{1}{4}a_2\right)e^{(0)}\right] \tag{4-71}$$

$$\begin{aligned}\frac{d\omega^{(0)}}{d\tilde{\tau}} = {} & \frac{\beta^{(0)4} r_e^2 n^{(0)}}{(1-e^{(0)2})^2}\left(2 - \frac{5}{2}\sin^2 i^{(0)}\right) + \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)}}{(1-e^{(0)2})^3}\Big[\frac{35}{4}e^{(0)} \sin i^{(0)} \sin\omega^{(0)} \cos^2 i^{(0)} \\ & - e^{(0)} \csc i^{(0)} \sin\omega^{(0)} + \frac{1}{e^{(0)}} \sin i^{(0)} \sin\omega^{(0)}\left(1 - \frac{5}{4}\sin^2 i^{(0)}\right)\Big] \\ & + \frac{n^{(0)} K^* (1-e^{(0)2})}{\beta^{(0)2} i^{(0)} \mu}\left[-\frac{1}{2}b_1 - \frac{3}{2}b_3 + \frac{1}{4}b_2 e^{(0)}\right]\end{aligned} \tag{4-73}$$

$$\frac{di^{(0)}}{d\tilde{\tau}} = \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} e^{(0)} \cos i^{(0)} \cos\omega^{(0)}}{(1-e^{(0)2})^3}\left(1 - \frac{5}{4}\sin^2 i^{(0)}\right) \tag{4-75}$$

$$\frac{d\Omega^{(0)}}{d\tilde{\tau}} = -\frac{\beta^{(0)4} r_e^2 n^{(0)} \cos i^{(0)}}{(1-e^{(0)2})^2} + \frac{\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)} e^{(0)} \sin\omega^{(0)}}{(1-e^{(0)2})^3}\left(\operatorname{ctn} i^{(0)} - \frac{15}{8}\sin 2i^{(0)}\right) \tag{4-77}$$

$$\frac{d\beta^{(0)}}{d\tilde{\tau}} = \frac{n^{(0)} K^*}{2\beta^{(0)}\mu}\left[\frac{1}{2}a_0 + \left(\frac{1}{2}a_1 - \frac{3}{2}b_3\right)e^{(0)}\right] \tag{4-79}$$

$$\frac{dm^{(0)}}{d\tilde{t}} = \frac{\beta^{(0)4} r_e^2 n^{(0)}}{(1-e^{(0)2})^{3/2}}\left(1-\frac{3}{2}\sin^2 i^{(0)}\right) \tag{4-82}$$

$$+ \frac{4\beta^{(0)6}(J_3/J_2) r_e^3 n^{(0)}}{3e^{(0)}(1-e^{(0)2})^{5/2}}\left[3e^{(0)2}\sin i^{(0)}\sin\omega^{(0)}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right)\right.$$

$$\left. - \frac{3}{4}\sin i^{(0)}\sin\omega^{(0)}\left(1-\frac{5}{4}\sin^2 i^{(0)}\right)\right]$$

$$+ \frac{n^{(0)}K^*(1-e^{(0)2})^{3/2}}{\beta^{(0)2}\mu}\left[\left(\frac{1}{2}b_1+\frac{3}{2}b_3\right)\frac{1}{e^{(0)}} + \left(\frac{1}{2}b_1+\frac{3}{4}b_2+\frac{3}{2}b_3\right)\right.$$

$$\left. + \left(\frac{9}{16}b_1 - \frac{1}{2}b_2 + \frac{27}{16}b_3\right)e^{(0)}\right]$$

Again, the solution procedure will be to solve simultaneously for $e^{(0)}$ and $\omega^{(0)}$, employing the same transformations and similar approximations as in Appendix E. These solutions will then be utilized to obtain solutions for the remaining elements. In consideration of this, the following constants (in addition to those given in Appendix E, i.e., $C_1 - C_8$) are defined:

$$\mathcal{D}_2 = \frac{n_o^{(0)} K^*}{\beta_o^{(0)2}\mu} \tag{F-1}$$

$$\alpha_0 = -\frac{1}{2}a_1 + \frac{3}{2}b_3 \tag{F-2}$$

$$\alpha_1 = -\frac{1}{4}a_0 + \frac{1}{4}a_2 \tag{F-3}$$

$$\beta_0 = \frac{1}{2}a_0 \tag{F-4}$$

$$\beta_1 = \frac{1}{2}a_1 - \frac{3}{2}b_3 \tag{F-5}$$

$$\gamma_0 = -\frac{1}{2}b_1 - \frac{3}{2}b_3 \tag{F-6}$$

$$\gamma_1 = \frac{1}{4}b_2 \tag{F-7}$$

$$\delta_{-1} = \frac{1}{2}b_1 + \frac{3}{2}b_3 \tag{F-8}$$

$$\delta_0 = \frac{1}{2} b_1 + \frac{3}{4} b_2 + \frac{3}{2} b_3 \qquad \text{(F-9)}$$

$$\delta_1 = -\frac{3}{16} b_1 - \frac{1}{2} b_2 - \frac{9}{16} b_3 \qquad \text{(F-10)}$$

In terms of these constants and those defined in Appendix E, the differential equations become

$$\frac{de^{(0)}}{d\tilde{t}} = C_1 \cos\omega^{(0)} + \mathcal{D}_2\left(\alpha_0 + \alpha_1 e^{(0)}\right) \qquad \text{(F-11)}$$

$$\frac{d\omega^{(0)}}{d\tilde{t}} = C_2 + C_2' e^{(0)} \sin\omega^{(0)} + \frac{C_2'' \sin\omega^{(0)}}{e^{(0)}} + \frac{\mathcal{D}_2}{e^{(0)}}\left(\gamma_0 + \gamma_1 e^{(0)}\right) \qquad \text{(F-12)}$$

$$\frac{di^{(0)}}{d\tilde{t}} = C_3 e^{(0)} \cos\omega^{(0)} \qquad \text{(F-13)}$$

$$\frac{d\Omega^{(0)}}{d\tilde{t}} = -C_4 + C_5 e^{(0)} \sin\omega^{(0)} \qquad \text{(F-14)}$$

$$\frac{d\beta^{(0)}}{d\tilde{t}} = \frac{\beta^{(0)}\mathcal{D}_2}{2}\left(\varrho_0 + \varrho_1 e^{(0)}\right) \qquad \text{(F-15)}$$

$$\frac{dM^{(0)}}{d\tilde{t}} = C_6 - 4C_7 e^{(0)} \sin\omega^{(0)} + \frac{C_7 \sin\omega^{(0)}}{e^{(0)}} + \mathcal{D}_2\left(\frac{\delta_{-1}}{e^{(0)}} + \delta_0 + \delta_1 e^{(0)}\right) \qquad \text{(F-16)}$$

Note that these equations are extensions of the oblateness-only equations (Appendix E) and are obtained by adding the drag perturbations to the respective oblateness perturbations.

F.2 SIMULTANEOUS SOLUTION FOR $e^{(0)}$ AND $\omega^{(0)}$

Applying the transformation equations

$$\xi = e^{(0)} \cos\omega^{(0)} \qquad \text{(E-15)}$$

$$\eta = e^{(0)} \sin\omega^{(0)} \qquad \text{(E-16)}$$

to Equations (4-71) and (4-73) results in

$$\frac{d\xi}{d\tilde{t}} = C_1 - C_2\eta + \mathcal{D}_2\left(\alpha_1\xi - \gamma_1\eta + \frac{\alpha_0\xi - \gamma_0\eta}{e^{(0)}}\right) \tag{F-17}$$

$$\frac{d\eta}{d\tilde{t}} = C_2\xi + \mathcal{D}_2\left(\alpha_1\eta + \gamma_1\xi + \frac{\alpha_0\eta + \gamma_0\xi}{e^{(0)}}\right) \tag{F-18}$$

These equations can be linearized by approximating $e^{(C)}$ by its epoch value $e_0^{(0)}$ (referred to as "backlining" $e^{(0)}$), resulting in

$$\frac{d\xi}{d\tilde{t}} = a + b\xi + c\eta \tag{F-19}$$

$$\frac{d\eta}{d\tilde{t}} = -c\xi + b\eta \tag{F-20}$$

where

$$a = C_1$$

$$b = \mathcal{D}_2\left(\frac{\alpha_1 e_0^{(0)} + \alpha_0}{e_0^{(0)}}\right)$$

$$c = -\left[\mathcal{D}_2\left(\frac{\gamma_1 e_0^{(0)} + \gamma_0}{e_0^{(0)}}\right) + C_2\right]$$

Differentiating the first equation, substituting from the second, and employing the differential operator notation of Appendix E yields

$$\left[\mathcal{D}^2 - 2b\mathcal{D} + (b^2+c^2)\right]\xi = -ab \tag{F-21}$$

The characteristic solution for this nonhomogeneous second-order equation is given as

$$\xi_c = \exp^{b\tilde{t}}\left[\lambda_1 \cos c\tilde{t} + \lambda_2 \sin c\tilde{t}\right]$$

where λ_1 and λ_2 are the integration constants, and the particular solution is

$$\xi_p = -\frac{ab}{b^2+c^2}$$

Hence, the complete solution to Equation (F-21) is given as

$$\xi = e_{xp}^{b\tilde{t}}\left[\lambda_1 \cos c\tilde{t} + \lambda_2 \sin c\tilde{t}\right] - \frac{ab}{b^2+c^2} \tag{F-22}$$

The solution for η is similarly obtained as

$$\eta = e_{xp}^{b\tilde{t}}\left[-\lambda_1 \sin c\tilde{t} + \lambda_2 \cos c\tilde{t}\right] - \frac{ac}{b^2+c^2} \tag{F-23}$$

The integration constants λ_1 and λ_2 can be obtained by evaluating Equations (F-22) and (F-23) at the epoch time t_0 resulting in

$$\lambda_1 = e_{xp}^{-b\tilde{t}_0}\left[\left(\xi_0 + \frac{ab}{b^2+c^2}\right)\cos c\tilde{t}_0 - \left(\eta_0 + \frac{ac}{b^2+c^2}\right)\sin c\tilde{t}_0\right] \tag{F-24}$$

$$\lambda_2 = e_{xp}^{-b\tilde{t}_0}\left[\left(\xi_0 + \frac{ab}{b^2+c^2}\right)\sin c\tilde{t}_0 + \left(\eta_0 + \frac{ac}{b^2+c^2}\right)\cos c\tilde{t}_0\right] \tag{F-25}$$

From Equations (E-15) and (E-16), the desired solutions are

$$e^{(0)} = \left(\xi^2 + \eta^2\right)^{1/2} \tag{4-72}$$

$$\omega^{(0)} = \tan^{-1}\left(\frac{\eta}{\xi}\right) \tag{4-74}$$

where ξ and η are given by Equations (F-22) and (F-23), respectively.

Recall that Equations (F-22) and (F-23), even though exact simultaneous solutions to Equations (F-19) and (F-20), are still approximate solutions to Equations (F-17) and (F-18) since these equations were linearized by backlining $e^{(0)}$. The validity of this approximation is demonstrated by the data in Table F-1, which consists of time-point comparisons between numerical integrations of Equations (F-17) and (F-18) and numerical evaluations of Equations (F-22) and (F-23). The data span a 60-day time period with an integration step size of 128 seconds and a print/evaluation step size of 0.5 day. The orbit eccentricity is 0.036.

In Reference 16, plots of the motion in the ξ, η-plane are given when considering oblateness only. The ξ, η solutions presented there are of the same form as those derived in Appendix E, i.e.,

t(days)	ξ	Numerical	Analytical	η	Numerical	Analytical
0		$-0.17073717\text{x}10^{-1}$	$-0.17073717\text{x}10^{-1}$		$0.32146791\text{x}10^{-1}$	$0.32146791\text{x}10^{-1}$
5		$-0.32414050\text{x}10^{-1}$	$-0.32415405\text{x}10^{-1}$		$0.13637429\text{x}10^{-1}$	$0.13638160\text{x}10^{-1}$
10		$-0.32363856\text{x}10^{-1}$	$-0.32367086\text{x}10^{-1}$		$-0.97996367\text{x}10^{-2}$	$-0.98002792\text{x}10^{-2}$
20		$0.41330305\text{x}10^{-2}$	$0.41331292\text{x}10^{-2}$		$-0.31283737\text{x}10^{-1}$	$-0.31290135\text{x}10^{-1}$
30		$0.30369370\text{x}10^{-1}$	$0.30377385\text{x}10^{-1}$		$-0.10532703\text{x}10^{-2}$	$-0.10545750\text{x}10^{-2}$
40		$0.87856799\text{x}10^{-2}$	$0.87895328\text{x}10^{-2}$		$0.27998938\text{x}10^{-1}$	$0.28007781\text{x}10^{-1}$
50		$-0.23692851\text{x}10^{-1}$	$-0.23703570\text{x}10^{-1}$		$0.13433323\text{x}10^{-1}$	$0.13440617\text{x}10^{-1}$
60		$-0.15714644\text{x}10^{-1}$	$-0.15727037\text{x}10^{-1}$		$-0.18930974\text{x}10^{-1}$	$-0.18943342\text{x}10^{-1}$

Table F-1. Comparison of Numerically Vs. Analytically Integrated Values of ξ, η

$$\xi = A \cos(Kt + \alpha) \tag{F-26}$$

$$\eta = A \sin(Kt + \alpha) + C/K \tag{F-27}$$

Plots of these solutions are circles centered at (0, C/K). The magnitude of $e^{(0)}$ (t) is given by the length of the line segment from the origin to the point (ξ(t), η(t)) on the circle, and the angle turned by this line (relative to the + ξ axis) measures $\omega^{(0)}$ (t).

Since drag perturbations tend to diminish $\underline{e}$, it was expected that the corresponding plots for $\underline{\xi}$ vs. $\underline{\eta}$, as given by Equations (F-22) and F-23), would depict this decaying effect while retaining the basic characteristics of the oblateness-only plots. Figure F-1 reveals that this is, indeed, true. The plot shows a 360-day variation of $\underline{\xi}$ with $\underline{\eta}$ for the orbit of 0.0364 eccentricity.

The solutions for $\underline{\xi}$, $\underline{\eta}$ given by Equations (F-22) and (F-23) represent the combined effects of oblateness and drag. Although these solutions are functionally different from the corresponding solutions for oblateness-only (as given by Equations (E-26) and (E-27)), they are identical in the limiting sense, i.e., as the drag perturbations tend to vanish. This can be seen by allowing $\underline{b}$ (or $\underline{D}_2$) to vanish in Equations (F-22) and (F-23).

$$\lim_{b \to 0} \xi = \lim_{b \to 0} \exp^{b\tau} \left[\lambda_1 \cos c\tau + \lambda_2 \sin c\tau\right] - \frac{ab}{b^2+c^2}$$

$$= \left[\xi_0 \cos C_2\tau_0 + \left(\eta_0 - \frac{C_1}{C_2}\right) \sin C_2\tau_0\right] \cos C_2\tau$$

$$+ \left[\xi_0 \sin C_2\tau_0 - \left(\eta_0 - \frac{C_1}{C_2}\right) \cos C_2\tau_0\right] \sin C_2\tau$$

which reduces to the oblateness solution

$$\xi = A \cos(C_2\tau + \alpha)$$

Similarly,

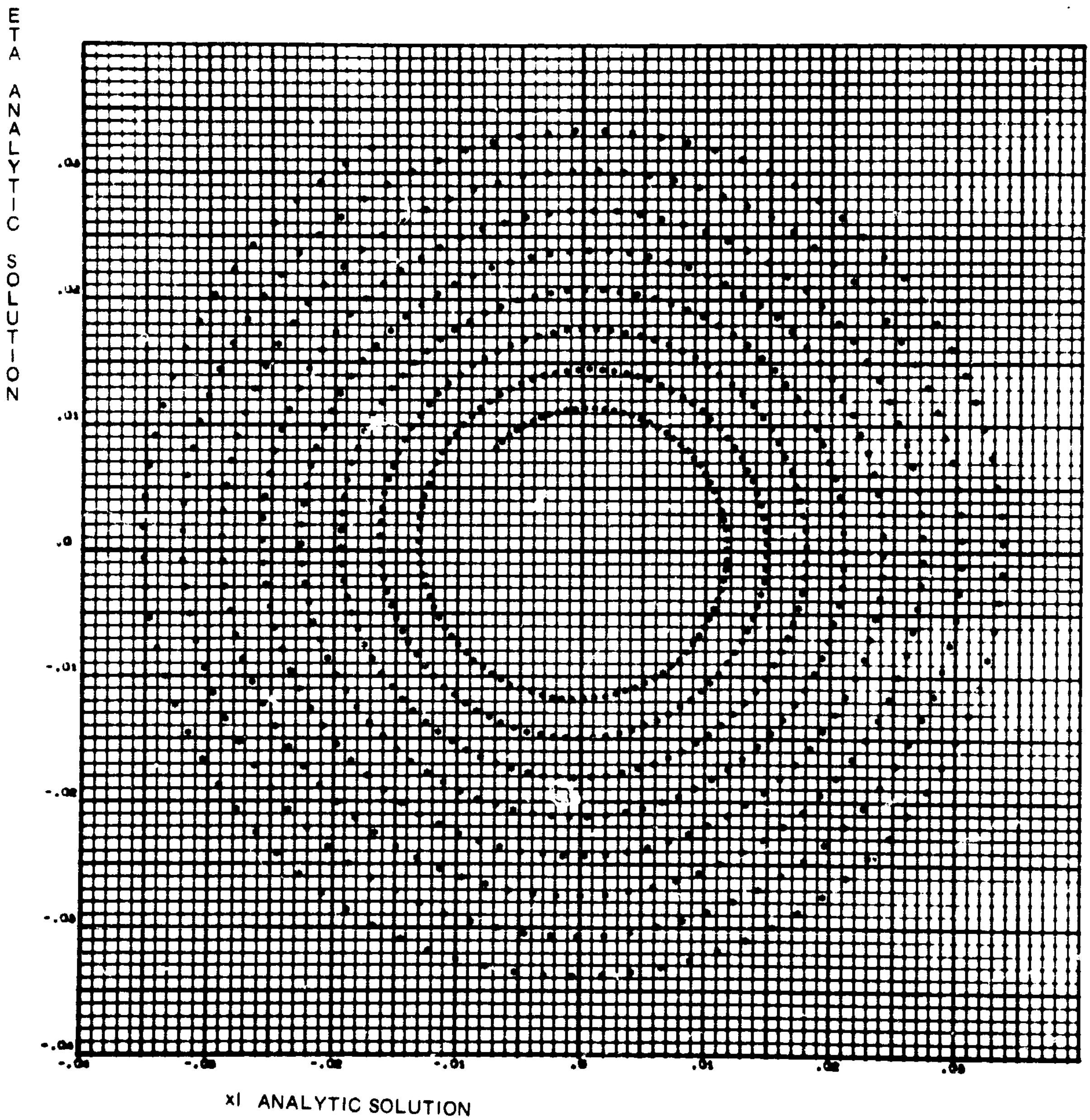


Figure F-1. Plot of ξ vs η as Given by Equations (F-22) and (F-23)

$$\lim_{b\to 0}\eta = \lim_{b\to 0} \exp^{b\tilde{\tau}}\left[-\lambda_1 \sin c\tilde{\tau} + \lambda_2 \cos c\tilde{\tau}\right] - \frac{ac}{b^2+c^2}$$

$$= \left[\xi_0 \cos C_2\tilde{\tau}_0 + \left(\eta_0 - \frac{C_1}{C_2}\right)\sin C_2\tilde{\tau}_0\right]\sin C_2\tilde{\tau}$$

$$+ \left[\xi_0 \sin C_2\tilde{\tau}_0 + \left(\eta_0 - \frac{C_1}{C_2}\right)\cos C_2\tilde{\tau}_0\right]\cos C_2\tilde{\tau} + \frac{C_1}{C_2}$$

which reduces to the oblateness solution

$$\eta = A \sin\left(C_2\tilde{\tau} + \alpha\right) + C_1/C_2$$

F.3 SOLUTION FOR $i^{(0)}$, $\Omega^{(0)}$

Since drag does not affect $\underline{i}$ and $\underline{\Omega}$ (at least when the drag model is tangential), the solutions for these elements remain the same as in the case for oblateness-only and are given by Equations (4-37) and (4-39), respectively.

F.4 SOLUTION FOR $B^{(0)}$

The solution for $B^{(0)}$ is obtained by approximating $e^{(0)}$ and $B^{(0)}$ by their epoch values on the right-hand side of Equation (F-15). The resulting equation can be directly integrated to yield

$$B^{(0)} = \frac{B_0^{0} D_2}{2}\left(\beta_0 + \beta_1 e_0^{(0)}\right)\left(\tilde{\tau} - \tilde{\tau}_0\right) \qquad (4\text{-}80)$$

(NOTE: Numerical results have verified the accuracy of using the backline epoch value to effect the integration.)

F.5 SOLUTION FOR $M^{(0)}$

Equation (F-16) can be rewritten (using Equation (E-15)) to yield

$$\frac{dM^{(0)}}{d\tilde{\tau}} = C_6 - \left(4C_7 - C_8\right)\eta + D_2\left(\frac{\delta_{-1}}{e^{(0)}} + \delta_0 + \delta_1 e^{(0)}\right)$$

The epoch value of $e^{(0)}$, i.e. $e_0^{(0)}$, will again be employed, and the solution form is given by

$$M^{(0)} = \left[C_6 + D_2\left(\frac{\delta_{-1}}{e_0^{(0)}} + \delta_0 + \delta_1 e_0^{(0)}\right)\right]\tilde{\tau} - \left(4C_7 - C_8\right)\int \eta \, d\tilde{\tau} + K$$

where $\underline{K}$ is the integration constant.

From Equation (F-23)

$$\eta = e_{xp}^{\,b\tilde{\tau}}\left[-\lambda_1 \sin c\tilde{\tau} + \lambda_2 \cos c\tilde{\tau}\right] - \frac{ac}{b^2+c^2}$$

so that

$$\int \eta \, d\tilde{\tau} = \frac{e_{xp}^{\,b\tilde{\tau}}}{b^2+c^2}\left[(\lambda_2 c - \lambda_1 b)\sin c\tilde{\tau} + (\lambda_2 b + \lambda_1 c)\cos c\tilde{\tau}\right] - \frac{ac\tilde{\tau}}{b^2+c^2}$$

Hence,

$$M^{(0)} = M_0^{(0)} + \left\{C_6 + (4C_7 - C_8)\frac{ac}{b^2+c^2} + D_2\left[\frac{d_{-1}}{e_0^{(0)}} + d_0 + d_1 e_0^{(0)}\right]\right\}(\tilde{\tau} - \tilde{\tau}_0) \tag{4-83}$$

$$- (4C_7 - C_8)\left\{\frac{e_{xp}^{\,b\tilde{\tau}}}{b^2+c^2}\left[(\lambda_2 c - \lambda_1 b)\sin c\tilde{\tau} + (\lambda_2 b + \lambda_1 c)\cos c\tilde{\tau}\right]\right.$$

$$\left. - \frac{e_{xp}^{\,b\tilde{\tau}_0}}{b^2+c^2}\left[(\lambda_2 c - \lambda_1 b)\sin c\tilde{\tau}_0 + (\lambda_2 b + \lambda_1 c)\cos c\tilde{\tau}_0\right]\right\}$$

where the integration constant $\underline{K}$ has been evaluated at epoch time $\underline{t}_0$. As mentioned in Paragraph 4.3.1.2, the Keplerian change in $\underline{M}$ that takes place in the time interval $(t-t_0)$ must be added to the above equation.

APPENDIX G - PROCEDURE FOR DETERMING FUNCTIONAL FORM OF $C(\tilde{t})$

G.1 INTRODUCTION

An outline of the procedures necessary for obtaining the $E^{(2)}$ solutions was given in Paragraph 4.3.3. Inherent within these procedures are the steps necessary for determining expressions for $C(\tilde{t})$. In this appendix, the details that must be considered in implementing these steps are discussed. Each step is illustrated by working out the solution of $C(\tilde{t})$ for the inclination i, denoted by $C_i(\tilde{t})$. Because of the complexity encountered in solving for $C(\tilde{t})$, a number of simplifying assumptions are made. However, the final expression for $C_i(\tilde{t})$ is shown to agree remarkably well with computed values from the GENPUR program, thereby indicating that highly refined equations may not be necessary in determining the functional form of $C(\tilde{t})$.

G.2 SECOND-ORDER EXPANSION OF THE BASIC DIFFERENTIAL EQUATIONS

The first step in the procedure is to expand, to order ϵ^2, the basic differential equations of perturbed motion. These equations for the gravity perturbation are Equations (2-34) through (2-39). Consider, for instance, the J_2 portion of the equation for i, Equation (2-36).

$$\frac{di}{dt} = -\frac{1}{2}\epsilon\sqrt{\mu}\, r_e^2 B^7 (1+e\cos\nu)^3 (1-e^2)^{-7/2} \sin 2i \sin 2u \tag{2-36a}$$

The required expansions may be obtained using the methods of Appendix B, i.e.,

$$\left.\begin{aligned}
B^7 &= B^{(0)7} + \epsilon\, 7B^{(0)6}B^{(1)} \\
(1+e\cos\nu)^3 &= (1+e^{(0)}\cos\nu^{(0)})^3 + \epsilon\left[3(1+e^{(0)}\cos\nu^{(0)})^2(e^{(1)}\cos\nu^{(0)} - e^{(0)}\nu^{(1)}\sin\nu^{(0)})\right] \\
(1-e^2)^{-7/2} &= (1-e^{(0)2})^{-7/2} + \epsilon\left[7(1-e^{(0)2})^{-9/2}e^{(0)}e^{(1)}\right] \\
\sin 2i &= \sin 2i^{(0)} + \epsilon\, 2i^{(1)}\cos 2i^{(0)} \\
\sin 2u &= \sin 2u^{(0)} + \epsilon\, 2u^{(0)}\cos 2u^{(0)}
\end{aligned}\right\} \tag{G-1}$$

These expanded expressions can be written in the form of binomials

$$\left.\begin{aligned}
\beta^7 &= X_1 + \epsilon Y_1 \\
(1+e\cos\nu)^3 &= X_2 + \epsilon Y_2 \\
(1-e^2)^{-7/2} &= X_3 + \epsilon Y_3 \\
\sin 2i &= X_4 + \epsilon Y_4 \\
\sin 2u &= X_5 + \epsilon Y_5
\end{aligned}\right\} \qquad \text{(G-2)}$$

where

$$\begin{aligned}
X_1 &= \beta^{(0)7} \\
X_2 &= (1+e^{(0)}\cos\nu^{(0)})^3 \\
X_3 &= (1-e^{(0)2})^{-7/2} \\
X_4 &= \sin 2i^{(0)} \\
X_5 &= \sin 2u^{(0)}
\end{aligned}$$

and

$$\begin{aligned}
Y_1 &= 7\beta^{(0)6}\beta^{(1)} \\
Y_2 &= 3(1+e^{(0)}\cos\nu^{(0)})^2(e^{(1)}\cos\nu^{(0)} - e^{(0)}\nu^{(1)}\sin\nu^{(0)}) \\
Y_3 &= 7(1-e^{(0)2})^{-9/2}e^{(0)}e^{(1)} \\
Y_4 &= 2i^{(1)}\cos 2i^{(0)} \\
Y_5 &= 2u^{(1)}\cos 2u^{(0)}
\end{aligned}$$

The basic differential equation expanded to order ϵ, in terms of these binomials, is merely

$$\frac{di}{d\tau} = -\frac{1}{2}\epsilon\sqrt{\mu}\, r_e^2 (X_1 X_2 X_3 X_4 X_5) \qquad \text{(2-36b)}$$

The same equation expanded to order ϵ^2 is

$$\frac{di}{dt} = -\frac{1}{2}\epsilon\sqrt{\mu}\, r_e^2 (X_1 X_2 X_3 X_4 X_5) - \frac{1}{2}\epsilon^2\sqrt{\mu}\, r_e^2 (Y_1 X_2 X_3 X_4 X_5 + X_1 Y_2 X_3 X_4 X_5 + X_1 X_2 Y_3 X_4 X_5 + X_1 X_2 X_3 Y_4 X_5 + X_1 X_2 X_3 X_4 Y_5) \tag{G-3}$$

or, in more concise notation

$$\frac{di}{dt} = \epsilon f_i + \epsilon^2 g_i$$

which is equivalent to Equation (4-115).

(NOTE: Equation (G-3) illustrates a convenient method for expanding the basic differential equation to order ϵ^2. First, convert the expansion of each term to the binomial form of Equations (G-2). Then, insert the y part of each binomial into the differential equation, one-at-a-time, to get the ϵ^2 expansion.)

The complex nature of Equation (G-3) is veiled by the notation. For instance, y_1 is

$$\begin{aligned}
Y_1 &= 7B^{(0)6}B^{(1)} = 7B^{(0)6}\left[f_N(B) + C_3(\tilde{E})\right] = 7B^{(0)6}\left\{K_2(B)\left[I_2(B) - S_2(B)n^{(0)}\tilde{t}\right] + C_8(\tilde{E})\right\} \\
&= 7B^{(0)6}\Big\{K_2(B)\Big[e^{(0)}\big(6\sin^2 i^{(0)} \sin v^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} - 6\sin^2 i^{(0)} \sin^2 v^{(0)} \sin^2\omega^{(0)} \cos v^{(0)} \\
&\quad + 3\sin^2 i^{(0)} \sin^2 v^{(0)} \cos v^{(0)} - 6\sin^2 i^{(0)} \sin^3 v^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} \\
&\quad + 3\sin^2 i^{(0)} \sin^2\omega^{(0)} \cos v^{(0)} - \cos v^{(0)}\big) + 2\sin^2 i^{(0)} \sin v^{(0)} \sin\omega^{(0)} \cos v^{(0)} \cos\omega^{(0)} \\
&\quad + \sin^2 i^{(0)} \sin^2 v^{(0)} - 2\sin^2 i^{(0)} \sin^2 v^{(0)} \sin^2\omega^{(0)}\Big] + C_8(\tilde{E})\Big\}
\end{aligned}$$

The remaining y_i terms are equally complex. Some simplification is possible, depending upon accuracy required, by neglecting terms containing certain powers of $e^{(0)}$ and by holding certain elements constant, such as i. (Use of the FORMAC program would allow retention of these terms in the expansion.)

One assumption that will be made at this point in working out the solution for the element i is that all terms involving powers of $e^{(0)}$ will be ignored (i.e., $e^{(0)}$, $e^{(0)2}$, etc.). Thus,

$$X_3 = (1 - e^{(0)2})^{-7/2} = 1.0$$

$$Y_3 = 0$$

The $\underline{f}_i$ and $\underline{g}_i$ functions become

$$f_i = -\frac{1}{2}\sqrt{\mu}\, r_e^2 (X_1 X_2 X_4 X_5)$$

$$g_i = -\frac{1}{2}\sqrt{\mu}\, r_e^2 (Y_1 X_2 X_4 X_5 + X_1 Y_2 X_4 X_5 + X_1 X_2 Y_4 X_5 + X_1 X_2 X_4 Y_5)$$

where, recall, $\underline{f}_i$ is the expansion of the basic differential equation to order $\underline{\epsilon}$, and $\underline{g}_i$ is the expansion to order $\underline{\epsilon}^2$. To further simplify for later analysis, let

$$d_1 = Y_1 X_2 X_4 X_5$$

$$d_2 = X_1 Y_2 X_4 X_5$$

$$d_4 = X_1 X_2 Y_4 X_5$$

$$d_5 = X_1 X_2 X_4 Y_5$$

so that

$$g_i = -\frac{1}{2}\sqrt{\mu}\, r_e^2 (d_1 + d_2 + d_4 + d_5)$$

G.3 PARTIAL DERIVATIVE OF $E^{(1)}$

The next step in the solution process is to compute the partial derivatives of the $E^{(1)}$ solutions with respect to $\underline{\tilde{t}}$. Recall from Equation (4-28) that these solutions have the form

$$E^{(1)} = K_2(E)\left[I_2(E) - S_2(E) m^{(0)} \tilde{t}\right] + \ldots + C_E(\tilde{t})$$

or merely

$$E^{(1)} = f_N(E) + C_E(\tilde{t})$$

The partial derivative is then

$$\frac{\partial E^{(1)}}{\partial \tilde{t}} = \frac{\partial f_N}{\partial \tilde{t}} + \frac{\partial C(\tilde{t})}{\partial \tilde{t}}$$

so that the crux of this step is in determining $\partial f_N/\partial \tilde{t}$. For those $E^{(1)}$ solutions that have no secular terms in $I_2(E)$ (or $I_3(E)$), such as $\underline{B}$, this step is relatively straightforward, but tedious. The element $\underline{i}$ has no secular term in $I_2(i)$. Neglecting $\underline{J}_3$ terms, its super-one solution is

$$i^{(1)} = K_2(i) I_2(i) + C_i(\tilde{t})$$

such that

$$f_N(i) = K_2(i) I_2(i)$$

The partial derivative of $f_N(i)$ is merely

$$\frac{\partial f_N(i)}{\partial \tilde{t}} = \frac{\partial}{\partial \tilde{t}} \left[K_2(i) I_2(i) \right]$$

One question that arises at this point is whether

$$K_2(i) = \frac{\beta^{(0)4} r_e^2}{2(1-e^{(0)2})^2}$$

can be considered constant. The $B^{(0)}$ term is constant if gravity perturbations only are being considered, and can be treated as constant for fairly long intervals even when drag is present. The $e^{(0)}$ term, however, has an important long-periodicity due to $\underline{J}_3$. It has not yet been determined whether the $e^{(0)}$ variation with $\tilde{\underline{t}}$ should be included. However, since the example is neglecting $e^{(0)^2}$ terms and is considering only gravity perturbations, $K_2(i)$ will be treated as a constant.

Evaluation of $\partial I_2(i)/\partial \tilde{t}$ requires further analysis. $I_2(i)$ is composed of terms such as

$$-2e^{(0)} \sin 2i^{(0)} \sin \omega^{(0)} \cos \omega^{(0)} \sin \nu^{(0)}$$

so that the partial derivative involves determining $de^{(0)}/d\tilde{t}$, $di^{(0)}/d\tilde{t}$, $d\omega^{(0)}/d\tilde{t}$, and $d\nu^{(0)}/d\tilde{t}$. The first three of these derivatives are available from Equations (4-32), (4-36), and (4-34), respectively. Determining $d\nu^{(0)}/d\tilde{t}$ is more complex, however, since the explicit expression for $\nu^{(0)}$ is not immediately available. To obtain such an expression, the Fourier-Bessel expansion may be used, i.e.,

$$\nu^{(0)} = M^{(0)} + 2e^{(0)} \sin M^{(0)} + \ldots$$

The $d\nu^{(0)}/d\tilde{t}$ derivative can then be expressed in terms of $dM^{(0)}/d\tilde{t}$ and $de^{(0)}/d\tilde{t}$

$$\frac{d\nu^{(0)}}{d\tilde{t}} = \frac{dM^{(0)}}{d\tilde{t}}\left(1 + 2e^{(0)} \cos M^{(0)}\right) + \frac{de^{(0)}}{d\tilde{t}}\, 2 \sin M^{(0)} + \ldots \qquad \text{(G-4)}$$

If only the J_2 gravity perturbation is being considered ($de^{(0)}/d\tilde{t} = 0$) and terms involving $e^{(0)}$ are neglected, then

$$\frac{d\nu^{(0)}}{d\tilde{t}} = \frac{dM^{(0)}}{d\tilde{t}}$$

An expression of $dM^{(0)}/d\tilde{t}$ is provided by Equation (4-43). Notice that Equation (4-43) does not include the Keplerian variation of $M^{(0)}$, which is dependent on $\underline{t}$ rather than $\underline{\tilde{t}}$.

Even with these simplifications, the partial derivative of only one term of $I_2(i)$ is very complex, i.e.,

$$\frac{dI_2(i)}{d\tilde{t}} = -\sin 2i^{(0)} \sin 2\omega^{(0)} \sin \nu^{(0)} \frac{de^{(0)}}{d\tilde{t}} - 2e^{(0)} \cos 2i^{(0)} \sin 2\omega^{(0)} \sin \nu^{(0)} \frac{di^{(0)}}{d\tilde{t}}$$
$$- 2e^{(0)} \sin 2i^{(0)} \cos 2\omega^{(0)} \sin \nu^{(0)} \frac{d\omega^{(0)}}{d\tilde{t}} - e^{(0)} \sin 2i^{(0)} \sin 2\omega^{(0)} \cos \nu^{(0)} \frac{d\nu^{(0)}}{d\tilde{t}} + \ldots$$

Substituting Equations (4-32), (4-36), (4-34), and (G-4) yields

$$\frac{dI_2(i)}{d\tilde{t}} = -\sin 2i^{(0)} \sin 2\omega^{(0)} \sin \nu^{(0)} \beta^{(0)6} (J_3/J_2) r_e^3 n^{(0)} \left(1 - e^{(0)2}\right)^{-2} \sin i^{(0)} \cos \omega^{(0)}$$
$$- 2e^{(0)} \cos 2i^{(0)} \sin 2\omega^{(0)} \sin \nu^{(0)} \beta^{(0)6} (J_3/J_2) r_e^3 n^{(0)} e^{(0)} \left(1 - e^{(0)2}\right)^{-3} \cos i^{(0)} \cos \omega^{(0)} \left(1 - \tfrac{5}{4} \sin^2 i^{(0)}\right)$$
$$- 2e^{(0)} \sin 2i^{(0)} \cos 2\omega^{(0)} \sin \nu^{(0)} \Big\{ \beta^{(0)4} r_e^2 n^{(0)} \left(1 - e^{(0)2}\right)^{-2} \left(2 - \tfrac{5}{2} \sin^2 i^{(0)}\right) + \beta^{(0)6} (J_3/J_2) r_e^3 n^{(0)}$$
$$\left(1 - e^{(0)2}\right)^{-3} \Big[\tfrac{35}{4} e^{(0)} \sin i^{(0)} \sin \omega^{(0)} \cos^2 i^{(0)} - e^{(0)} \csc i^{(0)} \sin \omega^{(0)} + \tfrac{1}{e^{(0)}} \sin i^{(0)} \sin \omega^{(0)}$$
$$\left(1 - \tfrac{5}{4} \sin^2 i^{(0)}\right) \Big] \Big\} - e^{(0)} \sin 2i^{(0)} \sin 2\omega^{(0)} \cos \nu^{(0)} \left(1 + 2e^{(0)} \cos M^{(0)}\right) \Big\{ \beta^{(0)4} r_e^2 n^{(0)}$$
$$\left(1 - e^{(0)2}\right)^{-3/2} \left(1 - \tfrac{3}{2} \sin^2 i^{(0)}\right) - \tfrac{4}{3} \beta^{(0)6} (J_3/J_2) r_e^3 n^{(0)} \frac{1}{e^{(0)}} \left(1 - e^{(0)2}\right)^{-5/2}$$
$$\Big[-3e^{(0)2} \sin i^{(0)} \sin \omega^{(0)} \left(1 - \tfrac{5}{4} \sin^2 i^{(0)}\right) + \tfrac{3}{4} \sin i^{(0)} \sin \omega^{(0)} \left(1 - \tfrac{5}{4} \sin^2 i^{(0)}\right) \Big] \Big\}$$

+ derivatives of 8 more terms.

As noted, this is only the expansion of one term in nine. Thus, evaluating the derivative of even the simple type $I_2(E)$, which has no secular terms, quickly becomes very involved.

To return to the more general form of an $I_2(E)$ integral, which contains secular terms also, consider the equation for $\Omega^{(1)}$, i.e.,

$$\Omega^{(1)} = K_2(\Omega)\left[I_2(\Omega) - S_2(\Omega)n^{(0)}\bar{t}\right] + \ldots + C_\Omega(\bar{t})$$

The terms within brackets are an expression of nonsecular (with respect to $\bar{t}$) parts of an integration. For the element $\underline{i}$, there were no secular terms in $I_2(i)$ so that $S_2(i)$ was equal to zero. For the element $\underline{\Omega}$, however, the $I_2(\Omega)$ integral contains a secular term, $-1/2\ v^{(0)} \cos i^{(0)}$. To remove this secularity from the brackets, $S_2(\Omega)$ is

$$S_2(\Omega) = -\frac{1}{2}\cos i^{(0)}$$

Assume for the moment that $I_2(\Omega)$ was composed only of the secular term, and call it $I'_2(\Omega)$. Then

$$\left[I_2'(\Omega) - S_2(\Omega)n^{(0)}\bar{t}\right] = -\frac{1}{2}\cos i^{(0)}\left(v^{(0)} - n^{(0)}\bar{t}\right)$$

To compute $\partial f_N(\Omega)/\partial\bar{t}$, the partial derivative of the right-hand side of this equation must be evaluated. Substituting Equation (4-45) for $v^{(0)}$ yields

$$\left(v^{(0)} - n^{(0)}\bar{t}\right) = M^{(0)} + 2e^{(0)}\sin M^{(0)} + \frac{5}{4}e^{(0)2}\sin 2M^{(0)} + \ldots - n^{(0)}\bar{t} \tag{G-5}$$

Next, using a functional form of Equation (4-44), i.e.,

$$M^{(0)} = n^{(0)}\bar{t} + M^{(0)}(\bar{t}) \tag{4-44a}$$

Substituting Equation (4-44a) into Equation (G-5)

$$\begin{aligned}(\nu^{(0)} - n^{(0)}\tilde{t}) &= n^{(0)}\tilde{t} + M^{(0)}(\tilde{t}) + 2e^{(0)}\sin M^{(0)} + \frac{5}{4}e^{(0)2}\sin 2M^{(0)} + \ldots - n^{(0)}\tilde{t} \\ &= M^{(0)}(\tilde{t}) + 2e^{(0)}\sin M^{(0)} + \frac{5}{4}e^{(0)2}\sin 2M^{(0)} + \ldots\end{aligned} \tag{G-6}$$

and taking the partial derivative of Equation (G-6) with respect to $\tilde{t}$

$$\frac{d}{d\tilde{t}}(\nu^{(0)} - n^{(0)}\tilde{t}) = \frac{dm^{(0)}}{d\tilde{t}}(1 + 2e^{(0)}\cos M^{(0)}) + \frac{de^{(0)}}{d\tilde{t}}2\sin M^{(0)} + \ldots$$

which is the same as Equation (G-4). Thus, the presence of secular terms within an $I_2(E)$ integral presents no new problems, but merely adds similar terms.

Performing the differentiation of all terms of each $E^{(1)}$ solution with respect to $\tilde{t}$ and making the necessary substitutions will result in extremely long expressions. (The need for an automatic manipulation language such as FORMAC in this step is indeed evident.)

To continue with the example for i, neglect terms of Equation (4-89) of order $e^{(0)}$ (and higher) and omit the J_3 terms

$$\begin{aligned}i^{(1)} = K_2(i)[&-2\sin 2i^{(0)}\sin\omega^{(0)}\cos\omega^{(0)}\sin\nu^{(0)}\cos\nu^{(0)} - \sin 2i^{(0)}\sin^2\nu^{(0)} \\ &+ 2\sin 2i^{(0)}\sin^2\omega^{(0)}\sin^2\nu^{(0)}] + C_i(\tilde{t})\end{aligned}$$

Assume also that

$$\frac{di^{(0)}}{d\tilde{t}} = 0$$

$$\frac{dK_2(i)}{d\tilde{t}} = 0$$

The $\partial f_N(i)/\partial\tilde{t}$ derivative is then

$$\frac{df_N(i)}{d\tilde{t}} = K_2(i)\Big\{\Big[-2\sin 2i^{(0)}\cos 2\omega^{(0)}\sin\nu^{(0)}\cos\nu^{(0)} + 2\sin 2i^{(0)}\sin^2\nu^{(0)}\sin 2\omega^{(0)}\Big]\frac{d\omega^{(0)}}{d\tilde{t}}$$

$$-\Big[\sin 2i^{(0)}\sin 2\omega^{(0)}\cos 2\nu^{(0)} + \sin 2i^{(0)}\sin 2\nu^{(0)}\cos 2\omega^{(0)}\Big]\frac{d\nu^{(0)}}{d\tilde{t}}\Big\} \qquad \text{(G-7)}$$

(To facilitate work in the next step, expressions for $d\omega^{(0)}/d\tilde{t}$ and $d\nu^{(0)}/d\tilde{t}$ will not be inserted at this point.)

G.4 SECULAR PARTS OF $\bar{t}$ INTEGRALS

A second application of the first uniformity condition requires determination of secular parts of the integration of $\underline{g}$ and $\partial f_N/\partial\tilde{t}$ with respect to $\underline{\bar{t}}$. However, $\underline{g}$ and $\partial f_N/\partial\tilde{t}$ are primarily functions of $\nu^{(0)}$ rather than $\underline{\bar{t}}$. Thus, it is more convenient to change the variable of integration from $\underline{\bar{t}}$ to $\nu^{(0)}$ by the relation

$$d\bar{t} = \frac{(1-e^{(0)2})^{3/2}\, d\nu^{(0)}}{\sqrt{\mu}\, \beta^{(0)3}(1+e^{(0)}\cos\nu^{(0)})^2} \qquad \text{(G-8)}$$

The following integrals must now be evaluated and the secular parts determined

$$\int g\, d\bar{t} = \int g\, \frac{(1-e^{(0)2})^{3/2}}{\sqrt{\mu}\, \beta^{(0)3}(1+e^{(0)}\cos\nu^{(0)})^2}\, d\nu^{(0)}$$

$$\int \frac{df_N}{d\tilde{t}}\, d\bar{t} = \int \frac{df_N}{d\tilde{t}}\, \frac{(1-e^{(0)2})^{3/2}}{\sqrt{\mu}\, \beta^{(0)3}(1+e^{(0)}\cos\nu^{(0)})^2}\, d\nu^{(0)}$$

This step is straightforward, but very tedious if no simplifying assumptions are made. FORMAC has been modified to evaluate these types of integrals and to then extract the secular terms.

To continue with the example for the element i, consider the integral of g_i

$$g_i = -\frac{1}{2}\sqrt{\mu}\, r_e^2 (d_1 + d_2 + d_4 + d_5)$$

There are some terms common to the d_i which may be extracted, i.e.,

$$g_i = -\frac{1}{2}\sqrt{\mu}\, r_e^2 \beta^{(0)6}(1+e^{(0)}\cos\nu^{(0)})^2(z_1 + z_2 + z_4 + z_5)$$

where

$$z_1 = 3\beta^{(0)}\sin 2i^{(0)}\sin 2u^{(0)}\left(e^{(1)}\cos\nu^{(0)} - e^{(0)}\nu^{(1)}\sin\nu^{(0)}\right)$$

$$z_2 = 7\beta^{(1)}(1+e^{(0)}\cos\nu^{(0)})\sin 2i^{(0)}\sin 2u^{(0)}$$

$$z_4 = 2\beta^{(0)}i^{(1)}(1+e^{(0)}\cos\nu^{(0)})\sin 2u^{(0)}\cos 2i^{(0)}$$

$$z_5 = 2\beta^{(0)}u^{(1)}(1+e^{(0)}\cos\nu^{(0)})\sin 2i^{(0)}\cos 2u^{(0)}$$

Then,

$$\int g_i\, d\bar{t} = -\frac{1}{2}\sqrt{\mu}\, r_e^2 \beta^{(0)6}\int(1+e^{(0)}\cos\nu^{(0)})^2(z_1 + z_2 + z_4 + z_5)\, d\bar{t} \qquad \text{(G-9)}$$

Neglecting the $(1-e^{(0)2})^{3/2}$ term in Equation (G-8) and substituting into Equation (G-9) yields

$$\int g_i\, d\bar{t} = -\frac{1}{2} r_e^2 \beta^{(0)3}\int\left[z_1 + z_2 + z_4 + z_5\right] d\nu^{(0)}$$

To evaluate integrals of each z_i term, without the use of FORMAC, all terms containing $e^{(0)}$ and higher powers of $e^{(0)}$ are ignored. The expression for z_1 becomes

$$z_1 = 3\beta^{(0)}\sin 2i^{(0)}\sin 2u^{(0)}e^{(1)}\cos\nu^{(0)}$$

In obtaining the value of $e^{(1)}$ to be used in $\underline{z}_1$, only the $\underline{J}_2$ portion is considered, i.e.,

$$
\begin{aligned}
e^{(1)} &= K_2(e)\, I_2(e) + C_e(\Xi) \\
&= \beta^{(0)4} r_e^2 \Big[-4\sin^2 i^{(0)} \sin\nu^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} + \tfrac{14}{3}\sin^2 i^{(0)} \sin^2\nu^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} \\
&\quad - \tfrac{7}{3}\sin^2 i^{(0)} \sin^2\nu^{(0)} \cos\nu^{(0)} + \tfrac{14}{3}\sin^2 i^{(0)} \sin^3\nu^{(0)} \sin\omega^{(0)} \cos\omega^{(0)} \\
&\quad - \tfrac{5}{3}\sin^2 i^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} - \tfrac{2}{3}\sin^2 i^{(0)} \cos\nu^{(0)} + \cos\nu^{(0)} \Big] \\
&\quad + C_e(\Xi)
\end{aligned}
$$

The expansion of $\cos\nu^{(0)} \sin 2u^{(0)}$ is

$$
\cos\nu^{(0)} \sin 2u^{(0)} = 2\cos 2\omega^{(0)} \sin\nu^{(0)} \cos^2\nu^{(0)} + \sin 2\omega^{(0)} \cos\nu^{(0)} - 2\sin 2\omega^{(0)} \cos\nu^{(0)} \sin^2\nu^{(0)}
$$

Hence,

$$
\begin{aligned}
z_1 = 3r_e^2 \beta^{(0)5} \sin 2i^{(0)} &\Big[2\cos 2\omega^{(0)} \sin\nu^{(0)} \cos^2\nu^{(0)} + \sin 2\omega^{(0)} \cos\nu^{(0)} - 2\sin 2\omega^{(0)} \cos\nu^{(0)} \sin^2\nu^{(0)} \Big] \\
&\Big[-2\sin^2 i^{(0)} \sin\nu^{(0)} \sin 2\omega^{(0)} + \tfrac{14}{3}\sin^2 i^{(0)} \sin^2\nu^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} - \tfrac{7}{3}\sin^2 i^{(0)} \sin^2\nu^{(0)} \cos\nu^{(0)} \\
&+ \tfrac{7}{3}\sin^2 i^{(0)} \sin^3\nu^{(0)} \sin 2\omega^{(0)} - \tfrac{5}{3}\sin^2 i^{(0)} \sin^2\omega^{(0)} \cos\nu^{(0)} - \tfrac{2}{3}\sin^2 i^{(0)} \cos\nu^{(0)} + \cos\nu^{(0)} \\
&+ \frac{C_e(\Xi)}{\beta^{(0)4} r_e^2} \Big]
\end{aligned}
$$

Hereafter, only those terms which are secular in $\nu^{(0)}$, after integrating with respect to $\nu^{(0)}$, will be retained. Such terms are of the form $\cos^M \nu^{(0)} \sin^N \nu^{(0)}$, where $\underline{M}$ and $\underline{N}$ are even integers (or zero). Thus, the secular terms in $\underline{z}_1$ are

$$
\begin{aligned}
z_{1_{SEC}} = 3 r_e^2 B^{(0)5} \sin 2i^{(0)} \Big[& -4 \cos 2\omega^{(0)} \sin 2\omega^{(0)} \sin^2 i^{(0)} \sin^2 \nu^{(0)} \cos^2 \nu^{(0)} \\
& + \frac{14}{3} \cos 2\omega^{(0)} \sin 2\omega^{(0)} \sin^2 i^{(0)} \sin^4 \nu^{(0)} \cos^2 \nu^{(0)} + \frac{14}{3} \sin^2 i^{(0)} \sin^2 \omega^{(0)} \sin 2\omega^{(0)} \sin^2 \nu^{(0)} \cos^2 \nu^{(0)} \\
& - \frac{7}{3} \sin^2 i^{(0)} \sin 2\omega^{(0)} \sin^2 \nu^{(0)} \cos^2 \nu^{(0)} - \frac{5}{3} \sin^2 i^{(0)} \sin^2 \omega^{(0)} \sin 2\omega^{(0)} \cos^2 \nu^{(0)} \\
& - \frac{2}{3} \sin^2 i^{(0)} \sin 2\omega^{(0)} \cos^2 \nu^{(0)} + \sin 2\omega^{(0)} \cos^2 \nu^{(0)} - \frac{28}{3} \sin^2 i^{(0)} \sin^2 \omega^{(0)} \sin 2\omega^{(0)} \sin^4 \nu^{(0)} \cos^2 \nu^{(0)} \\
& + \frac{14}{3} \sin^2 i^{(0)} \sin 2\omega^{(0)} \cos^2 \nu^{(0)} \sin^4 \nu^{(0)} + \frac{10}{3} \sin^2 i^{(0)} \sin^2 \omega^{(0)} \sin 2\omega^{(0)} \cos^2 \nu^{(0)} \sin^2 \nu^{(0)} \\
& + \frac{4}{3} \sin^2 i^{(0)} \sin 2\omega^{(0)} \cos^2 \nu^{(0)} \sin^2 \nu^{(0)} - 2 \sin 2\omega^{(0)} \cos^2 \nu^{(0)} \sin^2 \nu^{(0)} \Big]
\end{aligned}
$$

Integration and simplication yields

$$
\int z_{1_{SEC}} \, d\tilde{t} = \frac{3}{8} r_e^2 B^{(0)5} \sin 2i^{(0)} \sin 2\omega^{(0)} \left(2 - 3 \sin^2 i^{(0)}\right) \nu^{(0)}
$$

By similar procedures, it is found that

$$
\int z_{2_{SEC}} \, d\tilde{t} = 0
$$

$$
\int z_{4_{SEC}} \, d\tilde{t} = 0
$$

$$
\int z_{5_{SEC}} \, d\tilde{t} = 0
$$

where in the $\underline{z}_5$ integral $u^{(1)} = \nu^{(1)} + \omega^{(1)}$. The secular part of the $\tilde{\underline{t}}$ integration of $\underline{g}_1$ is then

$$
\int g_{i_{SEC}} \, d\tilde{t} = -\frac{3}{16} r_e^4 B^{(0)8} \sin 2i^{(0)} \sin 2\omega^{(0)} \left(2 - 3 \sin^2 i^{(0)}\right) \nu^{(0)} \qquad \text{(G-10)}
$$

Next, consider the integral of $\partial f_N/\partial \tilde{t}$ with respect to $\tilde{\underline{t}}$. Assume in Equation (G-7) that $d\omega^{(0)}/d\tilde{t}$ and $d\nu^{(0)}/d\tilde{t}$ are constant with respect to a $\tilde{\underline{t}}$ integration. Referring to Equation (4-34), it is obvious that the secular rate of change of $\omega^{(0)}$ with respect to $\tilde{\underline{t}}$ does not depend upon $\tilde{\underline{t}}$. It is not so obvious for $d\nu^{(0)}/d\tilde{t}$, since $\nu^{(0)}$ has a Keplerian variation that depends upon $\bar{t}$. In functional form

$$\nu^{(0)} = n^{(0)}\bar{t} + f(\tilde{t})$$

However, its derivative with respect to $\bar{t}$ is not a function of $\bar{t}$, i.e.,

$$\frac{d\nu^{(0)}}{d\bar{t}} = f'(\tilde{t})$$

Consquently, $d\nu^{(0)}/d\tilde{t}$ is constant for the $\bar{t}$ integration.

It is again convenient to change the variable of integration from $\bar{t}$ to $\nu^{(0)}$ by using Equation (G-8), where the $e^{(0)}$ term is neglected, i.e.,

$$d\bar{t} = \frac{d\nu^{(0)}}{\sqrt{\mu}\ \beta^{(0)3}}$$

The integral is then

$$\int \frac{df_N}{d\bar{t}} d\bar{t} = \frac{K_2(i)}{\sqrt{\mu}\ \beta^{(0)3}} \left\{ \frac{d\omega^{(0)}}{d\tilde{t}} \int \left(2 \sin 2i^{(0)} \sin 2\omega^{(0)} \sin^2 \nu^{(0)} - 2 \sin 2i^{(0)} \cos 2\omega^{(0)} \sin \nu^{(0)} \cos \nu^{(0)} \right) d\nu^{(0)} \right.$$

$$\left. - \frac{d\nu^{(0)}}{d\tilde{t}} \int \left(\sin 2i^{(0)} \sin 2\omega^{(0)} \left[1 - 2 \sin^2 \nu^{(0)} \right] + 2 \sin 2i^{(0)} \cos 2\omega^{(0)} \sin \nu^{(0)} \cos \nu^{(0)} \right) d\nu^{(0)} \right\}$$

Eliminating all but the secular terms gives

$$\int_{SEC} \frac{df_N(i)}{d\tilde{t}} d\bar{t} = \frac{K_2(i)}{\sqrt{\mu}\ \beta^{(0)3}} \nu^{(0)} \sin 2i^{(0)} \sin 2\omega^{(0)} \frac{d\omega^{(0)}}{d\tilde{t}} \tag{G-11}$$

G.5 INTEGRATION WITH RESPECT TO $\tilde{t}$

The differential equation for $C(\tilde{t})$ from Paragraph 4.3.3 is Equation (4-122)

$$\frac{dC_i(\tilde{t})}{d\tilde{t}} \bar{t} = \int_{SEC} g_i \, d\bar{t} - \int_{SEC} \frac{df_N(i)}{d\tilde{t}} d\bar{t} \tag{4-122}$$

The final step in the solution procedure is to solve this equation for $C(\tilde{t})$. Substituting Equations (G-10) and (G-11) into Equation (4-122) and then integrating yields

$$C_i(\tilde{t})\,\tilde{t} = \int -\frac{3}{16} r_e^4 \beta^{(0)K} \sin 2i^{(0)} \sin 2\omega^{(0)} (2 - 3\sin^2 i^{(0)}) \nu^{(0)}\, d\tilde{t}$$

$$- \int \frac{K_2(i)}{\sqrt{\mu}\,\beta^{(0)3}} \nu^{(0)} \sin 2i^{(0)} \sin 2\omega^{(0)} \frac{d\omega^{(0)}}{d\tilde{t}}\, d\tilde{t}$$

To evaluate the integrals, substitute

$$\nu^{(0)} = n^{(0)}\,\tilde{t}$$

$$K_2(i) = \frac{1}{2} \beta^{(0)4} r_e^2$$

so that

$$C_i(\tilde{t}) = \int -\frac{3}{16} r_e^4 \beta^{(0)K} \sin 2i^{(0)} \sin 2\omega^{(0)} (2 - 3\sin^2 i^{(0)}) n^{(0)}\, d\tilde{t}$$
$$- \int \frac{1}{2} \beta^{(0)4} r_e^2 \sin 2i^{(0)} \sin 2\omega^{(0)} \frac{d\omega^{(0)}}{d\tilde{t}}\, d\tilde{t} \qquad \text{(G-12)}$$

At this point, the $E^{(0)}$ solutions for each element (Equations (4-33), (4-35), (4-37), and (4-44)) should be substituted into Equation (G-12). The resulting integrals will be complex and very difficult to evaluate. For instance, the sin 2 $\omega^{(0)}$ term would have the form

$$\sin 2\omega^{(0)} = \sin\left\{2\tan^{-1}\left[\frac{A\sin(C_2\tilde{t}+\alpha) + C_1/C_2}{A\cos(C_2\tilde{t}+\alpha)}\right]\right\}$$

which must be combined with other functions of $\tilde{t}$ and then integrated.

For the inclination example, a different approach was taken. All elements within the integrals were assumed constant, except $\omega^{(0)}$. Then, from Equation (G-12)

$$C_i(\tilde{t}) = -\frac{3}{16} r_e^4 \beta^{(0)K} \sin 2i^{(0)} (2 - 3\sin^2 i^{(0)}) n^{(0)} \int \sin 2\omega^{(0)}\, d\tilde{t}$$

$$-\frac{1}{2} \beta^{(0)4} r_e^2 \sin 2i^{(0)} \int \sin 2\omega^{(0)} \frac{d\omega^{(0)}}{d\tilde{t}}\, d\tilde{t}$$

In the first integral it is easier to express $d\tilde{t}$ in terms of $d\omega^{(0)}$ than to express $\sin 2\,\omega^{(0)}$ in terms of $\tilde{t}$. Using the J_2 portion of Equation (4-34),

$$d\tilde{t} = \frac{d\omega^{(0)}}{B^{(0)4}\, r_e^2\, n^{(0)}\left(2 - \frac{5}{2}\sin^2 i^{(0)}\right)}$$

In the second integral, the $d\tilde{t}$'s cancel. Thus,

$$C_i(\tilde{t}) = -\frac{3}{16} r_e^2 B^{(0)4} \frac{\sin 2i^{(0)}\left(2 - 3\sin^2 i^{(0)}\right)}{\left(2 - \frac{5}{2}\sin^2 i^{(0)}\right)} \int \sin 2\omega^{(0)}\, d\omega^{(0)}$$

$$-\frac{1}{2} B^{(0)4} r_e^2 \sin 2i^{(0)} \int \sin 2\omega^{(0)}\, d\omega^{(0)}$$

Integrating and collecting terms gives

$$C_i(\tilde{t}) = \frac{1}{4} B^{(0)4} r_e^2 \sin 2i^{(0)} \cos 2\omega^{(0)} \left[\frac{3}{8}\left(\frac{2 - 3\sin^2 i^{(0)}}{2 - \frac{5}{2}\sin^2 i^{(0)}}\right) + 1\right] + K \qquad \text{(G-13)}$$

where K is a true constant of integration. Equation (G-13) is the final expression for $C_i(\tilde{t})$.

To test the validity of this solution, it will be compared with results from the GENPUR program. Appendix H presents solution components and associated constants as computed by GENPUR.

The plot of $\epsilon C_i(\tilde{t})$ is reproduced from Appendix H and shown in Figure G-1 for the low-eccentricity orbit case. Evaluating the constant K and inserting proper values for the initial conditions in Equation (G-13) results in

$$C_i(\tilde{t}) = 0.27 \cos 2\,\omega^{(0)} \quad \text{(rad)}$$

In Figure G-1, the values of $C_i(\tilde{t})$ have been multiplied by ϵ and expressed in degrees. The corresponding solution equation would be

$$\epsilon C_i(\tilde{t}) = 0.025 \cos 2\omega^{(0)} \quad \text{(deg)}$$

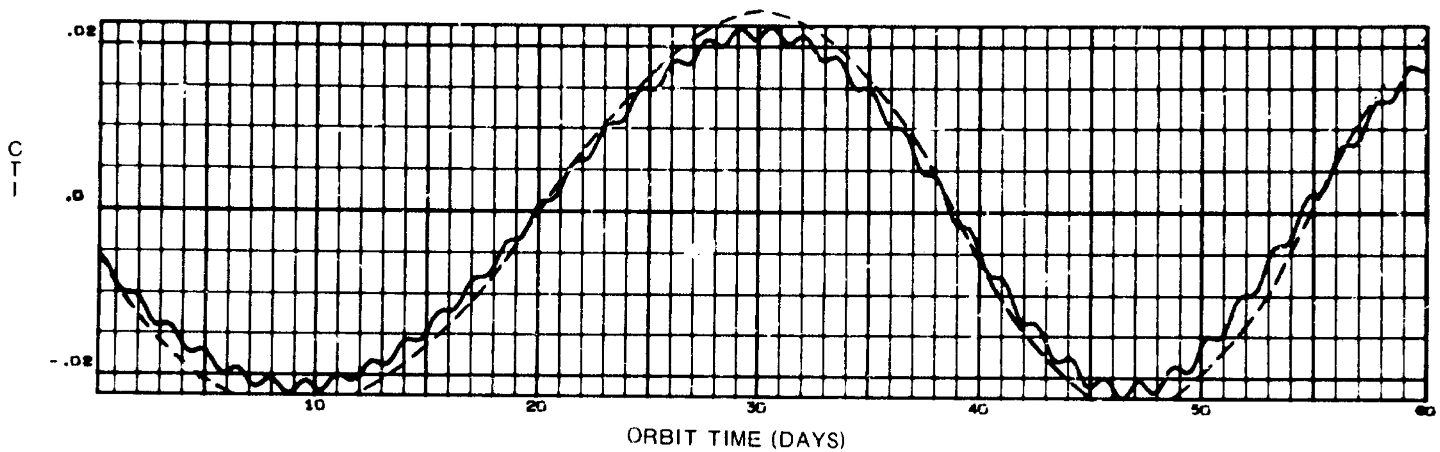


Figure G-1. $\epsilon C_i(\tilde{t})$ for the Low-eccentricity Orbit

The dashed line in Figure G-1 is a plot of this equation. It matches the GENPUR solution extremely well.

In summary, results of the simplified analysis to obtain the functional form of $C_i(\tilde{t})$ are very good. These results bring up the question of how much improvement is really needed or desired, especially since a great deal more effort would be required to eliminate the simplifying assumptions.

APPENDIX H - PLOTS OF SOLUTION COMPONENTS AND ASSOCIATED PARAMETERS

The second approximation solutions to the equations of motion result in expressions of the form

$$E(t) = E^{(0)} + \epsilon E^{(1)}$$

where $\underline{E}$ is any element in the set (e, i, Ω, ω, B, or M). The $E^{(0)}$ solutions are mean elements and contain the secular and long-periodic variations. The E(t) solutions (i.e., $E^{(0)} + \epsilon E^{(1)}$) are osculating elements and contain short-periodic in addition to secular and long-periodic variations. This appendix presents plots of these solution components for each element when considering a low-eccentricity orbit. The initial conditions for this orbit are shown in Table H-1.

Asymptotic series solutions were first derived when considering only the oblateness perturbation ($\underline{J}_2$ and $\underline{J}_3$ first-order effects). Later, combined solutions for oblateness and drag were obtained. Therefore, two sets of plots are presented in this appendix; one for the oblateness-only solutions and one for the combined solutions. In both sets, Brouwer's equations are used for second-order secular effects of $\underline{J}_2$ and $\underline{J}_4$ on $\underline{\Omega}$, $\underline{\omega}$, and $\underline{M}$.

Deeply involved in the solution procedure are assumptions that various parameters remain constant, at least relatively so for short intervals of time (2 or 3 days). It is known that some of these parameters (the $C(\tilde{t})$'s) contain important secular and long-periodic variations. Other parameters (C_1 through C_8) remain fairly uniform, at least for the oblateness-only solutions. Plots of these parameters are also presented in this appendix.

Figures H-1 through H-12 show the solution components and associated parameters for the oblateness-only solutions to the low-eccentricity case. Figures H-13 through H-24 show the corresponding solution components and associated parameters for the combined oblateness and drag solutions. Interesting differences can be seen in the behavior of some of the parameters when drag is added to the solution. Many

Mean Elements

B	($km^{-1/2}$)	0.01234014
a	(km)	6566.5731
e		0.00559414
i	(deg)	50.01120
Ω	(deg)	152.47131
ω	(deg)	52.62626
M	(deg)	2.91685

State Vector

x	(km)	-4872.6530
y	(km)	-1364.7471
z	(km)	4124.7041
$\dot{x}$	(km/sec)	4.41679
$\dot{y}$	(km/sec)	-5.51029
$\dot{z}$	(km/sec)	3.39451

Date

April 1, 1971

$C_D(A/m)$

0.0002 m^2/kg

Table H-1. Initial Conditions for the Low-Eccentricity Orbit of the parameters are functions of $B^{(0)}$, which is constant for oblateness only but has a secular variation when drag is added. Thus, the parameters take on a secular variation which was not present in the oblateness-only solutions.

In the plots of the solution components, the secular and long-periodic trends are easily recognized. However, since these are plots of points at 6-hour intervals, short-periodic trends cannot be distinguished because they appear as somewhat random fluctuations about the mean.

The units of the element solution components depicted in Figures H-1 through H-7 and H-13 through H-19 are as follows:

e - unitless

ω, i, Ω, M, ν -deg

B - $km^{-1/2}$

The units of the constants and functions depicted in Figures H-8 through H-12 and H-20 through H-24 are as follows:

$C_1, C_2, \ldots, C_7$ - rad/hr

α - rad

$C_e(\tilde{t})$ - unitless

$C_\omega(\tilde{t})$, $C_i(\tilde{t})$, $C_\Omega(\tilde{t})$, $C_M(\tilde{t})$ - deg

$C_B(\tilde{t})$ - $km^{-1/2}$

OBLATENESS ONLY

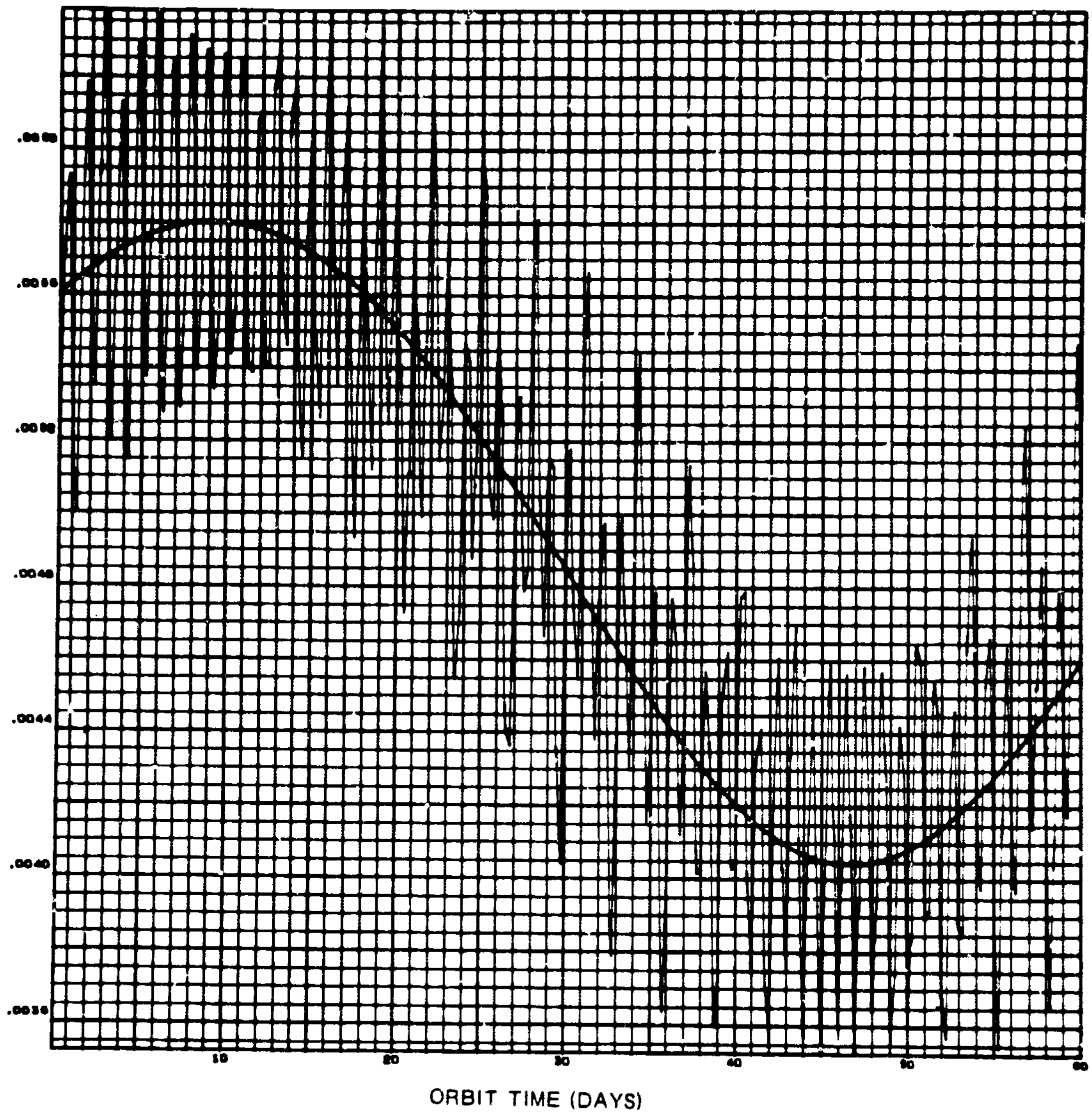


Figure H-1. Eccentricity Solution Components

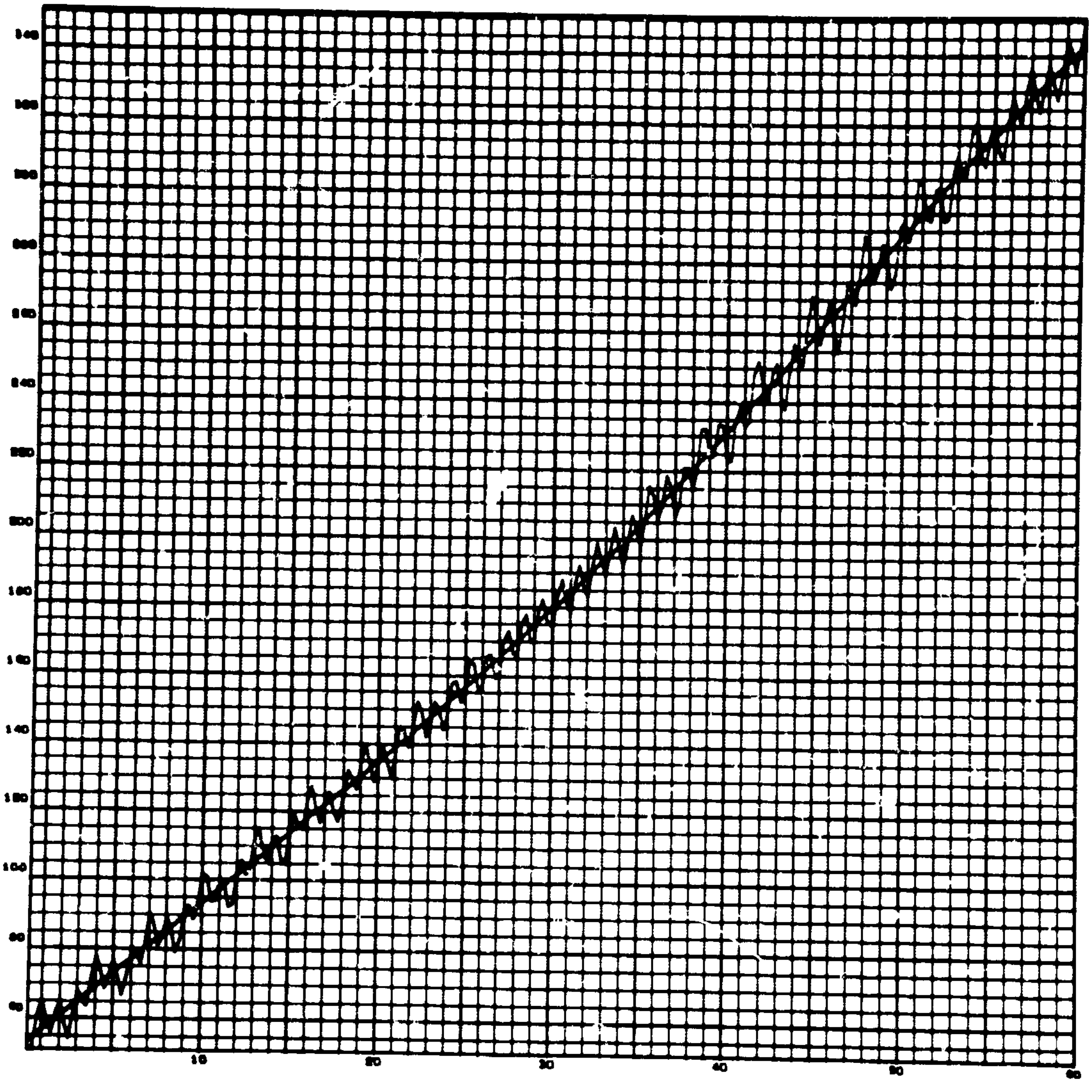


Figure H-2. Argument of Perigee Solution Components

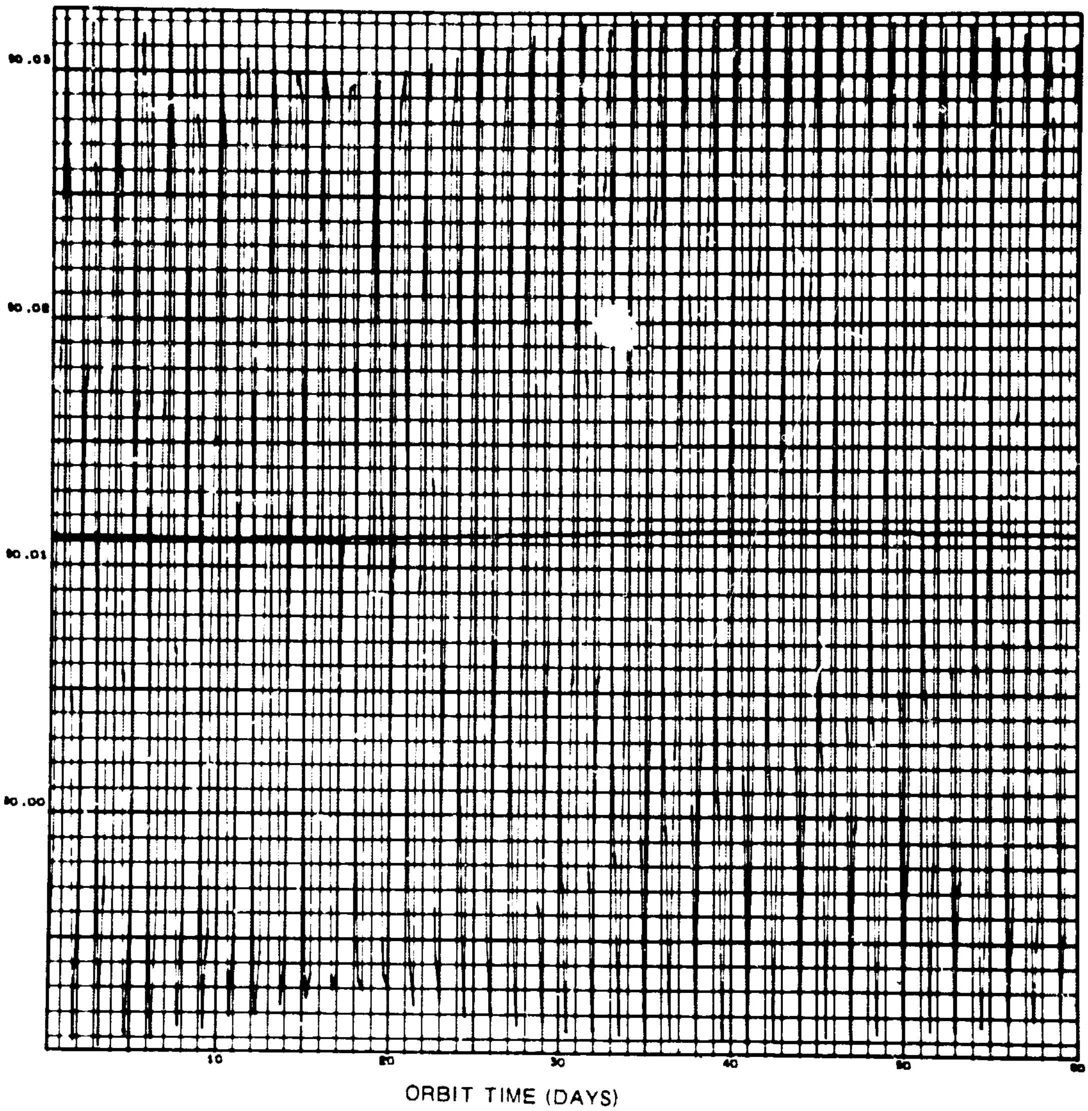


Figure H-3. Inclination Solution Components

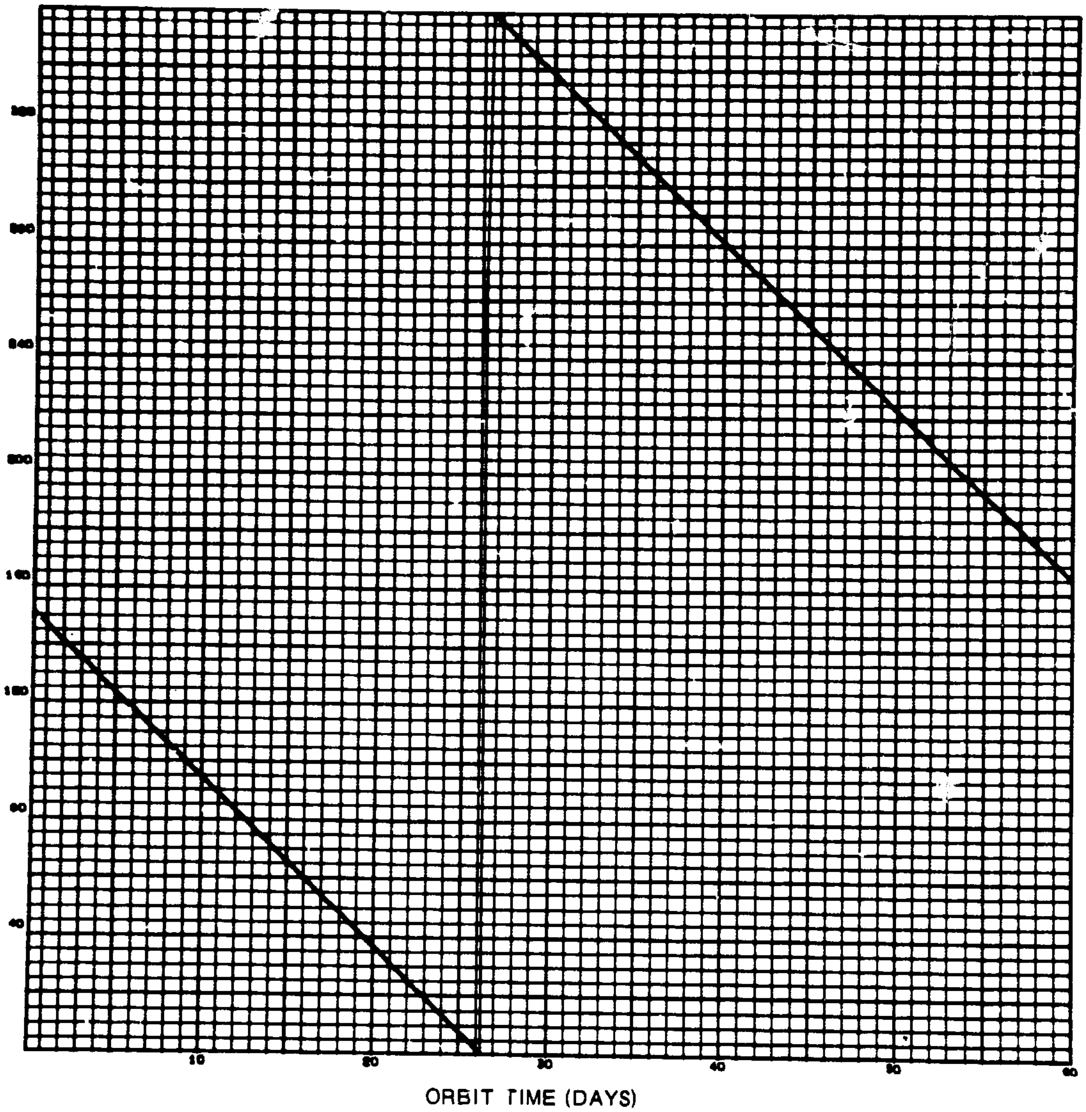


Figure H-4. Ascending Node Solution Components

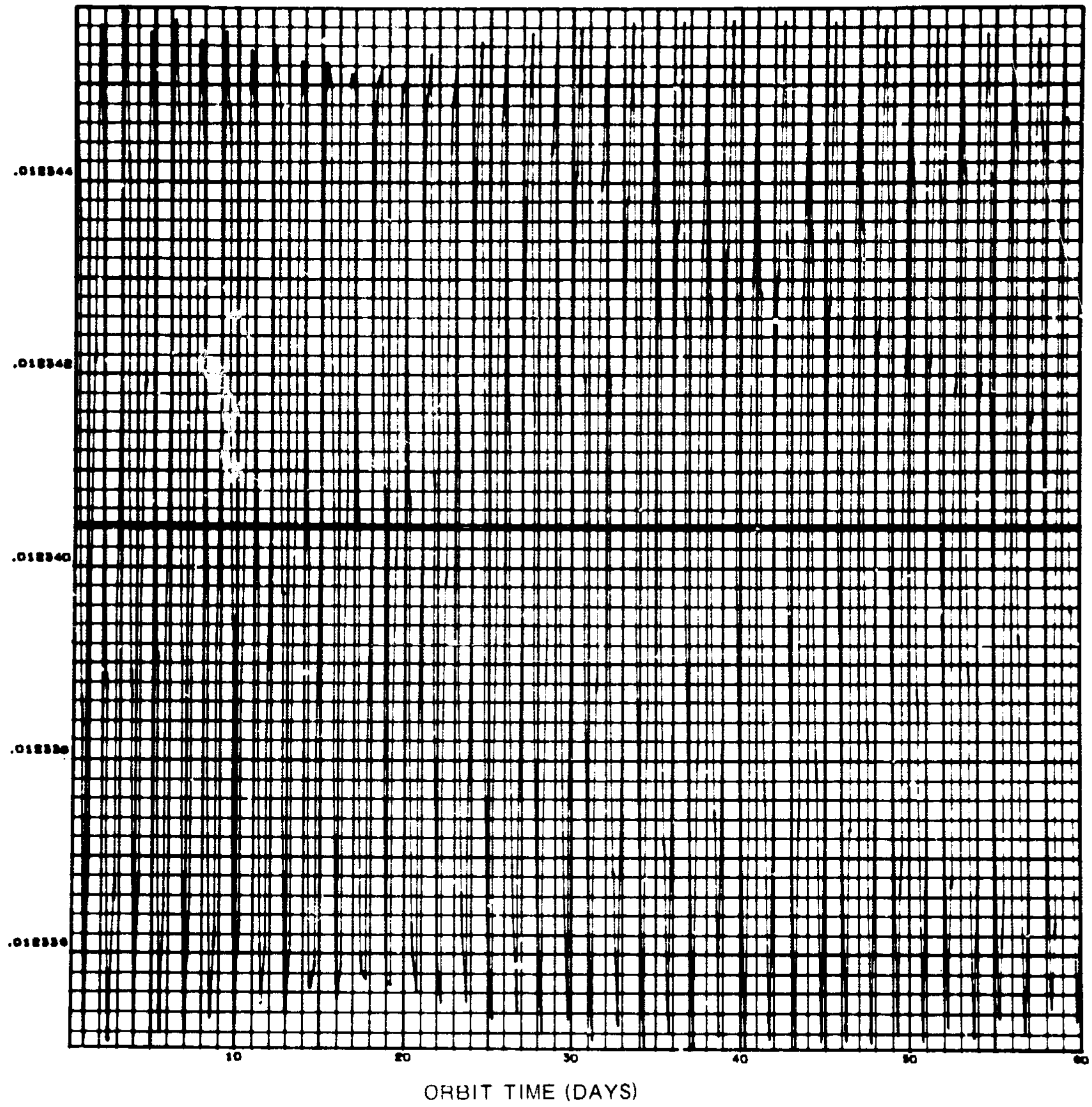


Figure H-5. Reciprocal Square Root of Semimajor Axis Solution Components

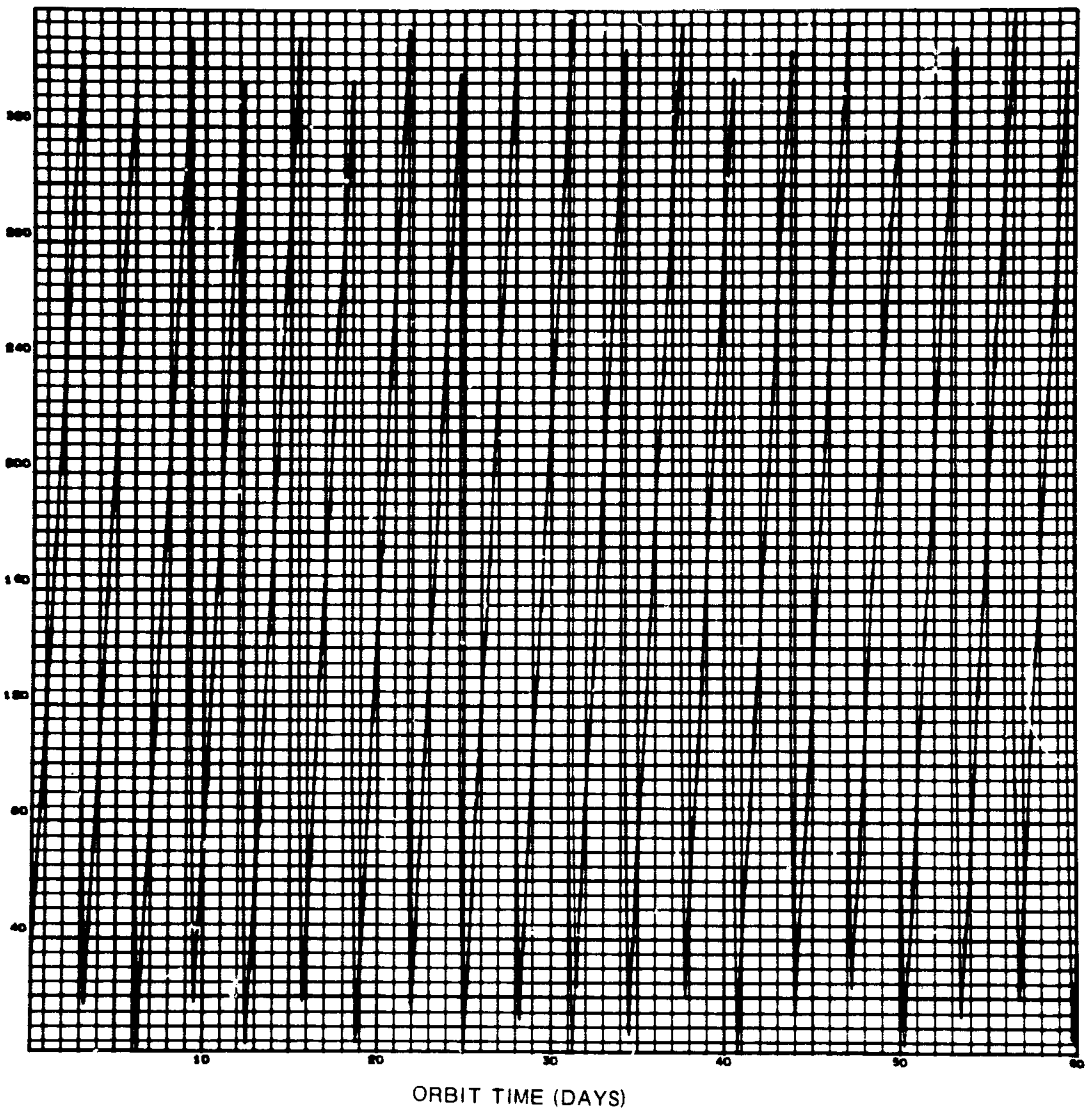


Figure H-6. Mean Anomaly Solution Components

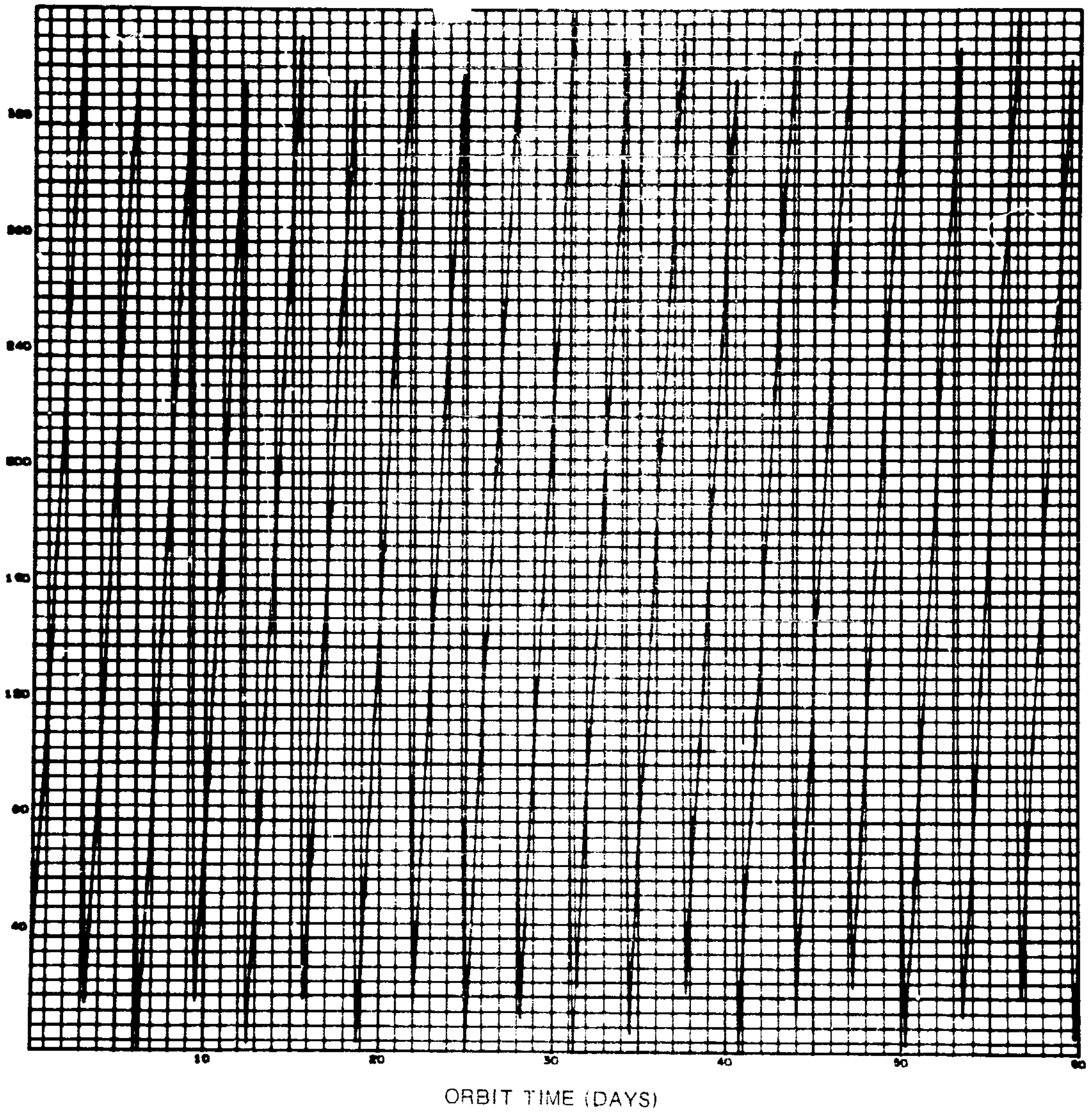


Figure H-7. True Anomaly Solution Components

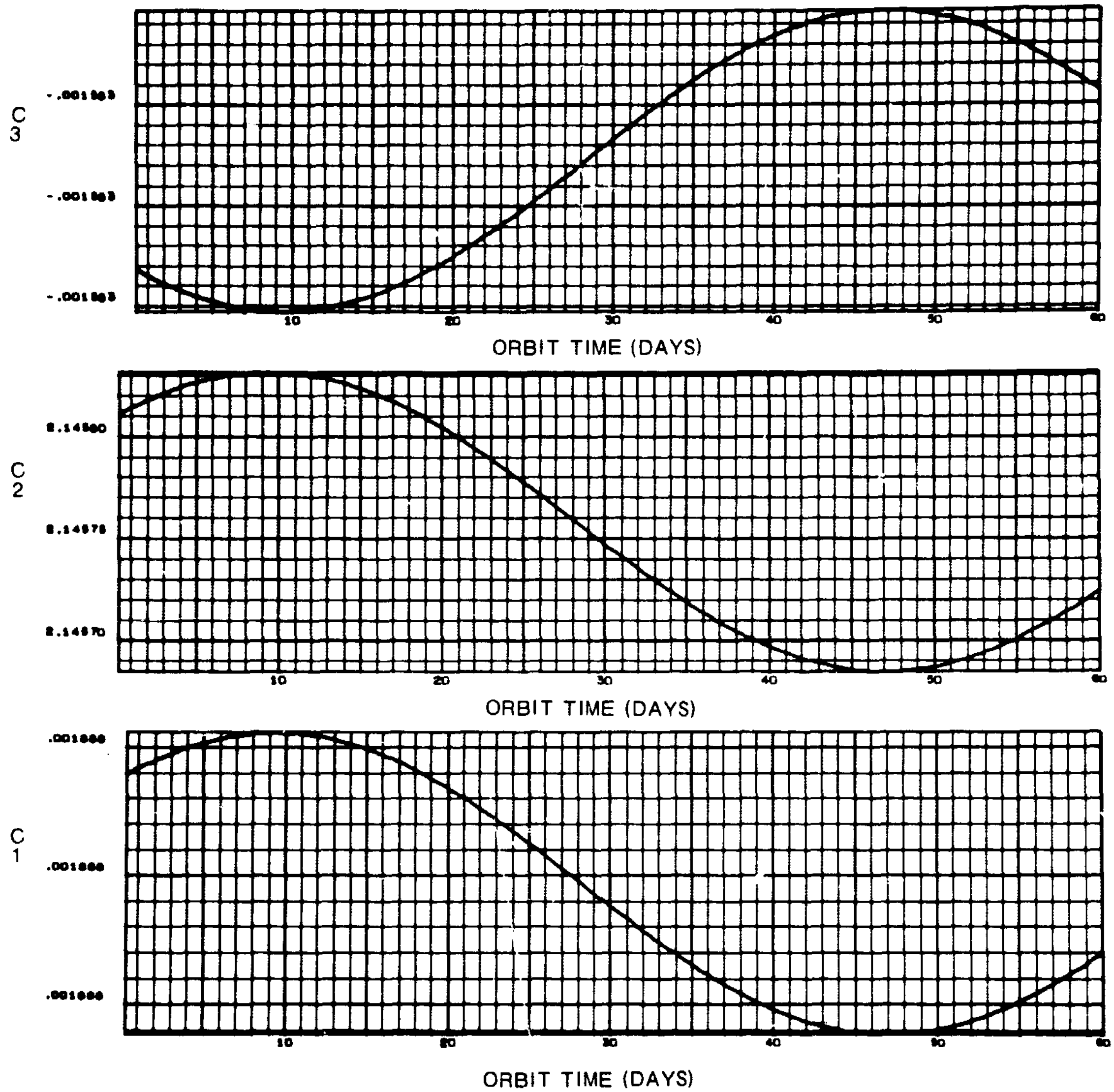


Figure H-8. C_1, C_2, and C_3 Constants

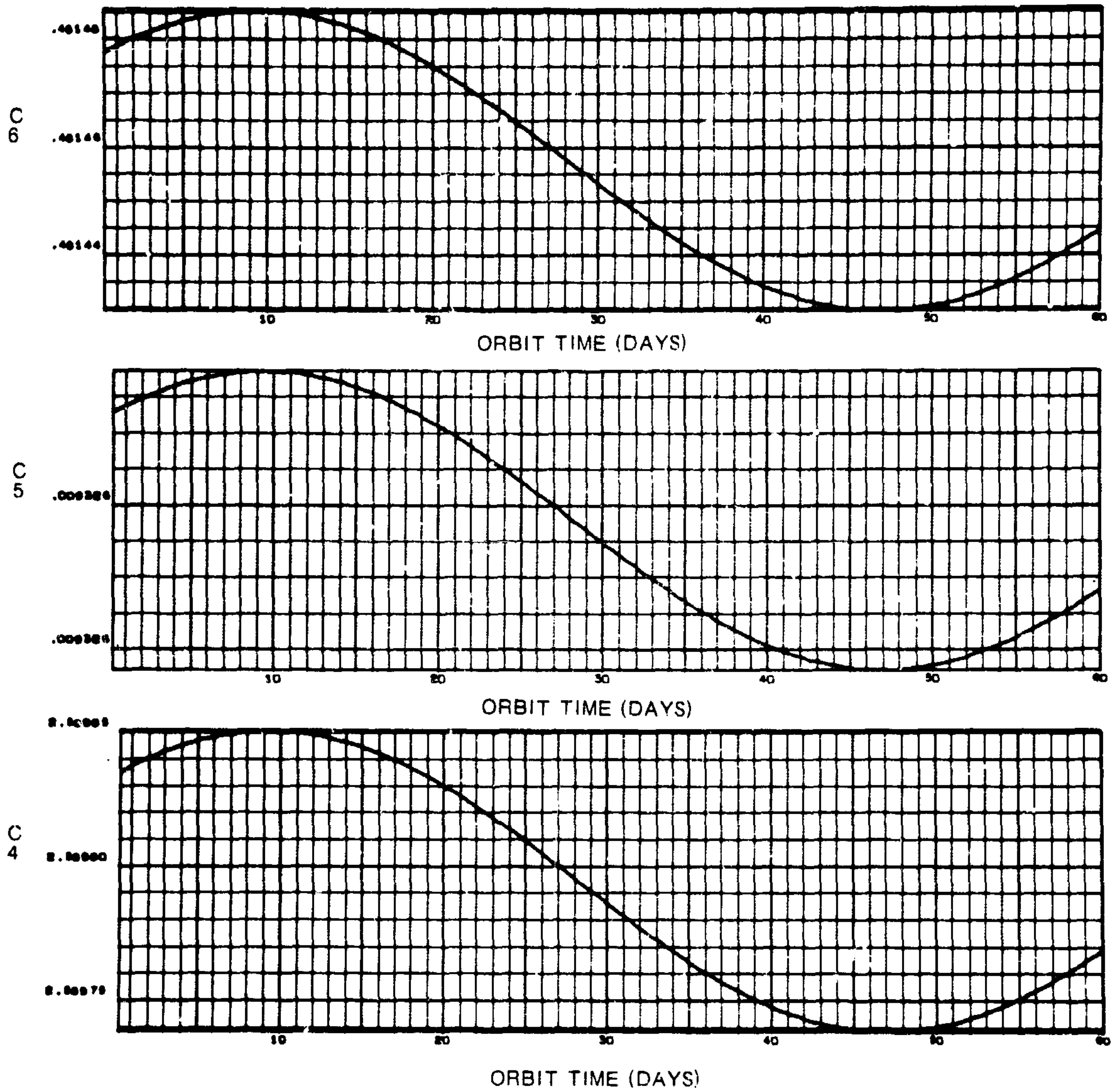


Figure H-9. C_4, C_5, and C_6 Constants

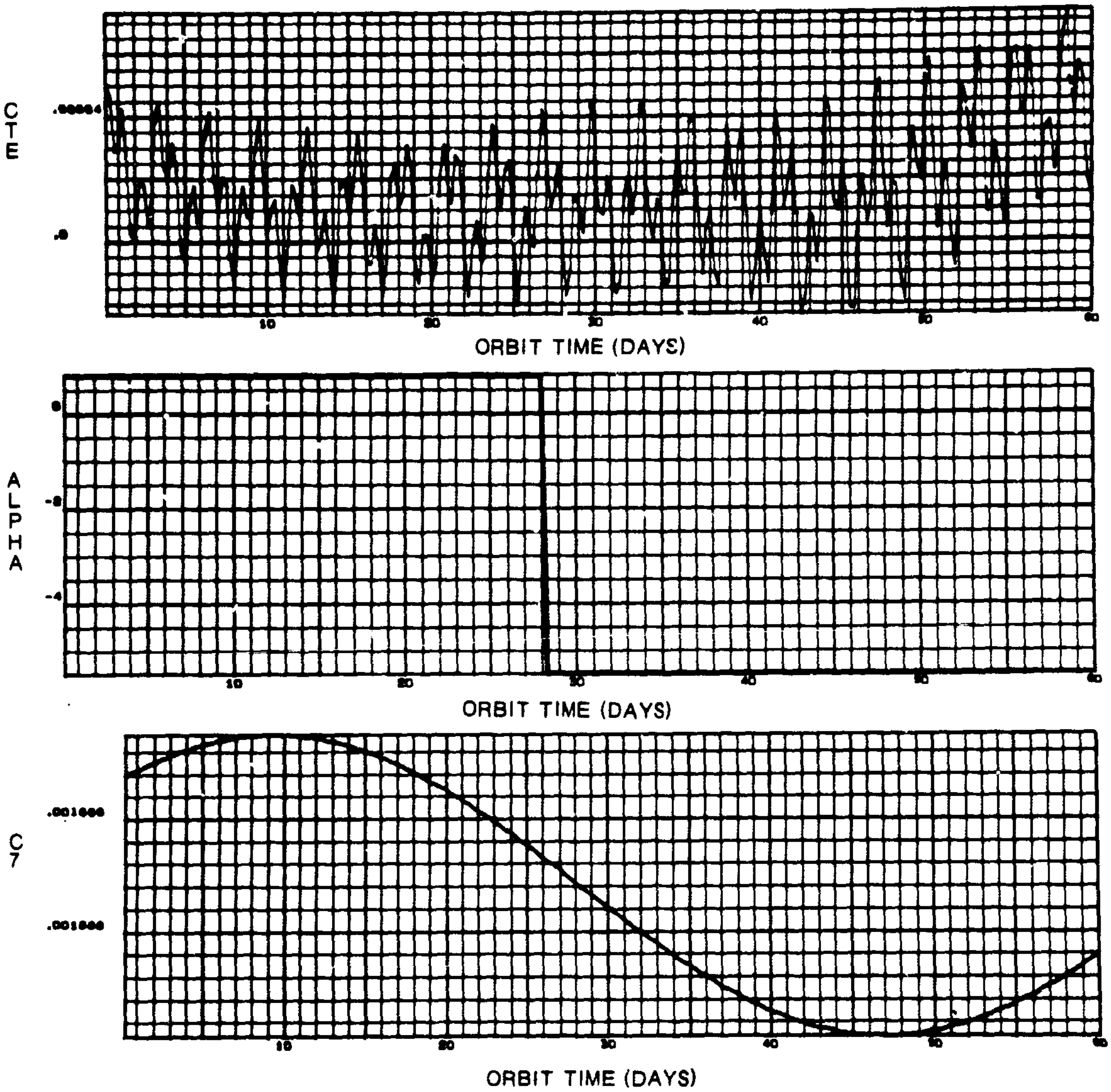


Figure H-10. C_7 and α Constants and $\epsilon\, C_e(\tilde{t})$ Function

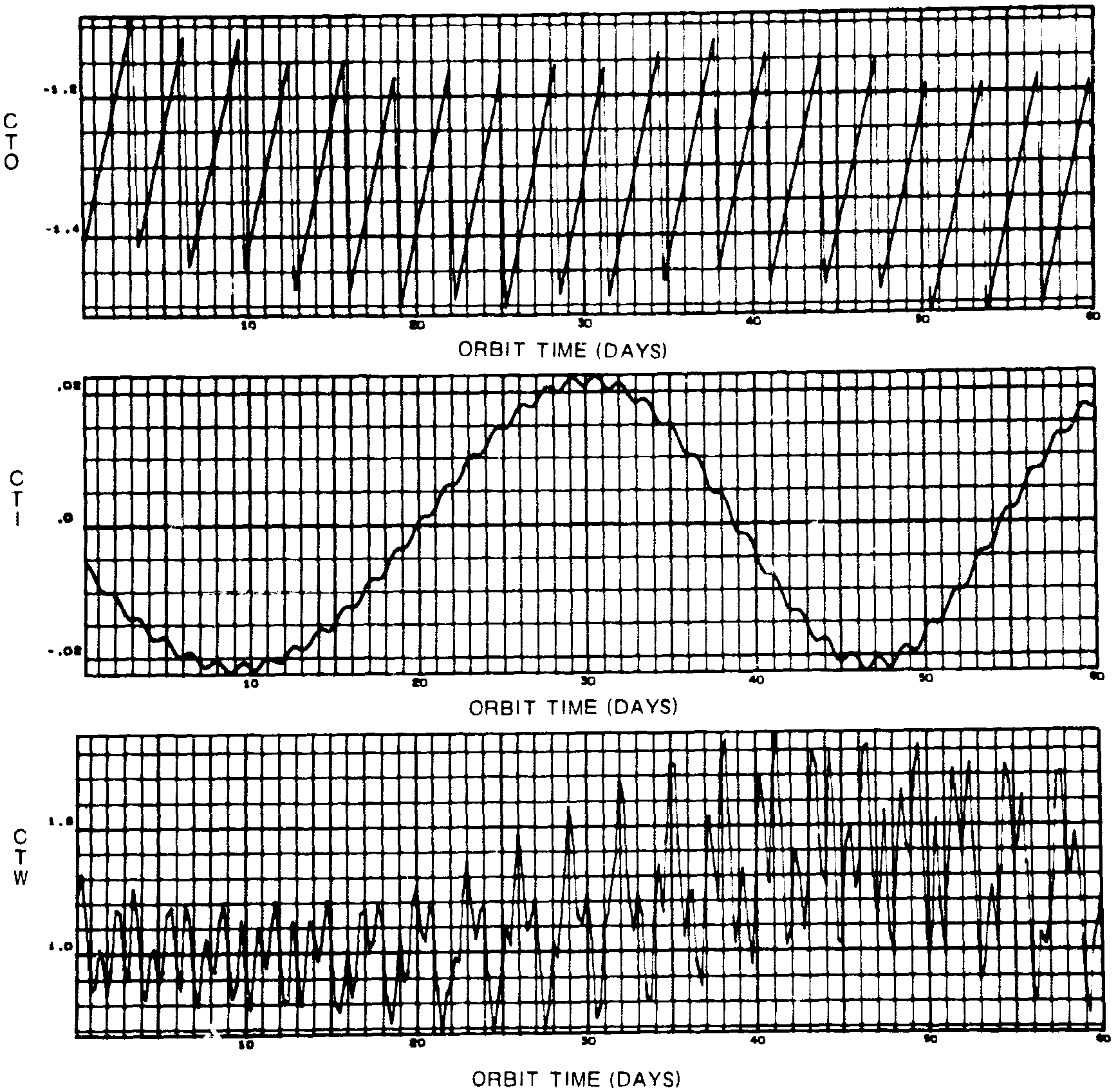


Figure H-11. $\epsilon C_{\omega}(\tilde{t})$, $\epsilon C_{i}(\tilde{t})$, and $\epsilon C_{\Omega}(\tilde{t})$ Functions

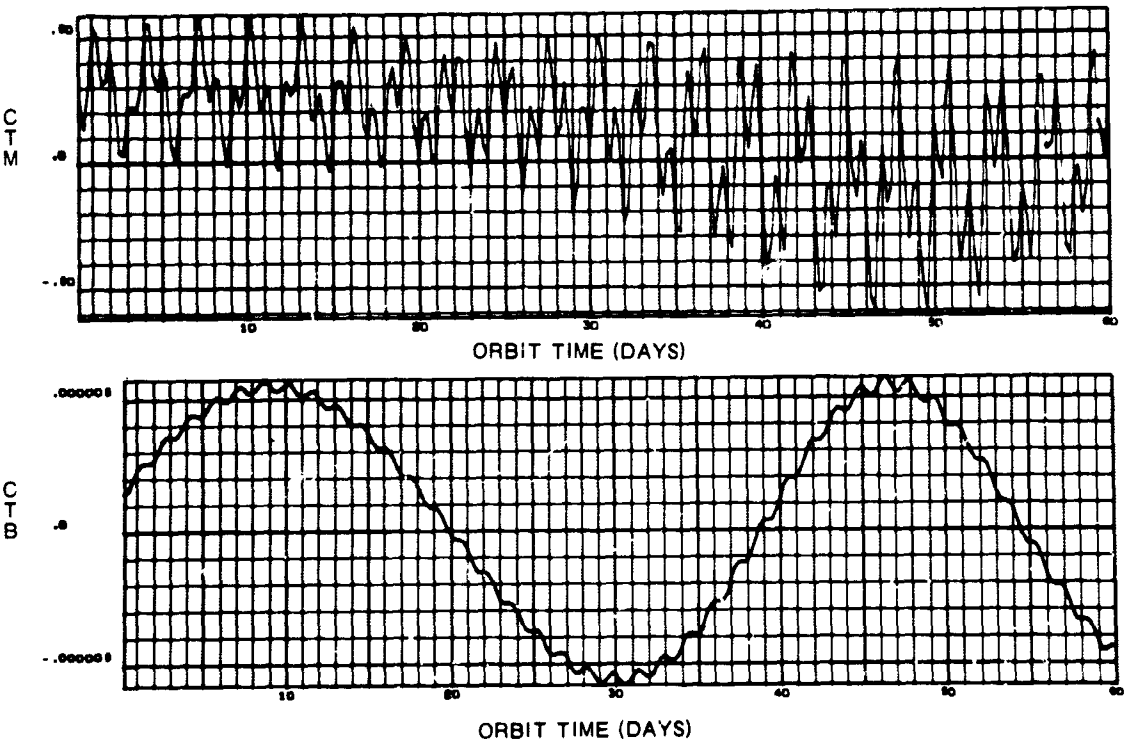


Figure H-12. $\epsilon C_B(\tilde{t})$ and $\epsilon C_M(\tilde{t})$ Functions

OBLATENESS AND DRAG

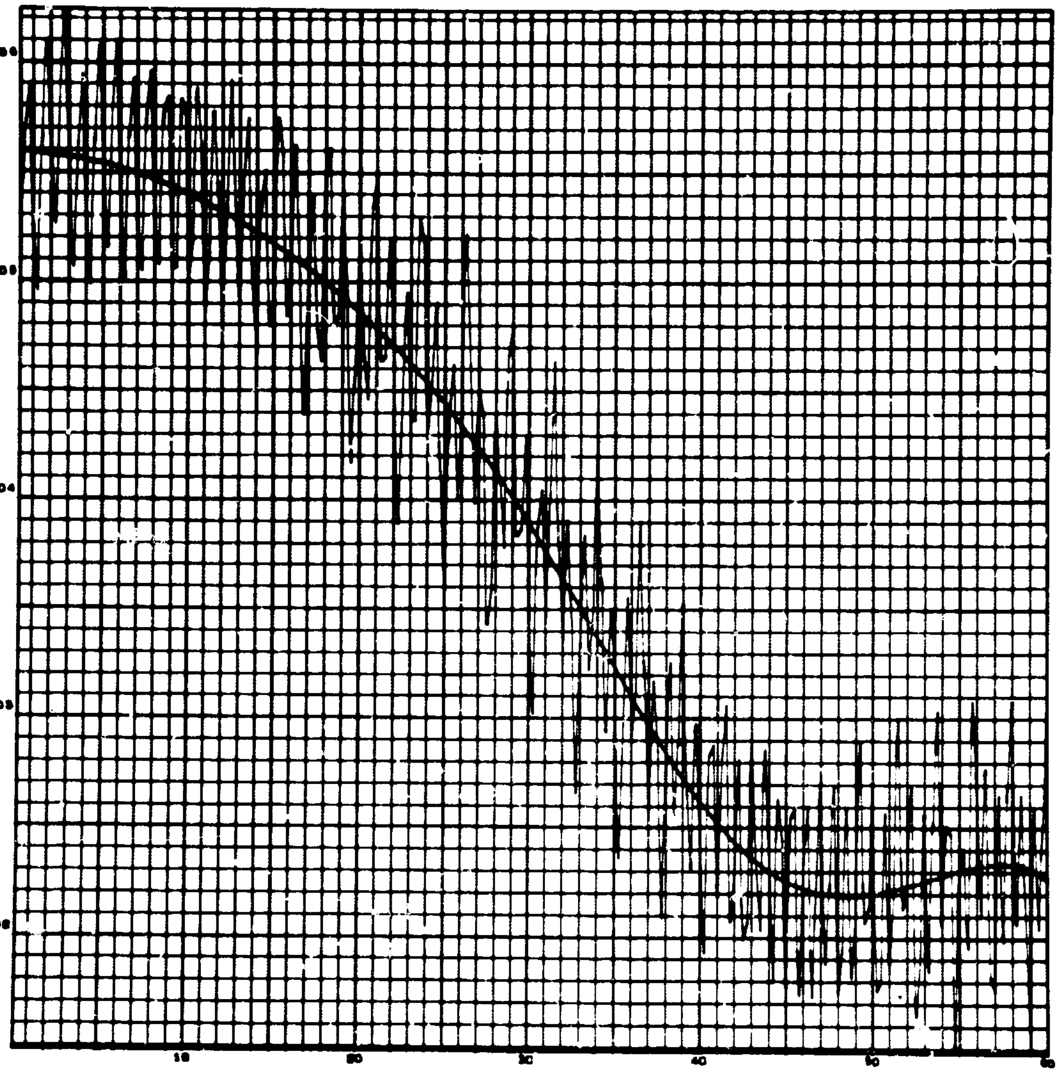


Figure H-13. Eccentricity Solution Components

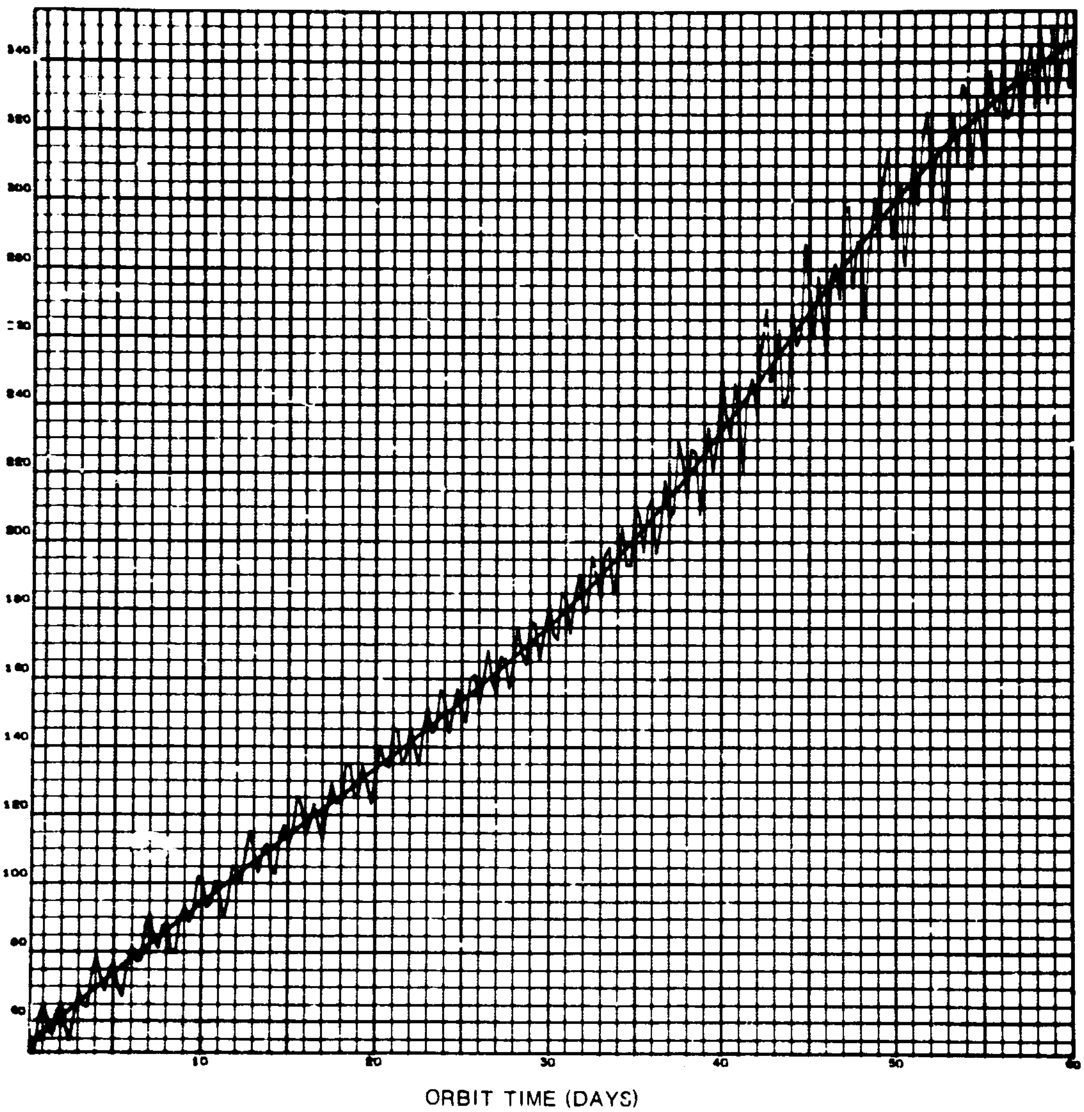


Figure H-14. Argument of Perigee Solution Components

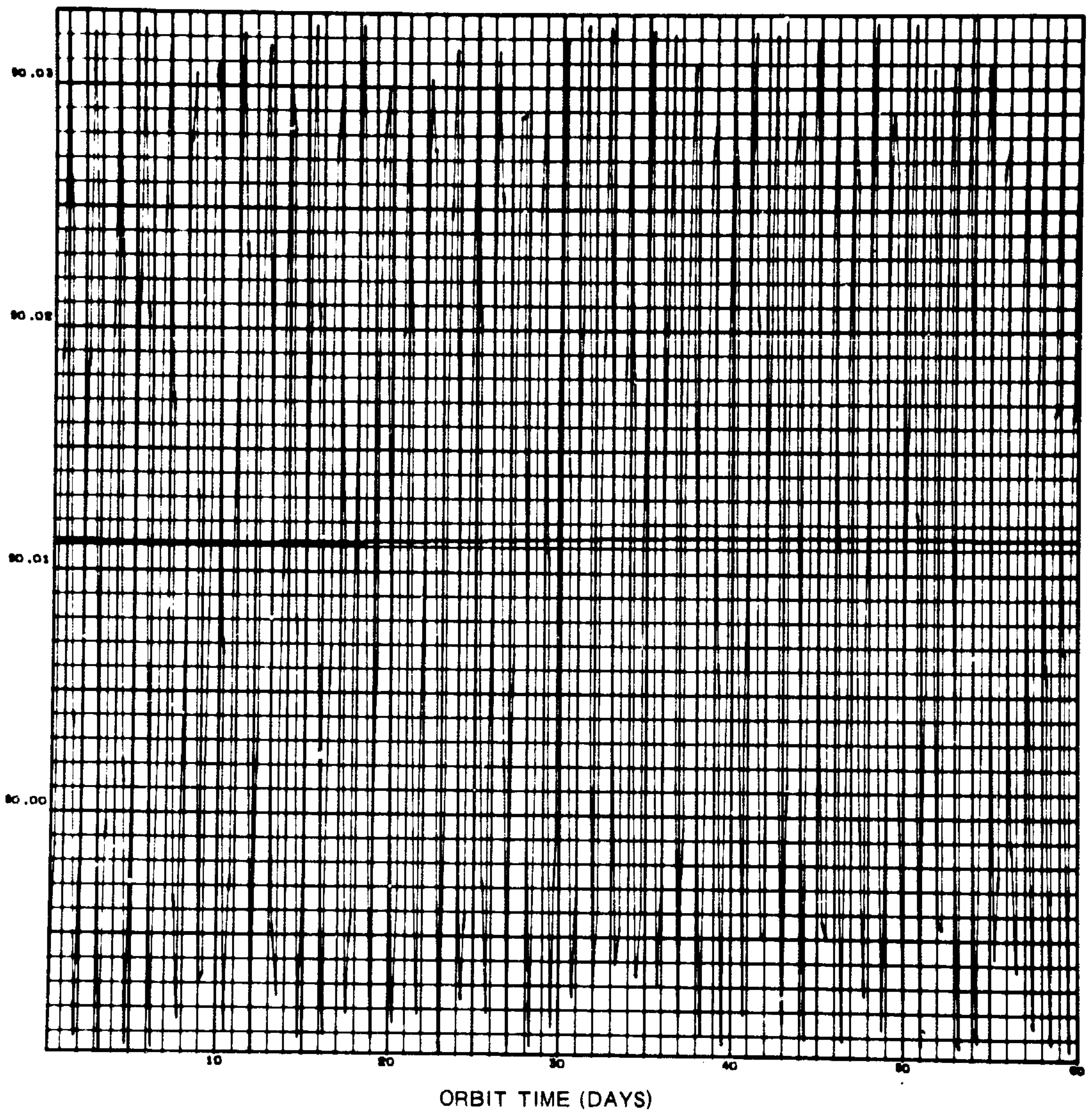


Figure H-15. Inclination Solution Components

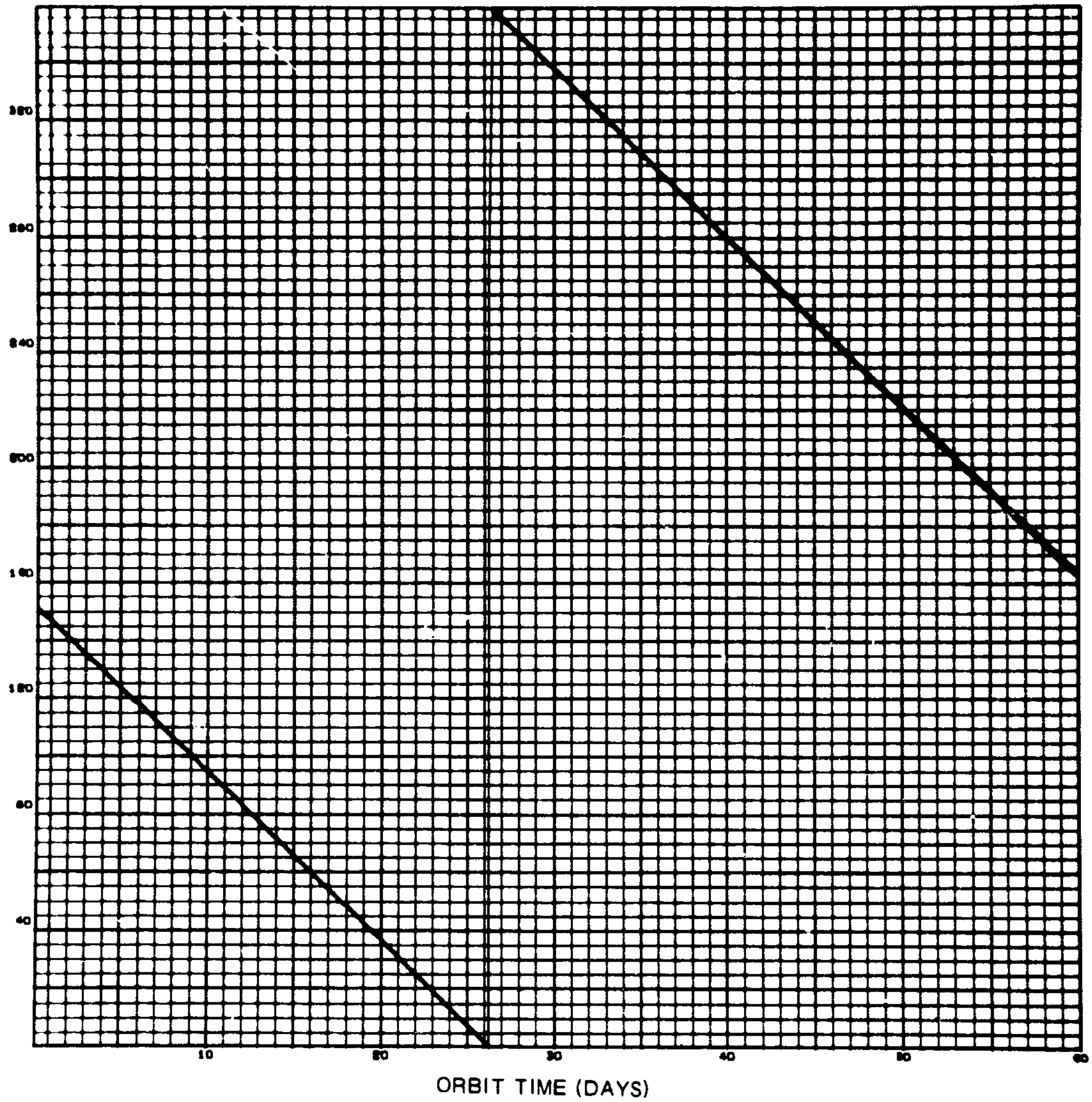


Figure H-16. Ascending Node Solution Components

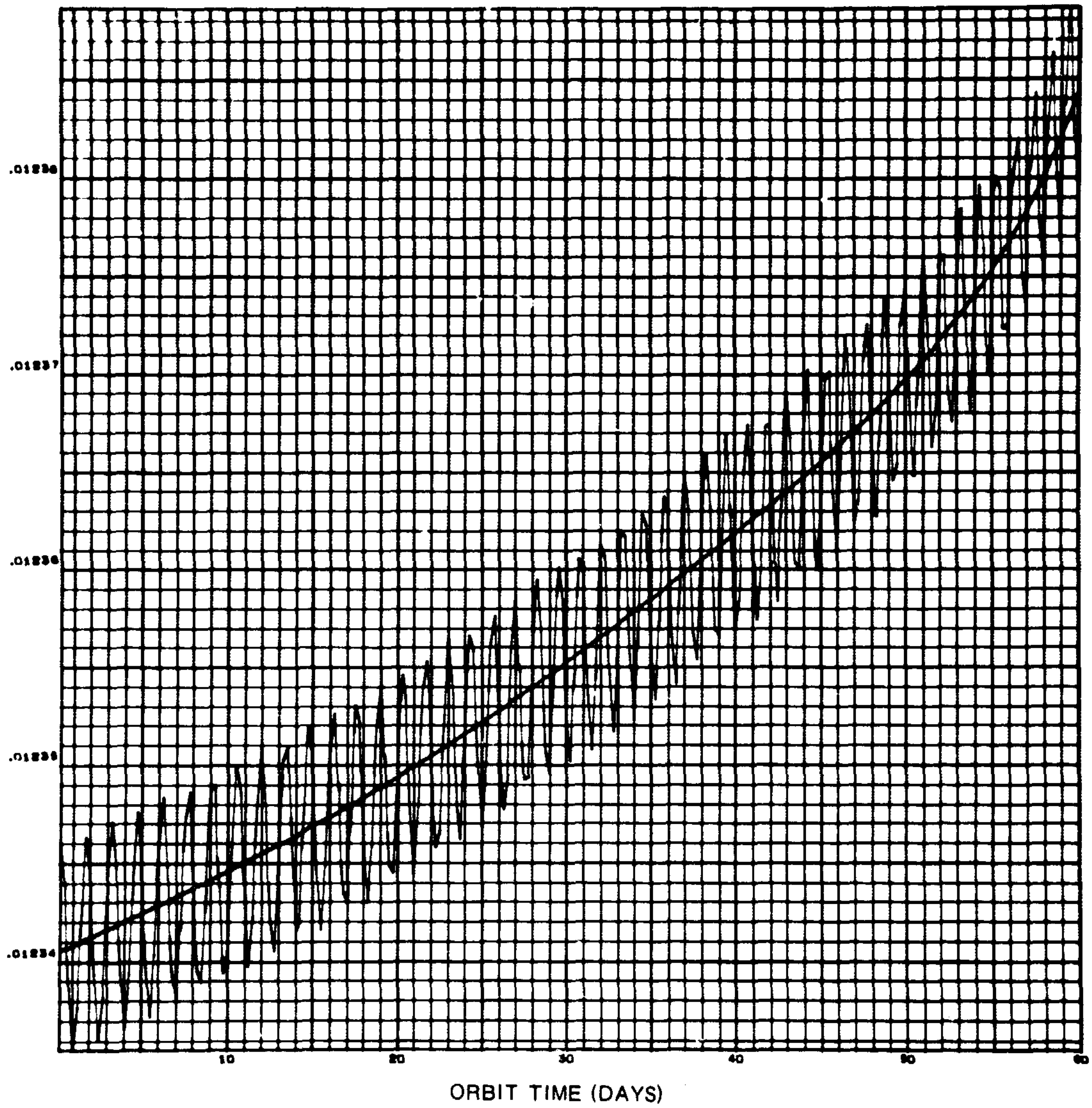


Figure H-17. Reciprocal Square Root of Semimajor Axis Solution Components

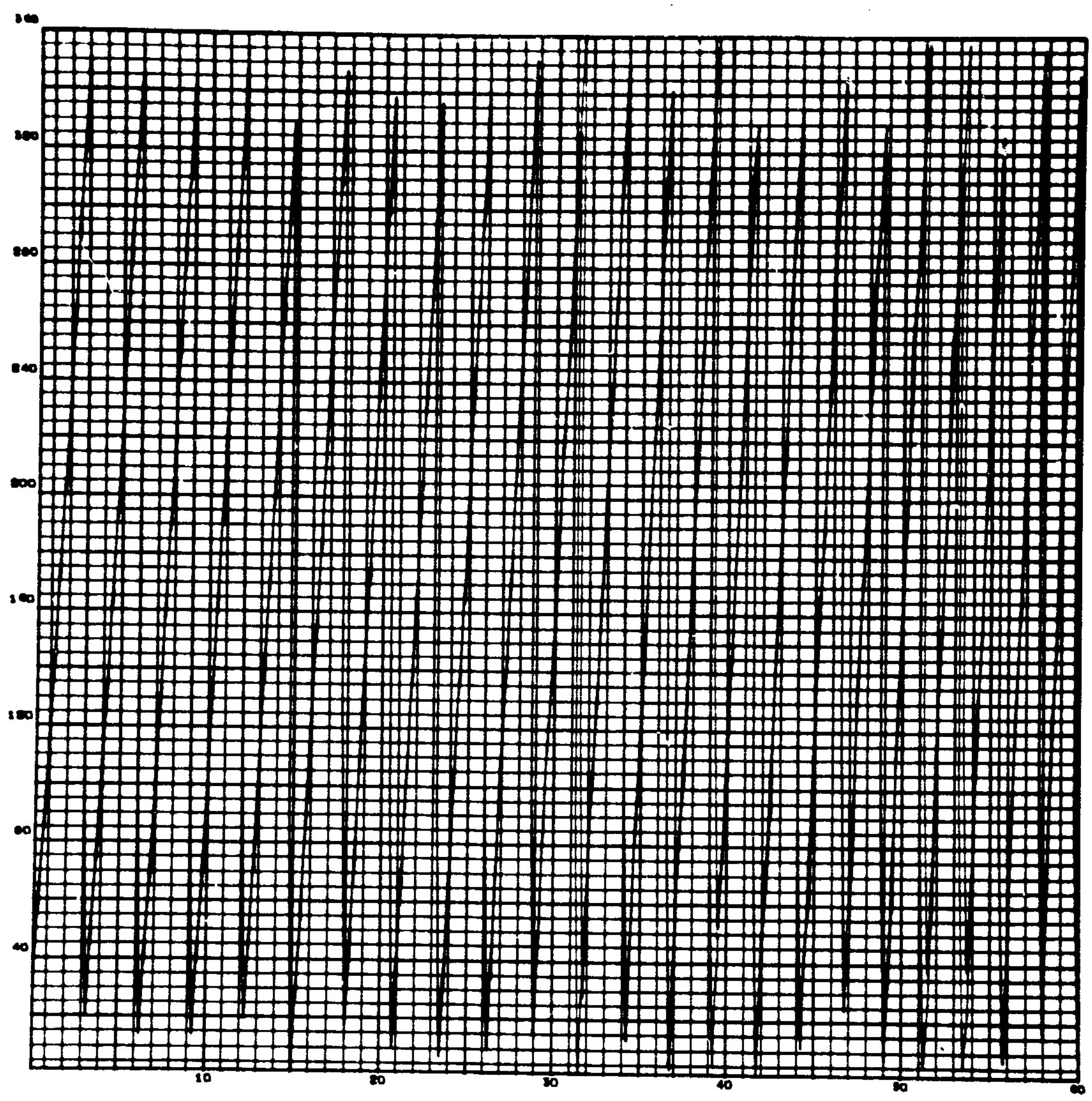


Figure H-18. Mean Anomaly Solution Components

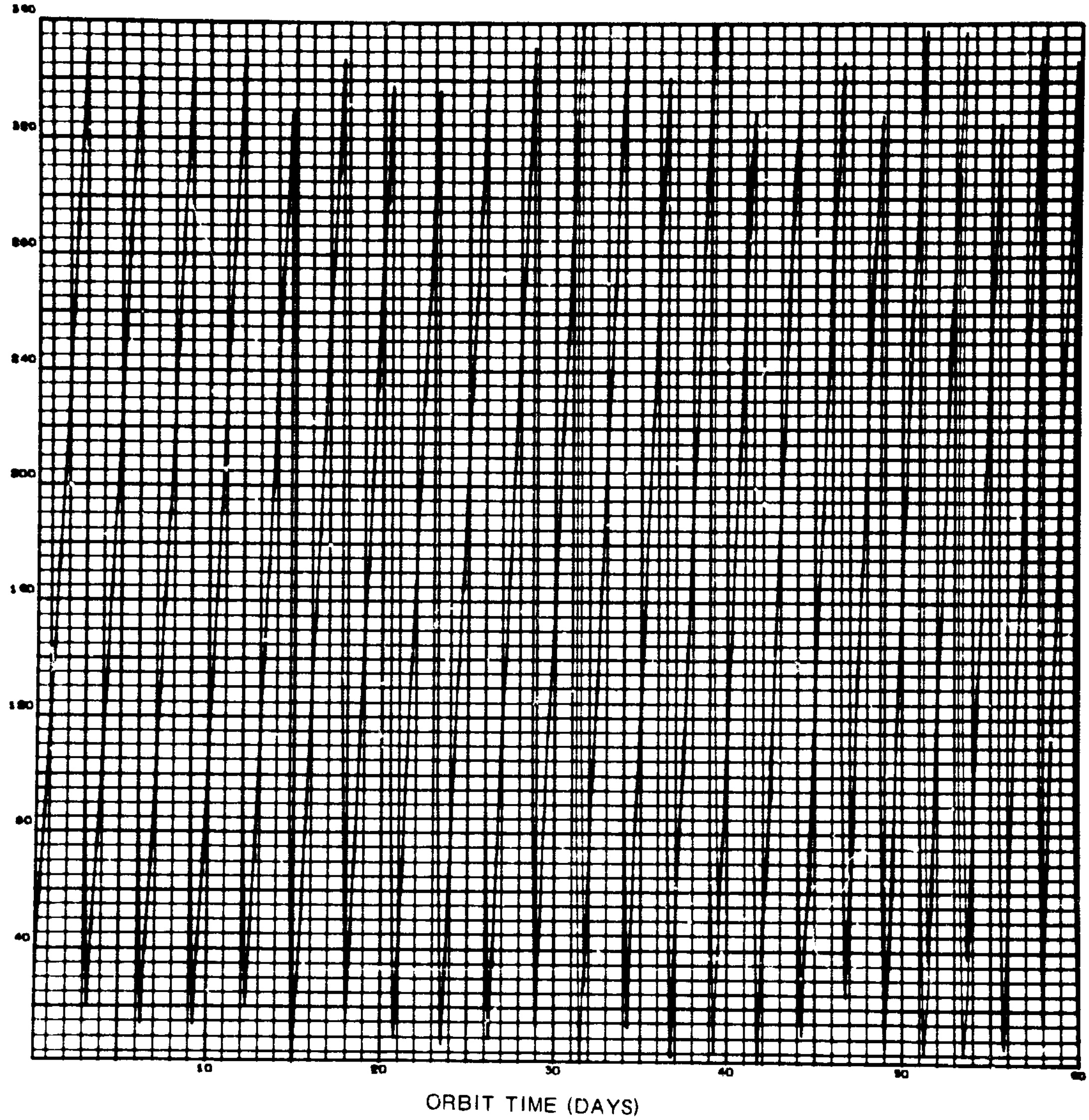


Figure H-19. True Anomaly Solution Components

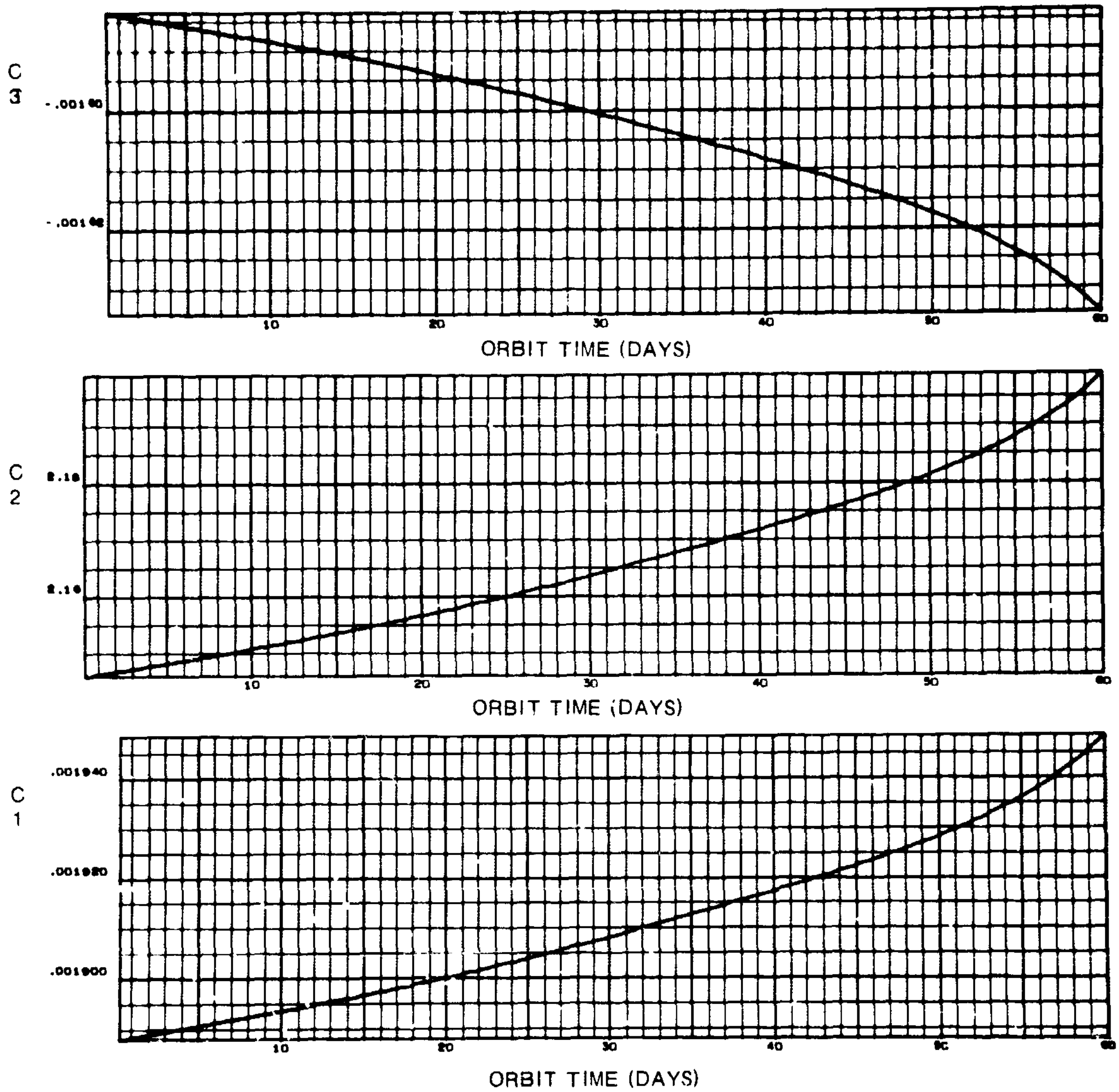


Figure H-20. C_1, C_2, and C_3 Constants

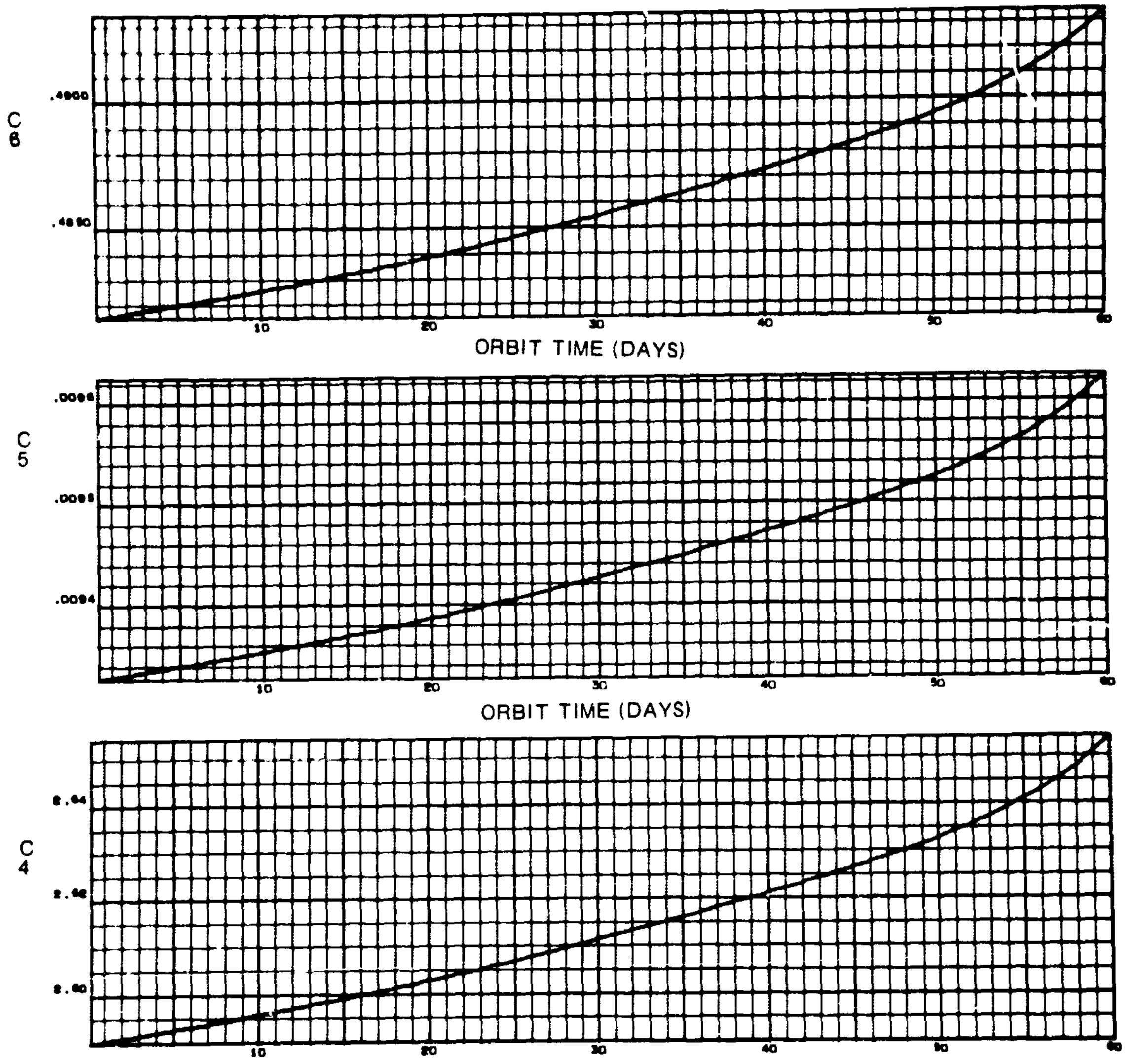


Figure H-21. C_4, C_5, and C_6 Constants

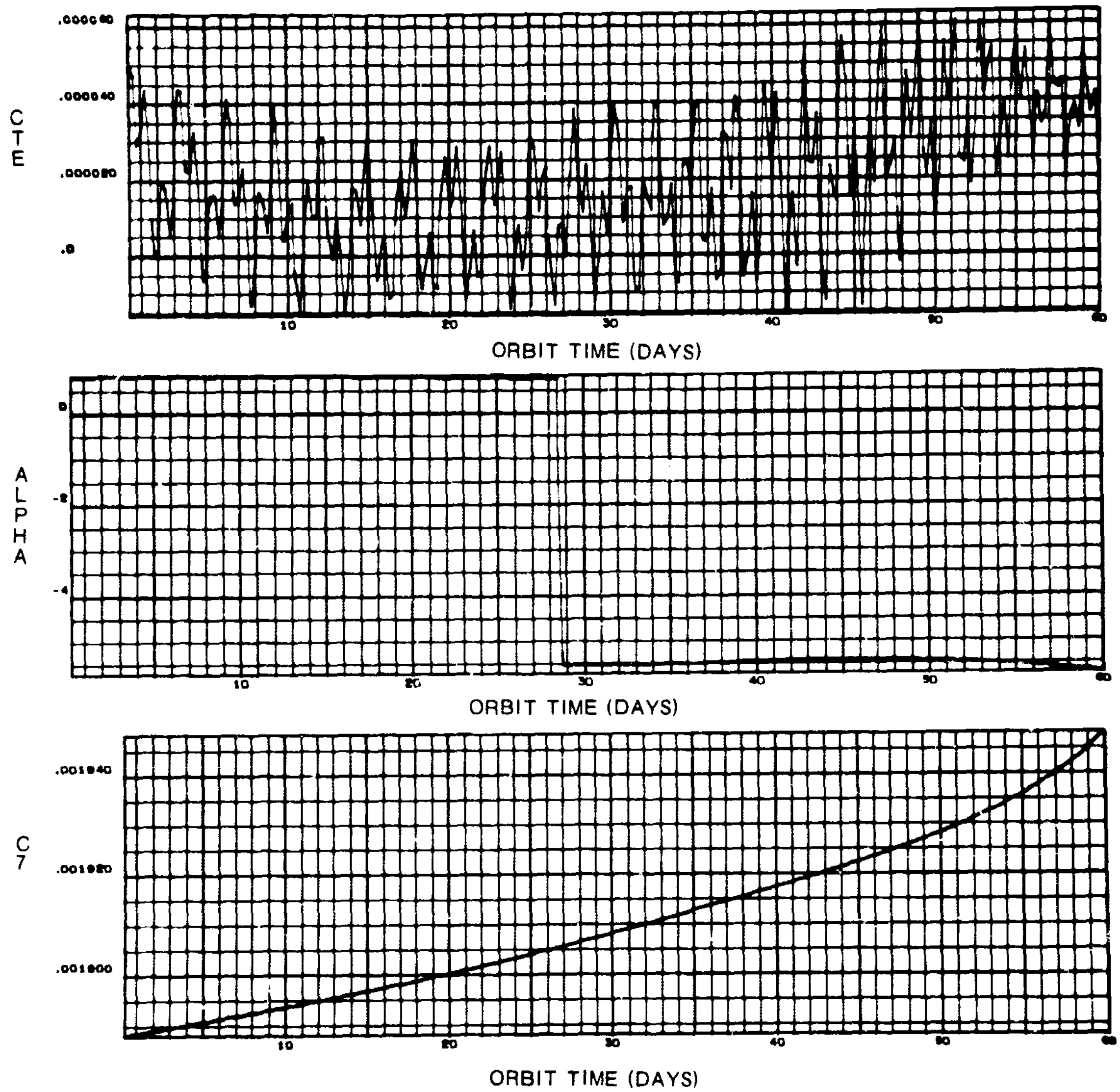


Figure H-22. C_7 and α Constants and $\epsilon\ C_e(\tilde{t})$ Function

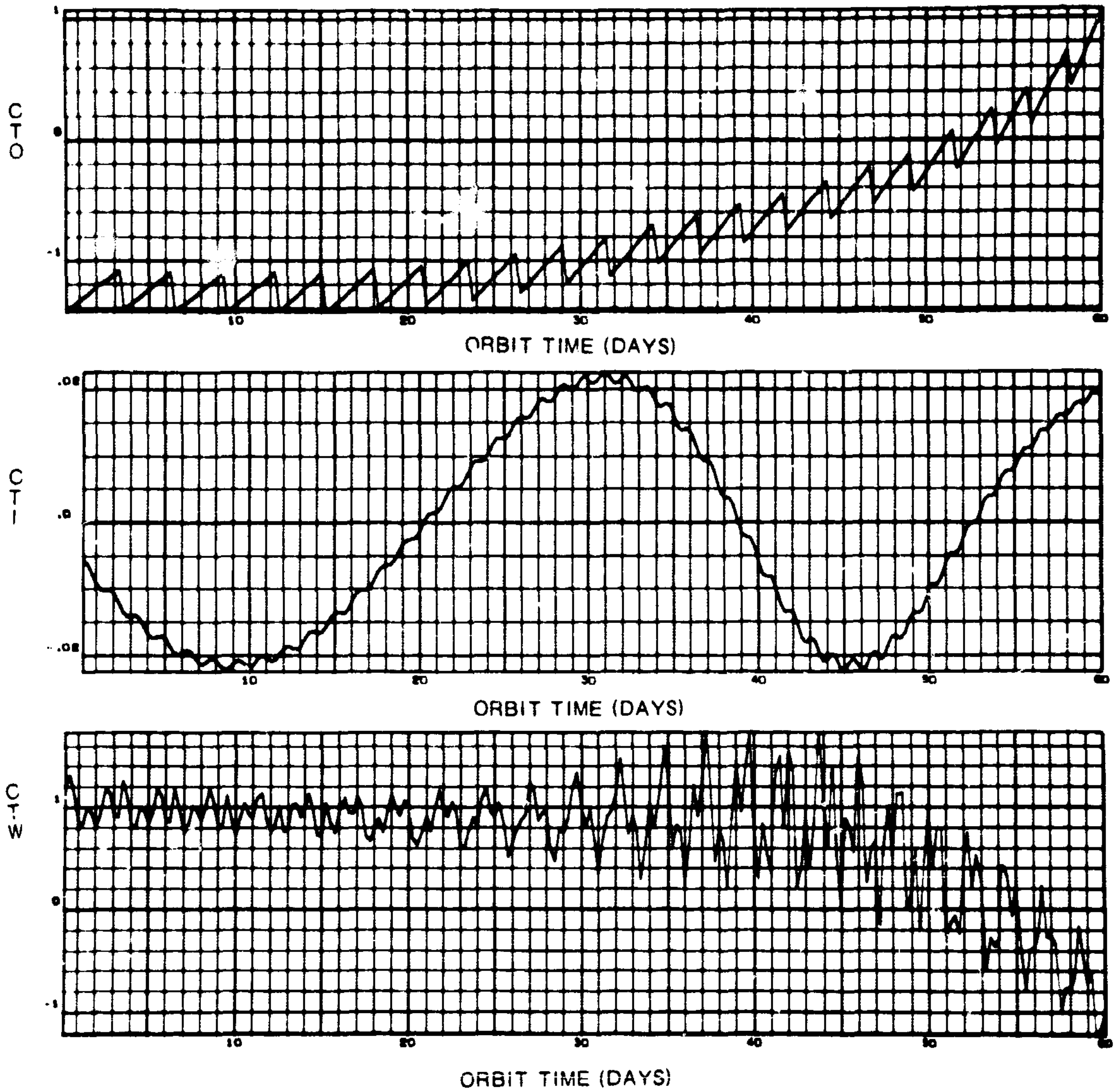
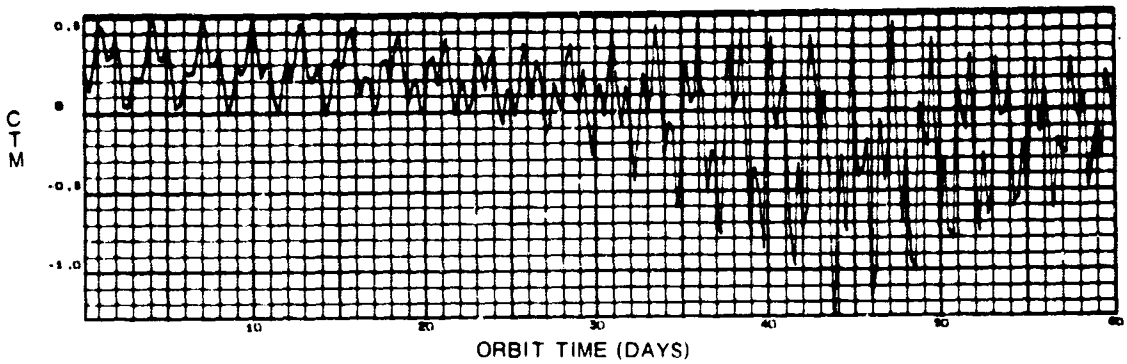


Figure H-23. $\epsilon C_{\omega}(\tilde{t})$, $\epsilon C_i(\tilde{t})$, and $\epsilon C_{\Omega}(\tilde{t})$ Functions



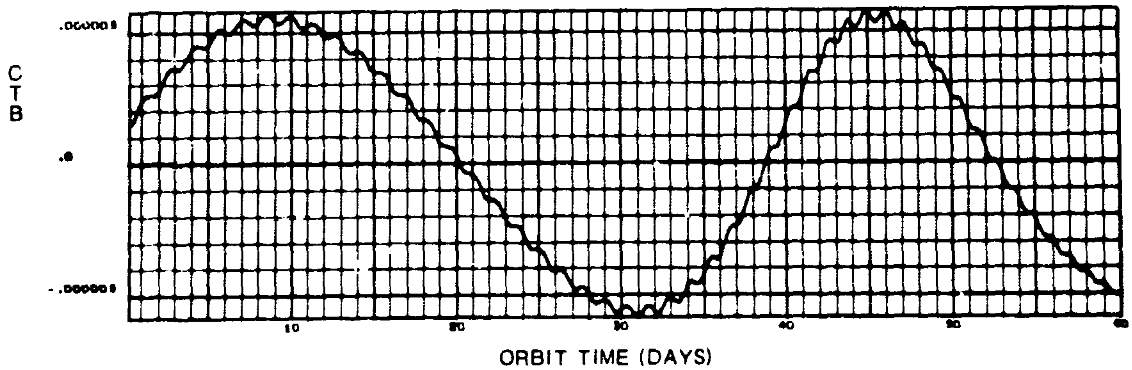


Figure H-24. $\epsilon C_B(\tilde{t})$ and $\epsilon C_M(\tilde{t})$ Functions

APPENDIX I - DISCUSSION OF OSCULATING AND MEAN ORBITAL ELEMENTS

There are two basic types of orbit elements, osculating and mean. A set of osculating elements represents an exact definition of one point on the orbit and, therefore, is equivalent to cartesian coordinates of position and velocity at that point. Though osculating elements may often be difficult to obtain, there is no ambiguity in their definition.

Mean elements, on the other hand, are not so well defined. As a satellite travels in its orbit around the earth, the osculating elements at each point in the orbit, when plotted versus time, will exhibit periodic fluctuations. These fluctuations may be either long-period (of the same period as $\underline{\omega}$) or short-period (of the same period as $\underline{\nu}$). A mean element, as the name implies, is an expression for the average value of the osculating element (without periodic fluctuations). Usually, a mean element is defined as the osculating minus only short-period fluctuations.

In engineering activities other than astrodynamics, the standard working elements are osculating elements; for instance, the powered flight trajectory analyst will usually present insertion conditions in terms of a position vector and a velocity vector. Various theories, therefore, have been devised to compute mean elements from a set of osculating elements. Some of these theories have peculiarities in that their definitions of mean elements are tailored toward use in a specific general perturbation theory. Consequently, these elements are not truly mean elements but, rather, starting conditions for that particular theory. The definition of "mean" semimajor axis by Kozai (Reference 23) is a good example of this type of definition. Assume that an initial osculating value of semimajor axis, $\underline{a}$, has been provided. To compute the mean semimajor axis, $\underline{\bar{a}}$, according to Kozai, first subtract short-period fluctuations:

$$a_0 = a - da_s$$

where

$$da_s = \frac{3}{2} J_2 \frac{r_e^2}{a} \left\{ \frac{2}{3}\left(1 - \frac{3}{2}\sin^2 i\right)\left[\left(\frac{a}{r}\right)^3 - (1-e^2)^{-3/2}\right] + \left(\frac{a}{r}\right)^3 \sin^2 i \cos 2(\nu + \omega)\right\}$$

At this point, one would have the standard value of mean semimajor axis (osculating minus short-periodic). However, Kozai continues and "conveniently" defines the mean value as

$$\bar{a} = a_0 \left[1 - \frac{3}{2} J_2 \left(\frac{r_e}{p} \right)^2 \left(1 - \frac{3}{2} \sin^2 i \right) \sqrt{1 - e^2} \right] \qquad \text{(I-1)}$$

which he needs for use in his general perturbation theory. The equations presently being used within the MSFC Orbit Lifetime Program (Reference 20) require the Kozai $\underline{\bar{a}}$ element. Moreover, it is believed that the "mean" elements given by SAO in their reports on past satellite histories (Reference 19) involve the Kozai $\underline{\bar{a}}$ definition. The GENPUR program, on the other hand, requires use of the $\underline{a_0}$ type definition, as do most other general perturbation theories. It might also be noted that output of the MSFC transformation program for "mean" elements uses $\underline{a_0}$ as the definition of semimajor axis.

Another theory which has been used to compute mean elements is that due to H. Small (Reference 24). For the sake of identification, results of his theory have been termed "smoothed" elements. Small's theory involves removing short-period fluctuations from the radius and velocity vectors. If $\underline{r}$, $\underline{v_R}$, and $\underline{v_L}$ are initial values of the osculating radius, radial velocity component, and normal velocity component, respectively, then the mean values are

$$\bar{r} = r - dr$$

$$\bar{v}_R = v_R - dv_R$$

$$\bar{v}_L = v_L - dv_L$$

These mean values are then used in a standard coordinate transformation procedure to compute mean (or rather "smoothed") elements. For instance,

$$\bar{a}_s = \frac{\bar{r}\mu}{2\mu - \bar{r}\bar{v}^2}$$

where

$$\bar{V}^2 = \bar{V}_R^2 + \bar{V}_L^2$$

In Reference 25, it was found that the essential difference between the Kozai a_0 and the Small $\bar{a}_s$ is

$$a_0 - \bar{a}_s = J_2 \frac{r_e^2}{a}\left(1 - \frac{3}{2}\sin^2 i\right)$$

The "smoothed" elements of Small are those used by the MSFC Orbit Lifetime Program for integration. However, it must be remembered that the A-E-P (Reference 26) equations within the Lifetime Program require the Kozai $\bar{a}$ given by Equation (I-1).

Other methods for computing mean elements are given in Reference 26. Initial mean elements for Brouwer's equations are computed by the following procedure. Sets of osculating elements over some time interval are required. The long-periodic and short-periodic variations are computed for each element in every osculating set by using Brouwer's equations. Then the secular motion is computed referencing every set to an arbitrarily chosen epoch time. Mean value sets are obtained by subtracting the secular and periodic terms from each respective osculating element set. These sets are then averaged, yielding one initial mean element set. Notice that, if only one set of osculating elements is available, the procedure is similar to that of Kozai. For instance, initial mean value of semimajor axis is

$$\bar{a} = a - a_p$$

where

$$a_p = \frac{1}{2} J_2 \frac{r_e^2}{a}\left\{\left(-1 + 3\cos^2 i\right)\left[\left(\frac{a}{r}\right)^3 - \left(1 - e^2\right)^{-3/2}\right] + 3\left(1 - \cos^2 i\right)\left(\frac{a}{r}\right)^3 \cos 2(v + \omega)\right\}$$

The Brouwer expression for a_p, in fact, is equal to the Kozai expression for da_s. However, there are no further steps in the Brouwer procedure, so his mean is truly osculating minus short-periodic terms.

The importance of having well-defined mean values cannot be overemphasized. For example, it has been found that an initial error of only 0.4 km in the initial mean value of semimajor axis can cause a 360° error in mean anomaly at the end of 330 days.