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Equicontrollability and the Model-Following Problem

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ABSTRACT

This dissertation introduces the idea of equicontrollability and studies its application to the linear time-invariant model-following problem. The problem is presented in the form of two systems: generically called the plant and the model. The requirement is to find a controller to apply to the plant so that the resultant compensated plant behaves, in an input-output sense, the same as the model. All systems are assumed to be linear and time-invariant.

The basic approach used is to find suitable equicontrollable realizations of the plant and model and to utilize feedback so as to produce a controller of minimal state dimension. The concept of equicontrollability (introduced here) is a generalization of control canonical (phase variable) form applied to multivariable systems. It allows one to visualize clearly the effects of feedback and to pinpoint the parameters of a multivariable system which are invariant under feedback.

The basic contributions contained in this work are: (1) the development of equicontrollable form; (2) solution of the model-following problem in an entirely algorithmic way, suitable for computer programming, and (3) resolution of some questions on system decoupling (along with the application of the above algorithm to accomplish decoupling, as shown in Appendix C).

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I. INTRODUCTION

A. Description of the Problem

The model-following problem has appeared periodically in the literature for some time. It most often is referenced in conjunction with proposed solutions for flight control problems. These have the property of having no *a priori* cost function to be minimized (contrary to most problems in modern control theory). Instead, a model is specified whose dynamic response is considered desirable. The problem then becomes finding a controller (compensator) which, when added to a given plant, will cause the resultant system to have a response as close to the model's as possible.

A typical example of this kind of problem might arise during the design of an SST. The future pilots of the SST would like to fly the aircraft before it is built. This clear contradiction is generally "solved" by construction of a simulator which does a credible job of reproducing the "feel" of the aircraft, but is still not quite the real thing. A recent proposal has been to take a small jet transport and build a complete SST nose section on its front. Then some sort of artificial feel system would connect the controls to the transport in such a way as to make it feel to the pilots as if it were an SST behind them. In this problem the model is the SST, the plant is the transport and the artificial feel system is the compensator to be designed. This problem was considered by Rynaski and Whitbeck [17] with some success.

Another problem of the same type occurs in the design of compensation for VTOL aircraft. In this case the plant is the helicopter or VTOL and the model is a mathematical description of what sort of system the pilot would like to fly. Such a description is probably no more detailed than the requirements of asymptotic stability, negligible overshoot and decoupling of lateral and longitudinal dynamics. Time constants are stated equally as grossly. Nonetheless, a model may be formulated which the pilot would accept. This type of problem has been considered by Wolovich and Shirley [21]. Unfortunately, they had to compromise the model substantially to get a solution.

In summary then, the model-following problem involves some sort of plant and a model whose response the compensated plant is to emulate. The key factor in every problem statement of this type is a lack of a measure of performance. No cost function is given to minimize, but rather the designer wants the model and compensated plant to be the same in response; or, barring that, "as close to the same as possible". This is the point on which the problem hinges. At such time as the measure of error is defined, the problem is half solved.

One might notice that there is a definite link between the model-following problem and classical control theory; at least in so far as the observation that virtually all classical design problems specify enough desired parameters to roughly define a model. For example, such criteria as overshoot, risetime, gain margin and so forth can be translated into the specification of a second or third order system. Certainly no cost function is given to be minimized and the designer would like the plant and such a model to behave as nearly the same as possible. Hence, the original form of the model-following problem acts as a sort of bridge between what is commonly called optimal control, i.e. problems <u>with</u> a cost functional, and classical control problems.

Strangely enough, this link was to some degree born out historically. At the time when a type of model-following problem was being considered in a frequency domain sense (see Freeman [8], Kavanaugh [11] and Morgan [14]),"modern" control theory was waxing; in particular Kalman's work was making the quadratic performance index <u>de riguer</u>. The net result was a loss of the link between the two approaches. Later some authors attempted to re-establish it (notably Brockett [3]), but most research continued in the direction of optimal control theory.

B. Proposed Solutions

1. The Frequency Domain Approach

The initial interest in the model-following problem came from the relative ease with which its scalar version could be solved. That version involved the block diagram in Fig. 1.1:

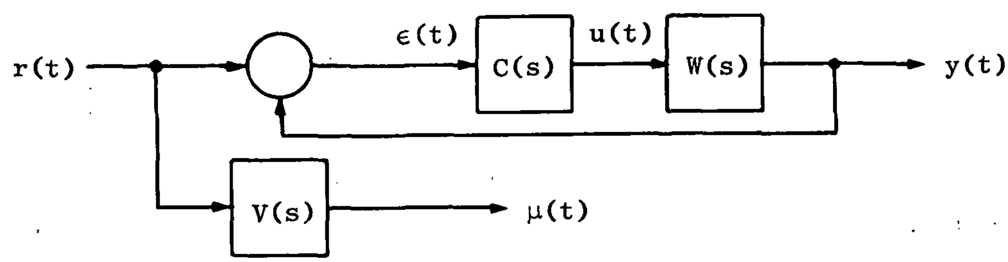

Fig. 1.1
The Scalar Problem

$r(t)$ is a scalar reference input, $V(s)$ is a given linear model and $W(s)$ is the plant. The plant has been imbedded into a unity feedback loop with the series compensator $C(s)$. The objective is to find $C(s)$ so that $y(t) = \mu(t)$. In other words so that:

$$\frac{C(s)W(s)}{1 + C(s)W(s)} = V(s)$$

or that:

$$C(s) = \frac{1}{W(s)} \cdot \frac{V(s)}{1-V(s)} \tag{1.1}$$

There were, of course, some restrictions placed on $C(s)$. Most often they would be:

1. the closed loop system using $C(s)$ must be stable and,
2. $C(s)$ must be realizable (or, more restrictively, low pass).

These two restrictions are easily dispatched. First, if the numerator of $W(s)$ has a root in the RHP, then clearly we must require that $V(s)$ also have one at the same location. This assumption will guarantee stability. The second restriction is no more difficult to deal with. If the numerator of $C(s)$ is of greater dimension than its denominator, one may modify the model in the form $\alpha^r V(s)/(s+\alpha)^r$ yielding:

$$C(s) = \frac{1}{W(s)} \cdot \frac{\alpha^r V(s)}{(s+\alpha)^r - \alpha^r V(s)} \tag{1.2}$$

We may certainly choose an r so that $C(s)$ will be realizable. Then α can be chosen so that the effect on the model will be as small as desired (at the cost of increased feedback gain of course).

The manipulations engendered by the scalar case are fairly transparent, while the multivariable case is quite opaque. The block diagram may be left the same, except that all variables would be vectors instead of scalars, and all transfer functions would be matrices. We may then formally derive the required compensator:

$$C(s) = [W(s)]^{-1} V(s)[I-V(s)]^{-1} \tag{1.3}$$

assuming that all inverses exist. Now, unfortunately, we cannot so glibly arrange to satisfy the two given requirements. Stability is not so clear, and realizability is more of a chore. Worse yet, the computation of the inverses is especially complicated. All in all, the problem, as given, is very difficult. It was only saved from extinction by the advent of the quadratic loss approach which circumvented the difficulties via a reformulation. Although we shall also reformulate it, Chapter V contains a consideration of a similar problem which the algorithm given in the sequel can solve.

2. The Time Domain Approach

The bulk of work in attempting to solve the model following problem has used optimal control theory. To do this requires the generation of a cost function. Since the actual problem statement does not include one, it is necessary for the designer to create one to suit both his convenience and to produce an acceptable answer. Most all of the papers written in the area specify a quadratic performance measure. More specifically, if the stateof the model is ξ and that of the plant is x, then they choose:

$$J = \int_0^\infty \left[(x-\xi)^T Q (x-\xi) + u^T R u \right] dt \qquad (1.4)$$

where u is the control applied, as their measure. Since the plant and model are assumed to be linear constant systems, the solution to such a problem is well known. Such an approach has been used by Kalman and Englar [10], Rynaski and Whitbeck [16], Rynaski et al [17], Tyler [18], Tyler and Tuteur [19], Asseo [1], and Winsor and Roy [20]. These have been, in some sense, ordered as to increasing sophistication in specifying the matrices Q and R ; i.e.: the relative cost or states and control effort. Coincidentally, they also appear in nearly correct chronological order and order of increasingly useful results (a judgment based on the author's personal bias and minimal data).

The key difficulty in all of these attempts is the lack of definition of Q and R . This is combatted by methods ranging from cut and try to the addition of side constraints. None is especially preferable to another, except that perhaps Winsor and Roy's approach appears to be slightly more of a science than an art. Therein lies the biggest fault in any of them. The production of the control law is more of an art depending on the wit, cunning and experience of the engineer rather than an algorithm which a computer can be taught to solve.

At the same time, the very nature of the solution lends itself to criticism. Those familiar with the solution to the linear quadratic loss regulator problem recall that it consists of constant feedback from all the states. Since the designer here had to use the states of both the system and the model in the cost function, he must build a realization of the model as part of the controller. Diagramatically, the compensated plant is shown in Fig. 1.2:

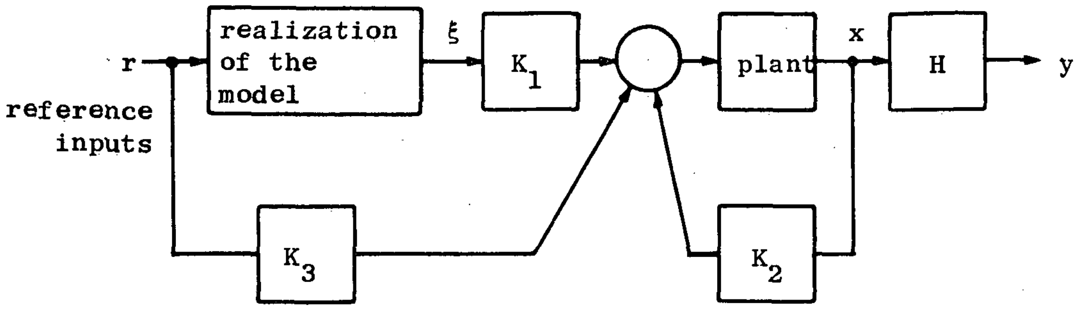

Fig. 1.2

Compensated Plant

The K_i would be produced by the solution to the quadratic loss problem created. Note that K_1 will not be zero, even in the case where a K_2 exists such that the system in Fig. 1.3

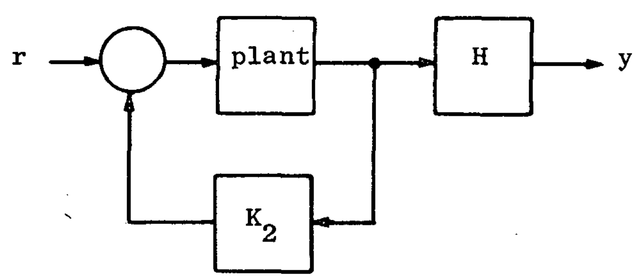

Fig. 1.3

Compensated Plant, Trivial Case

has the same response as the model! Similarly, intermediate solutions, such as a compensator needing only one state of dynamics to do the job perfectly, will not be uncovered. In summary, the solutions of this type suffer these difficulties:

1. Since the cost function is unspecified, it must be chosen ad hoc. Moreover, since the complete solution to the inverse problem is as yet unknown, iteration toward a solution is often a blind search.

2. Exact solutions, even when they exist, will not be found. The compensator must contain a realization of the model -- a real annoyance.
3. Although not mentioned here as yet, the solutions achieved by the above authors were none too good, unless the problems posed were virtually trivial. More on this below.

Quite a radical departure from the quadratic loss approach was suggested by Erzberger [6]. The essence of his approach is contained in the following observation. Of all conceivable happenstance, the most fortunate, from the point of view at hand, would be for a feedback to exist such that the plant and model would be identical. Suppose that the plant is governed by the vector differential equations:

$$\dot{x} = Fx + Gu \quad ; \quad y = Hx \tag{1.5}$$

and the model by:

$$\dot{\xi} = L\xi + Gr \quad ; \quad \mu = H\xi \quad . \tag{1.6}$$

Suppose there exists a K such that:

$$F + GK = L \tag{1.7}$$

then we may obtain the compensated system

$$\dot{x} = (F + GK)x + Gr \quad ; \quad y = Hx \tag{1.8}$$

by letting $u = Kx + r$. Now:

$$\dot{x} = Lx + Gr \quad ; \quad y = Hx \tag{1.9}$$

has the same transfer function as the model. Such a K exists if and only if $(I - GG^{\dagger})(L - F) = \theta^{*}$. The solution is then $K = G^{\dagger}(L - F)$ where $G^{\dagger}$ is the penrose pseudo-inverse of G . (See [10] or [12]). Close observation shows that for such a K to exist, great structural similarity must exist between F and L . Moreover, G must have sufficient rank. There have been at least two such examples given in the literature which were solved using the quadratic cost technique [17],

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* θ represents the null matrix.

[18]. Erzberger has shown that, for this limited class of problem, his direct approach involves <u>much</u> less computation than the quadratic cost approach. We have extended Erzberger's approach for use in cases in which the aforementioned structural correspondence is satisfied. Since this work is not directly related to our major exposition, it is treated in Appendix A.

The work of Asseo [1] is worthy of comment. Given the plant and model

$$\text{PLANT:} \quad \dot{x} = Fx + Gu \qquad y = Ix \tag{1.10a}$$

$$\text{MODEL:} \quad \dot{\xi} = \Phi\xi + \Gamma\omega \qquad \mu = I\omega \tag{1.10b}$$

the method given in [1] finds a feedback control: $u = K_v\omega + K_p x + K_m\xi$, such that K_v and K_m satisfy:

$$GK_v = \Gamma \tag{1.11a}$$

$$G(K_m - K_p) = \Phi - F \tag{1.11b}$$

where K_p is arbitrary. If (1.11) are satisfied, we may substitute into (1.10a) to obtain:

$$\dot{x} = Fx + G(K_v\omega + K_p x + K_m\xi) \tag{1.12}$$

hence:

$$(\dot{x} - \dot{\xi}) = (F + GK_p)(x - \xi) \tag{1.13}$$

and K_p may be chosen to make the error system as stable as is desired. (Under the usual controllability assumptions). The difficulty lies in trying to find a solution to (1.11). Such a solution is vital to the method. In [1] a canonical form (originally derived in [2]) is used as a tool to solve (1.11). The claim was that for any pair (F,G) , there exists a transformation T such that:

$$T^{-1}FT = \begin{array}{c} \left[\begin{array}{c|c} \theta & I \\ \hline A & B \end{array}\right] \end{array}\begin{array}{l} n-m \\ m \end{array} \quad \text{and} \quad T^{-1}G = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right]\begin{array}{l} n-m \\ m \end{array} \tag{1.14}$$

In fact, such a transformation does not, in general, exist even if we allow:

$$T^{-1}G = \begin{array}{c} \left[\begin{array}{c} \theta \\ \hline D \end{array}\right] \end{array} \begin{array}{l} n-m \\ m \end{array} \quad , \quad \det D \neq 0 \tag{1.15}$$

To check that this obviates the general solution of (1.11), we need only observe that first defining new states as $z = T_p x$ and $\eta = T_m \xi$ solving (1.11), in the sense of [1], is equivalent to finding P (nonsingular), K_1 and K_2 such that:

$$GK_1 = P^{-1}\Gamma \tag{1.16a}$$

$$GK_2 = P^{-1}\Phi - F \tag{1.16b}$$

Suppose (F,G) is in the form above. Then we need to solve:

$$GK_1 = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] K_1 = \left[\begin{array}{c} \theta \\ \hline K_1 \end{array}\right] = P^{-1}\Gamma \tag{1.17a}$$

$$\left[\begin{array}{c} \theta \\ \hline K_2 \end{array}\right] = P^{-1}\Phi P - \left[\begin{array}{c|c} \theta & I \\ \hline A & B \end{array}\right] \tag{1.17b}$$

But to solve (1.17) requires the existence of P as in (1.14), which is not, in general, possible [5]. Hence (1.11) does not have a solution for every plant and model. This implies, unfortunately, that Theorem 2, part b, [1] is false. (For reference, Theorem 2, part a, is concerned with the case in which G is nonsingular).

There is a further difficulty with the approach of [1]. The formulation given there attempts to match states. In a reasonable model-following problem, we would expect no net output transformation to be allowed. That is, if we desire to keep the error $\epsilon = y-\mu$ as small as possible (in some sense), to consider $\hat{\epsilon} = y-T\mu$ is just short of absurd. It is tantamount to matching, say, the pitch response of the model to the roll response of the aircraft. Since $x = y$ and $\xi = \mu$ in the formulation of [1], we must conclude that the plant and model transformation are the same $\left(T_p = T_m\right)$. Hence $P = T_p^{-1} T_m = I$ in (1.17) and the <u>same</u> transformation must take both the plant and the model to the <u>same</u> canonical form. This further reduces the number of cases which can be reasonably handled by this direct approach. Overall, the approach in [1], is an improvement on Erzberger's [6], but is not as general as needed for multivariable control.

Winsor and Roy's paper [20] errs in description but appears correct in mathematics. They falsely claim, in the text of the paper, that Erzberger's test is a necessary condition, whereas it is only sufficient, and quite restrictive at that. They do, however, use his results properly, which relegates their error to editing. Their approach, as optimal control approaches go, seems reasonable. At least they try to take up some of the slack in the problem by introducing side conditions, rather than ignoring it and plunging blindly ahead.

As a whole, then, the literature is none too satisfying on this problem. The solutions are sparse and essentially <u>ad hoc</u>. None is of the type where the problem may be simply packaged and fed to a computer which could return an answer. All have to interact with their programs in a series of cut and try iterations until, through chance or artistry, the solution emerges. It is believed that the present work eliminates these objections.

C. <u>Allied Problems</u>

The most closely allied problem found in the literature is that of decoupling [7], [15], [22]. In such a problem the designer is given a system and required to find a compensator which would decouple its transfer function. This is generally interpreted as meaning that each input should effect one output (or group of outputs) and no other input would effect that output (or group). If the transfer function matrix were square, we would wish to diagonalize it.

The chief difficulty in turning this into a model following problem is the lack of a specific model. It is possible to cut and try models until a suitable one can be found, but this becomes tedious. Nonetheless, the two problems are closely enough allied that workers in either area can profit from papers concerning the other. Perhaps the most interesting contrast between the two is that the 'decoupling' literature is almost exclusively theoretical in nature, while the 'model following' papers are devoted to computation and algorithms. This difference is perhaps most valuable to the workers in decoupling since many model following algorithms could be used to decouple whereas little of their theory can be used in a model following algorithm.

The present work has had considerable success in solving decoupling problems. Since 'exact' solutions are given, the results will indeed decouple the given plant. The only difficulty lies in finding a suitable model. In Chapter V we will show how the work of Wonham and Morse [15] and [22], is connected to the present discussion.

D. Contributions of this work

The essential contributions of this work are as follows:

1. The introduction of and development of equicontrollability (Chapter III).
2. Development of an algorithmic solution to the model-following problem which is suitable for computer programming (Chapter IV).
3. Resolution of some questions on the decoupling of multivariable systems (along with the application of the above algorithm to accomplish such decoupling) (Appendix C).

Attendant on these specific items are the techniques devised for the analysis of multivariable systems - particularly the use of non-minimal (equicontrollable) realizations, as noted in Chapters III and IV.

II. PROBLEM DEFINITION AND DISCUSSION

In view of the preceding discussion, it becomes apparent that some better approach must exist for solving the model following problem. It should not require a performance index as such, and should yield simple answers to simple problems. It should be reducible to an algorithm suitable for digital computation with little or no human interaction from input to solution. Hopefully, it should not be iterative. The solution to be proposed satisfies all of these requirements.

As has been pointed out, the definition of the problem essentially structures the solution. We suppose that we are given two linear constant dynamical systems in the differential equation form:

$$\text{PLANT:} \qquad \dot{x} = \underset{n\times n}{F}\, x + \underset{n\times m}{G}\, u \qquad y = \underset{m\times n}{H}\, x \tag{2.1}$$

$$\text{MODEL:} \qquad \dot{\xi} = \underset{p\times p}{\Phi}\, \xi + \underset{p\times m}{\Gamma}\, \omega \qquad \mu = \underset{m\times p}{\Delta}\, \xi \tag{2.2}$$

All capital letters are matrices. Note that the inputs and outputs of the plant and model have the same dimension. This is not an unreasonable assumption since it is hard to conceive of a problem having a plant with, say, three outputs attempting to track a model with four! This would be contradictory. The requirement that the number of inputs equal the number of outputs is more obtuse, but not without justification. In most aircraft problems, the number of control inputs is equal to the number of consciously controlled outputs. That is, if the pilot pushes the stick forward he has in mind a certain response in pitch. That this will, some integrations later, produce a change in altitude is not of immediate concern. Better, if the aircraft executes the proper motion in pitch, the other modes attendant on pitch will respond favorably.

Since the problem itself is linear, we might require that a linear controller be used. Moreover, since virtually all other methods involve state feedback, it is logical for us to do so also. This may be reduced to specifying that the controller be a linear function of both the states and external inputs. In the block diagram form shown in Fig. 2.1 we insist on the linear transfer function C(s) as the compensator.

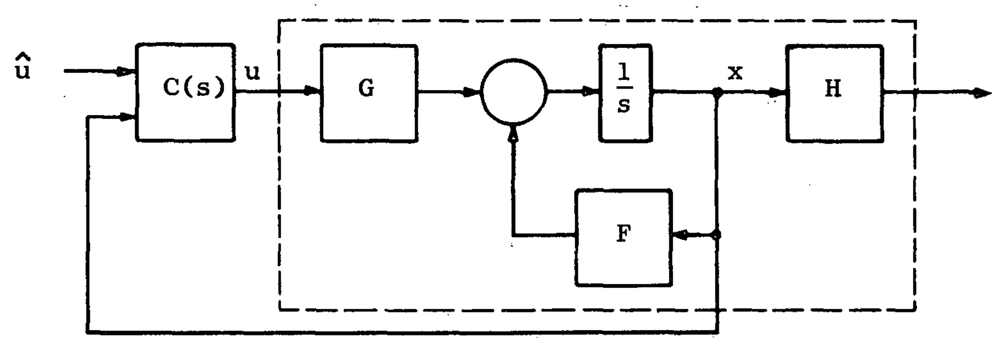

Fig. 2.1
Form of Compensation

The choice of a specific controller C(s) Must be based on some measure of goodness. We choose to make this measure in terms of acceptable modifications of the model instead of the minimization of some function. Although the advantages of this stratagem are legion, the clarity with which any performance degradation is displayed <u>before the computation is performed</u> is, in itself, sufficiently compelling to justify its use. More explicitly, we shall define the new model in Fig. 2.2:

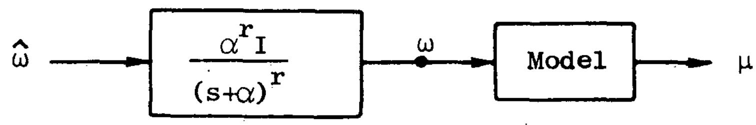

Fig. 2.2
Form of Model

where α is prespecified by the designer, and r is to be as small as possible.

α is a clear measure of closeness. As $\alpha \to \infty$, the above transfer function approaches V(s) . But the penalty for making α large is to have large feedback gains. Hence, one may conceivably balance gain magnitude against accuracy in matching, which is an anticipated tradeoff.

This discussion may be maneuvered into a definition of the problem, by way of listing the assumptions we shall now make.

1. The plant and model are finite dimensional linear constant systems.

2. Both the plant and model are completely controllable and observable.

3. The plant and model have the same number of inputs and outputs. (All four are the same).

4. The compensator should be a finite linear constant system, C(s) , taking $\hat{u}$, the command inputs, and x , the plant state, into u . This assumes that the state is available.

5. s = 0 is not a root of either num det $H(sI-F)^{-1}G$ or num det $\Delta(sI-\Phi)^{-1}\Gamma$. (Num det W(s) is defined below).

6. Num det $H(sI-F)^{-1}G$ has no roots in the right half plane.

7. The designer will accept a revised model of the form:

$$\Delta(sI-\Phi)^{-1}\Gamma \frac{\alpha^r}{(s+\alpha)^r}$$

where α is at his disposal, and r is to be as small as possible.

The assumptions, except for 5 and 6, are direct consequences of the previous discussion. The fifth is best taken on faith until Chapter 4 in which we shall use it as a guarantee that the algorithm will work. The symbol num det W(s) (where W(s) is a square transfer function matrix) represents the numerator of the determinant of W(s) . The numerator is extracted when the denominator equals the characteristic equation of W(s) . A more graphic description is: if W(s) = $H(sI-F)^{-1}G$, then:

$$\text{num det } W(s) \triangleq \det W(s)\cdot\det(sI-F) \tag{2.3}$$

For transfer functions of (2.1) or (2.2), num det is a polynomial in s of degree at most $n-1$. At the end of Chapter 3 we shall prove the fact that, for all conformable K, (i.e. for any K of the proper dimensions) num det $H(sI-F)^{-1}G$ = num det $(sI-F-GK)^{-1}G$. In other words, num det $W(s)$ is invariant under state feedback.

Since, in the scalar case, feedback is unable to move zeros, we will have to use a series compensator and pole-zero cancellation to achieve a perfect match if plant and model zeros differ. Clearly to cancel an RHP zero requires an unstable root, which would result in an unstable design. Hence the inclusion of Assumption 6, which is the analog of the above observation in the multivariable case. This assumption can be weakened, as we will see later.

Defining the problem via a set of assumptions is perhaps the clearest approach in this case. The original statement is so vague that no straightforward definition is really feasible. We do gain something for our interpretation of the problem. The ability to give the designer a remarkably lucid picture of what he can expect by way of a solution is especially useful. We do not have to present a vague idea of how well the compensated plant will do -- we can nearly describe its transfer function! Only the value of r is unknown. The remainder is specified by the given equations or is up to the designer's whim. To ask for more would border on the unreasonable.

III. EQUICONTROLLABILITY

A. Definition and Motivation

For a number of problems, the usual property of controllability does not provide sufficient information as to structure. A controllable system with m inputs may be controllable from only one input or require each and every one. Moreover, one cannot specifically give a canonical form for multi-input systems that is the same for all such systems. The best you can do is specify a procedure which will always yield some form, although many forms are possible. We shall consider here how we may trade one form of uncertainty for another to best serve our purposes.

DEFINITION: The pair (F,G) has controllability index ρ if ρ is the smallest integer such that:

$$C_\rho = \left[G, FG, \ldots, F^{\rho-1}G\right] \tag{3.1}$$

has full rank $n = \dim F$.

DEFINITION: The pair (F,G) is equicontrollable if $n = \rho m$ where ρ is the controllability index of (F,G), F is $n \times n$ and G is $n \times m$.

Note that the definition of equicontrollability implies that[1] $m|n$ and that the first $n = m\rho$ columns of the controllability matrix

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[1] $m|n$ is read "m divides n" and means that there exists a k such that $n = mk$ where k belongs to the same class as m and n, (e.g. if m and n are integers, k is an integer).

are independent. These conditions are enough to enable us to write such a system in a particularly useful canonical form -- equicontrollable form. Let us consider a slightly more general case.

<u>PROPOSITION 1</u>: Given the pair of real matrices (F,G) , F $n \times n$, G $n \times m$ and full rank, then there exists a similarity transformation T such that:

$$TFT^{-1} = \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \left[\begin{array}{c|c} \Theta & I \\ \hline \multicolumn{2}{c}{X} \end{array}\right] \end{array} = \begin{array}{cccccccc|l} r & m-r & r & m-r & & r & m-r & r & \\ \hline \Theta & \Theta & I & \Theta & & \Theta & \Theta & \Theta & r \\ \Theta & \Theta & \Theta & I & \cdots & \Theta & \Theta & \Theta & m-r \\ \Theta & \Theta & \Theta & \Theta & & \Theta & \Theta & \Theta & r \\ \Theta & \Theta & \ddots & & & I & \Theta & \Theta & r \\ \Theta & \Theta & & & & \Theta & I & \Theta & m-r \\ \Theta & \Theta & & & & \Theta & \Theta & I & r \\ X & X & & \cdots & & X & X & Y & m-r \\ X & X & & & & X & X & X & r \end{array} \quad [1] \tag{3.2}$$

and

$$TG = \begin{array}{c} m \\ \left[\begin{array}{c} \Theta \\ \hline S \end{array}\right] \begin{array}{l} n-m \\ m \end{array} \end{array} = \begin{array}{cc|l} m-r & r & \\ \hline \Theta & \Theta & r \\ \Theta & \Theta & m-r \\ \Theta & \Theta & r \\ \hline I & \Theta & m-r \\ A & I & r \end{array} \tag{3.3}$$

where the X's, Y, and A are unspecified entries determined by (F,G) ; and the parameter r is defined such that

$$n = (\rho - 1)m + r \qquad 0 \leq r < m \tag{3.4}$$

[1] All unmarked braces represent blocks of m rows or columns.

if and only if the last n columns of the controllability matrix:

$$C = F^{n-1}G\ ,\ F^{n-2}G\ ,\ldots,\ FG\ ,\ G \tag{3.5}$$

are linearly independent.

<u>NECESSITY:</u> Suppose the matrix T exists as specified and define the following matrices:

$$\begin{aligned} F_* &= TFT^{-1} \\ G_* &= TG \end{aligned} \tag{3.6}$$

let $$\bar{C} = \text{last } n \text{ columns of } C \tag{3.7}$$

$$C_* = \left[F_*^{n-1}G_*, F_*^{n-2}G_*, \ldots, F_*G_*, G_*\right] \tag{3.8}$$

$$\bar{C}_* = \text{last } n \text{ columns of } C_* \tag{3.9}$$

then by inspection, we see that

$$C = T^{-1}C_* \tag{3.10}$$

and hence that

$$\bar{C} = T^{-1}\bar{C}_* \tag{3.11}$$

Moreover, $\bar{C}_*$ is nonsingular by construction. More specifically:

$$\bar{C}_* = \begin{array}{c} \begin{array}{ccccc} r & \overbrace{m-r \quad r} & & \overbrace{m-r \quad r} & \end{array} \\ \left[\begin{array}{ccccc} I & \Theta & \Theta & \Theta & \Theta \\ X & I & \Theta & \Theta & \Theta \\ X & A & I \cdots & \Theta & \Theta \\ X & X & X & & \\ & & \vdots & I & \Theta \\ X & X & X \cdots & A & I \end{array}\right] \begin{array}{l} r \\ \left.\begin{array}{l} m-r \\ r \end{array}\right\} \\ \\ \left.\begin{array}{l} m-r \\ r \end{array}\right\} \end{array} \end{array} \tag{3.12}$$

thus $\bar{C}$ is nonsingular as required. ■

SUFFICIENCY: The proof is constructive. Assume that $\bar{C}$ is nonsingular. First we define partitions necessary to describe $\bar{C}$ in terms of F and G.

Let

$$G = \begin{array}{c} \begin{array}{cc} m-r & r \end{array} \\ \left[\begin{array}{c:c} G_1 & G_2 \end{array}\right] \end{array} \tag{3.13}$$

and

$$\bar{G} = \left[\begin{array}{c:c} FG_2 & G_1 \end{array}\right] \tag{3.14}$$

then

$$\begin{aligned} \bar{C} = & \left[\begin{array}{c:c:c:c:c:c} \overbrace{F^{k-1}G_2}^{r} & \overbrace{F^{k-2}G}^{m} & & \cdots & \overbrace{FG}^{m} & \overbrace{G}^{m} \end{array}\right] \\ & \left[\begin{array}{c:c:c:c:c:c:c:c:c} F^{k-1}G_2 & F^{k-2}G_1 & F^{k-2}G_2 & \cdots & FG_1 & FG_2 & G_1 & G_2 \end{array}\right] \\ & \left[\begin{array}{c:c:c:c:c} \underbrace{F^{k-2}\bar{G}}_{m} & \underbrace{F^{k-3}\bar{G}}_{m} & \cdots & \underbrace{\bar{G}}_{m} & \underbrace{G_2}_{r} \end{array}\right] \end{aligned} \tag{3.15}$$

Next we define the $m \times n$ matrix E from which the transformation is to be constructed.

$$E\bar{C} = \begin{array}{c} \begin{array}{ccccc} m & m & & m & r \end{array} \\ [\; I \quad \Theta \quad \ldots \; \Theta \quad \Theta \;] \end{array} \; m \tag{3.16}$$

E exists since $\bar{C}$ is nonsingular, and is simply the first m rows of $\bar{C}^{-1}$. We partition E so that:

$$E = \begin{array}{c} n \\ \left[\begin{array}{c} E_1 \\ \hdashline E_2 \end{array}\right] \end{array} \begin{array}{l} r \\ m-r \end{array} \tag{3.17}$$

We claim that the transformation T is:

$$T = \left[\begin{array}{l} E \\ EF \\ \vdots \\ EF^{k-2} \\ E_1F^{k-1} \end{array}\right] \begin{array}{l} m \\ m \\ \\ m \\ r \end{array} \tag{3.18}$$

In the first place, T is nonsingular as we can see by using the last partition given for $\bar{C}$:

$$T\bar{C} = \begin{bmatrix} EF^{k-2}\bar{G} & \cdots & E\bar{G} & EG_2 \\ EF^{k-1}\bar{G} & \cdots & EF\bar{G} & EFG_2 \\ \vdots & & \vdots & \vdots \\ EF^{2k-4}\bar{G} & \cdots & EF^{k-2}\bar{G} & EF^{k-2}G_2 \\ E_1F^{2k-3}\bar{G} & \cdots & E_1F^{k-1}\bar{G} & E_1F^{k-1}G_2 \end{bmatrix} \tag{3.19}$$

By the definition of E , many elements in this matrix can be evaluated to give:

$$T\bar{C} = \begin{array}{c} \begin{array}{ccccc} m & m & & m & r \end{array} \\ \left[\begin{array}{ccccc} I & \Theta & \cdots & \Theta & \Theta \\ EF^{k-1}\bar{G} & I & & \Theta & \Theta \\ \vdots & & & & \\ EF^{2k-4}\bar{G} & & \cdots & I & \Theta \\ E_1F^{2k-3}\bar{G} & & & E_1F^{k-1}\bar{G} & I \end{array}\right] \end{array} \begin{array}{c} \\ m \\ m \\ \\ m \\ r \end{array} \tag{3.20}$$

Hence T is nonsingular since $\bar{C}$ is nonsingular by hypothesis. Also we find the product:

$$TG = \begin{bmatrix} E_1G \\ E_2G \\ \vdots \\ E_1F^{k-2}G \\ \hdashline E_2F^{k-2}G \\ E_1F^{k-1}G \end{bmatrix} = \begin{array}{c} \overbrace{\begin{array}{cc} m-r & r \end{array}} \\ \left[\begin{array}{cc} \Theta & \Theta \\ \Theta & \Theta \\ & \\ \Theta & \Theta \\ \hdashline I & \Theta \\ A & I \end{array}\right] \end{array} \begin{array}{l} \\ \left.\begin{array}{l} r \\ m-r \end{array}\right\} \\ \\ r \\ \left.\begin{array}{l} m-r \\ r \end{array}\right\} \end{array} \tag{3.21}$$

where $A = E\ ,\ F^{k-1}G$. Similarly:

$$TFT^{-1} = \begin{bmatrix} E_1F \\ E_2F \\ \vdots \\ E_1F^{k-1} \\ \hdashline E_2F^{k-1} \\ E_1F^{k} \end{bmatrix} T^{-1} = \begin{array}{c} \begin{array}{ccccccc} \overbrace{r \quad m-r} & & \overbrace{r \quad m-r} & & & \overbrace{m-r \quad r} & \end{array} \\ \left[\begin{array}{ccccccc} \Theta & \Theta & I & \Theta & \cdots & \Theta & \Theta \\ \Theta & \Theta & \Theta & I & & \Theta & \Theta \\ & & & & \ddots & & \\ & & & & & I & \Theta \\ \Theta & \cdots & & & & \Theta & I \\ X & & & & & X & Y \\ X & & & & & X & X \end{array}\right] \end{array} \begin{array}{l} \\ r \\ m-r \\ \vdots \\ m-r \\ r \\ m-r \\ r \end{array} \qquad (3.22)$$

∎

Then as a special case we may write

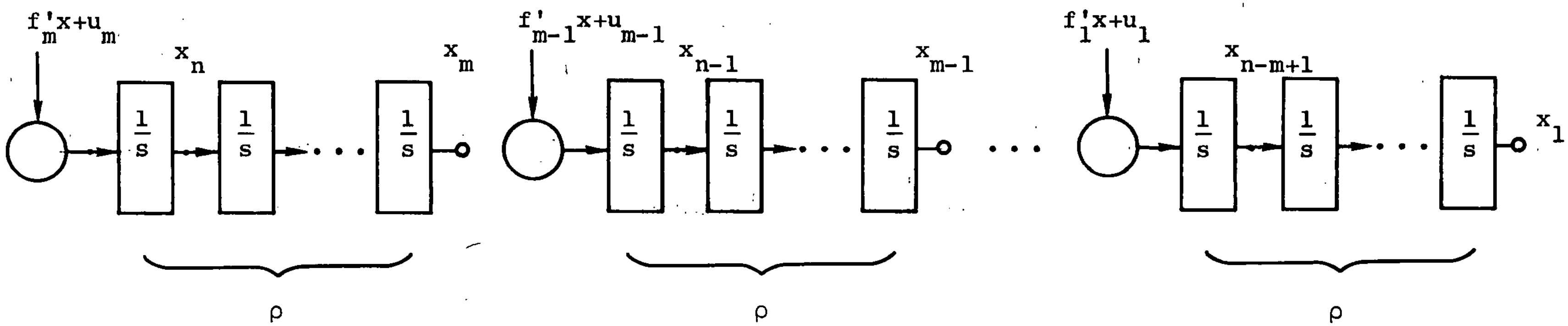
COROLLARY: Given the conditions in the previous proposition, and given that $r = 0$ (i.e. $m|n$) then there exists a transformation T such that:

$$TFT^{-1} = \begin{array}{c} \begin{array}{cc} m & n-m \end{array} \\ \left[\begin{array}{c:c} \Theta & I \\ \hdashline \multicolumn{2}{c}{X} \end{array}\right] \end{array} \ , \quad TG = \begin{array}{c} m \\ \left[\begin{array}{c} \Theta \\ \hdashline I \end{array}\right] \end{array} \begin{array}{l} \\ n-m \\ m \end{array} \qquad (3.23)$$

iff (F,G) is equicontrollable. The above is equicontrollable canonical form.

The name "equicontrollable" is now easy to justify. Fig. 3.1 shows a block diagram representative of such a system. Each input drives a string of m integrators, hence in some sense, the inputs control "equally". The figure also shows (as does the canonical form) the similarity of this form and control canonical form.

PROPOSITION 2: The form given in Proposition 1 is unique. That is, given two realizations of the same system: (F,G) and $(\hat{F},\hat{G})$, both in the canonical form specified in the statement of Proposition 1, then $F = \hat{F}$ and $G = \hat{G}$.



$$\begin{bmatrix} \quad X \quad \end{bmatrix} \triangleq \begin{bmatrix} \text{——} f_1' \text{——} \\ \vdots \\ \text{——} f_m' \text{——} \end{bmatrix}$$

Fig. 3.1

Equicontrollable Structure

The proof consists of two lemmas.

Lemma 1: The recipe given in Proposition 1 yields the same result regardless of the coordinates of the original system. That is, given (SFS^{-1}, SG), the procedure of Proposition 1 yields the same realization, for all S.

Proof of Lemma 1: In the sufficiency argument of Proposition 1, we constructed the transformation T which produced the desired realization. If we define

$$(F^s, G^s) \triangleq (SFS^{-1}, SG) \tag{3.24}$$

and:

$$\overline{C}^s \triangleq \left[(F^s)^{k-2}\overline{G}^s \,\vdots\, (F^s)^{k-3}\overline{G}^s \,\vdots\, \cdots \,\vdots\, G_2^s \right] \tag{3.25}$$

also:

$$E^s\overline{C}^s = [I \quad \theta \quad \dots \quad \theta] \tag{3.26}$$

and so forth, we find that:

$$\overline{C}^s = S\overline{C}, \quad E^s = ES^{-1}, \quad \text{and} \quad T^s = TS^{-1} \tag{3.27}$$

so that:

$$T^sF^s(T^s)^{-1} = TS^{-1}F^sST^{-1} = TFT^{-1} \tag{3.28a}$$

and

$$T^sG^s = TS^{-1}G^s = TG \tag{3.28b}$$

Q.E.D.

Lemma 2: If (F,G) is in the canonical form given in Proposition 1 then the transformation T given in the proof of sufficiency equals I.

Proof of Lemma 2: Consider $\overline{C}$. After some tedious computation:

$$
\overline{C} = \begin{array}{c}
\begin{array}{ccccccccc} r & m-r & r & m-r & r & & r & m-r & r \end{array} \\
\left[\begin{array}{c:cc:cc:c:ccc}
I & \theta & \theta & \theta & \theta & & \theta & \theta & \theta \\
\hdashline
\theta & I & \theta & \theta & \theta & & \theta & \theta & \theta \\
\hdashline
X & A & I & \theta & \theta & & & \vdots & \\
X & X & X & I & \theta & & & & \\
X & X & X & A & I & & & & \\
 & & & & & I \ \ddots & & & \\
X & X & X & & & & I & \theta & \theta \\
X & X & X & & \cdots & & \theta & I & \theta \\
X & X & X & & & & X & A & I
\end{array}\right]
\end{array}
\begin{array}{l} \\ \left.\begin{array}{l} r \\ m-r \end{array}\right\} \\ \\ \\ \\ \\ r \\ m-r \\ r \end{array}
\tag{3.29}
$$

where the X's are unspecified entries. This implies that:

$$
\begin{array}{rl}
 & \begin{array}{cccccc} m & m & m & & m & r \end{array} \\
E = & [\begin{array}{cccccc} I & \theta & \theta & \ldots & \theta & \theta \end{array}]m \\
EF = & [\begin{array}{cccccc} \theta & I & \theta & \ldots & \theta & \theta \end{array}]m \\
\vdots & \\
EF^{k-2} = & [\begin{array}{cccccc} \theta & \theta & \theta & \ldots & I & \theta \end{array}]m \\
E_1F^{k-1} = & [\begin{array}{cccccc} \theta & \theta & \theta & \ldots & \theta & I \end{array}]r
\end{array}
\tag{3.30}
$$

hence $T = I$.

Q.E.D.

<u>Proof of Proposition 2:</u> Lemma 2 shows that the recipe in Proposition 1 yields $T = I$ if the realization (F,G) is already in the given canonical form. But Lemma 1 shows that the procedure of Proposition 1 always gives the same answer, regardless of the coordinates of the original system. Hence any two realizations in that canonical form must be the same.

Q.E.D.

COROLLARY 1: Given two minimal equicontrollable systems of the same state dimension (H, F, G) and (Δ, Φ, Γ), there exists a feedback matrix K such that:

$$H(sI-F-GK)^{-1}G = \Delta(sI-\Phi)^{-1}\Gamma \tag{3.31}$$

if and only if $\hat{H}=\hat{\Delta}$ when the two systems are reduced to the respective equicontrollable canonical forms: $(\hat{H}, \hat{F}, \hat{G})$ and $(\hat{\Delta}, \hat{\Phi}, \hat{\Gamma})$.

Sufficiency is easy. If the last m rows of $\hat{F}$ are $[\hat{F}_1,\ldots,\hat{F}_\rho]$ and of $\hat{\Phi}$ are $[\hat{\Phi}_1,\ldots,\hat{\Phi}_\rho]$ then the feedback matrix K is:

$$K = [\Phi_1 - F_1,\ldots,\Phi_\rho - F_\rho]T^{-1} \tag{3.32}$$

where $T\Phi T^{-1} = \hat{\Phi}$; $T\Gamma = \hat{\Gamma}$, $\Delta T^{-1} = \Delta T^{-1} = \hat{\Delta}$ (= $\hat{H}$ by hypothesis).

So:

$$\begin{aligned} T(F + GK)T^{-1} &= \hat{\Phi} \\ &= \hat{\Gamma} \left(= \begin{bmatrix} \theta \\ \theta \\ \vdots \\ I \end{bmatrix} \right) \\ HT^{-1} &= \hat{\Delta} \end{aligned} \tag{3.33}$$

Hence, via the proposition, we are done.

Necessity is similar. Suppose K exists as required. Then there exists a T such that:

$$\begin{aligned} T\hat{\Phi}T^{-1} &= F + GK \\ T\hat{\Gamma} &= G \\ \hat{\Delta}T^{-1} &= H \end{aligned} \tag{3.34}$$

But $(H, F + GK, G)$ and $(\hat{\Delta},\hat{\Phi}, \hat{\Gamma})$ satisfy the hypothesis of the proposition, hence $T = I$ and $\hat{\Delta} = \hat{H}$. ∎

COROLLARY 2: In proposition 1, $Y = \theta$, i.e.:

$$
TFT^{-1} = \begin{bmatrix}
\Theta & \Theta & I & \Theta & \cdots & \Theta & \Theta \\
\Theta & \Theta & \Theta & I & & \Theta & \Theta \\
& \vdots & & & & & \\
\Theta & \Theta & & & & I & \Theta \\
\Theta & \Theta & & \cdots & & \Theta & I \\
X & X & & & & X & \Theta \\
X & X & & & & X & X
\end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow \text{N.B.} \\ \\ \end{matrix} \qquad (3.35)
$$

Proof:

$$
Y = E_2F^{k-1}T^{-1} \begin{array}{c|l} \begin{bmatrix} \Theta \\ \vdots \\ \Theta \\ \\ I \end{bmatrix} & \begin{matrix} m \\ \vdots \\ m \\ \\ r \end{matrix} \end{array} \qquad (3.36)
$$

and

$$
TG_2 = \begin{array}{c|l} \begin{bmatrix} \Theta \\ \vdots \\ \Theta \\ \\ I \end{bmatrix} & \begin{matrix} m \\ \vdots \\ m \\ \\ r \end{matrix} \end{array} \qquad (3.37)
$$

Hence:

$$
Y = E_2F^{k-1}T^{-1}(TG_2) = E_2F^{k-1}G_2 \qquad (3.38)
$$

but:

$$
\bar{E}F^{k-1}G_2 = \begin{array}{c|l} \begin{bmatrix} I \\ \hline \Theta \end{bmatrix} & \begin{matrix} r \\ m-r \end{matrix} \end{array} \qquad (3.39)
$$

so:

$$
Y = E_2F^{k-1}G_2 = \overset{r}{[\Theta]}\ m-r \qquad (3.40)
$$

Q.E.D.

As an interesting observation, if $m|n$ and the system has been placed in the canonical form given in Proposition 1 but $Y \neq \Theta$, then the transformation

$$
P = \begin{array}{c}
\begin{array}{cccccccc}
\overbrace{r \quad m-r} & & \overbrace{r \quad m-r} & & & \overbrace{r \quad m-r} & & r
\end{array} \\
\left[\begin{array}{cc|cc|c|cc|c}
I & \Theta & \Theta & \Theta & & \Theta & \Theta & \Theta \\
Q & I & \Theta & \Theta & \cdots & \Theta & \Theta & \Theta \\
\hline
\Theta & \Theta & I & \Theta & & \Theta & \Theta & \Theta \\
\Theta & \Theta & Q & I & & \Theta & \Theta & \Theta \\
\hline
\vdots & & \vdots & & & \vdots & & \\
\hline
\Theta & \Theta & \Theta & \Theta & & I & \Theta & \Theta \\
\Theta & \Theta & \Theta & \Theta & \cdots & Q & I & \Theta \\
\hline
\Theta & \Theta & \Theta & \Theta & & \Theta & \Theta & I
\end{array}\right]
\begin{array}{l}
r \\ m-r \\ r \\ m-r \\ \\ r \\ m-r \\ r
\end{array}
\end{array}
\tag{3.41}
$$

will leave the form the same, only affecting the last m rows of the system matrix. Hence, the form is surely not unique. That is, other transformations than (3.18) will yield a similar form. (Note that if $r = 0$ or $Y = \Theta$, then $Q \equiv \Theta$).

The utility of equicontrollable form is yet to be demonstrated. The reader could well interject that for $n = m\rho$ to be true requires that a substantial structural constraint be placed on the system. To widen the class of minimal realizations which can be placed in equicontrollable form has been shown above to be impossible. On the other hand we could allow the addition of states to the system, which do not modify the transfer function, but do alter the structure so as to achieve equicontrollability. Such states will have to be controllable but not observable. (One gets into some semantic difficulty here. Since we shall be physically constructing such states -- say in analog computer fashion -- they are "measurable", but from <u>the outputs specified</u>, they are not observable.) The following theorem will demonstrate that we can always add such states. The method to be used will then be displayed.

<u>THEOREM 1</u>: Given a system (F, G) with controllability index ρ, F $n \times n$, and G $n \times m$:

If $n < m\rho$, there exists matrices A and B and a number N such that the N-dimensional system:

$$\left(\left[\begin{array}{c|c} F & \Theta \\ \hline A & B \end{array}\right], \left[\begin{array}{c} G \\ \hline \Theta \end{array}\right] \right) \triangleq (\hat{F}, \hat{G}) \tag{3.42}$$

is equicontrollable iff $N = m(\rho + k)$ for some integer $k \geq 0$.

SUFFICIENCY: The proof will be essentially constructive. First the system is reduced to Luenberger form [13]. Fig. 3.2 shows a block diagram of the system, where we have renumbered states so as to draw the integrator strings in order of increasing length, left to right. Now we simply add integrators to the blocks so as to make them all the same length.

The smallest number we may add to do the job would be $\sum_{i=1}^{m} (q_m - q_i)$ where q_i is the number of integrators in the $i^{\underline{th}}$ block, and q_m is the length of the largest such block. But:

$$\sum_{i=1}^{m} (q_m - q_i) = mq_m - \sum_{i=1}^{m} q_i$$

$$= mq_m - n \tag{3.43}$$

Moreover, if we added another k integrators to each block we would have $m(q_m+k)-n$ states added, for a total of $m(q_m+k)$ states after all have been added. Hence, if we can show that $\rho = q_m$, we are done. By the definition of ρ ,

$$C_\rho = [G, FG, \ldots, F^{\rho-1}G] \tag{3.44}$$

will have n independent columns. Recall that in constructing the Luenberger canonical form, the length of the $i^{\underline{th}}$ block was determined by how many vectors of the form $F^{k-1}q_i$ were independent of vectors previously chosen from the controllability matrix. However, by the definition of C , $k \leq \rho$ regardless of how the vectors were chosen, i.e. $q_m \leq \rho$, but suppose $q_m < \rho$. Then $k < \rho$ which says that no block has length ρ so that

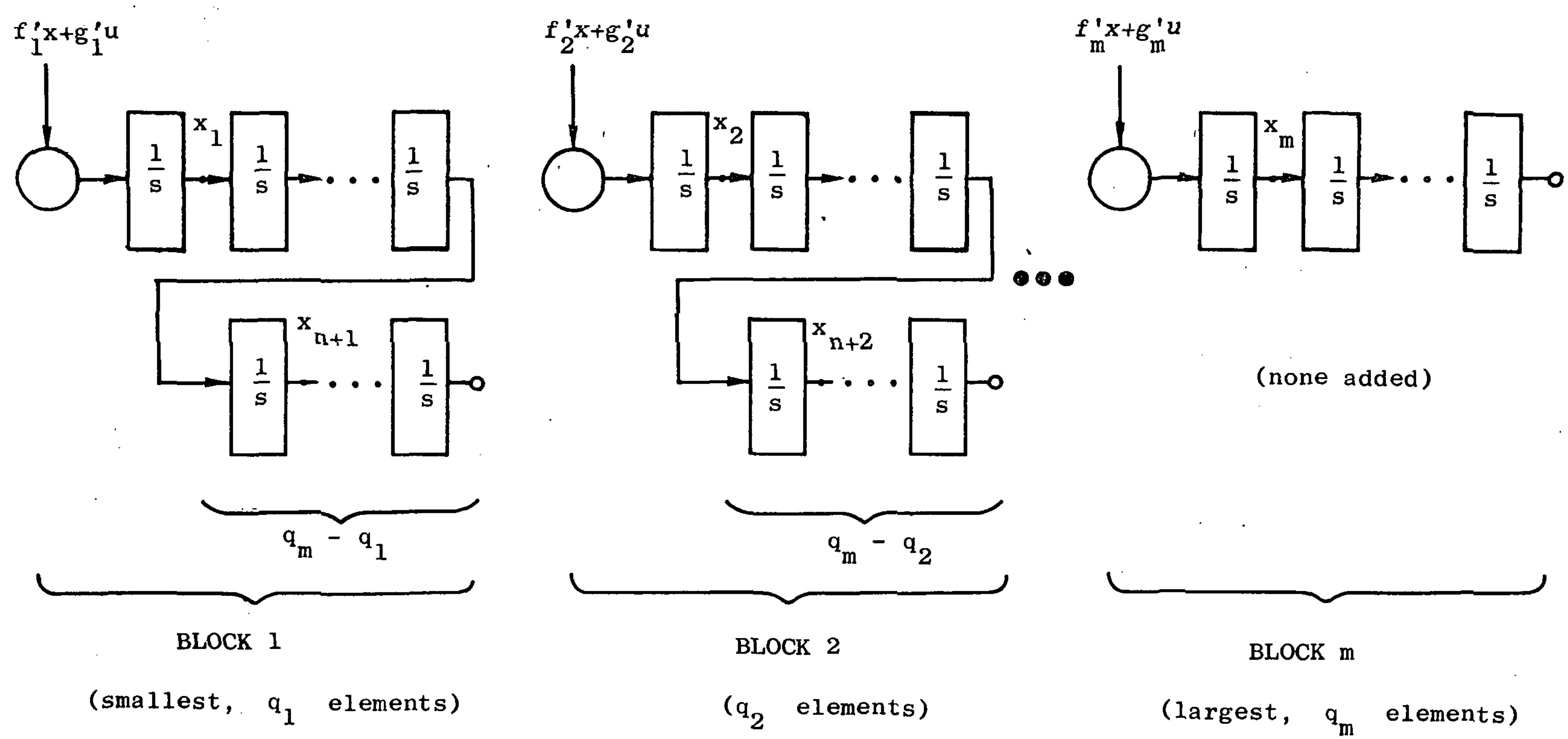


Fig. 3.2

Augmented System

$$C_{q_m} = [G,\ FG,\ \ldots,\ F^{q_m-1}G] \tag{3.45}$$

has full rank! But this contradicts the definition of ρ, hence $q_m = \rho$.

NECESSITY: We suppose that A and B exist, and that the controllability index of $(\hat{F},\hat{G})$ is ρ. If we show that $\hat{\rho} \geq \rho$, then $N \geq m\rho$. By appealing to Proposition 1, $m|N$ so $N = m(\rho+k)$ for k an integer ≥ 0.

Suppose $\hat{\rho} < \rho$. Consider the partial controllability matrix:

$$C_{\hat{\rho}} = \begin{bmatrix} G & FG & F^2G & \ldots & F^{\hat{\rho}}G \\ \theta & AG & AFG+BAG & & X \end{bmatrix} \tag{3.46}$$

$C_{\hat{\rho}}$ is $N \times N$ and full rank. Therefore,

$$[G,\ FG, \ldots, F^{\hat{\rho}}G]$$

is full rank. But this contradicts the definition of ρ, hence $\hat{\rho} \geq \rho$, and $N = m(\rho+k)$ as previously argued.

To demonstrate the nature of the sufficiency argument (and lead into an algorithm for implementing it) consider this example.

$$F = \begin{bmatrix} \multicolumn{6}{c}{\text{———} f_1' \text{———}} \\ \multicolumn{6}{c}{\text{———} f_2' \text{———}} \\ \multicolumn{6}{c}{\text{———} f_3' \text{———}} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} \multicolumn{3}{c}{\text{—} g_1' \text{—}} \\ \multicolumn{3}{c}{\text{—} g_2' \text{—}} \\ \multicolumn{3}{c}{\text{—} g_3' \text{—}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.47}$$

The system may be represented by the block diagram in Fig. 3.3

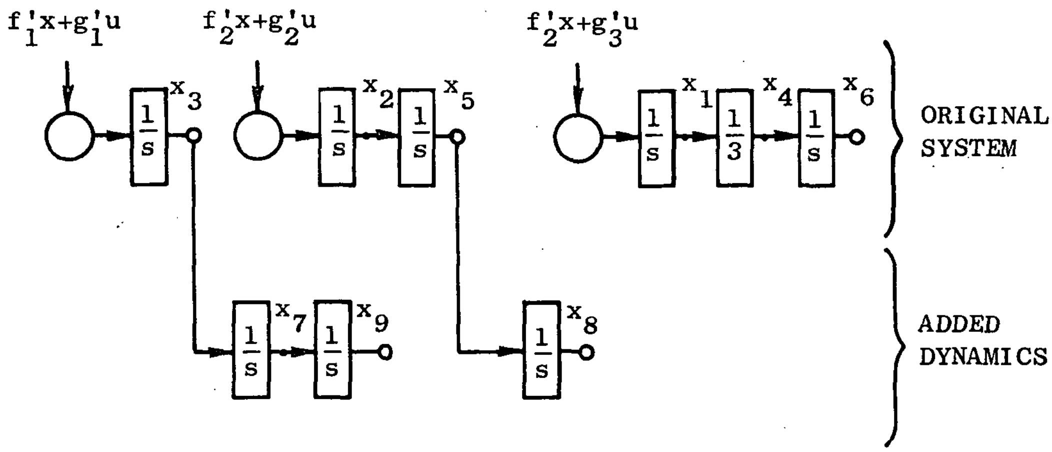


Fig. 3.3
Block diagram of
equicontrollable form

The resulting time domain matrices (after dynamics are added) are:

$$\hat{F} = \left[\begin{array}{cccccc|ccc} \multicolumn{6}{c|}{\text{———} f_1' \text{——————}} & 0 & 0 & 0 \\ \multicolumn{6}{c|}{\text{———} f_2' \text{———}} & 0 & 0 & 0 \\ \multicolumn{6}{c|}{\text{——————} f_3' \text{—}} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad \hat{G} = \left[\begin{array}{ccc} & & \\ & G & \\ & & \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{3.48}$$

Now the system may be put into equicontrollable form by renumbering the states. (Recall $\rho = q_m$ hence, since $n = q_m$, $n = \rho_m$).

Although not explicitly stated, the method of adding states used in Theorem 1 works equally as well if, instead of each added state being appended as $\frac{1}{s}$, it is appended as $\frac{1}{s+\alpha}$. A glance through the proof of sufficiency will confirm this. Since the resulting system is equicontrollable, we know via Proposition 2 how to reduce it to equicontrollable form.

In summary, we have shown:

1. That $n = \rho m$ is the key to equicontrollable form.
2. That there is some measure of uniqueness to such a form.
3. That we can find a non-minimal but nonetheless equicontrollable realization for any transfer function.

B. Properties

PROPERTY 1: If the system (H, F, G) has been converted to the equicontrallable form:

$$H=[H_1,H_2,\ldots,H_\rho]\ ,\quad F=\begin{bmatrix} \theta & I & \theta & \ldots & \theta \\ \theta & \theta & I & \ldots & \theta \\ & & \vdots & & \\ \theta & \theta & \theta & \ldots & I \\ F_1 & F_2 & F_3 & \ldots & F_\rho \end{bmatrix},\quad G=\begin{bmatrix} \theta \\ \theta \\ \vdots \\ \theta \\ I \end{bmatrix} \tag{3.49}$$

Then $W(s) = H(sI-F)^{-1}G = \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]\left[s^\rho I - \sum_{i=1}^{\rho} F_i s^{i-1}\right]^{-1}$

The last column of $(sI-F)^{-1}$ is easy to compute. To wit:

$$W(s) = \left[H_1, H_2, \ldots, H_\rho\right] \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\rho-1} \end{bmatrix} \left[s^\rho I - \sum_{i=1}^{\rho} F_i s^{i-1}\right]^{-1}$$

$$= \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right] \left[s^\rho I - \sum_{i=1}^{\rho} s^{i-1} F_i\right]^{-1} \quad \blacksquare \qquad (3.50)$$

<u>PROPERTY</u> 2: Num det $W(s) = \det\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]$

$$\text{num det } W(s) = \text{num det}\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right] \left[s^\rho I - \sum_{i=1}^{\rho} F_i s^{i-1}\right]^{-1} \qquad (3.51)$$

$$= \text{num}\left\{\det \sum_{i=1}^{\rho} H_i s^{i-1} \det\left[s^\rho I - \sum_{i=1}^{\rho} F_i s^{i-1}\right]^{-1}\right\} \qquad (3.52)$$

$$= \text{num}\left\{\frac{\det \sum_{i=1}^{\rho} H_i s^{i-1}}{\det\left[s^\rho I - \sum_{i=1}^{\rho} F_i s^{i-1}\right]}\right\} \qquad (3.53)$$

$$= \det \sum_{i=1}^{\rho} H_i s^{i-1} \qquad (3.54)$$

The last step follows from the definition of "num det", equation (2.3) and the observation that:

$$\chi_F(s) \triangleq \det\,(sI-F) = \det\left[s^\rho I - \sum_{i=1}^{\rho} s^{i-1} F_i\right] \quad \blacksquare \qquad (3.55)$$

PROPERTY 3: For any conformable matrix K,

$$\text{num det } H(sI-F)^{-1}G = \text{num det } H(sI-F-GK)^{-1}G \,. \tag{3.56}$$

PROOF: (cf: Morgan [14]). Without loss of generality, we may assume that:

$$F = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right], \quad G = \left[\begin{array}{c} \theta \\ \hline I \end{array}\right], \quad H = \left[\begin{array}{c|c} H_1 & H_2 \end{array}\right] \tag{3.57}$$

Let: $\text{num det } H(s-F)^{-1}G = \text{num det } W_F(s)$ and $T = \left[\begin{array}{c|c} I & (sI-F_{11})^{-1}F_{12} \\ \hline \theta & I \end{array}\right]$

$$\det W_F(s) = \det HTT^{-1}(sI-F)^{-1}G \tag{3.58}$$

$$= \det \left[\begin{array}{c|c} H_1 & H_1(sI-F_{11})^{-1}F_{12}+H_2 \end{array}\right] \left\{ \left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline -F_{21} & sI-F_{22} \end{array}\right] \left[\begin{array}{c|c} I & (sI-F_{11})^{-1}F_{12} \\ \hline \theta & I \end{array}\right]^{-1} \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \right\} \tag{3.59}$$

$$= \det \left[\begin{array}{c|c} H_1 & H_1(sI-F_{11})^{-1}F_{12}+H_2 \end{array}\right] \left[\begin{array}{c|c} sI-F_{11} & \theta \\ \hline -F_{12} & (sI-F_{22})-F_{12}(sI-F_{11})^{-}F_{12} \end{array}\right]^{-1} \left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \tag{3.60}$$

$$= \frac{\det\left[H_1(sI-F_{11})^{-1}F_{12}+H_2\right]}{\det\left[\ sI-F_{22}-F_{21}(sI-F_{11})^{-1}F_{12}\right]} \tag{3.61}$$

Observe that:

$$1.\ \det\left[\ H_1(sI-F_{11})^{-1}F_{12}+H_2\right] = \frac{\det\left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline H_1 & H_2 \end{array}\right]}{\det\ (sI-F_{11})} \tag{3.62}$$

$$2.\ \chi_F(s) = \det\ (sI-F_{11})\ .\ \det\ (sI-F_{22}-F_{21}(sI-F_{11})^{-1}F_{12}) \tag{3.63}$$

Therefore:

$$\det W_F(s) = \frac{\det\left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline H_1 & H_2 \end{array}\right]}{\chi_F(s)} \tag{3.64}$$

so that:

$$\text{num}\ \det W_F(s) = \det\left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline H_1 & H_2 \end{array}\right] \tag{3.65}$$

But if we compute num det $W_{F+GK}(s)$ in the same fashion,

$$\text{num}\ \det W_{F+GK}(s) = \text{num}\ \det\left[\begin{array}{c|c} H_1 & H_2 \end{array}\right]\left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline -F_{21}-K_1 & sI-F_{22}-K_2 \end{array}\right]^{-1}\left[\begin{array}{c} \theta \\ \hline I \end{array}\right] \tag{3.66}$$

$$= \det\left[\begin{array}{c|c} sI-F_{11} & -F_{12} \\ \hline H_1 & H_2 \end{array}\right] \tag{3.67}$$

$$= \text{num}\ \det W_F(s) \tag{3.68}$$

Q.E.D.

Note that property 3 is not only a property of equicontrollable systems, but of any system. We include it here since it fits in well with the properties of equicontrollable systems, making a well rounded picture.

PROPERTY 4: Suppose (H,F,G) is equicontrollable and completely observable, and (C,A,B) is equicontrollable. Further, F is $n \times n$, A is $N \times N$, $N \geq n$, and both have m inputs and r outputs. If

$$C(sI-A)^{-1}B = H(sI-F)^{-1}G \tag{3.69}$$

then there exist T,X,Y such that:

$$\begin{gathered} [H \,\vdots\, \theta]T = C \\ T \begin{bmatrix} G \\ \hdashline \theta \end{bmatrix} = B \\ T^{-1} \left[\begin{array}{c:c} F & \theta \\ \hdashline X & Y \end{array}\right] T = A \end{gathered} \tag{3.70}$$

Proof: Observe that since the transfer functions are equal,

$$HF^kG = CA^kB \quad \text{for any } k \tag{3.71}$$

Let p and q be the controllability indices of (F,G,H) and (A,B,C) respectively. Then we claim that:

$$T = \left[\begin{array}{c:c} G \;\; FG \; \ldots \; F^{p-1}G & F^pG \; \ldots \; F^{q-1}G \\ \hdashline \theta & I_{m(p-q)} \end{array}\right] \begin{bmatrix} B & AB & \ldots & A^{q-1}B \end{bmatrix}^{-1} \tag{3.72}$$

It is easy to check the first two equations above. To wit:

$$\left[H \,\vdots\, \theta\right]T = \left[HG \; HFG \ldots HF^{q-1}G\right]\left[B \;\; AB \; \ldots \; A^{q-1}B\right]^{-1} \tag{3.73}$$

but from our initial observation, we may write:

$$\left[H \,\vdots\, \Theta\right] T = \left[CB \quad CAB \ \ldots \ CA^{q-1}B\right]\left[B \quad AB \ \ldots \ A^{q-1}B\right]^{-1} \tag{3.74}$$

$$= C \ !!$$

$$TB = \left[\begin{array}{c|c} G \ FG \ \ldots \ F^{q-1}G & F^{q}G \ \ldots \ F^{q-1}G \\ \hline \Theta & I \end{array}\right]\begin{bmatrix} I \\ \Theta \\ \vdots \\ \Theta \end{bmatrix} = \left[\begin{array}{c} G \\ \hline \Theta \end{array}\right] \ !! \tag{3.75}$$

Finally we must check the last equation.

$$TAT^{-1} = \left[\begin{array}{c|c} \multicolumn{2}{c}{G \ FG \ \ldots \ F^{q-1}G} \\ \hline \Theta & I \end{array}\right]\left[B \ AB \ \ldots \ A^{q-1}B\right]^{-1}A \cdot$$

$$\left[B \ AB \ \ldots \ A^{q-1}B\right]\left[\begin{array}{c|c} \multicolumn{2}{c}{G \ FG \ \ldots \ F^{q-1}G} \\ \hline \Theta & I \end{array}\right]^{-1} \tag{3.76}$$

$$= \left[\begin{array}{c|c} \multicolumn{2}{c}{G \ FG \ \ldots \ F^{q-1}G} \\ \hline \Theta & I \end{array}\right]\begin{bmatrix} \Theta & \Theta & \ldots & \Theta & A_1 \\ I & \Theta & & \Theta & A_2 \\ \Theta & I & & \Theta & A_3 \\ \vdots & & & & \vdots \\ \Theta & \Theta & \ldots & I & A_q \end{bmatrix}\left[\begin{array}{c|c} \multicolumn{2}{c}{G \ FG \ \ldots \ F^{q-1}G} \\ \hline \Theta & I \end{array}\right] \tag{3.77}$$

where we have defined A_i :

$$\left[B \ AB \ \ldots \ A^{q-1}B\right]^{-1}A^{q}B = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_q \end{bmatrix} \tag{3.78}$$

<u>ASIDE:</u> From the above:

$$A^qB = \begin{bmatrix} B & AB & \ldots & A^{q-1}B \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_q \end{bmatrix} \tag{3.79}$$

Then:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} A^qB = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \ldots & A^{q-1}B \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix} \tag{3.80}$$

which implies that:

$$QF^qG \triangleq \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} F^qG = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \begin{bmatrix} G & FG & \ldots & F^{q-1}G \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix} \tag{3.81}$$

But Q is full rank (since (H,F,G) is completely observable). Hence we conclude that:

$$F^qG = \begin{bmatrix} G & FG & \ldots & F^{q-1}G \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix} \tag{3.82}$$

Now to return to the original train of thought. We may consider the first n rows of the matrix TAT^{-1} as follows.

$$\begin{bmatrix} G & FG & \ldots & F^{q-1}G \end{bmatrix} \begin{bmatrix} \Theta & \Theta & \cdots & \Theta & A_1 \\ I & \Theta & & \Theta & A_2 \\ \Theta & I & & \Theta & A_3 \\ & \vdots & & & \vdots \\ \Theta & \Theta & \cdots & I & A_q \end{bmatrix} \left[\begin{array}{c:c} \multicolumn{2}{c}{G \quad FG \quad \ldots \quad F^{q-1}G} \\ \hdashline \Theta & I \end{array}\right]^{-1}$$

$$= \begin{bmatrix} FG & F^2G & \ldots & F^{q-1}G & F^qG \end{bmatrix} \left[\begin{array}{c:c} \multicolumn{2}{c}{G \quad FG \quad \ldots \quad F^{q-1}G} \\ \hdashline \Theta & I \end{array}\right]^{-1} \tag{3.83}$$

$$= F\begin{bmatrix} G & F^2G & \ldots & F^{q-1}G \end{bmatrix} \left[\begin{array}{c:c} \multicolumn{2}{c}{G \quad FG \quad \ldots \quad F^{q-1}G} \\ \hdashline \Theta & I \end{array}\right] \tag{3.84}$$

$$= F\left[\begin{array}{c:c} I_n & \Theta \end{array}\right] \tag{3.85}$$

$$= \left[\begin{array}{c:c} F & \Theta \end{array}\right] \tag{3.86}$$

Hence:

$$TAT^{-1} = \left[\begin{array}{c:c} F & \Theta \\ \hdashline X & Y \end{array}\right] \text{ for some } X \text{ and } Y . \tag{3.87}$$

Q.E.D.

IV. THE ALGORITHM

A. Introduction to the Algorithm -- The Scalar Case

The basic difficulty in presenting the algorithm is in the large number of steps required. Any explanation tends to lose continuity when attempting to clarify each individual step of such a long procedure. To attempt to alleviate this problem, we shall try to proceed in a number of progressively more difficult stages, each cutting deeper into the wealth of detail involved. Extensive use will be made of simultaneous presentation of steps in sundry forms (block diagram, time domain equations, transfer functions, and text description) so that the mechanisms involved will be as clear as possible.

To begin the discussion let us consider a scalar problem in transfer function terms. The model and the plant might, for example, be given by:

$$\begin{aligned} &\text{PLANT}^0: \qquad \frac{(s+1)(s-2)}{(s+3)(s+4)(s+5)} \\ &\text{MODEL}^0: \qquad \frac{s+3}{(s+6)(s+7)} \end{aligned} \tag{4.1}$$

A moment's reflection will reveal that no realizable compensator can be added to the plant which will cause it to behave the same as the model. Further, if we are to be able to find a stable compensator, we shall have to require a change in the model, namely it must also have a zero at $s = 2$. These observations may be summed up by this modified problem.

$$\text{PLANT}^1: \quad \frac{(s+1)(s-2)}{(s+3)(s+4)(s+5)}$$

$$\text{MODEL}^1: \quad \frac{s+3}{(s+6)(s+7)} \cdot \frac{\alpha(s-2)}{(s+\alpha)} \tag{4.2}$$

α is arbitrary, but is prespecified by the designer. If this model's response were not acceptable, there would have to be negotiation with the designer until a suitable model were found. But we suppose here that this model was approved. Then the way to a solution is clear. First, we find the feedback which matches the dynamics (poles) of the two systems, yielding:

$$\text{PLANT}^2: \quad \frac{(s+1)(s-2)}{(s+6)(s+7)(s+\alpha)}$$

$$\text{MODEL}^2: \quad \frac{s+3}{(s+6)(s+7)} \cdot \frac{\alpha(s-2)}{(s+\alpha)} \tag{4.3}$$

Finally, we apply an input series compensator to the plant, which completes the solution.

$$\text{PLANT}^3: \quad \frac{(s+1)(s-2)}{(s+6)(s+7)(s+\alpha)} \cdot \frac{\alpha(s+3)}{(s+1)}$$

$$\text{MODEL}^3: \quad \frac{(s+3)}{(s+6)(s+7)} \cdot \frac{\alpha(s-2)}{(s+\alpha)} \tag{4.4}$$

The transfer functions are now the same, so we are done. Some comments are in order at this point.

1. Although this was a specific example, the procedure is quite general. Even if the plant had fewer states than the model, the same method would be used.
2. There is no doubt about the minimality of the solution, since the manipulations are quite transparent.
3. The need for adding RHP zeros of the plant to the model is also obvious, lest the compensator be unstable.

Let us do the same problem in the time domain. The format will be to place the associated block diagrams, time domain equations, etc., in an accompanying figure. Thus the steps may be more easily visualized during the discussion.

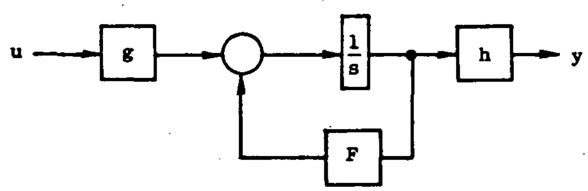
We begin with the original problem (Fig. 4.1). Since the systems were assumed to be controllable, we have chosen to write them in control canonical form. (Ordinarily the problem would be given in differential equation form, so it would first have to be reduced to such a form.) Exactly as before, the first step is a modification of the model.

PLANT^0:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \triangleq Fx + gu$$

$$y = [-2 \quad -1 \quad 1]\, x \triangleq hx$$

$$W(s) = \frac{(s+1)(s-2)}{(s+3)(s+4)(s+5)}$$



MODEL^0:

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ -42 & -13 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \omega \triangleq \Phi\xi + \gamma\omega$$

$$\mu = [\,3 \quad 1\,]\, \xi \triangleq \sigma\xi$$

$$V(s) = \frac{(s+3)}{(s+6)(s+7)}$$

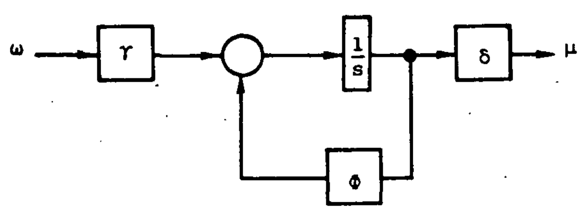


Fig. 4.1

Scalar Example: '0' Coordinates

Fig. 4.2 shows a method of appending the additional state to the model. This particular way of introducing them is used since only input changes are deemed reasonable (or realizable). Although such a restriction is not strictly necessary in the model, which is a mathematical fiction, it is essential in dealing with the plant. The outputs represent (usually) physical actions, such as pitch or roll of an aircraft, which <u>must</u> be the outputs. This observation is basic to the problem but has occasionally been overlooked, such as in [2]. Generally, this step would also include another transformation to control canonical form, but it was not shown here for the sake of better displaying the details of the state addition.

PLANT^1: unchanged

MODEL^1:

$$\dot{\bar{\xi}} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -42 & -13 & 1 \\ \hline 0 & 0 & -\alpha \end{array}\right] \bar{\xi} + \left[\begin{array}{c} 0 \\ \alpha \\ \hline -\alpha^2-2\alpha \end{array}\right] \omega \triangleq \overline{\Phi\xi} + \overline{\gamma\omega}$$

$$\mu = [3 \quad 1 \quad 0] \; \bar{\xi} \triangleq \overline{\sigma\xi}$$

$$V(s) = \frac{(s+3)}{(s+6)(s+7)} \cdot \frac{\alpha(s-2)}{(s+\alpha)}$$

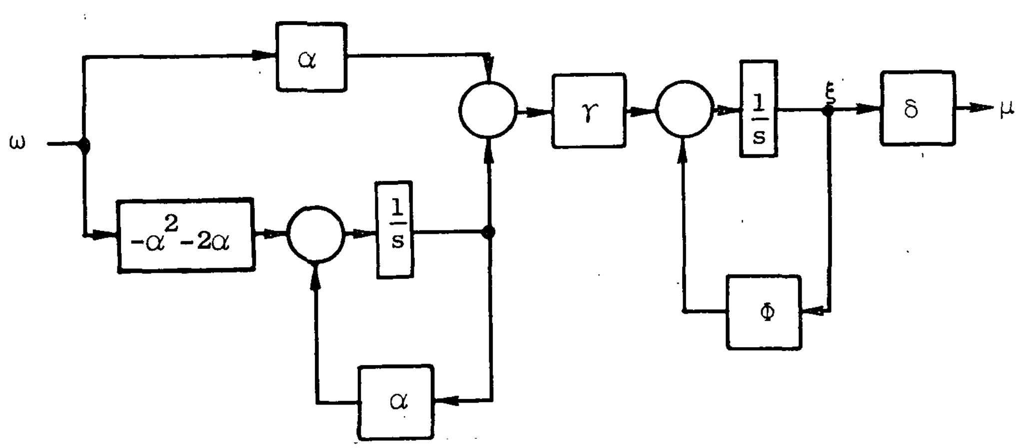


Fig. 4.2

Scalar Example: '1' Coordinates

At this stage (Fig. 4.3), three changes have been made. First, the <u>model</u> state has been redefined so as to place it in control canonical form. Second, the plant has had feedback (per fig. 4.4) applied, viz:

$$u = u + Kx \tag{4.5}$$

Third, a series compensator has been added. In the figure, the feedback is evidenced by the third row of the F matrix, to the left of the dashed line. The series compensator has state z and transfer function $\alpha(s+3)/(s+1)$. The result has been to give the plant the same transfer function as the model (after revision). The compensator is stable, as is the compensated system as a whole. Moreover, the compensator is minimal, whereas the resultant system is not, due to the cancellation of the (s+1) factor. Fig. 4.4 shows the series and parallel compensation as applied to the original plant. Note that, in this case, there is no feedback from the states of the plant to the input of the series compensator. This is not a general property of the method.

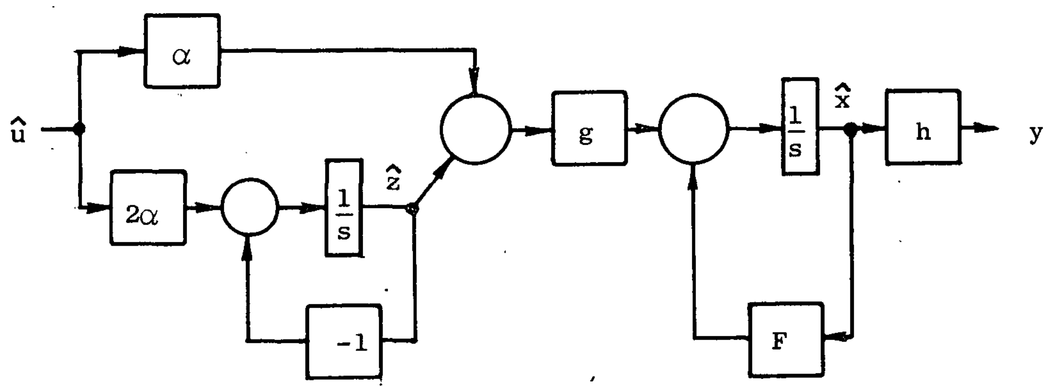
The changes in the model are neither unexpected or unreasonable. Were it not for the RHP zero of the original plant, the changes required would have been especially acceptable.

PLANT^2:

$$\begin{bmatrix} \dot{\hat{x}} \\ \hline \dot{\hat{z}} \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -42 & -(42+13\alpha) & -(\alpha+13) & 1 \\ \hline 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} \hat{x} \\ \hline \hat{z} \end{bmatrix} + \left[\begin{array}{c} 0 \\ 0 \\ \alpha \\ \hline 2\alpha \end{array}\right] \hat{u} = F \begin{bmatrix} \hat{x} \\ \hline \hat{z} \end{bmatrix} + \hat{G}\hat{u}$$

$$y = \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \end{array}\right] \begin{bmatrix} \hat{x} \\ \hline \hat{z} \end{bmatrix} \triangleq \hat{H} \begin{bmatrix} \hat{x} \\ \hline z \end{bmatrix}$$

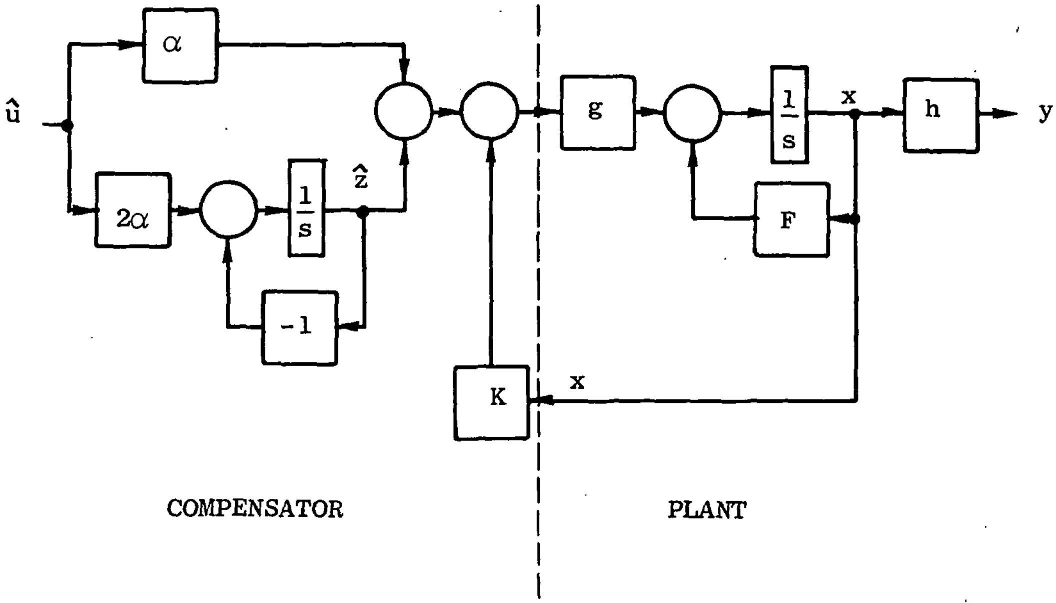
$$W(s) = \frac{(s+1)(s-2)}{(s+6)(s+7)(s+\alpha)} \cdot \frac{\alpha(s+3)}{(s+1)}$$



MODEL^2: unchanged

$$V(s) = \frac{(s+3)}{(s+6)(s+7)} \cdot \frac{\alpha(s-2)}{(s+\alpha)}$$

Fig. 4.3
Scalar example: '2' Coordinates



$$K = [-42\alpha+60 \,\vdots\, -(42+13\alpha)+47 \,\vdots\, -(\alpha+13)+12]$$

$$z = [-1]\hat{z} + [2 \,\vdots\, \theta] \begin{bmatrix} \hat{u} \\ \hdashline x \end{bmatrix}$$

$$u = [1]\hat{z} + [\alpha \,\vdots\, K] \begin{bmatrix} \hat{u} \\ \hdashline x \end{bmatrix}$$

Fig. 4.4

Scalar Example: Compensator Structure

In summary then, the steps involved in a scalar problem may be given as:

1. Get both the plant and the model into control canonical (equicontrollable) form.
2. By adding states to the plant and/or the model, correct unequal numbers of states in the plant and model.
3. Adjust pole locations by feedback.
4. Cancel and/or replace zeros in the plant as required, by series compensation.
5. Transform back to original coordinates and write the

compensator as:

$$\begin{aligned} \dot{z} &= F_c z + G_c \begin{bmatrix} u \\ x \end{bmatrix} \\ u &= H_c z + J_c \begin{bmatrix} u \\ x \end{bmatrix} \end{aligned} \tag{4.6}$$

thus completing the design.

Although the detail differences between the scalar and multivariable problems are legion, the basic steps are very similar, and the result is expressed in the same form.

B. The Multivariable Case -- Overview

The problem to be faced when looking at the multivariable case is the lack of a good canonical form for all possible systems. Since no one is suitable, the approach we shall use is to increase state dimension to simplify structure. More specifically we shall add states to any system which is not equicontrollable so as to make it equicontrollable. In chapter 3, theorem 1, we showed this to be always possible and subsequently described the procedure that we shall adopt to accomplish it. We lose simplicity in the sense of state dimension and we introduce some non-uniqueness, but we are more than repaid in structure by being able to write every matrix in a canonical form having all $m \times m$ partitions. Moreover, the sundry properties of the equicontrollable form may be invoked.

Even with this approach in mind, the path is not at all clear. A number of directions present themselves which seem at first to quickly solve the problem, but which actually create more difficulties than they remove. In this section we shall consider one of these, which, although it does not succeed, provides useful insights into the true nature of the problem and how we shall finally go about solving it.

To demonstrate the approach let us consider a simple multivariable problem (Fig. 4.5). Both systems have been chosen to be equicontrollable and of equal state dimension. Thus the detailed (but straightforward) step of achieving this form has been temporarily ignored. We intend to show that simply operating by analogy with the scalar case is not sufficient, and to do so would lead us astray, even in this simplified case.

For each system the transfer function's form is derived via Property 1, Chapter III.

The first step is a natural analog to what was done in the scalar case, but with a slight variation. If we wished to try a direct application of the scalar method, we would find a K such that:

$$\chi_{F+GK}(s) = \chi_{\Phi}(s) \tag{4.7}$$

This has two serious drawbacks:

1. The feedback is not unique (as it was in the scalar case), so choosing it now would remove a degree of freedom in the solution. Thus it might be better to wait until we have better grounds for the choice.
2. If we did choose K now, the next step could be impossible. Mainly since we will later add series compensation to the <u>input</u>, and we will need to have

$$\left[s^{\rho}I - \sum_{i=1}^{\rho} (F_i + K_i)s^{i-1}\right]^{-1}$$

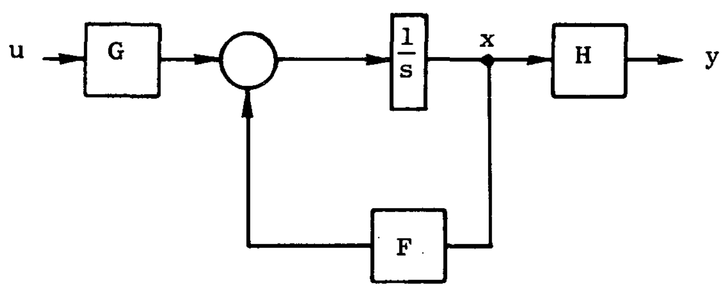
   and the compensator's transfer function commute. This is not always possible in the multivariable case, consistent with the requirement of prescribed characteristic equation. It was in the scalar case -- a fact which was not explicitly stated.

PLANT^0:

$$\dot{x} = \begin{bmatrix} \Theta & I & \cdots & \Theta \\ \Theta & \Theta & & \Theta \\ & \vdots & & \vdots \\ F_1 & F_2 & \cdots & F_\rho \end{bmatrix} x + \begin{bmatrix} \Theta \\ \Theta \\ \vdots \\ I \end{bmatrix} u \triangleq Fx + Gu$$

$$y = [H_1 \quad H_2 \quad \cdots \quad H_\rho] \triangleq Hx$$

$$W(s) = \left[\sum_{i-1}^{\rho} H_i s^{i-1} \right] \left[s^\rho I - \sum_{i-1}^{\rho} F_i s^{i-1} \right]^{-1}$$



MODEL^0:

$$\dot{\xi} = \begin{bmatrix} \Theta & I & \cdots & \Theta \\ \Theta & \Theta & & \Theta \\ & \vdots & & \vdots \\ \Phi_1 & \Phi_2 & \cdots & \Phi_\rho \end{bmatrix} \xi + \begin{bmatrix} \Theta \\ \Theta \\ \vdots \\ I \end{bmatrix} \omega \triangleq \Phi\xi + \Gamma\omega$$

$$\mu = [\Delta_1 \quad \Delta_2 \quad \cdots \quad \Delta_\rho] \; \xi \triangleq \Delta\xi$$

$$V(s) = \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1} \right] \left[s^\rho I - \sum_{i=1}^{\rho} \Phi_i s^{i-1} \right]^{-1}$$

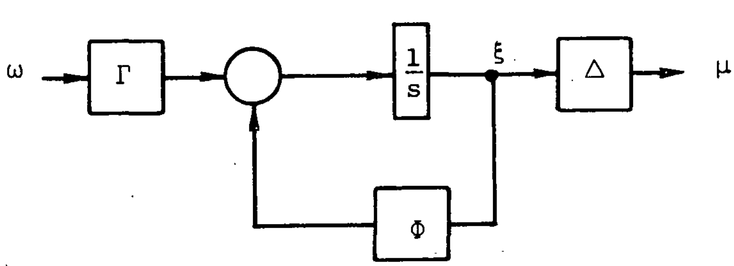


Fig. 4.5

Multivariable Example: '0' Coordinates

In view of observation number 2 above, we can choose a K which <u>will</u> allow the necessary commutation. In particular choose $K_i = -F_i$! The model is left unchanged, and the plant is still equicontrollable (a property invariant under feedback). Fig. 4.6 displays the result. Note that had $H_i = \Delta_i \forall i$, we could have let $K_i = \Phi_i - F_i$ and been done at the first step! The chances of this occurring are so slim that we ignore it. Now, at least in principle, the way is clear. We build a realization of

$$C(s) = \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]^{-1}\left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right] \tag{4.8}$$

if possible (we fail if num det $W(s)$ has a pole in the RHP, for example, since then the compensator is unstable, recalling that num det $W(s)$ divides det $\sum_{i=1}^{\rho} H_i s^{i-1}$). Then put that compensator in series with the plant to yield:

$$W(s)C(s) = \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]\frac{I}{s^{\rho}}\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]^{-1}\left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right] \tag{4.9}$$

$$= \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right]\left[s^{\rho} I\right]^{-1} \tag{4.10}$$

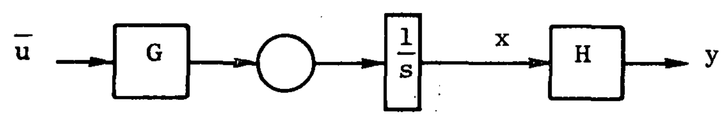
Now it would seem that we need only apply the feedback $\bar{K}_i = \Phi_i$ and we would be done. Unfortunately, this argument is specious. It is not true that after application of the series compensator $C(s)$, we may apply such a feedback. In particular, the cancellation of $\sum_{i=1}^{\rho} H_i s^{i-1}$ in eq. (4.9) above does not occur internally, hence the state has increased in size and $\bar{K}$ cannot be as claimed.

PLANT^1:

$$\dot{x} = \begin{bmatrix} \Theta & I & \Theta & \cdots & \Theta \\ \Theta & \Theta & I & \cdots & \Theta \\ \vdots & & & & \vdots \\ \Theta & \Theta & \Theta & \cdots & \Theta \end{bmatrix} x + \begin{bmatrix} \Theta \\ \Theta \\ \vdots \\ I \end{bmatrix} \bar{u} \triangleq (F+GK)x+G\bar{u}$$

$$y = [H_1 \quad H_2 \quad \cdots \quad H_\rho] \; x \triangleq Hx$$

$$W(s) = \left[\sum_{i=1}^{\rho} H_i s^{i-1} \right] \left[s^\rho I \right]^{-1}$$



MODEL^1: unchanged

$$V(s) = \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1} \right] \left[s^\rho I - \sum_{i=1}^{\rho} \Phi_i s^{i-1} \right]^{-1}$$

Fig. 4.6

Multivariable Example: '1' Coordinates

To demonstrate that this is the case, suppose $\rho = 2$. Then say that $C(s)$ has the (possibly nonminimal) realization:

$$[B_1 \; B_2] \;, \begin{bmatrix} \Theta & I \\ A_1 & A_2 \end{bmatrix}, \begin{bmatrix} \Theta \\ I \end{bmatrix} \qquad (4.11)$$

($C(s)$ may require more or less states than shown but the principle will remain). Then the series compensated plant will have the form:

$$[\Theta \quad I \quad \Theta \quad \Theta] \, , \left[\begin{array}{cc|cc} \Theta & I & \Theta & \Theta \\ \Theta & \Theta & B_1 & B_2 \\ \hline \Theta & \Theta & \Theta & I \\ \Theta & \Theta & A_1 & A_2 \end{array}\right] , \left[\begin{array}{c} \Theta \\ \Theta \\ \hline \Theta \\ I \end{array}\right] \tag{4.12}$$

Now we are somewhat at sea. Is the system above equicontrollable? Is it for some difficult realization of $C(s)$? Even if it is, how do we find a feedback to match it to the given model? These questions do not seem to have general satisfactory answers at this point.

On the other hand, the general approach is salvageable. We need only guarantee that a suitable feedback will exist to complete the compensation. This, in fact, can be done. In the next section we shall give such a construction. Briefly, it will proceed as follows:

1. Place both model and system in equicontrollable form with the same state dimension, even at the cost of adding states.
2. Apply a loop of feedback to the plant to place all of its eigenvalues at the origin. (In particular making the last m rows of its "F" matrix zeros.)
3. Synthesize a series compensator along the lines of the method given here, but modified in such a way that indeed feedback alone will be able to complete the job.
4. Find the aforementioned feedback, $\bar{K}$.
5. Untangle the compensator and return it to the original coordinates of the plant.
6. Find a minimal realization of the compensator -- considered as a linear system taking $(x,\hat{u})$ into u .

The next section will detail the algorithm.

C. The General Multivariable Case -- Detailed Description

Fig. 4.7 depicts the various operations which we propose to perform on the plant. This is essentially a pictorial version of the outline just given, but containing substantial detail. Each step is numbered along the left margin. These numbers will correspond with later

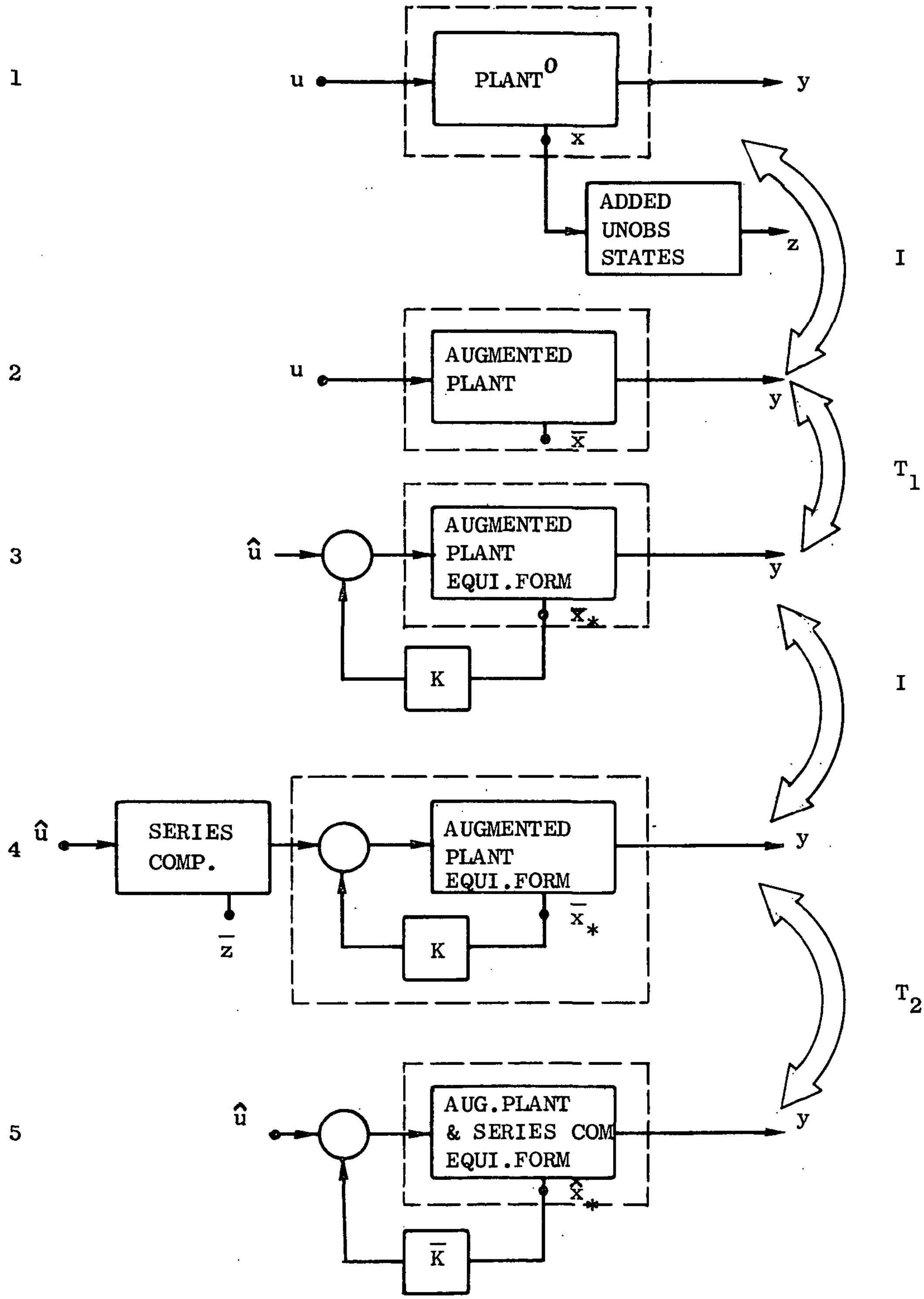


Fig. 4.7

Algorithm: Operations on the Plant

numbering of the plant in various coordinate systems. Again, each of these coordinate systems will be considered in detail in subsequent figures. The transformations indicated on the right show how the states of one version are derived from the previous one. In each case the block(s) within the dashed lines are derived through the indicated transformation from the previous diagram. The model will follow a similar history as we proceed, but it will not receive the various rounds of feedback. Fig. 4.8 gives the details for the model.

Fig. 4.9 shows the starting place for the plant and model. Note that the figure gives the block diagram, a state description, and the transfer function for each. This pattern will be continued in the remaining figures. In general, the plant and model are of different dimension. Neither is equicontrollable, and their controllability indices are usually different. To alleviate this uncertainty, we shall first add states to each to make them equicontrollable. (Alternatively one could say that we will find the smallest equicontrollable realization of each.) Then we shall increase the number of states in the smaller so that they both have the same state dimension. This addition will be made so as to create a new equicontrollable state realization of the appropriate state dimension. The reduction to equicontrollable form is accomplished in discrete steps.

First, we add states with roots at the prespecified location α . Fig. 4.10 shows the result of applying this procedure to the plant, as PLANT^1 . As the block diagram shows, the states z are totally isolated from the output. Hence they do not appear in the transfer function. J_{11} and J_{12} simply define the method of connection of those states, as per the recipe given in the proof of Theorem 1, Chapter III. In the example after that theorem, we may note that

$$J_{12} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad J_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{4.13}$$

and $\alpha = 0$. PLANT^2 is simply a redefinition of the previous coordinates where x and z are combined into the single vector $\bar{x}$.

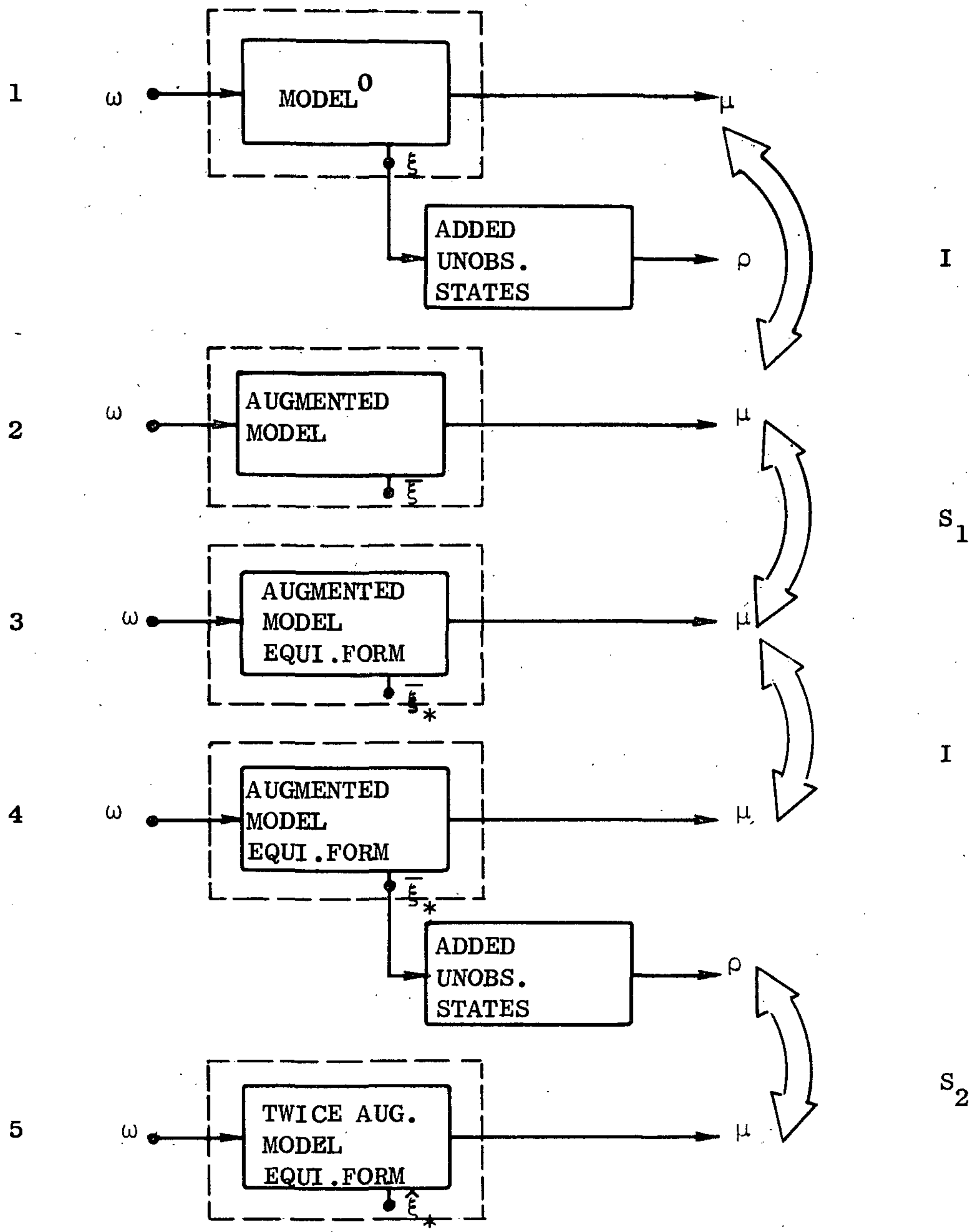
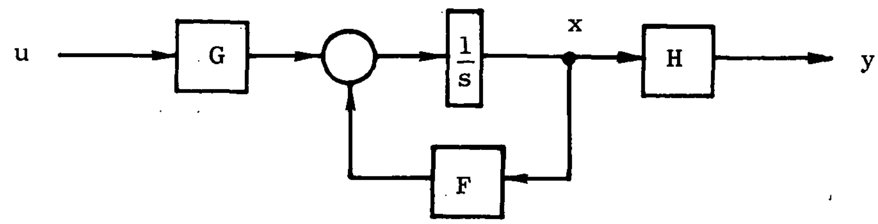


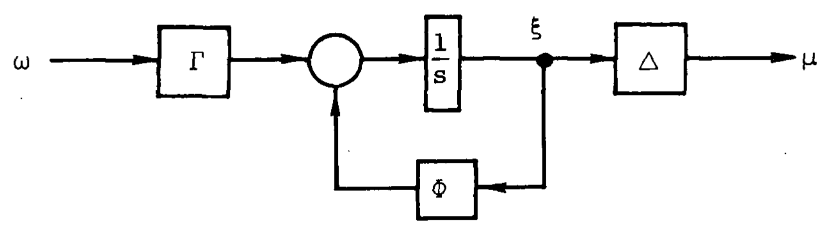
Fig. 4.8
Algorithm: Operations on the Model

PLANT^0:



$$\dot{x} = Fx + Gu$$
$$y = Hx$$
$$W_0(s) = H(sI - F)^{-1}G$$

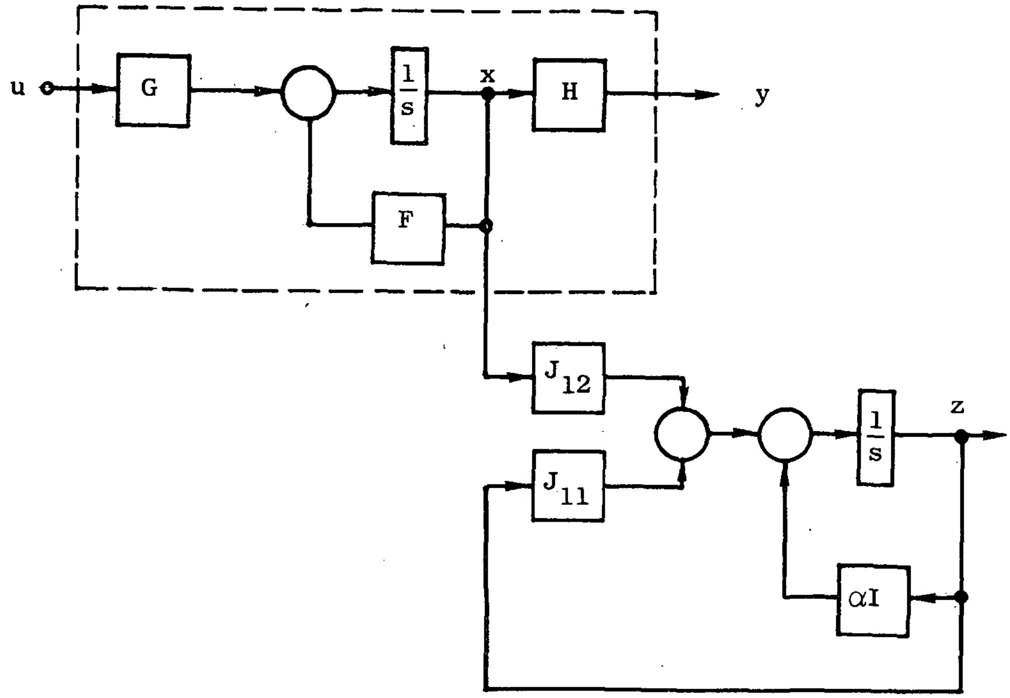
MODEL^0:



$$\dot{\xi} = \Phi\xi + \Gamma\omega$$
$$\mu = \Delta\xi$$
$$V_0(s) = \Delta(sI - \Phi)^{-1}\Gamma$$

Fig. 4.9
Algorithm: Plant and Model in
'0' Coordinates

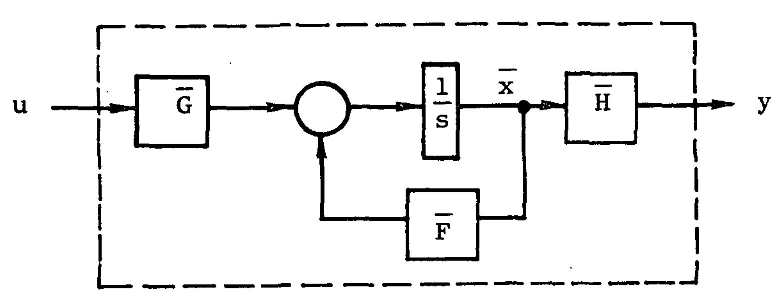
PLANT[1]:



$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \left[\begin{array}{c|c} F_L & \Theta \\ \hline J_{12} & \alpha I + J_{11} \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} G \\ \Theta \end{bmatrix} u$$

$$y = \left[\begin{array}{c|c} H & \Theta \end{array}\right] \begin{bmatrix} x_L \\ z \end{bmatrix}$$

PLANT[2]:



$$\dot{\bar{x}} = \bar{F}\bar{x} + \bar{G}u \qquad \bar{x} = \begin{bmatrix} x \\ z \end{bmatrix}$$

$$y = \bar{H}\bar{x}$$

Fig. 4.10

Algorithm : Plant in '1' and '2' Coordinates

Now the reduction to equicontrollable form, via the similarity transformation T_1 , is shown in Fig. 4.1. In our notation, a subscript "*" indicates equicontrollable form, and will be referred to as "star" coordinates.

The feedback KT_1 sets the last m rows of $\bar{F}_*$ to θ . We choose to define the feedback as KT_1 for notational convenience later, when we will unravel the transformations to give the compensator in the original coordinates of the plant. The result of the feedback is that:

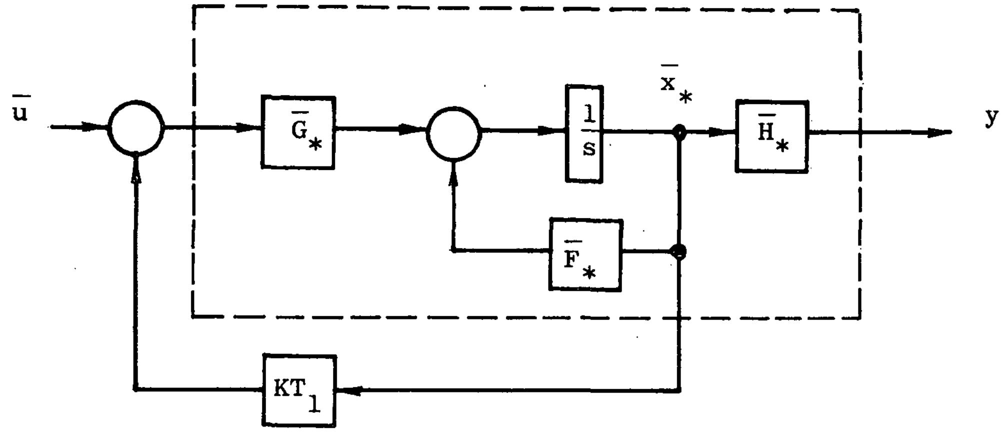
$$\bar{F}_* + \bar{G}_* KT_1 = \begin{bmatrix} \theta & I & \theta & \cdots & \theta \\ \theta & \theta & I & & \theta \\ & \vdots & & & \vdots \\ \theta & \theta & \theta & & I \\ \theta & \theta & \theta & \cdots & \theta \end{bmatrix} \tag{4.14}$$

Fig. 4.11 shows the external results of the feedback. The transfer function is reminiscent of the one found in the previous section (4-B). From this point on it would be well to keep in mind the lessons learned in that development.

Figs. 4.12 and 4.13 give the operations on the model so as to bring it to a set of coordinates comparable to the "bar-star" representation of the plant in Fig. 4.11, but without the feedback. No special indication is made as to whether the plant or model had a larger state dimension at the outset. Whichever it was, it no longer matters.

The discussion to follow is somewhat complicated, both in notation and in logic. Recall that if $\bar{H}_* = \bar{\Delta}_*$, we could complete the problem at this point by feedback alone. We desire to find a way to add states to the plant and model so that, when again reduced to equicontrollable form, the output matrices will indeed be the same. Moreover, we wish to do this consistent with the various restrictions which we placed on the original problem statement. Let us first look at the type of compensation we will add to the plant (series, in fact) and then go back and show how this answer works. Fig. 4.14 shows the application of the series compensator. J_{21} and J_{22} are defined, and simply show how we shall wire in the additional states. We shall define k matrices A_i and k scalars λ_i by the boxed equation.

PLANT^3:



$$\dot{\bar{x}}_* = (\bar{F}_* + \bar{G}_* K T_1)\bar{x}_* + \bar{G}_*\bar{u}$$

$$y = \bar{H}_*\bar{x}_*$$

$$\bar{F}_* = T_1^{-1}\bar{F}T_1 = \begin{bmatrix} \theta & I & \theta \dots & \theta \\ \theta & \theta & I & \theta \\ & \vdots & & \vdots \\ F_1 & F_2 & F_3 \dots & F_\rho \end{bmatrix}, \quad \bar{G}_* = T_1^{-1}\bar{G} = \begin{bmatrix} \theta \\ \theta \\ \vdots \\ I \end{bmatrix}$$

$$\bar{H}_* = \bar{H}T_1 = \begin{bmatrix} H_1 & H_2 & H_3 \dots H_\rho \end{bmatrix}$$

$$KT_1 = \begin{bmatrix} K_1 & K_2 & \dots & K_\rho \end{bmatrix} = \begin{bmatrix} -F_1 & -F_2 & \dots & -F_\rho \end{bmatrix}$$

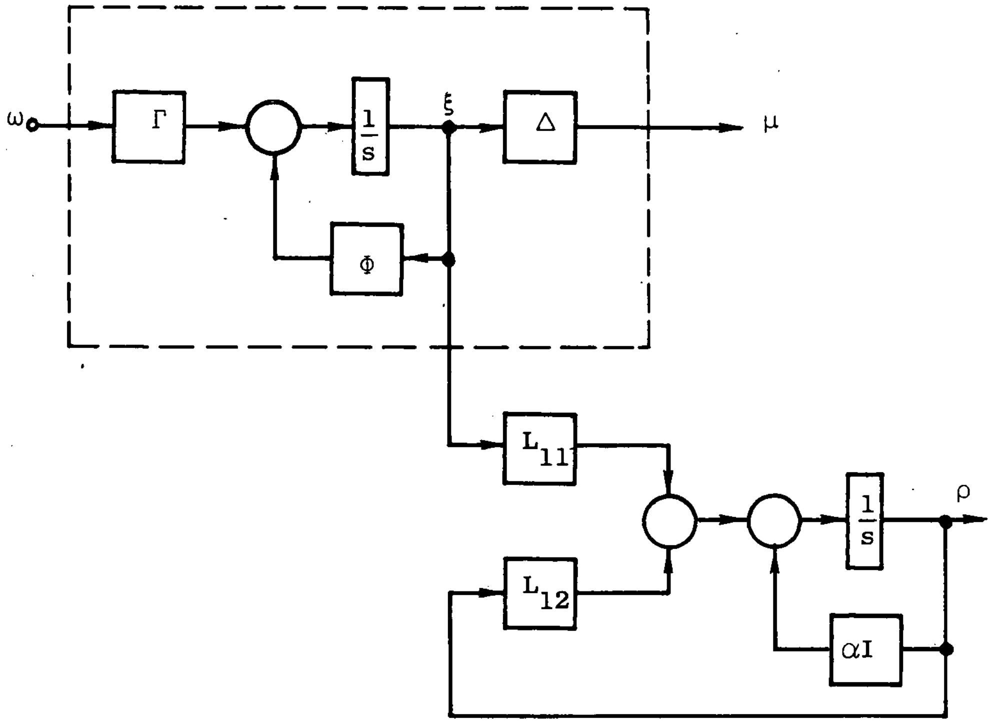
$$W_3(s) = \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]\left[s^\rho I - \sum_{i=1}^{\rho}(F_i - GK_i)s^{i-1}\right]^{-1}$$

$$= \left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]\frac{I}{s^\rho}$$

Fig. 4.11

Algorithm : Plant in '3' Coordinates

MODEL[1]:



$$\begin{bmatrix}\dot{\xi}\\ \hline \dot{\rho}\end{bmatrix} = \left[\begin{array}{c|c}\Phi & \Theta\\ \hline L_{11} & \alpha I + L_{12}\end{array}\right]\begin{bmatrix}\xi\\ \hline \rho\end{bmatrix} + \begin{bmatrix}\Gamma\\ \hline \Theta\end{bmatrix}\omega$$

$$\mu = \left[\begin{array}{c|c}\Delta & \Theta\end{array}\right]\begin{bmatrix}\xi_L\\ \hline \rho\end{bmatrix}$$

MODEL[2]:

$$\bar{\xi} = \begin{bmatrix}\xi\\ \hline \rho\end{bmatrix},\quad \bar{\Phi} = \left[\begin{array}{c|c}\Phi & \Theta\\ \hline L_{11} & \alpha I + L_{12}\end{array}\right],\quad \bar{\Gamma} = \begin{bmatrix}\Gamma\\ \hline \Theta\end{bmatrix},\quad \bar{\Delta} = \left[\begin{array}{c|c}\Delta & \Theta\end{array}\right]$$

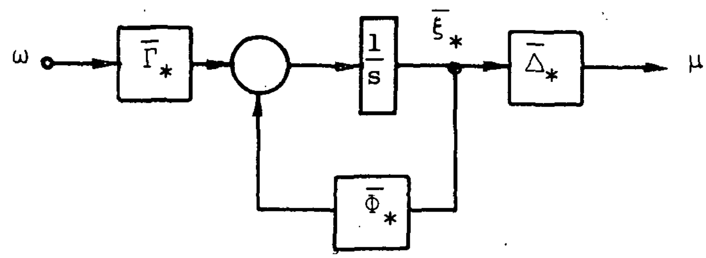
$$\dot{\bar{\xi}} = \bar{\Phi}\xi + \bar{\Gamma}\omega$$

$$\mu = \bar{\Delta}\,\bar{\xi}$$

Fig. 4.12

Algorithm: Model in '1' and '2' Coordinates

MODEL[3]:



$$\dot{\bar{\xi}}_* = \bar{\Phi}_* \bar{\xi}_* + \bar{\Gamma}_* \, \omega$$

$$\bar{\xi}_* = S_1 \bar{\xi}$$

$$\mu = \bar{\Delta}_* \bar{\xi}_*$$

$$\Phi_* = S_1^{-1} \bar{\Phi} S_1 = \begin{bmatrix} \Theta & I & \Theta & \cdots & \Theta \\ \Theta & \Theta & I & & \Theta \\ & \vdots & & & \vdots \\ \Phi_1 & \Phi_2 & \Phi_3 & \cdots & \Phi_\rho \end{bmatrix}, \quad \bar{\Gamma}_* = S_1^{-1} \bar{\Gamma} = \begin{bmatrix} \Theta \\ \Theta \\ \vdots \\ I \end{bmatrix}$$

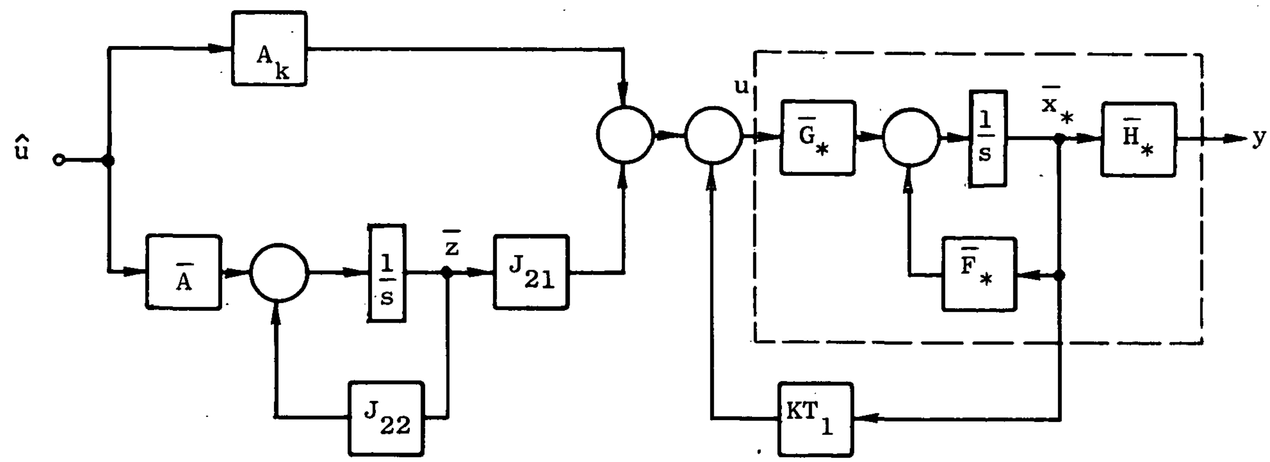
$$\bar{\Delta}_* = \bar{\Delta} S_1 = \begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_\rho \end{bmatrix}$$

$$V_3(s) = \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1} \right] \left[s^\rho I - \sum_{i=1}^{\rho} \Phi_i s^{i-1} \right]^{-1}$$

Fig. 4.13

Algorithm : Model in '3' Coordinates

PLANT[4]:



$$\begin{bmatrix} \dot{\bar{x}}_* \\ \hdashline \dot{\bar{z}} \end{bmatrix} = \left[\begin{array}{c:c} \bar{F}_* + \bar{G}_* KT_1 & J_{21} \\ \hdashline \Theta & J_{22} \end{array}\right] \begin{bmatrix} \bar{x}_* \\ \hdashline \bar{z} \end{bmatrix} + \begin{bmatrix} \bar{G}_* A_k \\ \hdashline \bar{A} \end{bmatrix} \hat{u} = \hat{F}\hat{x} + \hat{G}\hat{u}$$

$$y = \left[\begin{array}{c:c} \bar{H}_* & \Theta \end{array}\right] \begin{bmatrix} \bar{x}_* \\ \hdashline \bar{z} \end{bmatrix} = \hat{H}\hat{x}$$

$$\boxed{\frac{\sum_{i=1}^{k} A_i s^{i-1}}{\sum_{i=1}^{k} \lambda_i s^{i-1}} \triangleq \frac{\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]^{-1} \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right] \alpha^r}{(s+\alpha)^r}} \qquad (*)$$

where:

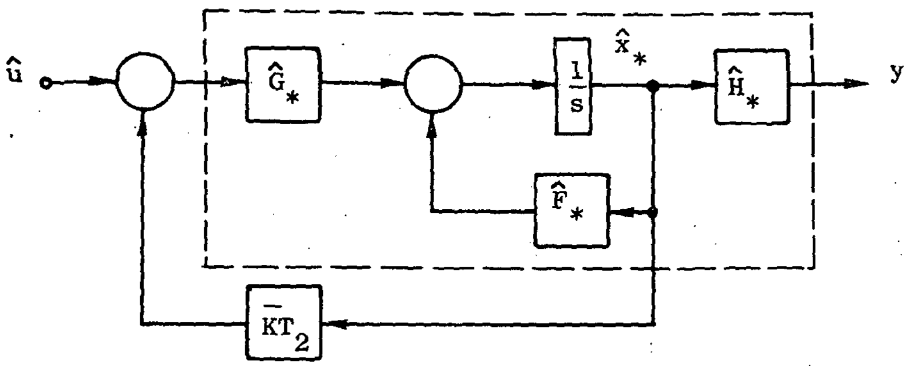
$$\bar{A} = \begin{bmatrix} A_{k-1} \\ \vdots \\ A_2 \\ A_1 \end{bmatrix} ; \quad J_{21} = \underbrace{\begin{bmatrix} \Theta & \Theta \ldots & \Theta \\ \Theta & & \vdots \\ \vdots & & \\ I & \Theta \ldots & \Theta \end{bmatrix}}_{\rho \text{ blocks}} \; \begin{matrix} k-1 \\ \text{blocks} \end{matrix} \qquad \begin{matrix} \lambda_k \triangleq 1 \\ \lambda_i \text{ scalar} \\ A_i \ m \times m \text{ matrix} \end{matrix}$$

$$J_{22} = \begin{bmatrix} \Theta & I & \Theta \ldots & \Theta \\ \Theta & \Theta & I & \Theta \\ & \vdots & & \vdots \\ \Theta & \Theta & \Theta \ldots & \Theta \end{bmatrix} \; \left.\begin{matrix} k-1 \\ \text{blocks} \end{matrix}\right\} \text{square}$$

Fig. 4.14

Algorithm: Plant in '4' Coordinates

PLANT[5]:



$$\dot{\hat{x}}_* = (\hat{F}_* + \hat{G}_*\overline{KT}_2)\hat{x}_* + \hat{G}_*\hat{u} \qquad \hat{x}_* = T_2 \left[\begin{array}{c} \bar{x}_* \\ \hline \bar{z} \end{array}\right]$$

$$y = \hat{H}_*\hat{x}_*$$

$$\hat{F}_* = T_2^{-1}\left[\begin{array}{c|c} \bar{F}_* + \bar{G}_*KT_1 & J_{21} \\ \hline \Theta & J_{22} \end{array}\right]T_2 \;, \quad \hat{G}_* = T_2^{-1}\left[\begin{array}{c} A_k\bar{G}_* \\ \hline \bar{A} \end{array}\right]$$

$$\hat{H}_* = \left[\begin{array}{c|c} \bar{H}_* & \Theta \end{array}\right]T_2$$

$$\overline{KT}_2 = \left[\hat{K}_1 \;\; \hat{K}_2 \;\ldots\; \hat{K}_{\rho+k+1}\right]$$

where:

$$\sum_{i=1}^{\rho+k-1} \hat{K}_i s^{i-1} = \left[s^{\rho}I - \sum_{i=1}^{\rho} \Phi_i s^{i-1}\right]\left[\sum_{i=1}^{k} \lambda_i s^{i-1}\right] - s^{\rho+k-1}I$$

Fig. 4.15

Algorithm: Plant in '5' Coordinates

(Note that the right hand side of the equation is reminiscent of the $C(s)$ given in section 4-B). We wish to use the smallest integers r and k for which that equation is satisfied for some A_i and λ_i. The value of k is increased and zero A_i are added if the numerator order is too low.

Fig. 4.15 completes the operations on the plant by applying a final round of feedback, $\overline{K}T_2$. T_2 represents the similarity transformation used to transform the plant in Fig. 4.14 to the equi-controllable form in Fig.4.15. T_2 is given by

$$T_2 = \begin{bmatrix} A_1 & A_2 & \cdots & A_k & \theta & \cdots & \theta \\ \theta & A_1 & A_2 & \cdots & A_k & \theta \cdots & \theta \\ \vdots & & & \ddots & & & \vdots \\ \theta & \cdots & A_1 & & A_2 & \cdots & A_k \\ \vdots & & & & & & \vdots \\ \theta & \theta & & \cdots & \theta & A_1 & A_2 \\ \theta & \theta & & & & \theta & A_1 \end{bmatrix} \tag{4.15}$$

We may also give the matrices in Fig. 4.14 specifically as follows.

$$\hat{F} = \left[\begin{array}{ccccc|cccc} \theta & I & \theta & \cdots & \theta & \theta & \theta & \cdots & \theta \\ \theta & \theta & I & & \theta & \theta & \theta & & \theta \\ & \vdots & & & \vdots & \vdots & & & \vdots \\ \theta & \theta & \theta & \cdots & \theta & I & \theta & \cdots & \theta \\ \hline \theta & \theta & \theta & \cdots & \theta & \theta & I & \theta \cdots & \theta \\ & \vdots & & & \vdots & \vdots & & & \vdots \\ \theta & \theta & \theta & \cdots & \theta & \theta & \theta & \theta \cdots & I \\ \theta & \theta & \theta & & \theta & \theta & \theta & \theta & \theta \end{array}\right] \qquad \hat{G} = \left[\begin{array}{c} \theta \\ \theta \\ \vdots \\ A_k \\ \hline A_{k-1} \\ \vdots \\ A_2 \\ A_1 \end{array}\right] \tag{4.16}$$

$$\hat{H} = \left[\begin{array}{cccc|cccc} H_1 & H_2 & \cdots & H_\rho & \theta & \theta & \theta \cdots & \theta \end{array}\right]$$

From which we may check that T_2 is indeed the correct transformation.

Note that T_2 will be non-singular if and only if A_1 is non-singular. That our assumptions are sufficient to guarantee the invertability of A_1 will be shown in Chapter V.

We now claim that the compensated plant has the correct transfer function. To prove that this is the case, consider the further expansion of the model shown in Fig. 4.16. The transfer function has not been affected, but it has now as many states as the plant in its '5' form. Further, the state transformation:

$$S_2 = \left[\begin{array}{cccc|cccc}
\Lambda_1 & \Lambda_2 & \cdots & & I & \theta & \cdots & \theta \\
\theta & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{k-1} & I & \cdots & \theta \\
\vdots & & \ddots & & & & & \vdots \\
\theta & \theta & \theta & \cdots & & \cdots & \Lambda_{k-1} & I \\
\hline
I & \theta & \cdots & \theta & \theta & \theta & \cdots & \theta \\
\theta & I & & \theta & \theta & \theta & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots \\
\theta & & \cdots & I & \theta & \theta & \cdots & \theta
\end{array}\right] = \left[\begin{array}{c|c} \multicolumn{2}{c}{\Lambda} \\ \hline I & \theta \end{array}\right] \tag{4.17}$$

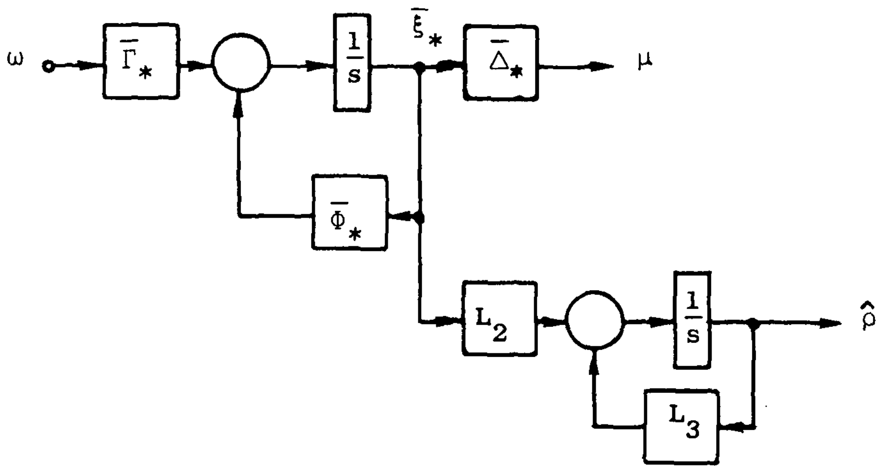
will place that model into equicontrollable form. Now consider the following digression, which will show that:

$$\sum_{i=1}^{\rho+k-1} \hat{H}_i s^{i-1} = \frac{\alpha^r}{(s+\alpha)^r} \sum_{i=1}^{\rho+k-1} \hat{\Delta}_i s^{i-1} \tag{4.18}$$

where:

$$\begin{aligned}
\hat{H}_* &= \hat{H}T_2 \triangleq \left[\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_{\rho+k-1}\right] \\
\hat{\Delta}_* &= \hat{\Delta}S_2 \triangleq \left[\hat{\Delta}_1, \hat{\Delta}_2, \ldots, \hat{\Delta}_{\rho+k-1}\right]
\end{aligned} \tag{4.19}$$

MODEL[4]:



$$\begin{bmatrix} \dot{\bar{\xi}} \\ \hline \dot{\hat{\rho}} \end{bmatrix} = \left[\begin{array}{c|c} \bar{\Phi}_* & \Theta \\ \hline L_2 & L_3 \end{array}\right] \begin{bmatrix} \bar{\xi} \\ \hline \hat{\rho} \end{bmatrix} + \begin{bmatrix} \bar{\Gamma}_* \\ \hline \Theta \end{bmatrix} \omega$$

$$\mu = \left[\begin{array}{c|c} \bar{\Delta}_* & \Theta \end{array}\right] \begin{bmatrix} \bar{\xi} \\ \hline \hat{\rho} \end{bmatrix}$$

$$V_4(s) = \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right]\left[s^{\rho} I - \sum_{i=1}^{\rho} \Phi_i s^{i-1}\right]^{-1}$$

$$L_2 = \begin{bmatrix} \Theta & \Theta & \dots & \Theta \\ \Theta & & & \vdots \\ \vdots & & & \vdots \\ \Lambda_k & & \dots & \Theta \end{bmatrix} \begin{array}{l} k-1 \\ \text{blocks} \end{array} \qquad \begin{array}{l} \Lambda_k \triangleq I \\ \Lambda_i \triangleq \lambda_i I \end{array}$$

ρ blocks

$$L_3 = \begin{bmatrix} \Theta & I & \Theta & \dots & \Theta \\ \Theta & \Theta & I & & \Theta \\ \vdots & & & & \\ -\Lambda_1 & -\Lambda_2 & & \dots & -\Lambda_{k-1} \end{bmatrix}$$

Fig. 4.16

Algorithm: Model in '4' Coordinates

By rearranging equation (*) in Fig. 4.14 we obtain:

$$(s+\alpha)^r \sum_{i=1}^{\rho} H_i s^{i-1} \sum_{i=1}^{k} A_i s^{i-1} = \alpha^r \sum_{i=1}^{\rho} \Delta_i s^{i-1} \sum_{i=1}^{k} \lambda_1 s^{i-1} \qquad (4.20)$$

equivalently:

$$(s+\alpha)^r \left[H_1, H_2, \ldots, H_\rho\right] \begin{bmatrix} \sum_{i=1}^{k} A_i s^{i-1} \\ s\sum_{i=1}^{k} A_i s^{i-1} \\ \vdots \\ s^{\rho-1}\sum_{i=1}^{k} A_i s^{i-1} \end{bmatrix} = \alpha^r \left[\Delta_1, \Delta_2, \ldots, \Delta_\rho\right] \begin{bmatrix} \sum_{i=1}^{k} \lambda_i s^{i-1} \\ s\sum_{i=1}^{k} \lambda_i s^{i-1} \\ \vdots \\ s^{\rho-1}\sum_{i=1}^{k} \lambda_i s^{i-1} \end{bmatrix}.$$

$$(s+\alpha)^r \left[H_1, H_2, \ldots, H_\rho\right] \begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_k & \theta & \cdots & \theta \\ \theta & A_1 & A_2 & A_3 & \cdots & A_k & \theta \cdots & \theta \\ \vdots & & \ddots & & & & & \vdots \\ \theta & \theta & A_1 & A_2 & & \cdots & & A_k \end{bmatrix} \begin{bmatrix} I \\ sI \\ \vdots \\ s^{k+\rho-2} I \end{bmatrix} \qquad (4.21)$$

$$= \alpha^r \left[\Delta_1, \Delta_2, \ldots, \Delta_\rho\right] \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_k & \theta & \cdots & \theta \\ \theta & \Lambda_1 & \Lambda_2 & \Lambda_3 & \cdots & \Lambda_k & \theta \cdots & \theta \\ \vdots & & \ddots & & & & & \vdots \\ \theta & \theta & \Lambda_1 & \Lambda_2 & & \cdots & & \Lambda_k \end{bmatrix} \begin{bmatrix} I \\ sI \\ \vdots \\ s^{k+\rho-2} I \end{bmatrix} \qquad (4.22)$$

Recalling that $\hat{H} = [H \vdots \theta]$ and $\hat{\Delta} = [\Delta \vdots \theta]$ and the definitions of T_2 and S_2 we may then write:

$$(s+\alpha)^r \hat{H} T_2 \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\rho+k-1} I \end{bmatrix} = \alpha^r \hat{\Delta} S_2 \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\rho+k-1} I \end{bmatrix} \tag{4.23}$$

or:

$$(s+\alpha)^r \hat{H}_* \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\rho+k-1} I \end{bmatrix} = \alpha^r \hat{\Delta}_* \begin{bmatrix} I \\ sI \\ \vdots \\ s^{\rho+k-1} I \end{bmatrix} \tag{4.24}$$

$$(s+\alpha)^r \sum_{i=1}^{\rho+k-1} \hat{H}_i s^{i-1} = \alpha^r \sum_{i=1}^{\rho+k-1} \hat{\Delta}_i s^{i-1} \tag{4.25}$$

But now note that:

$$S_2^{-1} \hat{\Phi}_* S_2 = \hat{\Phi}_* = \begin{bmatrix} \theta & I & \theta & \dots & \theta \\ \theta & \theta & I & & \theta \\ & \vdots & & & \vdots \\ \Phi_1 & \Phi_2 & \Phi_3 & & \Phi_{\rho+k-1} \end{bmatrix} \tag{4.26}$$

By computing the above in a similar vein to the previous development, we can show that:

$$\sum_{i=1}^{\rho+k-1} \hat{\Phi}_i s^{i-1} = \left[s^\rho I - \sum_{i=1}^{\rho} \Phi_i s^{i-1} \right] \left[\sum_{i=1}^{k} {}_i s^{i-1} \right] - s^{\rho+k-1} I \tag{4.27}$$

But this implies that $\hat{\Phi}_i = \bar{K}_i$! So we may conclude that:

$$\hat{H}_*(sI-\hat{F}_*)^{-1}\hat{G}_* = \hat{H}_*(sI-\hat{\Phi}_*)^{-1}\Gamma_* \tag{4.28}$$

$$= \hat{H}T_2(sI-\hat{\Phi}_*)^{-1} \tag{4.29}$$

$$= \frac{\alpha^r}{(s+\alpha)^r}\ \hat{\Delta}_*(sI-\hat{\Phi}_*)^{-1}\Gamma_* \tag{4.30}$$

$$= \frac{\alpha^r}{(s+\alpha)^r}\ \Delta(sI-\Phi)^{-1}\Gamma \quad ! \tag{4.31}$$

Hence the algorithm gives the desired result.

Unfortunately, we have two more tedious, but straightforward, steps to perform. First we must unravel the compensator and reduce it to original coordinates, and second we must find a minimal realization of that compensator. Both of these tasks are essentially bookkeeping and use standard techniques. The first may be accomplished by reversing the sundry steps shown in Fig. 4.7. The second can be achieved by using the technique given by Kalman [9], operating on the compensator shown in Fig. 4.17.

The essence of Kalman's method is as follows. Suppose the given system were described by:

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{4.32}$$

First check the rank of the controllability matrix:

$$C = [B, AB, \ldots, A^{n-1}B] \tag{4.33}$$

If rank $C < n$, find the transformation T such that:

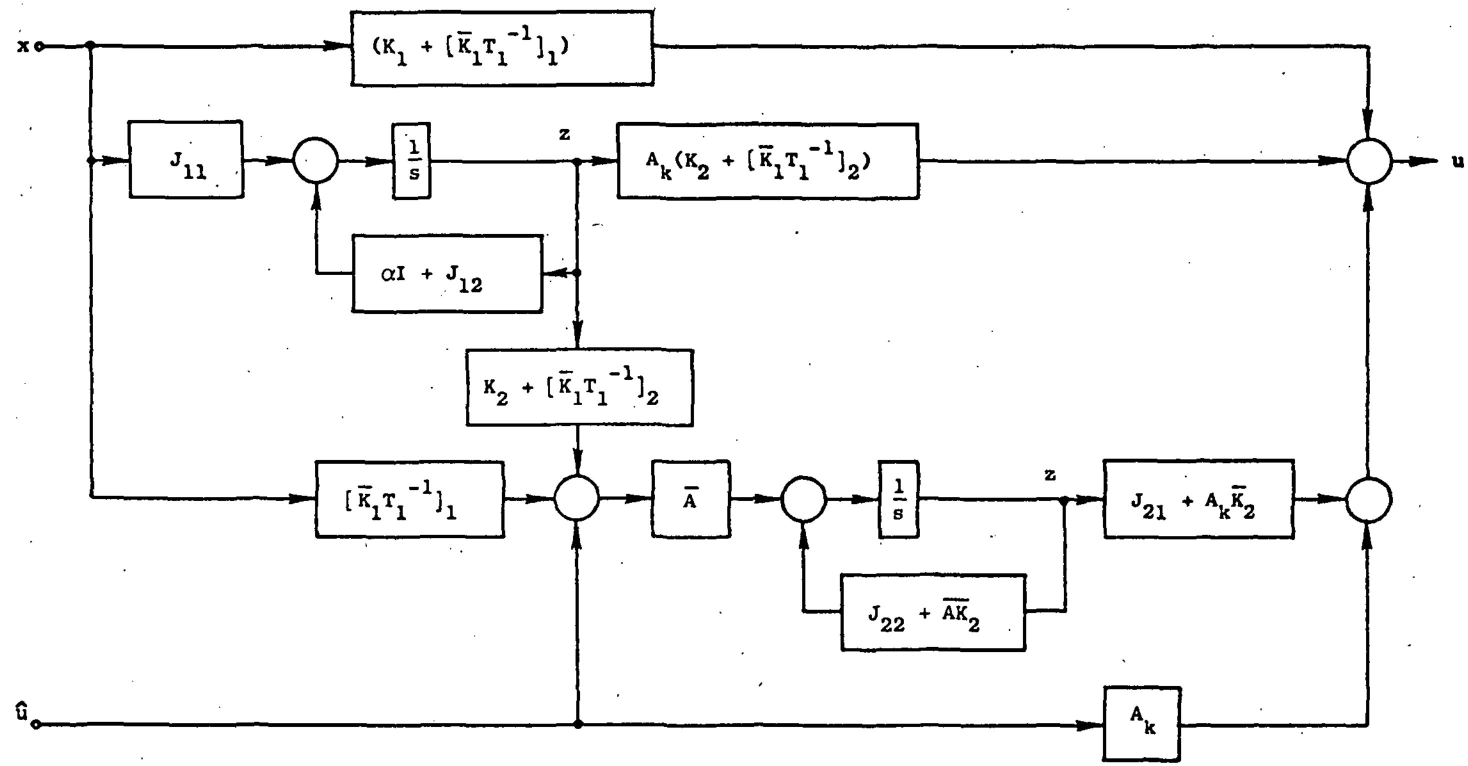
$$T^{-1}C = \left[\begin{array}{c|c} X & Y \\ \hline \theta & \theta \end{array}\right] \tag{4.34}$$

Then:

$$T^{-1}B = \left[\begin{array}{c} \bar{B} \\ \hline \theta \end{array}\right], \quad T^{-1}AT = \left[\begin{array}{c|c} \bar{A} & \bar{X} \\ \hline \theta & \bar{Y} \end{array}\right], \quad CT = [\bar{C} \,\vdots\, \bar{C}_1] \tag{4.35}$$

Therefore:

$$\bar{B}(sI-\bar{A})^{-1}\bar{C} = B(sI-A)^{-1}C \tag{4.36}$$



$$\begin{bmatrix} \dot{z} \\ \hline \dot{\bar{z}} \end{bmatrix} = \left[\begin{array}{c:c} \alpha I + J_{12} & \Theta \\ \hdashline \bar{A}\left(\left[\bar{K}_1 T_1^{-1}\right]_2 + K_2\right) & J_{22} + \overline{AK}_2 \end{array}\right] \begin{bmatrix} z \\ \hdashline \bar{z} \end{bmatrix} + \left[\begin{array}{c:c} \Theta & J_{11} \\ \hdashline \bar{A} & \bar{A}\left[\bar{K}_1 T_1^{-1}\right]_1 \end{array}\right] \begin{bmatrix} \hat{u} \\ \hdashline x \end{bmatrix}$$

$$u = \left[\begin{array}{c:c} \left(K_2 + A_r\left[\bar{K}_1 T_1^{-1}\right]_2\right) & A_r \bar{K}_2 + J_{21} \end{array}\right] \begin{bmatrix} z \\ \hdashline \bar{z} \end{bmatrix} + \left[\begin{array}{c:c} A_k & K_1 + A_k\left[\bar{K}_1 T_1^{-1}\right]_1 \end{array}\right] \begin{bmatrix} \hat{u} \\ \hdashline x \end{bmatrix}$$

Fig. 4.17
Controller in Detail

One then replaces the system (A,B,C) with the smaller dimension system $(\bar{A},\bar{B},\bar{C})$. The same operation may be repeated in the dual, observability case. The final result will be a minimal (i.e. completely controllable and observable) system. The approach has no formal difficulty, but finding T accurately is no mean numerical analysis task, for even moderate state dimension.

D. Conclusion

By a somewhat circuitous path we have succeeded in solving, algorithmically, the model-following problem as posed in Chapter II. Although the process is involved, the operations at each step are quite straightforward. The resultant compensated plant has the proper transfer function, although it usually represents a non-minimal realization of same. Our solution differs markedly from the quadratic loss, but bears a great deal in common with the classical approach. In a sense it is a hybrid, using the best features of each. One may visualize the process as finding feedback laws which take full advantage of existing plant structure so as to minimize the dynamics required in series. The classical approach insists on "unity" feedback, thereby obviating any potential advantage.

Even when programmed in an essentially brute-force manner, the algorithm is capable of generating the compensator in about the time required for one pass through a quadratic loss program using the Q-R algorithm (a very fast technique). Considering that the quadratic loss method has only begun after one pass, our algorithm is especially attractive.

Perhaps more significant is the potential of using non-minimal realizations (particularly equicontrollable or "equi-observable" ones) for the analysis of multivariable systems. Our previous claims as to the utility of such techniques should now be vindicated. Being able to write fairly explicit formulas for transfer functions and being able to see immediately the effects of feedback are compelling enough reasons to justify their use.

V. PROPERTIES OF THE ALGORITHM

A. Stability

Assumption 6 leads to stability in an easy way. Observing the resultant compensated plant leads to the observation that its characteristic equation's roots are: $-\alpha$, roots of the model's characteristic equation, or contained in num det $H(sI-F)^{-1}G$. By the assumption, num det $H(sI-F)^{-1}G$ has all of its roots in the left half plane, hence the compensated plant is surely stable if the model is. The condition is not, however, necessary. For example, if $p(s)$ equals a polynomial formed from all right half plane roots of num det $H(sI-F)^{-1}G$, and if $p(s)$ divides $\sum_{i=1}^{\rho} \Delta_i s^{i-1}$, we may cancel those terms in eq.(*),Fig.4.14, so that the roots will not appear in the result.

B. Existence

Assumption 5 guarantees that A_1 exists and is non-singular. To show this note that:

$$A_1 = H_1^{-1}\Delta_1\mu_1 \tag{5.1}$$

where $\det \Delta(sI-\Phi)^{-1}\Gamma \triangleq \sum_{i=1}^{\rho} \mu_i s^{i-1}$, if H_1 is non-singular. But:

$$\text{num det } H(sI-F)^{-1}G\Big|_{s=0} = \det H_1 \tag{5.2}$$

$$\text{num det } \Delta(sI-\Phi)^{-1}\Gamma\Big|_{s=0} = \det \Delta_1 = \mu_1 \tag{5.3}$$

Hence the assumption says that $\det H_1 \neq 0$ and $\det \Delta_1 \neq 0$. Thus A_1 exists and is non-singular.

Again, the condition is not necessary. For example, if $A_1 = \Theta$ and $\lambda_1 = 0$ we may still have success by cancelling the common factor of s in equation (*), Fig. 4.14, and renumbering (A_2 becomes A_1 , λ_2 becomes λ_1 and so forth). More complicated situations may also occur, but the essential ingredient is for T_2 to be non-singular. This would have been quite difficult to state at the outset, hence we chose to use an intuitively appealing sufficient condition.

C. Solution Invariance

In an interesting case, the solution is largely invariant under changes in the plant. Recall Fig. 4.16, which shows the compensator as a transfer function from u and x to u . Suppose that the plant and model are both given in equicontrollable forms of the same state dimension. Then if we had written the compensator as:

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{\bar{z}} \end{bmatrix} &= \begin{bmatrix} F_c \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} + \begin{bmatrix} G_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ x \end{bmatrix} \\ u &= \begin{bmatrix} H_c \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} + \begin{bmatrix} J_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ x \end{bmatrix} \end{aligned} \tag{5.4}$$

and the plant and model as:

$$\begin{aligned} \text{PLANT: } [\ H\ ] \ , \ &\left[\begin{array}{c:c} \Theta & I \\ \hdashline \multicolumn{2}{c}{F(q)} \end{array}\right] \ , \ \begin{bmatrix} \Theta \\ \hdashline I \end{bmatrix} \\ \text{MODEL: } [\ \Delta\ ] \ , \ &\left[\begin{array}{c:c} \Theta & I \\ \hdashline \multicolumn{2}{c}{\Phi} \end{array}\right] \ , \ \begin{bmatrix} \Theta \\ \hdashline I \end{bmatrix} \end{aligned} \tag{5.5}$$

where q is a vector of parameters, then:

$$\frac{\partial F_c}{\partial q_i} = \Theta \ , \ \frac{\partial G_c}{\partial q_i} = \Theta \ , \ \frac{\partial H_c}{\partial q_i} = \Theta \ , \ \frac{\partial J_c}{\partial q_i} = \left[\Theta \ \vdots \ \frac{\partial F(q)}{\partial q_i} \right] \tag{5.6}$$

since A_i , J_{ij} ; etc., depend only on H and not on F , and $T_0 = I$. Moreover, K was partitioned so that K_1 was $n \times m$! Hence K_2 vanishes. Therefore, the reduced compensator elements F_c, G_c, H_c are invariant under changes in q, whereas J_c is affected linearly. Note that we may relax the above restrictions to the point where:

1. The plant is equicontrollable, and of the form:

$$[\,H\,]\,,\ \left[\begin{array}{c|c} \Theta & I \\ \hline \multicolumn{2}{c}{F(q)} \end{array}\right],\ \left[\begin{array}{c} \Theta \\ \hline I \end{array}\right] \tag{5.7}$$

2. The original model's dimension is less than or equal to that of the plant.

The result of this observation is that one need only estimate accurately the parameters of H. The parameters of F do not as strongly effect the form of the compensator. Since J_c varies linearly in q_i, we know that small errors in estimating the parameters of F will result in small errors in the compensator. Moreover, if better estimates of F become available it is a trivial matter to adjust the compensator to match, since only the "feed-through" gains J_c need be adjusted.

D. Comparison with the Classical Approach

As discussed in Chapter I, the classical control problem, which led to the "modern" version we have considered, was not readily soluble. In fact, our algorithm cannot solve it either, since too much structure was placed on the solution. We shall, however, consider a compromise problem which we can solve, and which has many of the features of the classical problem. In itself, it is of some interest since it includes a reference input. Such a feature is common to many control problems. The block diagram of the problem we shall consider appears in Fig. 5.1:

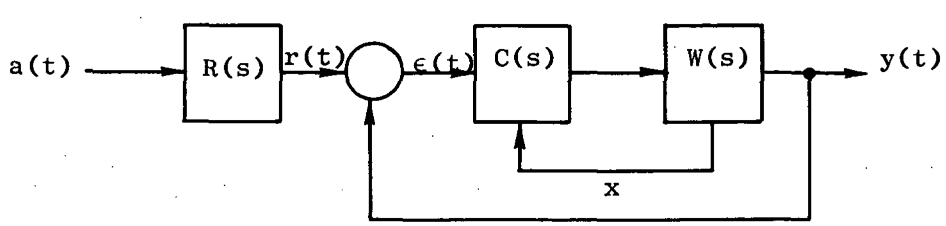


Fig. 5.1

A Classical Problem

$R(s)$ represents a linear system which generates the reference signal $r(t)$ in response to impulses at $a(t)$. ($a(t)$ can be thought to establish the initial conditions placed on the system). Our problem is to make $\epsilon(t)$ behave in some arbitrary manner by the choice of $C(s)$. For example, the designer might like $\epsilon(t) = \mathcal{L}^{-1}[Q(s)]$ for some transform $Q(s)$. Suppose that is indeed the case. For notational purposes, we might say that we wish the transfer function from a to ϵ to be $Q(s)$. Hence we wish to find $C(s)$ so that if the modified transfer function from ϵ to y were $V(s)$, then

$$Q(s) = (I+V(s))^{-1}R(s) \tag{5.8}$$

Hence, if we could make $V(s) = R(s)Q^{-1}(s) - I$, then the problem would be solved. So we have the model: $V(s) = R(s)Q^{-1}(s) - I$ and the plant: $W(s)$. This is then a problem which falls under the ability of our algorithm. We need only check that the various assumptions and conditions of that algorithm are either satisfied or easily accommodated.

A short study of the matrix $Q(s)$ will be of some help. $Q(s)$ represents the dynamic response which we would like the error to have. As such it is not unreasonable to suppose that an acceptable $Q(s)$ might be diagonal (noting that, by the definition used above, $Q(s)$ is indeed square). Moreover, a designer might well be pleased with a response of the form $Q(s) = I/(s+a)$. Although one usually would think of such a response as a bound on performance, we mean it here to be the actual dynamics of the error.

Another point worth a brief discussion is the transform $R(s)$. The reference system is simply a model of the specified reference input. In that case it is not unreasonable that $R(s)$ also be diagonal: that is, that the separate input (reference) signals $r_i(t)$ could be separately described by the diagonal entries of $R(s)$. Although $a(t)$ really only represents the initial conditions on the systems described by $R(s)$, it is not any problem to allow $a(t)$ to represent any input.

Putting these observations together, we find that:

$$V(s) = (s+a)R(s) - I$$

or

$$V_i(s) = (s+a)R_i(s) - 1 \tag{5.9}$$

If the state dimension of $R_i(s)$ is greater than or equal to one, there is no problem of having a non-realizable model. This does not appear to be a problem, since it makes little sense for the reference input to be non-dynamic!

In conclusion then, it is quite possible to wrestle the classical type of reference input control problem into a form which we may easily handle. The algorithm is used to solve an auxiliary problem which, in turn, makes the main problem trivial. Note that the nature of the problem involves a decoupling (since the defined model is diagonal). This points out the subserviance of the decoupling problem to that of model following, (see Appendix C).

When there is no reference input, the problem is amenable to our algorithm. The classical approach seems, at first, to be simpler if for no other reason than the compensator is easy to define. On the other hand, the indicated inverses are especially tedious and the resultant system is usually of over-large state dimension.

As an example of how the result of using the classical approach compares with our design, consider the following problem.

$$W(s) = \begin{bmatrix} \frac{1}{s+5} & \frac{s+4}{(s+1)(s+2.5)} \\ 0 & \frac{s+4}{(s+1)(s+2.5)} \end{bmatrix} \tag{5.10}$$

$$V(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ 0 & \frac{s+4}{(s+2)(s+3)} \end{bmatrix}$$

Using the procedure shown in Chapter I, we would find

$$C(s) = W(s)^{-1}V(s)[I-W(s)]^{-1}$$

$$= \begin{bmatrix} \frac{s+5}{s} & \frac{(s+5)(s^2+4s+6)}{s(s^2+4s+2)} \\ 0 & \frac{(s+1)(s+2.5)}{(s^2+4s+2)} \end{bmatrix} \tag{5.11}$$

This solution requires 3 states.

On the other hand, using our algorithm, we would find that only one dynamic element, namely an observer root, is required. If O(s) is an observer, then our solution is shown in Fig. 5.2:

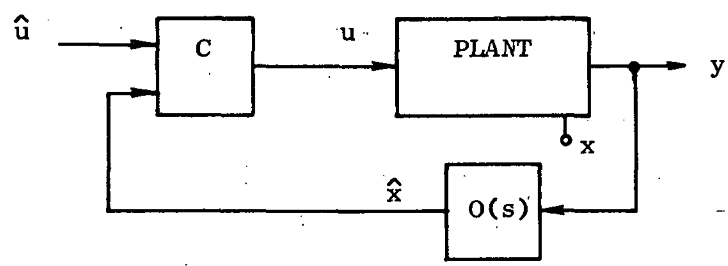


Fig. 5.2

Form of Solution to the Classical Problem

where C is defined by:

$$u = \left[\begin{array}{cc|ccc} 1 & 1 & 4 & -1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & 1/6 \end{array}\right] \begin{bmatrix} \hat{u} \\ \hline \hat{x} \end{bmatrix} \tag{5.12}$$

This feedback law was generated by our program using suitable realizations of V(s) and W(s) .

It is worth noting that our solution is much simpler than the classical design. A quadratic loss approach would result in a 4 state compensator (3 for the model realization, and 1 for an observer).

VI. COMPUTER PROGRAM AND EXAMPLES

A. The Computer Program

The foregoing algorithm has been programmed for an IBM 360/67 at Stanford University, in Fortran IV. The details of the program are hardly germain here - particularly in view of the brute force nature of the programming. Besides the computation of the controller, the program includes an impulse response simulation for the model and compensated plant. This provides a graphic check as to how well the solution actually matches the model's behavior. Some controls on roundoff error are included such as using double precision throughout, setting "canonical elements" in a Luenberger form to 1 or 0 as the case may be, and occasionally rounding off matrices to set small elements to zero.

A flow diagram is not particularly useful in describing the program since only the reduction to Luenberger form, a subroutine to find the independent columns of a matrix, and the simulation routine are iterative in any sense. Hence the flow is virtually always down such a chart. The program was a literal translation from the description of Chapter IV to Fortran. Subroutines were used heavily. The following is a rough scenario which is a reverse translation from the main program back to verbage.

1. Perform the requisite input, immediately echoing back most of what was read in as a check.
2. Reduce the plant and model to Luenberger canonical form, [13]. (It turns out to be easier to go through Luenberger form rather than to equicontrollable form directly - see Theorem 1, Chapter III.)

Print the results.

3. Add states as required and get the plant and model into equicontrollable forms of the same state dimension.
4. Perform a roundoff to set small entries to zero. Print the result.
5. Compute the polynomial matrix:

$$\left(\operatorname{adj}\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right]\right)\cdot\left[\sum_{i=1}^{\rho} \Delta_i s^{i-1}\right] \triangleq \sum_{i=1}^{k} A_i s^{i-1}$$

and the corresponding

$$\det\left[\sum_{i=1}^{\rho} H_i s^{i-1}\right] \triangleq \sum_{i=1}^{\rho} \lambda_i s^{i-1} .$$

(Note: k is determined and A_i adjusted as per Chapter IV, section C).

6. Again perform a roundoff.
7. Check for factors of s and stability of $\sum_{i=1}^{\rho} \lambda_i s^{i-1}$.

If assumptions fail -- exit the program with error message.

8. Print A_i and λ_i .
9. Reduce model to bar-star coordinates.
10. Invert A_1 . If it is singular -- exit with message.
11. Compute compensator. (This is a vast understatement. The code is lengthy, the essence is bookkeeping. Print the result.
12. Minimize the dimension of the compensator. Print it.
13. Find the compensated plant and simulate it.

Further details of the program may be gleaned from consulting the Appendix B. It gives a description of many of the matrices and variables computed.

B. Examples

1. Example 1.

Let us use the algorithm described above on a fairly simple problem.

PLANT:

$$F = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2.5 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{6.1}$$

MODEL:

$$\Phi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The first step is to reduce these to a Luenberger form.

PLANT[1]:

$$F_L = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2.5 & -3.5 \end{bmatrix}, \quad G_L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_L = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 4 & 1 \end{bmatrix} \tag{6.2}$$

MODEL[1]:

$$\Phi_L = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}, \quad \Gamma_L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Delta_L = \begin{bmatrix} 1 & 6 & 2 \\ 0 & 4 & 1 \end{bmatrix}$$

Now we add a root at $s = -15$ to each system ($\alpha = -15$) and again reduce to a canonical form similar to Luenberger's.

PLANT^2:

$$\bar{F} = \begin{bmatrix} -20 & 0 & 0 & -75 \\ 0 & 0 & 1 & 0 \\ 0 & -2.5 & -3.5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} 1 & 4 & 1 & 15 \\ 0 & 4 & 1 & 0 \end{bmatrix} \tag{6.3}$$

$$\bar{\Phi} = \begin{bmatrix} -16 & 0 & 0 & -15 \\ 0 & 0 & 1 & 0 \\ 0 & -6 & -5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{\Gamma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$\bar{\Delta} = \begin{bmatrix} 1 & 6 & 2 & 15 \\ 0 & 4 & 1 & 0 \end{bmatrix}$$

A simple state renumbering will reduce the above to equicontrollable form. Note that only one state was required since the previous block lengths of 1 and 2 (see Theorem 1, Chapter 3) indicate that 4 states will be needed. The equicontrollable form is then:

PLANT^3:

$$\bar{F}_* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -75 & 0 & -20 & 0 \\ 0 & -2.5 & 0 & -3.5 \end{bmatrix}, \quad \bar{G}_* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H_* = \begin{bmatrix} 15 & 4 & 1 & 1 \\ 0 & 4 & 0 & 1 \end{bmatrix} \tag{6.4}$$

MODEL^3:

$$\bar{\Phi}_* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -15 & 0 & -16 & 0 \\ 0 & -6 & 0 & -5 \end{bmatrix}, \quad \bar{\Gamma}_* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\bar{\Delta}_* = \begin{bmatrix} 15 & 6 & 1 & 2 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

Up to this point we have simply been maneuvering the plant and model into forms which are convenient, while maintaining their transfer functions. Now we proceed to find the compensation required. It is easy to compute that:

$$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 19 & 6 \\ 0 & 19 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 60 & 8 \\ 0 & 60 \end{bmatrix}$$
$$\lambda_3 = 1 \quad , \quad \lambda_2 = 19 \quad , \quad \lambda_1 = 60 \tag{6.5}$$

A_1 is clearly invertible. hence there will be no difficulty with getting the compensator. We can see that there are no zeros in the RHP in num det $W(s) = s + 3$, so stability of the solution is sure. Now via Fig. 16 we may simply write down the solution, after checking the definition of some assorted matrices. To wit:

$$J_{11} = [\ 0\ ] \qquad\qquad J_{12} = [1 \quad 0 \quad 0\ ] \tag{6.6}$$

$$J_{21} = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad J_{22} = \left[\begin{array}{cc:cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A simple observation aids in writing the solution. Consider:

$$H_c = \left[\left(K_2 + A_k\left[\bar{K}_1 T_1^{-1}\right]_2\right) \,\vdots\, \left(A_k \bar{K}_2 + J_{21}\right)\right] \tag{6.7}$$

By computation (hand or machine) we find that $H_c = [0]$. Thus only feedback will be required (along with a suitable input transformation). The solution is then:

$$u = \left[\begin{array}{cc:ccc} 1 & 1 & 4 & -2/3 & 2/3 \\ 0 & 1 & 0 & -2/3 & 1/6 \end{array}\right] \begin{bmatrix} \hat{u} \\ \hdashline x \end{bmatrix} \tag{6.8}$$

written in original coordinates. One may check that indeed this does the job. The above matrices were all taken from our computer program compiled by a load-and-go type processor, and run on an IBM 360/67. The execution took 1.34 seconds. (Compilation took 8.00 seconds). Execution time would have been of the order of 0.3 seconds if it had been compiled using the IBM Fortran IV Level H compiler-link editor, but compilation would have taken over a minute.

2. Example 2:

This example first appeared in a paper by Tuler and Tuteur [19]. The problem is defined by the equations:

$$
\begin{aligned}
F &= \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & -2.93 & -4.75 & 0.78 \\ 0.086 & -0.0 & -0.11 & -1.0 \\ 0.0 & -0.040 & 2.59 & -0.39 \end{bmatrix} \\
G &= \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & -3.91 \\ 0.035 & 0.0 \\ -2.53 & 0.31 \end{bmatrix} \\
H &= \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \\
\Phi &= \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & -73.14 & 3.18 \\ 0.086 & 0.0 & -0.11 & -1.0 \\ 0.0086 & 0.086 & 8.95 & -0.49 \end{bmatrix} \\
\Gamma &= \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & -3.91 \\ 0.035 & 0.0 \\ -2.53 & 0.31 \end{bmatrix} \\
\Delta &= \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix}
\end{aligned}
\tag{6.9}
$$

The solution we obtain by use of our program is:

$$
\begin{aligned}
\dot{z} &= F_c z + G_c \begin{bmatrix} \hat{u} \\ x \end{bmatrix} \\
u &= H_c z + J_c \begin{bmatrix} \hat{u} \\ x \end{bmatrix}
\end{aligned}
\tag{6.10}
$$

where:

$$F_c = \begin{bmatrix} -24.92975 \end{bmatrix}$$

$$G_c = \left[\begin{array}{cc:cc} 3.8066034 & -0.4247245 & 3.142755 & -0.2604497 \\ & & -18.09820 & -36.66145 \end{array} \right]$$

$$H_c = \begin{bmatrix} -2.441089 \\ 36.66145 \end{bmatrix} \qquad (6.11)$$

$$J_c = \left[\begin{array}{cc:cc} 1.000000 & 0.00000000 & 0.3074326 & -0.1369023 \\ 0.000000 & 1.00000000 & -4.6682222 & -0.09383566 \\ & & -0.9029511 & -3.715906 \\ & & 25.48522 & 54.65753 \end{array} \right]$$

If $F_c, G_c, H_c J_c$ are used to double precision (as they are computed, but not as they are listed above) then the model and the compensated plant agree to 14 decimal places in their respective impulse responses (over the first 4 time constants of the transient). Surely this is a good fit. Moreover the solution was obtained after <u>one</u> pass of the program (nee: algorithm), and took but 6 seconds of 360 time. The solution put forward by Tyler and Tuteur was only obtained after some unspecified number of man-machine iterations (i.e. new choices of the Q and R matrices), and involved 4 states in the compensator (by virtue of the quadratic loss approach). The observant reader might notice that no output matrices were named there. On the other hand, the solution they obtain, which purports to be matching all of the states, actually succeeds mainly in closely matching the characteristic equations rather than the transient responses, much less transfer functions. Our solution above, matches the first three states nearly perfectly in a transfer function sense. Should it be required to match the third state, say, instead of the fourth, one would simply make:

$$H = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix} \qquad (6.12)$$

This may then be computed to have the solution:

$$F_c = \begin{bmatrix} -72.62536 \end{bmatrix}$$

$$G_c = \begin{bmatrix} 0.03243403 & 0.150421 & -0.0001956313 \\ 0.08903046 & 2.771866 & -0.1186034 \end{bmatrix}$$

$$H_c = \begin{bmatrix} -2.771866 \\ -0.07890249 \end{bmatrix} \tag{6.13}$$

$$J_c = \begin{bmatrix} 1.00000000 & 0.00000000 & -0.00295 \\ 0.00000000 & 1.00000000 & -0.00008405342 \\ -0.1098584 & -0.3698480 & -0.04065118 \\ -0.4967333 & 17.48052 & 0.6149679 \end{bmatrix}$$

This solution is not surprisingly quite different from the first. However, one may as easily solve for any compromise between the two or any other desired output set - so long as it is limited to two directly. (Since one state is the integral of the other, it will be matched up automatically if its associate is).

3. <u>Example 3:</u>

Rynaski and Whitbeck [17] present a similar problem. To wit:

$$F = \begin{bmatrix} -0.751 & 0.0000046 & 0.000572 & -1.604 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -32.2 & -0.0296 & 17.45 \\ 1.0 & -0.0000214 & -0.0009599 & -0.681 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.0015 & -2.65 \\ 0.0 & 0.0 \\ 8.46 & 0.0 \\ -0.0069 & -0.0326 \end{bmatrix} \tag{6.14}$$

$$H = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -1.28 & 0.0000044 & 0.0005067 & -0.2558 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -32.2 & -0.0263 & 30.58 \\ 1.0 & -0.0000086 & -0.0009859 & -0.8584 \end{bmatrix}$$

/Contd.....

/Contd...(6.14)

$$\Gamma = \begin{bmatrix} 0.0048 & -0.898 \\ 0.0 & 0.0 \\ 17.4 & 0.0 \\ -0.00934 & -0.129 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

Again, the answer yields remarkably good results, namely 7 decimal places accuracy as measured by the simulation over a similar time period as the previous example. The output matrices had to be contrived, but exactly the same arguments may be put forward as for Example 2. Also it is possible to match 3 of the 4 states nearly exactly, and there is considerable freedom as to which states or combination of states will be matched. The solution obtained is:

$$\begin{aligned}
F_c &= \begin{bmatrix} -0.8044424 \end{bmatrix} \\
G_c &= \left[\begin{array}{cc|c} -0.0115643 & 0.0751778 & -0.8354341 \\ 0.0443522 & 0.0008648 & 0.5827508 \end{array}\right] \\
H_c &= \begin{bmatrix} -0.8354341 \\ -0.0227829 \end{bmatrix} \\
J_c &= \left[\begin{array}{cc|c} 2.0587376 & 0.0000000 & 0.4112750 \\ -0.0006471 & 0.3388679 & 0.2127252 \\ 0.0424707 & 0.0002993 & 1.0295431 \\ 0.0011583 & 0.0000224 & -0.5221243 \end{array}\right]
\end{aligned} \qquad (6.15)$$

APPENDIX A

A MINIMUM NORM APPROACH

A. **The linear problem.**

Given the plant and model:

$$\text{PLANT:} \quad \dot{x} = Fx + Gu$$
$$\text{MODEL:} \quad \dot{\xi} = \Phi\xi + G \tag{6.16}$$

We might wish to find a feedback control law $u = Kx$ $\ni$:

$$J = \|e^{\Phi\tau} - e^{(F+GK)\tau}\|_Q^2 \quad ; \quad \|A\|^2 = \text{trace } [A^T QA] \tag{6.17}$$

is minimized over choices of K. This is a generalization of Erzberger's [6] approach. We refer the reader to his paper for motivation of the formulation. It incorporates two distinctive features not possessed by his, however,

1. A weighting matrix Q is included $(Q \geq 0)$ so that one's relative interest in matching the states can be expressed.
2. A parameter τ is included which is a gross measure of closeness. As $\tau \to 0$, this problem becomes the same as Erzberger's.

The main hurdle to be overcome in solving this problem is finding the gradient of the cost function with respect to K. Further we would like the gradient to be linear in K. To that end, let us make a model of the plant.

$$x(n + 1) = e^{F\tau}x(n) + F^{-1}(e^{F\tau} - I)Gu(n) \tag{6.18}$$

(Note: F^{-1} need not exist for the term to exist, as a series expansion will confirm).

Suppose we choose τ to be small so that the above is a good representation and further that the feedback law:

$$u(n) = Kx(n) + \hat{u}(n) \tag{6.19}$$

is desired. Then:

$$x(n+1) = \left[e^{F\tau} + F^{-1}(e^{F\tau} - I)GK\right]x(n) + F^{-1} + F^{-1}(e^{F\tau} - I)G\hat{u}(n) \tag{6.20}$$

$\hat{u}(n)$ represents external inputs. Now, the continuous system, after such feedback, is:

$$\dot{x} = (F + GK)x + G\hat{u} \tag{6.21}$$

But this system is virtually equivalent to the preceding (for small τ), hence:

$$e^{F\tau} + F^{-1}(e^{F\tau} - I)G \simeq e^{[F+GK]\tau} \tag{6.22}$$

Then:

$$J = \|e^{\Phi\tau} - e^{F\tau} - F^{-1}(e^{F\tau} - I)GK\|_Q^2 = \|e^{\Phi\tau} - e^{F\tau}\|_Q^2 \tag{6.23}$$

$$-2\langle \Gamma^T Q(e^{\Phi t} - e^{Ft}), K\rangle + \langle \Gamma^T Q\Gamma K, K\rangle$$

where $\Gamma = F^{-1}(e^{Ft} - I)GK$ and $\langle A,B\rangle = \sqrt{\text{trace } A^T QB}$

Then taking $\nabla_K(J)$ formally, we obtain:

$$\nabla_K(J) = -2\Gamma^T Q(e^{\Phi\tau} - E^{F\tau}) + 2\Gamma^T Q\Gamma K \tag{6.24}$$

Setting $\Delta_K(J) = \theta$ we obtain:

$$K = [\Gamma^T Q\Gamma]^{-1}\Gamma^T Q[e^{\Phi\tau} - e^F] \tag{6.25}$$

This is then a solution to a form of model following problem. It suffers from the basic difficulty of Erzberger's approach, namely that the plant and model must have vast structural similarity for it to be successful. Fortunately, however, there are many problems having this trait. In aircraft control problems we have a great deal of information as to structure. One often finds the plant and model are different only in a few parameters, while they have the same essential organization of dynamics. Hence the success of Tyler's problem (among others) is not surprising.

One might notice that, as in the quadratic loss formulation, we have an unknown weighting matrix Q . No attempt will be made to minimize the problems engendered by it. Suffice to say that the solution here takes less than one second to perform, hence a time-share mode might be quite feasible, wherein the designer would rapidly cut and try various Q's until an acceptable solution appeared. That such an approach can yield spectacular results will be shown by example.

Rynaski, et al [16] give an interesting problem, depicted in Fig. A.1. They give the usual solution, but indicate that improvement in accuracy would involve substantial increase in control effort. However, trying our approach yields an interesting result. Not only can we find a pure feedback control which does slightly better than theirs, but it does so with <u>uniformly</u> less control effort. For an increase of about 20% in maximum effort we may do a spectacularly close match. Clearly we may choose any tradeoff in between that we desire. These results are depicted in Figs. A.2 and A.3. Plotted are the pitch responses of the open loop aircraft, the model, Rynaski's solution, and the two solutions obtained via our method. The associated control efforts are plotted to verify our claim.

$$
\Phi = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-3.72 & -2.156 & 0.05626 & 0.317 & 0 & 0 \\
-8.54 & 0 & -2.56 & 2.50 & 0 & 0 \\
3.72 & 2.0 & 0 & -3.17 & 0 & 0 \\
0 & 0 & 0 & 0 & -25 & 0 \\
0 & 0 & 00 & 0 & 0 & -6.67
\end{bmatrix}
$$

$$
F = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-2.959 & -0.0119 & 0.00906 & 0.0103 & 1.36 & 0.250 \\
0.514 & 0 & -0.0625 & 0.0376 & -2.33 & -5.12 \\
2.959 & 0 & -0.0037 & -0.0103 & -1.423 & -0.25 \\
0 & 0 & 0 & 0 & -25 & 0 \\
0 & 0 & 0 & 0 & 0 & -6.67
\end{bmatrix}
$$

$$
G = \begin{bmatrix}
0 & 0 \\
0.0667 & 0 \\
0 & 0 \\
0 & 0 \\
25 & 0 \\
0 & 6.67
\end{bmatrix}
$$

Fig. A.1

Rynaski Problem

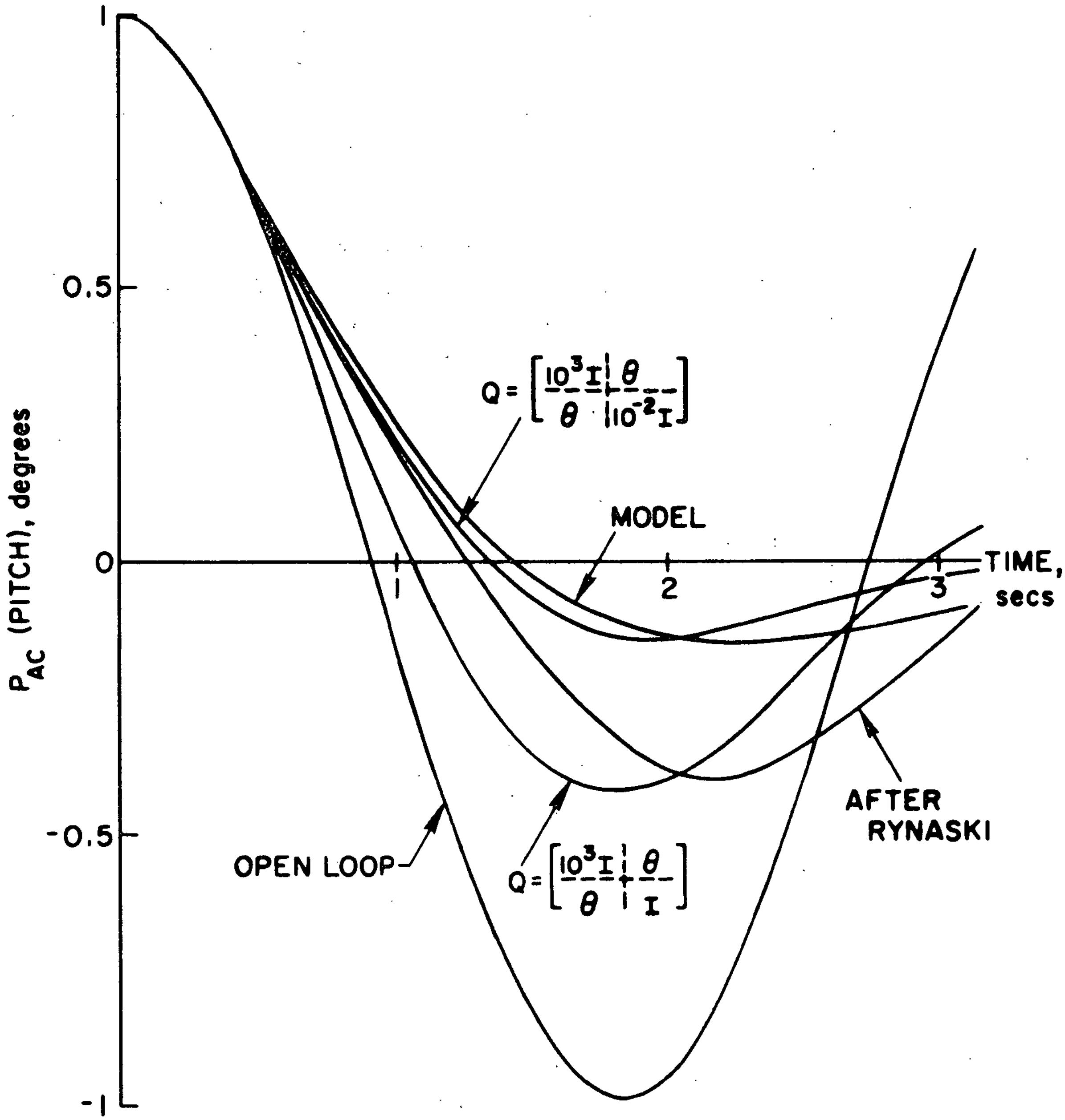


Fig. A.2

Pitch Response

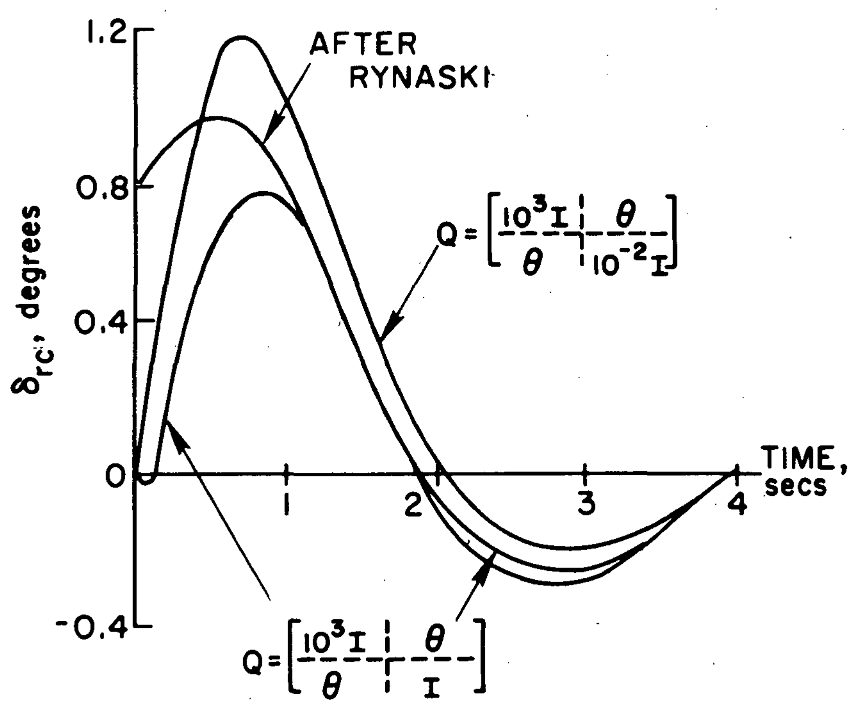


Fig. A.3

Control Effort

A final example, from Rynaski and Whitbeck [17], is almost trivial.

$$F = \begin{bmatrix} -0.751 & 0.0 & 0.0 & -1.60 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -32.2 & -0.029 & 17.45 \\ 1.0 & 0.0 & 0.0 & -0.68 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} -1.285 & 0.0 & 0.0 & -0.26 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -32.2 & -0.026 & 30.6 \\ 1.0 & 0.0 & 0.0 & -0.86 \end{bmatrix} \qquad (6.26)$$

$$G = \begin{bmatrix} 0.0015 & -2.65 \\ 0 & 0 \\ 8.5 & 0 \\ -0.0069 & -0.033 \end{bmatrix}$$

with $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the resultant system

was

$$F + GK = \begin{bmatrix} -1.275 & 0.0 & 0.0 & -0.264 \\ 1.0 & 0.0 & 0.0 & 0.0 \\ 0.066 & -32.2 & -0.0260 & 30.53 \\ 0.993 & 0.0 & 0.0 & -0.674 \end{bmatrix} \qquad (6.27)$$

Enough said!

In conclusion then, if a problem with such a high degree of structural consistence between plant and model appears, this method is worth trying. The computation times involved are so short that it might be used as first cut in any event. If five or so tries fail to yield an acceptable solution, it is perhaps best to switch to the more general algorithm.

B. <u>A non-linear extension</u>

Suppose we have the plant:

$$\dot{x} = F\underline{f}(x) + Gu$$

where where

$$\underline{f}(\underline{x}) = \begin{bmatrix} f_1(x_1) \\ f_2(x_2) \\ \vdots \\ f_n(x_n) \end{bmatrix} \tag{6.28}$$

and F and G are the usual matrices, and a model: $\dot{\xi} = \Phi\hat{\underline{f}}(\xi)$
where

$$\hat{\underline{f}}(\xi) = \begin{bmatrix} f_1(\xi_1) \\ \vdots \\ f_m(\xi_m) \end{bmatrix} \quad m \leq n \tag{6.29}$$

and further assume that for the output matrix H,

$$\hat{\underline{f}}(Hx) = H\underline{f}(x) \tag{6.30}$$

Then say we desire $\xi(t) = Hx(t)$ given that $\xi(0) = Hx(0)$ implies:

$$\begin{aligned} \xi(t) &= H\dot{x}(t) \\ &= HF\underline{f}(x) + HG\,u \end{aligned} \tag{6.31}$$

But:

$$\begin{aligned} \dot{\xi} &= \Phi\hat{\underline{f}}(\xi) \\ &= \Phi\hat{\underline{f}}(Hx) \\ &= \Phi H\underline{f}(\underline{x}) \end{aligned} \tag{6.32}$$

So that:

$$HF\underline{f}(x) + HGu = \Phi H\underline{f}(x) \tag{6.33}$$

or

$$HGu = (\Phi H - HF)\underline{f}(x)$$

define

$$u = K\hat{\hat{f}}(Dx)$$

where

$$\hat{\hat{f}}(Dx) = \begin{bmatrix} f_1(d_1x) \\ \vdots \\ f_p(d_px) \end{bmatrix} = D\underline{f}(x) \tag{6.34}$$

(we assume such a D exists).

Then we desire a $K \ni$:

$$[HGKD - (\Phi H - HF)]\underline{f}(x) = \theta \quad \text{for any} \quad x . \tag{6.35}$$

It is sufficient that:

$$HGKD = \Phi H - HF \tag{6.36}$$

Then the solution, if it exists is:

$$K = (HG)^{\dagger}(\Phi H - HF)D^{\dagger} \tag{6.37}$$

For example, the systems:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \underline{f}(x) = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2^2 \end{bmatrix} \tag{6.38}$$

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{so} \quad \underline{f}(Hx) \overset{!}{=} \underline{f}(x)$$

and $D = I$ is also reasonable. So

$$K = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\dagger} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = [2 \quad 0] \tag{6.39}$$

So

$$u = [2 \quad 0]\,\hat{\hat{f}}(Dx) = [2 \quad 0] \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix} = 2x_1 \tag{6.40}$$

which is correct, although trivial. Nonetheless, the idea is that in cases in which the previous linear approach is used, we may allow some non-linearity into the problem without changing the solution. Such a condition is often present in aircraft problems; i.e. a small amount of nonlinearity. This might be effectively modeled in the form we suggest which would assure us that if the linear problem could be solved, then the non-linear one could be also.

APPENDIX B

DEFINITION OF SOME VARIABLES

Our purpose here is to describe those variables which are used in the algorithm which are particularly obtuse in definition. They will be considered in order of occurrence in the original.

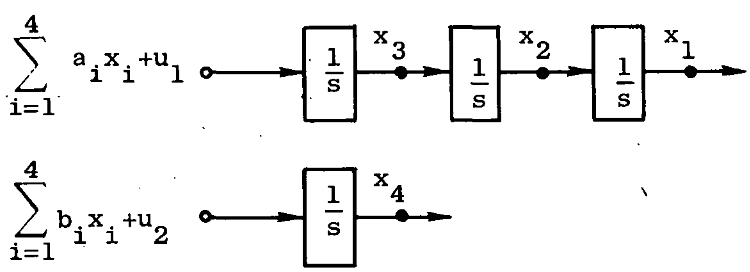
J_1 (L_1)

The number of states added in z depends on:

1. The number of states n_1 needed to make $m|n+n_1$. The smallest such n_1 is chosen.

2. Whether or not the model has more than $n+n_1$ states. If it does, say n_2 more, then n_2 extra states are added.

We suppose that the system is in Luenberger form and that a state map is available. It has ρ rows and m columns ($\rho = (n+n_1+n_2)/m$). It represents how the current block diagram of the plant is numbered. For example:



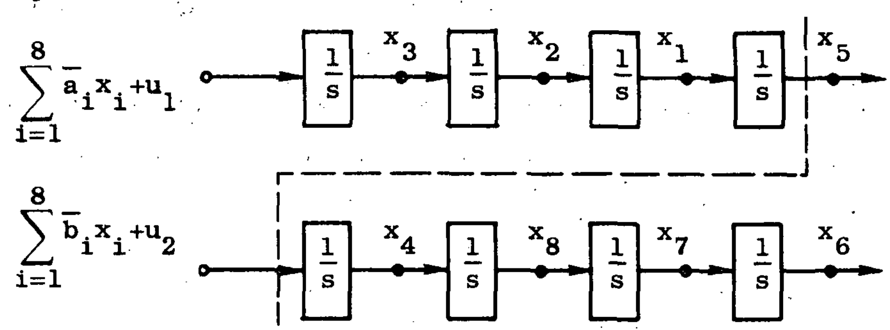
This would yield the state map:

$$\begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}$$

The zeros are merely place keepers. To illustrate how states are added, and in turn how J_1 is created, suppose in the above example we wish to add 4 states. The new map would be:

$$\begin{bmatrix} 5 & 6 \\ 1 & 7 \\ 2 & 8 \\ 3 & 4 \end{bmatrix}$$

The procedure is to "push down" the short blocks and to add states into the spaces required, numbering left to right and top to bottom. The block diagram then becomes:



This is reminiscent of the discussion after Theorem 1, Chapter III, we may define:

$$J_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] = \left[J_{12} \,\vdots\, J_{11}\right]$$

The partition is made between the n^{th} and $(n+1)^{st}$ columns of J_1. Even though no α has been mentioned, the dynamics associated are contained in $\bar{a}_i$ and $\bar{b}_i$ rather than in J_1. In actuality the added states would have been $1/(s+\alpha)$ rather than $1/s$, but it is an easy matter to separate those dynamics from the structural information of J_1 (which is invariant under α).

L_1 is entirely analogous for the model. The notational changes should be apparent.

K In "bar-star" coordinates:

$$\bar{F}_* = \begin{bmatrix} \theta & I & \theta & \cdots & \theta \\ \theta & \theta & I & & \theta \\ \vdots & & & & \vdots \\ F_1 & F_2 & \cdots & & F_\rho \end{bmatrix}$$

then

$$KT_1 = -\begin{bmatrix} F_1 & F_2 & \cdots & F_\rho \end{bmatrix}$$

KT_1 is defined rather than K for notational convenience.

A_k
$\bar{A}$
λ_i

$$\bar{A} = \begin{bmatrix} A_k \\ A_{k-1} \\ \vdots \\ A_2 \\ A_1 \end{bmatrix} \qquad A_i \text{ are } m \times m \text{ matrices}$$

Formally, A_i are computed as follows. In bar-star coordinates:

$$\bar{H}_* = \begin{bmatrix} H_1 & H_2 & \cdots & H_\rho \end{bmatrix} \quad ; \quad \bar{\Delta}_* = \begin{bmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_q \end{bmatrix}$$

$$\frac{\sum_{i=1}^{k} A_i s^{i-1}}{\sum_{i=1}^{k} \lambda_i s^{i-1}} \triangleq \left[\sum_{i=1}^{\rho} H_i s^{i-1} \right]^{-1} \left[\sum_{i=1}^{\rho} \Delta_i s^{i-1} \right] \frac{\alpha^r}{(s+\alpha)^r}$$

This is subject to the restrictions:

1. $\lambda_k = 1$
2. r is to be as small as possible.
3. The left side of the above is to be minimal. That is, any cancellations possible are made between numerator and denominator.
4. The numerator and denominator on the left are to be the same length (as shown).

L_2 L_3	$L_3 = \begin{bmatrix} \Theta & I & \Theta & \cdots & \Theta \\ \Theta & \Theta & I & & \Theta \\ & \vdots & & & \vdots \\ -\Lambda_1 & -\Lambda_2 & -\Lambda_3 & \cdots & -\Lambda_{k-1} \end{bmatrix}$ all blocks are m×m $\Lambda_i = \lambda_i I$ $L_2 = \begin{bmatrix} \Theta & \Theta & \Theta & \cdots & \Theta \\ \Theta & \Theta & \Theta & & \Theta \\ & \vdots & & & \vdots \\ I & \Theta & \Theta & \cdots & \Theta \end{bmatrix}$ k-1 blocks ρ blocks
J_{21} J_{22}	$J_{21} = [\, I \quad \Theta \quad \Theta \, \cdots \, \Theta \,]$ all blocks are m×m $J_{22} = \begin{bmatrix} \Theta & I & \Theta & \cdots & \Theta \\ \Theta & \Theta & I & & \Theta \\ & \vdots & & & \vdots \\ \Theta & \Theta & \Theta & \cdots & \Theta \end{bmatrix}$ k-1 blocks square
$\overline{K}$	In bar-star coordinates: $\overline{\Phi}_{*}^{*} \triangleq \begin{bmatrix} \Theta & I & \Theta & \cdots & \Theta \\ \Theta & \Theta & I & & \Theta \\ & \vdots & & & \vdots \\ \overline{\Phi}_1 & \overline{\Phi}_2 & \overline{\Phi}_3 & \cdots & \overline{\Phi}_\rho \end{bmatrix}$ If: $\overline{KT}_2 = \begin{bmatrix} \overline{K}_1 & \overline{K}_2 & \cdots & \overline{K}_{\rho+k-1} \end{bmatrix}$ then: $\sum_{i=1}^{\rho+k-1} \overline{K}_i s^{i-1} = \left[s^\rho I - \sum_{i=1}^{\rho} \overline{\Phi}_i s^{i-1} \right] \left[\sum_{i=1}^{k} \lambda_i s^{i-1} \right] - s^{\rho+k-1} I$

$$T_2 \qquad \begin{bmatrix}
A_1 & A_2 & \cdots & A_k & \Theta & \cdots & \Theta & \Theta \\
\Theta & A_1 & A_2 & \cdots & A_k & \Theta & \cdots & \Theta \\
\vdots & & & & & & & \vdots \\
 & & \Theta & A_1 & \cdots & & A_{k-1} & A_k \\
\Theta & & & & & & & A_{k-1} \\
 & & & & & & & \vdots \\
\Theta & \Theta & & & & \Theta & A_1 & A_2 \\
\Theta & \Theta & \Theta & \cdots & & \Theta & \Theta & A_1
\end{bmatrix}$$

$\rho + k - 1$ blocks square

Note that T_2 is nonsingular iff A_1 is nonsingular. Also T_2 is $(\rho+k-1)m$ dimensional.

APPENDIX C

DECOUPLING

After becoming familiar with the technique of this dissertation, it becomes clear that the decoupling problem (see Wonham and Morse [22], and Morse and Wonham [15]) is closely allied to model-following. To wit, the model is decoupled. Granted the decoupling problem specifies no particular model. On the other hand, we can easily see that if the plant satisfies the conditions given in Chapter II, then virtually <u>any</u> decoupled model can be followed perfectly. This observation is borne out in theory. In [22] a necessary and sufficient condition is given such that a system may be decoupled by dynamic compensation. We have shown (see [23]) that their condition is equivalent to ours under the input and output restrictions given in Chapter II. Hence our intuition is nicely fulfilled in reality.

Specifically we may state the following theorem, proven in [23]:

<u>Theorem</u>: Given the system (H,F,G) with H $m \times n$, F $n \times n$, G $n \times m$ and full rank, then (H,F,G) may be decoupled by a dynamic compensator $C(s)$, of the form given in Fig. 2.1, if and only if num det $H(sI - F)^{-1}G$ has none of its roots at the origin of the s-plane.

This theorem constitutes a simple test for decoupling in the m-input, m-output case. The design is as per the previous solution (Chapter IV) with the model virtually arbitrary. Note further that the comments in Chapter V also may be applied to decoupling. In particular, Wonham and Morse and the above theorem do not specify that the resultant decoupling be stable. If that is required, the further restriction of the roots of num det $H(sI - F)^{-1}G$ to the left half plane will suffice.

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