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Technical Report No. 69
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ABSTRACT

This work presents concepts and results in the fields of mathematical modelling, economics and stability analysis. A coupled renewal-differential equation structure is presented as a modelling form for systems possessing hereditary characteristics, and this structure is applied to a model of the Austrian theory of business cycles. For realistic conditions, the system is shown to have an infinite number of poles, and conditions are presented which are both necessary and sufficient for all poles to lie strictly in the left half plane.

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I. INTRODUCTION

The Time Delay Structure

In order to describe or predict the behavior of a physical system, a common procedure is to represent the various physical variable values at any time t as a finite dimensional vector x(t) and to assume that the rate of change of x(t) depends only on quantities measurable at time t, leading to the ordinary differential equation system

$$\frac{dx(t)}{dt} \overset{\Delta}{=} x'(t) = f(x(t), t), \quad x(0) = x_0. \tag{1.1}$$

The class of physical systems which can adequately be described by (1.1) is large, yet systems in physiology, economics or sociology, among others, often require a more sophisticated mathematical structure. For example, the number of human births at any time clearly depends on the number of women of childbearing age alive nine months previously. Systems of the above form lead to a modelling structure of equations with time delay terms and which may be algebraic, differential or integral in form with single or multiple delays.

There are many examples in the literature of the use of such time delay equations in modelling diverse problems. Lotka [1] considered the problem of industrial replacement with the model

$$p(t) + \int_0^t f(s) \, p(t - s) \, ds = 1, \tag{1.2}$$

where p(t) is the probability that an individual machine survives at least t years, f(t) is the replacement rate and the number of machines in use is constrained to being a constant. Wangersky and Cunningham [2] analyzed the effects of reaction time delay on prey-predator relationships by using as a model the equations

$$x'(t) = A x(t) - A x^2(t) / K_x - K_1 x(t) y(t) \tag{1.3}$$

$$y'(t) = K_2 x(t - \tau) y(t - \tau) - my(t), \tag{1.4}$$

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where $x(t)$ and $y(t)$ are respectively the number of prey and predator at time t, τ is the reaction time delay and A, K_x, K_1, K_2 and m are various constants. A more general form of (1.2), the renewal equation, was used by Lotka [3] in his fundamental work on the dynamic behavior of the age structure of a general population. Denoting by $b(t)$ the number of births at any time t, by $b*(t)$ the number of births due to the population alive at time zero and by $m(t)$ the maternity function, "Lotka's Equation" is given by

$$b(t) = b*(t) + \int_0^t b(t - \tau)\, m(\tau)\, d\tau. \qquad (1.5)$$

The structure of (1.5) seems to be especially broad in its applications, having been used for studies in solid state physics [4], plasma dynamics [5] and biology [6]. Such a (vector) form should be considered in any attempt to model large economic processes, since many certainly possess a preponderant hereditary nature. Unfortunately, the analysis of renewal systems becomes increasingly difficult as the system order increases. One purpose of this work is to illustrate how certain simplifying assumptions can be made about a system to reduce this complexity. Specifically, the equation structure to be examined is of the form

$$\underline{x}(t) = \underline{g}(t) + \int_0^t B(t - \tau)\, \underline{x}(\tau)\, d\tau + C\underline{v}(t) \qquad (1.6)$$

$$\underline{v}'(t) = D\underline{x}(t) + E\underline{v}(t), \quad \underline{v}(0) = \underline{v}_0, \qquad (1.7)$$

where $\underline{x}(t)$ and $\underline{g}(t)$ are n-vectors, $\underline{v}(t)$ is an m-vector and $B(t)$, C, D and E are $n \times n$, $m \times n$, $n \times m$ and $m \times m$ matrices respectively. While at first glance equations (1.6) and (1.7) seem more complex than an ordinary vector renewal equation, they should be considered conceptually as an nth order renewal system coupled to an mth order differential system to model an $(m + n)^{th}$ order renewal process. The following section gives the motivation for an economic problem to which the above structure will be applied.

Motivation

The business cycle is unarguably a physical process of paramount importance. A "run-away" inflation in Germany at the close of World War I

effectively wiped out the life savings of millions of Germans and the Great Depression caused unbelievable world-wide social agony. Whether the evils of depression exceed those of inflation is a question of at most academic interest; the problem is to control a country's economic behavior between these two extremes. As governments' powers and influence have grown, their abilities to influence a country's economic posture have increased, and the monetary control procedures are now hopefully available. The problem, then, is to recognize at an early time the advent of inflationary or deflationary factors. As an example that such recognition is not at all simple, exactly one week before the crash of 1929, eminent Yale economist Irving Fisher announced that the American economy was moving on a "... permanently high plateau" [7].

Modern economists apply one of two methods in their "prognostic" attempts. The first of these involves the use of econometric models of an empirical-statistical nature. All such models have in common a relatively short-term forecastive ability, which may or may not be a liability, and an almost overwhelming complexity, certainly a liability. The Wharton Econometric and Forecasting Unit model, for example, consists of 52 stochastic equations, 29 identities, 144 statistically determined variables and 117 endogenous and exogenous variables [8]. Such complexity, besides causing computational problems, also tends to obscure the possible existence of fundamental economic factors which, if completely understood, might lead to a more basic understanding of the economic process. These substrative concepts form the basis of the second forecastive approach, a more qualitative treatment.

Qualitative economic theories attempt to present a foundation that simply describes the system's aggregate behavior, yet is comprehensive enough to allow explanation of the myriad of peripheral activities. Since Adam Smith's Wealth of Nations [9], countless theories have been propounded,

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and some, like Smith's, are completely valid given the circumstances under which they were formulated. Changing conditions, however, have shown even the best of these to be dated. To discover whether or not any particular theory will prove independent of the events surrounding its inception, it should be modelled mathematically. This allows a testing procedure far more general than a mere comparison with past history. Although some attempts have been made along these lines (see in particular reference [8], Chapter 14), there still exist many qualitative theories which have not been quantitatively expressed. This work will present a simple model of one of these, the Austrian theory [10] of business cycles.

The Austrian theory, as expounded by Ludwig von Mises [11] and F. A. Hayek [12], is basically a monetary theory of the cycle. If an economy has a fixed supply of money, that amount available for investment is a function of the populace's time preference for consumption. If the people are in a buying mood, only a small percentage of their income is "saved" through savings accounts, insurance policies, stocks and bonds, and the interest rate will rise. If, on the other hand, the time preference is slanted toward future consumption, larger savings are made and the interest rate drops. Turning now to the other half of the investment process, the Austrian theory divides industry into two sections, the capital goods industries and the consumer-goods industries. The capital-goods section is characterized mainly by its long-term aspects. The building of a steel mill, for example, requires considerable time and continued investments as work progresses. Such investments, then, should be made at low interest rate conditions, since these indicate future consumer desire. The consumer-goods industries, conversely, are more quickly expanded, and should expand when the interest rate is high, indicating an immediate demand for goods. The overall process seems stable, since the availability of consumer goods is increased when the demand for them is high, and the entrepreneur builds for future consumption when society is saving.

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Suppose now that the government intrudes into this investment process by credit expansion through federal banks. As credit is created, the interest rate drops and it appears to the entrepreneur that consumer time preferences have lengthened. This clearly calls for a shifting to capital-goods expansion, but as the excess money filters down to the consumer, he attempts to assert this increase in buying power in accordance with his old saving/consumption proportions. This, of course, completely startles the business community, which had been misled by the drop in interest rates into believing that these proportions had changed. As soon as the government decides to end its inflationary credit policies, as it must sooner or later, the continued investments required for the capital-goods industries are not obtainable at profitable interest rates, and previous investments are found to have been ill-conceived. The only recourse is liquidation of certain projects, signalling the beginning of the depression. The recovery period begins only when the capital-goods/consumer-goods investment proportions are in agreement with the consumption characteristics of the individual buyer.

While this is only a brief summary of the Austrian theory, it does cover the main concepts and provides a starting point for the mathematical modelling process.

Formulation of Model and Summary of Results

The model to be presented will characterize the Austrian theory at least locally. That is, a linear model will be formulated in an attempt to describe small deviations of the physical process from the trend. Suppose x(t) is defined to be the investment in capital-goods industries differing from the trend, and consider the viewpoint the entrepreneur takes in deciding whether or not to invest in x(t). Recalling that such industries have long completion times, our entrepreneur would certainly consider the activity in this sector over the past several years. The existence of several steel mills under construction would, of course, tend to dampen interest in this area. This could be expressed mathematically by

$$x(t) \propto - \int_0^t \rho(\tau)\, x(t - \tau)\, d\tau, \tag{1.8}$$

where $\rho(t)$, non-negative in general, might be called an "influence" function. If $i(t)$ is defined as the difference between the current rate of interest and the trend, the theory states that $x(t)$ will tend to be positive when $i(t)$ is negative; thus,

$$x(t) \propto - i(t). \tag{1.9}$$

Turning now to the investment in consumer-goods industries as differing from the trend, denoted by $v(t)$, it would be possible to structure a modelling equation exactly as (1.8) and (1.9). However, the time-delays in such industries are of comparatively short duration, and the system model is simplified if it is assumed that this faster response process can be modelled by an ordinary differential equation. That is, since an investor in a consumer-goods industry should change his position with respect to the instantaneous levels of investment activity, the model should express

$$v'(t) \propto - v(t). \tag{1.10}$$

This investment behavior as a function of the interest rate is also of a somewhat different character. Since consumer-goods production can increase so rapidly, a high interest rate should signal further increases; that is, the derivative of $v(t)$ should be proportional to $i(t)$,

$$v'(t) \propto i(t). \tag{1.11}$$

While the system of proportionalities (1.8) - (1.11) could now be combined in a mathematical model, a more appropriate system of equations results from an additional consideration. The changes are to be concerned with $i(t)$, since the most modern economic theories are not so directly tied to the interest rate "per se". There is certainly a question of the entrepreneur's ability to recognize the interest rate trend, and thus, use deviations from this trend in any decision process. Perhaps the best approximation to the

business community's behavior is to assume that information concerning the interest rate is actually gained through a study of the existing market investment structure. That is, increases in capital-goods investment are tied more directly to decreases in consumer-goods investment than to any interest rate deviations. This line of reasoning leads to a change in proportionalities (1.9) and (1.11) to

$$x(t) \propto - v(t) \tag{1.12}$$

and

$$v'(t) \propto - x(t). \tag{1.13}$$

The government's credit expansion policies may now be denoted separately by $g(t)$ and considered as a system input,

$$x(t) \propto g(t) \tag{1.14}$$

Combining (1.8), (1.10), (1.12), (1.13) and (1.14) with appropriate constants of proportionality, there results

$$x(t) = g(t) - \int_0^t \rho(t - \tau) \, x \, (\tau) d\tau - \alpha v(t) \tag{1.15}$$

$$v'(t) = -\gamma \, x(t) - \beta \, v(t), \quad v(0) = v_0. \tag{1.16}$$

These equations are structurally identical to (1.6) and (1.7), and the modelling process has indicated the reduction to a minimum of the order of the renewal equation portion of the mathematical model.

With the qualitative Austrian theory now quantitatively expressed, the questions become mathematical in nature. In Chapter II, equations (1.6) and (1.7) are imbedded in a more general structure and existing theorems utilized to provide conditions which insure the existence, uniqueness and transformability of solutions. Some discussion is given concerning the calculation of numerical results, with the differential character of (1.6) - (1.7) emphasized. Chapter III is devoted to a system stability analysis. With the determinental equation expressed as a finite exponential series, necessary

and sufficient conditions for all zeros to lie strictly in the left half plane are presented. The results are a generalization of those of Pontryagin [13] for exponential polynomials. Finally, an application is presented to illustrate the use of the stability conditions, and additional economic implications are mentioned.

II. EXISTENCE, UNIQUENESS AND
NUMERICAL RESULTS

This section will present conditions which will insure the existence, uniqueness and transformability of solutions. To utilize existing theorems, system (1.6) - (1.7) will be imbedded in a more general structure. Equation (1.7) may be integrated on both sides and, with the appropriate initial condition, becomes

$$\underline{v}(t) - \underline{v}_0 = \int_0^t \{ D \underline{x}(\tau) + E \underline{v}(\tau) \} \, d\tau. \tag{2.1}$$

Solving (2.1) for $\underline{v}(t)$ and substituting the result in (1.6) yields

$$\underline{x}(t) = \underline{g}(t) + C \underline{v}_0 + \int_0^t B(t - \tau) \underline{x}(\tau) \, d\tau + C \int_0^t \{ D \underline{x}(\tau) + E \underline{v}(\tau) \} \, d\tau$$

and thus

$$\begin{bmatrix} \underline{x}(t) \\ \underline{v}(t) \end{bmatrix} = \begin{bmatrix} \underline{g}(t) + C \underline{v}_0 \\ \underline{v}_0 \end{bmatrix} + \int_0^t \begin{bmatrix} B(t - \tau) + CD & CE \\ D & E \end{bmatrix} \begin{bmatrix} \underline{x}(\tau) \\ \underline{v}(\tau) \end{bmatrix} d\tau. \tag{2.2}$$

Hence, equations (1.6) - (1.7) may be expressed in the vector form

$$\underline{z}(t) = \underline{h}(t) + \int_0^t A(t - \tau) \underline{z}(\tau) \, d\tau. \tag{2.3}$$

It is of interest to note that the initial condition, $\underline{v}_0$, appears in (2.1) in a manner analogous to $\underline{g}(t)$ in (1.6). Although $\underline{g}(t)$ is just the input, $\underline{g}(0) + C\underline{v}_0$ is also the initial value of $\underline{x}(t)$. If it is desired to arbitrarily set an initial condition for $\underline{x}(t)$, $\underline{g}(t)$ must be defined appropriately for $t = 0$ or $\underline{x}(0)$ will be multiply defined. Thus, the input set for (2.3) also defines the space of applicable initial conditions.

The proofs of the following theorems are extensions to the vector case of available proofs [14] for the scalar form of (2.3), and will not be presented here. Introduce the vector and matrix norms

$$\|\underline{z}\| = \sum_{i=1}^{n} |z_i|$$

and

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|.$$

Theorem 2.1. If

(i) $\|\underline{h}(t)\| \leq c$ for all $t \in [0, t_0]$ and

(ii) $\|A(t)\| \leq m$ for all $t \in [0, t_0]$,

then the solution to (2.3) exists and is unique

for $t \in [0, t_0]$, and is continuous if $\underline{h}(t)$ is continuous.

Theorem 2.2. Let $\underline{h}(t)$ be continuous. If for some

a there exist two positive constants c_1 and

c_2 such that

(i) $\|\underline{h}(t)\| \leq c_1 e^{at}$ for all $t > 0$, and

(ii) $\int_0^\infty e^{-at} \|A(t)\| \, dt = c_2 < 1$,

then

$$\|\underline{z}(t)\| \leq \frac{c_1 e^{at}}{1 - c_2}$$

for all $t \geq 0$, and the Laplace transform of $\underline{z}(t)$ exists.

The basic tool in the proof of the above theorems is the technique of successive approximations. This technique may also be used to find conditions which insure that $\underline{z}(t)$ is Lesbeque integrable, Riemann integrable or of bounded variation ([14], pp. 217-29).

The foregoing imbedding has emphasized the integral characteristics of equations (1.6) and (1.7). Their differential character is perhaps best illustrated by noting the simplifications in numerical solution techniques which the differential character of (1.7) makes possible. Figure 1 illustrates

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THE NUMERICAL SOLUTION

Solution desired for $0 \leq t \leq T$. Choose Δt for

rectangular integration accuracy; $N = T/\Delta t$.

Read v_0, α, β and γ. Store

$g(t_i)$ and $\rho(t_i)$, $i = 0, 1, \ldots, N$.

Set DEQB initial and maximum step

size and error specification; $i = 0$.

Enter DEQB to extend solution

from t_i to t_{i+1}

Intermediate exit; $t = t_0$, $t_i < t_0 < t_{i+1}$.

Initial guess available for $v(t_0)$;

$v'(t_0)$ required.

Final exit; $t = t_{i+1}$;

$v(t_{i+1})$ and $x(t_{i+1})$ output;

$x(t_{i+1})$ stored; $i = i + 1$.

$g(t_0) = g(t_i) + [g(t_{i+1}) - g(t_i)][t_0 - t_i] / \Delta t$;   Integral

$= \Delta t \sum_{j=0}^{i} x(t_j) \, \rho(t_{i-j+1}) + \Delta t \rho_0 \{ x(t_i) + [x(t_i) - x(t_{i-1})][t_0 - t_i] / \Delta t \}$;
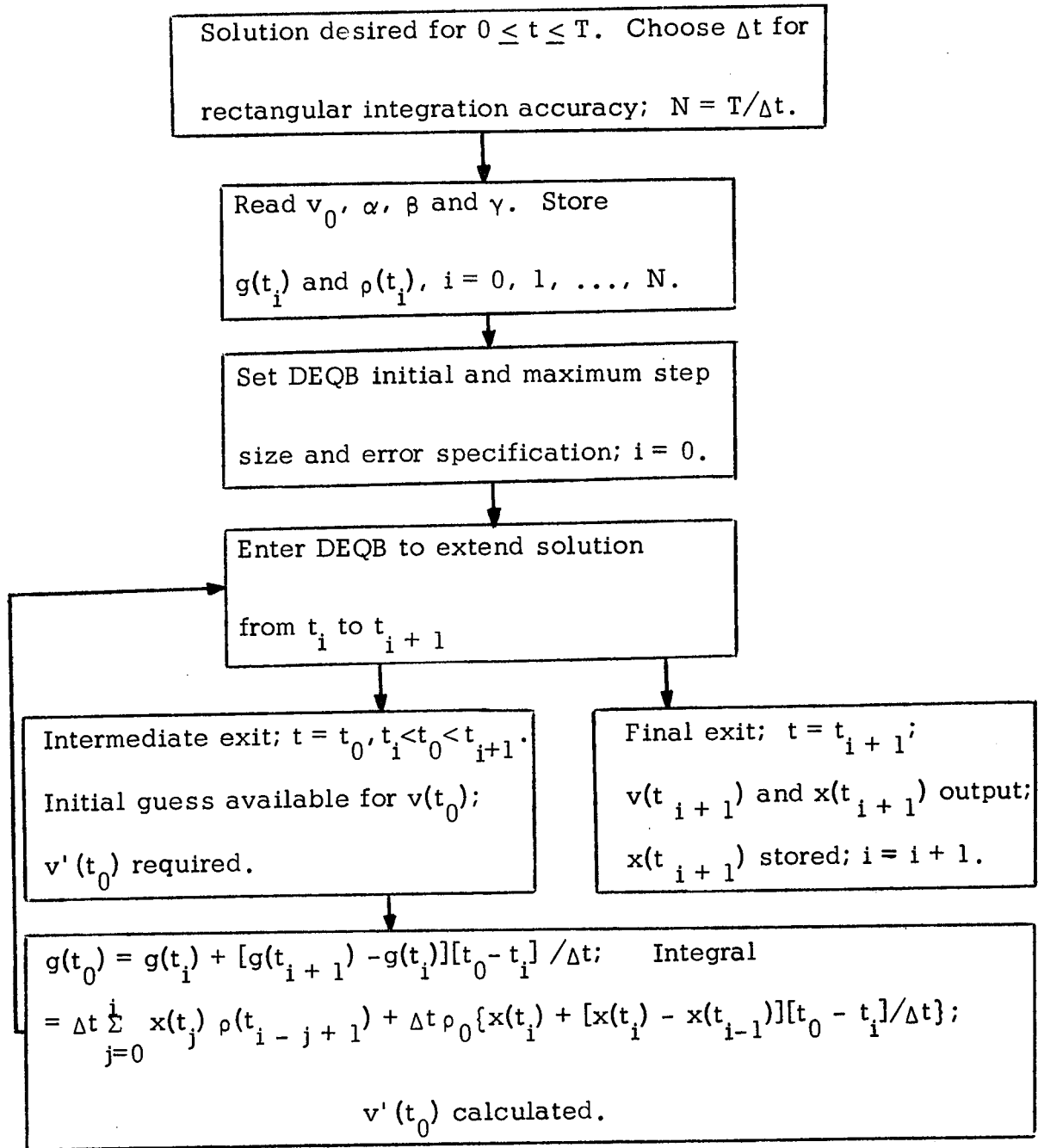
$v'(t_0)$ calculated.

Figure 1

the utilization made of a sophisticated, fast differential equation solving subroutine [15] in solving (1.15) and (1.16) for particular values of $g(t)$, $\rho(t)$, α, β, γ and v_0. For a solution desired for $0 \leq t \leq T$, an N is chosen such that $t_0 = 0$, $t_n = T$ and Δt, defined by $\Delta t = t_{i+1} - t_i$, $i < N$, is sufficiently small to allow good approximations to (1.15) by a rectangular integration process with rectangles of width Δt. The input and influence function values are stored for appropriate values of i, and an error specification set for the subroutine (DEQB). DEQB is then entered with the initial value v_0 and with $i = 0$, and instructed to find $v(t_{i+1})$ under the constraints of (1.16). At various times t_0, $t_i < t_0 < t_{i+1}$, control is returned to the main program to evalutate $v'(t_0)$. The input, $g(t_0)$, is found by linear interpolation and the convolution integral approximated by

$$\Delta t \sum_{j=0}^{i} x(t_j)\, \rho(t_{i-j+1}) + \Delta t \rho_0 (x(t_i) + (x(t_i) - x(t_{i-1}))(t_0 - t_i)/\Delta t).$$

The subroutine itself provides an initial guess for $v(t_0)$, so $v'(t_0)$ is now calculated and control returned to DEQB. As the exit is made from DEQB at $t_0 = t_{i+1}$, $v(t_{i+1})$ and $x(t_{i+1})$ are available for output, $x(t_{i+1})$ is stored, i is incremented and control is returned to the subroutine for the next iteration.

While memory storage needs are large for evaluation of the convolution integral, additional differential equations could be added to (1.15) and (1.16) with only linearly increasing requirements. The subroutine provides automatic step size correction to minimally meet specified error contraints, which leaves only the evaluation of the integral for the programmer. While only a crude rectangular approximation was used for demonstration purposes, even rather critical constant solutions were evaluated within acceptable limits. These and other solutions will be presented later as examples.

III. STABILITY CONDITIONS

With conditions now having been given insuring the existence, uniqueness and exponential boundedness of solutions, the main questions remaining concern system stability. The results to be presented are quite general, applying to (1.6) - (1.7), but algebraic considerations will require use of equations (1.15) and (1.16) in illustrative examples.

If $g(t)$ and $\rho(t)$ in (1.15) - (1.16) satisfy the conditions which allow application of the Laplace Transform, and if $v_0 = 0$, then

$$\begin{bmatrix} X(z) \\ V(z) \end{bmatrix} = \begin{bmatrix} 1 + P(z) & \alpha \\ \gamma & z + \beta \end{bmatrix}^{-1} \begin{bmatrix} G(z) \\ 0 \end{bmatrix} ,$$

and the system stability is clearly a function of the location of the zeros of the determinantal equation.

$$D(z) = z + z\,P(z) + \beta\,P(z) + (\beta - \alpha\,\gamma). \tag{3.1}$$

Since the difficulty in finding the zero locations of (3.1) depends on the form of $P(z)$, a particular $\rho(t)$,

$$\rho(t) = \begin{cases} \sum_{j=0}^{m} \rho_j t^j , & 0 \le t \le T \\ 0 , & T < t \end{cases} \tag{3.2}$$

will be considered. This form is chosen for two reasons. First, the hereditary character of the system can certainly be considered to extend back in time only some finite length. Second, by the Stone-Weierstrass theorem [16], any function continuous on any closed and bounded interval [a, b] may be approximated uniformly there by a polynomial. Thus,

$$P(z) = \sum_{j=0}^{m} \int_{0}^{T} \rho_i t^i e^{-zt}\,dt$$

$$= -\frac{\rho_0 e^{-zT}}{z} + \frac{\rho_0}{z} - \rho_1 \{ \frac{T\ e^{-zT}}{z} + \frac{e^{-zT}}{z^2} \} + \frac{\rho_1}{z^2}$$

$$- \ldots - \rho_m e^{-zT} \sum_{i=0}^{m} \frac{i!\,T^{m-1}}{z^{i+1}} + \frac{\rho_m m!}{z^{m+1}}$$

$$= \sum_{j=0}^{m} \{ -\rho_j e^{-zT} \sum_{i=0}^{j} \frac{i!\,T^{j-i}}{z^{i+1}} + \frac{\rho_j j!}{z^{j+1}} \}. \tag{3.3}$$

Then

$$D(z) = z + (z + \beta) \{ \sum_{j=0}^{m} [-\rho_j e^{-zT} \sum_{i=0}^{j} \frac{i!\,T^{j-i}}{z^{i+1}} + \frac{\rho_j j!}{z^{j+1}}] \} + (\beta - \alpha\gamma). \tag{3.4}$$

Since the zeros of $D(z)$ are the quantities of interest, define $\bar{D}(z) = e^{zT} D(z)$ and the zeros of $\bar{D}(z)$ are just those of $D(z)$. Hence,

$$\bar{D}(z) = \{ z + (\beta - \alpha\gamma) + (z + \beta) \sum_{j=0}^{m} \frac{\rho_j j!}{z^{j+1}} \} e^{zT}$$

$$- (z + \beta) \{ \sum_{j=0}^{m} \rho_j [\sum_{i=0}^{j} \frac{i!\,T^{j-i}}{z^{i+1}}] \}. \tag{3.5}$$

It is easily seen that $\bar{D}(z)$ may, in general, have an infinite number of zeros. This causes two serious obstacles to arise in stability studies. First, of course, is the difficulty in locating "all" of the zeros. Then, even if it is known that all the zeros have strictly negative real parts, they may approach the imaginary axis asymptotically, giving rise to undamped solutions. Compared to the first problem, this latter one might be considered somewhat pathological, although it will not be neglected in the ensuing work.

Perhaps the most useful results available in the area of zero location for transcendental functions are those due to Pontryagin [13]. Basically, he gives necessary and sufficient conditions for an exponential polynomial to have all of its zeros strictly in the left half plane. Considering (3.5), it is seen that $\bar{D}(z)$ is not an exponential polynomial, and the obvious procedure would be to examine the zeros of $z^{m+1} \bar{D}(z)$. This approach would certainly be valid if the zeros of $\bar{D}(z)$ and $z^{m+1}\bar{D}(z)$ were coincident; that is, if $\bar{D}(z)$ had an $(m+1)^{st}$ order pole at the origin. A simple counterexample shows, however, that this supposition is not always true. Consider (3.5) for a $\rho(t)$ of the form $\rho_n t^n$ for $0 \le t \le T$ and zero elsewhere. Then an evaluation at the origin yields

$$
\bar{D}(z)\Big|_{z=0} = \lim_{z \to 0} \left\{ \frac{[z^{n+2} + (\beta - \alpha\gamma) z^{n+1} + n!\rho_n z + n!\beta\rho_n] e^{zT}}{z^{n+1}} \right.
$$

$$
\left. - \frac{(\beta\rho_n + z\rho_n)(\sum\limits_{i=0}^{n} i!\, T^{n-i} z^{n-i})}{z^{n+1}} \right\}
$$

which is indeterminate of the form 0/0. Applying L'Hospital's Rule,

$$
\bar{D}(z)\Big|_{z=0} = \lim_{z \to 0} \left\{ T\,[z^{n+2} + (\beta - \alpha\gamma) z^{n+1} + n!\rho_n z + n!\beta\rho_n] e^{zT} \right.
$$

$$
+ [(n+2) z^{n+1} + (n+1)(\beta - \alpha\gamma) z^n + n!\,\rho_n] e^{zT}
$$

$$
- (\beta\rho_n + z\rho_n) \sum\limits_{i=0}^{n-1} (n-i)i!\, T^{n-i} z^{n-i-1}
$$

$$
\left. - \rho_n \sum\limits_{i=0}^{n} i!\, T^{n-i} z^{n-i} \right\} / \lim_{z \to 0} \{(n+1) z^n\}
$$

$$
= \lim_{z \to 0} \left\{ \frac{[n! - (n-1)!]\, T \beta\rho_n}{(n+1) z^n} \right\}, \quad n \ne 0,
$$

and $\bar{D}(z)$ is seen to have at most an nth order pole at the origin. Thus,

$z^{n+1} \bar{D}(z)$ will have at least a first order zero at the origin that $\bar{D}(z)$ does not have. Unfortunately, use of Pontryagin's results will only imply that under no conditions can $z^{n+1} \bar{D}(z)$ possess only left half plane zeros, and there is no way provided to treat the extraneous zero separately.

Since exponential polynomials have been shown to be of insufficient generality for the study at hand, consideration will now be given to the zero location problem of the next more general category of transcendental functions, the finite exponential series. That is, transcendental functions of the form

$$H(z) = \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} a_{mn} z^m e^{nz} \qquad (3.6)$$

will be considered. This set of functions clearly includes the determinental equation of (1.6) - (1.7) for influence functions of the form of (3.2), and in fact is of sufficient generality to allow influence functions sectionally describable as (3.2) and hence only piecewise continuous. Such time sections, however, must be rationally related, since the restriction of n in (3.6) to integer values will require all time intervals to be integer multiples of some base interval. Finite exponential series, as defined by (3.6), are closely related to exponential polynomials, and the following results will be a generalization of Pontryagin's basic work. The proofs of some of his theorems will be utilized directly, others changed and still others corrected.

Two basic lemmas which will be used later will now be proven.

Lemma 3.1. If $v^{(s)}(t)$ is any non-zero polynomial in t of degree s, then there exists a real number ϵ such that $v^{(s)}(\exp(x + i(\epsilon + 2n\pi)))$ is non-zero for all real x and all integer n.

Proof. Since $v^{(s)}(\exp z)$ is periodic with period $2\pi i$, the result will hold for all integer n if it holds for n = 0. There clearly exist only s complex numbers t_0 which satisfy

$$v^{(s)}(t_0) = 0.$$

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Then $\nu^{(s)}(\exp z_0) = 0$ if and only if

$\exp(z_0) = t_0$. For y $(z = x + iy)$ in the interval

$y_0 \le y \le 2\pi + y_0$, any y_0, there are at most two such

z_0's for each t_0. Hence, $\nu^{(s)}(\exp z)$ has only 2s

zeros in any semi-infinite horizontal strip, of width

2π, in the complex plane. Thus, there exists an

infinity of numbers ϵ, and in particular one, such that

$\nu^{(s)}(\exp(x + i\epsilon)) \neq 0$ for all real x.   Q.E.D.

Lemma 3.2.   If $\delta(z)$ is any polynomial in cos z and sin z, of

degree s, not identically zero, then there exists

an ϵ such that $\delta(\epsilon + 2n\pi + iy)$ is non-zero for all

real y and integer n.

Proof. $\delta(z)$ is clearly periodic with period 2π, and if the

result holds for n = 0, it therefore holds for all n.

Using a common variable substitution, let

$$u = (t + 1/t)/2, \quad v = (t - 1/t)/2i \qquad (3.7)$$

and for $t = \exp(iz)$, $u = \cos z$ and $v = \sin z$.

Then substituting (3.7) for u and v in $\delta(u,v)$, the

resultant $\delta(t)$ clearly has 2s zeros. Every zero, z_0,

of $\delta(z)$ must thus satisfy $\exp(iz_0) = t_0$, where t_0 is a

zero of $\delta(t)$. Hence, for x in the interval $x_0 \le x \le x_0 + 2\pi$,

$\delta(z)$ has at most 4s zeros, and there exists an infinite

set of numbers ϵ such that $\delta(\epsilon + iy) \neq 0$ for all y.   Q.E.D.

Now define h(z, t), a finite series in the two variables z and t, by

$$h(z,t) \stackrel{\Delta}{=} \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} a_{mn} z^m t^n.$$

The finite exponential series, H(z), may thus be defined

$$H(z) \stackrel{\Delta}{=} h(z, \exp(z)). \qquad (3.8)$$

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From the discussions of the previous section, it will be assumed without loss of generality that $H(z)$ possesses no pole at the origin. It must also be assumed throughout the remaining discussions that $H(z)$ is not degenerate; i.e., not only the coefficients a_m are non-zero and there exists an $m_o > 0$ such that $a_{m_o n} \neq 0$ for some $n \neq 0$.

If there exists a non-zero coefficient a_{rs} such that for all other non-zero a_{mn}, $r > m$ and $s \geq n$ or $r \geq m$ and $s > n$, then $a_{rs} z^r t^s$ will be called the principal term of the finite series. This definition leads to the following instability result.

Theorem 3.1.　If $h(z,t)$ has no principal term, then $H(z)$ has an infinite number of zeros with arbitrarily large positive real parts.

Proof.　　　Write

$$z^q h(z,t) = \sum_{m=q}^{p_1} \sum_{n=0}^{p_2} a_{mn} z^{m+q} t^n;$$

$z^q H(z)$ is thus an exponential polynomial. If $h(z,t)$ has no principal term, then neither does $z^q h(z,t)$. By Pontryagin's first theorem, $z^q H(z)$ thus has an infinite number of zeros with arbitrarily large positive real parts, and $H(z)$ certainly possesses the same property.　　　　　　　　　　　　　Q.E.D.

If $h(z,t)$ does possess a principal term, a consideration of the behavior of $H(z)$ along the imaginary axis will yield information concerning the location of the zeros. For z imaginary, $H(z)$ may be separated into its real and imaginary parts by

$$H(iy) = \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} a_{mn} i^m y^m \exp(iny) \tag{3.9}$$

$$= \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} a_{mn} i^m y^m (\cos y + i \sin y)^n. \tag{3.10}$$

Then H(iy) may be written

$$H(iy) = \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} y^m [\varphi_m^{(n)} (\cos y, \sin y) + i\psi_m^{(n)} (\cos y, \sin y)] \tag{3.11}$$

where $\varphi_m^{(n)}(u,v)$ and $\psi_m^{(n)}(u,v)$ are polynomials of degree n, homogeneous in u and v. F(y) and G(y) are now defined by

$$F(y) \overset{\Delta}{=} \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} y^m \varphi_m^{(n)} (\cos y, \sin y), \tag{3.12}$$

$$G(y) \overset{\Delta}{=} \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} y^m \psi_m^{(n)} (\cos y, \sin y), \tag{3.13}$$

and hence

$$H(iy) = F(y) + i G(y). \tag{3.14}$$

Examining the structure of F(y) and G(y), certain observations can be made which will be useful later. F(y) and G(y) will clearly have no pole at the origin. Suppose a function p(z, u, v) is defined by

$$p(z, u, v) \overset{\Delta}{=} \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} z^m \gamma_m^{(n)} (u, v) \tag{3.15}$$

where $\gamma_m^{(n)}(u,v)$ is a polynomial of degree n, homogeneous in u and v, and $\gamma_m^{(n)}(\cos y, \sin y)$ is formed as $\varphi_m^{(n)}(\cos y, \sin y)$ and $\psi_m^{(n)}(\cos y, \sin y)$ in (3.13). Then F(y) and G(y) may be expressed as f(y, cos y, sin y) and g(y, cos y, sin y) where f(z, u, v) and g(z, u, v) are of the form of p(z, u, v). Suppose further that a principal term for p(z, u, v) is defined as expected. Then if h(z, t) has a principal term $a_{rs} z^r t^s$, f(z, u, v) and g(z, u, v) will have principal terms $z^r \varphi_r^{(s)}(u, v)$ and $z^r \psi_r^{(s)}(u, v)$ and conversely.

Suppose p(z, u, v) has a principal term $z^r \gamma_r^{(s)}(u, v)$. Then define

$$\gamma_*^{(s)}(u, v) \overset{\Delta}{=} \sum_{n=0}^{s} \gamma_r^{(n)}(u, v) \tag{3.16}$$

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and

$$\bar{\gamma}_*^{(s)}(z) \overset{\Delta}{=} \gamma_*^{(s)}(\cos z, \sin z).$$ (3.17)

Such definitions lead to the following fundamental theorem.

Theorem 3.2. Let $p(z, u, v)$ have a principal term $z^r \gamma_r^{(s)}(u, v)$, and choose an ϵ such that $\bar{\gamma}_*^{(s)}(\epsilon + zn\pi + iy)$ is never zero for any real y. Define $P(z) \overset{\Delta}{=} p(z, \cos z, \sin z)$ and assume that $P(z)$ has no pole at the origin. Then for k sufficiently large, $P(z)$ will have exactly $4sk + r$ zeros in the strip $-2k\pi + \epsilon \leq x \leq 2k\pi + \epsilon$.

Proof. If $\bar{\gamma}_*^{(s)}(z)$ can be shown to be not identically zero for all z, Lemma 2 will imply the existence of an appropriate ϵ. Since $\bar{\gamma}_*^{(s)}(z)$ merely represents either $\bar{\varphi}_*^{(s)}(z)$ or $\bar{\psi}_*^{(s)}(z)$, it may be shown that neither of these functions is zero for all z. Recalling (3.9), it is clear that an alternate form for (3.10) may be given by

$$H(iy) = \sum_{m=-q}^{p_1} \sum_{n=0}^{p_2} a_{mn} i^m y^m (\cos(ny) + i\sin(ny)).$$ (3.18)

This implies the following:

if $m = \ldots, -4, 0, 4, \ldots$ then

$$\varphi_m^{(n)}(\cos y, \sin y) = a_{mn} \cos(ny)$$

and

$$\psi_m^{(n)}(\cos y, \sin y) = a_{mn} \sin(ny);$$

if $m = \ldots, -1, 3, 7, \ldots$ then

$$\varphi_m^{(n)}(\cos y, \sin y) = a_{mn} \sin(ny)$$

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and

$$\psi_m^{(n)}(\cos y, \sin y) = -a_{mn}\cos(ny);$$

if $m = \ldots, -2, 2, 6, \ldots$ then

$$\varphi_m^{(n)}(\cos y, \sin y) = -a_{mn}\cos(ny)$$

and

$$\psi_m^{(n)}(\cos y, \sin y) = -a_{mn}\sin(ny);$$

if $m = \ldots, -3, 1, 5, \ldots$ then

$$\varphi_m^{(n)}(\cos y, \sin y) = -a_{mn}\sin(ny)$$

and

$$\psi_m^{(n)}(\cos y, \sin y) = a_{mn}\cos(ny).$$

Suppose the principal term of $H(iy)$ is $a_{rs}i^r y^r e^{isy}$, and for definiteness suppose $r = 4k$, some k. Then

$$\varphi_*^{(s)}(z) = a_{r0} + a_{r1}\cos z + \ldots + a_{rs}\cos(sz) \qquad (3.19)$$

For $\overline{\varphi}_*^{(s)}(z)$ to be zero for all z, it clearly must be zero for $z = ix$. Equation (3.19) then becomes

$$\overline{\varphi}_*^{(s)}(z) = a_{r0} + a_{r1}\cosh x + \ldots + a_{rs}\cosh(sx). \qquad (3.20)$$

But since $\cosh x$ is monotonically increasing for $x \geq 0$ and $\cosh n_1 x > \cosh n_2 x$ for $n_1 > n_2$ and $x > 0$, (3.20) clearly cannot be identically zero. Similar arguments hold for $\overline{\psi}_*^{(s)}(z)$ and for $r = 4k+1$, $4k+2$ and $4k+3$, any k. With the existence of an appropriate ϵ now shown, and proceeding in a fashion similar to that in the proof of Theorem 3.1, write

$$z^q p(z, u, v) = \sum_{m=-q}^{p_1}\sum_{n=0}^{p_2} z^{m+q}\gamma_m^{(n)}(u, v).$$

Then since $p(z, u, v)$ has a principal term $z^r \gamma_r^{(s)}(u, v)$, $z^q p(z, u, v)$ will have a principal term $z^{r+q}\gamma_r^{(s)}(u, v)$.

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Pontryagin's third theorem may then be used to imply that $z^q P(z)$ has, for sufficiently large k, $4sk+r+q$ zeros in the strip $-2k\pi+\epsilon \leq x \leq 2k\pi+\epsilon$. Since $P(z)$ has no pole at the origin, it thus has $4sk+r$ zeros in the same strip, again for sufficiently large k.

<div align="right">Q.E.D.</div>

A similar extension of Pontryagin's second theorem leads to the following result.

Theorem 3.3.   If $p(z,u,v)$ has no principal term, then $P(z)$ has an infinite number of non-real zeros.

Suppose now that $h(z,t)$ has a principal term $a_{rs} z^r t^s$. Denote by $\eta_*^{(s)}(t)$ the polynomial coefficient of z^r. Then $h(z,t)$ may be written

$$h(z,t) = z^r \eta_*^{(s)}(t) + \sum_{m=-q}^{r-1} \sum_{n=0}^{s} a_{mn} z^m t^n. \qquad (3.21)$$

Next, denote by N_k the number of zeros of $H(z)$ in the semi-infinite strip defined by $x \geq 0$, $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$. Denote by $\omega^*(a,b)$ the overall angular variation of the vector $\omega = H(iy)$ about the origin as y ranges from a to b.

Theorem 3.4.   Let $h(z,t)$ have a principal term $a_{rs} s^r t^s$ and choose an ϵ such that $\eta_*^{(s)}(\exp(x + i(2n\pi + \epsilon)))$ is non-zero for all positive, real x and integer n. Assume that $H(z)$ has no zeros on the imaginary axis and no pole at the origin. Then

$$\omega^*(-2k\pi + \epsilon, 2k\pi + \epsilon) = 2\pi(2sk - N_k + r/2) + \delta_k \qquad (3.22)$$

where $\delta_k \to 0$ as $k \to \infty$.

Proof.   By the fundamental theorem of algebra, $\eta_*^{(s)}(t)$ can be identically zero only if it is the zero polynomial. But this would imply that $a_{rs} = 0$, a contradiction by

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definition of a principal term. By Lemma 3.1 then, an appropriate ϵ may be chosen. Since for z large the leading term of H(z) is dominant, and H(z) has been previously assumed to be non-degenerate, H(z) can be written

$$H(z) = z^r \eta_*^{(s)}(\exp(z)) \; [1 + \delta_1] \qquad (3.23)$$

where $\delta_1 \to 0$ as $|z| \to \infty$. By Lemma 3.1, there are clearly no zeros of $z^r \eta_*^{(s)}(\exp(z))$ along any horizontal line $y = \pm 2k\pi + \epsilon$, k an integer, and the proof of Lemma 1 implies that there will exist a positive real number a such that $z^r \eta_*^{(s)}(\exp(z))$ will have no zero on or to the right of the vertical line $x = a$. Consider the rectangle R_{ka} defined by $0 \le x \le a$, $-2k\pi + \epsilon \le y \le 2k\pi + \epsilon$. Then the angular variation of ω, as z tranverses the upper, lower and right sides of R_{ka}, will differ from that of $\bar{\omega} = z^r \eta_*^{(s)}(\exp(z))$ only by a number δ_2 where $\delta_2 \to 0$ as $k \to \infty$. Since the boundary of R_{ka} has been shown to be free of poles and zeros, Cauchy's index theorem easily implies the desired result for $\bar{\omega}$, and hence for ω^*.   Q.E.D.

The following lemmas will complete the work preparatory to the final results.

Lemma 3.3.   Let h(z, t) have a principal term. Then

$$\omega^*(a + \epsilon, \, b + \epsilon) = \omega^*(a, \, b) + \delta_3 \qquad (3.24)$$

where $\delta_3 \to 0$ as $a \to \pm \infty$ and $b \to \pm \infty$.

Proof.   From the definition of $\omega^*(a, b)$, it is clear that $\omega^*(a, b) = \omega^*(a, c) + \omega^*(c, b)$ for any c. It is also clear from (3.23) that for $|a|$ large, $\omega^*(a, a + \epsilon)$ differs little from the angular variation due to $y^r \eta_*^{(s)}(\exp(y))$. But this variation is just $\sum\limits_{n=0}^{s} n\epsilon$. Thus

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$$\omega^*(a, a + \epsilon) = \sum_{n=0}^{s} n\epsilon + \delta_4$$

where $\delta_4 \to 0$ as $a \to \pm\infty$. Similarly,

$$\omega^*(b, b + \epsilon) = \sum_{n=0}^{s} n\epsilon + \delta_5$$

where $\delta_5 \to 0$ as $b \to \pm\infty$. Then

$$\omega^*(a + \epsilon, b + \epsilon) = \omega^*(a, b) - \omega^*(a, a+\epsilon) + \omega^*(b, b+\epsilon)$$

which implies

$$\omega^*(a + \epsilon, b + \epsilon) = \omega^*(a, b) + \delta_3$$

where $\delta_3 \to 0$ as $a \to \pm\infty$ and $b \to \pm\infty$.   Q.E.D.

Lemma 3.4. Let $h(z, t)$ have a principal term $a_{rs} z^r t^s$, and $f(z, u, v)$ and $g(z, u, v)$ have principal terms $z^r \varphi_r^{(s)}(u, v)$ and $z^r \psi_r^{(s)}(u, v)$ respectively. Then for λ and μ arbitrary real numbers not both zero, there exists a number ϵ such that $\lambda \overline{\varphi}_*^{(s)}(\epsilon + iy) + \mu \overline{\psi}_*^{(s)}(\epsilon + iy) \neq 0$ for all real y.

Proof. By definition, $\lambda \overline{\varphi}_*^{(s)}(z) + \mu \overline{\psi}_*^{(s)}(z)$ is a polynomial in $\cos z$ and $\sin z$ and if not identically zero for all z, Lemma 2 will imply the desired result immediately, If either $\overline{\varphi}_*^{(s)}(z)$ or $\overline{\psi}_*^{(s)}(z)$ were zero for all z, an appropriate choice of λ and μ could be made such that the polynomial would be always zero. However, the proof of Theorem 3.2 clearly shows that neither $\overline{\varphi}_*^{(s)}(z)$ nor $\overline{\psi}_*^{(s)}(z)$ can be zero for all z. The remaining possibility is for $\overline{\varphi}_*^{(s)}(z)$ and $\overline{\psi}_*^{(s)}(z)$ to satisfy

$$\overline{\varphi}_*^{(s)}(z) = -\mu/\lambda \, \overline{\psi}_*^{(s)}(z) \tag{3.25}$$

for some λ and μ, neither zero, and all z. This could obviously occur if $\varphi_r^{(n)}(\cos z, \sin z)$ and $\psi_r^{(n)}(\cos z, \sin z)$

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were non-zero only for $n = 0$, and this would imply a principal term for $H(z)$ of the form $a_{r0}z^r$. But this can occur only if $H(z)$ is degenerate, a contradiction to a fundamental hypothesis. With s now shown to be non-zero, the proof of Theorem 3.2 is again examined, and it is noted that (3.18), (3.19) and (3.20) imply that for (3.25) to hold, it is necessary that

$$\varphi_r^{(n)}(\cos z, \sin z) = -\mu/\lambda \; \psi_r^{(n)}(\cos z, \sin z) \qquad (3.26)$$

for all $n \leq s$. But this implies that

$$\text{Re} \{ a_{rn}i^r(\cos(n z) + i \sin (s z)) \}$$
$$= -\mu/\lambda \; \text{Im} \{a_{rn}i^r(\cos(n z) + i\sin(n z)) \} \qquad (3.27)$$

for all $n \leq s$. For any value of r, however, (3.27) can be shown to imply that $a_{rs} = 0$, a final contradiction. Q.E.D.

Lemma 3.5. Let $h(z, t)$ have a principal term and write $H(iy) = F(y) + i\,G(y)$. Suppose $\Delta \omega^*(-2k\pi, 2k\pi) = 4sk\pi + \pi r + \delta_6$ where $\Delta = \pm 1$ and $-\pi/2 \leq \delta_6 \leq \pi/2$. Then for λ and μ arbitrary real numbers not both zero, $\lambda F(y) + \mu G(y)$ has only real and simple zeros and

$$\Delta \left[G'(y) F(y) - G(y) F'(y) \right] > 0 \qquad (3.28$$

for all y.

Proof. Let λ and μ be given and consider the curve traced by the vector $\omega = H(iy)$ as it subtends an angle within $\pi/2$ radians of $4sk\pi + \pi r$ radians in the positive or negative direction as y varies from $-2k\pi$ to $2k\pi$. This curve intersects the line $\lambda\omega' + \mu\omega'' = 0$ $(\omega' + i\omega'' = \omega)$ for at least $4ks + r$ distinct values y, which implies that $\lambda F(y) + \mu G(y)$ has at least $4ks + r$ real zeros for $-2k\pi \leq y \leq 2k\pi$. By Lemma 3.3, the same result holds for $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$, any ϵ, if k is sufficiently large.

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By Lemma 3.4, choose an ϵ such that $\lambda \, \overline{\varphi}_*^{(s)}(\epsilon + iy)$ $+ \mu \, \overline{\psi}_*^{(s)}(\epsilon + iy)$ is non-zero for all real y. Then by Theorem 3.2, $\lambda F(y) + \mu G(y)$ will have, for sufficiently large k, no more than $4sk + r$ zeros for $-2k\pi + \epsilon \leq y \leq 2k\pi + \epsilon$, and hence all the zeros are real and simple. The simplicity of the zeros implies that the curve traced by ω always moves in a constant direction, and since the velocity of rotation is given by

$$\frac{d\omega^*}{dy} = \frac{G'(y)\, F(y) - G(y)\, F'(y)}{F^2(y) + G^2(y)} \, ,$$

clearly

$$\Delta[\, G'(y)\, F(y) - G(y)\, F'(y)\,] > 0$$

for all y.

<div align="right">Q.E.D.</div>

The final results are now presented.

Theorem 3.5. Suppose all the zeros of H(z) are strictly in the left half plane, and write $H(iy) = F(y) + iG(y)$. Then the zeros of F(y) and G(y) are real, simple and alternate and $G'(y)\, F(y) - F'(y)\, G(y) > 0$ for all y.

Proof. The contrapositive of Theorem 3.1 implies that h(z, t) has a principal term. By Lemma 3.1, choose an ϵ such that $\eta_*^{(s)}(\exp(x + i\,(2n\pi + \epsilon)))$ is non-zero for all positive real x and integer n. Applying Theorem 3.4, and since $N_k = 0$,

$$\omega^*(-2k\pi + \epsilon, \; 2k\pi + \epsilon) = 4ks\pi + \pi r + \delta'_k$$

where $\delta'_k \to 0$ as $k \to \infty$. For $\lambda = 1$ and $\mu = 0$, Lemmas 3.3 and 3.5 imply that the zeros of F(y) are real and simple. For $\lambda = 0$ and $\mu = 1$, the zeros of G(y) are similarly real and simple. Lemma 3.5 also implies that

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$$F(y) \; G'(y) - F'(y) \; G(y) > 0$$

for all real y, which,in turn, implies that the zeros of
F(y) and G(y) alternate.

Q.E.D.

Theorem 3.6. Let H(z) be non-degenerate with no pole at the origin
and write $H(iy) = F(y) + i \, G(y)$. Then in order for all
of the zeros of H(z) to lie strictly in the left half plane,
each of the following conditions is both necessary and
sufficient:

(i) All of the zeros of F(y) and G(y) are real and alternate
and $F(y) \; G'(y) - F'(y) \; G(y) > 0$ for some y.

(ii) All of the zeros of F(y) are real and at each zero, y_0,
of F(y), $F'(y_0) \; G(y_0) < 0$.

(iii) All the zeros of G(y) are real and at each zero, y_0,
of G(y), $F(y_0) \; G'(y_0) > 0$.

Proof. Necessity.

The necessity of each condition is immediately obvious
from a consideration of Theorem 3.5.

Sufficiency.

(i). Since the zeros of F(y) and G(y) are real, the
contrapositive of Theorem 3.3 implies that f(z, u, v) and
g(z, u, v) both have principal terms, and hence so does
h(z, t). By Lemma 3.2, an ϵ exists completing the
hypothesis of Theorem 3.2 and F(y) and G(y) thus each
have 4sk + r real zeros for $-2k\pi + \epsilon \le y \le 2\pi + \epsilon$,
k sufficiently large. The rotation of H(iy) is thus at
least $\pm(rks\pi + \pi r) \pm \pi/2$ radians and hence

$$\Delta\omega^*(-2k\pi + \epsilon, \; 2k\pi + \epsilon) \ge 4ks\pi + \pi r - \pi/2.$$

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Then by Lemma 3.3,

$$\Delta \omega^*(-2k\pi, 2k\pi) \geq 4ks\pi + \pi r - \pi/2$$

for k sufficiently large, and the results of Lemma 3.5
obviously hold implying

$$\Delta [F(y) \ G'(y) - F'(y) \ G(y) \ ] > 0$$

for all y. Since $F(y) \ G'(y) - F'(y) \ G(y) > 0$ for some y
by hypothesis, $\Delta = +1$. Thus,

$$\omega^*(-2k\pi, 2k\pi) \geq 4sk\pi + \pi r - \pi/2 \qquad (3.29)$$

Now since the zeros of F(y) and G(y) alternate, they
are simple and H(z) has no imaginary zero. Use of
Lemma 3.1 completes the hypothesis of Theorem 3.4,
and

$$\omega^*(-2k\pi + \epsilon, 2k\pi + \epsilon) = 2\pi(2sk - N_k + r/2) + \delta_k$$

where $\delta_k \to 0$ as $k \to \infty$. By Lemma 3.3,

$$\omega^*(-2k\pi, 2k\pi) = 4sk\pi + \pi r - 2\pi N_k + \delta_k' \qquad (3.30)$$

where $\delta_k' \to 0$ as $k \to \infty$. Then for large k, (3.29) and
(3.30) imply that $N_k = 0$.   Q.E.D.

(ii) and (iii). Proof of the sufficiency of conditions (ii)
and (iii) follows that of condition (i) closely, with the
two inequalities insuring the absence of imaginary zeros
and setting the direction of rotation of the vector H(iy).

                                                    Q.E.D.

The results of Theorem 3.6 have left only two problems to be solved
before definitive statements can be made concerning the asymptotic stability
of any particular system. The first of these, illustrated in some detail in a
following chapter, is the analytical difficulty involved in parametrically
satisfying one of the three conditions. The second, mentioned earlier in this
work, concerns the insufficiency of left half plane zero locations in implying
global asymptotic stability. The following lemma will answer certain questions
concerning this problem.

Lemma 3.6. Let $P(z) = \mathcal{L}\{p(t)\}$ and $p(t) = \mathcal{L}^{-1}\{P(z)\}$ be a well-defined Laplace transform pair and suppose all poles of $P(z)$ lie to the left of the line $R_e\{z\} = -c$, where c is a strictly positive real number. Then

$$\lim_{t \to \infty} p(t) = 0.$$

Proof. By definition, $p(t)$ is given by

$$p(t) = \frac{1}{2\pi j} \lim_{y \to \infty} \int_{\sigma - jy}^{\sigma + jy} P(z) e^{zt} dz$$

where the path of integration is any line $z = \sigma$, a constant damping insuring the convergence of $P(z)$. Clearly $\sigma = -c/2$ will suffice, and thus

$$p(t) = \frac{1}{2\pi j} \lim_{y \to \infty} \int_{-\frac{c}{2} - jy}^{-\frac{c}{2} + jy} P(z) e^{zt} dz.$$

Since the integration is for $z = -c/2 + j\omega$, $-\infty < \omega < \infty$, write

$$p(t) = \frac{1}{2\pi j} \lim_{y \to \infty} \int_{-\frac{c}{2} - jy}^{-\frac{c}{2} + jy} P(-\frac{c}{2} + j\omega)\ e^{\frac{-ct}{2}} e^{j\omega t} d(j\omega)$$

and

$$|p(t)| \leq \frac{1}{2\pi} \lim_{y \to \infty} \int_{-\frac{c}{2} - jy}^{-\frac{c}{2} + jy} |P(-\frac{c}{2} + j\omega)| e^{\frac{-ct}{2}} d\omega.$$

But this implies

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$$|p(t)| \leq e^{\frac{-c}{2} t} \frac{1}{2\pi} \lim_{y \to \infty} \int_{\frac{-c}{2} - jy}^{\frac{-c}{2} + jy} |P(-\frac{c}{2} + j\omega)| \, d\omega,$$

and since the path of integration is free of singularities of P(z), the result immediately follows.   Q.E.D.

This lemma implies the following theorem.

Theorem 3.7.   Given the system of (1.6) – (1.7), suppose the deter-minantal equation is expressable as (3.6) with the usual aforementioned restrictions. Then the solutions of (1.6)–(1.7) will approach zero as $t \to \infty$ if there exists a real number $e > 0$ such that $H(z - e)$ has all its zeros strictly in the left half plane.

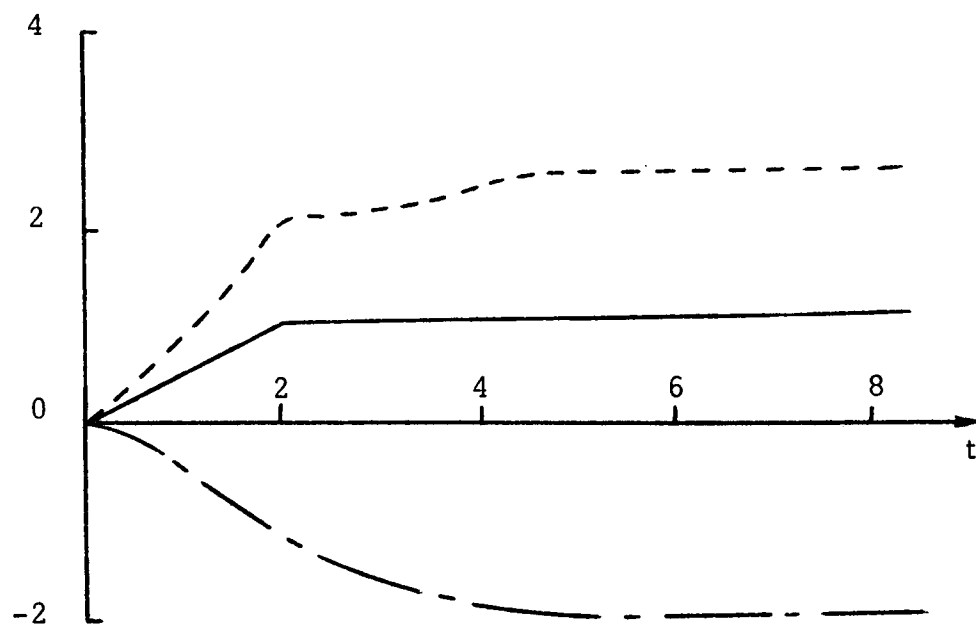# IV. CHARACTERISTICS OF MATHEMATICAL MODEL SOLUTIONS

Qualitative Characteristics

E. C. Bratt ([17], p. 204) in summarizing the contributions of the Austrian school, states that the "...monetary overinvestment theorists have shown little interest in statistical verification." Rothbard ([10], pp. 3-5), a major Austrian supporter, believes that "economic theories cannot be 'tested' by historical or statistical fact.... . Theory cannot emerge, Phoenix-like, from a cauldron of statistics; neither can statistics be used to test an economic theory." This dearth of factual numerical support for the Austrian theory makes difficult the testing of the system mathematical model, yet it can be shown that the model does admit to solutions which agree qualitatively with the Austrian conclusions. These conclusions may be summarized as follows:

(i)   The boom period could continue indefinitely if credit could be expanded indefinitely   ([8], p. 332).

(ii) The capital-goods industries are capable of fluctuating more widely than the consumer-goods industries ([10], pp. 16-17).

(iii) The capital-goods industries expand, in general, at the expense of expansion in the consumer-goods sector, leading to an out-of-phase cyclical behavior ([8], pp. 332-3).

Figure 2 illustrates a particular numerical example whose solution agrees with conclusion (i). Although this particular conclusion is often attacked as "...seriously in error" ([8], pp. 332-3), it is a basic tenet of the Austrian theory and any valid model must admit to this type solution. Solutions satisfying conclusion (ii) are easily obtained, merely increasing the damping effects on v(t) and sharply truncating the credit expansion input, as illustrated in Figure 3. Conclusion (iii) also is often the subject of criticism, since at first glance it seems to deny the possibility of depressed conditions existing in both industrial sectors. Figure 4, however, illustrates the proper implications of this conclusion, since the

$$x(t) = g(t) - \int_0^1 0.5 \; x(t-\tau)d\tau - 1.5 \; v(t)$$

———— g(t)

- - - - - x(t)

$$v'(t) = -2.0 \; v(t) - 1.5 \; x(t), \; v(0) = 0$$

—·— v(t)
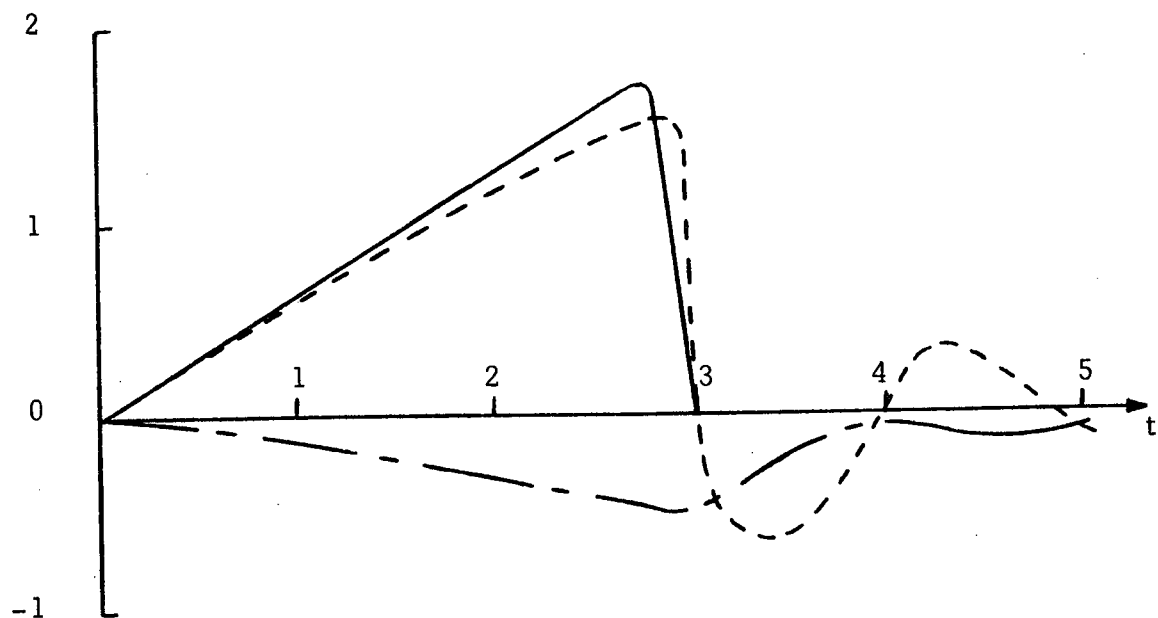
Figure 2

$$x(t) = g(t) - \int_0^1 0.5\, x(t - \tau)d\tau - 1.0\, v(t)$$

$$v'(t) = -3.0\, v(t) - 1.0\, x(t), \quad v(0) = 0$$

———— g(t)

- - - - x(t)

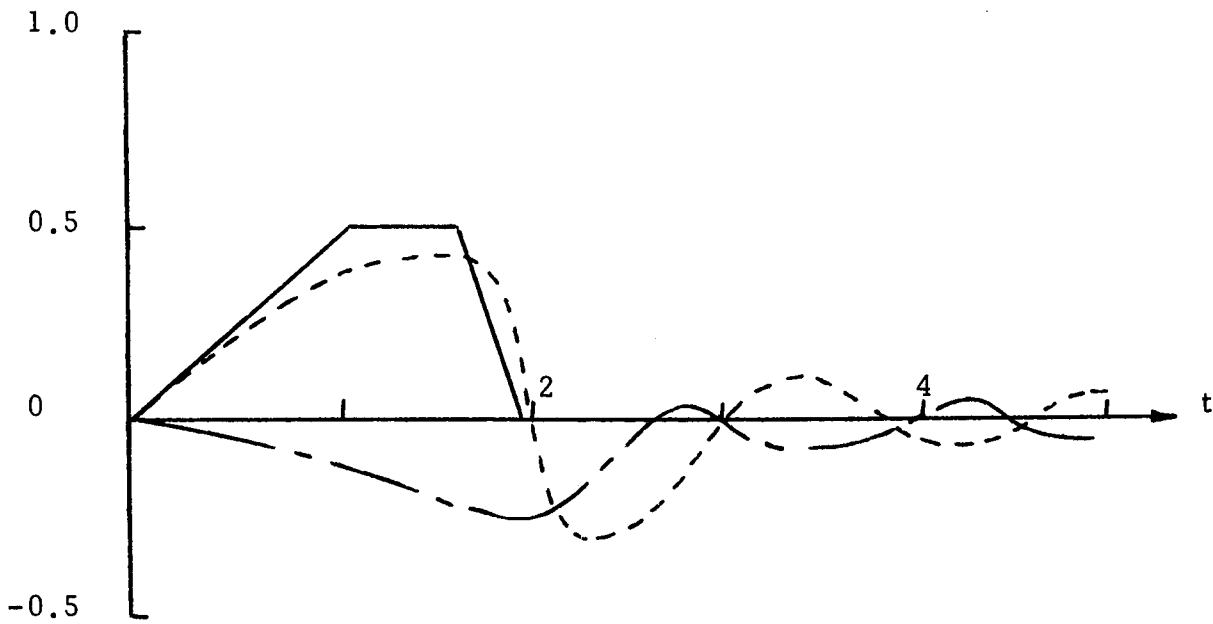—— - — v(t)

Figure 3

$$x(t) = g(t) - \int_0^t 0.5 \, x \, (t - \tau) d\tau - 0.3 \, v(t)$$

$$v'(t) = -2.0 \, v(t) - 0.5 \, x \, (t), \quad v(0) = 0$$

—————— g(t)

- - - - x(t)

—·— v(t)

Figure 4

capital-goods investment might become rapidly depressed before the consumer-goods investment could recover. This implies an overall depressed period followed by the out-of-phase cyclical behavior as the system returns to its normal (zero) solution.

One final solution type concerns the existence of non-zero constant trajectories under zero input conditions. Prior to the Great Depression, the economy was considered a dynamic, always changing phenomenon. A constantly depressed economy, for example, was considered an impossibility. The events of the thirties proved an embarrassing counterexample, as year after year idle men loafed next to idle machines. It was John Maynard Keynes [18] who explained this paradox, simply noting that a depressed economy could be in perfect economic balance even though in the depths of social agony. Thus, any business cycle theory must admit solutions which exhibit this economic balance/social imbalance characteristic. Figure 5 illustrates such a constant solution, with a continuing investment in the capital-goods industries above the norm and a corresponding continuing investment in consumer-goods industries below the norm. This, of course, corresponds to a continually inflated economy with no continuing credit expansion.

Quantitative Characteristics

As previously mentioned, the quantitative aspects of the Austrian theory are not presented by its proponents. While the required statistics are available for individual industries, the Austrian theorists have not attempted the dichotomization and compilation of such statistics necessary to support their conjectures. The following is an attempt to characterize the cyclical patterns of the two investment sectors (capital-goods and consumer goods) by two specific quantities, the value of building permits and department store sales, in a brief examination of the minor business cycle of 1921-1924.

The value of building permits issued was used to characterize the investment in capital-goods industries. Since the construction industry

A CONSTANT SOLUTION
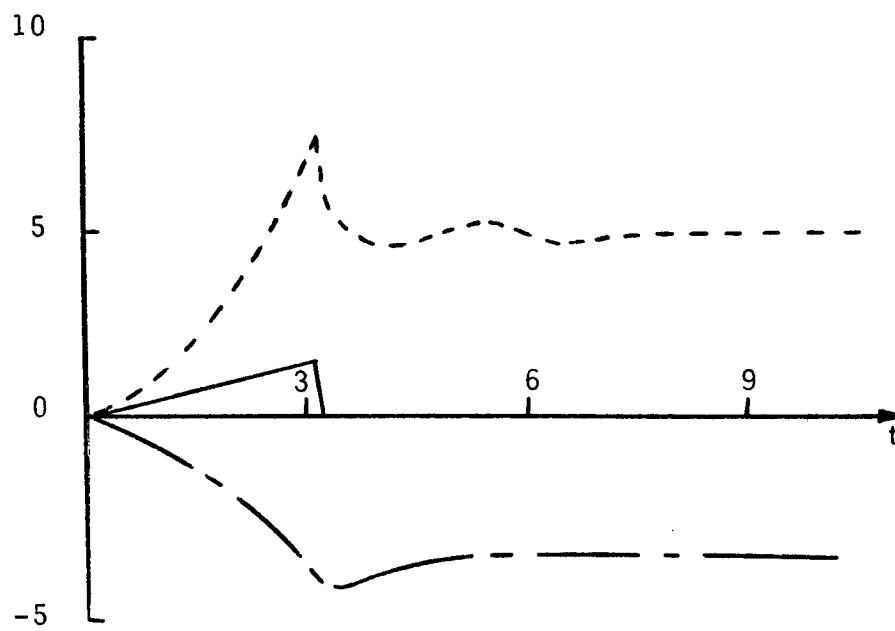
$$x(t) = g(t) - \int_0^1 0.5\, x(t - \tau)d\tau - 2.0\, v(t)$$

$$v'(t) = -2.0\, v(t) - 1.5\, x(t),\quad v(0) = 0$$

———	g(t)
- - - -	x(t)
—·—	v(t)

Figure 5

exhibits long delays and is historically hurt by tight-money situations, it should approximate to some degree the overall investment nature of the capital-goods industries. To approximate the consumer-goods sector, retail department store sales was chosen. As the department stores vary their stock levels to compensate for sales fluctuations, a rapid, short term investment process is carried out, typical of investment in consumer-goods industries.

The raw data for the value of building permits issued and department store sales (from [19], pp. 194-6) is presented in Figures 6 and 7, with the trend chosen in each case as a linear least squares fit. The governmental credit policies are presented in Figure 8, compiled from Rothbard([10], pp. 101 - 5). In the mathematical model, the influence function was chosen as a constant for eighteen months duration, and the influence function amplitude α, β, and γ varied until the model solution minimized a least squares error criterion for the period January, 1921, to July, 1922. With the parameters then fixed at $\rho = 0.5$, $\alpha = 0.5$, $\beta = 3.5$ and $\gamma = 1.0$, the solution was extended under the actual credit policy input to January, 1924. The model solution is compared with the actual values in Figures 9 and 10.

The mathematical model solution does not fit the actual quantities well in two respects; the building slump from January to July, 1922, is not fitted and the rising department store sales after July, 1922, is lagged by several months. It should be noted, however, that the Austrian theory itself does not explain these results. Since government credit expansion monotonically increased until July, 1922, neither a drop in building nor an increase in consumer demand could be predicted. Either the Austrian theory is at fault or, more likely, the use of relatively small portions of the economic process to characterize larger segments is of limited accuracy. For example, factors such as lumber prices and weather could cause fluctuations in the construction industry but would not greatly affect the overall capital-goods investment

INDEX OF VALUE OF BUILDING PERMITS

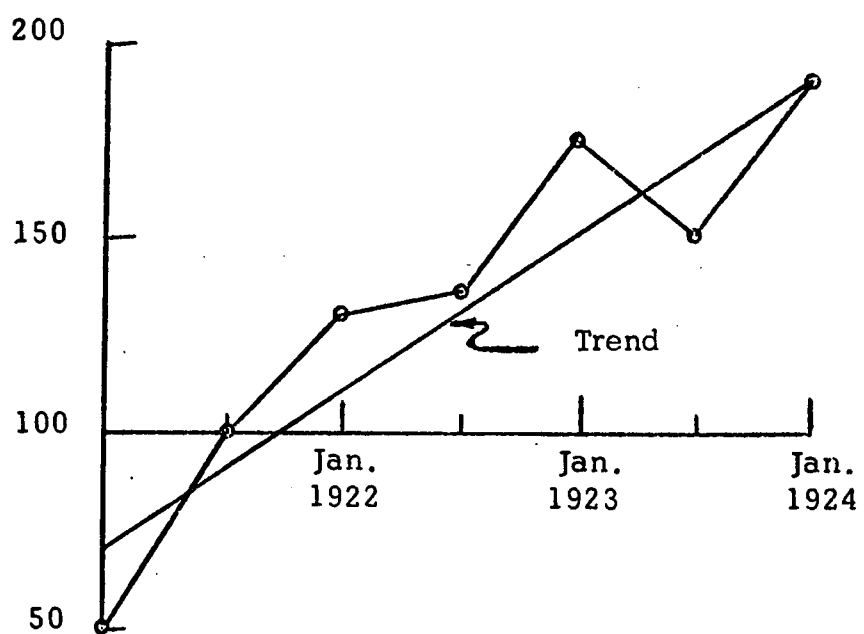JANUARY 1921 – JANUARY 1924

(July 1921 = 100)

Figure 6

DEPARTMENT STORE SALES

JANUARY 1921 – JANUARY 1924
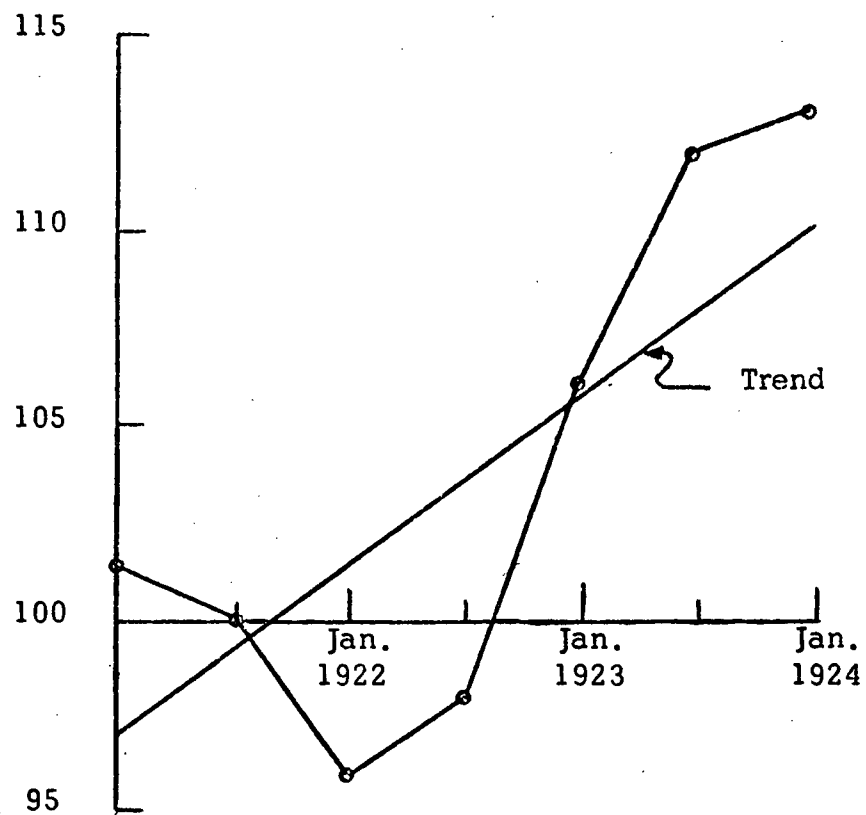
(July 1921 = 100)

Figure 7

GOVERNMENT CREDIT EXPANSION

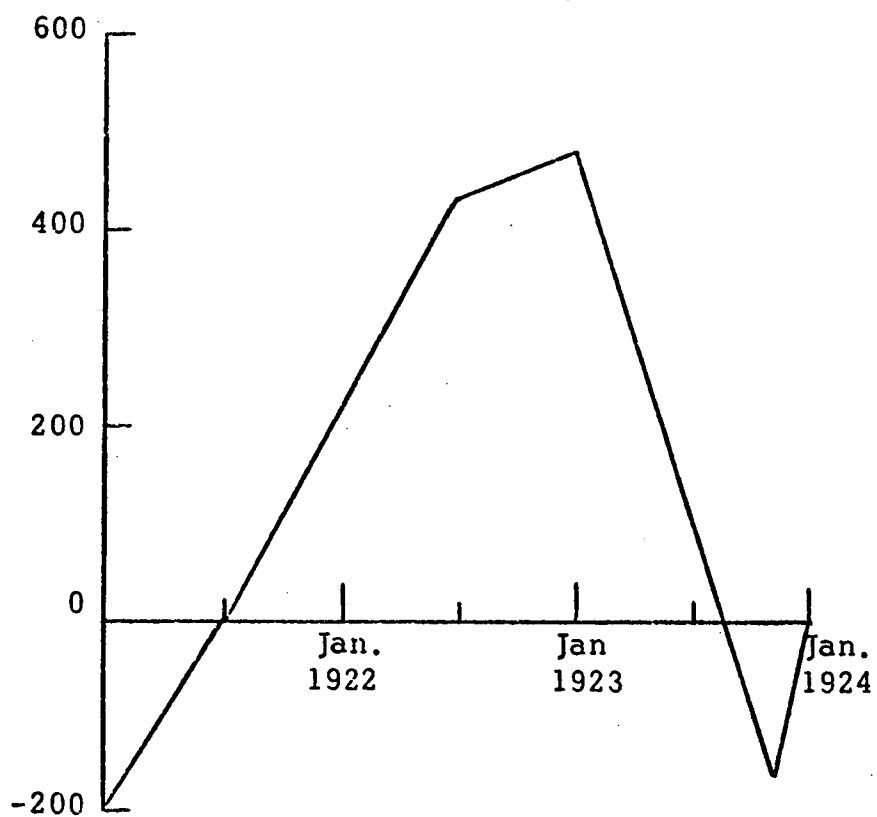IN MILLIONS OF DOLLARS

JANUARY 1921 – JANUARY 1924

Figure 8

VALUE OF BUILDING PERMITS DEVIATION FROM
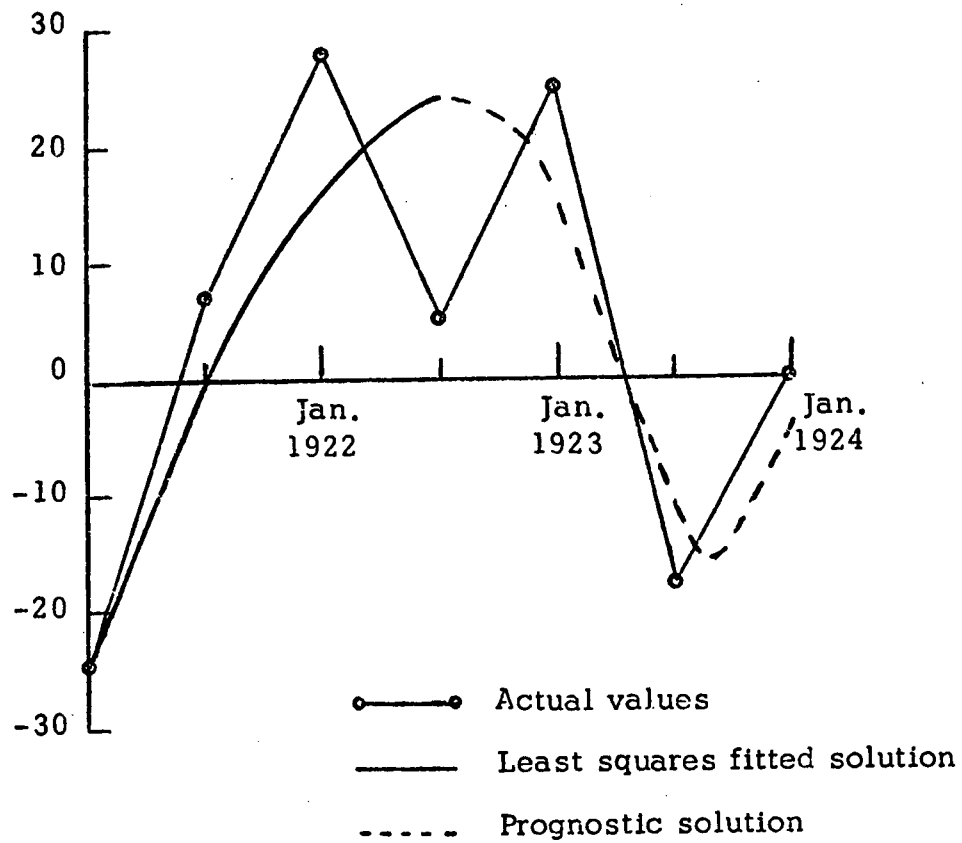THE TREND: IN PERCENT OF JULY 1921 VALUE

Figure 9

DEPARTMENT STORE SALES DEVIATION FROM THE
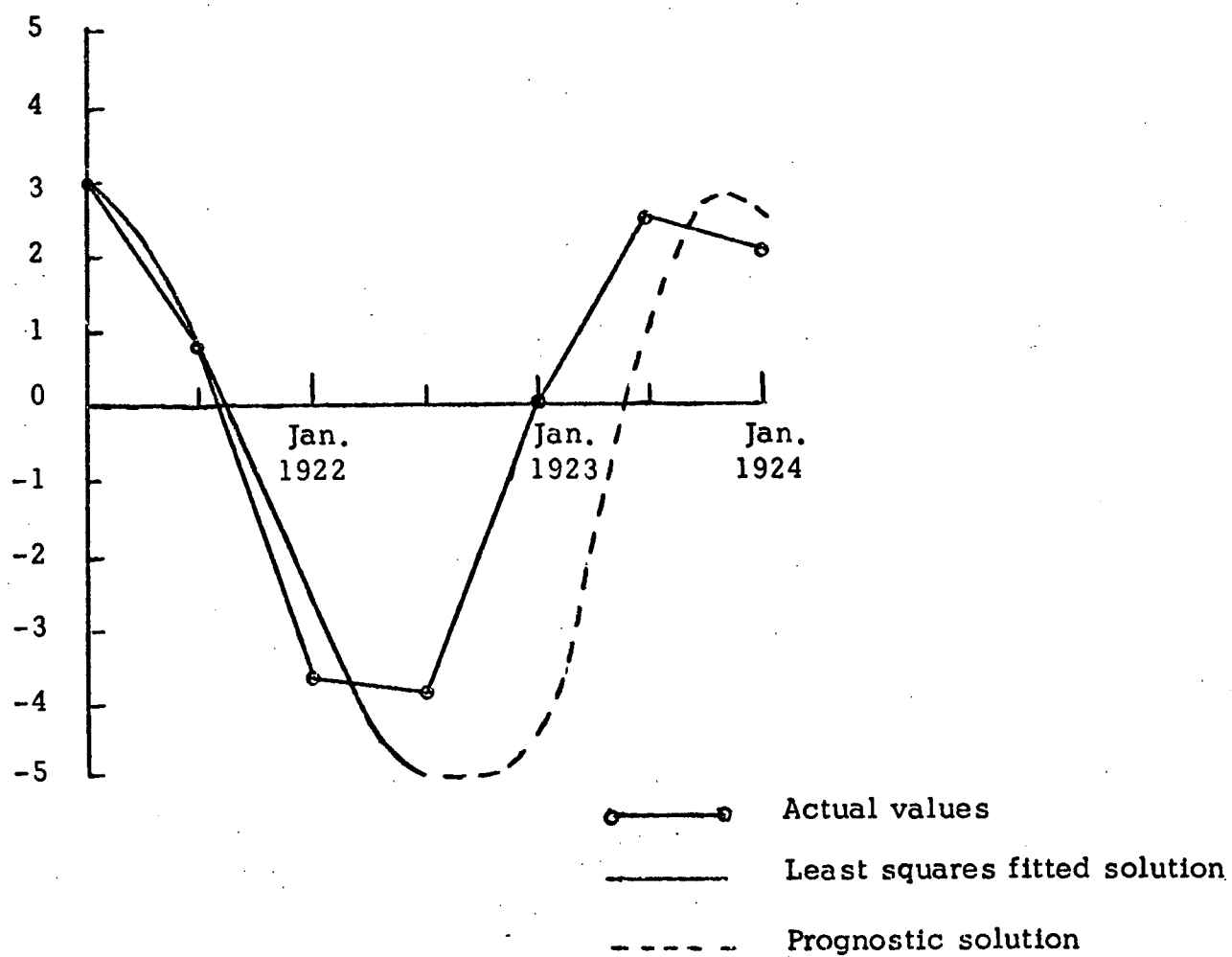## TREND: IN PERCENT OF JULY 1921 SALES

Figure 10

process. But even with these inaccuracies, a prognostic solution under proposed credit policies would, in July of 1922, have indicated the results of such a precipitous drop in monetary supply.

V. A ZERO LOCATION PROBLEM

With some correspondence having been established between the theory conclusions and the model solutions, this chapter will present a specific numerical example illustrating the numerical applications of Theorem 3.6. Suppose the system equations are given by

$$x(t) = g(t) - \int_0^1 0.5 \, x \, (t-\tau) d\tau - \alpha \, v(t)$$

$$v'(t) = -2.0 \, v(t) - \gamma \, x(t), \quad v(0) = 0,$$

and the system stability is to be studied as a function of the coupling parameters α and γ. The zero locations are determined by

$$H(z) = [z + (2.5 - \alpha \gamma) + 1/z] e^z - 0.5 - 1/z,$$

and condition (i) will be utilized. Let $z = x + iy$, $\xi = 2.5 - \alpha \gamma$ and

$$H(iy) = [i(y - 1/y) + \xi][\cos y + i \sin y] - 0.5 + 1/y,$$

which implies

$$F(y) = \xi \cos y - (y - 1/y) \sin y - 0.5$$

$$G(y) = \xi \sin y + (y - 1/y) \cos y + 1/y.$$

The zeros of $F(y)$ will be examined first, utilizing the fundamental Theorem 3.2. The principal term of $F(y)$ is $-y \sin y$, hence

$$\varphi_*^{(s)}(\cos y, \sin y) = -\sin y$$

and

$$\bar\varphi_*^{(s)}(z) = -\sin z.$$

Then

$$\bar\varphi_*^{(s)}(\epsilon + 2n\pi + iy) = -\sin(\epsilon + 2n\pi + iy)$$

is clearly non-zero for all real y and integer n if $\epsilon = \pi/2$. It must now be shown that $F(y)$ has, for sufficiently large k, exactly $4k + 1$ zeros (s = 1 in this case) for $-2k\pi + \pi/2 \le y \le 2k\pi + \pi/2$. Figure 11 graphically illustrates the following observations:
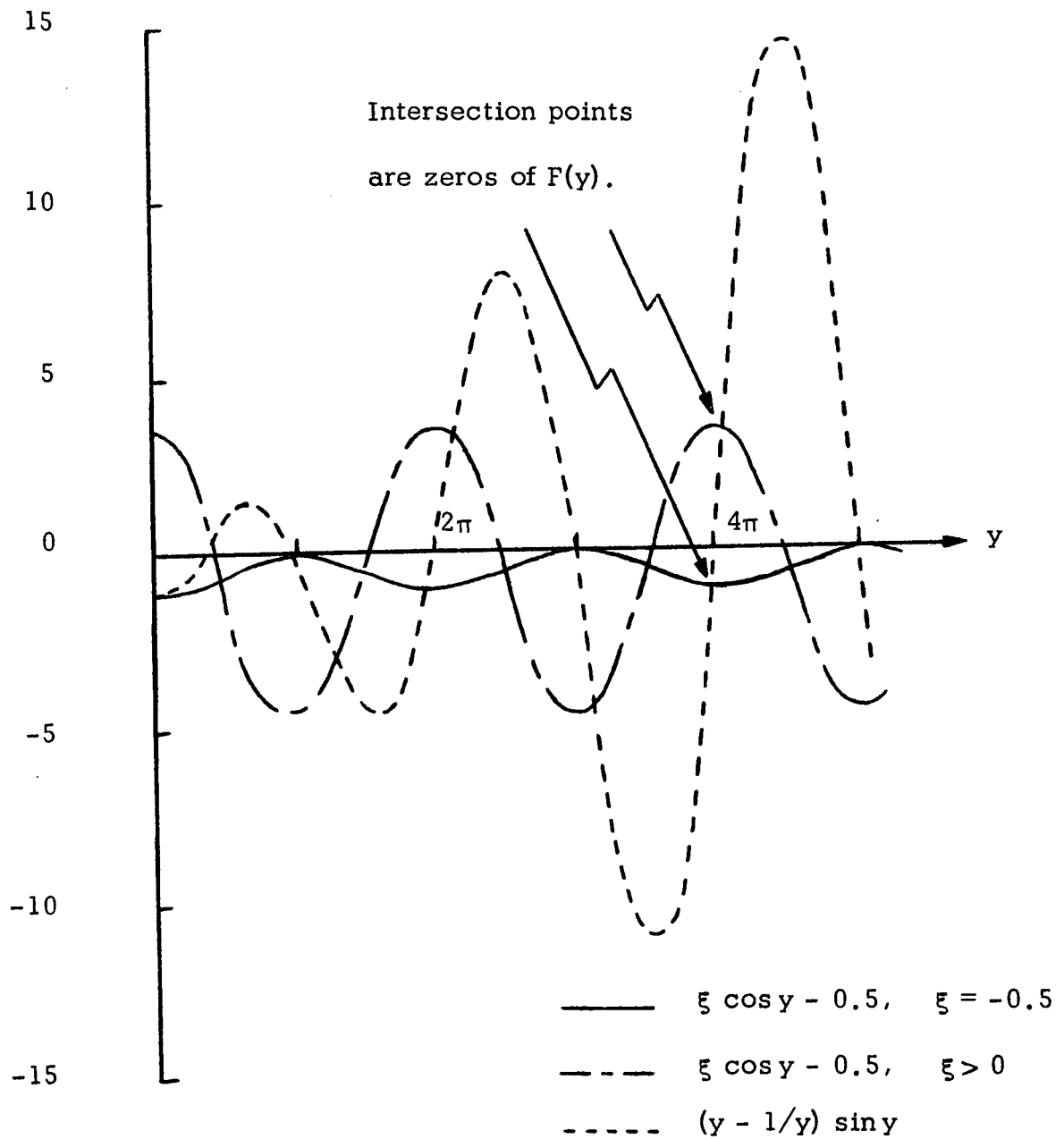
THE ZEROS OF F(y)

Figure 11

(i)   For ξ less than -0.5, $F(y)$ will have no zero for $0 \leq y \leq \pi/2$.

(ii)   For ξ exactly -0.5, $F(y)$ will have a double zero at $y = 0$.

(iii)   For ξ greater than -0.5, $F(y)$ will have exactly one zero for $0 < y \leq \pi/2$.

(iv)   For $\xi \geq -0.5$, $F(y)$ will have exactly one zero in each open interval $n\pi + \pi/2 < y < (n+1)\pi + \pi/2$, $n = 1, 2, \ldots$.

The preceding observations, and noting that $(y - 1/y)\sin y$ and $\cos y - 0.5$ are even functions of y, imply that $F(y)$ has, for sufficiently large k, exactly $4k + 1$ zeros for $-2k\pi + \pi/2 \leq y \leq 2k\pi + \pi/2$ if and only if $\xi \geq -0.5$. The zeros of $G(y)$ are next examined.

The principal term of $G(y)$ is $y\cos y$, and thus

$$\bar{\psi}_*^{(s)}(\epsilon + 2n\pi + iy) = \cos(\epsilon + 2n\pi + iy)$$

is non-zero for all y and integer n if $\epsilon = 0$. Noting that $\xi \geq -0.5$ has already been shown to be a necessary condition, it may now be examined only as a sufficiency requirement. Figure 12 illustrates the following:

(i)   For $\xi \geq -0.5$, $G(y)$ has exactly one zero in each interval $n\pi < y \leq (n+1)\pi$, $n = 0, 1, 2, \ldots$.

(ii)   For any ξ, $G(y)$ has a zero at $y = 0$, and that zero is simple.

The preceding observations, and noting that $\xi \sin y$ and $-1/y - (y - 1/y)\cos y$ are both odd functions of y, imply that $\xi \geq -0.5$ is sufficient to insure that $G(y)$ has, for sufficiently large k, exactly $4k + 1$ zeros for $-2k\pi \leq y \leq 2k\pi$.

The relative positions of the zeros of $F(y)$ and $G(y)$ must next be examined. It is first noted that for $\xi = -0.5$, $F(y)$ has a double root at the origin. Since this obviously would not allow the zeros of $F(y)$ and $G(y)$ to alternate, the condition must be changed to $\xi > -0.5$. Suppose now that $\xi > 0.5$. The zero locations of $F(y)$ may then be described by
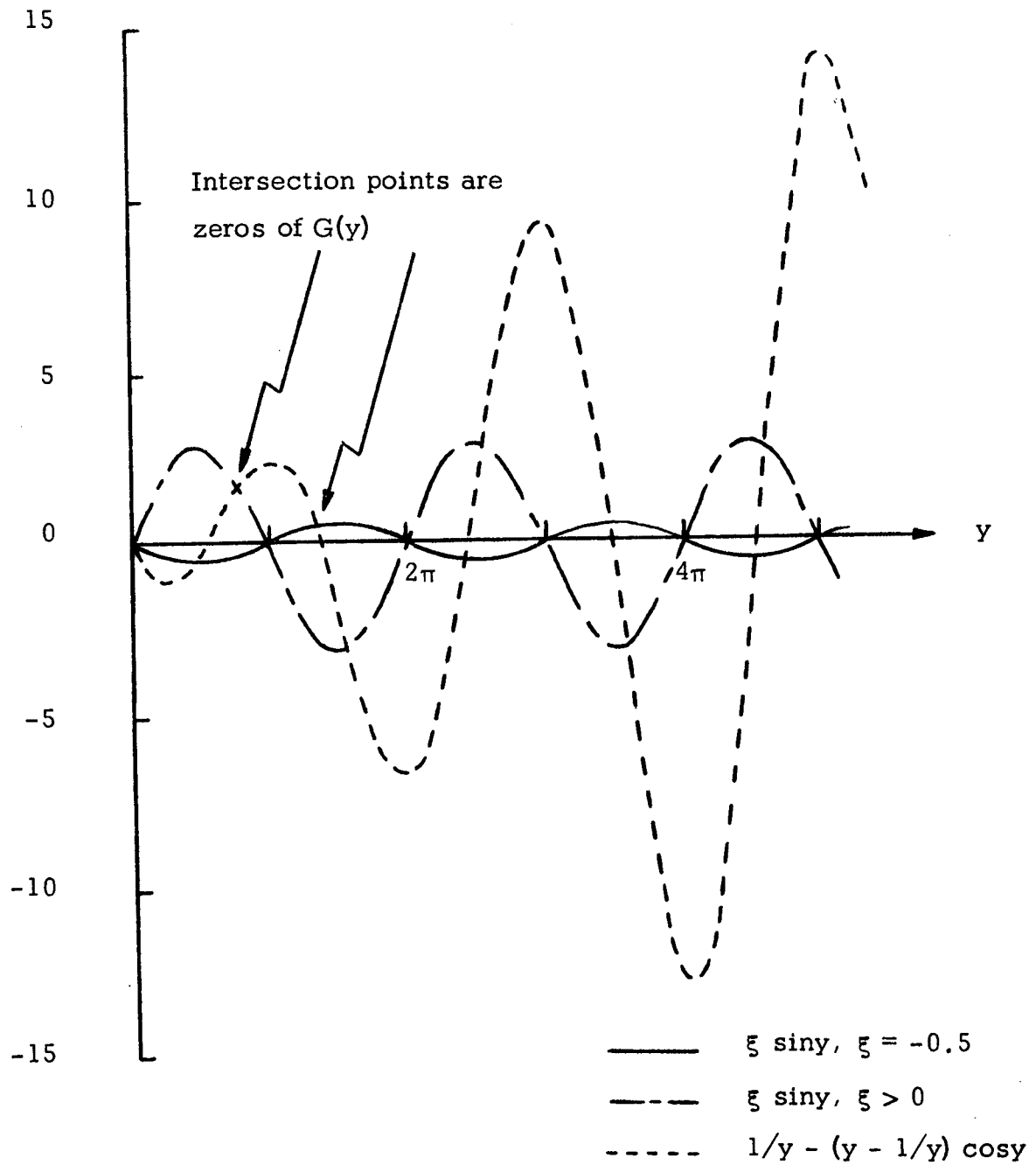
Figure 12

47

$$0 < y_{10} < \pi/2$$

$$y_{20} = \pi + \theta_2$$

$$\vdots$$

$$(n - 1)\pi \leq y_{n0} < (n - 1)\pi + \pi/2 , \text{ n odd}$$

$$y_{n0} = (n - 1)\pi + \theta_n, \quad \text{n even}$$

$$\vdots$$

where $0 < \theta_n < \pi/2$ and θ_n is defined by

$$\xi \cos((n - 1)\pi + \theta_n) - 0.5 = ((n - 1)\pi + \theta_n$$

$$- 1/((n - 1)\pi + \theta_n)) \sin((n + 1)\pi \theta_n) , \text{ n even.}$$

Similarly, the zero locations for $G(y)$ are given by

$$y'_{10} = 0$$

$$\pi/2 < y'_{20} < \pi$$

$$y'30 = \pi + \theta_3'$$

$$\vdots$$

$$(n - 2)\pi + \pi/2 < y'_{n0} < (n - 1)\pi , \text{ n even}$$

$$y'_{n0} = (n - 2)\pi + \theta_n', \text{ n odd,}$$

where θ_n' is defined by

$$\xi \sin((n - 2)\pi + \theta_n') = - 1/((n - 2)\pi + \theta_n') - ((n - 2)\pi + \theta_n'$$

$$- 1/((n - 2)\pi + \theta_n'))\cos((n - 2)\pi + \theta_n') , \text{ n odd.}$$

It is clear that if $\theta_n' > \pi/2$, y_{n0}' lies to the right of $y_{n-1,0}$. If $\theta_n' < \pi/2$, it must be shown that

$$\theta_n < \theta_n'$$

for all appropriate n. While a lengthy and complicated analysis is required to prove this result rigorously, the following geometric argument can be briefly presented. A short study of Figure 11 shows that θ_n will be large (slightly less than $\pi/2$ radians) only if ξ is in some sense large and y in

some sense small. Figure 12, however, indicates that these same conditions result in θ_n' being very large, approaching π radians. On the other hand, a brief calculation shows that θ_n' is at least 1.5 radians, and the minimal values occur for ξ small. But if ξ is small, the zeros of $F(y)$ lie only slightly to the right of odd multiples of π, implying that θ_n is indeed less than 1.5 radians.

Similar (but somewhat simpler) arguments for $-0.5 < \xi \le 0.5$ show that $\xi > -0.5$ is indeed a sufficient condition for the zeros of $F(y)$ and $G(y)$ to alternate. For the final step, it must be shown that $\xi > -0.5$ implies that

$$F(y)G'(y) - F'(y)G(y) > 0$$

for some y. The appropriate derivatives are given by

$$F'(y) = -a \sin y - (y - 1/y) \cos y - (1 + 1/y^2) \sin y,$$

$$G'(y) = a \cos y - (y - 1/y) \sin y + (1 + 1/y^2) \cos y - 1/y^2.$$

Suppose $y = 0$ is chosen. Then $G(y) = 0$,

$$F(0) = \xi + \lim_{y \to 0} \frac{\sin y}{y} - 0.5$$

$$= \xi + 0.5$$

and

$$G'(0) = \xi + \lim_{y \to 0} \frac{\sin y}{y} + \lim_{y \to 0} \left\{ \frac{\cos y}{y^2} - \frac{1}{y^2} \right\} + 1$$

$$= \xi + 1.5.$$

Thus

$$F(0)G'(0) = (\xi + 0.5)(\xi + 1.5)$$

and $F(0)G'(0) > 0$ if $\xi > -0.5$. Finally, then, the zeros are all strictly in the left half plane if and only if $\alpha \gamma < 3.0$.

VI. CONCLUSIONS

Evans ([8], p. 415), in speaking of the multiplier-accelerator econometric model, states that "...second-order difference equations are no longer adequate to determine the solution in models with more realistic and complicated lag structures." To overcome these problems, he suggests that an appropriate approach would be to "...state a more general theory of the cycle, estimate this theory empirically, and examine its behavior by actual simulation." While it might certainly be true that a more general theory of economy is needed, it is not so clear that any such new theory should be examined from an empirical point of view. It is certainly possible that a mathematical structure more complex than second-order difference equations might very accurately model such a new theory.

To support this conjecture, this work has presented a new modelling form, the coupled renewal-differential equation system, and with this structure modelled the Austrian theory of business cycles.

Mathematical results presented include conditions insuring the existence, uniqueness and Laplace transformability of solutions, and an illustration of the utilization of a differential equation solving subroutine in calculating actual numerical solutions. Stability results include a generalization of results due to Pontryagin [13] which gives necessary and sufficient conditions for all the system poles to lie strictly in the left half plane. Further numerical results include a qualitative comparison of the Austrian theory conclusions and the types of solutions the mathematical model possesses, and finally an illustration of the numerical use of the stability conditions.

While the Austrian theory of business cycles has been the only economic theory modelled, no attempt has been made to either support or reject, through the numerical results, the conclusions of the Austrian school. This particular theory was used only in an illustrative capacity to indicate the possible use of the modelling structure.

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