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TECHNICAL REPORT #1

THE GENERATION OF RANDOM NUMBERS ON THE IBM 360/44

DEPARTMENT OF
MATHEMATICS

TEXAS A&I UNIVERSITY

RESEARCH GRANT #NGR 44-073-003

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by

F. M. Speed and L. C. Laurito

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THE GENERATION OF RANDOM NUMBERS ON THE IBM 360/44 COMPUTER

A Thesis
Presented
In Partial Fulfillment
Of the Requirements for the Degree
Master of Science
Texas A. & I. University

by
Lewis C. Laurito
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PREFACE

In recent years there has been a need for "reliable" random numbers which have a uniform distribution over the unit interval U(0,1). At the present time there are three modes of providing random numbers for use on digital computers, specifically the IBM 360/44: external provision, internal generation by a random process, and internal generation of sequences of digits by a recurrence relation. The most common mode in use today is the internal generation of sequences of digits by a recurrence relation.

The use of random numbers in important large scale simulation programs, often referred to as Monte Carlo applications, has increased the need for good quality random numbers.

In the last few years there has been speculation from noted authorities in the field of random numbers that one of the established random number generators employed on many IBM 360 computers is inadequate for scientific results.

This paper is a summary of the generation and testing of random numbers on the IBM 360/44 computer.

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CHAPTER I

INTRODUCTION

The need and use of random numbers has a wide variety of applications. One would also guess that there have been many techniques used to generate random numbers. Some examples for uses of random numbers are as follows:

a) Sampling. Experiments often require the generation of large numbers. A random sample will often provide insight into some questions.

b) Numerical Analysis. By using random numbers resourceful techniques have been devised to solve difficult numerical problems.

c) Decision Making. In the theory of games randomness is an essential part of optional strategies.

d) Simulation. The random simulation of a problem either of a deterministic nature or some direct simulation of a random process is often referred to as the Monte Carlo method. Extensive use of this method is now employed in a variety of situations, for example: the study of nuclear fields, cell growth, traffic flow, business, and even social-economic systems. The Monte Carlo method has also been used in computing multiple integrals, solving differential equations, finding eigenvalues, and inverting matrices. However, in order to employ the Monte Carto method successfully, one must have two things, a high speed computer and a "good" source of random numbers.

The literature is full of contradictions concerning random number generators, specifically the congruential methods. Noted individuals in the field of random number generators have stated that the method known as RANDU employed by the IBM 360 is completely unsatisfactory. One such scholar is Dr. Gates of Texas A & M University. Dr. Gates believes the methods presently used are good for nothing other than "piddling around in the classroom." [19]

This paper reports the results of extensive testing of random number generators, specifically the multiplicative congruential and the mixed congruential generators. This was carried out to determine if the present method, RANDU, can be used for important scientific and research investigations.

A random number generator is a procedure for producing within a computer a sequence of numbers $X_1, X_2, \ldots$ which is supposed to represent a sequence of independent uniform random variables. Usually, the X_i's are given by:

$$X_{i+1} = (\lambda X_i + \mu) \bmod M$$

where M is taken as 2^n for an n-bit binary machine and 10^n for an n-digit decimal machine. The generator is called multiplicative congruential if $\mu = 0$, and mixed congruential if $\mu \neq 0$.

RANDU is a multiplicative congruential generator with $\lambda = 2^{16} + 3$. The numbers are computed by:

$$X_{i+1} = [(2^{16} + 3)\, X_i] \bmod M$$

where M = 2^{31} for the IBM 360/44 computer.

Over 190 different generators of the mixed and multiplicative congruential method were tested. The statistical tests used to judge the sequences generated were cited in the literature as satisfactory and reasonable. A summary and tabulation of the pertinent data collected during the investigation is given at the end of this paper.

CHAPTER II

DEFINITIONS

In discussing the concept of generating random numbers it is necessary to make the following formal definitions.

Random Variable [12, p. 54]

Let $\mathcal{E}$ be an experiment and S a sample space associated with the experiment. A function x assigning to every element $s \in S$, a real number, x(s), is called a random variable $A = \left\{x;\ x = x(s),\ s \in S\right\}$.

Discrete Random Variable [12, p. 59]

Let X be a random variable. If the number of possible values of X is finite or countably infinite, we call X a discrete random variable.

Continuous Random Variable [12, p. 66]

X is said to be a continuous random variable if there exists a function f, called the probability density function (pdf) of X, satisfying the following conditions

(a) $f(x) \geq 0$, for all x

(b) $\int_{-\infty}^{+\infty} f(x)dx = 1$

(c) For any a,b, with $-\infty < a < b < +\infty$, we have $P(a \leq X \leq b) = \int_a^b f(x)dx$.

Uniformly Distributed Random Variables [12, p. 74]

Suppose that X is a continuous random variable assuming all values in the interval [a,b], where both a and b are finite. If the pdf of x is given by

$$f(x) = \frac{1}{b - a}, \quad a \leq x \leq b$$

$$= 0 \quad \text{elsewhere,}$$

we say that X is uniformly distributed over the interval [a,b].

Mathematical Expectation [12, p. 119]

Let X be a discrete random variable with possible values $x_1, \ldots, x_n, \ldots$. Let $p(x_i) = P(x = x_i)$, $i = 1, 2, \ldots, n \ldots$.

Then the expected value of X (or mathematical expectation of X), denoted be E(X), is defined as

$$E(X) = \sum_{i=1}^{\infty} x_i p(x_i).$$

Expected Value [12, p. 121]

Let X be a continuous random variable with pdf f. The expected value of x is defined as

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx.$$

If this improper integral does not converge, then we say that E(X) exists if and only if

$$\int_{-\infty}^{+\infty} |x| f(x)dx$$

is finite.

Confidence Intervals:

If one is going to estimate a parameter it is more meaningful to have some measure of the possible error in the estimate. In other words, an estimate $\bar{B}$ of a parameter B should also have some interval about $\bar{B}$, possibly of the form $\bar{B} - d$ to $\bar{B} + d$, accompanied by some measure of assurance that the true parameter B does lie within the interval. Thus, the ejection seat of a jet aircraft is said to subject the pilot's body to 20 ± 3 g forces with the idea that the g force is unlikely to be less than 17 g's or greater than 23 g's.

Confidence intervals enable one to obtain useful information concerning population parameters without treating such parameters as statistical variables.

In discussing what is termed a two-sided confidence interval one first begins by finding a random variable, call it Z, that involves the desired parameter B but the distribution of which does not depend upon any other unknown parameters. Then Z_1 and Z_2 are chosen such that

$$P\left\{Z_1 < Z < Z_2\right\} = \theta$$

where θ is the desired conficence interval coefficient, such as

0.90, which was the coefficient used throughout this investigation. This can then be written so that the probability statement is of the form

$$P\left\{ L < \gamma < U \right\} = \theta$$

where L and U are random variables depending on Z but not involving γ. One can then determine numerical values for L and U to obtain the desired confidence interval.

In order to determine the desired confidence interval a cumulative chi-square distribution table can be employed. However, for very large values of N, it may be necessary to compute the confidence interval. This was the case for the frequency, matrix, and gap test in this thesis.

In the frequency test the unit interval was divided into 1024 equal subintervals. The calculation of a 90 per cent confidence interval follows: The desired probability statement is

$$P\left\{ L \leq X^2_{K-1} \leq U \right\} = .90$$

where L is the lower value and U is the upper. The X^2 distribution in this case calls for K-1 degrees of freedom, i. e. K-1 = 1023

$$E\ [X^2_{1023}] = 1023 = r$$

$$V\ [X^2_{1023}] = 2046 = 2r.$$

Since N is large, the X^2 distribution can be approximated by the normal distribution; hence the probability statement is

$$P\left\{\frac{L - E[X^2_{1023}]}{\sqrt{V[X^2_{1023}]}} \leqslant \frac{X^2_{1023} - E[X^2_{1023}]}{\sqrt{V[X^2_{1023}]}} \leqslant \frac{U - E[X^2_{1023}]}{\sqrt{V[X^2_{1023}]}}\right\} = .90$$

and

$$P\left\{\frac{L - 1023}{\sqrt{2046}} \leqslant \frac{X^2_{1023} - 1023}{\sqrt{2046}} \leqslant \frac{U - 1023}{\sqrt{2046}}\right\} = .90$$

then

$$\frac{L - 1023}{\sqrt{2046}} = -1.65 \quad \text{(from a table of N(0,1) values).}$$

Solving for L:

$$L = 948.1.$$

Also

$$\frac{U - 1023}{\sqrt{2046}} = 1.65 \quad \text{(from a table of N(0,1) values)}$$

and solving for U:

$$U = 1097.9.$$

One can now say that a 90% confidence interval is 948.1 to 1097.9. It is usually written as such:

$$948.1 \leqslant X^2 \leqslant 1097.9.$$

CHAPTER III

GENERATORS

Section 1: Uniform Random Number Generators

At the present time there are a number of methods by which random numbers can be generated. Some of the more popular methods include: 1). the manual method, 2). stored tables, 3). Analog computers, and 4). Digital computers. The following discussion is a summary from [7, pp. 3-10], [6, pp. 20-40], and [14, pp. 43-57].

The manual methods are probably the simplest and the least practicable. They include such techniques as coin tossing, dice throwing, roulette wheels and card shuffling. People who needed random numbers used these techniques as well as special mechanical machines. The first of these machines was used by M. G. Kendall and B. Babington-Smith to produce 100,000 random digits in 1939. Later the Rand Corporation published a table containing one million random digits produced from an analog machine. The British Premium Savings Bonds lottery uses a machine called ERNIE to pick its winners. The main disadvantage to this method is that the numbers or sequence of numbers can not be reproduced.

The library tables are more advantageous than the manual method because of its reproducibility. However, the table could be a problem if it were to short or was a nuisance to prepare and maintain.

Not long after computers were introduced, people began to investigate efficient ways to obtain random numbers from computer programs. For example, the Rand Corporation's random numbers were generated by an analog computer. Numbers generated by an analog computer are deemed "truly random" because they depend on some physical process (ie, pulses from electronic circuits). This method is considerably faster than either manual methods or library tables but is also nonreproducible which would make checking the program impossible. In large scale calculations it is necessary to produce random numbers in volume. If one gets two answers to a problem, he cannot tell which one is correct if he cannot reproduce the same data in order to check his calculations.

On digital computers there are presently three modes of generating random numbers: (1) external provision, (2) internal generation by a random physical process, (3) and internal generation of sequences of digits by a recurrence relation.

External provision is exactly what it means. Random number tables are recorded on magnetic tape, on disk, or in punched cards for input into a digital computer which are then treated as data for the problem under consideration. The major objection to this method is that it takes longer to read one character of information into a computer than it does to perform an arithmetic operation on a single character.

Internal generation by a random physical process involves the use of a special component to a digital computer that can register the result of a random process. This method includes the decay of radioactive material and the thermal noise in an electronic valve circuit. The method is also nonreproducible, so it is impossible to check calculations. Also, a check on the random process producing the digits requires large sets of data which may limit the memory capacity of a computer.

The third mode is internal generation by a recurrence relation. This involves the generation of "pseudorandom numbers" by the "indefinitely continued transformation of a group of arbitrarily chosen numbers." L. D. H. Lehmer [18, p. 142] defined pseudorandom numbers as "a vague notion embodying the idea of a sequence in which each term is unpredictable to the uninitiated and whose digits pass a certain number of tests, traditional with statisticians and depending somewhat on the use to which the sequence is to be put." The prime advantage to this method is that it is reproducible. Tocker [17, p. 41] has stated that, "the principal objection to this solution is on the rather philosophical grounds that a sequence of digits generated by a purely deterministic rule is the direct antithesis of a random sequence." However, a sequence of numbers may be considered random if it satisfies some predetermined set of statistical tests of randomness.

The main concern of this paper is with the digital computer methods. Historically, one of the first arithmetic methods was developed by Neumann and Methropolis in 1946. This was called the mid-square method. Each number in the sequence is defined as the middle set of digits of the square of the previous number. For example, the first number of the sequence is 2111. This number squared is 04456321, and the second number in the sequence is 4563, the middle four digits of the squared number. However, this method was difficult to analyze, relatively slow, and statistically unsatisfactory [7, p. 3]. The mid-square method was abandoned in favor of congruential methods which at the present time is widely accepted and was originally developed by Lehmer [18, pp. 141-146].

It would seem obvious that certain criteria must be met in order for the generating method to be considered acceptable. They must be (1) uniformly distributed, (2) statistically independent, (3) reproducible, and (4) nonrepeated for any desired length. (5) The method must also be capable of generating random numbers at great speed, while (6) using a minimum of the computers memory. Techniques specifically designed to fulfill many of these requirements will be discussed in the next section.

Section 2: The Congruential Generators

The better or most successful pseudorandom-number generators are based on congruence arithmetic. There are two major types: the multiplicative congruential method and the mixed congruential method. Both congruential methods are based on a fundamental congruence relationship expressed by the following formula:

$$x_i = (\lambda x_{i-1} + \mu) \bmod m \qquad 3.0$$

where x_i, λ, μ, and m are non-negative integers. By expanding equation 3.0 for $i = 0, 1, 2, \ldots$ one obtains the following:

$$x_1 = \lambda x_0 + \mu \pmod{m}$$

$$x_2 = \lambda x_1 + \mu = \lambda^2 x_0 + (\lambda + 1)\mu \pmod{m}$$

$$\vdots$$

$$x_i = \lambda^i x_0 + \frac{\mu(\lambda^i - 1)}{(\lambda - 1)} \pmod{m} \qquad 3.1$$

Given an initial starting value x_0, a constant multiplier λ, and an additive constant μ, then for any value of i over the sequence $\{x_1, x_2, x_3, \ldots, x_i, \ldots\}$ equation 3.1 yields a congruence relationship (modulo m). Then the subsequent terms of $\{x_i\}$, which are determined by equation 3.1 above, are integers forming a sequence of residues modulo m. This implies that $x_i < m$ for all x_i. Then by constructing the sequence

$$\{r_i\} = \left\{\frac{x_i}{m}\right\},$$

rational numbers in the unit interval (0,1) can be obtained from the integers in the sequence $\{x_i\}$.

An important characteristic of this generator is the period. We now ask the question whether there exists a smallest positive value of i, where $i = h$, such that $x_h = x_0$, where h is the period of the sequence $\{x_i\}$. It can be shown [5, p. 50] that in fact h does exist; and one is then concerned with what conditions can be imposed on the parameters x_0, λ, μ, and m to insure that the period of $\left\{x_i\right\}$ is a maximum. Therefore, if

$$x_i = x_0 \qquad \text{for some } i = h$$

then

$$\begin{aligned} x_{h+1} &= x_1 \\ x_{h+2} &= x_2 \\ x_{h+3} &= x_3 \\ &\vdots \\ &\text{etc.} \end{aligned}$$

That is to say, the sequence will repeat itself after a period which is equal to h.

If one wishes to show that such an h always exists and its maximum value depends on m, there are theorems available. In other words, by congruential methods it is impossible to obtain nonrepeating sequences. However, by selecting the right combination of parameters the period of the sequence can be set very high.

Section 3: The Multiplicative Method

The form of the multiplicative generator is

$$x_i = (\lambda x_{i-1}) \bmod m.$$

For a binary computer, the best choice of m is 2^b where b = 31 for the IBM 360/44. It can be shown [5, pp. 50-51] that the maximum perild is $h = 2^{b-2} = 2^{29}$. However, λ must be relatively prime to m. Hence λ must be odd. It was shown [5, pp. 51-52] that such an λ must be of the form

$$\lambda = 8t \pm 3,$$

where t is a positive integer. A computer procedure for generating numbers by such a scheme is given below:

1. Let x_0 be any odd number.

2. Let integer $\lambda = 8t \pm 3$, where t is any positive integer for a constant multiplier.

3. Using fixed point integer arithmetic, calculate λx_0. This product will consist of 26 bits, from which the high-order b bits are discarded, and the low-order b bits represent x_1.

4. Compute $r_1 = x_1/2^b$ to obtain a uniformly distributed variable defined in the unit interval U(0,1).

5. Then each successive number x_{i+1} is obtained from the low-order bits of the product λx_i.

Section 4: The Mixed Method

The form of this method is $\mu \neq 0$, i. e.

$$x_i = (\lambda x_{i-1} + \mu) \bmod m.$$

If the following conditions are satisfied, then the period is equal to m:

1. μ is relatively prime to $m = 2^{31}$.
2. $\lambda = 2^s + 1$, where $s \geqslant 2$.

The following are procedures to generate numbers by this method:

1. Choose any value for x_0.
2. Choose μ relatively prime to 2^{31}.
3. Set $\lambda = 2^s + 1$, $s \geqslant 2$.
4. Compute $\lambda x_0 + \mu$ with fixed point integer arithmetic.
5. Find $r = x/2^{31}$.
6. Repeat.

Section 5: Summary

It must be pointed out that the research for this paper was done on a binary computer and the procedures for a decimal computer are different. However, the information is readily available in the literature.

Many random-number generators in use today are not very good. While the theory has been developed for some time, good generators for the IBM 360/44 are not available. There is empirical evidence that the multiplicative and mixed congruential methods produce acceptable pseudorandom numbers that pass selected statistical tests.

CHAPTER IV

OTHER GENERATORS

It goes without saying that most of the systems analyzed by the Monte Carlo method are based upon distributions other than the uniform. However, most of the other distributions that are needed, such as the normal, exponential, t, etc., can be expressed as a function of the uniform distribution. This chapter will give one method for obtaining these distributions.

Probably, the most popular method is the one known as the "Inverse Transformation Method". This method is based upon the following considerations. Suppose one wants to generate $y_1, y_2, \ldots, y_n$ from a distribution with cumulative density function F(y), where the inverse F, say F^{-1}, is also a function. It is known that $Z = F(y)$ is distributed U(0,1). Now let $Y = F^{-1}(Z)$, where Z is U(0,1). We want to show that Y has a cumulative distribution function F(y).

Now

$$\begin{aligned} P\left\{Y \leqslant y\right\} &= P\left\{F^{-1}(Z) \leqslant y\right\} \\ &= P\left\{Z \leqslant F(y)\right\} \\ &= F(y). \end{aligned}$$

Consider the following example. Suppose we want to generate $y_1, y_2, \ldots, y_n$ from a distribution with density function

$$f(y) = y/2, \quad 0 < y < 2.$$

Hence

$$f(y) = y^2/4, \quad 0<y<2.$$

Let

$$F(y) = Z, \text{ where } Z \text{ is } U(0,1).$$

Hence

$$y^2/4 = Z \quad \text{or} \quad y = 2\sqrt{Z},$$

where $\sqrt{Z}$ is a function since F(y) is defined on (0,2). Thus the y's can be obtained by generating uniform random numbers and then setting

$$y = 2\sqrt{Z}.$$

The remaining part of this chapter will be devoted to the derivation of the more popular generators.

1.) X is distributed U(a,b).

In this case

$$F(X) = \frac{X-a}{b-a}.$$

Hence setting

$$\frac{X-a}{b-a} = Z$$

we have

$$X = (b-a)Z + a, \text{ for } 0<Z<1.$$

2.) X is a distribution exponential with parameter λ , i.e. X is distributed EXP (λ). Hence,

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < X < \infty.$$

Hence

$$F(x) = 1 - e^{-\lambda x}.$$

Thus, setting

$$1 - e^{-\lambda x} = Z,$$

we have

$$e^{-\lambda x} = 1 - Z$$

or

$$-\lambda x = \ln (1 - Z)$$

or

$$x = -1/\lambda \ln(1 - Z).$$

3.) X is distributed "Cauchy." i.e.

$$f(x) = 1/\pi \left(\frac{1}{1 + (x-u)^2}\right).$$

Thus

$$F(X) = 1/2 + 1/\pi \text{ arc tan } (x-u).$$

Therefore

$$1/2 + 1/\pi \ \text{arc tan } (x-u) = Z$$

or

$$1/\pi \ \text{arc tan } (x-u) = Z-1/2.$$

Hence

$$X = \tan [\pi (Z - 1/2)] + u.$$

4.) X is distributed Gamma with parameters k and α. It can be shown [12, p. 54] that if k is an integer, then gamma is a sum of k exponentials with parameter α. i.e. $Y = X_1 + \dots + X_k$ is $G(k, \alpha)$ if X_i is $EXP(\alpha)$. Hence,

$$Y = \sum_{i=1}^{K} - 1/\alpha [\ln(Z_i)],$$

where Z_i's are U(0,1), is $G(K, \alpha)$. The case when h is not an integer will not be discussed here.

5.) X is distributed N(0,1). It can be shown [12, p. 55] that if Z_1 and Z_2 are U(0,1), then

$$X_1 = (-2\ln Z_1)^{1/2} \cos 2\pi Z_2$$

and

$$X_2 = (-2\ln Z_1)^{1/2} \sin 2\pi Z_2$$

are normally and independently distributed.

6.) Discrete distributions.

The following is a general procedure for generating random numbers from a discrete distribution. Suppose X has following distribution:

b_i	$P\{x=b_i\} = P_i$
b_1	P_1
b_2	P_2
.	.
.	.
.	.
b_t	P_t

Then if we generate Z from a U(0,1), we choose b_j if

$$P_1 + P_2 = \dots = P_{j-1} < Z \leqslant P_1 + P_2 + \dots + P_j$$

For example, if X is distributed B(4,1/2) we have

b_i	$P\{X\} = b_i$
0	1/16
1	4/16
2	6/16
3	4/16
4	1/16

Suppose the generated Z is .5; then the binomial random number is 2 since

$$1/16 + 4/16 < .5 < 1/16 + 4/16 + 6/16.$$

In summary, it is clear that the choice of the generator that is used to produce the U(0,1) random number is very critical. For if this generator is "suspect", then so are the other generators. It is for this reason that so much time and effort has been spent trying to find an adequate generator.

CHAPTER V

STATISTICAL TESTS

By using the theoretical concepts of statistics we are provided with some good qualitative measures for determining the randomness, independence, and distributive properties of numbers. There are many tests which can be performed; however, this paper will discuss only those tests which proved to be most useful and readily adapted to high speed computer calculation.

The statistical properties desired for uniform random numbers are those which would be produced by a device which selected numbers from the unit interval, $0 \leqslant x \leqslant 1$, independently and with each number equally possible. However, the numbers produced by a high speed computer are not random, since they are completely determined by the starting data and the method used to generate successive numbers.

When employing Monte Carlo application, one is more interested in quality random numbers than in random digits. In random number generators random numbers are used as a multiplier or additive constant, since only a few digits of the result are significant. Therefore, the usual tests for randomness of bits or digits need not be considered.

Many computer programs were written in FORTRAN language for the IBM 360/44 computer in order to do extensive testing of random number generators. Several programs consisted of tests known to give good results when employed on a sequence of suspected uniform random numbers. One generator, claimed by highly reputable sources, to be unsatisfactory for large N was also found to be unsatisfactory in this paper. The input to the programmed generators consisted of the multiplier λ, the constant μ, the initial random number X_0, and the sample size N. The sample size varied from N = 100 to N = 1,024,000.

A discription of each test follows, and the results are contained in table form in the back of this paper.

FREQUENCY TEST

The first requirement that a sequence must meet is that its numbers are, in fact, uniformly distributed between zero and one. A good random generator is expected to distribute the numbers "uniformly" on the unit interval. To do this we divide the unit interval into k subintervals; then generate N numbers and count the number of times a certain number appears in each subinterval.

The expected number in each subinterval is $E_j = N/k = E$, $j = 1, 2, 3, \ldots, k$. To test the deviation from this expected value, compute the chi-square value

$$x^2 = \sum_{j=1}^{k} \frac{(f_j - E_j)^2}{E_j}$$

where f_j is the observed frequency in the subinterval j. This chi-square value is compared with a chi-square distribution with k - 1 degrees of freedom and a desired confidence interval can be determined. Since too close an agreement can be just as suspect as too much deviation it is typical to use a two-sided confidence interval.

In all of our frequency tests a sample size N of 10,240 was used, and k was set at 1024. The expected number of numbers in each subinterval would then be 10. The results of these tests are noted in the tables. Extensive tests were run with larger samples, and some had sample sizes of 1,024,000.

The results of this test indicate that the frequency test is rather insensitive. Of the one hundred and ninety three generators tested 85 per cent or better than one hundred and fifty successfully passed the test. A few of the generators gave X^2 values of zero which would cause one to conclude that the numbers were too "random". A 90 per cent confidence interval was used which establishes the X^2 critical region, $948.1 \leqslant X^2 \leqslant 1097.9$.

KOLMOGOROV — SMIRNOV TEST

In this section the Kolmogorov — Smirnov Test for goodness of fit will be discussed. Suppose, for example, $x_1, x_2, x_3, \ldots, x_n$ is a sample of size n taken from some distribution specified by a continuous function F(x). In this case the distribution is U(0,1). Let the sample density function $S_n(x)$ be defined as follows:

$$S_n(x) = k/n, \quad \text{where } x_k \leqslant x < x_{k+1}$$

based on n observations. The maximum deviation D_n is defined by

$$D_n = \max_x \left| F(x) - S_n(x) \right| .$$

In order to apply the test, the quantity

$$\sqrt{n}\ D_n$$

is calculated and compared with tabulated values. If the value of $\sqrt{n}\ D_n$ is in excess of the specified critical values then the hypothesis that the data came from U(0,1) is rejected.

The Kolmogorov - Smirnov test was one of the most discriminating tests employed in that all but nineteen generators failed to pass the test. It was also noted that the generators which failed the X^2 frequency test also failed this test. For a discussion of the relative merits of chi-square and Kolmogorov - Smirnov tests see Lilliefors [9, pp. 399-402].

MATRIX TEST

In order to investigate the degree of randomness between successive numbers in a sequence the matrix test was employed. This test was proposed by M. L. Tuncosa and suggests one construct a k by k matrix whose elements x_{ij} represent the number of times a number in the i^{th} interval is followed by a number in the j^{th} interval. A sequence of M consecutive sets of N random numbers is generated, and equal values are expected for all the matrix elements. The chi-square statistic

$$X^2 = \sum_{i=1}^{k} \left(\sum_{j=1}^{k} \frac{(x_{ij} - N/k^2)^2}{N/k^2} \right)$$

is computed and compared with expected chi-square distribution with k^2-1 degrees of freedom. A 90 per cent confidence interval was established as in the frequency test and is 948.1 and 1097.9. All generators with chi-square values in this range were considered acceptable.

In order to compare our results with published results in the literature a 32 by 32 matrix was selected which gives 1024 elements.

Of the 193 generators tested only 68 successfully passed this test. The results of this test on each generator are shown in table (II).

GAP TEST

The tests preceding this one have been concerned with the randomness of sequences of numbers, where each number contains come fixed number of digits; this test is concerned with the randomness of the digits in a sequence of numbers. Given any non-zero positive digit A, then one is concerned with the lengths of gaps of non-A digits between any two of the given digits. The gap length L occurs when L non-A digits occur between two A's. One would expect two consecutive A's to give a gap length L = 0.

For any given sequence of numbers, the number of gaps occurring for each length are counted. The actual number of gaps of length L can be compared with the expected number by using the chi-square goodness of fit test.

The gap test was not quite as discriminating as the matrix test or the Kolmogorov - Smirnov (K - S) test but was considerably more selective than the frequency test.

AUTO-CORRELATION COEFFICIENT TEST

The auto-correlation coefficient test provides us with another way to measure the independence of the numbers x_n of a sequence. The auto-correlation coefficient is defined by

$$c_k = \frac{1}{N} \sum_{i=1}^{N} x_i \cdot x_{i+k}, \quad k = 0, 1, \ldots .$$

For a random sample it can be shown that if there is no correlation between x_i and x_{i+k}, then the values of c_k are normally distributed about the mean 1/4 for $k > 0$, and mean 1/3 for $k = 0$, and standard deviation equal

$$\frac{\sqrt{13N - 19k}}{12(N - k)}, \quad \text{for } k > 0$$

or standard deviation = 0.3/N in both cases and the variance is 0.09/N.

In this test k took on the values from 0 through 9. The coefficients were calculated and are contained in table (IV).

CHAPTER VI

CONCLUSION

There is one fact associated with random numbers which is not sufficiently recognized: random numbers are not as easy to come by as one would suspect. On the basis of this test, which may be viewed as preliminary but certainly indicative, the random number generator now employed on the IBM 360/44 is suspect for important research investigations. This generator is called RANDU and is a multiplicative congruential generator of the form

$$X_{i+1} = [\lambda X_i + \mu] \bmod M,$$

where $M = 2^{31}$ for the IBM 360/44, $\lambda = 2^{16} + 3$, and $\mu = 0$. RANDU was tested with three different starting values: 1) $X_0 = 3$, 2) $X_0 = \text{DCBABEB}$, a hexacedimal number, and 3) $X_0 = 1111111111$. It is noted from the data that when the starting value was 3, the generator was rejected by the frequency test, and the K-S test. When started with $X_0 = \text{DCBABEB}$, the generator successfully passed all the statistical tests. The third starting value, $X_0 = 1111111111$, which was used in classroom simulation, did not pass the frequency test or K-S test. It is quite evident that the selection of a starting value, X_0, is critical. After discovering that $X_0 = \text{DCBABEB}$ was a good starting value, the generator was tested with N = 120,000, i.e. 120,000 numbers were generated; then the next 10,240 were subjected to the tests. The generator passed the frequency test but failed the K-S test. The results of this test paralleled the remarks made in the literature. In other works, RANDU appears to be adequate for N less than 100,000 but inadequate for larger values of N.

While testing RANDU, another generator was found to be a good substitute. The generator is a mixed congruential type of the form

$$X_{i+1} = [\lambda X_i + \mu] \bmod M,$$

where $M = 2^{31}$ for the IBM 360/44, $\lambda = 2^9 + 1$, and $\mu = 1$. This generator passed all the statistical tests for small as well as very large values of N. Of the 193 generators tested, this one proved to be superior to any other tested.

It would seem that the congruential generators are not as good a generating source as one would like. However, it appears to be the most popular method employed by the IBM 360. Until a better method is perfected, we recommend the generator discussed previously.

In conclusion, we would like to point out that with the high speeds of modern computers it is no longer important to look for the fastest possible random number generator because all generators are reasonably fast, but to find the generator which will give the desired results.

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TABLE I-A FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 1$	1	1064.3	$2^{17} + 1$	1	383.7
$2^{2} + 1$	1	1023.9	$2^{18} + 1$	1	151.7
$2^{3} + 1$	1	1032.1	$2^{19} + 1$	1	99.7
$2^{4} + 1$	1	972.1	$2^{20} + 1$	1	599.9
$2^{5} + 1$	1	1090.7	$2^{21} + 1$	1	0.0
$2^{6} + 1$	1	1035.5	$2^{22} + 1$	1	10240.0
$2^{7} + 1$	1	1013.1	$2^{23} + 1$	1	30720.0
$2^{8} + 1$	1	981.5	$2^{24} + 1$	1	71680.0
$2^{9} + 1$	1	958.7	$2^{25} + 1$	1	153600.0
$2^{10} + 1$	1	1203.1	$2^{26} + 1$	1	317440.0
$2^{11} + 1$	1	1074.5	$2^{27} + 1$	1	645120.0
$2^{12} + 1$	1	1152.1	$2^{28} + 1$	1	1300480.0
$2^{13} + 1$	1	921.3	$2^{29} + 1$	1	2611200.0
$2^{14} + 1$	1	903.5	$2^{30} + 1$	1	5232638.0
$2^{15} + 1$	1	834.7	$2^{31} + 1$	1	10475520.0
$2^{16} + 1$	1	709.1	$2^{32} + 1$	1	10475520.0

a.) DATA is for N = 10,240, with 1024 frequencies.

The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leq X^2 \leq 1097.9$$

b.) $X_0 = 3$

TABLE IB FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1}+1$	1	1066.9	$2^{17}+1$	1	1398.2
$2^{2}+1$	1	1038.5	$2^{18}+1$	1	753.7
$2^{3}+1$	1	1043.1	$2^{19}+1$	1	304.1
$2^{4}+1$	1	1045.5	$2^{20}+1$	1	203.3
$2^{5}+1$	1	1022.5	$2^{21}+1$	1	0.0
$2^{6}+1$	1	1052.3	$2^{22}+1$	1	10240.0
$2^{7}+1$	1	984.5	$2^{23}+1$	1	30720.0
$2^{8}+1$	1	905.1	$2^{24}+1$	1	71680.0
$2^{9}+1$	1	1025.7	$2^{25}+1$	1	153600.0
$2^{10}+1$	1	950.9	$2^{26}+1$	1	317440.0
$2^{11}+1$	1	1046.5	$2^{27}+1$	1	645120.0
$2^{12}+1$	1	965.3	$2^{28}+1$	1	1300480.0
$2^{13}+1$	1	1039.5	$2^{29}+1$	1	2.6×10^{6}
$2^{14}+1$	1	1071.1	$2^{30}+1$	1	5.2×10^{6}
$2^{15}+1$	1	850.1	$2^{31}+1$	1	1.0×10^{7}
$2^{16}+1$	1	694.3	$2^{32}+1$	1	1.8×10^{7}

a.) DATA is for N = 10,240, with frequencies.

The 90 percent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^{2} \leqslant 1097.9$$

b.) X_o = DCBABEB

TABLE IC FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^1 + 1$	1	1088.5	$2^{17} + 1$	1	1068.9
$2^2 + 1$	1	1074.1	$2^{18} + 1$	1	1055.5
$2^3 + 1$	1	1052.5	$2^{19} + 1$	1	864.7
$2^4 + 1$	1	977.1	$2^{20} + 1$	1	1004.5
$2^5 + 1$	1	1084.5	$2^{21} + 1$	1	1022.5
$2^6 + 1$	1	1054.3	$2^{22} + 1$	1	1048.5
$2^7 + 1$	1	971.1	$2^{23} + 1$	1	1031.1
$2^8 + 1$	1	1020.3	$2^{24} + 1$	1	980.9
$2^9 + 1$	1	982.9	$2^{25} + 1$	1	994.7
$2^{10} + 1$	1	1157.1	$2^{26} + 1$	1	1024.9
$2^{11} + 1$	1	964.3	$2^{27} + 1$	1	1096.5
$2^{12} + 1$	1	1074.7	$2^{28} + 1$	1	1125.7
$2^{13} + 1$	1	1022.5	$2^{29} + 1$	1	1097.9
$2^{14} + 1$	1	1105.9	$2^{30} + 1$	1	1104.1
$2^{15} + 1$	1	1031.3	$2^{31} + 1$	1	1066.9
$2^{16} + 1$	1	.039.3	$2^{32} + 1$	1	1066.9

a.) DATA is for N = 10,240, with 1024 frequencies.

The 90 per cent confidence interfal for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leq X^2 \leq 1097.9$$

b.) X_0 = DCBABEB

TABLE IID CONTINUED FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETER μ	CHI-SQUARE VALUES
$2^{1}+3$	0	1018.7	$2^{17}+3$	0	1121.7
$2^{2}+3$	0	1037.9	$2^{18}+3$	0	1010.3
$2^{3}+3$	0	1086.3	$2^{19}+3$	0	1041.9
$2^{4}+3$	0	1026.9	$2^{20}+3$	0	974.9
$2^{5}+3$	0	1006.5	$2^{21}+3$	0	1056.3
$2^{6}+3$	0	1019.7	$2^{22}+3$	0	1006.3
$2^{7}+3$	0	1047.5	$2^{23}+3$	0	1090.3
$2^{8}+3$	0	1048.9	$2^{24}+3$	0	971.7
$2^{9}+3$	0	1048.7	$2^{25}+3$	0	992.7
$2^{10}+3$	0	967.1	$2^{26}+3$	0	988.5
$2^{11}+3$	0	1029.5	$2^{27}+3$	0	1026.9
$2^{12}+3$	0	1026.1	$2^{28}+3$	0	955.7
$2^{13}+3$	0	1008.3	$2^{29}+3$	0	995.3
$2^{14}+3$	0	1082.3	$2^{30}+3$	0	1105.1
$2^{15}+3$	0	1028.1	$2^{31}+3$	0	1067.5
$2^{16}+3$	0	1140.5	$2^{32}+3$	0	1067.5

a.) DATA is for $N = 10{,}240$, with 1024 frequencies.

The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) $X_o = 3$

TABLE IE CONTINUED FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALVES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALVES	GENERATOR λ	PARAMETER μ	CHI-SQUARE VALVES
$2^1 + 3$	0	1027.7	$2^{17} + 3$	0	1002.9
$2^2 + 3$	0	1044.3	$2^{18} + 3$	0	925.3
$2^3 + 3$	0	961.7	$2^{19} + 3$	0	1029.3
$2^4 + 3$	0	1021.5	$2^{20} + 3$	0	973.1
$2^5 + 3$	0	1022.1	$2^{21} + 3$	0	1038.1
$2^6 + 3$	0	972.5	$2^{22} + 3$	0	928.1
$2^7 + 3$	0	1130.7	$2^{23} + 3$	0	1019.1
$2^8 + 3$	0	959.5	$2^{24} + 3$	0	965.1
$2^9 + 3$	0	1018.3	$2^{25} + 3$	0	1028.1
$2^{10} + 3$	0	961.3	$2^{26} + 3$	0	1049.9
$2^{11} + 3$	0	1024.5	$2^{27} + 3$	0	1041.1
$2^{13} + 3$	0	1141.7	$2^{28} + 3$	0	967.3
$2^{13} + 3$	0	1051.9	$2^{29} + 3$	0	1087.1
$2^{14} + 3$	0	1078.3	$2^{30} + 3$	0	1013.3
$2^{15} + 3$	0	1060.3	$2^{31} + 3$	0	1080.3
$2^{16} + 3$	0	1073.9	$2^{32} + 3$	0	1080.3

a.) DATA is for N = 10,240, with 1024 frequencies. The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) X_0 = DCBABEB

TABLE IF FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETER μ	CHI-SQUARE VALUES
$2^1 + 3$	1	1023.9	$2^{17} + 3$	1	1053.9
$2^2 + 3$	1	1055.5	$2^{18} + 3$	1	1044.5
$2^3 + 3$	1	1079.3	$2^{19} + 3$	1	1030.9
$2^4 + 3$	1	1051.1	$2^{20} + 3$	1	912.7
$2^5 + 3$	1	1050.5	$2^{21} + 3$	1	1041.3
$2^6 + 3$	1	1004.5	$2^{22} + 3$	1	1089.3
$2^7 + 3$	1	986.5	$2^{23} + 3$	1	1046.7
$2^8 + 3$	1	1022.3	$2^{24} + 3$	1	1117.1
$2^9 + 3$	1	1051.1	$2^{25} + 3$	1	1088.5
$2^{10} + 3$	1	1015.7	$2^{26} + 3$	1	1051.5
$2^{11} + 3$	1	986.1	$2^{27} + 3$	1	1052.3
$2^{12} + 3$	1	947.3	$2^{28} + 3$	1	1066.3
$2^{13} + 3$	1	958.5	$2^{29} + 3$	1	1042.1
$2^{14} + 3$	1	1005.1	$2^{30} + 3$	1	1104.1
$2^{15} + 3$	1	1078.7	$2^{31} + 3$	1	1064.3
$2^{16} + 3$	1	986.9	$2^{32} + 3$	1	1064.3

a.) DATA is for N = 10,240, with 1024 frequencies. The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) $X_o = 3$

TABLE IG FREQUENCY TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 3$	0	1026.8	$2^{17} + 3$	0	1002.2
$2^{2} + 3$	0	1043.4	$2^{18} + 3$	0	924.2
$2^{3} + 3$	0	960.8	$2^{19} + 3$	0	1031.6
$2^{4} + 3$	0	1021.2	$2^{20} + 3$	0	972.8
$2^{5} + 3$	0	1020.6	$2^{21} + 3$	0	1037.8
$2^{6} + 3$	0	972.4	$2^{22} + 3$	0	928.6
$2^{7} + 3$	0	1129.6	$2^{23} + 3$	0	1019.8
$2^{8} + 3$	0	960.6	$2^{24} + 3$	0	964.4
$2^{9} + 3$	0	1016.4	$2^{25} + 3$	0	1026.4
$2^{10} + 3$	0	961.8	$2^{26} + 3$	0	1048.4
$2^{11} + 3$	0	1026.4	$2^{27} + 3$	0	1039.6
$2^{12} + 3$	0	1142.4	$2^{28} + 3$	0	968.6
$2^{13} + 3$	0	1053.0	$2^{27} + 3$	0	1088.8
$2^{14} + 3$	0	1080.4	$2^{30} + 3$	0	1011.6
$2^{15} + 3$	0	1060.6	$2^{31} + 3$	0	1081.4
$2^{16} + 3$	0	1073.4	$2^{32} + 3$	0	1081.4

a.) DATA is for the sequence of numbers between 120,000 and 130,240. N = 10,240. The 90 per cent confidence interval for CHI-Square distribution with 1023 degree of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) X_o = DCBABEB

GENERATOR PARAMETERS VS CHI SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR	PARAMETERS	CHI-SQUARE VALUES
$2^{16}+3$	0	930.5			
$*2^{16}+3$	0	976.8			
$**2^{16}+3$	0	978.4			
$***2^{16}+3$	0	980.4			

a.) Data is for N = 10,240, with 1024 frequencies. The 90 per cent confidence interval for the CHI-SQUARE distribution with 1023 degrees of freedom is: $948.1 \leqslant X^2 \leqslant 1097.9$

b.) X_o = 1111111111

$*X_o$ = DCBABEB, N = 1,024,000

$**X_o$ = DCBABEB, N = 102,400

$***X_o$ = 3 , N = 102,400

TABLE IIA MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 3$	1	56268.3	$2^{17} + 3$	1	1045.7
$2^{2} + 3$	1	37735.6	$2^{18} + 3$	1	960.1
$2^{3} + 3$	1	20674.5	$2^{19} + 3$	1	1049.5
$2^{4} + 3$	1	8065.0	$2^{20} + 3$	1	1050.3
$2^{5} + 3$	1	1868.5	$2^{21} + 3$	1	961.7
$2^{6} + 3$	1	1238.3	$2^{22} + 3$	1	986.7
$2^{7} + 3$	1	1052.9	$2^{23} + 3$	1	972.1
$2^{8} + 3$	1	1080.1	$2^{24} + 3$	1	985.7
$2^{9} + 3$	1	1036.7	$2^{25} + 3$	1	2094.5
$2^{10} + 3$	1	964.7	$2^{26} + 3$	1	4455.0
$2^{11} + 3$	1	1015.3	$2^{27} + 3$	1	18188.5
$2^{12} + 3$	1	1033.5	$2^{28} + 3$	1	45440.7
$2^{13} + 3$	1	1035.5	$2^{29} + 3$	1	100135.1
$2^{14} + 3$	1	1070.1	$2^{30} + 3$	1	100205.1
$2^{15} + 3$	1	1023.3	$2^{31} + 3$	1	100429.3
$2^{16} + 3$	1	1075.3	$2^{32} + 3$	1	100429.3

a.) DATA is for N = 10,240, with a 32 X 32 MATRIX. The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) $X_0 = 3$

TABLE II-B MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 1$	1	100429.3	$2^{17} + 1$	1	2292.9
$2^{2} + 1$	1	56258.3	$2^{18} + 1$	1	2477.7
$2^{3} + 1$	1	27237.2	$2^{19} + 1$	1	3260.9
$2^{4} + 1$	1	10040.6	$2^{20} + 1$	1	7760.0
$2^{5} + 1$	1	1311.5	$2^{21} + 1$	1	10240.0
$2^{6} + 1$	1	1137.3	$2^{22} + 1$	1	21760.0
$2^{7} + 1$	1	1096.3	$2^{23} + 1$	1	46080.0
$2^{8} + 1$	1	935.1	$2^{24} + 1$	1	71680.0
$2^{9} + 1$	1	1045.9	$2^{25} + 1$	1	15360.0
$2^{10} + 1$	1	994.3	$2^{26} + 1$	1	317440.0
$2^{11} + 1$	1	1227.1	$2^{27} + 1$	1	645120.0
$2^{12} + 1$	1	1290.7	$2^{28} + 1$	1	1300480.0
$2^{13} + 1$	1	1085.3	$2^{29} + 1$	1	2611200.0
$2^{14} + 1$	1	1785.5	$2^{30} + 1$	1	
$2^{15} + 1$	1	6596.7	$2^{31} + 1$	1	
$2^{16} + 1$	1	7190.1	$2^{32} + 1$	1	

a.) DATA is for N = 10,240, with a 32 by 32 matrix. The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) $X_0 = 3$

MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^1 + 1$	1	99911.5	$2^{17} + 1$	1	
$2^2 + 1$	1	56151.9	$2^{18} + 1$	1	
$2^3 + 1$	1	27177.1	$2^{19} + 1$	1	
$2^4 + 1$	1	10065.7	$2^{20} + 1$	1	
$2^5 + 1$	1	1198.7	$2^{21} + 1$	1	
$2^6 + 1$	1	1204.1	$2^{22} + 1$	1	
$2^7 + 1$	1	1077.5	$2^{23} + 1$	1	
$2^8 + 1$	1	119.5	$2^{24} + 1$	1	
$2^9 + 1$	1	1031.3	$2^{25} + 1$	1	
$2^{10} + 1$	1	1100.9	$2^{26} + 1$	1	
$2^{11} + 1$	1	972.1	$2^{27} + 1$	1	
$2^{12} + 1$	1	952.5	$2^{28} + 1$	1	
$2^{13} + 1$	1	1060.9	$2^{29} + 1$	1	
$2^{14} + 1$	1		$2^{30} + 1$	1	
$2^{15} + 1$	1		$2^{31} + 1$	1	
$2^{16} + 1$	1		$2^{32} + 1$	1	

a.) DATA is for N = 10,240, with a 32 X 32 matrix. The 90 per cent confidence interval for the CHI-Square distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^2 \leqslant 1097.9$$

b.) X_0 = DCBABEB

TABLE II-D MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 3$	0	56232.1	$2^{17} + 3$	0	1003.1
$2^{2} + 3$	0	37766.0	$2^{18} + 3$	0	1017.3
$2^{3} + 3$	0	20668.3	$2^{19} + 3$	0	959.5
$2^{4} + 3$	0	80905.3	$2^{20} + 3$	0	951.5
$2^{5} + 3$	0	1819.7	$2^{21} + 3$	0	1037.1
$2^{6} + 3$	0	1151.1	$2^{22} + 3$	0	996.7
$2^{7} + 3$	0	1102.9	$2^{23} + 3$	0	972.3
$2^{8} + 3$	0	967.1	$2^{24} + 3$	0	1184.3
$2^{9} + 3$	0	1005.5	$2^{25} + 3$	0	2003.5
$2^{10} + 3$	0	973.5	$2^{26} + 3$	0	18142.1
$2^{11} + 3$	0	965.5	$2^{27} + 3$	0	45204.3
$2^{12} + 3$	0	1091.7	$2^{28} + 3$	0	100050.7
$2^{13} + 3$	0	982.5	$2^{29} + 3$	0	99879.4
$2^{14} + 3$	0	1124.9	$2^{30} + 3$	0	100038.1
$2^{15} + 3$	0	1017.9	$2^{31} + 3$	0	100038.1
$2^{16} + 3$	0	1003.7	$2^{32} + 3$	0	100038.1

a.) DATA is for N = 10,240, with a 32 X 32 Matrix. The 90 per cent confidence interval for the CHI-SQUARE distribution with 1023 degrees of freedom is:

$$948.1 \leqslant X^{2} \leqslant 1097.9$$

b.) $X_{0} = 3$

TABLE II-E MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 3$	0	56245.3	$2^{17} + 3$	0	949.5
$2^{2} + 3$	0	37553.8	$2^{18} + 3$	0	972.5
$2^{3} + 3$	0	20640.3	$2^{19} + 3$	0	992.9
$2^{4} + 3$	0	7978.1	$2^{20} + 3$	0	1024.1
$2^{5} + 3$	0	1890.9	$2^{21} + 3$	0	977.3
$2^{6} + 3$	0	1201.3	$2^{22} + 3$	0	946.1
$2^{7} + 3$	0	1072.7	$2^{23} + 3$	0	934.1
$2^{8} + 3$	0	1062.3	$2^{24} + 3$	0	1245.5
$2^{9} + 3$	0	1041.5	$2^{25} + 3$	0	2152.0
$2^{10} + 3$	0	976.5	$2^{26} + 3$	0	18215.5
$2^{11} + 3$	0	955.1	$2^{27} + 3$	0	45471.1
$2^{12} + 3$	0	1052.9	$2^{28} + 3$	0	100078.8
$2^{13} + 3$	0	974.5	$2^{29} + 3$	0	100353.8
$2^{14} + 3$	0	1000.9	$2^{30} + 3$	0	100073.3
$2^{15} + 3$	0	1009.9	$2^{31} + 3$	0	100073.3
$2^{16} + 3$	0	1071.5	$2^{32} + 3$	0	100073.3

a.) DATA for N = 10,240, with a 32 X 32 Matrix. The 90 per cent confidence interval for the CHI-SQUARE distribution with 1023 degrees of freedom is:

$$948.1 \leq X^{2} \leq 1097.9$$

b.) X_{0} = DCBABEB

TABLE II-F MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES	GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUES
$2^{1} + 3$	1	56151.9	$2^{17} + 3$	1	1027.5
$2^{2} + 3$	1	37571.8	$2^{18} + 3$	1	960.1
$2^{3} + 3$	1	20567.9	$2^{19} + 3$	1	1020.3
$2^{4} + 3$	1	8028.5	$2^{20} + 3$	1	860.7
$2^{5} + 3$	1	1724.5	$2^{21} + 3$	1	978.1
$2^{6} + 3$	1	1213.1	$2^{22} + 3$	1	1047.3
$2^{7} + 3$	1	1002.9	$2^{23} + 3$	1	1042.7
$2^{8} + 3$	1	1029.1	$2^{24} + 3$	1	1066.5
$2^{9} + 3$	1	1009.7	$2^{25} + 3$	1	2162.7
$2^{10} + 3$	1	1079.9	$2^{26} + 3$	1	4418.2
$2^{11} + 3$	1	1060.7	$2^{27} + 3$	1	18181.7
$2^{12} + 3$	1	945.5	$2^{28} + 3$	1	45410.9
$2^{13} + 3$	1	1040.9	$2^{29} + 3$	1	99935.8
$2^{14} + 3$	1	1012.3	$2^{30} + 3$	1	99912.6
$2^{15} + 3$	1	946.3	$2^{31} + 3$	1	99911.5
$2^{16} + 3$	1	1019.7	$2^{32} + 3$	1	99911.5

a.) DATA for $N = 10{,}240$, with a 32 X 32 Matrix. The 90 per cent confidence interval for the CHI-SQUARE distribution with 1023 degrees of freedom is:

$$948.1 \leq X^2 \leq 1097.9$$

b.) X_0 = DCBABEB

MATRIX TEST DATA

GENERATOR PARAMETERS VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETERS μ	CHI-SQUARE VALUE	GENERATOR	PARAMETER	CHI-SQUARE VALUE
$2^{16} + 3$	0	1001.5			

a.) DATA for N = 10,240, with 32 X 32 Matrix. The 90 per cent confidence interval for the CHI-SQUARE distribution with 1023 degree of freedom is:

$$948.1 \leq X^2 \leq 1097.9$$

b.) $X_0 = 1111111111$

TABLE III-A GAP TEST

GENERATOR PARAMETER VS CHI-SQUARE VALUES

GENERATOR λ	PARAMETER μ	CHI-SQUARE VALUES 100	 110	 120	 130	 140	 150
$2^3 + 3$	1	3.38	1.79	1.46	1.01	1.69	2.62
$2^4 + 3$	1	5.20	4.96	6.08	5.47	7.80	7.09
$2^5 + 3$	1	5.80	5.19	4.94	5.01	6.19	6.94
$2^6 + 3$	1	6.65	4.28	4.64	6.02	6.14	6.26
$2^7 + 3$	1	3.92	5.10	4.74	3.27	7.87	8.14
$2^8 + 3$	1	4.14	4.03	2.96	3.22	3.53	4.22
$2^9 + 3$	1	3.11	2.94	2.94	3.41	3.43	4.95
$2^{12} + 3$	1	1.24	2.84	1.79	2.02	1.69	3.62
$2^{13} + 3$	1	1.58	1.26	1.24	1.46	1.46	1.21
$2^{14} + 3$	1	5.44	3.16	2.11	2.10	3.32	2.46
$2^{17} + 3$	1	5.53	5.90	7.90	6.72	6.20	6.74
$2^{18} + 3$	1	1.14	6.70	1.09	7.21	1.09	1.13
$2^{20} + 3$	1	2.65	1.99	1.99	3.56	6.39	6.82
$2^{21} + 3$	1	1.58	1.94	3.33	3.22	4.54	5.82
$2^{22} + 3$	1	1.76	2.19	2.11	3.02	4.67	4.81
$2^{27} + 3$	1	4.28	5.37	5.31	3.52	4.10	5.79
$2^{28} + 3$	1	1.14	6.70	1.15	1.18	1.33	1.26
$2^{29} + 3$	1	5.96	5.01	3.03	2.47	2.79	3.22

a.) Data for N= 4 . The 90 per cent confidence interval for the CHI-SQUARE distribution with 3 degrees of freedom is: $.352 \leq X^2 \leq 7.81$.

b.) $X_o = 3$

TABLE III-B GAP TEST

GENERATOR PARAMETER VS CHI-SQUARE VALUES

GENERATOR PARAMETER		CHI-SQUARE VALUES					
λ	μ	100	110	120	130	140	150
$2^{5} + 3$	0	1.74	1.81	5.16	4.38	3.60	3.41
$2^{16} + 3$	0	2.10	3.16	2.86	3.70	5.64	5.38
$2^{25} + 3$	0	2.56	2.76	3.90	4.72	5.09	5.14
$2^{28} + 3$	0	.174	3.52	4.14	3.89	4.10	3.62
$2^{6} + 3$	1	3.02	2.57	2.58	3.56	3.60	5.49
$2^{9} + 3$	1	2.56	2.94	2.16	2.38	3.22	4.67
$2^{17} + 3$	1	3.76	3.39	3.04	2.41	2.70	3.10
$2^{18} + 3$	1	3.42	3.28	3.11	3.92	2.77	3.77
$2^{19} + 3$	1	4.42	5.74	4.39	4.79	4.97	6.58
$2^{22} + 3$	1	3.14	1.99	3.58	3.49	3.92	3.49
$2^{26} + 3$	1	4.94	5.68	8.43	7.72	6.77	7.70
$2^{27} + 3$	1	2.62	2.70	3.08	3.18	6.04	5.02
$2^{28} + 3$	1	3.28	3.63	3.34	2.73	3.89	3.02

a.) Data for $N = 4$. The 90 per cent confidence interval for the CHI-Square distribution with 3 degrees of freedom is:

$$.352 \leqslant X^2 \leqslant 7.81$$

b.) X_o = DCBABEB

GENERATOR PARAMETER VS CHI-SQUARE VALUES

GENERATOR PARAMETER		CHI-SQUARE VALUES					
λ	μ	100	110	120	130	140	150
$*2^7 + 1$	1	2.77	2.67	5.91	8.02	7.79	6.35
$*2^9 + 1$	1	6.32	5.19	4.83	6.25	5.96	6.02
$*2^{10} + 1$	1	7.41	6.36	6.79	6.22	6.59	4.95
$*2^{15} + 1$	1	2.24	2.83	2.70	3.24	3.26	3.82
$*2^{23} + 1$	1	6.29	1.87	3.33	2.59	2.70	4.17
$*2^{24} + 1$	1	5.76	2.94	2.49	2.79	2.57	3.34
$*2^{25} + 1$	1	5.40	4.03	5.64	3.95	6.32	6.89
$*2^{26} + 1$	1	4.10	5.54	6.63	5.79	6.57	7.82
$**2^{16} + 3$	0						
$2^3 + 3$	0	4.64	4.88	7.08	6.93	5.60	5.23
$2^{15} + 3$	0	5.96	5.94	5.18	4.02	4.23	3.74
$2^{26} + 3$	0	3.19	3.77	4.61	7.12	6.60	4.89
$2^9 + 1$	1	4.84	5.59	6.59	5.82	4.99	8.46
$2^{23} + 1$	1	6.30	5.01	3.51	3.24	3.24	6.38

a.) Data for N = 40. The 90 per cent confidence interval for the CHI-SQUARE distribution with 3 degrees of freedom is:

$$.352 \leq X^2 \leq 7.81 \,.$$

b.) $X_o = 3$

$**X_o = 1111111111$

$*X_o = \text{DCBABEB}$

TABLE IV-A AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO-CORRELATION COEFFICIENTS

GENRATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^3 + 1$	1	3288	2566	2488	2474	2475	2473	2492	2490	2477	2492
$2^5 + 1$	1	3322	2536	2491	2502	2501	2510	2492	2512	2493	2493
$2^6 + 1$	1	3324	2504	2499	2501	2499	2531	2511	2503	2506	2524
$2^7 + 1$	1	3368	2554	2552	2564	2547	2546	2551	2527	2543	2534
$2^8 + 1$	1	3367	2536	2536	2530	2528	2523	2518	2532	2547	2533
$2^9 + 1$	1	3377	2541	2555	2541	2547	2536	2536	2529	2543	2538
$2^{10} + 1$	1	3238	2437	2438	2456	2438	2441	2452	2438	2465	2446
$2^{19} + 1$	1	3334	2511	2485	2512	2491	2514	2497	2512	2499	2513
$2^{20} + 1$	1	3323	2472	2476	2481	2483	2487	2489	2491	2496	2502
$2^{21} + 1$	1	3335	2487	2493	2495	2499	2503	2508	2499	2505	2501
$2^{22} + 1$	1	3326	2482	2487	2487	2477	2484	2485	2485	2497	2513
$2^{23} + 1$	1	3311	2470	2459	2448	2446	2459	2481	2456	2480	2455

a.) All the values of C_h are $X\,10^{-4}$. The expected value of Co is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325, 341] for Co, and [242, 258] for C_h.

b.) $X_0 = 3$

TABLE IV-B AUTO-CORRELATION TEST DATA

GENERATOR PARAMETER VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^3 + 1$	1	3333	2587	2505	2500	2489	2496	2488	2469	2487	2491
$2^4 + 1$	1	3351	2574	2517	2529	2524	2537	2512	2516	2527	2514
$2^5 + 1$	1	3332	2529	2515	2523	2539	2516	2537	2513	2530	2513
$2^6 + 1$	1	3334	2492	2508	2505	2520	2513	2501	2494	2501	2509
$2^7 + 1$	1	3330	2474	2484	2465	2483	2486	2468	2472	2481	2471
$2^8 + 1$	1	3318	2499	2482	2491	2491	2478	2493	2493	2486	2482
$2^9 + 1$	1	3347	2519	2539	2540	2534	2537	2512	2521	2496	2532
$2^{10} + 1$	1	3324	2478	2478	2487	2458	2496	2503	2479	2504	2471
$2^{11} + 1$	1	3345	2515	2503	2505	2511	2514	2536	2513	2442	2508
$2^{12} + 1$	1	3344	2511	2424	2494	2496	2507	2485	2518	2463	2513
$2^{20} + 1$	1	3312	2490	2466	2486	2473	2484	2473	2477	2478	2471
$2^{21} + 1$	1	3332	2487	2496	2492	2512	2500	2491	2493	2512	2483
$2^{22} + 1$	1	3329	2495	2498	2507	2505	2503	2484	2522	2487	2508
$2^{23} + 1$	1	3336	2510	2491	2493	2493	2504	2436	2497	2512	2499
$2^{24} + 1$	1	3349	2512	2541	2531	2525	2481	2591	2421	2479	2479
$2^{16} + 3$	0	3372	2546	2530	2552	2563	2530	2539	2534	2545	2535

X_o = 1111111111

a.) All the values of C_h are X 10^{-4}. The expected value of Co is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325, 341] for Co, and [242, 258] for C_h.

b.) X_o = DCBABEB

TABLE IV-C AUTO-CORRELATION TEST DATA

GENERATOR PARAMETER VS AUTO-CORRELATION COEFFIECIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^3 + 3$	1	3348	2594	2534	2535	2521	2516	2541	2521	2549	2525
$2^4 + 3$	1	3327	2548	2498	2495	2495	2492	2493	2500	2488	2495
$2^5 + 3$	1	3351	2546	2530	2529	2533	2515	2510	2527	2509	2529
$2^6 + 3$	1	3379	2531	2530	2513	2512	2529	2511	2521	2506	2510
$2^7 + 3$	1	3305	2484	2462	2471	2460	2478	2484	2490	2487	2475
$2^8 + 3$	1	3347	2531	2510	2527	2492	2515	2545	2518	2523	2515
$2^9 + 3$	1	3376	2560	2527	2546	2538	2550	2549	2547	2544	2553
$2^{10} + 3$	1	3328	2517	2482	2487	2494	2486	2485	2500	2496	2502
$2^{11} + 3$	1	3289	2466	2470	2479	2444	2476	2477	2470	2466	2486
$2^{12} + 3$	1	3331	2503	2487	2486	2504	2488	2477	2476	2484	2481
$2^{13} + 3$	1	3349	2503	2498	2519	2510	2507	2508	2507	2494	2494
$2^{14} + 3$	1	3295	2442	2448	2465	2447	2452	2446	2448	2442	2447
$2^{15} + 3$	1	3346	2495	2505	2507	2497	2502	2525	2510	2497	2525
$2^{16} + 3$	1	3311	2471	2487	2461	2481	2473	2487	2468	2445	2474
$2^{17} + 3$	1	3319	2512	2493	2488	2477	2480	2487	2487	2482	2500
$2^{18} + 3$	1	3314	2474	2487	2492	2485	2470	2481	2479	2493	2480
$2^{19} + 3$	1	3321	2482	2489	2500	2487	2516	2482	2482	2477	2497
$2^{21} + 3$	1	3359	2502	2520	2500	2535	2503	2517	2535	2537	2489
$2^{23} + 3$	1	3340	2497	2517	2508	2499	2513	2505	2515	2504	2489
$2^{24} + 3$	1	3351	2541	2519	2518	2539	2515	2526	2515	2514	2504
$2^{25} + 3$	1	3383	2528	2533	2529	2513	2539	2526	2537	2547	2512
$2^{27} + 3$	1	3306	2467	2501	2484	2491	2482	2488	2492	2470	2502

TABLE IV-C Cont. AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{28} + 3$	1	3305	2446	2443	2468	2467	2473	2483	2457	2486	2466
$2^{29} + 3$	1	3349	2432	2510	2533	2511	2511	2518	2520	2515	2505

a.) All the values of C_h are X 10^{-4}. The expected value of Co is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325, 341] for Co, and [242, 258] for C_h.

b.) X_o = DCBABEB

TABLE IV-D AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^4 + 3$	1	3327	2534	2491	2496	2500	2472	2507	2489	2472	2478
$2^5 + 3$	1	3290	2486	2463	2441	2450	2454	2465	2443	2454	2542
$2^6 + 3$	1	3325	2500	2489	2474	2487	2473	2482	2469	2488	2476
$2^7 + 3$	1	3348	2498	2511	2518	2516	2519	2502	2510	2500	2493
$2^8 + 3$	1	3360	2516	2517	2515	2527	2538	2531	2527	2499	2524
$2^9 + 3$	1	3365	2522	2549	2524	2532	2523	2537	2543	2525	2523
$2^{10} + 3$	1	3282	2445	2452	2462	2442	2451	2449	2447	2450	2454
$2^{12} + 3$	1	3375	2528	2544	2529	2534	2536	2524	2529	2512	2538
$2^{13} + 3$	1	3345	2487	2520	2507	2513	2500	2505	2495	2507	2503
$2^{14} + 3$	1	3319	2493	2480	2482	2499	2497	2493	2490	2507	2498
$2^{15} + 3$	1	3357	2519	2513	2522	2522	2509	2498	2509	2528	2512
$2^{16} + 3$	1	3321	2455	2474	2489	2455	2475	2469	2465	2460	2471
$2^{17} + 3$	1	3328	2472	2496	2483	2476	2502	2490	2495	2487	2478
$2^{18} + 3$	1	3331	2488	2485	2495	2504	2476	2497	2475	2501	2486
$2^{19} + 3$	1	3331	2517	2478	2477	2482	2516	2489	2482	2508	2491
$2^{20} + 3$	1	3331	2516	2504	2516	2517	2508	2501	2508	2511	2508
$2^{21} + 3$	1	3307	2485	2464	2464	2471	2464	2458	2478	2466	2484
$2^{22} + 3$	1	3304	2468	2468	2468	2464	2441	2475	2470	2474	2454
$2^{24} + 3$	1	3391	2541	2546	2541	2547	2555	2551	2539	2542	2546
$2^{25} + 3$	1	3390	2544	2551	2529	2552	2552	2547	2558	2534	2549
$2^{26} + 3$		3363	2521	2534	2531	2513	2510	2530	2511	2517	2511
$2^{27} + 3$	1	3290	2455	2457	2453	2450	2461	2444	2464	2482	2449

TABLE IV-D AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO-CORRELATION COEFFIECIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{28} + 3$	1	3360	2512	2507	2544	2508	2519	2543	2530	2519	2527
$2^{29} + 3$	1	3308	2409	2466	2528	2481	2482	2504	2478	2491	2506

a.) All the values of C_h are X 10^{-4}. The expected value of Co is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325, 341] for Co, and [242, 258] for Ch.

b.) $X_o = 3$

TABLE IV-E AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{4} + 3$	0	3333	2513	2499	2491	2488	2487	2491	2492	2478	2480
$2^{5} + 3$	0	3277	2472	2446	2446	2438	2439	2443	2440	2450	2446
$2^{6} + 3$	0	3322	2478	2488	2462	2467	2469	2467	2479	2475	2482
$2^{7} + 3$	0	3376	2539	2545	2544	2548	2536	2532	2523	2522	2543
$2^{8} + 3$	0	3296	2473	2479	2458	2457	2468	2456	2495	2483	2479
$2^{9} + 3$	0	3295	2464	2480	2468	2473	2456	2471	2449	2455	2456
$2^{10} + 3$	0	3337	2514	2490	2498	2492	2519	2494	2508	2486	2482
$2^{11} + 3$	0	3283	2452	2445	2439	2450	2447	2455	2449	2439	2433
$2^{12} + 3$	0	3302	2487	2469	2481	2462	2473	2470	2466	2474	2441
$2^{13} + 3$	0	3352	2526	2524	2523	2520	2500	2515	2534	2517	2515
$2^{14} + 3$	0	3341	2498	2507	2496	2510	2491	2500	2513	2492	2505
$2^{15} + 3$	0	3311	2484	2480	2487	2488	2494	2467	2493	2465	2480
$2^{16} + 3$	0	3389	2556	2565	2553	2546	2566	2568	2541	2549	2574
$2^{17} + 3$	0	3290	2462	2486	2478	2471	2463	2474	2461	2466	2456
$2^{18} + 3$	0	3331	2515	2495	2498	2518	2519	2515	2530	2523	2521
$2^{19} + 3$	0	3320	2493	2480	2488	2493	2508	2472	2475	2494	2472
$2^{20} + 3$	0	3307	2481	2496	2478	2492	2488	2499	2482	2490	2472
$2^{21} + 3$	0	3359	2523	2517	2527	2549	2517	2549	2527	2525	2500
$2^{22} + 3$	0	3363	3523	2517	2544	2538	2524	2526	2545	2538	2532
$2^{23} + 3$	0	3328	2532	2482	2492	2490	2478	2491	2496	2515	2495
$2^{24} + 3$	0	3357	2503	2526	2544	2516	2510	2516	2526	2530	2520
$2^{25} + 3$	0	3341	2524	2502	2504	2509	2516	2522	2517	2515	2492

TABLE IV-E Cont. AUTO-CORRELATION TEST DATA

GENERATOR PARAMETERS VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{27}+3$	0	3306	2459	2474	2467	2467	2465	2492	2475	2458	2475
$2^{28}+3$	0	3313	2480	2475	2460	2463	2459	2468	2478	2478	2468
$2^{29}+3$	0	3361	2512	2490	2523	2522	2507	2535	2536	2518	2536

a.) All the values of C_h are X 10^{-4}. The expected value of C_0 is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325,241] for C_0, and [242,258] for C_h.

b.) X_0 = DCBABEB

GENERATOR PARAMETER VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^3 + 3$	0	3345	2578	2484	2494	2493	2494	2496	2495	2491	2491
$2^4 + 3$	0	3312	2537	2479	2499	2464	2501	2492	2489	2479	2479
$2^5 + 3$	0	3322	2521	2493	2492	2505	2524	2515	2494	2496	2517
$2^6 + 3$	0	3362	2536	2526	2523	2540	2512	2519	2526	2529	2525
$2^7 + 3$	0	3311	2480	2474	2491	2484	2478	2468	2476	2477	2495
$2^8 + 3$	0	3356	2499	2498	2508	2524	2511	2509	2496	2531	2522
$2^9 + 3$	0	3286	2449	2437	2448	2459	2448	2449	2445	2470	2443
$2^{10} + 3$	0	3340	2523	2514	2522	2512	2497	2512	2522	2521	2512
$2^{11} + 3$	0	3342	2412	2509	2487	2506	2512	2515	2524	2508	2503
$2^{12} + 3$	0	3321	2483	2485	2479	2475	2468	2469	2490	2489	2484
$2^{13} + 3$	0	3309	2480	2494	2468	2484	2476	2477	2483	2478	2476
$2^{14} + 3$	0	3328	2517	2497	2499	2509	2497	2512	2517	2508	2526
$2^{15} + 3$	0	3308	2483	2460	2470	2494	2479	2482	2488	2497	2485
$2^{16} + 3$	0	3376	2524	2538	2508	2537	2539	2532	2545	2525	2515
$2^{17} + 3$	0	3356	2518	2537	2523	2519	2518	2529	2509	2543	2530
$2^{18} + 3$	0	3363	2520	2525	2502	2504	2535	2525	2528	2521	2516
$2^{19} + 3$	0	3332	2503	2486	2511	2520	2514	2506	2504	2466	2508
$2^{20} + 3$	0	3363	2518	2544	2521	2527	2498	2520	2512	2538	2506
$2^{21} + 3$	0	3317	2500	2481	2489	2493	2478	2487	2479	2497	2502
$2^{24} + 3$	0	3338	2498	2523	2515	2504	2428	2529	2513	2525	2525
$2^{25} + 3$	0	3315	2471	2471	2476	2460	2482	2475	2482	2502	2463

TABLE IV-F CON'T AUTO-CORRELATION TEST DATA

GENERATOR PARAMETER VS AUTO-CORRELATION COEFFICIENTS

GENERATOR PARAMETER		h VALUES									
λ	μ	0	1	2	3	4	5	6	7	8	9
$2^{26} + 3$	0	3297	2495	2471	2485	2461	2447	2444	2466	2461	2461
$2^{27} + 3$	0	3273	2461	2450	2437	2448	2466	2461	2475	2459	2469
$2^{28} + 3$	0	3357	2508	2525	2519	2509	2548	2500	2516	2518	2532
$2^{31} + 3$	0	3304	2759	2557	2485	2471	2470	2469	2468	2477	2470
$2^{32} + 3$	0	3304	2759	2557	2485	2471	2470	2469	2468	2477	2470

a.) All the values of C_h are X 10^{-4}. The expected value of C_0 is .3333 while that of C_h is .2500. The critical region for this test is two standard deviations, that is [325,341] for C_0, and[242,258] for C_h.

b.) $X_o = 3$

TABLE V-A

KOLMOGOROV SMIRNOV TEST

GENERATOR PARAMETER VS PROBABILITY

GENERATOR	PARAMETER	VALUES		GENERATOR	PARAMETER	VALUES	
λ	μ	D	°	λ	μ	D	°
$2^7 + 1$	1	.6301	.8219	$2^9 + 1$	1	.6301	.8219
$2^2 + 3$	1	.6385	.8095	$2^3 + 1$	1	.5936	.8726
$2^6 + 3$	1	.5080	.9585	$2^7 + 1$	1	.5626	.9095
$2^9 + 3$	1	.7761	.5833	$2^{10} + 1$	1	.4179	.9948
$2^{11} + 3$	1	.5640	.9080	$2^{12} + 3$	1	.5508	.9219
$2^{16} + 3$	1	.6675	.7643	$2^4 + 3$	0	.6066	.8553
$2^8 + 3$	0	.6857	.7348	$2^9 + 3$	0	.4629	.9828
$2^{11} + 3$	0	.6103	.8503	$2^{12} + 3$	0	.5759	.8944
$2^{16} + 3$	0	1.1116	.1652	$2^{20} + 3$	0	.5579	.9146
$2^{26} + 3$	0	.6019	.8616	$2^{27} + 3$	0	.6645	.7692
$2^{30} + 3$	0	.5211	.9488	$*2^9 + 1$	1	.5825	.8865
$**2^{18} + 3$	0	.5724	.8985	$**2^{22} + 3$	1	.7578	.6139

a.) D = maximum deviation. °1 - Pro (D) gives the probability of making an error by saying the data is not U(0,1).

b.) $X_0 = 3$

* N = 121,000

** N = 121,000, X_0 = DCBABEB

TABLE V-B KOLMOGOROV SMIRMOV TEST

GENERATOR PARAMETER VS PROBABILITY

GENERATOR λ	PARAMETER μ	VALUES D	°	GENERATOR λ	PARAMETER μ	VALUES D	°
$2^{5} + 3$	0	.6844	.7369	$2^{7} + 3$	0	.6886	.7299
$2^{10} + 3$	0	.4533	.9863	$2^{14} + 3$	0	.5592	.9131
$2^{16} + 3$	0	.5937	.8725	$2^{3} + 3$	1	.7062	.7007
$2^{5} + 3$	1	.7634	.9220	$2^{9} + 3$	1	.6026	.8608
$2^{16} + 3$	1	.7232	.6722	$2^{18} + 3$	1	.3757	.9989
$2^{20} + 3$	1	.5381	.9342	$2^{22} + 3$	1	.5501	.9638
$2^{24} + 3$	1	.5331	.9386	$2^{3} + 1$	1	.7036	.7051
$2^{4} + 1$	1	.6491	.7932	$2^{8} + 1$	1	.4689	.9804
$2^{9} + 1$	1	.7817	.5741	$2^{11} + 1$	1	.5153	.9532
$2^{12} + 1$	1	.3408	.9998	$2^{13} + 1$	1	.3684	.9992
*$2^{16} + 3$	0	1.0313	.2378	**$2^{16} + 3$	0	.1044	.2251
***$2^{16} + 3$	0	.9639	.3106				

a.) D = maximum deviation. °1 - Pro (D) gives the probability of making an error by saying the data is not U(0,1).

b.) X_0 = DCBABEB

* X_0 = 1111111111

** X_0 = 1111111111, N = 121,000

*** X_0 = DCBABEB, N = 121,000