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CONTENTS

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I. INTRODUCTION

The problem of determining theoretically the heat transport through a convecting layer of fluid bounded by horizontal conducting surfaces has received much attention lately. Theoretical approaches to this problem may be placed in two categories; (1) statistical theories, and (2) direct integration of the fluid's equations of motion. Statistical theories seek to determine some of the statistically averaged properties of the flow field without computing completely the turbulent temperature and velocity fields. Its virtue is that average properties are smooth functions of space-time coordinates so that fewer parameters are needed for their description than for the chaotic velocity and temperature fields. On the other hand, direct integration of the equations of motion is certainly to be preferred in those situations in which the flow field is non turbulent, and spacially regular; it is also of interest to attempt a solution of the turbulent flow situation by this second avenue. A reason for this statement is that current statistical theories are not able to estimate accurately their errors. A suitably designed integration procedure may thus be used not only to study the flow fields directly, but also to assess the accuracy of proposed statistical theories.

Applications of some of the statistical theories to the thermal convection problem has been made for the case in which the Prandtl, number, σ, is put equal to infinity (Herring, 1969). The statistical procedures studied included the direct interaction approximation (Kraichnan (1964)) and the quarinormal approximation (an extension of that proposed by Tatsumi, Proudman, and Reid (1954), and (1957)). Of these two, only the direct interaction appears to be a physically acceptable procedure. It was found to reproduce accurately some statistical results constructed from the direct integration approach. The study was restricted to moderately small values of Rayleigh number ($R \lesssim 10^4$) and to the free boundary conditions.

An important question raised by the above study is which of the two methods – the statistical approximation or a direct integration of the amplitude equations of motion is most economical from the computational point of view. For the $\sigma \to \infty$ limit, it appears that the direct integration method may be at least as feasible as the statistical approach. The reason for this statement is that in this limit the velocity field does not boundary layer, and consequently the statistical wave number cascade of entropy and energy to large horizontal wave numbers is not very severe. The velocity and temperature fields may then be represented by relatively few horizontal wave number vectors.

A meaningful amplitude calculation must not only contain a sufficient number of mesh points (or Fourier modes) to accurately represent the small scale structure of the turbulent flow, but must as well represent its homogeniety and isotropy. It appears that present day computors can deal with such a problem provided the degree of turbulent activity is not too intense.

The present paper presents some results of a direct integration of the Boussinesq-Navier-Stokes equations for the thermal convection problem. The Fourier transform of the equations of motion are integrated forward in time until certain time average properties in the system become steady in time. The initial data for the temperature and velocity field are selected from a Gaussian distribution having horizontal homogeniety and isotropy. Results are obtained for the case in which the Prandtl number, σ, is put equal to infinity, which simplifies the dynamics of the flow by effectively eliminating the $\vec{v}\nabla\vec{v}$ nonlinearity from the momentum equation. The resulting equations should be valid provided the Reynolds number of the flow is small. This condition does not preclude turbulent flows, since the Pechlet number may be large. Our calculations here is a preliminary step toward considering the more complicated case of finite σ convection.

The use of Fourier mode techniques permits an entirely energetically consistent treatment of the nonlinear terms, and, more importantly affords a convenient basis for representing flow fields having dimensions of homogeniety. The present problem is set up for a horizontally infinite system, and the initial data are consistent with horizontal homogeniety.

Previous numerical calculations for thermal convection have been made by Fromm (1965) and Deardorff (1964). Our calculations differ from theirs in several respects. First, the present results for $\sigma = \infty$, whereas Fromm and Deardorff put $\sigma = 1$. The system we study is therefore, in this respect, simpler. However, under these simplified conditions, we carry out a three dimensional integration of the equations of motion, where the above authors restrict their results to two dimensions. Moreover, the method of integration is designed to treat (approximately) the flow situation in which the velocity and temperature fields have horizontal isotropy and homogeniety.

The results may be compared to experiments for high Prandtl number fluids. Recent experiments for such fluids have been reported by Rossby (1966), Somerscales and Dropkin (1966), Dropkin and Somerscales (1965), and most recently by Somerscales and Gazda (1968). The results of Somerscales and Gazda are of particular interest to us in that they present detailed measurements of the mean temperature field and (r.m.s.) temperature fluctuation fields for a range of Rayleigh numbers ($7.39 \times 10^5 \leq R \leq 3.21 \times 10^8$). A curious feature of the latter authors results is that the measured mean temperature field is asymetric about the midplane of the flow. Such an asymetry is compatible with the Boussinesq equations for a horizontally infinite system, but it is difficult to rationalize on the basis of theories of turbulent convection. For example, it is well known that hexagonal cellular convection produces an asymetric mean temperature field, but if the flow is time-fluctuating (as indeed the authors report) it is difficult to see how the basic hexagonal plan form can persist for a long time. Even if the flow is static, the hexagonal plan form may not be stable. Asymetrics of the sort reported by Somerscales and Gazda have also been reported by Somerscales and Dropkin, and by Dropkin, but not by Rossby.

The results of the present calculation are in good agreement with the above cited experiments with respect to the Nusselt number. Our results with respect to the r.m.s. temperature fluctuation field, θ, are in only fair agreement; the theoretical estimate of $< \theta^2 >^{1/2}$ is too large near the boundary layer by $\sim 40\%$. This overestimate of θ may be connected with the low level of turbulence activity predicted theoretically. It is not clear to what extent this particular error is due to the use of $\sigma = \infty$, and how much is due to an inadequate numerical treatment of the problem.

Our conclusions regarding the theoretical existence of an asymetric mean temperature field (of the sort found experimentally by Somerscales and Gazda) are not so simple. The calculations indicate that an initial asymetry of the mean field persists for a time long compared to a circulation time of a fluid particle. Nevertheless, there is a clear tendency in the present calculation for the asymetry to disappear, or at least to be reduced to a level much below that found by the above experimenters. For those runs for which the initial data does not induce any initial asymetry in the mean temperature field no measurable asymetries developed. Such cases include those runs started near the conduction state with completely random initial data. These results indicate that the hexagonal type plan form is not stable (against finite perturbation). The predicted plan form (insofar as one can speak of such) appears to be one in which the motion is three dimensional, and consists of two interacting rolls, and suggests that found by Rossby for large σ fluids if $R > 10^4$. The shape of the mean temperature field is similar to that predicted by the quasilinear theory. The results at large R ($R > 0.5 \times 10^6$) have the overshoot region just exterior to the boundary layer. However, the region is more diffuse in the present calculations that in the mean field calculation.

II. THEORY AND INTEGRATION PROCEDURE

(a) The equations of motion and their Fourier decomposition.

The Boussinesq-Navier-Stokes equations at infinite Prandtl number may be reduced to the following nondimensional form:

$$\nabla \cdot \vec{v} = 0 \tag{1}$$

$$-\nabla^4 \vec{v} = R\left\{\nabla \frac{\partial}{\partial Z} - \hat{k}\,\nabla^2\right\}\theta \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = \beta w - \nabla \cdot (\vec{v}\theta - \overline{v\theta}) \tag{3}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial Z^2}\right)\overline{T} = -\frac{\partial}{\partial Z}\overline{w\theta} \tag{4}$$

Here, $\vec{v}$ in the fluid velocity field, $T = \theta + \overline{T}$ is the temperature field, $\overline{T}$ is the horizontal average of the temperature field, R is the Rayleigh number, $\hat{k}$ is a unit vector in the vertical direction, β is $(-d\overline{T}/dz)$, and w is v_z. The infinite conduction – rigid boundary conditions imply that (Chandrasekhar (1960)):

$$T(0,\ t) = 0$$

$$T(Z = 1,\ t) = -1$$

$$w(0,t) = \frac{\partial w}{\partial Z}(0,t) = w(1,t) = \frac{\partial w}{\partial Z}(1,t) = 0.$$

These equations for T are appropriate for perfectly conducting boundaries. The nondimensionalization is such that the top plate is at Z = 1. We do not impose (explicitly) any lateral boundaries, in view of the fact that we are interested in the solution of (1) - (4) under the condition of statistical (horizontal) homogenuity and isotropy. A convenient way of treating such a system is to use Fourier transform techniques, in the horizontal. We use it also in the vertical. Let θ_n^α denote the component of $\theta(\vec{r})$ proportional to $\sin n\pi Z\, e^{r\pi\vec{\alpha}\cdot(\hat{i}x+\hat{j}g)}$, and let β_n be the component of $-(d\overline{T}/dZ)$ proportional to $\cos n\pi Z$. Then θ_n^α and β_n satisfy,

$$\left(\frac{\partial}{\partial t} + \nu_n^\alpha\right)\theta_n^\alpha = \sum_m B_{nm} w_m^\alpha + \frac{1}{2}\sum_{\alpha=\alpha'+\alpha''} C_{npq}^{\alpha\alpha'\alpha''}\, w_p^{\alpha'}\theta_q^{\alpha''} \tag{5}$$

$$\left(\frac{d}{dt} + n^2\right)\beta_n = -\frac{n^2\pi^2}{2}\sum_m\sum_\alpha\left(w_m^{-\alpha}\ \theta_{m+n}^\alpha - \sigma|n-m|\,\theta_{|n+m|}^\alpha\right) \tag{6}$$

where

$$w_n^\alpha = \sum_m J_{nm}^\alpha\ \theta_m^\alpha\,,$$

$$\nu_n^\alpha = n^2 + \alpha^2$$

$$J^{\alpha}_{nm} = \frac{R}{\pi 4} \frac{\alpha^2}{(n^2 - \alpha^2)^2} \left[\delta_{nm} - \frac{4 n m \alpha \left((-1)^{n+m} + 1 \right) \left((-1)^n \cosh \pi \alpha - 1 \right)}{\pi (m^2 + \alpha^2)^2 \left((-1)^n \sinh \pi \alpha - \pi \alpha \right)} \right],$$

and

$$\frac{2}{\pi} C^{\alpha\alpha'\alpha''}_{npq} = \left(q + \frac{p}{2} - \frac{p}{2} \cdot \frac{\alpha' - \alpha'' \alpha}{\alpha' \alpha} \right) \delta_{n,p+q}$$

$$+ \left(p - \frac{p}{2} + \frac{p}{2} \cdot \frac{\alpha^2 - \alpha''^2}{\alpha'^2} \right) \sigma(p - q) \left(\delta_{p,n+q} + \delta_{q,n+p} \right).$$

A derivation and discussion of these equations have been presented elsewhere (Herring (1964)). Here we only note that the J^{α}_{nm} factor serves to eliminate the velocity field from the $\sigma = \infty$ system. The last term in the definition of J represents the effect of the rigid boundary conditions.

(b) Initial Value Problem for Homogenious and Isotropic System

Consider those solutions of (1), (2), (3), and (4) which possess horizontal homogeniety and isotropy. By this we mean that the spacial average of the $\vec{v}$ - T fields (averaged over a finite but large area A) are independent of the horizontal position of A, and depend only on the vertical distance from the lower boundary. The linear dimensions of A should be large compared to correlation lengths such as

$$\ell = \int T(x)\, T(x + L)\, dL / \int (T(x))^2\, dL.$$

To set up such solutions as an initial value problem we employ the Fourier amplitude equations for θ_n^α and T_n. First the wave number space $\vec{\alpha}$ is descritized to $\vec{\alpha}$ i, and (5) and (6) are replaced by a finite set of N equations for $\theta_n^{\alpha i}$ and T_n. At t = 0, the $\theta_n^{\alpha i}$ are assigned random values by selecting each complex number $\theta_n^{\alpha i}$ from a Gaussian set of random numbers. The randomization is with respect to $\vec{\alpha}$i only and correlations between modes θ_n^α $\theta_m^{-\alpha}$, ($n \neq m$) are permitted. Indeed, such correlations are necessary if the initial temperature field boundary layers. If we now take the limit of such a system as $N \to \infty$ (in such a way that the $\vec{\alpha}$i's "fill up" the entire finite wave number plane) then the initial temperature field T ($\vec{r}$, 0) becomes spacially homogenious and isotropic. We note that this method of posing the problem is one for which horizontal averaging is identical to ensemble averaging over the initial set of randomly selected amplitudes.

In practice we must always deal with a finite number of the $\vec{\alpha}$i's, and n's (vertical wavenumber). With the above procedure for selecting initial data, the truncated system can only achieve approximate isotropy and homogeniety. Moreover the subsequent development of the system will be spuriously affected by the finite dimensionally of the system (N). Suppose, for example, a particular wave number in a finite system is (accidentally) strongly excited at the initial time. This mode (or perhaps another to which this mode transfers its energy by nonlinear interactions) will eventually have a dominant role in the system. If the number of degrees of freedom of the system were very large, then the chance over excitation of any mode would be correspondingly small; consequently it would take a very long time for it or the mode to which it transfers its energy it feeds to become dominant.

In the mean time the average properties of the flow come to quasi-steady equilibrium. It is the properties of the system during this quasi-steady state which interests us. In the limit as the number of modes becomes infinite, this quasi-steady state becomes a time stationary state. No mention has been made here as to whether the flow is turbulent. Presumably, if the experimental flow is turbulent the above method will give the turbulent results. If however the flow is cellular and static, the present calculation may differ in principle from experiments.

Because of practical limitations in the size of the calculations, it is important to have some approximate measure of the degree of horizontal isotropy of the T-field. It is measured by a two dimensional tensor,

$$I_{ij} = \int d\Omega \, \frac{\partial T}{\partial Xi} \, \frac{\partial T}{\partial Xi} \Big/ \int d\Omega \, (\nabla_I T)^2 .$$

Here,

$$\nabla_I \equiv \hat{i}\, \partial/\partial i + \hat{j}\, \partial/\partial Xj \text{ , and } Xi \text{ , } Xj$$

are horizontal orthogonal coordinates. The volume of integration is over the entire field. An elipse may be associated with I in the usual way. If the flow is isotropic in the X_1, X_2 plane, the semimajor and semiminor axes are equal, for flows which only depends on one dimension, the semiminor axis is zero. We define an isotropy parameter p as

$$p = 1 - (a - b)/(\alpha + b).$$

Here a is the semimajor axis and b is the semiminor axis. The p-parameter does not completely characterize the angular distribution of the flow, since $p = 0$ for the nonisotropic hexagonal or square plan form also. p characterizes the second moment of the angular distribution only. A complete characterization of the angular distribution, needs a specification of the rest of the moments.

(c) Method of Truncating the Wave Number Spectrum.

The truncation procedure used here to reduce the infinite system of equations for θ_n^α and β_n to a finite set guarantees two important properties of the flow: (1) entropy conservation and (2) the prescription must be appropriate to describe an approximate horizontally isotropic system. Entropy conservation is simplest to guarantee: one may verify from equations (5) and (6) that any truncation prescription which selects an arbitrary set of $\vec{\alpha}\,i$ and the first N vertical wave number for $\theta_n^{\alpha i}$, and the first 2N vertical wave number for β_n satisfies a finite wave number version of the entropy conservation law:

$$\frac{d}{dt} \sum_{n,\alpha}^{N} |\theta_n^\alpha|^2 + \sum_{n}^{2N} |\overline{T}|^2 + \sum_{n,\alpha}^{N} v_n^\alpha \, |\theta_n^\alpha|^2 + \sum_{n=1}^{2N} v_n^\circ \, |\overline{T}_n|^2 = 1 + \pi \sum_{n=1}^{2N} n\overline{T}_n \tag{7}$$

In applying the above truncation prescription, the convolution term in (5) is taken to be non zero only if the three wave numbers α, α', α'' satisfy $\vec{\alpha} = \vec{\alpha}' + \vec{\alpha}''$. How to choose the set $\vec{\alpha}i$ so as to best achieve the goal of isotropy in the horizontal is not so clear. Two possible methods suggest themselves.

The first method is prescribed by generating a set of $\vec{\alpha}_1$'s by the equation

$$\vec{\alpha}_i = (\hat{i}n + \hat{j}m)C, \ (n, m) \equiv i = (0, 1, 2, \cdots), \tag{8}$$

and then discarding any $\vec{\alpha}i$ for which

$$|\vec{\alpha}_1| < \alpha_0, \text{ or } |\vec{\alpha}_1| > \alpha_1.$$

Here α_0 and α_1 are arbitrary cut off wave numbers and c is an arbitrary constant. For a calculation to be realistic α_0 and α_1 must be outside the entropy containing region of the spectrum. It is clear that for a finite system the above procedure favor square plan forms.

A second method consists in first generating a set of constant length vector by means of the formula,

$$\vec{\alpha}p = \left(\hat{i}\cos\frac{p\pi}{N^*} + \hat{j}\ \sin\frac{p\pi}{N^*}\right)\alpha_0, \ p = 0, 1, \ \cdots N^*$$

and then constructing all sums and differences $\vec{\alpha}_q$ of this set for which $\alpha_0 \leq |\alpha_q| \leq \alpha_r$ This gives exact isotropy under a finite rotation of angle $(P/N^*)\pi$, $P = 0, 1, 2, \ldots N^*$, and does not favor any plan form unless N^* is divisible by 4 or 6, for which square or triangular (hexagonal) plan forms are preferred. This second method gives a larger number of nonlinear terms than the first for a given total number of modes included in the system.

(d) Numerical Integration Procedure

The numerical integration method consists in a modified second order predictor corrector technique. This relatively low order of the scheme is dictated by machine storage considerations. The modification referred to consists chiefly of a stabilization of the high wave number modes. Equations (5) and (6) may be symbolically written as

$$\frac{d}{dt} + \nu \quad X = F(x).$$

A formal integration of this equation yields

$$X(t + \Delta) = X(t)\, e^{-v\Delta} + \int_0^{\Delta} dt'\, e^{-vt'}\, F(X(t')).$$

From this equation we extract the following (constant Δ) second order predictor correction scheme:

$$y(t + \Delta) = X(t)\, e^{-v\Delta} + \frac{(1 - e^{-v\Delta})}{2v} \left\{(1 - a)F(X(t) - \Delta)) + aF(X(t))\right\} \tag{10}$$

$$X(t + \Delta) = X(t)\, e^{-v\Delta} + \frac{(1 - e^{-v\Delta})}{2v} \left\{F(X(t)) + F(y(t + \Delta))\right\} \tag{11}$$

Here $y(t + \Delta)$ is the predicted value of $X(t + \Delta)$; $\underline{a}$ is an as yet arbitrary number, whose value affects the stability (but not the order of accuracy) of the scheme. If $a = 1$ the above procedure (except for the $e^{-v\Delta}$-factor) is the standard Adams-Moulton method.

Methods based on (10) and (11) are not entirely compatible with the entropy constraint (7). To see this, consider the simple case in which $\nu = 0$, and $\overline{T} = 0$. Then (7) implies that

$$\sum_{n,\alpha} |\theta_n^\alpha|^2$$

is a constant of motion, while (10) and (11) give

$$\Sigma\, |\theta_n^\alpha (t + \Delta)|^2 = \Sigma\, |\theta_n^\alpha (t)|^2 + \theta(\Delta^3 X).$$

We chose $\underline{a}$ so that the long time average of the entropy constraint is most nearly satisfied. Clearly one way of doing this is to pick a value of $\underline{a}$ each time step which forces (7) to be satisfied. However, this is too complicated to be practical, and some numerical experimentation persuaded us that $a = -1/2$ is the best, simplest overall choice. Nordsieck has pointed out that the choice $a = -1/2$ (for second order integration schemes) best preserves the finite time step reversibility condition for reversible systems. He also points out that this scheme is the most unstable. However, in our system, the present scheme is stabilized by conductivity factors $e^{-v\Delta}$ serve to stabilize.

An integration technique which preserves the entropy constraints exactly is to transform dependent variables from the (complex) θ_n^α field to variables E and ψ, where $\theta = \sqrt{E}\, e^{i\psi}$. Such a scheme was tried, but gave excessively large error for E near zero because of stability problems.

IV. RESULTS AND DISCUSSION

(a) Mean Properties of the Flow

Results for the mean temperature field $\bar{T}(Z)$ and the r.m.s. temperature field $\psi(Z) = (\langle T^2 (r,t)\rangle - \langle T\rangle^2)^{1/2}$ are presented in figures (1) - (2) for a range of Rayleigh numbers $R = 4 \times 10^3$, 10^4, 10^5, and 7.39×10^5. These results are both horizontally and time averaged. The horizontal wave number spectrum is determined by the second method described in section IIIC, and the parameters are, $N^* = 6$, $\alpha_0 = 1 \sqrt{2}$, $\alpha_1 = \sqrt{2}$. The number of vertical wave numbers ranges from 5 for $R = 4000$ to 20 for $R = 10^6$. We postpone till section IVC a discussion and estimation of the wave number truncation error. The time averaging of the ψ and $\bar{T}$ fields was carried out only over the last portion of the time displacement of the system - after the excursion in N_u had become less than 10%. Time integrations were stopped when the time average of N_u become steady to three significant figures. The isotropy parameters for the $R = 4 \times 10^3$ and $R = 10^5$ runs are shown in figure 3.

Results for the averaged Nusselt number are shown in figure 4. The range of Rayleigh number covered does not warrant an extraction of a power law from the data. However, if one assumes a $N_u \sim R^{1/3}$ law (as has been analytically forecast by Roberson for two dimensional convection) then a continued fraction extrapolation of the data predicts

$$N_u \to 0.945\, R^{1/3} \quad \text{for} \quad R \to \infty .$$

The points in figure 4 give the experimental results of the author cited in the introduction.

(b) Comparison of Present Calculation with Quasilinear Theory and with Experiments

It is of interest to compare the present results with those of the quasilinear theory, which simplifies the basic equations (1), (2), and (3) by deleting the last term in equation (3). A comparison of the two calculations is presented in figure (5) for the mean temperature field $\overline{T}(z)$, at $R = 10^5$. For the quasilinear system, $N_u = 11.5$ while for the complete system $N_u = 9.18$. The main difference between the two calculations is seen to be that the overshoot region, just exterior to the boundary layer is much more diffuse according to the present calculation than according to the quasilinear calculation. The inclusion of the fluctuating-self interactions does not completely remove the overshoot region, as was one time thought (Herring (1964)).

The numerical results may also be compared to experiments at high Prandtl number. Figures 6 and 7 give a comparison of the present results for $\overline{T}$ and

$\psi^{1/2}$ at R = 7.39 × 10^5 with the recent experimental results of Somerscales and Gazda. The most important differences between the present results and the experimental results is the fact that the present results for $\bar{T}$ and ψ are very nearly symmetric about the mid-plane of the flow. Such a high degree of symmetry is not a result of time averaging but is also obtained at any instant provided the system is near the steady state. This disparity is curious. Within the framework of the Boussinerg approximation, an asymmetric $\bar{T}$ and ψ fields may be expected if the large scale part of the flow field is nearly statis and if the plan form of the motion is more or less hexagonal. By this we mean that the θ(x, y, z)-field has its large scale (vertical) structure dominated by modes θ^{α}, $\theta^{\alpha'}$, and $\theta^{\alpha''}$ ($\alpha = \alpha' + \alpha''$) such that θ^{α}, $\theta^{\alpha'}$ and $\theta^{\alpha''}$ are in phase (for an exact hexagonal plan form, $\theta^{\alpha} = \theta^{\alpha'} = \theta^{\alpha''}$). The asymmetry corresponds to ascending (descending) columns of fluid of high velocity but low crossectional area and descending (ascending) columns of fluid of small velocity but large crossectional area. The direction of asymmetry is arbitrary in the Boussinerg approximation and is determined by the slight dependence of viscosity on the density. Our calculations admit such a solution-provided these solutions are stable to finite perturbations. Apparently these solutions are not stable in as much as the solution we find passes a near symmetric $\bar{T}$ field. Our solutions compound to having the large scale part of the θ(x, y, z) field discribable by modes θ^{α}, $\theta^{\alpha'}$, and $\theta^{\alpha''}$ ($\alpha = \alpha' + \alpha''$) such that $\theta^{\alpha'} \simeq \theta^{\alpha''}$, and θ^{α} being (approximately) the harmonic of $\theta^{\alpha'}$ and $\theta^{\alpha''}$. If only these three modes are included in the calculation, and if the solution is static, then it may be verified that the solution has exact mid-plane symetry. This solution corresponds to two roll structures superimposed at an angle of 30°. A possible explanation of the disparity is that Somerscales and Gazda actually sampled the temperature along a line in a convective cell, which remained fixed in space during the course of the measurement.

The computed ψ field shown in figure 7 overestimates the r.m.s. temperature fluctuation field near the boundary layer by about 40%. This overestimation is probably due to the absence of strong small scale turbulence in the numerical experiment, and to its presence in the real experiment. What is not clear is how much of the difference is due to the fact that our numerical results are at infinite σ, and the experiment at finite σ, and how much is due to the inadequate numerical representation of the small scale structures.

The computed values of the Nusselt number are in good agreement with experiment (N_v ($R = 7.39 \times 10^5$) = 9.03 compared to our 9.18). However, this agreement is somewhat accidental, and depends on our arbitrary choice of α_0. We shall examine in section IVC the dependence of the $\overline{T}$-field on α_0; it turns out that the value of $N_v = -(d\overline{T}/dz)_{z=0}$ depends more sensitively on α_0 than $(d\overline{T}/dz)$ at other values of Z.

(c) Discussion of Accuracy of the Calculation

The two important sources of numerical errors is the present calculation are time descritization error and wave number truncation errors. To eliminate the time step errors, the time step was decreased until the evolution of certain selected quantities whose accurate evolution was desired were predicted with tolerable accuracy. These quantities were the Nusselt number, the isotropy parameter, p, and the entropy spectrum defined by

$$S_n^\alpha = \sum_{\vec{\alpha}=|\alpha|} \theta_n^{\vec{\alpha}} \, \theta_n^{-\vec{\alpha}} \, .$$

Naturally, this means that certain other quantities, for example θ_n^α for α and n in large, are not predicted with great accuracy. The time steps were, $\Delta t = 0.01$, for R = 4,000; $\Delta t = 0.025$, for $R = 10^4$; $\Delta t = 0.0025$ for $R = 10^5$, and $\Delta t = 0.001$, for $R = 7.39 \times 10^5$.

Wave number truncation errors exist both with respect to the vertical cut of wave number, n_0, and with respect to the horizontal wave number cut offs α_0 and α_1. The appearance of significant vertical wave number truncation error is manifest by spurious oscilations in the mean field of frequency πn_0. Such errors thus easily recognized are easily eliminated.

Assessment of horizontal wave number truncation error is more difficult, since such errors do not manifest themselves in terms of easily recognizable unphysical behavior for $\theta(\vec{r}, t)$ or $\overline{T}(\vec{r}, t)$. For example, the θ and $\overline{T}$ fields for the quasilinear system behave quite reasonably despite the fact that these amplitude contain only one α. It appears plausable that these errors are eliminated for a given vertical wave number n if $S_n^{\alpha_0}$ and $S_n^{\alpha_1}$ are much smaller than the peak value of S_n^{α}: in other words the spectrum S_n^{α} should be adequately represented within a wave number band α_0, α_1. Data on this point is presented in Table I for $R = 7.39 \times 10^5$. From these data, it appears that there is little truncation error near α_0. Also, there appears to be little error in the large scale part (the odd modes) of the S_n^{α}-field. However, the even modes appear to have an appreciable error, since near α_1 these intensities are maximum. These errors, however produce little change in the mean field or $\psi(z)$-field.

Another assessment of the α-truncation error may be obtained by rescaling the α-spectrum by multiplying each value of $\vec{\alpha}$ by a constant factor. Such a scale change amounts to selecting a new (and in principle exact) solution to the $\overline{T}$, θ system. If both of the $\vec{\alpha}$ spectra cover the energetically significant portion of α-span one expects little difference in the horizontally averaged properties of the flow. However, our results cover only a small annulus in α-space, so that it is important to have some measure of the difference. Figure 8 and 9 give a comparison of the $\overline{T}$ and ψ-fields for $\alpha_0 = 1/\sqrt{2}$ and $\alpha_0 = 1.0$. The respective values of N_u are 9.18 and 10.21. We observe that the two mean fields are indistinguishable, with only slight differences in the ψ fields. Apparently N_u is the most sensitive to a rescaling of the α spectrum of all the mean field quantities. This fact seems to vitiate any claim to predicting accurately the N_u versus R curve at large R. However, we note that of $\overline{T}$ and ψ are insensitive to the above variations of α_0.

V. CONCLUDING COMMENTS

In the present paper we have developed numerical solutions to the full scaled three dimensional thermal convection problem, at infinite Prandtl number, and for rigid boundaries. The procedure used was designed to give an approximate account of horizontally homogeneous and isotropic flow situations. To implement this we used the Fourier transform of the equations of motion. Our results, which included a maximum of 36 horizontal wave number vectors and 20 vertical wave numbers appear to adequately describe the flow fill up to $R = 10^5$; beyond this R the results appear to show horizontal wave number truncation error. This error seems to affect the boundary slope of the mean temperature field more than other mean quantities, such as $\overline{T}$ and ψ. Despite some numerical uncertainties, certain of the qualitative features of the flow fill are predicted with reasonable confidence.

The mean temperature field $\overline{T}$ developes a negative gradient just exterior to the boundary layer, similar to that predicted by the quasilinear method. Thus the presence of the fluctuating-self-interactions does not entirely remove the negative gradient region, but does weaken it and makes it more diffusive. This type of mean temperature field is probably characteristic of large σ fluids, and appears to result from the fact that at large σ the flow is characterized by large scale velocity field sweeping small scale temperature structures into the boundary layer.

We do not find the large asymetrics in the mean temperature field reported by Somerscales and Gazda. Moreover, it does not appear likely that this result depends on wave number truncation error, since this behavior is primarily connected with the large scale part of the T-field. Possible reasons for this discrepancy are (1) the possibility that Somerscales and Gazda measured not the horizontal mean temperature field but rather the temperature field at a fixed vertical position in a slowly evolving convective cell, (2) in the experiment, the vertical boundaries play a significant role, (3) the finite (but small) value of Reynolds number in the experiment affects the results. At any rate, it appears that the hexagonal type asymetry in the mean field is definitely not stable with respect to finite perturbations.

The dynamic of the flow as emerge from the numerical calculation consists of almost static large scale cells with more rapidly fluctuating small scale structures superimposed on them. The flow field is three dimensional (it does not degenerate into rolls) large scale part converts (roughly speaking) of rolls superimposed at an angle of 30°. The precise value of this angle depends on our method of selecting the α-spectrum, and no great significance shall be attacked to its value.

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Table 1

$\psi(\dot{n}, \alpha)$

n/α 0.7425	1.0253	1.2374	1.3789	Total
0.3877E-04	0.7914E-04	0.3942E-04	0.1644E-04	0.1738E-03
0.1707E-04	0.2105E-04	0.4508E-04	0.3409E-02	0.3492E-02
0.6827E-05	0.1808E-03	0.5760E-04	0.5694E-04	0.3022E-03
0.1734E-04	0.3224E-04	0.7151E-04	0.8746E-03	0.9957E-03
0.6862E-04	0.1072E-03	0.8646E-04	0.1060E-03	0.3683E-03
0.6419E-05	0.1641E-04	0.1684E-04	0.5760E-03	0.6156E-03
0.1486E-04	0.1586E-04	0.3266E-04	0.2946E-04	0.9284E-04
0.3482E-05	0.1072E-04	0.1386E-04	0.4128E-03	0.4409E-03
0.2083E-05	0.8242E-05	0.4788E-05	0.4889E-05	0.2000E-04
0.3422E-05	0.1008E-05	0.5775E-05	0.1768E-03	0.1870E-03
0.1507E-05	0.2558E-05	0.2116E-05	0.3358E-05	0.9539E-05
0.5182E-06	0.1501E-05	0.1066E-05	0.6728E-04	0.7036E-04
0.5152E-06	0.8817E-06	0.7412E-06	0.5385E-06	0.2677E-05
0.4823E-06	0.6162E-06	0.5470E-06	0.1336E-04	0.1501E-04
0.1824E-06	0.5188E-06	0.2070E-06	0.1040E-06	0.1012E-05
0.1501E-06	0.6226E-07	0.1203E-06	0.2557E-05	0.2890E-05
0.5348E-07	0.1044E-07	0.6024E-08	0.1086E-06	0.1786E-06
0.2157E-07	0.4123E-07	0.1550E-07	0.1700E-06	0.2483E-06
0.4877E-07	0.1338E-07	0.2043E-08	0.2604E-07	0.9024E-07
0.1041E-07	0.2876E-08	0.6730E-08	0.1404E-06	0.1604E-06

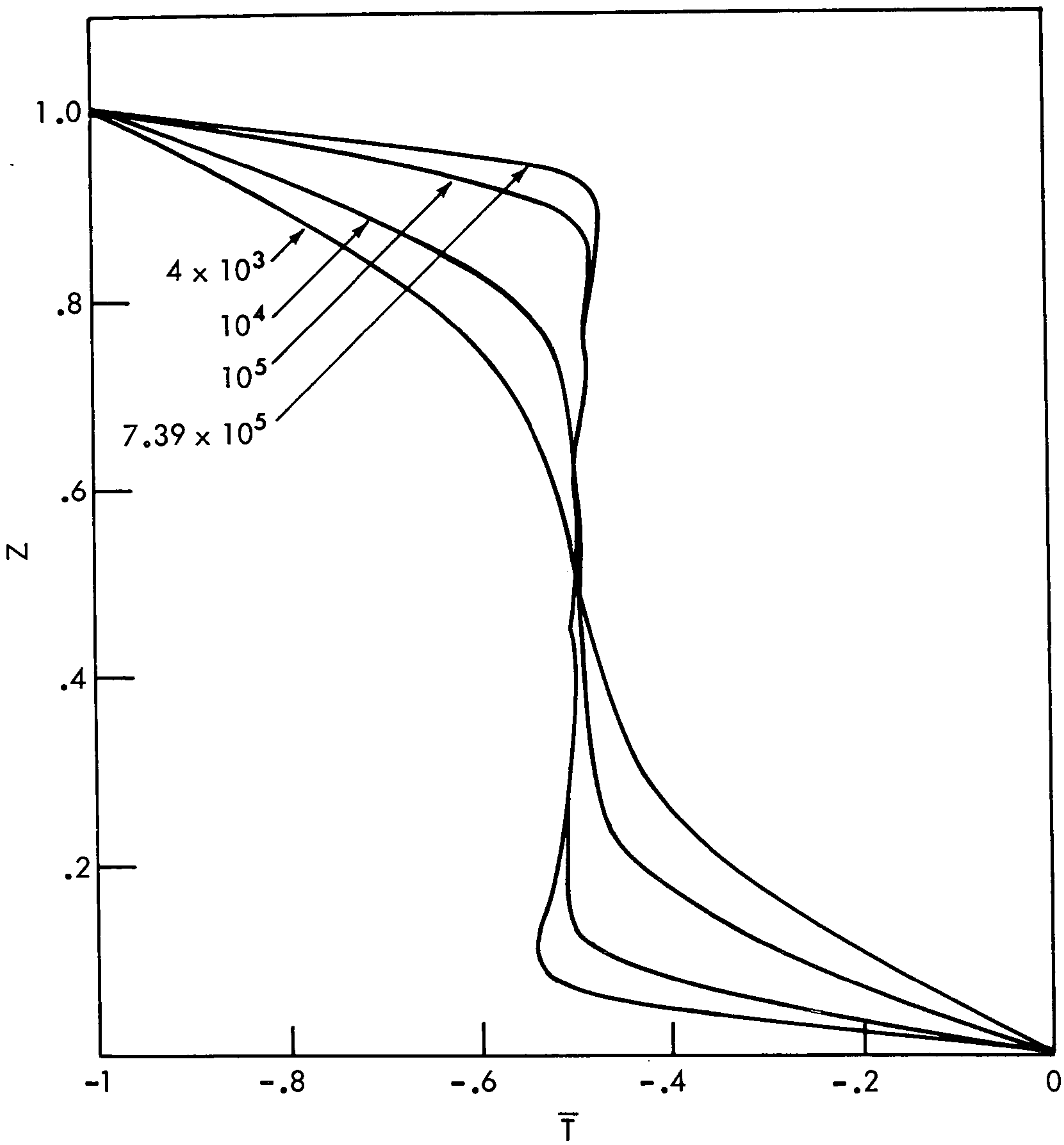

Figure 1. Mean temperature, $\overline{T}$, as a function of z for $R = 4.10^3$, 10^4, 10^5, and $7.39* 10^5$.

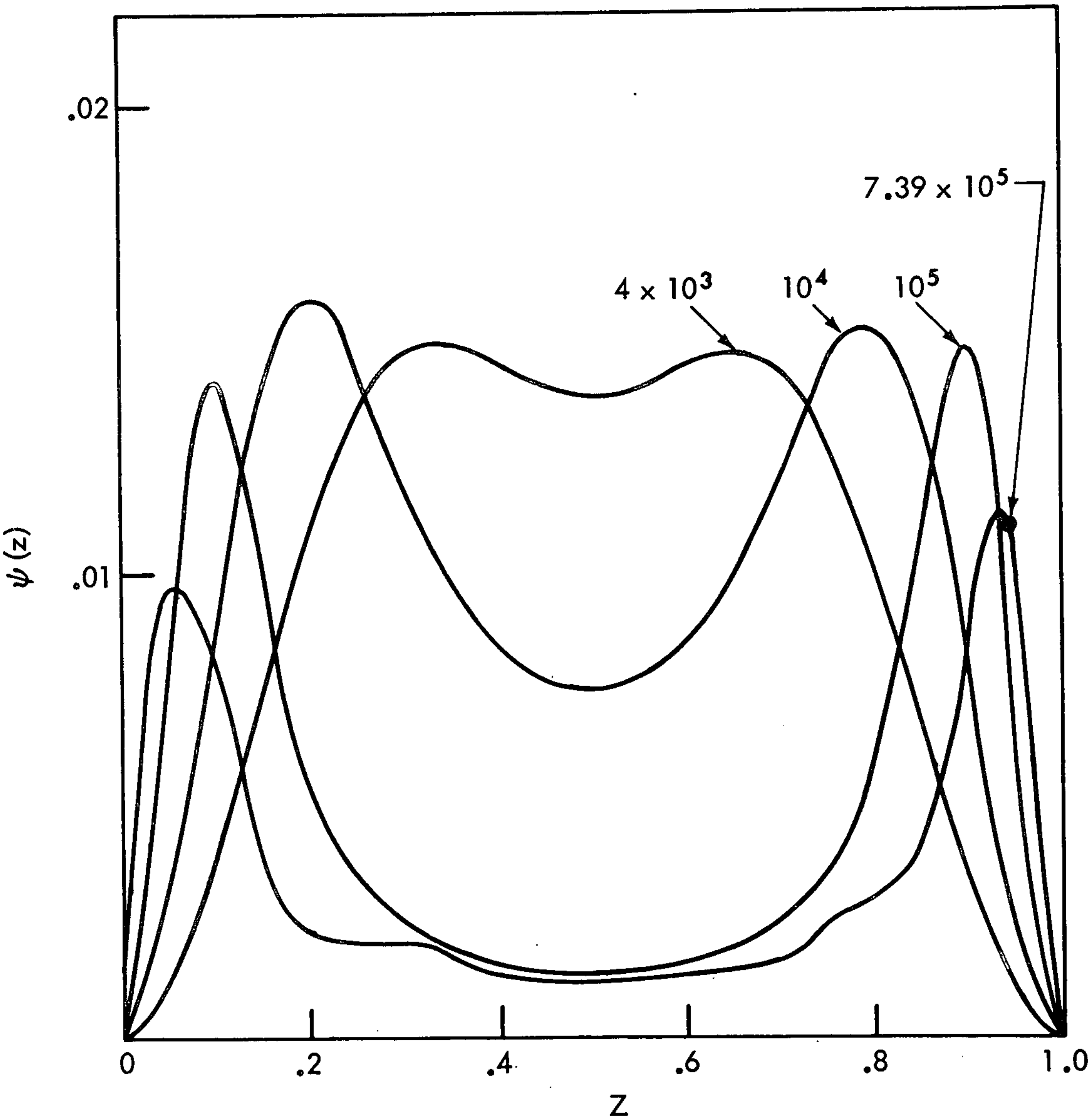

Figure 2. Mean squared temperature fluctuation ψ, as a function of z $R = 4{*}10^3$, 10^4, 10^5, and $7.39{*}10^5$.

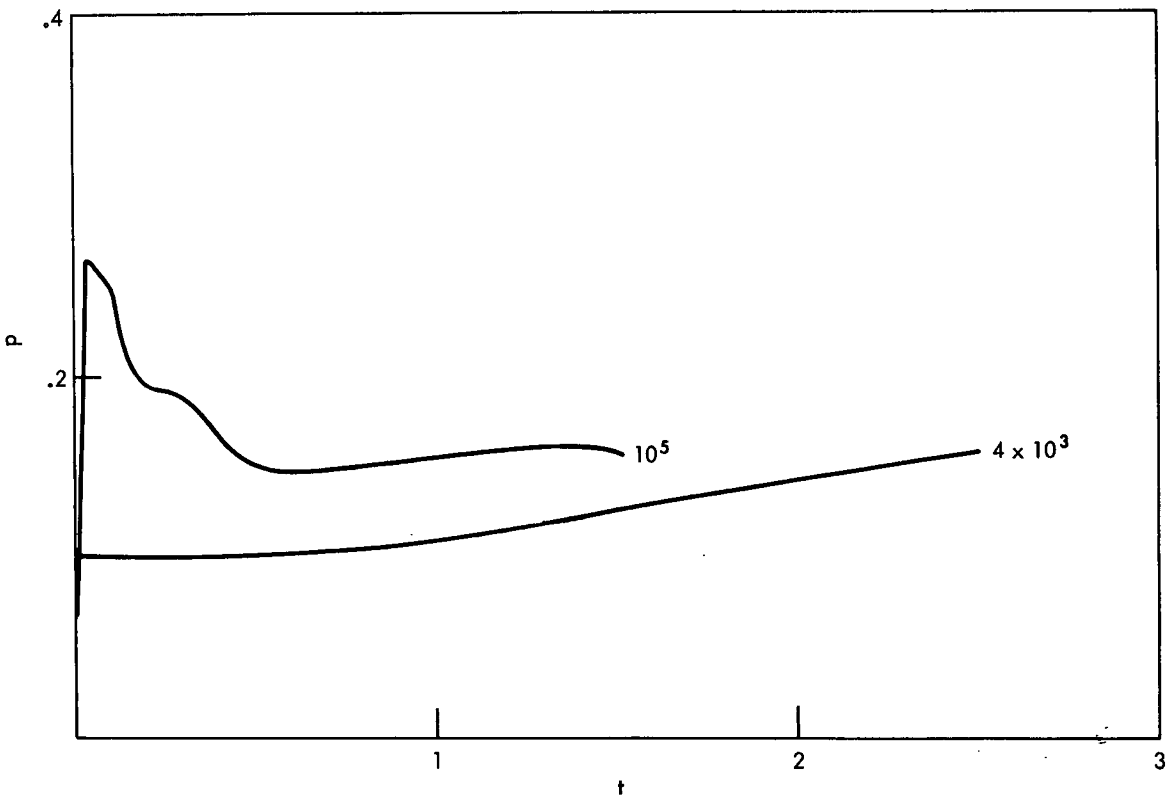

Figure 3. Isotropy parameter (see text), p as a function of time for $R = 4{*}10^3$, and 10^5.

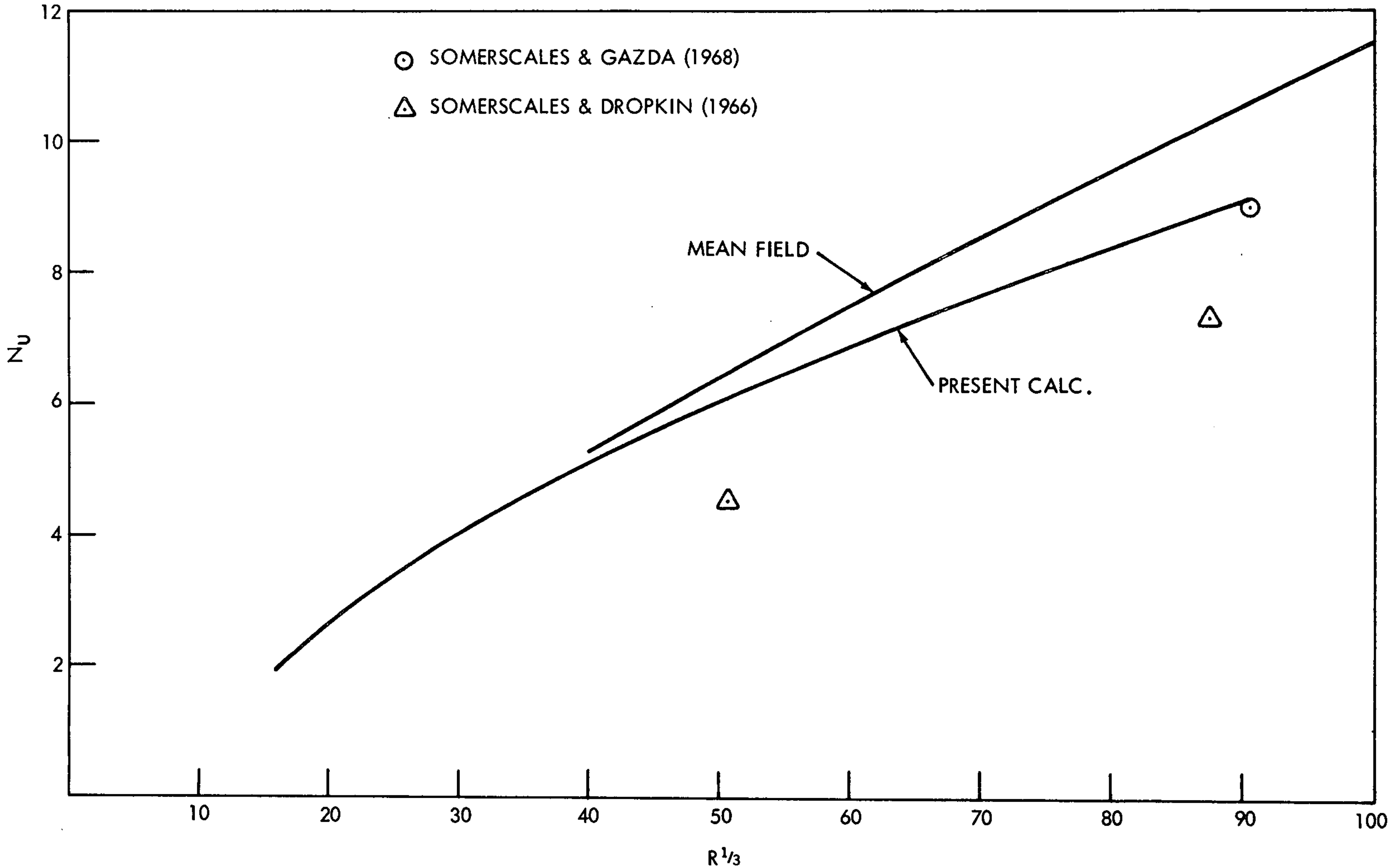

Figure 4. Nusselt number as a function of R for quasi-linear and present calculation. Points, ⊙, give Somerscales and Gazada (1968) experiments, and points, △, give Somerscales and Dropkin (1966) experiment.

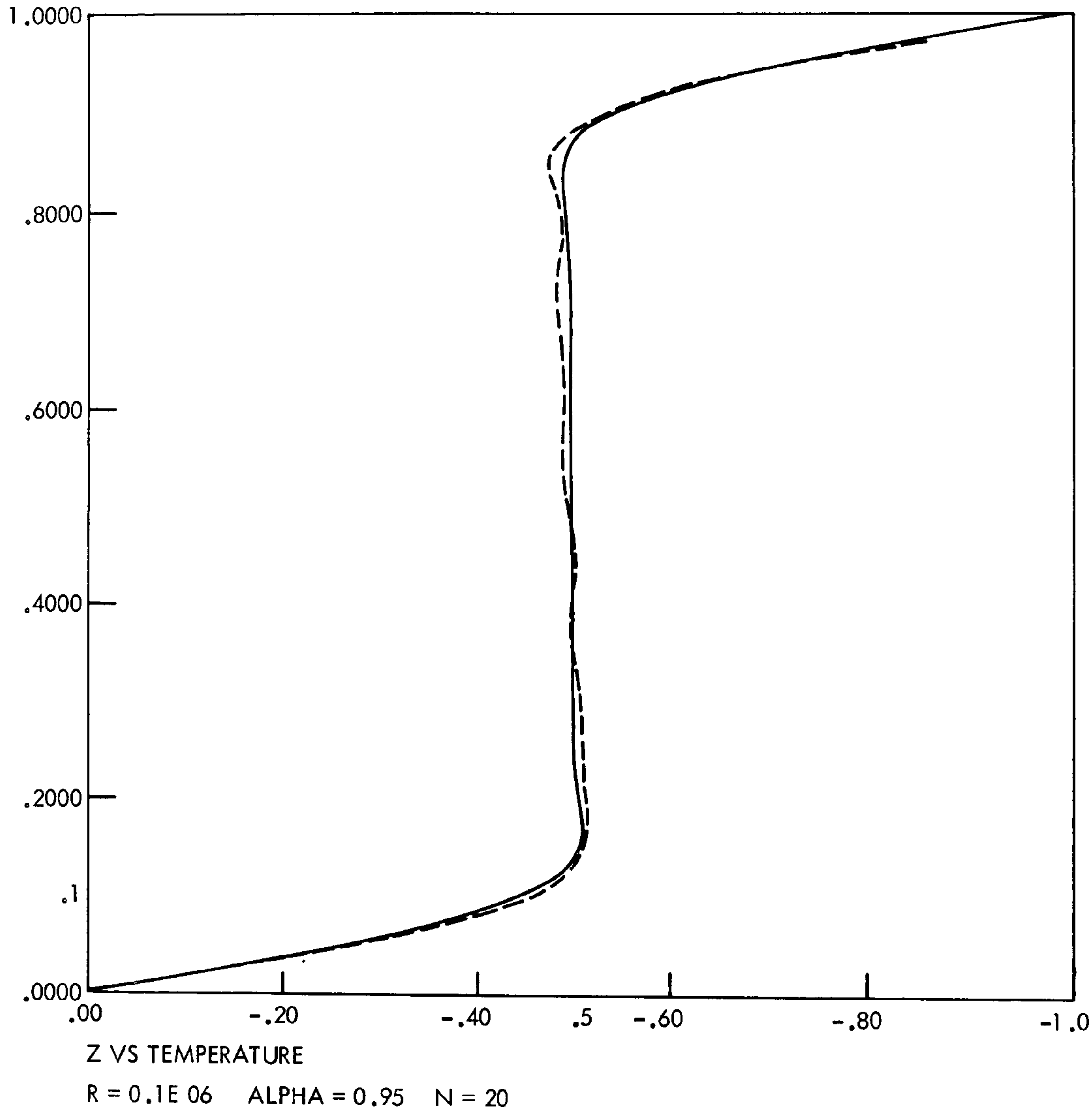

Figure 5. Comparison of present mean temperature profile, $\overline{T}$, with quasi-linear theory. Here $R = 10^5$.

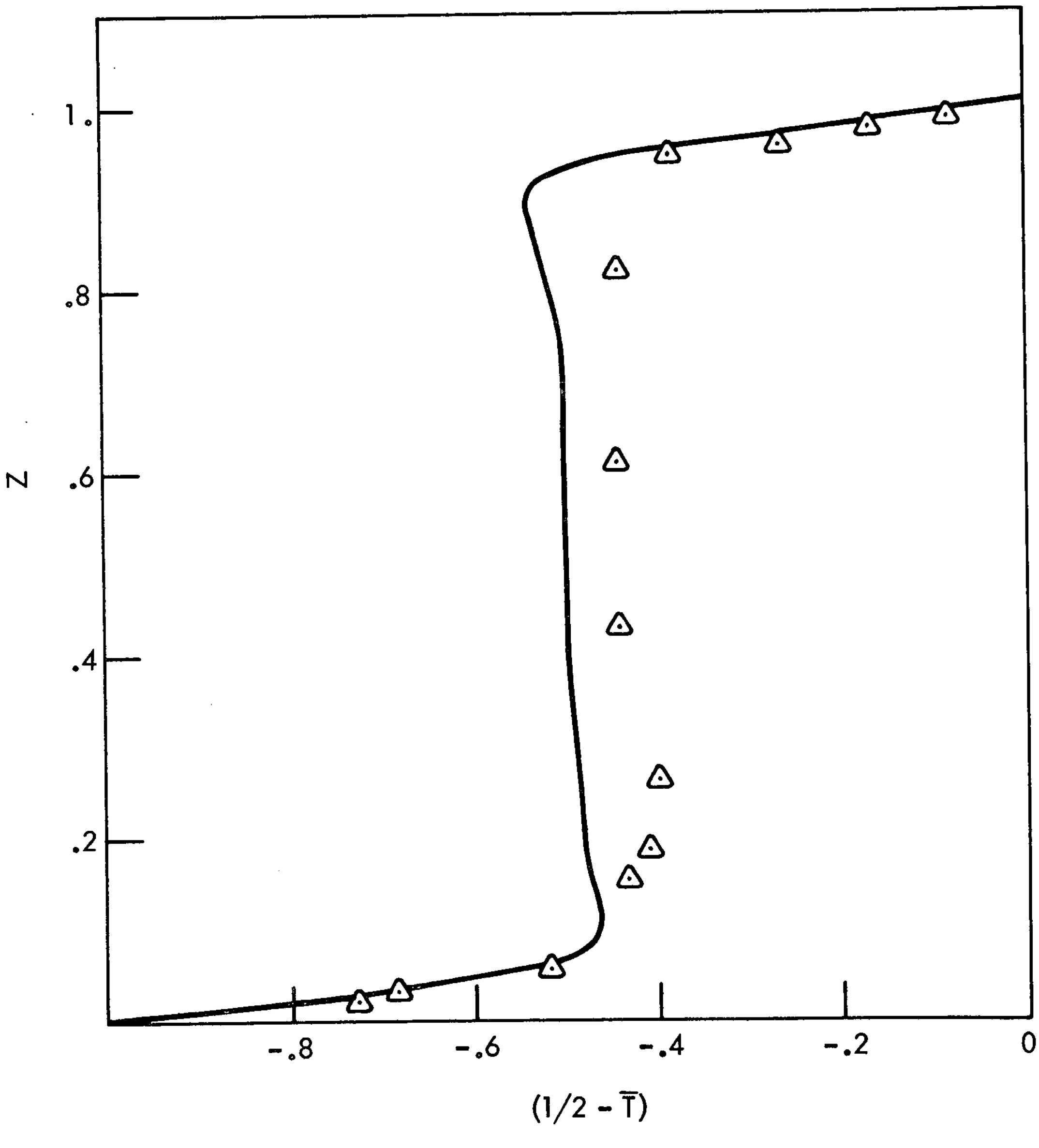

Figure 6. Comparison of present calculation of $\overline{T}$ with experiment of Somerscale and Gazda (1968).

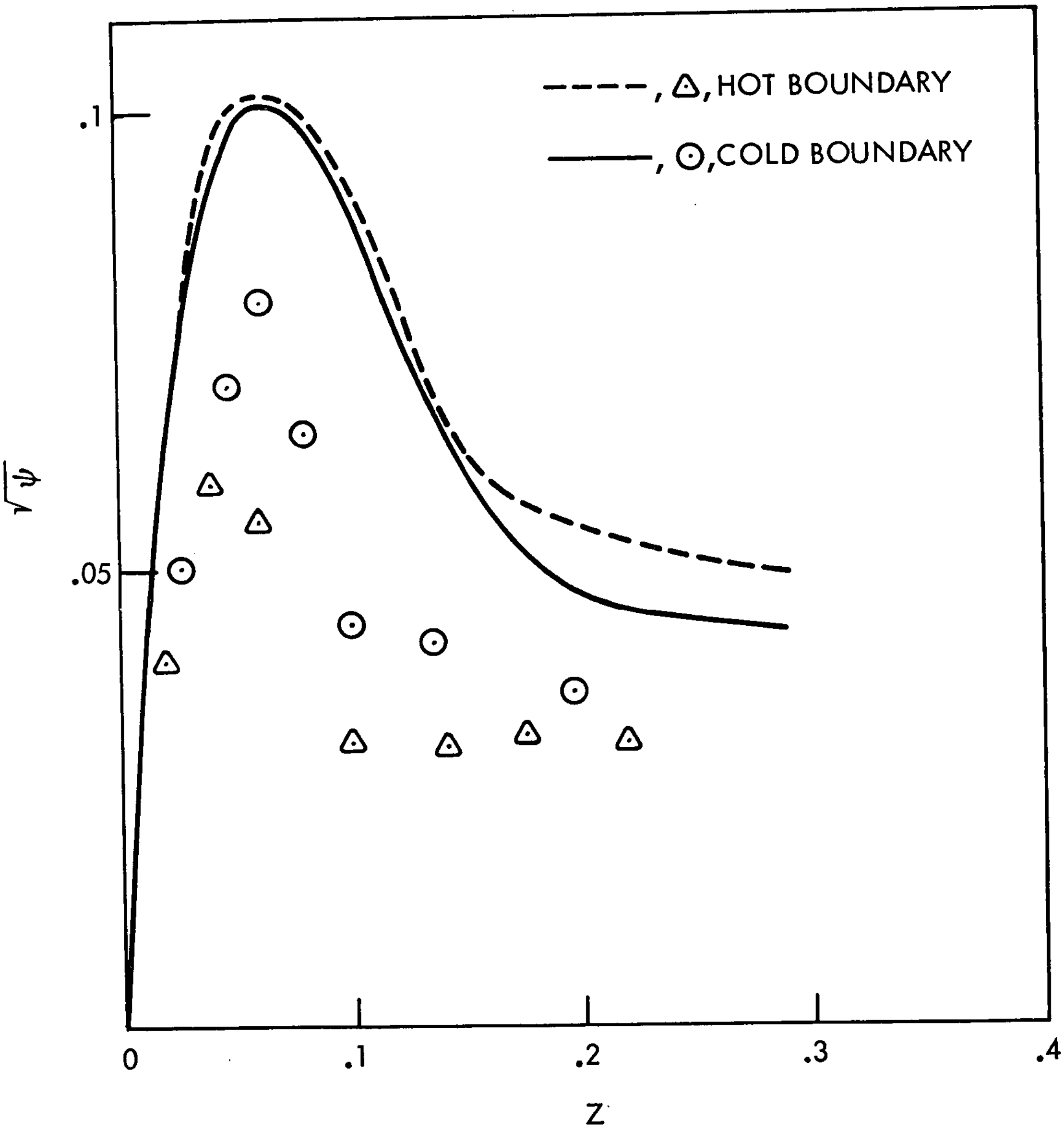

Figure 7. Comparison of r.m.s. temperature fluctuation field with experiment of Somerscales and Gazda (1968).

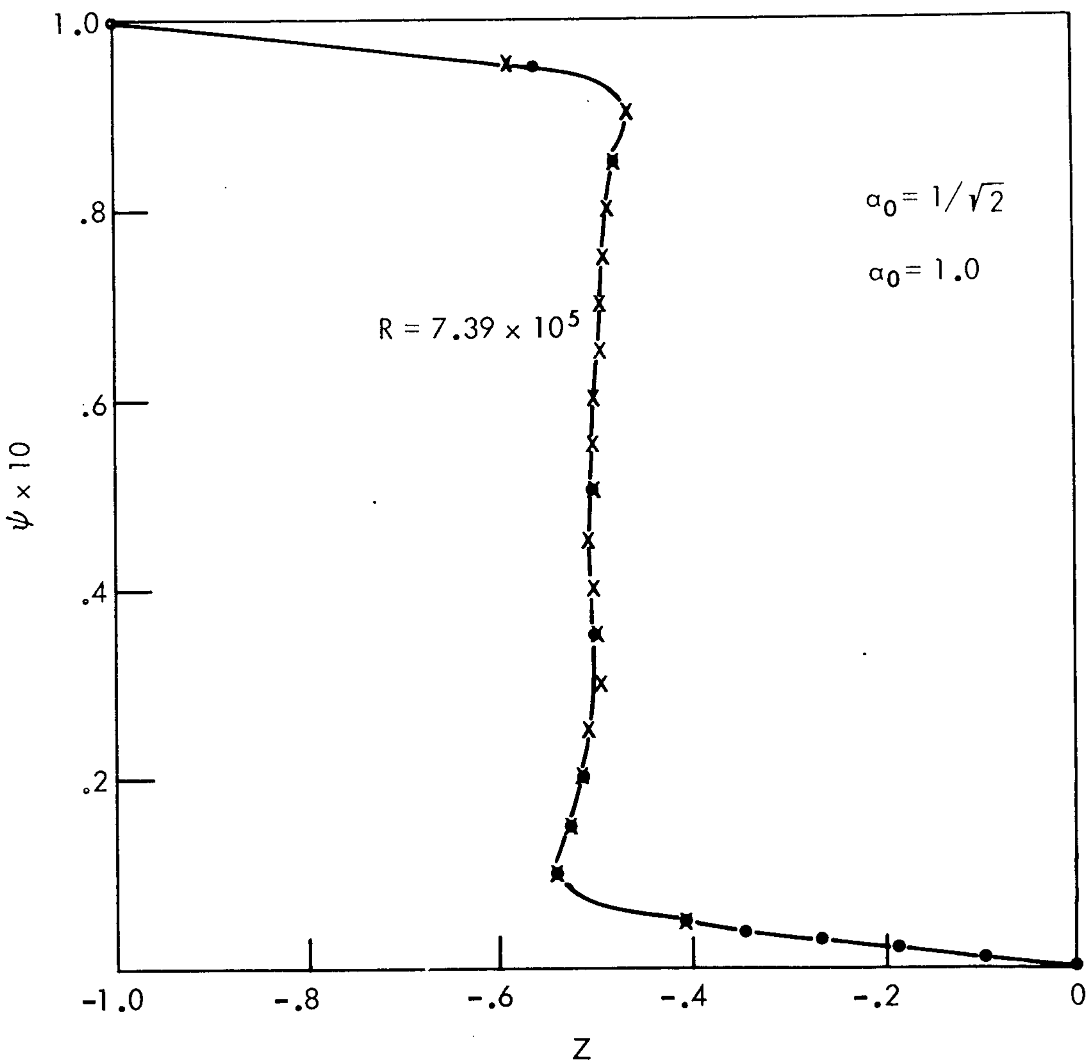

Figure 8. Comparison of $\overline{T}$ for $\alpha_0 = 2^{-1/2}$, indicated by x, with $\alpha_0 = 1$, indicated by line, for $R = 7.39*10^5$.

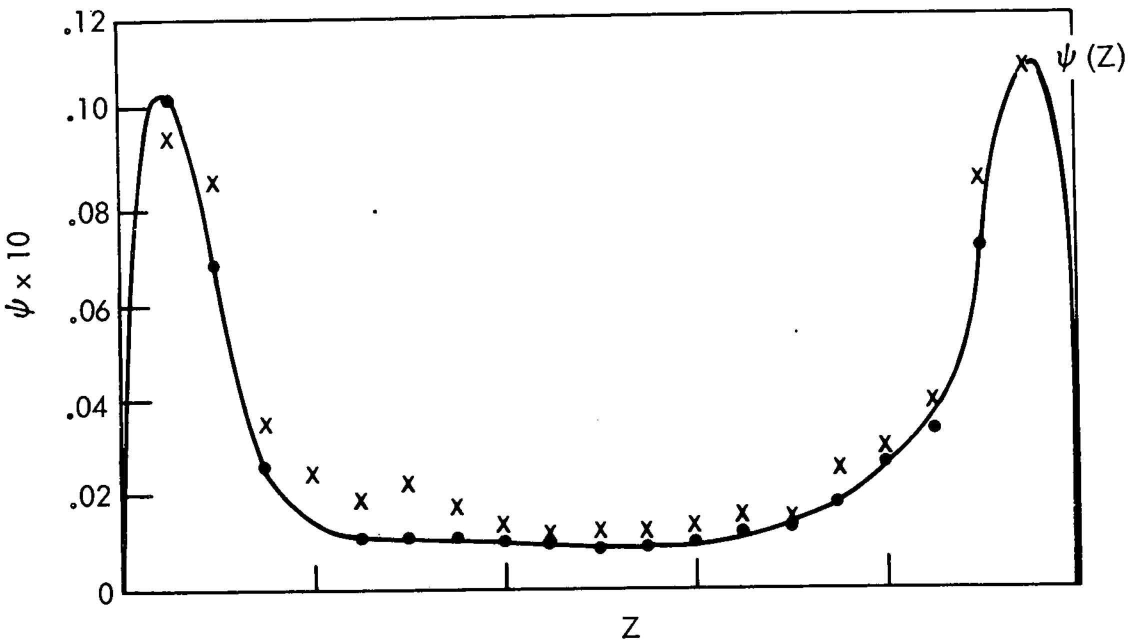

Figure 9. Comparison of ψ for $\alpha_0 = 2^{-1/2}$ with ψ for $\alpha_0 = 1$, for $R = 7.39*10^5$.