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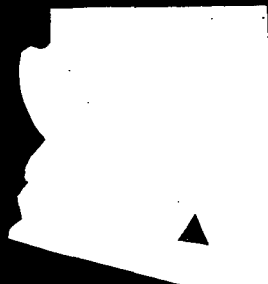
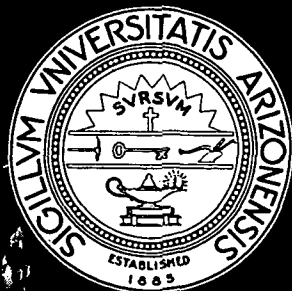
DESIGN OF NON-INTERACTING MULTIVARIABLE
DISCRETE TIME SYSTEMS

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DESIGN OF NON-INTERACTING MULTIVARIABLE
DISCRETE TIME SYSTEMS

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TABLE OF CONTENTS

	Page
LIST OF ILLUSTRATIONS	iv
ABSTRACT	v
CHAPTER	
1 INTRODUCTION	1
2 PROBLEM STATEMENT	5
System Notation	5
System Representation in Discrete Time	5
State Variable Closed Loop Feedback	9
Delay Decoupling	9
3 CANONICAL DECOUPLING	17
4 DESIRED SUBSYSTEM RESPONSE	25
Separation of Subsystems	25
Desired Subsystem Transfer Function	28
Use of Phase Variable Transformation Techniques	29
5 INACCESSIBLE STATE VARIABLES	35
Time Considerations	35
6 EXAMPLES	40
Part I: Simple Example	40
Problem Formulation	40
Step-by-Step Solution	44
Subsystem 1	58
Subsystem 2	61
Inaccessible States	63
Part II: Nuclear Rocket Engine Control	65
Problem Formulation	65

TABLE OF CONTENTS (Continued)

	Page
The Computer Program	69
Results of the Desired Decoupling of the Nuclear Reactor in Continuous Time	69
Results of Decoupling the Nuclear Reactor in Discrete Time Using a Zero Order Hold	71
Results of Discrete Time Decoupling With Impulse Sampling	74
Inaccessible States	81
7 CONCLUSIONS	83
Step-by-Step Summary	85
Further Research	87
APPENDIX A: COMPUTER PROGRAM FOR DESIRED RESPONSE	88
APPENDIX B: COMPUTER PROGRAM FOR FINDING INACCESSIBLE STATES	100
LIST OF REFERENCES	103

LIST OF ILLUSTRATIONS

Figure	Page
2.1 Closed Loop State Variable Feedback	10
3.1 Open Loop Delay Decoupled System	19
3.2 Block Diagram of System with All the Feedbacks and Gains and the Equivalent Original System	23
4.1 Phase Variable Transformed System	33
6.1 Block Diagram of the System in s-domain	41
6.2 Block Diagram of the System in z-domain	45
6.3 Resultant Block Diagram of Delay Decoupled System	49
6.4 Block Diagram of the Delay Decoupled System with Feedback Shown	50

ABSTRACT

The theory of decoupling a multivariable system with the help of state variable feedback has been applied to discrete time systems. The system differential equations in the continuous time domain are converted to a discrete time representation. A discrete time controller is designed to control the overall input to the system such that a specified desired decoupled output response is obtained. The main advantage of such a decoupling is that the decoupled system behaves as a set of single-input, single-output systems and, hence, state variable techniques can be easily applied to control each output individually.

Control is achieved by periodically sampling the inputs and all the states and feeding them back through a controller. For linear time invariant systems the control happens to be time invariant. The basic theory of decoupling involves feeding back all the states twice, first to decouple the system and then to achieve a desired response. The states of the system are computed with the help of a state estimator which calculates all the present states of the system based upon only the past values of inputs and outputs so that on-line control can be applied to real time systems.

Finally, the above theory is applied to a model of a nuclear rocket engine. In doing so, the nonlinear equations of the model have been linearized about the operating point.

CHAPTER 1

INTRODUCTION

With smaller, cheaper and faster computers, digital control is a useful and sometimes more flexible alternative to continuous time control. In most non-trivial, continuous optimal control problems, the design depends upon the solution of nonlinear differential equations. Since these equations must be solved on the computer, they must be discretized in time and an approximate solution sought. The alternative is to view the problem as a discrete-time problem to start with. Then only algebraic equations occur, theoretical difficulties are reduced, digital computer solutions are straightforward, and computation time is usually reduced dramatically [Williams, 1969].

The specific concern of this study is to discretize the systems which have multiple inputs and outputs with no interaction, i.e., to design the system such that one input affects one and only one output. (It is assumed that the system has an equal number of inputs and outputs.) One very practical example of such a system is a nuclear rocket engine where the typical inputs are reactivity and pressure ratio across the turbo pumps and the outputs are temperature and pressure. We will consider this example in more detail in later sections of this study. Several other examples are listed in Slivinsky [1969] and Ferg [1971].

One advantage of decoupling is that the decoupled system appears to function in the simplest possible manner when seen from the input-output point of view. Because the decoupled system behaves as a set of single input-single output systems, well-known state variable feedback design techniques can be applied to these subsystems to achieve the desired transfer function response.

Attempts to achieve non-interaction for the continuous time systems started as early as 1950. Until the early 1960's the conventional frequency domain techniques were the only basis of design [Boksenbom and Hood, 1949; Horowitz, 1960; Brockett and Mesarovic, 1962]. Morgan [1963] first studied state variable techniques to achieve non-interaction. However, Falb and Wolovich [1967] were the first to formulate and prove the necessary and sufficient conditions for decoupling a multivariable system.

In the first part of Chapter 2 an algorithm is provided for discretizing a continuous time system. The method discussed for finding a discrete time equivalent for a continuous time system has been utilized throughout this study. The second portion of Chapter 2 draws a parallel to Falb and Wolovich's work in discrete time. The complicated proof for necessary and sufficient conditions for decoupling has been omitted but the procedure is identical to that of Falb and Wolovich.

Chapter 3 draws a parallel to the work of Gilbert [1969] for discrete time systems. Gilbert utilized Falb and Wolovich's standard procedure for decoupling and then changed variables in the decoupled system to establish a canonical form for multivariable systems. The

use of this canonical form permits the identification of the poles and zeros of each subsystem. The prior theory only identified the pole-zero excess of each subsystem. (The number of subsystems equals the number of inputs which equals the number of outputs.)

In Chapter 4 we separate the subsystems of the CD (canonically decoupled) system and find the feedback and gain coefficients for each of the above subsystems to realize a specified subsystem transfer function. The feedback and gain matrices for the whole CD system are formed by assembling the feedback and gain coefficients of all the subsystems in partitioned matrix form. Once the feedback and gain matrices for a desired decoupled transfer function response are known for the CD system the equivalent feedback and gain matrices for the original system are easily calculated.

In Chapter 5 we deviate from the main topic in order to pay attention to inaccessible state variables. In many practical systems it is not possible to measure all the states of the system needed for state variable feedback. Thus, assuming no noise is present, the concept of observability is used to estimate the present values of the states from the past and/or present outputs and inputs. Since any computer will take a finite amount of time to calculate the proper control based upon states, we may not be able to use present inputs and outputs to calculate present states for on-line control. Also, a complete algorithm is developed to compute the present values of all the states from the past inputs and outputs [Williams, 1969].

Finally, in Chapter 6 we present a simple third order example in detail and then a more practical example concerning a nuclear rocket

engine. Since the computations are very involved, computer programs were used throughout this latter example and only the results of the various steps are presented.

The last chapter presents conclusions and a brief review of the important steps involved in decoupling a discrete time system. Suggestions for further research are also given.

The Appendix lists the computer programs which were developed for this study.

CHAPTER 2

PROBLEM STATEMENT

System Notation

Unless specifically mentioned:

1. A capital letter in a system equation denotes a matrix whose dimensions are compatible with the other variables in the equation.
2. A lower case letter denotes either a scalar quantity or an element of a vector.
3. A system equation refers to the discrete time system or a discrete time equivalent system.
4. A subscript i on a vector is used to denote the i^{th} element of the vector, and similarly a subscript i on a matrix is used to denote i^{th} row of the matrix.
5. m is the number of inputs and also the number of outputs of the multivariable system.
6. n is the number of states of the system in question.
7. The system under study is both controllable and observable [Kalman, 1960].
8. $n \geq m$.

System Representation in Discrete Time

In modern control theory the linear differential equations describing the behavior of a continuous time system are represented

in vector matrix notation as a set of n first order differential equations as follows

$$\dot{X}(t) = AX(t) + BU(t) \quad (2.1)$$

$$y(t) = CX(t) \quad (2.2)$$

where $(\dot{})$ indicates the vector derivative w.r.t time t .

$X(t)$ is an n element column vector representing the states of the system.

$U(t)$ is an m element column vector representing the control inputs to the system.

$Y(t)$ is an m element column vector representing the outputs of the system.

A is $n \times n$ system state matrix.

B is $n \times m$ system control matrix.

C is $m \times n$ output matrix.

In short hand notation Eq. (2.1) and (2.2) are written as $S(A,B,C)$.

It is well known [Schultz and Melsa, 1967] that Eq. (2.1) has the solution

$$X(t) = e^{A(t-t_0)} X(t_0) + \int_{t_0}^t e^{A(t-\tau)} BU(\tau) d\tau \quad (2.3)$$

where $X(t_0)$ represents the values of states at the reference time t_0 .

With T as a sampling time and k an integer, if t is replaced by $(k+1)T$ and t_0 by KT , we get

$$X((k+1)T) = e^{AT}X(kT) + \int_{kT}^{(k+1)T} e^{A(\overline{(k+1)T} - \tau)} BU(\tau) d\tau \quad (2.4)$$

with a change in variable $(k+1)T - \tau = \lambda$, we get

$$X((k+1)T) = e^{AT}X(kT) + \int_0^T e^{A\lambda} BU(\overline{(k+1)T} - \lambda) d\lambda \quad (2.5)$$

The evaluation of the above integral depends upon the way in which the input is formed. In this report two separate methods are considered for converting the number sequence from the digital controller into a time function. One of the most intriguing aspects of this thesis is the substantial differences in the ease of decoupling which occur between the two methods of data reconstruction.

1. Impulse approximation: In this method the input in the interval of integration is an impulse.
2. Polynomial approximation: In this method the input in the interval is a polynomial.

Both methods have their relative advantages and disadvantages. Method 1 gives a simple answer and preserves the form of the B matrix, i.e., preserves zeros in the matrix, which has a marked effect on the decoupling as we shall see in the further developments. However, the impulse is difficult to approximate in practice. Method 2 is more practical but has the disadvantage of changing the form of the B matrix, which may introduce uncontrollable poles in the system.

In method 1 $U(\tau)$ in the interval $kT - (k+1)T$ is $TU(kT)\delta(\overline{k+1}T - \tau)$. From Eq. (2.4) the state equation becomes

$$X((k+1)T) = e^{AT} X(kT) + BTU(kT) \quad (2.6)$$

In method 2 there are several possibilities. If we use a zero order hold (ZOH), the control input $U(\overline{k+1}T - \lambda)$ of Eq. (2.5) in the interval $0-T$ is the constant $U(kT)$. Equation (2.5) becomes

$$X((k+1)T) = e^{AT} X(kT) + \left[\int_0^T e^{A\lambda} d\lambda \right] BU(kT) \quad (2.7)$$

Commonly the matrix e^{AT} is represented by the symbol $\Phi(T)$. We will omit the argument T for notational convenience everywhere in the equation. Also, we denote by B' the matrix $\left[\int_0^T e^{A\lambda} d\lambda \right]$ so that Eq. (2.6) and (2.7) can be rewritten in the simplified form as follows:

$$X(k+1) = \Phi X(k) + BTU(k) \quad (2.8)$$

$$X(k+1) = \Phi X(k) + B'BU(k) \quad (2.9)$$

Both of the above equations can be represented by a single equation

$$X(k+1) = \Phi X(k) + HU(k) \quad (2.10)$$

where H is the control transition which depends upon the approximation used in the evaluation of the integral. Substituting $t = k$ in Eq. (2.2) we get

$$y(k) = CX(k) \quad (2.11)$$

where the argument T is omitted for convenience. Equations (2.10) and (2.11) define the multivariable system in discrete time and form the basis for the rest of our study.

State Variable Closed Loop Feedback

We now want to find out the feedback and gain matrices F and G such that the overall control input to the system $U(k)$ [see Fig. 2.1] is given by

$$U(k) = FX(k) + GR(k) \quad (2.12)$$

The object of the rest of the report is to find F and G which decouples the multivariable system in the desired manner. Here F is a $m \times n$ matrix and G is an $m \times m$ matrix. $R(k)$ is the $m \times 1$ reference input.

Substituting for $U(k)$ in Eq. (2.10) we get

$$X(k+1) = \Phi X(k) + H[FX(k) + GR(k)]$$

The closed-loop system equation in terms of the reference input is

$$X(k+1) = (\Phi + HF)X(k) + HGR(k) \quad (2.13)$$

Delay Decoupling

Falb and Wolovich [1967] were the first to develop a procedure which integrator decouples the continuous time system; that is, they place all the poles of the decoupled system at the origin. We proceed on the same lines but with a discrete time system, we delay decouple the system. That is, we place all the poles of the decoupled system at the origin in the z -domain.

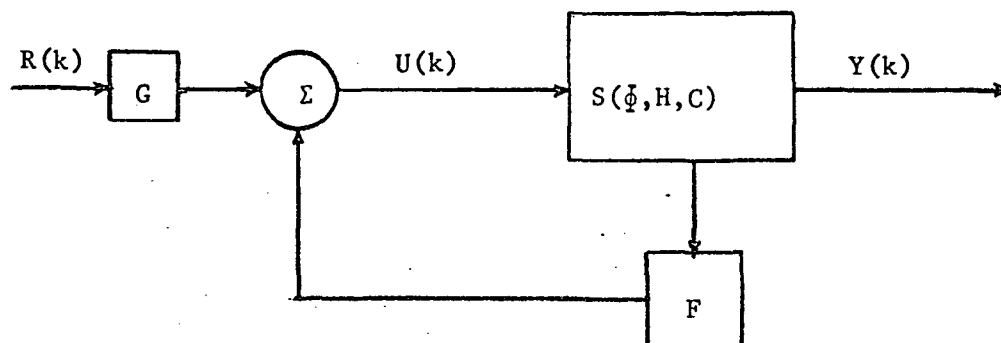


Fig. 2.1 Closed Loop State Variable Feedback

Let us define

$$d_i = \text{minimum } [j; C_i \phi^j H \neq 0, j = 0, 1 \dots n-1]$$

and $d_i = n-1$ if the above is not satisfied for any j less than n . As mentioned in the beginning C_i means i^{th} row of the output matrix C .

The scalar d_i is the smallest integer j such that $C_i \phi^j H \neq 0$. As we will see in the next chapter, $d_i + 1$ represents the pole zero excess for the i^{th} subsystem.

For the i^{th} subsystem

$$\begin{aligned} Y_i(k+1) &= C_i X(k+1) = C_i [(\phi + HF) X(k) + HGR(k)] \\ &= C_i \phi X(k) + C_i HF X(k) + C_i HGR(k) \\ &= C_i \phi X(k) \end{aligned}$$

if $d_i > 0$

$$\begin{aligned} Y_i(k+2) &= C_i \phi X(k+1) = C_i \phi [(\phi + HF) X(k) + HGR(k)] \\ &= C_i \phi^2 X(k) + C_i \phi HF X(k) + C_i \phi HGR(k) \\ &= C_i \phi^2 X(k) \end{aligned}$$

if $d_i > 1$

and, similarly

$$Y_i(k+d_i) = C_i \phi^{d_i-1} X(k+1) = C_i \phi^{d_i} X(k)$$

Hence

$$\begin{aligned}
 Y_i(k+d_i+1) &= C_i \phi^{d_i} x(k+1) = C_i \phi^{d_i} [(\phi + HF)x(k) + HGR(k)] \\
 &= (C_i \phi^{d_i+1} + C_i \phi^{d_i} HF) x(k) + C_i \phi^{d_i} HGR(k)
 \end{aligned} \tag{2.15}$$

For the sake of notational convenience we define

$$\phi_i^* = C_i \phi^{d_i+1}$$

$$H_i^* = C_i \phi^{d_i} H$$

$$Y_i^* = y_i(k+d_i+1)$$

so that we can rewrite Eq. (2.15) as follows

$$Y_i^* = [\phi_i^* + H_i^* F] x(k) + H_i^* GR(k) \tag{2.16}$$

The value of d_i+1 determines the pole zero excess of the i^{th} subsystem.

We will have m such equations for $i = 1 \dots m$ and we can combine these

m equations and represent them by one single matrix equation as follows

$$\begin{bmatrix} Y_1^* \\ Y_2^* \\ \vdots \\ Y_m^* \end{bmatrix} = \begin{bmatrix} \phi_1^* \\ \phi_2^* \\ \vdots \\ \phi_m^* \end{bmatrix} + \begin{bmatrix} H_1^* \\ H_2^* \\ \vdots \\ H_m^* \end{bmatrix} F \begin{bmatrix} x(k) \end{bmatrix} + \begin{bmatrix} H_1^* \\ H_2^* \\ \vdots \\ H_m^* \end{bmatrix} GR(k) \tag{2.17}$$

or, in its simplified form

$$Y^* = (\Phi^* + H^* F) x(k) + H^* G R(k) \quad (2.18)$$

We want to choose F and G such that we can achieve a decoupling between the outputs Y and inputs R.

Choose

$$\begin{aligned} F = F^* &= -H^{*-1} \Phi^* \\ G = G^* &= -H^{*-1} \end{aligned} \quad (2.19)$$

So that Eq. (2.18) reduces to

$$Y^* = R(k) \quad (2.20)$$

or, in its expanded form

$$\begin{bmatrix} Y_1(k + d_1 + 1) \\ Y_2(k + d_2 + 1) \\ \vdots \\ Y_m(k + d_m + 1) \end{bmatrix} = \begin{bmatrix} R_1(k) \\ R_2(k) \\ \vdots \\ R_m(k) \end{bmatrix}$$

and separating them

$$Y_i(k + d_i + 1) = R_i(k) \quad (2.21)$$

Equation (2.21) shows that i^{th} output is a function of only the i^{th} input and the output follows the input delayed by $(d_i+1)T$ secs. Thus, in a way, our system has been decoupled.

Falb and Wolovich prove that the necessary and sufficient condition for a multivariable system to be decoupled is the non-singularity of matrix H^* [1967].

The F and G matrices of Eq. (2.19) are not the only matrices which decouple the system. Actually this is a very restricted way of decoupling. To show this, let us take the z transform of Eq. (2.20)

$$z^{d_i+1} y_i(z) = R_i(z)$$

or

$$\frac{Y_i(z)}{R_i(z)} = \frac{1}{z^{d_i+1}} \quad (2.22)$$

This subsystem has all its poles at the origin. Our system may be too sluggish. However, as shown in the next chapter, the above form of decoupling forms the basis for further discussion and restores our confidence in state variable feedback.

To displace the poles from the origin, let us start from Eq. (2.18) which is

$$Y^* = (\Phi^* + H^* F) X(k) + H^* G R(k) \quad (2.18)$$

let us choose

$$G = H^{*-1} W \quad \text{where } W \text{ is } m \times m \text{ diagonal matrix}$$

$$F = H^{*-1} \left[\sum_{j=0}^{\delta} M_j C \Phi^j - \Phi^* \right] \quad (2.23)$$

where $\delta = \max [d_1, d_2, \dots, d_m]$ and M_j 's are $m \times m$ diagonal matrices for each j from 0 to δ . Then substituting (2.23) into (2.10) we get

$$Y^* = \left[\sum_{j=0}^{\delta} M_j C \Phi^j \right] X(k) + WR(k) \quad (2.24)$$

where $\sum_{j=0}^{\delta} M_j C \Phi^j$ is a sum of δ $m \times n$ matrices and, hence, has a dimension of $m \times n$.

Separating out the i^{th} subsystem (i^{th} row of Eq. (2.24)), we get

$$Y_i(k+d_i+1) = \sum_{j=0}^{d_i} M_{ij} C_i \Phi^j X(k) + W_{ii} R_i(k)$$

we know from the development of Eq. (2.15)

$$C_i \Phi^j X(k) = Y_i(k+j) \quad \text{for } j \leq d_i$$

therefore

$$\begin{aligned} Y_i(k+d_i+1) &= m_{i0} Y_i(k) + m_{i1} Y_i(k+1) + m_{i2} Y_i(k+2) + \dots \\ &\quad + m_{id_i} Y_i(k+d_i) + W_{ii} R_i(k) \end{aligned}$$

where m_{ij} is the i^{th} diagonal term of M_j matrix. Taking the z transform of the above equation

$$z^{d_i+1} y_i(z) = m_{i0} + m_{i1} z + m_{i2} z^2 + \dots + m_{id_i} z^{d_i} + W_{ii} R(z)$$

$$[z^{d_i+1} - m_{id_i} z^{d_i} - \dots - m_{i1} z - m_{i0}] Y_i(z) = W_{ii} R(z)$$

or

$$\frac{Y_i(z)}{R_i(z)} = \frac{W_{ii}}{z^{d_i+1} - m_{id_i} z^{d_i} - \dots - m_{i1} z + m_{i0}}$$

The coefficients m_{ip} where $p = 0, 1, \dots, d_i$ can be chosen as desired and, hence, the poles of the decoupled subsystem can be placed at any desired position.

However, because the absence of zeros restricts our choice of transfer functions, a more general approach is considered in the next two chapters.

CHAPTER 3

CANONICAL DECOUPLING

In this chapter a canonical representation of the multivariable system is found. The main advantage of such a representation is that the internal structure of the system is recognized which in turn may be used to separate the subsystems of the system.

Gilbert [1969] worked with the integrator decoupled system of Falb and Wolovich [1967]. Since the decoupling procedure changes the system dynamics, the canonical representation of a decoupled system gives only the approximate structure of the subsystem. However, this is a significant improvement over the DD (Delay Decoupled) system of the last chapter where only the pole-zero excess of the subsystems were available. By finding a canonical representation of the DD system, we are able to determine both the poles and the zeros of the system. We have more flexibility in positioning the poles of the subsystems which were cancelled by any open loop zeros in the subsystem in the previous development.

In Chapter 2 the matrices F^* and G^* (Eq. 2.19) delay decouple the system. Since we reserve the notation R for the open loop control input of the original system, let us denote the open loop input to DD system by $\bar{U}$. Thus the closed-loop input to the original system is given by

$$U(k) = F^* X(k) + G^* \bar{U}(k) \quad (3.1)$$

Substituting the above into the system Eq. (2.21) and (2.11), we get the DD system equations as follows

$$X(k+1) = \Phi X(k) + H[F^* X(k) + G^* \bar{U}(k)]$$

$$Y(k) = C X(k)$$

or

$$X(k+1) = (\Phi + HF^*) X(k) + HG^* \bar{U}(k)$$

$$Y(k) = C X(k)$$

Let us define

$$\bar{\Phi} = \Phi + HF^*$$

$$\bar{H} = HG^*$$

$$\bar{C} = C$$

then the DD system equations are given by

$$X(k+1) = \bar{\Phi} X(k) + \bar{H} \bar{U}(k) \quad (3.2)$$

$$Y(k) = \bar{C} X(k) \quad (3.3)$$

The original system $S(\Phi, H, C)$, the DD system $\bar{S}(\bar{\Phi}, \bar{H}, \bar{C})$ and the feedback and gain matrices F^* and G^* which convert $S(\Phi, H, C)$ to $\bar{S}(\bar{\Phi}, \bar{H}, \bar{C})$ are shown in Fig. 3.1.

The transformation of the delay decoupled system into the canonical form is done with the help of a change of variable

$$\hat{X}(k) = Q X(k) \quad (3.4)$$

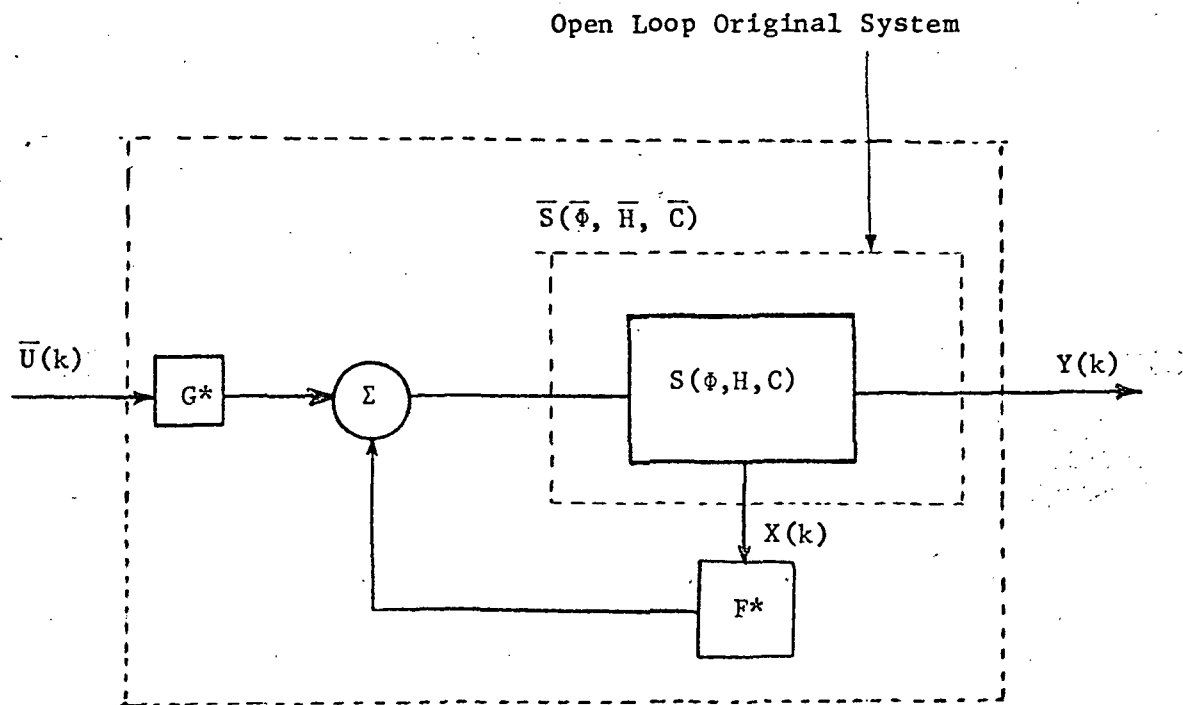


Fig. 3.1 Open Loop Delay Decoupled System

where the matrix Q takes the DD system $\bar{S}(\bar{\Phi}, \bar{H}, \bar{C})$ into the canonical form $\hat{S}(\hat{\Phi}, \hat{H}, \hat{C})$. The transformation matrix Q is found as follows [Gilbert, 1969].

The key point in the computation of the Q matrix is to investigate the $\bar{Q}$ matrix defined as follows

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ \vdots \\ \bar{Q}_m \end{bmatrix}$$

where $\bar{Q}_i$ is an n element row vector $[q_{i1} \ q_{i2} \ \dots \ q_{in}]$

where the row vector Q_i satisfies the following $(m-1) \times n$ relations

$$[q_{i1} \ q_{i2} \ \dots \ q_{in}] (\Phi)^j H_k = 0$$

$k = 1, 2, \dots, m$ but $k \neq i$: $(m-1)$ values, and

$j = 0, 1, \dots, n-1$: n values of j .

There are only n unknowns in $\bar{Q}_i$ and we have $(m-1) \times n$ equations, but a lot of these equations are redundant.

Actually, the real values of $q_{i\gamma}$ $\gamma = 1 \dots n$ are not of interest. Rather, the number of nonzero different elements in the vector $\bar{Q}_i$ is the dimension of Q_i and is denoted by n_i . n_i determines the order of the i^{th} subsystem in the system. The transformation matrix Q is given by

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{m+1} \end{bmatrix}$$

where Q_i is a $n_i \times n$ matrix for $i = 1, 2 \dots m$ and Q_{m+1} is a $\left[n - \sum_{i=1}^m n_i \right] \times n$ matrix. Thus the rank of Q is determined. The actual elements of Q are given as follows.

The first $d_i + 1$ rows of each Q_i matrix $\{n_i \times n\}$ for $i = 1, \dots, m$ are $C_i \bar{\Phi}$, $C_i (\bar{\Phi})^2$, $\dots$, $C_i (\bar{\Phi})^{d_i}$ and the remaining $n_i - d_i - 1$ rows are any linearly independent rows. The rows of Q_{m+1} are also any linearly independent rows independent of all the rows of all the Q_i matrices $i = 1 \dots m$. Thus the Q matrix is fully determined.

A computer program has been written by Gilbert and Pivnichny [1969] which computes the transformation matrix Q and the matrices representing the canonically transformed system $\hat{S}(\hat{\Phi}, \hat{H}, \hat{C})$. Consequently, one need not do by hand the tedious calculations involved. However, in our simple example of Chapter 6 we illustrate the basic calculations.

To find the equations for the canonical representation we simply substitute for $X(k)$ from $\hat{X}(k) = Q X(k)$ in Eq. (3.2) and (3.3)

$$Q^{-1} \hat{X}(k+1) = \bar{\Phi} Q^{-1} \hat{X}(k) + \bar{H} \bar{U}(k) \quad (3.5)$$

$$Y(k) = \bar{C} Q^{-1} \hat{X}(k) \quad (3.6)$$

or

$$\hat{X}(k+1) = Q \bar{\Phi} Q^{-1} \hat{X}(k) + Q \bar{H} \bar{U}(k) \quad (3.7)$$

$$Y(k) = \bar{C} Q^{-1} \hat{X}(k) \quad (3.8)$$

Let us define

$$\begin{aligned}\hat{\Phi} &= Q \bar{\Phi} Q^{-1} \\ \hat{H} &= Q \bar{H} \\ \hat{C} &= \bar{C} Q^{-1} = CQ^{-1}\end{aligned}$$

Then

$$\hat{X}(k+1) = \hat{\Phi} \hat{X}(k) + \hat{H} \bar{U}(k) \quad (3.9)$$

$$\hat{Y}(k) = \hat{C} \hat{X}(k) \quad (3.10)$$

Figure 3.2 illustrates such a transformation and the transformed system. The figure also illustrates the closed loop input to this canonically transformed system.

Suppose we are able to find out a set of matrices $\hat{F}$ and $\hat{G}$ such that overall input to the canonical system

$$\hat{U}(k) = \hat{F} \hat{X}(k) + \hat{G} R(k) \quad (3.11)$$

forces the system to give the desired decoupled response. Then by proceeding backwards we can find out a set of matrices F and G which when fed back to the original system will produce the same response as the canonically decoupled system with $\hat{F}$ and $\hat{G}$, as shown below.

The input $\bar{U}(k)$ to the DD system is given by

$$\bar{U}(k) = \hat{U}(k) = \hat{F} Q \hat{X}(k) + \hat{G} R(k) \quad (3.12)$$

since $\hat{X}(k) = Q X(k)$ (see Fig. 3.2). But the total closed loop input to the original system is given by $U(k)$ where

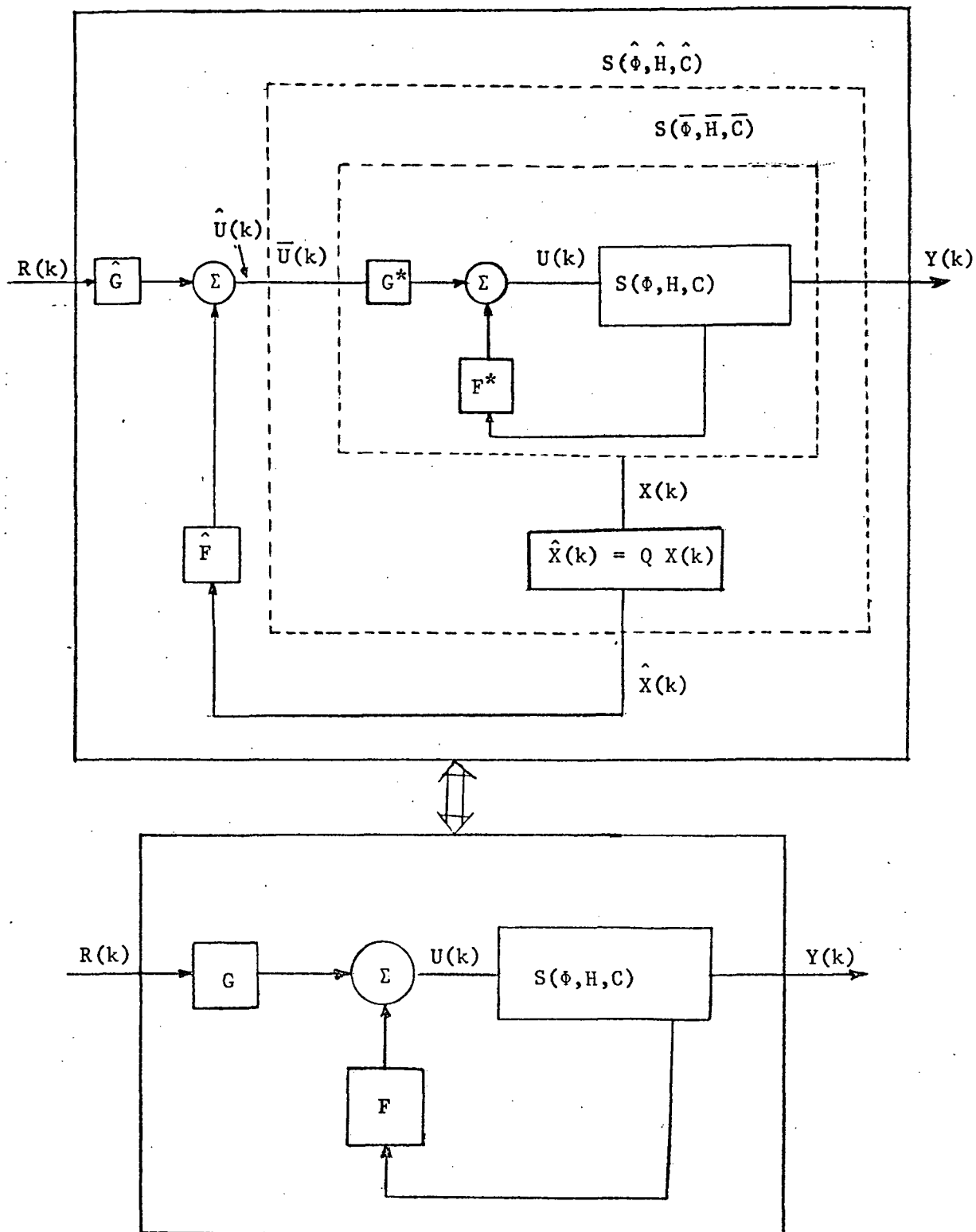


Fig. 3.2 Block Diagram of System with All the Feedbacks and Gains and the Equivalent Original System

$$U(k) = F^* X(k) + G^* \bar{U}(k) \quad [\text{see Fig. 3.2 and Eq. (3.1)}]$$

and substituting for $\bar{U}(k)$ from Eq. (3.12)

$$\begin{aligned} U(k) &= F^* X(k) + G^* [\hat{F} Q X(k) + \hat{G} R(k)] \\ &= (F^* + G^* \hat{F} Q) X(k) + G^* \hat{G} R(k) \end{aligned} \quad (3.13)$$

We know that the above input gives the desired decoupled response. But we wanted to find out F and G such that the closed loop input

$$U(k) = F X(k) + G R(k)$$

to the original system gives the desired decoupled response.

Hence

$$F = F^* + \hat{G} \hat{F} Q \quad (3.14)$$

$$G = G^* \hat{G} \quad (3.15)$$

We know from Chapter 2 how to find F^* and G^* and, hence, if we can find $\hat{F}$ and $\hat{G}$, the matrices for the CD system F and G are completely known for the original system. How to find $\hat{F}$ and $\hat{G}$ for the CD system is the subject matter of the next chapter.

CHAPTER 4

DESIRED SUBSYSTEM RESPONSE

Separation of Subsystems

Gilbert [1969] has shown that the matrices $\hat{\Phi}$, $\hat{H}$ and $\hat{C}$ in the CD system $\hat{S}(\hat{\Phi}, \hat{H}, \hat{C})$ equations

$$\hat{X}(k+1) = \hat{\Phi}X(k) + \hat{H}U(k) \quad (4.1)$$

$$\hat{Y}(k) = \hat{C}X(k) \quad (4.2)$$

will have a canonical form in which the subsystems of the system appear in partitioned matrix form. Without loss of generality the canonical form of $\hat{\Phi}$, $\hat{H}$, and $\hat{C}$ can be demonstrated by an example of a fourth order system with two inputs and two outputs. Let n_1 (the order of first subsystem) = 2 and $n_2 = 1$. Thus, $4 - n_1 - n_2 = 1 =$ the number of uncontrollable poles in the system. The superscripts 1, 2, and c will denote subsystems 1, 2, and uncontrollable part, respectively. Then $\hat{\Phi}$, $\hat{H}$, and $\hat{C}$ will have the form as shown below and also they can be partitioned to separate the subsystems as shown.

$$\hat{\Phi} = \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 & 0 & 0 \\ \phi_{21}^1 & \phi_{22}^1 & 0 & 0 \\ 0 & 0 & \phi_{11}^2 & 0 \\ \phi_1^c & \phi_2^c & \phi_3^c & \phi_4^c \end{bmatrix} \quad \hat{H} = \begin{bmatrix} H_1^1 & 0 \\ H_2^1 & 0 \\ 0 & H_1^2 \\ H_1^c & H_2^c \end{bmatrix}$$

$$\hat{C} = \left[\begin{array}{cc|c|c} c_{11}^1 & c_{12}^1 & 0 & q \\ \hline 0 & 0 & c_{11}^2 & 0 \end{array} \right]$$

Thus, the subsystems can be separated as follows

$$\hat{X}^1(k+1) = \begin{bmatrix} \phi_{11}^1 & \phi_{12}^1 \\ \phi_{11}^1 & \phi_{22}^1 \end{bmatrix} \hat{X}^1(k) + \begin{bmatrix} H_1^1 \\ H_2^1 \end{bmatrix} \hat{U}^1(k) \quad (4.3)$$

$$Y_1(k) = [c_{11}^1 \quad c_{12}^1] \hat{X}^1(k)$$

$$\hat{X}^2(k+1) = \phi_{11}^2 \hat{X}^2(k) + H_1^2 \hat{U}^2(k) \quad (4.4)$$

$$Y_2(k) = c_{11}^2 \hat{X}^2(k)$$

Now we want to find out the feedback and gain matrices, $\hat{F}$ and $\hat{G}$, such that the closed loop input

$$\hat{U}(k) = \hat{F} \hat{X}(k) + \hat{G} R(k) \quad (4.5)$$

to the CD system produces the desired decoupled response. It has also been shown that the system will preserve decoupling if $\hat{F}$ and $\hat{G}$ can also be partitioned in the same form as $\hat{\Phi}$, $\hat{H}$, and $\hat{C}$, i.e., for our above fourth order example we must have

$$\hat{F} = \begin{bmatrix} F_{11}^1 & F_{12}^1 & 0 & 0 \\ \hline 0 & 0 & F_{11}^2 & 0 \end{bmatrix} \quad \hat{G} = \begin{bmatrix} G_{11}^1 & 0 \\ \hline 0 & G_{11}^2 \end{bmatrix}$$

$\hat{U}(k)$ can also be partitioned into systems. Here

$$\hat{U}^1(k) = [F_{11}^1 \ F_{12}^1] X^1(k) + G_{11}^1 r_1(k) \quad (4.6)$$

$$\hat{U}^2(k) = F_{11}^2 X^2(k) + G_{11}^2 r_2(k) \quad (4.7)$$

We now have m single input-single output subsystems which are decoupled. Thus, the familiar state variable feedback techniques can be applied to position the poles of each subsystem at the desired positions. In vector matrix equation form the i^{th} subsystem can be represented as follows

$$\hat{X}^i(k+1) = \hat{\Phi}^i \hat{X}^i(k) + \hat{H}^i \hat{U}^i(k) \quad (4.8)$$

$$Y_i(k) = \hat{C}^i \hat{X}^i(k) \quad (4.9)$$

$$\hat{U}^i(k) = \hat{F}^i \hat{X}^i(k) + \hat{G}^i r_i(k) \quad (4.10)$$

where the dimensions of the matrices correspond to an n_i th order single input-single output system.

We want to find $\hat{F}^i$ and $\hat{G}^i$ for each of the m subsystems so that by combining them into partitioned matrix form $\hat{F}$ and $\hat{G}$ can be formed. Since finding of F^i and G^i is similar for all the subsystems, we will demonstrate the method by considering one general case of an n^{th} order

subsystem. For the sake of convenience, let us drop the subscript from Eq. (4.8) and (4.9) and assume that the equations describing the i^{th} subsystem are

$$X(k+1) = \Phi X(k) + HU(k) \quad (4.10)$$

$$Y(k) = C X(k) \quad (4.11)$$

$$U(k) = F X(k) + G R(k) \quad (4.12)$$

Desired Subsystem Transfer Function

First, an input to output transfer function for each subsystem is selected. The subsystem transfer functions form the diagonal terms in the overall transfer function of the system and since the system has been decoupled, the nondiagonal terms are zero. Thus, picking a transfer function for the i^{th} subsystem is picking an i^{th} diagonal term in the overall transfer function of the system.

Since we are using only state-variable feedback techniques, the following two limitations must be kept in mind:

1. The order of the subsystem response cannot be increased.
2. The position of the subsystem zeros cannot be changed.

If the subsystem zero happens to be outside the unity circle in the z -domain, then the system will go unstable at high gains. Also, if the sum of the order of the subsystems is less than the order of the main system, then there are poles of the system which are not controllable by state variable feedback. If any of those poles happens

to be in an unstable region, the system will be unstable after decoupling even though the system might be stable before decoupling.

Use of Phase Variable Transformation Techniques

Suppose that our subsystem has a transfer function of the form

$$\frac{Y(z)}{R(z)} = C(ZI - \Phi)^{-1}H = \frac{k_1(Z^\ell + a_1Z^{\ell-1} + a_{\ell-1}Z^{\ell-2} + \dots + a_2Z + a_1)}{Z^n + p_nZ^{n-1} + p_{n-1}Z^{n-2} + \dots + p_2Z + p_1} \quad (4.13)$$

and keeping in mind the limitations of state variable feedback, our desired transfer function must be

$$\left. \frac{Y(z)}{R(z)} \right|_{\text{desired}} = \frac{k(Z^\ell + a_\ell Z^{\ell-1} + a_{\ell-1}Z^{\ell-2} + \dots + a_2Z + a_1)}{Z^n + q_n Z^{n-1} + q_{n-1}Z^{n-2} + \dots + q_2Z + q_1} \quad (4.14)$$

where a_i , p_i and q_i are any general coefficients and $\ell < n$. The feedback and gain coefficients to achieve the above desired transfer function are very easily found if our subsystem is represented in phase variable form because the coefficients of the transfer function numerator and denominator polynomial, directly appear in the C and Φ matrices of the system, respectively. Such a representation is highly desirable because the feedback coefficients are then directly found by the difference of polynomial coefficients of the similar power of Z in the two transfer functions.

Let the matrix T be such that the transformation $X'(k) = T^{-1}X(k)$ brings the system into phase variable form. Then with the above substitution the phase variable representation of our subsystem is given by

$$X'(k+1) = \Phi' X'(k) + H' U(k) \quad (4.15)$$

$$Y(k) = C' X'(k) \quad (4.16)$$

$$U(k) = F' X'(k) + G' r(k) \quad (4.17)$$

where

$$\Phi' = T^{-1} \Phi T$$

$$H' = T^{-1} H$$

$$C' = C T$$

It has been shown that such a transformation matrix is nonsingular for a controllable and observable system and an algorithm for calculating T has been given by various writers. The one used during this study was as given by Johnson and Wonham [1966] which is as follows.

Let $T = [T_1 \quad T_2 \quad \dots \quad T_n]$ where T_1 are n element column vectors.

Then

$$T_n = H$$

$$T_{n-1} = \Phi T_n - p_n T_n$$

$$T_{n-2} = \Phi T_{n-1} - p_{n-1} T_n$$

$$\vdots$$

$$T_2 = \Phi T_3 - p_2 T_n$$

$$T_1 = \Phi T_2 - p_1 T_n$$

Thus, the matrix T can be calculated from the system matrices and the coefficients of the characteristic polynomial of the system. Thus, the matrices Φ' , H' and C' must be of the form:

$$\Phi' = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -p_1 & -p_2 & -p_3 & \dots & \dots & -p_n \end{bmatrix} \quad H' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C' = k' [a_1 \quad a_2 \quad \dots \quad a_\ell \quad 0 \quad 0 \dots 0]$$

Let

$$F' = [f'_1 \quad f'_2 \quad f'_3 \dots f'_n] \text{ and } G' = g'$$

So that substituting $U(k) = [f'_1 \quad f'_2 \dots f'_n] X'(k) + g'(k)$ in Eq. (4.15)

we get

$$X'(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & 0 \\ 0 & \cdot & & & & \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 \\ -p_1+f_1 & -p_2+f_2 & \cdot & \cdot & \cdot & -p_n+f_n \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ g' \end{bmatrix} r(k)$$

$$r(k) = k' [a_1 \quad a_2 \quad \dots \quad a_n]$$

Since even after feedback the system is still in phase variable form, the new transfer function is given by

$$\frac{Y(z)}{R(z)} = \frac{g'k'[Z^n + a_1 Z^{n-1} + \dots + a_2 Z + a_1]}{Z^n + (p_n - f_n)Z^{n-1} + \dots + (p_2 - f_2)Z + p_1 f_1} \quad (4.18)$$

If the transfer function of Eq. (4.18) is to be the same as the desired transfer function of Eq. (4.14), f_i and g' must satisfy:

$$g'k' = k$$

$$g' = k/k'$$

$$p'_n - f'_n = q_n$$

$$f'_n = p'_n - q_n$$

$$p'_{n-1} - f'_{n-1} = q_{n-1}$$

$$f'_{n-1} = p'_{n-1} - q_{n-1}$$

$$\vdots$$

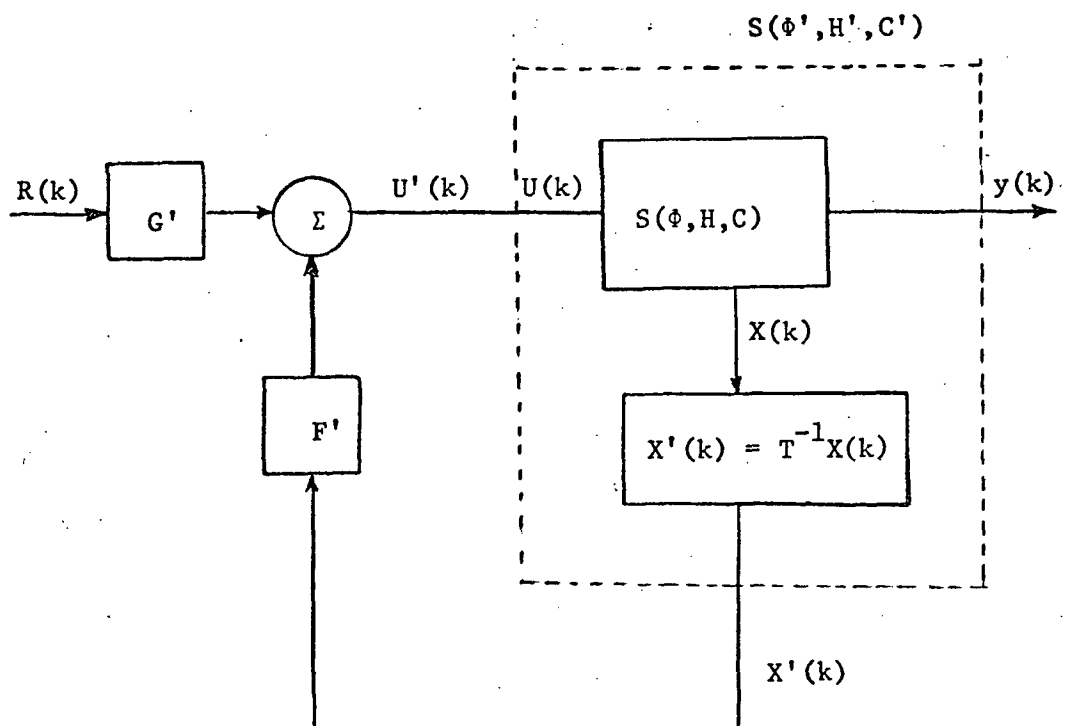
$$\vdots$$

$$p'_2 - f'_2 = q_2$$

$$f'_2 = p'_2 - q_2$$

$$p'_1 - f'_1 = q_1$$

$$f'_1 = p'_1 - q_1$$



Note: $U'(k)$ = closed loop feedback for subsystem in phase variable form $S(\phi', H', C')$.

$U(k)$ = closed loop feedback for original subsystem $S(\phi, H, C)$.

Fig. 4.1 Phase Variable Transformed System

Thus, $U(k) = [f'_1 \ f'_2 \ \dots \ f'_n] X'(k) + g' r(k)$ can be determined. Substituting $X'(k) = T^{-1} X(k)$, we get

$$U(k) = [f'_1 \ f'_2 \ \dots \ f'_n] T^{-1} X(k) + g' r(k)$$

In order to have the desired transfer function of Eq. (4.14), the closed loop feedback to the subsystem of the CD system must satisfy [see Fig. 4.1]

$$U(k) = F X(k) + G R(k)$$

where $F = [f'_1 \ f'_2 \ \dots \ f'_n] T^{-1}$ and $G = g'$. Reinserting the subscripts which we had dropped in Eq. (4.12) for convenience

$$\hat{U}^i(k) = \hat{F}^i \hat{X}^i(k) + \hat{G}^i Y_i(k).$$

Thus, $\hat{U}_i$ can be calculated for all the m subsystems. Hence, by assembling all $\hat{F}^i$ and $\hat{G}^i$ in partitioned matrix form, $\hat{F}$ and $\hat{G}$ in Eq. (4.5) for the whole CD system can be calculated. We have seen in the last chapter that once $\hat{F}$ and $\hat{G}$ for the CD system are known, F and G for the original system can be calculated as given by Eq. (3.14) and (3.15).

CHAPTER 5

INACCESSIBLE STATE VARIABLES

So far the desired transfer function is based upon state variable feedback given by:

$$U(k) = F X(k) + G R(k).$$

Obviously, this requires that all the states be available for feedback at all times k . This is seldom true in practice. In most systems either the outputs or a noisy measurement of the internal state variables is all that is available.

If the system is observable and no noise is present, the states can be estimated from the input and output measurements which is relatively simple in discretized systems. The problem of estimating states reduces to solving a set of n linear simultaneous equations. Although in practice, some noise is always generated because of roundoff error, the effect can be considerably reduced by increasing the word length of the computer. The following development assumes that the word length available is sufficient for all practical purposes.

Time Considerations

We assume that the present state and output only depend upon the past values of the input, i.e., no direct path exists from the input to the output. This assumption is true for systems which cannot respond

instantaneously. Since all computers take a finite time to make a computation, the current values of the output will not be used in estimating the current values of the states. Thus, the states will be available when they are needed provided that the estimation time is less than the sample period. This is necessary for on-line control. Once again the system equations in discrete time are given by

$$X(k+1) = \Phi X(k) + H U(k) \quad (5.1)$$

$$Y(k) = CX(k) \quad (5.2)$$

Since the system is observable and since Φ^{-1} exists, we can multiply both sides of Eq. (5.1) by Φ^{-1} to get

$$\Phi^{-1} X(k+1) = X(k) + \Phi^{-1} HU(k) \quad (5.3a)$$

Shifting one step in time backward

$$\Phi^{-1} X(k) = X(k-1) + \Phi^{-1} HU(k-1) \quad (5.3b)$$

and similarly,

$$\Phi^{-1} X(k-1) = X(k-2) + \Phi^{-1} HU(k-2) \quad (5.3c)$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \Phi^{-1} X(k-p) & = X(k-p-1) + \Phi^{-1} HU(k-p-1) & (5.3d) \end{array}$$

for all integer values of p . Multiplying the equation in (5.3a) by C on both sides and noting that $C X(k-1) = Y(k-1)$, we get

$$C\Phi^{-1} X(k) = Y(k-1) + C\Phi^{-1} HU(k-1) \quad (5.4a)$$

For a system of order n there are n unknown states and, hence, we need at least n simultaneous linear equations to solve for the n unknowns. But for a multiple input-multiple output system Eq. (5.4a) is a set of m simultaneous equations so we need p such equations where

$$p = n/m \text{ if } n/m \text{ is an integer}$$

$$p = \text{next greatest integer if } n/m \text{ is not an integer}$$

In the first case, we will have n simultaneous equations and n unknowns and in the second case we will have n unknowns and a little more than n equations. Multiply Eq. (5.3a) by ϕ^{-1} on both sides and substituting from Eq. (5.3c)

$$\begin{aligned}\phi^{-2} X(k) &= \phi^{-1} X(k-1) + \phi^{-2} HU(k-1) \\ &= X(k-2) + \phi^{-1} HU(k-2) + \phi^{-2} HU(k-1)\end{aligned}$$

Multiplying both sides by C and noting that $Y(k-2) = CX(k-2)$, we get

$$C\phi^{-2} X(k) = Y(k-2) + C\phi^{-2} HU(k-1) + C\phi^{-1} HU(k-2) \quad (5.4b)$$

Similarly by multiplying (5.3b) by ϕ^{p-1} and substituting from Eq. (5.3) we get

$$\begin{aligned}\phi^{-p} X(k) &= \phi^{-(p-1)} X(k-1) + \phi^{-p} HU(k-1) \\ &= \phi^{-(p-2)} X(k-2) + \phi^{-p} HU(k-1) + \phi^{-(p-1)} HU(k-2) \\ &= X(k-p) + \phi^{-p} HU(k-1) + \phi^{-(p-1)} HU(k-2) + \dots \phi^{-1} HU(k-p)\end{aligned}$$

Multiplying by C on both sides and noting that $Y(k-p) = CX(k-p)$, we get

$$C\Phi^{-p}X(k) = Y(k-p) + C\Phi^{-p}HU(k-1) + C\Phi^{-(p-1)}HU(k-2) + \dots C\Phi^{-1}HU(k-p) \quad (5.4c)$$

Rewriting Eq. (5.4a) through (5.4c) in partitioned matrix form

$$\begin{bmatrix} C\Phi^{-1} \\ C\Phi^{-2} \\ \vdots \\ C\Phi^{-p} \end{bmatrix} X(k) = \begin{bmatrix} Y(k-1) \\ Y(k-2) \\ \vdots \\ Y(k-p) \end{bmatrix} + \begin{bmatrix} C\Phi^{-1}H & 0 & \dots & 0 \\ C\Phi^{-2}H & C\Phi^{-1}H & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C\Phi^{-p}H & C\Phi^{-(p-1)}H & \dots & C\Phi^{-1}H \end{bmatrix} \begin{bmatrix} U(k-1) \\ U(k-2) \\ \vdots \\ U(k-p) \end{bmatrix} \quad (5.5a)$$

or, in a more concise form,

$$[MCO] X(k) = [YKP] + [MCOH][UKP]$$

where MCO , YKP , $MCOH$ and UKP each represent a matrix in Eq. (5.3b)

corresponding to a matrix of Eq. (5.5a) in the same position. Hence,

$$X(k) = [MCO]^{-1} \left[[YKP] + [MCOH][UKP] \right] \quad (5.6)$$

Thus, in Eq. (5.6) all the quantities are known by time $(k-1)$ and hence $X(k)$ the states at time k are known some time after the time $k-1$, so that they are available for computing control at time k .

It is to be noted that if n/m is not an integer, then $[MCO]$ is not a square matrix and it has a dimension of $(p \times m) \times n$. One has to be very careful in calculating $[MCO]^{-1}$ in case $[MCO]$ is not square. In writing the computer program for this study we used the left inverse so that

$$[MCO]_{n \times (p \times m)}^{-1} = [MCO]_{n \times (p \times m)}^T \left[[MCO]_{(p \times m) \times n} \times [MCO]_{n \times (p \times m)}^T \right]^{-1}$$

So that

$$[X(k)]_{n \times 1} = [MCO]_{n \times (p \times m)}^{-1} \left[[YKP]_{(p \times m) \times 1} + [MCOH]_{(p \times m) \times (p \times m)} [UKP]_{(p \times m) \times 1} \right]$$

Hence, a consistent set of equations is obtained.

A separate computer program has been written to calculate the matrices $[MCO]^{-1}$ and $[MCOH]$ given the system matrices Φ , C , H and n and m for the system. The program has been named subroutine INACCESS.

CHAPTER 6

EXAMPLES

Part I. Simple Example

Problem Formulation

The theory of the previous chapters is illustrated in this chapter by a simple example. This example has been taken from Slivinsky [1969]. The example which he worked in continuous time will be worked in discrete time with our desired transfer function in z domain. We start with the problem in continuous time and convert it into a discrete time problem. A block diagram of the system in the S domain is shown in Fig. 6.1.

The system in the S domain is described in state variable form by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 10 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

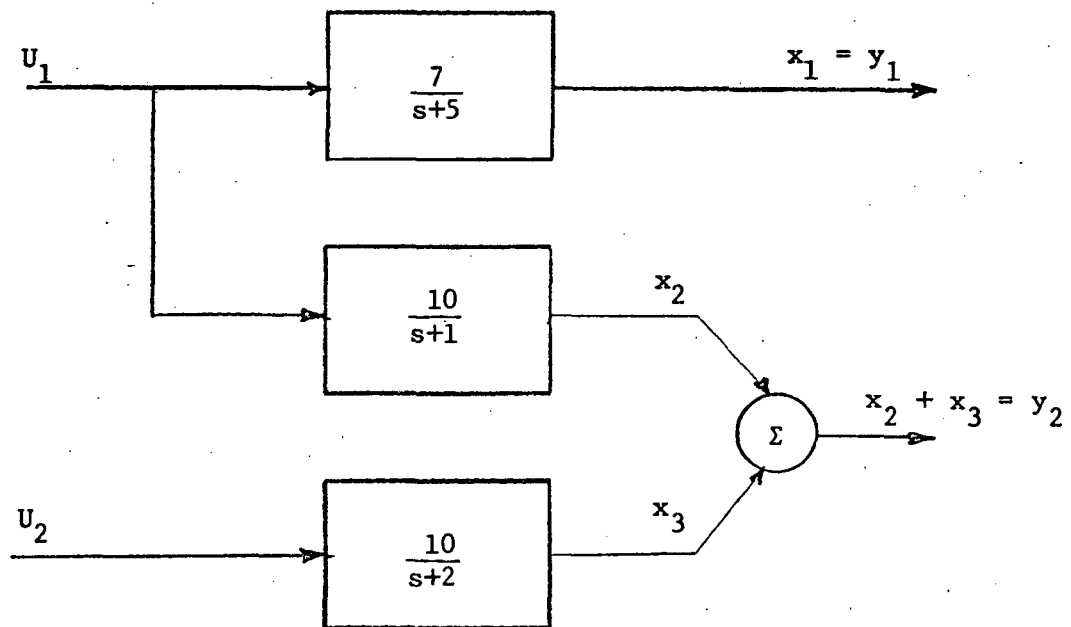


Fig. 6.1 Block Diagram of the System in s-domain

In shorthand notation the system represented by Eq. (2.1) and (2.2) is

$$\begin{aligned}\dot{X}(t) &= AX(t) + BU(t) \\ Y(t) &= C X(t)\end{aligned}\tag{6.1}$$

First of all a choice of the sampling time T has to be made. From the block diagram in Fig. 6.1, the system has time constant of $1/5$, 1 , and $1/2$ secs. So a choice of $T = 1/20$ (this choice is somewhat arbitrary) will be adequate for most purposes since the sampling rate is four times faster than the fastest time constant in the system. Prefiltering of high frequency inputs is necessary to prevent aliasing errors. By definition of the state transition matrix

$$\Phi(T) = \Phi\left(\frac{1}{20}\right) \equiv I + \frac{1}{20} A + \frac{\left(\frac{1}{20} A\right)^2}{2!} + \dots$$

we have

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

For a diagonal A matrix the Φ matrix is also diagonal,

$$\Phi(T) = \begin{bmatrix} e^{-5T} & 0 & 0 \\ 0 & e^{-T} & 0 \\ 0 & 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} .7788 & 0 & 0 \\ 0 & .9512 & 0 \\ 0 & 0 & .9124 \end{bmatrix}$$

The control transition matrix is defined by

$$H = \left[\int_0^T \Phi(\lambda) d\lambda \right] \quad B = \int_0^T \begin{bmatrix} e^{-5\lambda} & 0 & 0 \\ 0 & e^{-\lambda} & 0 \\ 0 & 0 & e^{-2\lambda} \end{bmatrix} d\lambda \quad [B]$$

which in this case is

$$= \begin{bmatrix} \frac{1}{5} e^{-5\lambda} & 0 & 0 \\ 0 & -e^{-\lambda} \\ 0 & 0 & -\frac{1}{2} e^{-2\lambda} \end{bmatrix}_0^T [B] = \begin{bmatrix} \frac{1}{5}(1-e^{-5T}) & 0 & 0 \\ 0 & (1-e^{-T}) & 0 \\ 0 & 0 & \frac{1}{2}(1-e^{-2T}) \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{5}(1-e^{-5T}) & 0 \\ 10(1-e^{-T}) & 0 \\ 0 & 5(1-e^{-2T}) \end{bmatrix} = \begin{bmatrix} .3096 & 0 \\ .4877 & 0 \\ 0 & .4771 \end{bmatrix}$$

Thus, the discrete time system equations to two decimal places are given as shown below.

$$X(k+1) = \begin{bmatrix} .78 & 0 & 0 \\ 0 & .95 & 0 \\ 0 & 0 & .91 \end{bmatrix} X(k) + \begin{bmatrix} .31 & 0 \\ .49 & 0 \\ 0 & .48 \end{bmatrix}$$

$$Y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} X(k)$$

The new discrete time system can now be put into a block diagram format by utilizing the Z transform of the state equations as shown in Fig. 6.2. At this point the feedback and gain matrices which will decouple the system may be calculated. For this case $n=3$ and $m=2$.

Step-by-Step Solution

First determine the scalar d_i where $d_i = \text{minimum } [j: c_i \phi^j H \neq 0 \text{ } j = 0, 1, 2]$ calculations for d_1 :

$$\text{With } j = 0 : C_1 \phi^0 H = [1 \quad 0 \quad 0] \begin{bmatrix} .31 & 0 \\ .49 & 0 \\ 0 & .48 \end{bmatrix} \neq 0$$

therefore

$$d_1 = 0 \quad (6.3)$$

For d_2

$$\text{With } j = 0 : C_2 \phi^0 H = [0 \quad 1 \quad 1] \begin{bmatrix} .31 & 0 \\ .49 & 0 \\ 0 & .48 \end{bmatrix} \neq 0$$

therefore

$$d_2 = 0 \quad (6.4)$$

Step 2: Now check to see if the system can be decoupled:

$$H^* = \begin{bmatrix} C_1 \phi^{d_1} H \\ C_2 \phi^{d_2} H \end{bmatrix} = CH = \begin{bmatrix} .31 & 0 \\ .49 & .48 \end{bmatrix}$$

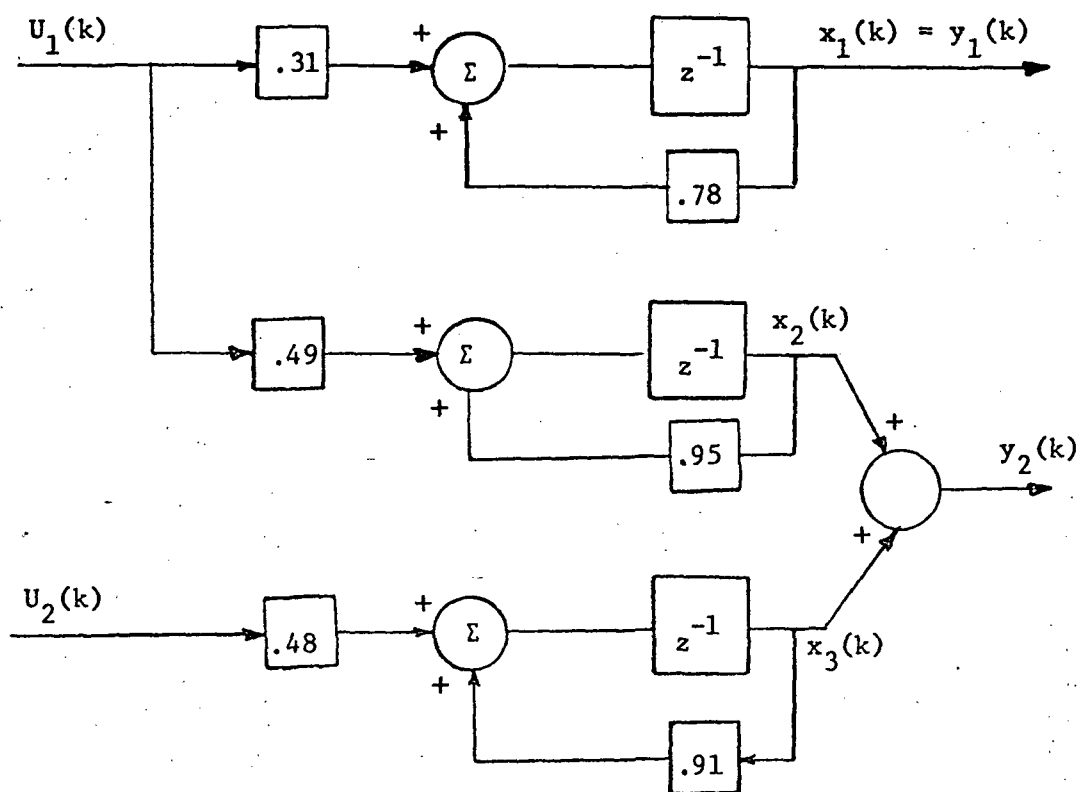


Fig. 6.2 Block Diagram of the System in z -domain

In this case H^* is nonsingular and the system can be decoupled.

Step 3: Next find the matrices which delay decouple the system

[Eq. (2.19)]

$$\begin{aligned}\phi^* &= \begin{bmatrix} C_1 \phi^{d1+1} \\ C_2 \phi^{d2+1} \end{bmatrix} = \begin{bmatrix} C_1 \phi \\ C_2 \phi \end{bmatrix} = C\phi \\ &= \begin{bmatrix} .78 & 0 & 0 \\ 0 & .95 & .91 \end{bmatrix}\end{aligned}$$

By definition

$$G^* = H^{*-1} = \begin{bmatrix} 3.22 & 0 \\ -3.30 & 2.09 \end{bmatrix}$$

and also

$$\begin{aligned}F^* &= -H^{*-1} \phi^* \\ &= \begin{bmatrix} -2.51 & 0 & 0 \\ 2.56 & -1.99 & -1.91 \end{bmatrix}\end{aligned}$$

Step 4: Next find the system equations of the DD system after the above feedback is applied. The effect of the feedback is found from

$$HF^* = \begin{bmatrix} -0.78 & 0 & 0 \\ -1.22 & 0 & 0 \\ 1.22 & -.95 & -.91 \end{bmatrix}$$

The new state transition matrix

$$\bar{\Phi} = \Phi + HF^* = \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 1.22 & -.95 & 0 \end{bmatrix}$$

and the new control transition matrix

$$\bar{H} = HG^* = \begin{bmatrix} 1.0 & 0 \\ 1.57 & 0 \\ 1.01 & 1.0 \end{bmatrix}$$

Thus, the equivalent delay decoupled system is given by the equations

$$X(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ -122 & .95 & 0 \\ 122 & .95 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1.01 & 0 \\ 1.58 & 0 \\ -1.58 & 1.0 \end{bmatrix} \bar{U}(k) \quad (6.5)$$

The input output z domain transfer function of the DD system is given by Eq. (2.22)

$$\frac{y_1(z)}{r_1(z)} = \frac{1}{z^{d_1+1}} = \frac{1}{z}$$

$$\frac{y_2(z)}{r_2(z)} = \frac{1}{z^{d_2+1}} = \frac{1}{z}$$

In matrix form, the transfer function is

$$\frac{Y(z)}{R(z)} = \begin{bmatrix} \frac{1}{z} & 0 \\ 0 & \frac{1}{z} \end{bmatrix}$$

so that the desired delay decoupled response has been obtained introducing a pole zero cancellation [as shown in Fig. 6.3].

The diagram of Fig. 6.4 shows the original system, the feedback F^* and the gain G^* to delay decouple the system.

Step 5: Next canonically decouple the DD system. As discussed in Chapter 3, the key point in searching for a transformation matrix is to find the dimensions of a matrix $\bar{Q}$ which determines the elements of the transformation matrix Q . Since $m=2$ and $n=3$, the form of $\bar{Q}$ is

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1 \\ \bar{Q}_2 \end{bmatrix} \quad \text{where } \bar{Q}_1 = [q_{11} \ q_{12} \ q_{13}]$$

The number of different non zero elements in each row is found from the equations

$$\bar{Q}_1 \phi^0 H_2 = 0 \quad (a) \qquad \bar{Q}_2 \phi^0 H_1 = 0 \quad (d)$$

$$\bar{Q}_1 \phi H_2 = 0 \quad (b) \qquad \bar{Q}_2 \phi H_1 = 0 \quad (e)$$

$$\bar{Q}_1 \phi^2 H_2 = 0 \quad (c) \qquad \bar{Q}_2 \phi^2 H_1 = 0 \quad (f)$$

From Eq. (a)

$$[q_{11} \ q_{12} \ q_{13}] \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix} = 0$$

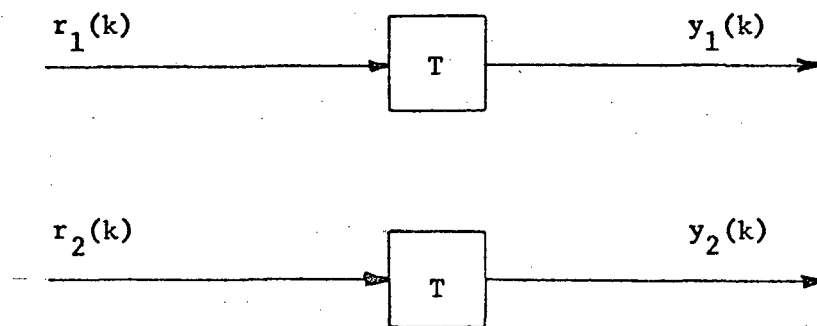


Fig. 6.3 Resultant Block Diagram of Delay Decoupled System

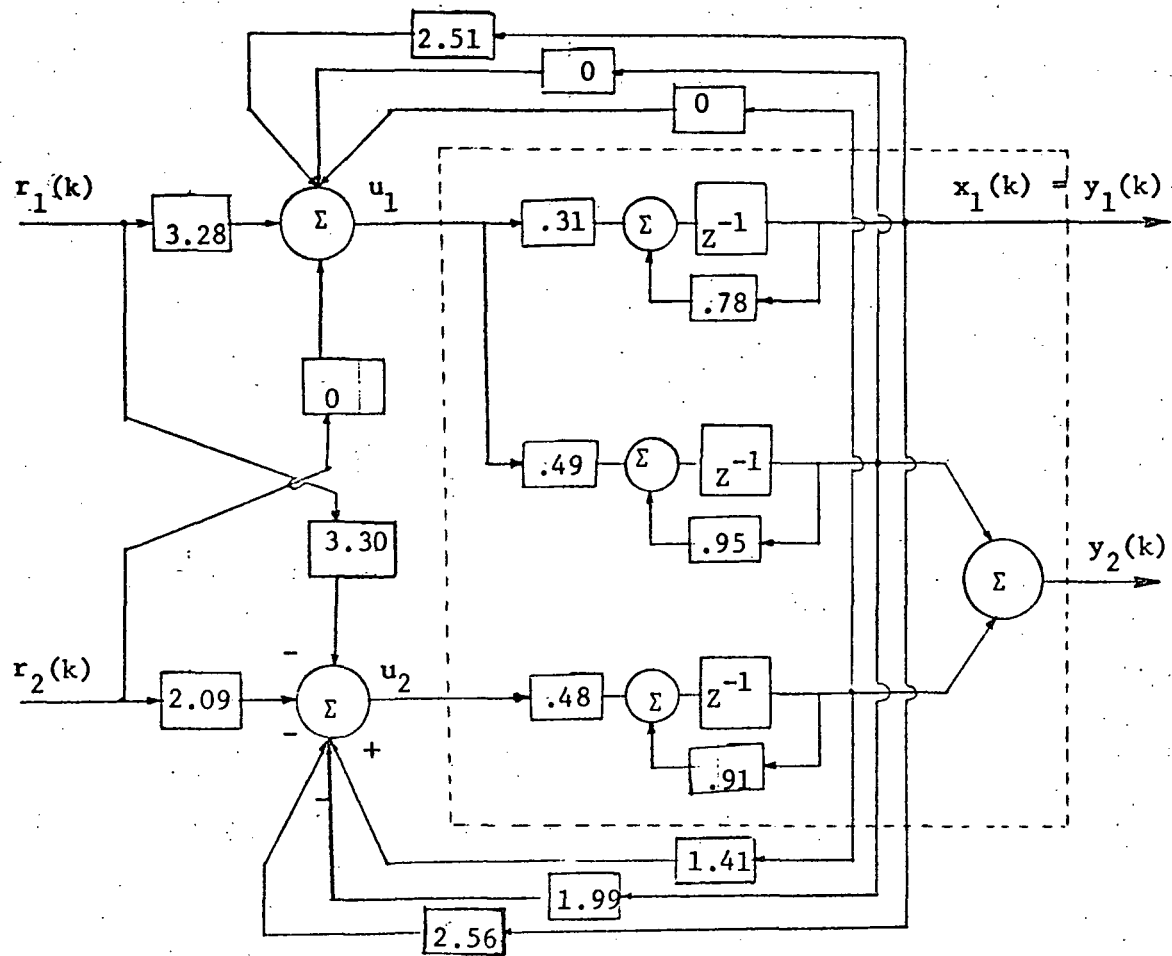


Fig. 6.4 Block Diagram of the Delay Decoupled System with Feedback Shown

Thus

$$q_{13} = 0.$$

From Eq. (b)

$$[q_{11} q_{12} q_{13}] \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 1.22 & -.95 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix} = 0$$

No new information

From Eq. (c)

$$[q_{11} q_{12} q_{13}] \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 1.22 & -.95 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix} = 0$$

No new information

Therefore, we conclude

$$\bar{Q}_1 = [q_{11} \quad q_{12} \quad 0]$$

$$\dim \bar{Q}_1 = 2 = n_1 = \text{order of subsystem 1.}$$

Similarly, from Eq. (d)

$$[q_{21} q_{22} q_{23}] \begin{bmatrix} 1.0 \\ 1.58 \\ -1.58 \end{bmatrix} = 0$$

Thus

$$q_{21} + 1.58(q_{22} - q_{23}) = 0 \quad (g)$$

From Eq. (e)

$$[q_{21} q_{22} q_{23}] \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 1.22 & -.95 & 0 \end{bmatrix} \begin{bmatrix} 1.0 \\ 1.58 \\ -1.58 \end{bmatrix} = 0$$

Thus

$$[q_{21} q_{22} q_{23}] \begin{bmatrix} 0 \\ -.607 \\ +.607 \end{bmatrix} = 0$$

and

$$q_{22} = q_{23} \quad (b)$$

From Eq. (f)

$$[q_{21} q_{22} q_{23}] \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 1.22 & -.95 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1.58 \\ -1.58 \end{bmatrix} = 0$$

Thus

$$(q_{22} - q_{23})(.016) = 0 \quad \text{and} \quad q_{22} = q_{23} \quad (i)$$

which is the same as Eq. (h). Substituting Eq. (h) in Eq. (g), we get

$$q_{21} = 0,$$

thus

$$\bar{Q}_2 = [0 \quad q_{22} \quad q_{22}]$$

and since the $\dim \bar{Q}_2$ is the number of non zero different elements the order of subsystem 2 is

$$n_2 = 1$$

The transformation matrix Q is of the form
$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}$$

where

the number of rows of $Q_1 = n_1 = 2$

the number of rows of $Q_2 = n_2 = 1$

the number of rows of $Q_3 = n - n_1 - n_2 = 3 - 2 - 1 = 0$

From Eq. (6.4) and (6.5) $d_1 + 1 = 1$ and $d_2 + 1 = 1$. Now referring back to Chapter 3, the first $d_1 + 1 = 1$ rows of Q_1 is C_1 , and the first $d_2 + 1 = 1$ row of Q_2 is C_2 . But Q_2 has only one row, hence Q_2 is completely known. The remaining rows ($2 - 1 = 1$) of Q_1 are any row vectors independent of the rows of Q_1 and Q_2 . Thus, we have

$$Q = \begin{bmatrix} C_1 \\ \text{any independent row vector} \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \text{any independent row vector} \\ 0 & 1 & 1 \end{bmatrix}$$

A simple choice of the second row such as $[0 \quad 1 \quad 0]$ yields

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

This transformation to the DD systems determines the canonical representation of the system as given in Eq. (3.8) and (3.9).

The new states after transformation are defined as $\hat{X}$ and the new state equations are

$$\begin{aligned} \hat{X}(k+1) &= \hat{\Phi} \hat{X}(k) + \hat{H} \bar{U}(k) \\ Y(k) &= \hat{C} \hat{X}(k) \end{aligned} \quad (6.6)$$

where

$$\hat{X} = Q^{-1} X$$

and

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

then the new state transition matrix is

$$\hat{\Phi} = Q \bar{\Phi} Q^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the new control transition matrix is

$$\hat{H} = QH = \begin{bmatrix} 1 & 0 \\ 1.58 & 0 \\ 0 & 1 \end{bmatrix}$$

and the new output matrix is

$$C = C Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Writing Eq. (6.6) in partitioned form

$$\hat{X}(k+1) = \begin{bmatrix} 0 & 0 & 0 \\ -1.22 & .95 & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{X}(k) + \begin{bmatrix} 1 & 0 \\ 1.58 & 0 \\ 0 & 1 \end{bmatrix} U(k) \quad (6.7)$$

$$Y(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \hat{X}(k) \quad (6.8)$$

Thus, the $\hat{\Phi}$, $\hat{H}$ and $\hat{C}$ matrices are partitioned as shown in Chapter 4 on Page 25. This partitioning shows the separation of the system into the subsystems as shown below. A superscript of 1 denotes subsystem 1, etc. In this case subsystem 1 is given by

$$\hat{X}^1(k+1) = \begin{bmatrix} 0 & 0 \\ -1.22 & .95 \end{bmatrix} \hat{X}^1(k) + \begin{bmatrix} 1 \\ 1.58 \end{bmatrix} U_1(k) \quad (6.9)$$

$$Y_1(k) = [1 \quad 0] \hat{X}^1(k) \quad (6.10)$$

Subsystem 2 is given by

$$\hat{X}^2(k+1) = [0] \hat{X}^2(k) + [1] U_2(k) \quad (6.11)$$

$$Y_2(k) = \hat{X}^2(k) \quad (6.12)$$

The input output transfer function of subsystem 1 is given by

$$\begin{aligned} H^1(z) &= \hat{C}^1 (zI - \hat{\Phi}^1)^{-1} \hat{H}^1 \\ &= [1 \ 0] \begin{bmatrix} z & 0 \\ 1.22 & z-.95 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1.58 \end{bmatrix} \\ &= [1 \ 0] \begin{bmatrix} z-.95 & 0 \\ -1.22 & z \end{bmatrix} \begin{bmatrix} 1 \\ 1.58 \end{bmatrix} \\ &\quad \underline{z(z-.95)} \\ &= \frac{(z-.95)}{z(z-.95)} \end{aligned} \quad (6.13)$$

Similarly

$$H^2(z) = [1][z]^{-1}[1] = \frac{1}{z} \quad (6.14)$$

It is worth noticing the presence of both a pole and a zero in subsystem 1 which was not apparent before canonically decoupling the system.

The DD system of Eq. (6.5) did not show the internal behavior of the system. The transfer function of the subsystem 1 was $1/z$ which is the same as subsystem 2 as far as the input-output properties are

concerned so that the cancellation of a zero of subsystem 1 was not obvious.

The total transfer function of the CD system is given by

$$H(z) = \begin{bmatrix} \frac{z-.95}{z(z-.95)} & 0 \\ 0 & \frac{1}{z} \end{bmatrix}$$

Step 5: Find feedback and gain matrices to achieve a given (desired) decoupled response.

Let us use the desired response

$$H_d(z) = \begin{bmatrix} \frac{2(z-.95)}{(z-.5)^2 + .5} & 0 \\ 0 & \frac{5}{z+.8} \end{bmatrix}$$

One limitation on the choice of a desired transfer function as discussed in Chapter 4 is the position of the subsystem zero because zeros are not affected by state variable feedback. Also, the order of each subsystem must not be increased. Otherwise, the choice is arbitrary.

Clearly $H_d(z)$ is made of two parts. Part 1 is

$$H_d^1(z) = \frac{2(z-.95)}{(z-.5)^2 + 5} \quad (6.15)$$

to be realized by subsystem 1 and

$$H_d^2(z) = \frac{5}{z+.8} \quad (6.16)$$

to be realized from subsystem 2. We will now treat the subsystems as a single input, single output systems and realize the desired transfer function by state variable feedback.

Subsystem 1

One computationally feasible method uses a transformation matrix T which transforms the subsystem into phase variable form. Using the algorithm given in Chapter 4, Page 30, T is determined from the subsystem matrices as follows

$$T = [T_1 \quad T_2]$$

where

$$T_2 = \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.58 \end{bmatrix}$$

$$\begin{aligned} T_1 = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} &= AT_2 + p_1 T_2 = \begin{bmatrix} 0 & 0 \\ -1.22 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ 1.58 \end{bmatrix} + (-.95) \begin{bmatrix} 1 \\ 1.58 \end{bmatrix} \\ &= \begin{bmatrix} -.95 \\ -1.22 \end{bmatrix} \end{aligned}$$

$$T = \begin{bmatrix} -.95 & 1 \\ -1.22 & 1.58 \end{bmatrix}$$

and

$$T^{-1} = \frac{\begin{bmatrix} 1.58 & -1 \\ 1.22 & -.95 \end{bmatrix}}{-1.58 \times .95 + 1.22 \times 1} = \begin{bmatrix} -5.62 & 3.55 \\ -4.34 & 3.38 \end{bmatrix}$$

With the transformation

$$\hat{X}^1(k) = TX'(k)$$

the system in phase variable form is given in Eq. (6.16). Note that in this case the system representation in phase variable form could have been done just by inspection of the system transfer function.

$$X'(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & .95 \end{bmatrix} X'(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_1(k) \quad (6.15)$$

$$Y(k) = [-.95 \quad 1] X'(k)$$

Let our state variable feedback be

$$U_1(k) = f'_1 X'_1(k) + f'_2 X'_2(k) + g r_1(k)$$

Substituting in Eq. (6.15) and rearranging

$$X'(k+1) = \begin{bmatrix} 0 & 1 \\ f'_1 & f'_2 + .95 \end{bmatrix} X'(k) + \begin{bmatrix} 0 \\ g \end{bmatrix} r_1(k)$$

Then the new transfer function is given by

$$H^1(z) = \frac{g(z-.95)}{z^2 - (f'_2 + .95)z - f'_1} \quad (6.17)$$

In order for this transfer function to be the same as our desired transformation of Eq. (6.15)

$$H_d^1(z) = \frac{2(z-.95)}{(z-.5)^2 + (.5)^2} = \frac{2(z-.95)}{z^2 - z + .5}$$

we must equate the coefficients in the two cases,

$$g = 2$$

$$-(f' + .95) = -1$$

$$-f'_1 = .5$$

thus

$$f'_2 = .05$$

$$f'_1 = -0.5$$

the control input to the plant is

$$U_1(k) = [-0.5 \quad .05] X'(k) + 2 r_1(k)$$

Substituting for $X'(k)$

$$\begin{aligned} U_1(k) &= [-0.5 \quad .05] T^{-1} \hat{X}^1(k) + 2 r_1(k) \\ &= [-0.5 \quad .05] \begin{bmatrix} -5.62 & 3.55 \\ -4.34 & 3.38 \end{bmatrix} \hat{X}^1(k) + 2 r_1(k) \end{aligned}$$

so that the control input in terms of the CD variables are

$$= [2.59 \quad -1.61] \hat{X}^1(k) + 2 r_1(k)$$

Subsystem 2

Since subsystem 2 is a first order system, it is already in phase variable form so that $T = 1$. The CD system is

$$\hat{X}^2(k+1) = [0] \hat{X}^2(k) + U_2(k)$$

and the control input is

$$U_2(k) = f \hat{X}^2(k) + g_2 r_2(k)$$

substituting into the above equation we get

$$\hat{X}^2(k+1) = [f] \hat{X}^2(k) + g_2 r_2(k)$$

and

$$Y_2(k) = [1] \hat{X}^2(k)$$

The actual transfer function is

$$H^2(z) = \frac{g_2}{z-f}$$

Comparing this with our desired transfer function, $H_d^2(z) = \frac{5}{z+.8}$ we get

$$f_2 = -.8 \qquad g_2 = 5$$

$$U_2 = [-.8] \hat{X}^2(k) + 5 r_2(k)$$

The feedbacks and gains of the subsystems are now combined in partitioned matrix form as shown in Chapter 4, Page 27 to give the overall feedback and gain matrices for the CD system.

$$\hat{F} = \begin{bmatrix} \hat{F}^1 & 0 \\ 0 & \hat{F}^2 \end{bmatrix} = \begin{bmatrix} 2.59 & -1.61 & 0 \\ 0 & 0 & -0.8 \end{bmatrix}$$

$$\hat{G} = \begin{bmatrix} \hat{G}^1 & 0 \\ 0 & \hat{G}^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Thus the closed loop feedback around the CD system is

$$\hat{U}(k) = \begin{bmatrix} 2.59 & -1.61 & 0 \\ 0 & 0 & -0.8 \end{bmatrix} \hat{X}(k) + \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} R(k)$$

Once $\hat{F}$ and $\hat{G}$ are known, the combined feedback (F) and gain (G) matrices for the original system are readily obtained from Eq. (3.13) and (3.14).

$$F = F^* + G^* \hat{F} Q = \begin{bmatrix} 5.93 & -5.23 & 0 \\ -6.0 & 1.67 & 22 \end{bmatrix}$$

$$G = G^* \hat{G} = \begin{bmatrix} 6.5 & 0 \\ -6.6 & 10.4 \end{bmatrix}$$

[F^* and G^* have been calculated in Step 3]

Thus, the combined closed loop feedback around the system to obtain the desired decoupled response is given by

$$U(k) = F X(k) + G R(k) \quad (6.18)$$

Inaccessible States

This part demonstrates how system states can be calculated based only upon the observation of past outputs and inputs. Since $n = 3$ and $m = 2$, as discussed in Chapter 5, Page 37 two observations are needed to reconstruct the states X for the output Y and the input U .

$$Y(k) = \begin{bmatrix} C\phi^{-1} \\ C\phi^{-2} \end{bmatrix}^{-1} \begin{bmatrix} Y(k-1) \\ Y(k-2) \end{bmatrix} + \begin{bmatrix} C\phi^{-1}H & 0 \\ C\phi^{-2}H & C\phi^{-1}H \end{bmatrix} \begin{bmatrix} U(k-1) \\ U(k-2) \end{bmatrix} \quad (6.19)$$

where C , ϕ , H , and Y are as given in Eq. (6.2) and U is as given in Eq. (6.18), the inverses of the state transition matrix is

$$\phi^{-1} = \begin{bmatrix} 1.29 & 0 & 0 \\ 0 & 1.05 & \\ 0 & 0 & 1.10 \end{bmatrix}$$

and

$$C\phi^{-1} = \begin{bmatrix} 1.29 & 0 & 0 \\ & 1.05 & 1.10 \end{bmatrix}$$

$$C\phi^{-1}H = \begin{bmatrix} .40 & 0 \\ .58 & .52 \end{bmatrix}$$

$$C\Phi^{-2} = C\Phi^{-1}(\Phi^{-1}) = \begin{bmatrix} 1.66 & 0 & 0 \\ 0 & 1.1 & 1.2 \end{bmatrix}$$

$$C\Phi^{-2}_H = \begin{bmatrix} .51 & 0 \\ .54 & .57 \end{bmatrix}$$

Rewriting Eq. (6.19) in the symbolic form used in Eq. (5.6), page 38

$$X(k) = [MCO]^{-1} [[Ykp] + [MCOH][Ukp]]$$

$\begin{matrix} 3 \times 1 & 3 \times 4 & 4 \times 1 & 4 \times 4 & 4 \times 1 \end{matrix}$

We have

$$[MCO] = \begin{bmatrix} 1.29 & 0 & 0 \\ 0 & 1.05 & \\ 1.65 & 0 & 0 \\ 0 & 1.1 & 1.20 \end{bmatrix}$$

$$[MCOH] = \begin{bmatrix} .40 & 0 & 0 & 0 \\ .51 & .52 & 0 & 0 \\ .51 & 0 & .40 & 0 \\ .54 & .57 & .51 & .52 \end{bmatrix}$$

Since $[MCO]$ is not a square matrix, its left inverse is found by the following formula

$$([MCO][MCO]^T)([MCO][MCO]^T)^{-1} = I \text{ the identity matrix,}$$

Multiplying both sides by $[MCO]^{-1}$ we get

$$[MCO]^{-1} = [MCO]^T ([MCO][MCO]^T)^{-1}$$

we have

$$[MCO][MCO]^T = \begin{bmatrix} 1.65 & 0 & 2.12 & 0 \\ 0 & 2.30 & 0 & 2.47 \\ 2.12 & 0 & 2.73 & 0 \\ 0 & 2.47 & 0 & 2.66 \end{bmatrix}$$

Taking the inverse and multiplying by $[MCO]^T$, we get

$$[MCO]^{-1} = \begin{bmatrix} 0 & 0 & 0.6 & 0 \\ 0 & 23.27 & 0 & -21.23 \\ 0 & -21.48 & 0 & 20.36 \end{bmatrix}$$

Since everything in Eq. (6.19) is known, the state $X(k)$ can be calculated as soon as the previous values of the output and the input are available.

Part II. Nuclear Rocket Engine Control

Problem Formulation

We will now take up a very practical example of a nuclear rocket engine. This problem is being studied by many people, viz., L. E. Kendrick, M. R. Johnson. At present, the nonlinear Smith-Stenning [1961] model of a representative nuclear rocket engine is being studied. The state equations in continuous time domain are given as follows.

$$\begin{aligned}\dot{T} &= 2(N - P\sqrt{T}) \\ \dot{N} &= (65 U_d - .065 \sqrt{T} + 19.5/T - 65)N + 65C \\ \dot{C} &= 0.1(N-C) \\ \dot{P} &= 0.4 (U_p P\sqrt{T} - P^2/\sqrt{T})\end{aligned}$$

where

T = normalized temperature

N = normalized neutron density

C = normalized precursor density

P = normalized pressure

U_p = normalized pressure ratio across turbo pump
(a function of the bypass valve position)

U_d = normalized reactivity input

All the four states are normalized to unity at full power conditions and the value of controls at the operating point is obtained by setting the derivatives of the states to be zero. The outputs to be controlled are T (temperature) and P (pressure). The controller inputs are U_p and U_d .

The aim is to remove the coupling between inputs and outputs and force the system to the desired transfer function.

Define the state variables, inputs and outputs as follows:

$$T = X_1 = Y_1$$

$$N = X_2$$

$$C = X_3$$

$$P = X_4 = Y_2$$

$$U_p = u_1$$

$$U_d = u_2$$

The state x_1 should only be a function of r_1 and x_4 should only be a function of r_2 . First, let us linearize the state equations about the full power operating point. If $\dot{x} = f(x,u)$ is the set of nonlinear equations, then from Meditch [1970]

$$\dot{x} = Ax + BU$$

represents the system equations in the vicinity of operating point, where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial u_1} & \frac{\partial f_m}{\partial u_2} & \dots & \frac{\partial f_m}{\partial u_m} \end{bmatrix}$$

All the derivatives inside the matrices are evaluated at the operating point.

In this example

$$\dot{x}_1 = f_1 = 2(x_2 - x_4\sqrt{x_1}) \quad (6.20)$$

$$\dot{x}_2 = f_2 = (65u_2 - .065\sqrt{x_1} + 19.5 x_4/x_1 - 65)x_2 + 65x_3 \quad (6.21)$$

$$\dot{x}_3 = f_3 = 0.1(x_2 - x_3) \quad (6.22)$$

$$\dot{x}_4 = f_4 = 0.4(u_1 x_4 \sqrt{x_1} - x_4^2 / \sqrt{x_1}) \quad (6.23)$$

In order for the system to be in equilibrium at the operating point, the values of the controls at the operating point are selected so that the derivatives of the states are zero. From Eq. (6.21) and (6.23) the controls are

$$u_1 = 1$$

and

$$u_2 = -19.435/65.0 \quad (6.24)$$

The linearized continuous time equations of the system are found by evaluating the partials at the operating point.

$$\dot{x} = \begin{bmatrix} -1.0 & 2.0 & 0 & -2.0 \\ -19.4675 & -65 & 65 & 19.5 \\ 0 & 0.1 & -0.1 & 0 \\ 0.4 & 0 & 0 & -0.4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 65 \\ 0 & 0 \\ 0.4 & 0 \end{bmatrix} u \quad (6.25)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x \quad (6.26)$$

Since this is a fourth order system, the calculations involved in various steps are too lengthy to be presented here. The results of each step as described in the presentation of the previous example is given with two significant digits while the actual calculations are based on 15 digit accuracy.

The Computer Program

The computer program used during this study consists of four main parts described as follows:

Part 1 finds the discrete time representation of the system.

Part 2 canonically decouples a given multivariable system using the program of Gilbert and Pivnichny [1969].

Part 3 obtains the feedback and gain matrices for the original system to find a desired decoupled transfer function response. The program uses the results of Part 2 and also prints out the transfer function of the system before and after decoupling in pole-zero form.

Part 4 finds the estimator matrices ($[MCO]^{-1}$ and $[MCOH]$ of Eq. (5.7)) to predict the states based only upon the past values of inputs and outputs.

To demonstrate the validity of the results and also for the purpose of comparison, the problems were done both in continuous time and in discrete time. However, several interesting distinctions were noted for this problem.

First, the main results of continuous time decoupling are presented and then two cases of discrete time decoupling are presented.

Results of the Desired Decoupling of the Nuclear Reactor in Continuous Time

From the integrator decoupling program

$d_1 = 1$ The subsystem 1 has two more poles than zeros.

$d_2 = 0$ The subsystem 2 has one more pole than zeros.

From the canonical decoupling program

$n_1 = 2$ The subsystem 1 is second order and, hence, has no zeros.

$n_2 = 1$ The subsystem 2 is first order and, hence, has no zeros.

Since $N - n_1 - n_2 = 4 - 2 - 1 = 1$, there is one uncontrollable pole.

The desired decoupled transfer function was chosen to be

$$\frac{Y(s)}{R(s)} = \begin{bmatrix} \frac{5}{(s+1)(s+10)} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix} \quad (6.27)$$

If the desired response included a zero or if a higher order response was required, additional series compensation would be required. The required feedback and gain matrices for the original system to obtain the above desired response are

$$F = \begin{bmatrix} -1.0 & 0 & 0 & -1.5 \\ 0.29 & 0.84 & -1.0 & -0.16 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 12.5 \\ 0.04 & 0.07 \end{bmatrix}$$

The transfer function of the system with the above feedback is

$$\frac{Y(s)}{R(s)} = \begin{bmatrix} \frac{5(s+0.1)}{(s+0.1)(s+1)(s+10)} & 0 \\ 0 & \frac{5(s+0.1)}{(s+0.1)(s+1)} \end{bmatrix}$$

There is an uncontrollable pole and an uncontrollable zero of the system both at -0.1 . It is not clear with which subsystem the above pole and zeros are associated. No amount of feedbacks and gains can displace the positions of the uncontrollable poles and zeros if decoupling is to be preserved. However, we are lucky that both the pole and the zero are in a stable region and, hence, cause no problem.

Results of Decoupling the Nuclear Reactor in
Discrete Time Using a Zero Order Hold

Keeping in mind the time constants of the system, the sampling time was chosen to be $T = .01$ secs. with the above sampling time the Φ and H matrices for the discrete time system are

$$\Phi = \begin{bmatrix} 9.8E-01 & 1.4E-02 & 5.2E-03 & -1.8E-02 \\ -1.4E-01 & 5.2E-01 & 4.8E-01 & 1.4E-01 \\ -7.8E-05 & 7.3E-01 & 9.9E-01 & 7.8E-05 \\ 3.9E-03 & 3.2E-04 & 7.39E-06 & 9.9E-01 \end{bmatrix}$$

$$H = \begin{bmatrix} -3.7E-05 & 5.2E-03 \\ 3.1E-04 & 4.8E-01 \\ 1.1E-07 & 2.6E-04 \\ 4.0E-03 & 7.4E-06 \end{bmatrix}$$

$d_1 = 0$ The subsystem 1 has one more pole than zeros

$d_2 = 0$ The subsystem 2 has one more pole than zeros.

From the canonical decoupling program

$n_1 = 1$ The subsystem 1 is first order, hence, there are
no zeros.

$n_2 = 1$ The subsystem 2 is first order, hence, there are
no zeros.

In this $N - n_1 - n_2 = 4 - 1 - 1 = 2$; thus, there are two uncontrollable poles and with state variable feedback only a first order subsystem response can be realized. However, if higher order subsystem response is needed, it can be obtained by using series compensation. Series compensation will not be used in this example.

Before there was one uncontrollable pole of the continuous time system whereas now there are two uncontrollable poles. Since our input matrix H has fewer zeros than the input matrix B of the continuous time system, the inputs are coupled to each state of the system and, hence, the decoupling is much more difficult. It seems as though we pay the price of losing control over one of the poles of the system.

The ZOH approximation introduces additional zeros in the equivalent Z domain desired transfer function. For example, if we have a second order system without any zeros in the continuous time domain, then an equivalent discrete time system sampled with a ZOH will have two poles and one zero in its Z domain transfer function. This zero may require series compensation in realizing a desired response by state variable feedback.

For this problem we have two first order subsystems so only a first order response can be obtained by state variable feedback. The desired transfer function in the continuous time and the equivalent desired transfer function in discrete time for ZOH sampling is shown below.

$$\frac{Y(s)}{R(s)} = \begin{bmatrix} \frac{5}{s+10} & 0 \\ 0 & \frac{5}{s+1} \end{bmatrix} \equiv \frac{Y(z)}{R(z)} = \begin{bmatrix} \frac{0.4755}{z-.9048} & 0 \\ 0 & \frac{.05}{z-.9900} \end{bmatrix}$$

To obtain the above desired response in discrete time, the requisite feedback and gain matrices are

$$F = \begin{bmatrix} -9.65E-01 & -2.98E-03 & -1.70E-07 & -1.48E-00 \\ -1.58E+01 & -2.77E-00 & -9.49E-01 & 3.45E-00 \end{bmatrix}$$

$$G = \begin{bmatrix} -1.67E-01 & 1.25E+01 \\ 9.01E+01 & 8.92E-02 \end{bmatrix}$$

With the above feedback and gain, the new transfer function of the system is given by

$$\frac{Y(z)}{R(z)} = \begin{bmatrix} \frac{5(z-.9990)(z+.8030)}{(z-.9990)(z+.8030)(z-.9048)} & 0 \\ 0 & \frac{5(z-.9990)(z+.8030)}{(z-.9990)(z+.8030)(z-.9900)} \end{bmatrix}$$

The above transfer function is the same as the desired one except that it has two pole-zero cancellations. It is worth noticing that the above pole-zero cancellation is not under our control because, if decoupling is to be preserved, no feedback and gain matrices can affect their position. This differs from the continuous time case where we had only one uncontrollable pole and zero and, hence, subsystem 1 was a second order subsystem. Now we have only a first order subsystem. However, we are fortunate that all the uncontrollable poles and zeros are in the stable region.

Results of Discrete Time Decoupling
With Impulse Sampling

The main advantage of such a sampling is that the form of B matrix is preserved in the discrete time case. The zeros in the B matrix appear as zeros in the H matrix and thus no further coupling is introduced and, hence, all the controllable poles in the continuous time decoupling remain controllable in discrete time decoupling. The Φ matrix for the system remains the same as for the ZOH case and the H matrix for the system is given by Eq. (2.8)

$$H = BT = \begin{bmatrix} 0 & 0 \\ 0 & 0.65 \\ 0 & 0 \\ 0.04 & 0 \end{bmatrix}$$

The results of decoupling are:

Step 1:

$d_1 = 1$ The subsystem 1 has two more poles than zeros.

$d_2 = 0$ The subsystem 2 has one more pole than zero.

Step 2:

$$H^* = \begin{bmatrix} -7.30E-05 & 9.50E-03 \\ 4.00E-03 & 0.0 \end{bmatrix}$$

which is nonsingular.

Step 3: The feedback and gain matrices which delay decoupled the system are:

$$F^* = \begin{bmatrix} -9.92E-01 & -8.10E-03 & -1.85E-03 & -2.49E+02 \\ -1.02E+02 & -2.32E+00 & -1.84E+00 & 1.68E+00 \end{bmatrix}$$

$$G^* = \begin{bmatrix} 0 & 2.50E+02 \\ 1.05E+02 & 1.92E+0 \end{bmatrix}$$

Step 4: The system matrices of the DD system are:

$$\bar{\Phi} = \begin{bmatrix} 9.88E-01 & 1.46E-02 & 5.27E-03 & -1.82E-02 \\ -6.68E+01 & -9.88E-01 & -7.17E-01 & 1.23E+00 \\ -7.88E-05 & 7.34E-04 & 9.99E-01 & 7.97E-05 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{H} = \begin{bmatrix} 0 & 0 \\ 6.84E+01 & 1.25E+00 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The resulting transfer function is

$$\frac{Y(z)}{R(z)} = \begin{bmatrix} \frac{1}{z^2} & 0 \\ 0 & \frac{1}{z} \end{bmatrix}$$

Step 5: From the canonical decoupling we get

$n_1 = 2$ The subsystem 1 is second order, hence, has no zeros.

$n_2 = 1$ The subsystem 2 is first order, hence, has no zeros.

$n_3 = 1$ The system has 1 uncontrollable pole.

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 9.88E-01 & 1.46E-02 & 5.27E-03 & -1.82E-02 \\ 0 & 0 & 0 & 1 \\ 0 & -3.60E-01 & 1 & 0 \end{bmatrix}$$

After the transformation the matrices of the CD system as:

$$\hat{\Phi} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline -2.44E+01 & 4.90E+01 & 4.51E-01 & 9.99E-01 \end{array} \right]$$

$$\hat{H} = \left[\begin{array}{c|c} 0 & 0 \\ 1 & 0 \\ \hline 0 & 1 \\ \hline -2.46E+01 & -4.50E-01 \end{array} \right]$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & | & 0 & | & 0 \\ \hline 0 & 0 & | & 1 & | & 0 \end{bmatrix}$$

Partitioning $\hat{\Phi}$, $\hat{H}$ and $\hat{C}$ as shown above, the subsystems are subsystem 1

$$\hat{X}^1(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{X}^1(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_1(k) \quad (6.28)$$

$$Y_1(k) = [1 \quad 0] \hat{X}^1(k)$$

and subsystem 2

$$\hat{X}^2(k+1) = [0] \hat{X}^2(k) + [1] U_2(k)$$

$$Y_2(k) = [1] \hat{X}^2(k) \quad (6.29)$$

The transfer functions of the above subsystems of the CD system are

$$H^1(z) = \frac{1}{z^2}$$

$$H^2(z) = \frac{1}{z}$$

Step 6: Now we select a desired transfer function which is the discrete time equivalent of the transfer function for this system in continuous time [see Eq. (6.27)]. To obtain the state equations, we first write the desired transfer function in partial fraction form

$$H_d(s) = \frac{5}{(s+1)(s+10)} = \frac{5/9}{s+1} - \frac{5/9}{s+10}$$

Hence, the state equations representing the above transfer function are

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 5/9 \\ -5/9 \end{bmatrix} R$$

Using impulse sampling the equivalent discrete time system equations are

$$\mathbf{X}(k+1) = \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-10T} \end{bmatrix} \mathbf{X}(k) + \begin{bmatrix} 5/9 \\ -5/9 \end{bmatrix} T R(k)$$

$$Y(k) = [1 \quad 1] \mathbf{X}(k)$$

The transfer function in the z domain is $C(ZI - \Phi)^{-1}H$

$$H_d(z) = \frac{5/9(e^{-T} - e^{-10T})T}{(z - e^{-10T})(z - e^{-T})}$$

With a sample time, $T = .01$

$$H_d(z) = \frac{4.734E-04}{z^2 - 1.892z + .895} \quad (6.30)$$

Similarly for subsystem 2, $\frac{5}{s+1}$ has a transfer function in the z domain given by

$$H_d(z) = \frac{5T}{z-e^{-T}} = \frac{.05}{z-.99} \quad (6.31)$$

Hence, for subsystem 1, the feedback and gain for the CD subsystem 1 of Eq. (6.28) to have the desired transfer function of Eq. (6.29) is given by

$$U_1(k) = [-.89 \quad 1.89] \hat{X}^1(k) + .0004734 r_1(k)$$

and, similarly, for subsystem 2

$$U_2(k) = [.99] \hat{X}^2(k) + .05 r_2(k)$$

Combining the feedback and gain matrices in partitioned matrix form, we get feedback and gain for the overall CD system

$$\hat{F} = \left[\begin{array}{c|c|c} \hat{F}^1 & 0 & 0 \\ \hline 0 & \hat{F}^2 & 0 \end{array} \right] = \left[\begin{array}{cc|cc} -.89 & 1.89 & 0 & 0 \\ \hline 0 & 0 & .99 & 0 \end{array} \right]$$

$$\hat{G} = \left[\begin{array}{c|c} \hat{G}^1 & 0 \\ \hline 0 & \hat{G}^2 \end{array} \right] = \left[\begin{array}{c|c} 4.734E-04 & 0 \\ \hline 0 & 5.0E-02 \end{array} \right]$$

The overall feedback and gain for the original system is therefore as given below.

$$F = F^* + G^* \hat{F} Q$$

$$= \begin{bmatrix} -9.99E-01 & -8.10E-03 & -1.4E-03 & -1.48E-00 \\ 2.32E-01 & 5.92E-01 & -7.86E-01 & -6.16E-02 \end{bmatrix}$$

$$G = G^* \hat{G}$$

$$= \begin{bmatrix} 0 & 12.5E+0 \\ 4.98E-02 & 9.61E-02 \end{bmatrix}$$

Step 7: The transfer function of the system with above feedback and gain is given by

$$\frac{Y(z)}{R(z)} = \begin{bmatrix} \frac{4.734E-04(z-.9990)}{(z-.999)(z-.990)(z-.904)} & 0 \\ 0 & \frac{.05(z-.9990)}{(z-.9990)(z-.904)} \end{bmatrix}$$

Thus, the desired transfer function has been achieved. The system has one additional uncontrollable pole and zero at $z = .9990$ or $z = e^{-.001} = e^{-.1T}$, which exactly corresponds to the system pole at $-.1$ in continuous time.

Also both the continuous time system and the discrete time system were seen to have exactly the same desired decoupled response. This helps to verify the calculations made in both the discrete time and continuous time.

Inaccessible States

Results of this section are based upon the algorithm developed in Chapter 5. Since $n = 4$ and $m = 2$ as discussed in Chapter 5, page 37 exactly two observations are needed to reconstruct the states X from the output Y and input U . Referring to Eq. (5.5) the states X are given by

$$X(k) = \begin{bmatrix} C\Phi^{-1} \\ C\Phi^{-2} \end{bmatrix}^{-1} \begin{bmatrix} Y(k-1) \\ Y(k-2) \end{bmatrix} + \begin{bmatrix} C\Phi^{-1}H & 0 \\ C\Phi^{-2}H & C\Phi^{-1}H \end{bmatrix} \begin{bmatrix} U(k-1) \\ U(k-2) \end{bmatrix} \quad (6.32)$$

From part 4 of the computer program (see page 69) we get

$$\begin{bmatrix} C\Phi^{-1} \\ C\Phi^{-2} \end{bmatrix}^{-1} = \begin{bmatrix} 6.41E+00 & -2.21E+03 & 3.41E+00 & 2.20E+03 \\ 3.42E+02 & -1.46E+05 & 2.41E+02 & 1.45E+05 \\ 5.15E+02 & -2.32E+05 & 4.13E+02 & 2.31E+05 \\ 1.37E-02 & -2.90E+00 & 5.79E-03 & 3.88E+00 \end{bmatrix}$$

$$\begin{bmatrix} C\Phi^{-1} & 0 \\ C\Phi^{-2} & C\Phi^{-1}H \end{bmatrix} = \begin{bmatrix} 9.03E-05 & -1.83E-02 & 0 & 0 \\ 4.01E-03 & 3.28E-05 & 0 & 0 \\ 2.12E-04 & -5.36E-02 & 9.03E-05 & -1.83E-02 \\ 4.13E-03 & 1.69E-04 & 4.01E-03 & 3.28E-05 \end{bmatrix}$$

Since everything in Eq. (6.32) is known, the states $X(k)$ can be calculated as soon as the previous value of output $Y(k-1)$ and input $U(k-1)$ are known and, hence, on-line control can be applied.

CHAPTER 7

CONCLUSIONS

This study concerns the problem of designing a discrete-time controller for a linear time-invariant multivariable system. Any arbitrary non-interacting desired transfer-function may be achieved by the design technique. State variable feedback serves as the basis for the design. Also, an algorithm estimates the present states of the system based only upon the past inputs and outputs so that the method is applicable to on-line control. Several interesting comparisons can be made between decoupling in discrete time and decoupling in continuous time. State variable feedback is applied first to decouple the discrete time system and then once again to achieve the desired transfer function in the z -domain rather than the s -domain. The above two results are combined to produce the required overall feedback and gain matrices. The state estimation for on-line control is based upon the observability criteria of the system and on the assumption that the system cannot respond instantaneously.

Unlike the continuous time system, the behavior of the decoupled system, also depends upon the discrete time control transition matrix. Depending upon the type of sampling used, zero elements of input matrix of the continuous time system may not remain zero in

the control transition matrix of the equivalent discrete time system representation. Introducing nonzero elements in the control transition matrix makes the decoupling much more difficult. In some examples some of the system poles become uncontrollable. That is, after decoupling, some of the poles remain unaffected by state variable feedback.

If the system has uncontrollable poles in continuous time, they will appear as uncontrollable poles in the equivalent discrete time system. Further, the discrete time system may have additional uncontrollable poles. If the uncontrollable poles of the system happen to be in an unstable region of the z -domain, then the system will be internally unstable after decoupling. This does not necessarily mean that the system is inherently unstable since in most cases state variable feedback can position all the poles of the system, if decoupling is not a criteria.

The above theory of decoupling of a multivariable system in discrete time was applied to the control of a nuclear rocket engine. Since the system was highly nonlinear, a linearized set of equations representing the system in the vicinity of operating point (full power conditions) was obtained. The discrete time system representation was obtained using a zero order hold. The system has one uncontrollable pole in continuous time. In discrete time, after decoupling, one more pole became uncontrollable. Fortunately, both uncontrollable poles lie in the stable region of the z -domain. Uncontrollable poles are always cancelled by a zero at the same position so that they do not affect the transfer function of the system. By using the impulse sampling, only one uncontrollable pole corresponding to the one in the continuous time occurred. Throughout the solution the results of discrete time

decoupling were in one-to-one correspondence with the continuous time decoupling. Also, an equivalent desired decoupled transfer function was achieved. Finally, the state estimation matrices were obtained in a straight forward manner for the case when all the state variables are not available.

Step-by-Step Summary

1. The system differential equations are written in standard matrix form so that the number of states and the number of inputs and outputs of the system are evident.

2. The discrete time representation of the system is found by making a suitable choice of sampling time T and a suitable method of sampling. The ZOH is the most common mode of sampling but it does not give as straight forward results as an impulse approximation.

3. The decoupling matrix H^* is calculated as discussed in Chapter 2. If it is nonsingular, we can proceed further; otherwise, the system cannot be decoupled.

4. The feedback and gain matrices F^* and G^* are found which delay decouple the system and then the system representation for the DD system is obtained by substituting for the input $U(k)$ in terms of F^* and G^* in the original system equations. In the DD system each output is just the corresponding input delayed by a few sample periods depending upon the pole zero excess of the corresponding subsystem.

5. The system is then canonically decoupled (CD) by finding a transformation matrix Q . In the canonically decoupled system the

subsystems can be separated in a partitioned matrix form. Thus, the subsystem structure is determined. This is a remarkable improvement over the DD system where only the pole-zero excess of the subsystem is known and all the poles are lumped at the origin. The decoupled subsystems are separated and treated separately as single-input single-output systems.

6. State variable feedback techniques are applied to each individual subsystem to achieve a desired transfer function in the z -domain. This is done with the help of a transformation matrix T which transforms the subsystem into phase variable form. This transformation facilitates the computation of the feedback and gain coefficients for a given transfer function response. After the transformation is used, the system transfer function coefficients appear directly in the system matrix equations. Thus, the feedback coefficients to achieve a given transfer function are directly obtained by equating what we have with what we want.

7. The feedback and gain coefficients for each individual subsystem are assembled in the partitioned matrix form to give $\hat{F}$ and $\hat{G}$ the feedback and gain matrices for the CD system which produce the given decoupled response.

8. For the original system to have the same desired decoupled response as that of CD system, the overall feedback and gain matrices are easily obtained by combining the decoupling matrices with the $\hat{F}$ and $\hat{G}$ matrices. The design of a discrete time multivariable system is thus complete. The new system behaves as a set of subsystems each with a desired transfer function.

Further Research

The following topics are suggested as extensions of the design technique presented in this study.

1. The decoupling of a multivariable system depends upon the method of reconstruction of the input to the plant from the discrete output of the controller. Only a few guidelines have been suggested in this thesis which relate the mode of reconstruction to the problems of decoupling.
2. At present the position of the uncontrollable poles which may result from decoupling cannot be altered. It would be of value to know how to insure the stability of these poles.
3. Control of multivariable nonlinear systems may be effected by linearizing about an operating point where the linearization is updated constantly by an on-line computer control scheme.
4. The sensitivity of the closed-loop system to the control system parameters must be determined to assess the effect of round-off errors in the system.
5. Series compensation could be used to achieve a higher order closed loop response and possibly even to modify the position or the number of uncontrollable poles.
6. Rather than decoupling the system, a more general approach would allow an arbitrary system transfer function which may include coupling.

APPENDIX A

COMPUTER PROGRAM FOR DESIRED RESPONSE

Program thesis finds the required feedback and gain matrices to achieve the given decoupled transfer function response (used for canonically decoupled multivariable systems).

Input to the program consists of the following respectively.

1. From the original multivariable system

M and N = Number of inputs to the system and number of states of the system.

FEE = Control transition matrix of the system

H = Input matrix of the system

C = Output matrix of the system

2. From the output of canonical decoupling program

Q = Transformation matrix which canonically decouples the system

FCON = Feedback matrix which delay decouples the system

GCON = Gain matrix which delay decouples the system

3. From each of the subsystems of canonically decoupled system

n and l = Number of states and number of zeros

A = Control transition matrix

B = Input matrix

DECOEF = Coefficients of the desired denominator polynomial
in the subsystem transfer function (constant term on top row).

GM = Desired gain for the subsystem.

In addition to all the inputs, the output of the program consists of the following:

1. The coefficients of transfer function polynomials of the original system and their roots (positions of poles and zeros).
2. Coefficients of characteristic polynomials of each subsystem.
3. Phase variable transformation matrix for each subsystem.
4. Inverse of phase variable transformation matrix.
5. Feedback matrix for the subsystem in phase variable form.
6. Feedback for the subsystem in original form.
7. Gain for the subsystem in original form.
8. FHAT = Feedback matrix for the canonically decoupled subsystem.
9. GHAT = Gain matrix for the canonically decoupled subsystem.
10. Feedback matrix for the original system.
11. Gain matrix for the original system.
12. New equivalent control transition matrix of the overall system.
13. New equivalent input matrix of the overall system.
14. New desired decoupled (achieved) transfer function polynomials of the overall system.
15. The roots of overall transfer function of the system (indicating poles and zeros of overall decoupled system).

The input data format is discussed in the comment lines in the beginning of the program.

The system supplied subroutine MATRIX (CDC 6400) has been used throughout this program. The subroutine CHREQ and PROOT are taken from Melsa [1967].

```

      PROGRAM THESIS (INPUT,OUTPUT)
C PROGRAM THESIS FINDS THE REQUIRED FEEDBACK AND GAIN MATRICES TO ACHIEVE THE
C GIVEN DECOUPLED TRANSFER FUNCTION RESPONSE AND PRINTS OUT ALL THE DATA INPUTS.

C THE INPUT DATA CONSISTS OF THE FOLLOWING RESPECTIVELY. M,N AND FEE,H,C,Q,
C FCON AND GCON MATRICES (ALL THE ABOVE DATA FROM THE MAIN SYSTEM)
C THE ABOVE IS FOLLOWED BY THE SUBSYSTEMS DATA THE NO. OF STATES,THE NO. OF ZERO
C -S,A,B,DCOEFF AND CM MATRICES. THE ABOVE DATA FOR THE SUBSYSTEM IS REPEATED TIL
C -L ALL THE SUBSYSTEM ARE DONE.

C THE INPUT DATA FORMAT FOR ALL REAL NUMBERS IS 4E20.8 WITH EACH ROW OF A MATRIX
C OR VECTOR STARTING ON A NEW CARD. FORMAT FOR ALL INTEGERS IS 2I2.
C NOTATIONS USED IN THIS PROGRAM ARE AS FOLLOWS.
C FOR MORE DETAILED EXPLANATION OF THE NOTATIONS REFER (AGPAWAL,THESIS)
C IN=M=NO. OF INPUTS=NO. OF OUTPUTS. JN=N= NO. OF STATES OF THE MAIN SYSTEM
      DIMENSION FEE(10,10),HH(10,10),CC(10,10),FEHF(10,10),HG(10,10)
      DIMENSION FHAT(10,10),GHAT(10,10), Q(10,10),FCON(10,10)
      DIMENSION GCON(10,10),TEMP(10,10),FF(10,10),FG(10,10)
      COMMON A(10,10),B(10),DCOEFF(10),COEF(11),FM(10),FSUB(10),A1(9,9)
C TO READ NO. OF SUBSYSTEMS=IN=NO.OF INPUTS =NO.OF OUTPUTS
      READ9, IN,JN
      PRINT7, IN,JN
      PRINT600
600  FORMAT(55H THE FOLLOWING OUTPUT CROSSPONDS TO NUCLEAR REACTOR /
      ,62H PROBLEM. FEE AND H CROSSPOND TO SAMPLE TIME=0. )
      PRINT26
      DO 60 I=1,JN
      READ10, (FEE(I,J),J=1,JN)
60   PRINT25,(FEE(I,J),J=1,JN)
      PRINT27
      DO 61 I=1,JN
      READ10, (HH(I,J),J=1,IN)
61   PRINT25,(HH(I,J),J=1,IN)
      PRINT29
      DO 62 I=1,IN
      READ10, (CC(I,J),J=1,JN)
62   PRINT25,(CC(I,J),J=1,JN)
      PRINT31
      DO 65 I=1,JN
      READ10, (Q(I,J),J=1,JN)
65   PRINT25,(Q(I,J),J=1,JN)
      PRINT32
      DO 66 I=1,IN
      READ10, (FCON(I,J),J=1,JN)
66   PRINT25,(FCON(I,J),J=1,JN)
      PRINT33
      DO 67 I=1,IN
      READ10, (GCON(I,J),J=1,IN)
67   PRINT25,(GCON(I,J),J=1,IN)
C
C TO FIND THE TRANSFER FUNCTION
      CALL TRANSF (IN,JN,FEE,HH,CC)
      D054I=1,10
      D054J=1,10
      FHAT(I,J)=0
54   GHAT(I,J)=0
      J1=0
      D051M=1,IN
C TO READ ORDER OF SUBSYSTEM =N. NO.OF ZEROES INTHE SUBSYSTEM=IDP
      READ9,N,IDP

```

```

M=N-IDP
PRINT5,MN,N,IDP
PRINT41
DO 70 I=1,N
  READ10, (A(I,J),J=1,N)
70 PRINT25,(A(I,J),J=1,N)
PRINT42
DO 71 I=1,N
  READ10, R(I)
71 PRINT25,R(I)
PRINT43
DO 72 I=1,N
  READ10,DCOEF(I)
72 PRINT25,DCOEF(I)
PRINT44
READ10,GM
PRINT25,GM
CALL FGM (N,IDP,M,GM)
DO55J=1,N
55 FHAT(MN,J+J1)=FM(J)
  GHAT(MN,MN)=GM
51 J1=J1+N
C FHAT AND GHAT AT THIS POINT ARE TOTAL FEEDBACK AND GAIN MATRICES FOR CD SYSTEM
PRINT35
DO 75 I=1,IN
75 PRINT25, (FHAT(I,J),J=1,JN)
PRINT36
DO 76 I=1,IN
76 PRINT25, (GHAT(I,J),J=1,IN)
C
C THE PROGRAM BELOW FINDS F AND G MATRICES FOR THE ORIGINAL SYSTEM
C AND ALSO FINDS THE CORRESPONDING NEW SYSTEM MATRICES
DO56I=1,IN
DO56J=1,IN
56 TEMP(I,J)=GCON(I,J)
  CALL MATRIX(20,IN,JN,JN,FHAT,10,0,10,FHAT,10)
  PRINT37
  DO 77 I=1,IN
77 PRINT25, (FHAT(I,J),J=1,JN)
    CALL MATRIX(20,IN,IN,JN,GCON,10,FHAT,10,FHAT,10)
    PRINT38
    DO 78 I=1,IN
78 PRINT25, (FHAT(I,J),J=1,JN)
      CALL MATRIX(21,IN,JN,0,FCON,10,FHAT,10,FF,10)
      PRINT39
      DO 79 I=1,IN
79 PRINT25, (FF(I,J),J=1,JN)
        CALL MATRIX(20,IN,IN,IN,TEMP,10,GHAT,10,GHAT,10)
        PRINT40
        DO 80 I=1,IN
80 PRINT25, (GHAT(I,J),J=1,IN)
          DO 800 I=1,JN
          DO 800 J=1,JN
800 HG(I,J)=HH(I,J)
            CALL MATRIX(20,JN,IN,JN,HH,10,FF,10,HH,10)
            CALL MATRIX(21,JN,JN,0,FEF,10,HH,10,FEHF,10)
            PRINT400
            DO 81 I=1,JN
81 PRINT25, (FEHF(I,J),J=1,JN)
              CALL MATRIX (20,JN,IN,IN,HG,10,GHAT,10,HG,10)

```

```

POINT401
DO 82 I=1,JN
PRINT25.(PG(I,J),J=1.IN)
C
C TO FIND THE NEW TRANSFER FUNCTION
CALL TRANSF (IN,JN,FEHF,HG,CC)
5   FORMAT(1H1'SUBSYSTEM NUMBER=*I1,* ORDER OR SUBSYSTEM=*I1,2X
     *NO. OF ZEROES IN THE SUBSYSTEM=*I1)
7   FORMAT(1H1'NUM OF INPUTS=NUM OF OUTPUTS=*I2,5X'NUM OF STATES=*I2)
9   FORMAT(2I2)
10  FORMAT(4E20,12)
25  FORMAT(1X5F20,8)
26  FORMAT(1X *FEED MATRIX FOR THE SYSTEM*)
27  FORMAT(1X *FH MATRIX FOR THE SYSTEM*)
28  FORMAT(1X *CC MATRIX FOR THE SYSTEM*)
31  FORMAT(1X *C-MATRIX WHICH TRANSFORMS THE SYSTEM FROM DD TO CD*)
32  FORMAT(1X *F-CONJUGATE MATRIX*)
33  FORMAT(1X *G-CONJUGATE MATRIX*)
35  FORMAT(1H1'CHAT MATRIX FOR CD SYSTEM*)
36  FORMAT(1X *CHAT MATRIX*)
37  FORMAT(1X *IN BETWEEN STEPS MATRIX*)
38  FORMAT(1X *IN BETWEEN STEPS MATRIX*)
39  FORMAT(1X *FEED BACK MATRIX FOR THE ORIGINAL SYSTEM*)
40  FORMAT(1X *GAIN MATRIX FOR THE ORIGINAL SYSTEM*)
400 FORMAT (* FEE * HF MATRIX FOR THE ORIGINAL SYSTEM*)
401 FORMAT(40H *HG MATRIX FOR THE ORIGINAL SYSTEM)
41  FORMAT(1X *A MATRIX FOR THE ABOVE SUBSYSTEM*)
42  FORMAT(1X *B MATRIX FOR THE ABOVE SUBSYSTEM*)
43  FORMAT(1X *DESIRED CHARACTERSTIC POLYNOMIAL COEFFICIENTS FOR THE SUB
     *SYSTEM *COEFFICIENT OF CONSTANT TERM FIRST*)
44  FORMAT(1X *GAIN FOR THE SUBSYSTEM *)
STOP
END

SUBROUTINE FGM (N, IOP, M, GM)
C SUBROUTINE FGM FINDS THE FEEDBACK AND GAIN FOR EACH SUBSYSTEM OF THE SYSTEM.
COMMON A(10,10),B(10),DCOEF(10),COEF(11),FM(10),FSUB(10),AI(9,9)
COMMON T(6,6)
IF(N.EQ.1)GO TO 18
CALL CHREQ (A,N,COEF)
PRINT45.(COEF(I),I=1,N)
CALL TMAT (N,A,B,COEF)
PRINT24
DO 54 I=1,N
PRINT25.(T(I,J),J=1,N)
CALL MATRIX(10,N,N+1,T,6,8)
PRINT30
DO 55 I=1,N
PRINT25.(T(I,J),J=1,N)
DO 56 I=1,N
FSUB(I)=COEF(I)-DCOEF(I)
PRINT40.(FSUB(J),J=1,N)
DO 15 I=1,N
FM(I)=0
DO 15 J=1,N
FM(I)=FM(I)+FSUB(J)*T (J,I)
GO TO 21
15 COEF(I)=A(N,N)
FM(I)=COEF(I)-DCOEF(I)

```

```

21 PRINT 20,(FM(I),I=1,N)
   PRINT22,GH
45 FORMAT(1X*COEFS OF CHARAC. POLYNOMIAL OF SUBSYSTEM*COEFF. OF CONST
   .ANT TERM FIRST IN ROW*/(6(4XF16.8)))
24 FORMAT (1X*PHASE VARIABLE TRANSFORMATION MATRIX*)
25 FORMAT(4F20.8)
30 FORMAT(1H *I-INVERSE MATRIX*)
40 FORMAT(1X*FSUB MATRIX*/(6(4XF16.8)))
20 FORMAT (1X*FEEDBACK MATRIX TO ACHIEVE DESIRED SUBSYSTEM RESPONSE*/
   .(6(4XF16.8)))
22 FORMAT(1X*REQUIRED GAIN FOR SUBSYSTEM*/1XF19.8)
   RETURN
   END

SUBROUTINE TMAT(N,A,B,COEF)
C SUBROUTINE TMAT FINDS THE PHASE VARIABLE TRANSFORMATION MATRIX FOR A SYSTEM
COMMON A(10,10),B(10),DCOEF(10),COEF(11),FM(10),FSUB(10),A1(9,9)
COMMON T(6,6)
DO 5J=1,N
5 T(J,N)=B(J)
N1=N-1
NP=N+1
DO 25 I=1,N1
DO 25 J=1,N
T(J,N-I)=0
DO 25 K=1,N
25 T(J,N-I)=T(J,N-I)+A(J,K)*T(K,NP-I)
25 T(J,N-I)=T(J,N-I)+COEF(NP-I)*T(J,N)
RETURN
END

SUBROUTINE TRANSF(IN,JN,FEE,HH,CC)
C SUBROUTINE TRANSF FINDS THE INPUT OUTPUT TRANSFER FUNCTION FOR THE SYSTEM
DIMENSION FEE(10,10),HH(10,10),CC(10,10),D(10),CO(10)
DIMENSION C(10),H(10)
DIMENSION X(10),Y(10)
N1=JN+1 N2=JN-1
D(N1)=1.0 C(N1)=0.0
PRINT290
DO 100 J1=1,IN
DO 100 J2=1,IN
DO 101 I1=1,JN
101 H(I1)=HH(I1,J1)
C(I1)=CC(I1,J1)
CALL CONV(FEE,H,C,D,CO,JN)
PRINT29,J2,J1
PRINT 25,(CO(I),I=1,N1)
IF (J1.EQ.J2)50,100
50 PRINT26
DO 51 KL=1,JN
IF(N2.LT.1)GO TO 100
IF(CO(N1-KL).EQ.0.0)51,52
51 N2=N2-1
52 CALL PROOT(N2,CO,X,Y,1)
PRINT25,(X(I),I=1,N2)
PRINT25,(Y(I),I=1,N2)
100 N2=JN-1
PRINT291

```

```

PRINT 25,(D(I),I=1,N1)
PRINT 26
CALL PROOT(JN,D,X,Y,1)
PRINT 25,(X(I),I=1,JN)
PRINT 25,(Y(I),I=1,JN)
PRINT 79
79  FORMAT(/54H NOTE-DENO. POLY. IS THE SAME FOR ALL THE TRA. FUNC. )
25  FORMAT(1X5(F20.8,1X))
26  FORMAT(1H *THE ROOTS OF THE ABOVE POLYNOMIAL ARE GIVEN BELOW R/I*)
29  FORMAT(// * TRANSFER FUNCTION H(*I1.*I1.*)*)
290 FORMAT(//45X46H NUM. POLY. A(1)+A(2)X+A(3)X**2+A(4)X**3+... )
291 FORMAT(//45X48H DENO. POLY. A(1)+A(2)X+A(3)X**2+A(4)X**4+...? /)
RETURN
END

```

```

SUBROUTINE CONV(A,R,C,D,CC,N)
C   FOR 1970 CHANGES TO IMRED IN ZTRAN, AND FOR LIBRARY
DIMENSION A(10,10),B(10),H(10),C(10),AA(10,10),E(10)
C   DIMENSION D(10),P(10,10),CC(10),HH(10),PIN(10,10),U(10),V(10)
C   RATIONAL FUNCTION CALCULATIONS
9   NN=N+1
CALL CHREQ(A,N,D)
DO11 I=1,N
11  P(I,N)=B(I)
IF(N-1) 15,15,14
14  DO12 JJ=2,N
DO12 I=1,N
J=N-JJ+1
K=J+1
P(I,J)=D(K)*B(I)
DO12 L=1,N
12  P(I,J)=P(I,J)+A(I,L)*P(L,K)
49  DO13 I=1,N
CC(I)=0.
DO13 J=1,N
13  CC(I)=P(J,I)*C(J)+CC(I)
RETURN
15  CC(1)=C(1)*R(1)
CC(2)=0.0
D(1)=A(1)
D(2)=1.0
RETURN
END

```

```

SUBROUTINE CHREQ(A,N,C)
DIMENSION J(11),C(10),R(10,10),A(10,10),D(10)
NN=N+1
DO20 I=1,NN
20  C(I)=0.
C(NN)=1.
DO14 M=1,N
K=0
L=1
J(1)=1
GO TO 2
1  J(L)=J(L)+1
2  IF(L-M) 3,5,50
3  MM=M-1

```

```

      DO 4 I=L,M
      IF(I=1)
      4 J(I)=J(I)+1
      5 CALL FORM(J,M,A,B)
      K=K+1
      O(K)=DET(B,M)
      DO 6 I=1,M
      L=M-I+1
      IF(J(I)-(N-M+L)) 1,6,50
      6 CONTINUE
      M1 = M-M+1
      DO 14 I=1,K
      14 C(M1)=C(M1)+O(I)*(-1.)**M
      RETURN
      50 PRINT 2000
      2000 FORMAT (1H0,5X,14HERROR IN CHREQ)
      RETURN
      END

```

```

      FUNCTION DET(A,N)
      C      FUNCTION DET DETERMINES THE DETERMINANT OF THE MATRIX #A#
      DIMENSION A(10,10),B(10,10)
      C      SET #A# EQUAL TO #A# BECAUSE WE DESTROY #A# IN THE PROCESS
      DO 1 K=1,N
      DO 1 JK=1,N
      B(K,JK) = A(K,JK)
      1 CONTINUE
      NN = N-1
      D = 1.0
      C      IF N=1 THEN BYPASS PIVOT PROCEDURE. GO DIRECTLY TO CALCULATION OF DET.
      IF(NN)50,59,3
      C      START PIVOT SEARCH PROCEDURE
      3 DO 100 L=1,NN
      LL = L+1
      AMAX = A(L,L)
      IM = L
      JM = L
      DO 15 I=L,N
      DO 15 J=L,N
      IF (AMAX-ABS(A(I,J))) 10,15,15
      10 IM = I
      JM = J
      AMAX = ABS(A(I,J))
      15 CONTINUE
      C      FOUND PIVOT AT ROW #IM# AND AT COLUMN #JM#
      C      IF ALL REMAINING TERMS IN THE MATRIX ARE ZERO, SET DET=0. AND RETURN.
      IF (AMAX) 70,70,14
      C      NOW CHANGE ROWS AND COLUMNS IF NECESSARY
      14 IF(IM-L) 16,20,16
      16 DO 17 J=1,N
      T = A(IM,J)
      A(IM,J) = A(L,J)
      17 A(L,J) = T
      D = -D
      20 IF(JM-L) 21,25,21
      21 DO 22 I=1,N
      T = A(I,JM)
      A(I,JM) = A(I,L)
      22 A(I,L) = T

```

```

      D = -D
C      PIVOT NOW AT A(L,L). DIVIDE ROW #L# BY A(L,L).
25  DO 30 K1=LL,N
      A(L,K1) = A(L,K1)/A(L,L)
30  CONTINUE
C      NOW PRODUCE ZERGES BELOW MAIN DIAGONAL, A(J,L)=0.
      DO 50 J=LL,N
      DO 50 K=LL,N
      A(J,K) = A(J,K)-A(J,L)*A(L,K)
50  CONTINUE
100 CONTINUE
C      MULTIPLY MAIN DIAGONAL ELEMENTS TO GET VALUE OF THE DETERMINANT
69  DO 200 I=1,N
      D = D*A(I,I)
200 CONTINUE
      DFT = D
C      NOW RESTORE THE VALUES OF THE #A# MATRIX
      DO 2 IK=1,N
      DO 2 JK=1,N
      A(IK,JK) = B(IK,JK)
2  CONTINUE
      RETURN
70  DFT=0.
      DO 4 JX=1,N
      DO 4 IK=1,N
      A(IK,JK) = B(IK,JK)
4  CONTINUE
      RETURN
      END

```

```

SUBROUTINE FORM(J,M,A,B)
  DIMENSION A(10,10),B(10,10),J(11)
  DO1 I=1,M
  DO1 K=1,M
  NR=J(I)
  NC=J(K)
1  S(I,K)=A(NR,NC)
  RETURN
  END

```

```

SUBROUTINE PROOT(N,A,U,V,IR)
  DIMENSION A(10),U(10),V(10),H(10),B(10),C(10)
  IDEV=IR
  NC=N+1
  DO1 I=1,NC
  H(I)=A(I)
1  CONTINUE
  P=0.
  Q=0.
  R=0.
3  IF(H(1))4,2,4
2  NC=NC-1
  V(NC)=0.
  U(NC)=0.
  DO16021=1,NC
  H(I)=H(I+1)
1602 CONTINUE
  GOT03

```

LAP00050
LAP00070

LAP00100
LAP00110
LAP00120
LAP00160
LAP00170
LAP00180
LAP00190

4	IF(NC-1)5,100,5	
5	IF(NC-2)7,6,7	
6	P=-H(1)/H(2)	LAP00260
	GO TO 50	LAP00270
7	IF(NC-3)9,8,9	LAP00280
8	P=H(2)/H(3)	LAP00290
	Q=H(1)/H(3)	LAP00300
	GO TO 70	LAP00310
9	IF(ABS(H(NC-1)/H(NC))-ABS(H(2)/H(1)))10,19,19	
10	IDEV=-IRFV	LAP00340
	M=NC/2	LAP00350
	DO 11 I=1,M	LAP00360
	N1=NC+1-I	LAP00370
	F=H(N1)	LAP00380
	H(N1)=H(I)	LAP0039
0	11 H(I)=F	LAP0040
0	IF(Q)13,12,13	LAP0041
0	12 C=0.	LAP0042
0	GO TO 15	LAP0043
0	13 P=P/Q	LAP0044
	Q=1./Q	LAP00450
15	IF(P)16,19,16	LAP00460
16	P=1./P	LAP00470
19	E=5.E-10	
	B(NC)=H(NC)	LAP00500
	C(NC)=H(NC)	LAP00510
	B(NC+1)=0.	LAP00520
	C(NC+1)=0.	LAP00530
	NP=NC-1	LAP00540
20	DO 49 J=1,1000	LAP00550
	DO 21 I=1,NP	LAP00560
	I=NC-I	LAP00570
	R(I)=H(I)+P*B(I+1)	LAP00580
21	C(I)=B(I)+P*C(I+1)	LAP00590
	IF(ABS(B(1)/H(1))-E)50,50,24	
24	IF(C(2))23,22,23	
22	R=P+1.	LAP00610
	GO TO 30	LAP00620
23	P=P-R(1)/C(2)	LAP00630
30	DO 37 I=1,NP	LAP00660
	I=NC-I	LAP00670
	R(I)=H(I)-P*R(I+1)-Q*R(I+2)	LAP00680
37	C(I)=B(I)-P*C(I+1)-Q*C(I+2)	LAP00690
	IF(H(2))32,31,32	LAP00790
31	IF(ABS(R(2)/H(1))-E)33,33,34	
32	IF(ABS(R(2)/H(2))-E)33,33,34	
33	IF(ABS(R(1)/H(1))-E)70,70,34	
34	CBAR=C(2)-B(2)	
	D=C(3)*2-CBAR*C(4)	LAP00720
	IF(D)36,35,36	
35	P=P-2.	
	Q=Q*(Q+1.)	
	GO TO 49	
36	P=P*(R(2)*C(3)-B(1)*C(4))/D	
	Q=Q*(-B(2)*CBAR+B(1)*C(3))/D	
49	CONTINUE	
	E=E*10.	
	GO TO 20	
50	NC=NC-1	
	V(NC)=0.	

IF (IREV) 51, 52, 52	LAP00890
51 U(NC)=1./R	LAP00900
GO TO 53	LAP00910
52 U(NC)=R	LAP00920
53 DO 54 I=1, NC	LAP00930
H(I)=B(I+1)	
54 CONTINUE	
GO TO 4	LAP00950
70 NC=NC-2	LAP00970
IF (IREV) 71, 72, 72	LAP00980
71 QP=1./Q	LAP0099
0 PP=P/(Q*2.0)	LAP0100
0 GO TO 73	LAP0101
0 72 QP=Q	LAP0102
0 PP=P/2.0	LAP0103
0 73 F=(PP)**2-QP	LAP0104
0 IF (F) 74, 75, 75	LAP0105
0 74 U(NC+1)=-PP	LAP0107
0 H(NC)=-PP	LAP0108
0 V(NC+1)=SQRT(-F)	
V(NC)=-V(NC+1)	LAP01100
GO TO 76	LAP01110
75 U(NC+1)=- (PP/ABS(PP)) * (ABS(PP) + SQRT(F))	
V(NC+1)=0.	LAP01140
U(NC)=QP/U(NC+1)	LAP01150
V(NC)=0.	LAP01160
76 DO 77 I=1, NC	LAP01180
H(I)=B(I+2)	
77 CONTINUE	
GO TO 4	LAP01200
100 RETURN	
END	
7/8/9	

APPENDIX B

COMPUTER PROGRAM FOR FINDING INACCESSIBLE STATES

```

SUBROUTINE INACES(IN,JN,A,HH,CC,M1,MCOH,MM)
C SUBROUTINE INACES FINDS THE MCO-INVERSE AND MCOH MATRICES WHICH ARE NEEDED
C FOR CALCULATING STATES OF A SYSTEM (USING PAST INPUTS AND OUTPUTS)
C INPUT TO THE SUBROUTINE IS AS FOLLOWS.
C IN=NUMBER OF INPUTS IN THE SYSTEM
C JN=NUMBER OF OUTPUTS OF THE SYSTEM
C A=CONTROL TRANSITION MATRIX OF THE SYSTEM
C HH= INPUT MATRIX OF THE SYSTEM
C CC=OUTPUT MATRIX OF THE SYSTEM
C OUT PUT OF THE SUBROUTINE CONSISTS OF THE FOLLOWING
C M1= MCO-INVERSE MATRIX OF THE STATE ESTIMATION EQUATION
C MCOH= MCOH MATRIX IN THE STATE ESTIMATION EQUATION
C MM=NUMBER OF OBSERVATIONS NEEDED TO CALCULATE THE STATES
      DIMENSION FEE(10,10),HH(10,10),CC(10,10),A(10,10)
      REAL      MCO(10,10),MCOH(10,10),M1(10,10),M2(10,10),M3(10,10)
C MIN=NO. OF OBSERVATIONS NECESSARY
      K1=JN/IN
      K2=K1*IN
      IF(K2.EQ.JN)100,200
100  MIN=K1
      GO TO 50
200  MIN=JN/IN+1
50   DO 51 I=1,JN
      DO 51 J=1,JN
      FFE(I,J)=A(I,J)
51   MCOH(I,J)=0
C TO FORM MCO MATRIX FROM C-FEE-INVERSE MATRICES
      CALL MATRIX (10,JN,JN,1,FFE,10,A1)
      PRINT26
      DO 60 I=1,JN
60   PRINT26, (FFE(I,J),J=1,JN)
      DO 52 I=1,MIN
C STORE FEE-INVERSE AND HH MATRICES BECAUSE THEY ARE DESTROYED IN CALL MATRIX
      DO 53 J=1,JN
      DO 53 K=1,JN
      M1(J,K)=FFE(J,K)
      M2(J,K)=HH(J,K)
53   CALL MATRIX (20,IN,JN,JN,CC,10,M1,10,CC,10)
      DO 54 J=1,IN
      DO 54 K=1,JN
      IA=(I-1)*IN+J
54   MCO(IA,K)=M3(J,K)=CC(J,K)
      PRINT27
      DO 61 J=1,IN
61   PRINT 25, (CC(J,K),K=1,JN)
      CALL MATRIX (20,IN,JN,IN,M3,10,M2,10,M3,10)
C TO FORM MCOH MATRIX FROM C-FEE-INVERSE-H MATRICES
      MC=MIN-1
      DO 56 JK=1,MC
      DO 56 J=1,IN
      DO 56 K=1,IN
      JJ=J+(JK-2)*IN
      KK=K+(JK-1)*IN
56   MCOH(JJ,KK)=M3(J,K)
52   CONTINUE
      MM=MIN*IN
      PRINT29
      DO 62 I=1,MM
62   PRINT25, (MCO(I,J),J=1,JN)
      PRINT30
      DO 63 I=1,MM

```

```

63  PRINT25, (MCOH(I,J),J=1,MM)
•   TO FIND INVERSE OF MCOMATRIX=A-TRA*((A*A-TRAS)INVERSE)
    IF(K2.FO.JN)GO TO 101
    DO 58 I=1,MM
    DO 58 J=1,MM
    M1(I,J)=MCO(I,J)
58  M1(I,J)=M2(I,J)=MCO(J,I)
    CALL MATRIX (20,MM,JN,MM,M3,10,M1,10,M1,10)
    PRINT31
    DO 64 I=1,MM
64  PRINT 25, ( M1(I,J),J=1,MM)
    CALL MATRIX (10,MM,MM,1,M1,10,A2)
    PRINT32
    DO 65 I=1,MM
65  PRINT 25, ( M1(I,J),J=1,MM)
    CALL MATRIX(20,JN,MM,MM,M2,10,M1,10,M1,10)
    PRINT33
    DO 66 I=1,JN
66  PRINT 25, ( M1(I,J),J=1,MM)
    GO TO 201
101 CALL MATRIX (10,JN,JN,1,MCO,10,A2)
    PRINT34
    DO 67 I=1,JN
67  PRINT25,(MCO(I,J),J=1,JN)
    DO 68 I=1,JN
    DO 68 J=1,JN
68  M1(I,J)=MCO(I,J)
25  FORMAT(1X5F20.8)
26  FORMAT(1X* FEE-INVERSE MATRIX*)
27  FORMAT(1X*CFEE-INVERSE MATRIX*)
29  FORMAT(1X* MCOMATRIX *)
30  FORMAT(1X* MCOHMATRIX *)
31  FORMAT(55H MCO*MCO-TRAN MATRIX )
32  FORMAT(55H (MCO*MCO-TRAN)-INVERSE MATRIX )
33  FORMAT(55H MCO-TRAN(MCO*MCO-TRAN)INVER=MCO-INVERSE MATRIX )
34  FORMAT(55H MCO IS SQARE MATRIX SO INVERSE IS DIRECTLY FOUND )
201 RETURN
    END
    SUBROUTINE MATMUL (N,M,L,A,M1,B,M2,C,M3)
    DIMENSION A(M1,M),B(M2,L),C(M3,L)
    DO 10 I=1,N
    DO 10 J=1,L
    C(I,J)=0
    DO 10 K=1,M
    C(I,J)=C(I,J)+A(I,K)*B(K,J)
10  RETURN
    END
    SUBROUTINE MATADD (N,M,A,M1,B,M2,C,M3) &
    DIMENSION A(M1,M),B(M2,M),C(M3,M)
    DO 10 I=1,N
    DO 10 J=1,M
10  C(I,J)=A(I,J)+B(I,J)
    RETURN
    END

```

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