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DETERMINATION OF ATMOSPHERIC STRUCTURE FUNCTION BY USING A SINGLE COHERENT DETECTOR

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I. INTRODUCTION

Fried[1] has shown that atmospheric turbulence can impose severe restrictions on the design of telescopes for optical heterodyne detection systems. The atmospheric turbulence causes amplitude and phase fluctuations in the received signal which limit the effective telescope aperture size. This limitation is directly connected with the wave structure function and thus a knowledge of this structure function becomes important.

Measurements of structure function have been proposed that use dual optical systems and dual coherent detectors[2]. Many times, however, the equipment available is a laser heterodyne receiver which contains a single beam optical system and a single coherent detector. In this paper we will describe and analyze a technique for using the single beam laser receiver to measure the wave structure function.

The measurement procedure is as follows: first, mask the telescope aperture with an opaque screen having two small holes, second, cover one hole and measure the output of the laser receiver and, finally, measure the output of the laser receiver with both holes uncovered. By repeating these two measurements for different hole separations, one can simply determine the wave structure function.

II. LASER RECEIVER DESCRIPTION

A block diagram of the laser heterodyne receiver to be considered is shown in Figure 1. The receiver consists of a telescope used to collect the incident signal. The beam from the telescope is then beat with a local oscillator beam in a photomixer. The output of the photomixer is then passed through an I.F. amplifier and A.M. detected.

The collection aperture of the telescope is located in the plane AA' as shown in Figure 1. The aperture is masked so that the incident signal can only enter through two small holes. The signal from these two holes is then focussed on the photomixer surface located in plane BB'. Concurrently, a local oscillator beam is directed onto the surface of the photomixer by means of a beam splitter.

The photomixer which is shown in Figure 1, gives an output current which is proportioned to the power incident upon its surface. More quantitatively, we have

$$i_P = \gamma \int_P |E_P|^2 \, dA, \qquad \gamma - \text{constant} \tag{1}$$

where E_P is the total electric field on the surface and P is the surface area of the photomixer. The output of the photomixer is followed by an I.F. amplifier. This amplifier passes all frequencies on or near the beat frequency of the local oscillator and the incoming signal. Other frequencies are rejected. The output of the I.F. amplifier, i_{IF}, is then A.M. detected. If we have

$$i_{IF}(t) = A(t) \cos [\Delta \omega t + \phi(t)] \tag{2}$$

where $\Delta\omega$ is the beat frequency and, A (t) and ϕ (t) are slowly varying with

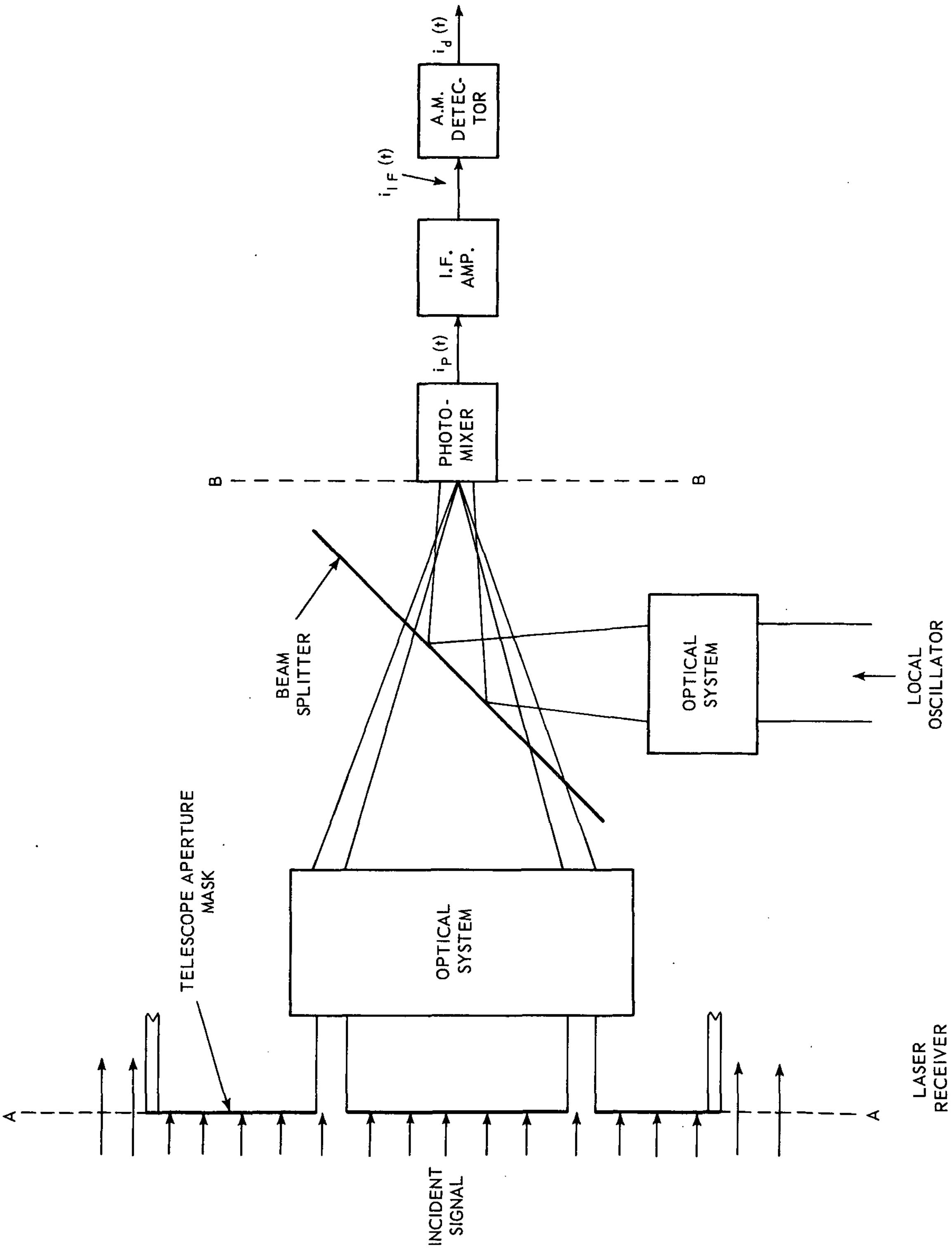
A
A
B
B
INCIDENT SIGNAL
TELESCOPE APERTURE MASK
LASER RECEIVER
OPTICAL SYSTEM
BEAM SPLITTER
OPTICAL SYSTEM
LOCAL OSCILLATOR
PHOTO-MIXER
i_P (t)
I.F. AMP.
i_{IF} (t)
A.M. DETEC-TOR
i_d (t)

Figure 1

respect to $\Delta\omega$, then

$$i_d(t) = \sigma A(t) \qquad \sigma - \text{constant} \tag{3}$$

where i_d is the output of the A.M. detector.

III. THEORETICAL ANALYSIS OF RECEIVER RESPONSE

We will now proceed with the theoretical analysis of the receiver input. The analysis will be kept as general as possible in this and the next section; then, in Section V, certain simplifying assumptions will be made to make the experimental measurement tractable. This procedure is useful since, if certain of the simplifying assumptions are not met exactly, one can use the general formulation to see how errors are introduced into the measurement.

We will now calculate the field due to the incident signal in the plane of the photomixer. In Figure 2, we show a simplified drawing of the receiver optics without including the local oscillator subsection. The plane z = 0 is the aperture plane and corresponds to the plane AA' in Figure 1. It is an opaque plane with two arbitrarily shaped holes in it. The plane $z = z_p$ is the plane of the photomixer. The focusing and relaying optics are located between the aperture and photomixer planes, however, they are not shown for simplification purposes.

If the opaque screen is illuminated by quasi-monochromatic light, $E_{inc}(\underline{\rho}, t)$, then the field in the detector plane, $E_s(\underline{\rho}, t)$, is given by

$$E_s(\underline{\rho}, t) = \int_{A_1 + A_2} h(\underline{\rho}, \underline{\rho}')\, E_{inc}(\underline{\rho}', t)\, d\rho' \tag{4}$$

where

$$\underline{\rho} = x\underline{a}_x + y\underline{a}_y, \qquad \underline{\rho}' = x'\underline{a}_x + y'\underline{a}_y \tag{5}$$

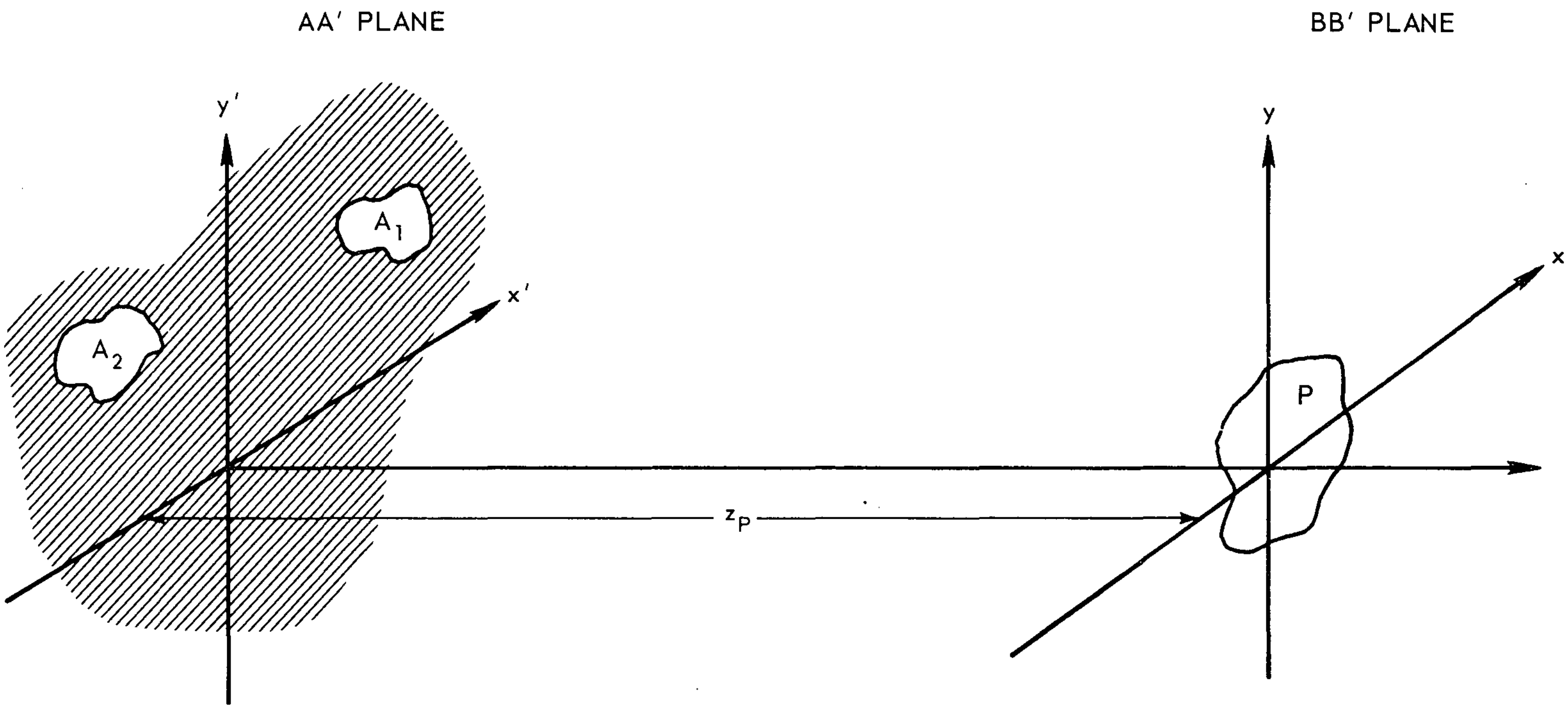

Figure 2

and $h(\underline{\rho}, \underline{\rho}')$ is the two dimensional transfer junction from the $z = 0$ plane to the $z = z_P$ plane[3]. We have adopted this general formulation since it is not necessary at this point to specify $h(\underline{\rho}, \underline{\rho}')$ in a more detailed manner. We should note that $h(\underline{\rho}, \underline{\rho}')$ also depends upon z_P, however, this dependence has not been explicitly shown. The integration in Eq. (4) is an area integral over the two holes. This has been shown symbolically by writing $A_1 + A_2$.

In the absence of turbulence, we will assume that a plane wave is normally incident upon the aperture of the telescope, i.e.,

$$E_{inc}(\underline{\rho}, t) = A e^{j(\omega_s t + \phi)} \tag{6}$$

where A and ϕ are constants. The effect of turbulence will be taken into account by assuming that the amplitude and phase of the incident wave are distorted. The incident field becomes

$$E_{inc}(\underline{\rho}, t) = A(\underline{\rho}, t) e^{j[\omega_s t + \phi(\rho, t)]} \tag{7}$$

where the amplitude variation can be expressed in terms of the log-amplitude function $\ell(\rho, t)$ i.e.,

$$A_s(\underline{\rho}, t) = \overline{A}_s e^{\ell(\underline{\rho}, t)} . \tag{8}$$

Here $\overline{A}_s$ is the r.m.s. amplitude of $A_s(\underline{\rho}, t)$. We have chosen to normalize $A_s(\underline{\rho}, t)$ to $\overline{A}_s$ rather then to A since $\overline{A}_s$ is more accessible to measurement. The time dependence of 1 and ϕ is due to the relatively slow changes in the gross properties of the turbulence. The fact that these changes are slow compared with the time constants of the optical waves is what allows us to use Eq. (4) which was derived under monochromatic assumptions.

We will assume that the two holes in the opaque screen are small enough so that $\ell(\underline{\rho}, t)$ and $\phi(\underline{\rho}, t)$ are approximately constant across each hole. Now plugging Eq. (7) into Eq. (4) and using the above assumptions, we obtain

$$E_s(\underline{\rho}, t) = M_1(\underline{\rho})\, e^{\ell_1(t) + j[\omega_s t + \phi_1(t)]} + M_2(\underline{\rho})\, e^{\ell_2(t) + j[\omega_s t + \phi_2(t)]} \tag{9}$$

where

$$M_{\frac{1}{2}}(\underline{\rho}) = \overline{A}_s \cdot \int_{A_{\frac{1}{2}}} h(\underline{\rho}, \underline{\rho}')\, d\underline{\rho}' \tag{10}$$

and

$$\ell_{\frac{1}{2}}(t) = \ell(\underline{\rho}_{\frac{1}{2}}, t), \tag{11}$$

$$\phi_{\frac{1}{2}}(t) = \phi(\underline{\rho}_{\frac{1}{2}}, t). \tag{12}$$

The field, E_P, in the plane of the photomixer consists of the sum of the signal field E_s plus the oscillator field E_0, i.e.,

$$E_P = E_0 + E_s. \tag{13}$$

It will be assumed that the local oscillator field in the plane of the photomixer is given by

$$E_0(\underline{\rho}, t) = A_0(\underline{\rho})\, e^{j[\omega_0 t + \phi_0(\underline{\rho}, t)]}. \tag{14}$$

Here we have included a time dependence in $\phi_0(\underline{\rho}, t)$ to allow for the phase jitter of the oscillator signal. If we now use Eq. (13) in Eq. (1), we have

$$i_P(t) = \gamma \int_D (E_0 + E_s)(\overline{E}_0 + \overline{E}_s)\, d\underline{\rho} \tag{15}$$

$$i_P(t) = \gamma \int_D E_0 \overline{E}_0\, d\underline{\rho} + \mathrm{Re}\, 2\gamma \int_D E_s \overline{E}_0\, d\underline{\rho} + \gamma \int_D E_s E_s\, d\underline{\rho} \tag{16}$$

where a bar over a symbol means the complex conjugate of that symbol. For low noise heterodyne receivers, $|E_0| >> |E_s|$, therefore

$$i_P(t) = \gamma \int_D E_0 \bar{E}_0 \, d\underline{\rho} + 2\gamma \, \mathrm{Re} \int_D E_s \bar{E}_0 \, d\underline{\rho}. \tag{17}$$

Now plugging Eq. (9) and (14) in Eq. (17) gives

$$i_P(t) = \gamma \int_D A_0^2(\underline{\rho}) \, d\underline{\rho} + \mathrm{Re}\left\{ B_1(t)\, e^{\ell_1(t) + j[\Delta\omega t + \phi_1(t)]} + B_2(t)\, e^{\ell_2(t) + j[\Delta\omega t + \phi_2(t)]} \right\} \tag{18}$$

where

$$B_{\frac{1}{2}}(t) = 2\gamma \int_D M_{\frac{1}{2}}(\underline{\rho})\, A_0(\underline{\rho})\, e^{j\phi(\underline{\rho},t)} \, d\underline{\rho} \tag{19}$$

and

$$\Delta\omega = \omega_s - \omega_0 .$$

Next we write $B_{\frac{1}{2}}(t)$ in polar form:

$$B_{\frac{1}{2}}(t) = |B_{\frac{1}{2}}(t)|\, e^{j\psi_{\frac{1}{2}}(t)} \tag{20}$$

and use this expression in Eq. (18) to obtain

$$i_P(t) = \gamma \int_D A_0^2(\underline{\rho}) \, d\underline{\rho} + |B_1(t)|\, e^{\ell_1(t)} \cos[\Delta\omega t + \phi_1(t) + \psi_1(t)] + |B_2(t)|\, e^{\ell_2(t)} \cos[\Delta\omega t + \phi_2(t) + \psi_2(t)] \tag{21}$$

Since the function $B_{\frac{1}{2}}(t)$, $\ell_{\frac{1}{2}}(t)$, and $\phi_{\frac{1}{2}}(t)$ vary slowly compared with $\Delta\omega$, Eq. (21) can be viewed as a sum of a dc term plus two amplitude and phase modulated waves of frequency $\Delta\omega$. Upon passing the signal through the I.F., we just obtain

$$i_{IF}(t) = |B_1(t)|\, e^{\ell_1(t)} \cos[\Delta\omega t + \phi_1(t) + \psi_1(t)]$$
$$+ |B_2(t)|\, e^{\ell_2(t)} \cos[\Delta\omega t + \phi_2(t) + \psi_2(t)]. \quad (22)$$

Next the I.F. signal goes into an A.M. detector which has been described in the previous section by Eqs. (2) and (3).

In order to apply Eqs. (2) and (3), we combine the cosines in Eq. (22). By using this result, which is derived in Appendix A, we find

$$i_{IF}(t) = A(t) \cos[\Delta\omega t + \theta(t)] \quad (23)$$

where

$$A(t) = \sqrt{|B_1|^2 e^{2\ell_1} + |B_2| e^{2\ell_2} + 2|B_1|\,|B_2|\, e^{(\ell_1+\ell_2)} \cos(\phi_1 - \phi_2 + \psi_1 - \psi_2)} \quad (24)$$

The expression for θ (t) is given in Appendix A but is not presented here since θ(t) information is lost in A.M. detection.

Now passing i_{IF} (t) through the A.M. detector we find

$$i_d(t) = A(t). \quad (24a)$$

This signal is now amplified and recorded on tape for later processing.

It should be recalled that the measurement of the total structure function will require not only a knowledge of the receiver output from the two hole aperture, but also the output when one hole is covered. This result can readily be obtained from Eq. (24) by setting $B_1 = 0$ or $B_2 = 0$.

IV. STATISTICAL CHARACTERIZATION OF INCIDENT WAVE

The transmitted beam becomes statistical in nature after passing through the random or turbulent medium. As a result, the wave's log amplitude $\ell(\underline{\rho})$ and phase $\phi(\underline{\rho})$ become statistical quantities that must be described by probability distributions or characterized by statistical moments. In the past experimental and theoretical results have shown that under certain conditions ℓ and ϕ obey independent Gaussian statistics[4]. This information can be used to calculate the wave structure function by using Eq. (24).

If we assume that the statistics of $\phi(\underline{\rho})$ and $\ell(\underline{\rho})$ are locally homogenous and isotropic, then one can define the log-amplitude and phase structure functions as

$$D_\ell(r) = \left\langle [\ell(\underline{\rho}_1) - \ell(\underline{\rho}_2)]^2 \right\rangle \tag{25}$$

and

$$D_\phi(r) = \left\langle [\phi(\underline{\rho}_1) - \phi(\underline{\rho}_2)]^2 \right\rangle \tag{26}$$

In the above $\langle u \rangle$ is the ensemble average of u and $r = |\underline{\rho}_1 - \underline{\rho}_2|$. The wave structure function D(r) is then defined as

$$D(r) = D_\ell(r) + D_\phi(r) \tag{27}$$

Now squaring Eq. (24) and taking the ensemble average, we have

$$\begin{aligned} \langle A^2 \rangle = {} & |B_1|^2 \langle e^{2\ell_1} \rangle + |B_2|^2 \langle e^{2\ell_2} \rangle \\ & + 2\,|B_1|\,|B_2| \left\langle e^{(\ell_1+\ell_2)} \cos(\phi_1 - \phi_2 + \psi_1 - \psi_2) \right\rangle \end{aligned} \tag{28}$$

By using Fried's[1] results:

$$\exp\,[2\,\ell\,(\underline{\rho}_{\frac{1}{2}})] = 1 \tag{29}$$

and

$$\exp\left\langle\{\ell\,(\underline{\rho}_1) + \ell\,(\underline{\rho}_2) \pm [\phi\,(\underline{\rho}_1) - \phi\,(\underline{\rho}_2)]\}\right\rangle = e^{-\frac{1}{2}D(r)} \tag{30}$$

in Eq. (28) , we have

$$\langle A^2 \rangle = |B_1|^2 + |B_2|^2 + 2\,|B_1|\,|B_2|\,\cos\,(\psi_1 - \psi_2)\,e^{-\frac{1}{2}D(r)} \tag{31}$$

Therefore, we see that the mean square ensemble average of the A.M. detector output is directly related to the wave structure function.

V. EXPERIMENTAL DETERMINATION OF STATISTICAL QUANTITIES

We will now use the results of the previous two sections to devise a method of measurement of the wave structure function by using the output signal from the A.M. detector.

As a first step, the output signal A (t) is sampled and $\langle A^2 \rangle$ is determined. We would now like to use Eq. (31) to determine D (r) from $\langle A^2 \rangle$, however, the parameters $|B_1|$, $|B_2|$ and $\psi_1 - \psi_2$ are unknown. In order to simplify the measurement, we will require that the laser receiver be designed so $B_1 = B_2 =$ B. Under these conditions, Eq. (31) reduces to

$$\langle A^2 \rangle = 2\,|B|^2\,[1 + e^{-\frac{1}{2}D(r)}] \tag{32}$$

The measurement of D (r) can now be carried out as follows: first, cover one hole in the opaque screen and measure the output of the A.M. detector. This output will be denoted by A_1 (t). We now sample A_1 (t) and compute $\langle A_1^2 \rangle$. By setting $B_1 = 0$ or $B_2 = 0$ in Eq. (31) we obtain

$$|B|^2 = \langle A_1^2 \rangle \tag{33}$$

Next we measure the output of the A.M. detector with both holes open. Since we already know $|B|$ from Eq. (33) we can readily find D (r) from Eq. (32). Now move the holes in the aperture mask so a new value of r is obtained and repeat the above procedure.*

There are several optical arrangements which will make $B_1 = B_2$. One particular arrangement which is fairly insensitive to design inaccuracies is as follows:

1. Adjust the local oscillator beam so that its amplitude and phase are constant across the photomixer surface P.

2. Require that the signal passing through the holes in the aperture mask be focused in the plane of the photomixer.

3. Make the small holes in the mask circular and of the same radius.

4. Require that the diffraction pattern (Airy disc) from each hole be completely within the photomixer surface P.

5. Place the two circular holes so that they are an equal distance from the center of the telescope collection aperture.

That conditions (1) - (5) produce $B_1 = B_2$ is verified in Appendix B.

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*The value of $|B|$ will not have to be remeasured if the holes in the opaque screen are kept at a constant distance from the center of the telescope aperture.

In addition to determining the total structure function, the above procedure can also be used to determine the instantaneous phase difference if amplitude fluctuations are negligible. Let us assume $\ell_1 = \ell_2 = 0$ and $B_1 = B_2$. Then Eq. (24) becomes

$$A(t) = \sqrt{2}\,|B|\,\sqrt{1 + \cos \Delta\phi(t)} \tag{34}$$

where

$$\Delta\phi(t) = \phi_1(t) - \phi_2(t) \tag{35}$$

Again, by covering a hole, the value of $|B|$ is obtained and then when used in Eq. (34), the two hole measurement will yield the phase difference.

APPENDIX A

We will now prove that two sinusoids with different amplitudes and phases can be combined into one sinusoid, i.e.,

$$\alpha_1 \cos(\omega t + \beta_1) + \alpha_2 \cos(\omega t + \beta_2) = A \cos(\omega t + \theta). \qquad (1A)$$

To show that the above is true and find A and θ in terms of $\alpha_{1,2}$ and $\beta_{1,2}$, we rewrite the sinusoids on the left in their exponential form. Upon doing this, we obtain

$$\alpha_1 \left[\frac{e^{j(\omega t + \beta_1)} + e^{-j(\omega t + \beta_1)}}{2}\right] + \alpha_2 \left[\frac{e^{j(\omega t + \beta_2)} + e^{-j(\omega t + \beta_2)}}{2}\right] = A \cos(\omega t + \theta)$$

or

$$\frac{1}{2} \gamma e^{j\omega t} + \frac{1}{2} \bar{\gamma} e^{-j\omega t} = A \cos(\omega t + \theta) \qquad (3A)$$

where

$$\gamma = \alpha_1 e^{j\beta_1} + \alpha_2 e^{+j\beta_2}$$

$$= (\alpha_1 \cos\beta_1 + \alpha_2 \cos\beta_2) + j(\alpha_1 \sin\beta_1 + \alpha_2 \sin\beta_2) \qquad (4A)$$

$$|\gamma| = [(\alpha_1 \cos\beta_1 + \alpha_2 \cos\beta_2)^2 + (\alpha_1 \sin\beta_1 + \alpha_2 \sin\beta_2)^2]^{1/2} \qquad (5A)$$

$$= [\alpha_1^2 \cos^2\beta_1 + 2\alpha_1 a_2 \cos\beta_1 \cos\beta_2 + \alpha_2^2 \cos^2\beta_2 \qquad (6A)$$

$$+ \alpha_1^2 \sin^2\beta_1 + 2\alpha_1\alpha_2 \sin\beta_1 \sin\beta_2 + \alpha_2^2 \sin^2\beta_2]^{1/2}$$

$$|\gamma| = [\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cos(\beta_1 - \beta_2)]^{1/2} \qquad (7A)$$

$$< \gamma = \tan^{-1} \frac{\alpha_1 \sin \beta_1 + \alpha_2 \sin \beta_2}{\alpha_1 \cos \beta_1 + \alpha_2 \cos \beta_2} \quad (8A)$$

Since $\gamma = |\gamma| \, e^{j < \gamma}$, we have

$$\frac{1}{2} |\gamma| \, e^{j<\gamma} e^{j\omega t} + \frac{1}{2} |\gamma| \, e^{-j<\gamma} e^{-j\omega t} = A \cos(\omega t + \theta) \quad (9A)$$

$$|\gamma| \cos(\omega t + < \gamma) = A \cos(\omega t + \theta) \quad (10A)$$

$$\therefore \quad A = |\gamma| \quad , \quad \theta = < \gamma + 2\pi n \quad , \quad n = 0, \pm 1, \pm 2, \quad (11A)$$

APPENDIX B

We would now like to show that $B_1 = B_2$, when conditions (1) - (5) of Section V are satisfied. By rewriting this condition and by using the definition of B_1 and B_2 given in Eq. (19), we have

$$\int_D M_1(\underline{\rho})\, A_0(\underline{\rho})\, e^{j\phi(\underline{\rho},t)}\, d\underline{\rho} = \int_D M_2(\underline{\rho})\, A_0(\underline{\rho})\, e^{j\phi(\underline{\rho},t)}\, d\underline{\rho}. \tag{1B}$$

If we now assume the oscillator beam amplitude and phase are constant across the photomixer surface, Eq. (1B) reduces to

$$\int_D M_1(\underline{\rho})\, d\underline{\rho} = \int_D M_2(\underline{\rho})\, d\underline{\rho}. \tag{2B}$$

Next we assume the system is focused, i.e., $z_P = f$ where f is the focal length of the system. Then the field in the photomixer plane is proportional to the Fourier transform of the aperture field. Therefore $h(\underline{\rho}, \underline{\rho}')$, given by Eq. (3), is

$$h(\underline{\rho}, \underline{\rho}') = \sigma e^{-j\frac{k}{2f}\underline{\rho}\cdot(\underline{\rho}-2\underline{\rho}')}, \qquad \sigma = -\frac{e^{-jkf}}{j\lambda f} \tag{3B}$$

and then using Eq. (3B) in Eq. (10), we obtain

$$M_{\frac{1}{2}}(\underline{\rho}) = \bar{A}_s\, \sigma\, e^{-j\frac{k}{2f}\rho^2} \int_{A_{\frac{1}{2}}} e^{+j\frac{k}{f}\underline{\rho}\cdot\underline{\rho}'}\, d\underline{\rho}' \tag{4B}$$

where

$$\rho = |\underline{\rho}|.$$

If we now assume the holes are circular holes of radius a with centers located at $\underline{\rho}_1$ and $\underline{\rho}_2$, Eq. (4B) becomes

$$M_2^1(\underline{\rho}) = \bar{A}_s \sigma e^{-j \frac{k}{2f} \rho^2} \int P_a(\underline{\rho}' - \underline{\rho}_{\frac{1}{2}}) e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}'} d\underline{\rho}' \qquad (5B)$$

where

$$P_a(\underline{\rho}) = \begin{cases} 1, & |\rho| \leq a \\ 0, & |\rho| > a \end{cases} \qquad (6B)$$

and the integral is over the whole $\underline{\rho}'$ plane.* If we make the change of variable $\rho'' = \underline{\rho}' - \underline{\rho}_{\frac{1}{2}}$, Eq. (5B) becomes

$$M_2^1(\underline{\rho}) = \bar{A}_s \sigma e^{-j \frac{k}{f} \left[\frac{\rho^2}{2} - \underline{\rho} \cdot \underline{\rho}_{\frac{1}{2}}\right]} \int P_a(\rho'') e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}''} d\underline{\rho}' \qquad (7B)$$

Performing the two dimensional integral in cylindrical coordinates, we obtain 5

$$M_2^1(\underline{\rho}) = f(\rho) e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}_{\frac{1}{2}}} \qquad (8B)$$

where

$$f(\rho) = \frac{k a f \sigma}{\pi \rho} J_1\left(\frac{k a \rho}{2f}\right) e^{-j \frac{k}{2f} \rho^2} \qquad (9B)$$

and $J_1(z)$ is the Bessel function of first order. Now using Eq. (8B) in Eq. (2B), we obtain

$$\int_D f(\rho) e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}_1} d\underline{\rho} = \int_D e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}_2} f(\rho) d\underline{\rho} \qquad (10B)$$

If we now assume that the diffraction patterns of the circular holes, $f(\rho)$ lie completely within P, then Eq. (10B) is unchanged if we extend the integration limits to infinity. Doing this (10B) becomes

*Whenever integration limits are not specified, the limits should be taken are infinite.

$$\int f(\rho)\, e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}_1} \, d\underline{\rho} = \int e^{+j \frac{k}{f} \underline{\rho} \cdot \underline{\rho}_2} f(\rho)\, d\underline{\rho} \,. \tag{11B}$$

Now introducing cylindrical coordinates

$$\begin{aligned} x &= \rho \cos \theta & x_{\frac{1}{2}} &= \rho_{\frac{1}{2}} \cos \theta_{\frac{1}{2}} \\ y &= \rho \sin \theta & y_{\frac{1}{2}} &= \rho_{\frac{1}{2}} \sin \theta_{\frac{1}{2}} \end{aligned} \tag{12B}$$

into Eq. (11B), we have

$$\int_0^\infty \int_0^{2\pi} e^{+j \frac{k}{f} \rho \rho_1 \cos(\theta - \theta_1)} f(\rho)\, \rho \, d\rho \, d\theta$$

$$= \int_0^\infty \int_0^{2\pi} e^{+j \frac{k}{f} \rho \rho_2 \cos(\theta - \theta_2)} f(\rho)\, \rho \, d\rho \, d\theta. \tag{13B}$$

By using the integral representation for the Bessel function[5]

$$2\pi J_0(\alpha) = \int_0^{2\pi} e^{+j \alpha \cos(\theta - \gamma)} \, d\theta \tag{14B}$$

the angular integration in (13B) can be performed. The result is

$$\int_0^\infty J_0\left(\frac{k}{f} \rho \rho_1\right) f(\rho)\, \rho \, d\rho = \int_0^\infty J_0\left(\frac{k}{f} \rho \rho_2\right) f(\rho)\, \rho \, d\rho. \tag{15B}$$

We see if $\rho_1 = \rho_2$, the two integrals will be equal and as a result

$$B_1 = B_2 \,.$$

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