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Charged Particle Motion in the Vicinity of a Neutral Plane

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The motion of a charged particle in a 2-dimensional neutral sheet with linear magnetic field variation is analyzed by Hamiltonian methods. From the basic Hamiltonian formulation, results of Sonnerup (J. Geophys. Res. 76, 8211, '71) are recovered and given intuitive interpretation. The transformed Hamiltonian is used to derive the correct frequency of oscillation and serves as the basis of analytical treatment of perturbed versions of the motion, e.g. with electric field added. For 2-dimensional fields with slightly differing configurations – in particular, with added small field component orthogonal to the sheet, with an X-type null or with a chain of null points – two alternative methods are developed, reducing the problem either to motion in a 2-dimensional potential or to a pair of coupled oscillators.

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The Hamiltonian

Let a magnetic field $\underline{B}$ be given, directed along the cartesian y-axis with the field intensity proportional to x and vanishing on the plane $x = 0$:

$$\underline{B} = (B_0/L)\, x\, \hat{\underline{y}} \tag{1}$$

This somewhat resembles the field in the tail of the magnetosphere: the plane $x = 0$ corresponds to the neutral plane and L is a scale length, giving the distance from the neutral plane at which the field intensity reaches some standard value B_0 (The notation so far corresponds to that of E.N.Parker, Phys. Rev. 107, 924 (1957), who included motion in such a field as an example of non-adiabatic motion at the end of his article on the guiding-center plasma; the cartesian axes, however, correspond to the choice made in "Adiabatic Particle Orbits in a Magnetic Null Sheet" by B.U.Ö. Sonnerup, JGR 76, 8211, '71).

By introducing Euler potentials (α, β) one easily derives a vector potential $\underline{A}$ of the form $\alpha \nabla\beta$:

$$\underline{B} = (-x B_0/L)(\nabla x \times \nabla z) = -(B_0/2L)(\nabla x^2 \times \nabla z)$$

$$= \nabla \times (-x^2 B_0/2L)\, \hat{\underline{z}} \tag{2}$$

The Hamiltonian is thus

$$H = \frac{1}{2m}\left\{ p_x^{\,2} + p_y^{\,2} + \left(p_z + (eB_0/2Lc)\, x^2\right)^2 \right\} \tag{3}$$

Two things at once stand out. First of all, the motion separates into an unconstrained motion along the field's direction (y-axis) and a motion orthogonal to it. If we abbreviate

$$eB_0/2cL = \sigma \tag{4}$$

we get for the latter motion

$$H_\perp = (1/2m)\left\{ p_x^{\,2} + (p_z + \sigma x^2)^2 \right\} = E_\perp = \text{const.} \tag{5}$$

Secondly, since z is absent, p_z is a constant of the motion which we will denote by P_z . In terms of the particle's velocity

$$P_z = m\dot{z} + \sigma x^2 = \text{const.} \tag{6}$$

It is also convenient to define a constant $P_\perp$ equal to the momentum component orthogonal to $\underline{B}$

$$P_\perp^2 = 2\,m\,E_\perp \tag{7}$$

Qualitative Aspects

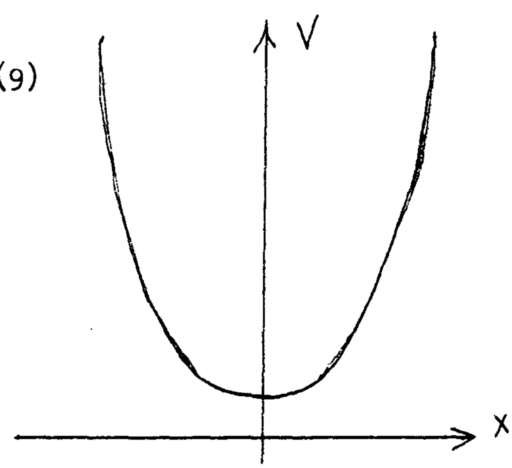
Because of the constancy of P_z , the motion produced by the Hamiltonian (5) may be viewed as a one-dimensional motion in a potential

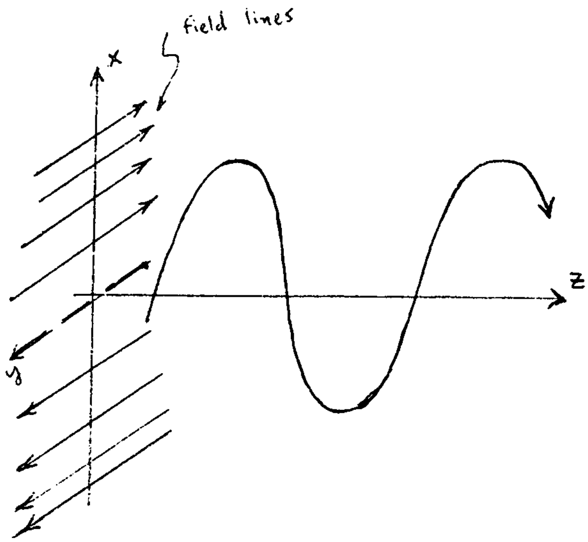
$$V = (1/2m)\,(P_z + \sigma x^2)^2 \tag{8}$$

Two possibilities exist, depending on whether P_z and σ have identical or opposite signs. In the first case we may define a suitable constant x_0 and write (note that V is in any case positive-definite)

$$V = (\sigma^2/2m)\,(\,x^2 + x_0^{\,2}\,)^2 \tag{9}$$

In that case it is found that V has a single minimum at $x = 0$, and its value there is $\sigma^2 x_0^{\,2}/2m$. The potential well is then parabola-shaped (but given by a 4th degree curve) as shown in the sketch:



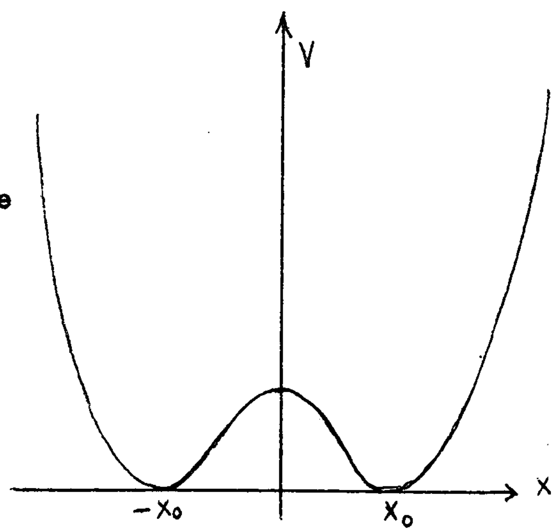


In this case the particle oscillates from one side of the well to the other. With each oscillation, it crosses the neutral plane $x = 0$: the motion is therefore as drawn in the sketch on the right. Following Sonnerup we will call this a meandering mode of motion — specifically, meandering without looping, that is, without the orbit crossing itself at any point.

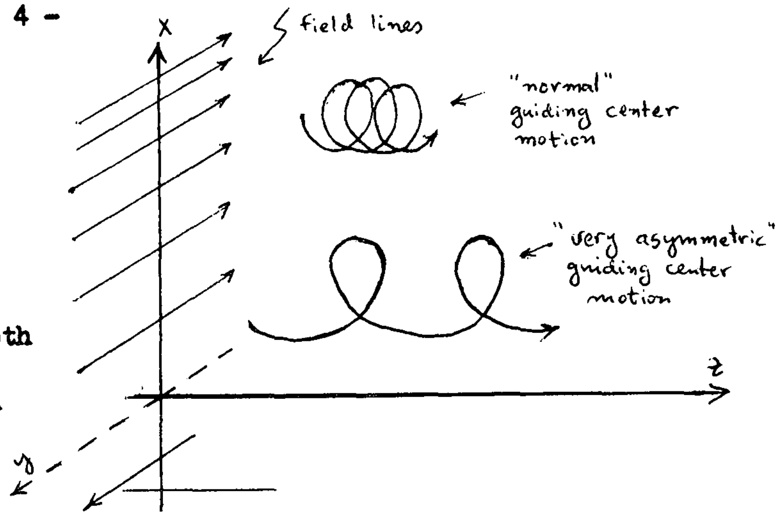
If P_z and σ have opposite signs, a suitable constant x_0 may be defined so that the potential assumes the form

$$V = (\sigma^2/2m)(x^2 - x_0^2) \qquad (10)$$

V then assumes its smallest possible value, namely zero, at $x = \pm x_0$. If one examines $\partial V/\partial x$ one confirms that these are true extrema, and that there also exists an extremum at $x = 0$, which clearly must be a relative maximum. The graph of V therefore has the general appearance of the curve drawn on the right.

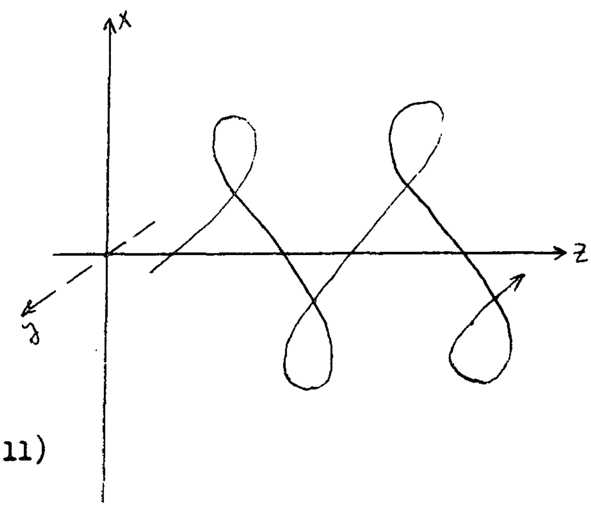


Two possibilities exist: if the particle is trapped in one of the "side pockets", it will oscillate back and forth without crossing the neutral sheet $x = 0$. The motion then has the general nature of guiding-center motion, especially for orbits far away from the neutral sheet. For orbits that



come close to the neutral plane (i.e. those oscillating rather high up in the "pocket") there will exist a marked asymmetry: both cases are shown in the drawing on the right:

If the particle clears the central peak, on the other hand, it will again oscillate across the plane $x = 0$ and again the result will be a meandering mode of motion. This mode, however, will have orbits crossing themselves ("meandering with looping"). To show that this indeed occurs, we use the Hamiltonian (5) to derive $\dot{z}$. We then get



$$\dot{z} = \pm(\sigma/m)\,(x^2 \pm x_0^2) \qquad (11)$$

the inner sign being positive if P_z and are of the same sign and negative if they are not. In the first case, $\dot{z}$ is always of the same sign, i.e. the particle steadily advances in one direction on the z-axis, as in meandering without looping. In the second case, $\dot{z}$ can assume either sign — it changes sign as the particle passes $x = \pm x_0$ — and this implies looping.

As Sonnerup has shown, there exists a continuum of solutions ranging from regular guiding-center modes to meandering modes, in the field given here. One may look at the situation from the following angle. Far from the neutral sheet the motion is of the guiding-center type and the particle's drift velocity has the order of magnitude

$$v_{drift} \cong \text{constant} \cdot (P_\perp^2 \, |\nabla \underline{B}| / \omega B) \qquad (12)$$

where ω is the angular velocity of gyration and all quantities are evaluated at the guiding center. In our particular example, the enumerator is constant ($\nabla \underline{B}$ has only one component, equal to B_0/L) while each term in the denominator is proportional to B . Thus as the guiding center approaches the neutral plane, the above expression diverges as B^{-2} .

Physically, the formula for the drift velocity is expected to break down somewhat before this situation is reached, because a particle cannot drift at a velocity larger than some fraction of $P_\perp/m$. This indeed happens when the guiding-center mode of motion evolves into the meandering mode, in which the z-component of the velocity indeed is some appreciable fraction of $P_\perp/m$.

Analytical Treatment

Let us follow the Hamilton-Jacobi method for generating a canonical transformation

$$(z, p_z, x, p_x) \rightarrow (Q_1, P_1, Q_2, P_2) \qquad (13)$$

such that the new momenta are constants and the new coordinates are linear in time. Since p_z already is a constant we may write the generating function, tentatively, as

$$W = z P_1 + W_1(P_1, P_2, x) \qquad (14)$$

The Hamilton-Jacobi equation then gives $(P_1 = P_z$, obviously)

$$(\partial W_1/\partial x)^2 + (P_1 + \sigma x^2)^2 = P_\perp^2 \qquad (15)$$

$$W = z P_1 + \int \sqrt{P_\perp^2 - (P_1 + \sigma x^2)^2} \, dx \qquad (16)$$

This contains $P_\perp$ which depends directly on the constant of energy $E_\perp$. In general, $P_\perp$ will also be some function of (P_1, P_2) ; this functional relationship may be specified more or less arbitrarily and every choice leads to a different definition of P_2. Any of these definitions leads to a solution of the Hamilton-Jacobi equation; however, if we plan to deal with problems differing from the given one by some small perturbation, it is advantageous to make a special choice, namely, to define P_2 as the <u>action variable</u> J, evaluated as an integral over one period of the system:

$$J = \oint p_x \, dx = \oint \frac{\partial W}{\partial x} dx$$

$$= \oint \sqrt{P_\perp^2 - (P_1 + \sigma x^2)^2} \, dx \qquad (17)$$

This choice arises naturally if one demands that its canonical conjugate Q_1 — denoted in this case by Ω and called the <u>angle variable</u> — increases by unity each period:

$$1 = \oint d\Omega = \oint \frac{\partial \Omega}{\partial x} dx = \oint \frac{\partial}{\partial x} \frac{\partial W}{\partial J} dx$$

$$= \oint \frac{\partial}{\partial J} \frac{\partial W}{\partial x} dx = \frac{\partial}{\partial J} \oint p_x \, dx \qquad (18)$$

In our case this leads to a general type of solution

$$\oint p_x \, dx = J + F(P_1, \sigma) \qquad (19)$$

but it will be presently seen that it is advantageous to set the added arbitrary function equal to zero. Intuitively, this choice of Q_1 means that when the Hamiltonian is expressed in terms of the new variables, Ω will always appear in functions that are periodic in Ω with period unity, that is, $\sin(2\pi\Omega)$, $\cos(2\pi\Omega)$ and their higher harmonics, and functions which can (in principle) be expanded in fourier series in such functions.

Mathematically, the result may be viewed as follows. When all dependence on Ω is of the type described above, then any function U of the variables can be separated into the sum of two terms: the "average" of U (over Ω)

$$\langle U \rangle = \oint U \; d\Omega \equiv \int_0^1 U \; d\Omega \tag{20}$$

which does not depend on Ω at all, and the "purely periodic" part of U

$$U_{per} = U - \langle U \rangle \tag{21}$$

which has zero average and therefore represents a purely oscillating quantity.

The significance of the choice of J as in (17), with the additional function allowed by (19) set equal to zero, is that when it is used, all partial derivatives of W_1 are <u>purely</u> periodic, except for $\partial W_1/\partial J$ which increases by unity each period. As an example, consider the derivative by P_1 : for this to be purely periodic, one requires

$$\begin{aligned} \oint d\left(\frac{\partial W}{\partial P_1}\right) &= 0 = \oint \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial P_1}\right)dx = \oint \frac{\partial}{\partial P_1} \; p_x \, dx \\ &= \frac{\partial}{\partial P_1} \oint p_x \, dx = \partial J/\partial P_1 = 0 \end{aligned} \tag{22}$$

The last equality follows from J and P_1 being independent (new) variables. The fact that $\partial/\partial P_1$ may be taken out before the integral follows from the fact that the integration is over a closed loop: it could be broken up into two or more sections, and the limits of integration in each section would then be spelled out. These limits may well depend on P_1 , but it is easy to see that the contribution of each limit appears then in two terms that cancel each other, an upper limit term and a lower limit term.

Some consequences of this may be pointed out. First of all, Q_1 will differ from z only by purely periodic terms, for

$$Q_1 = \frac{\partial W}{\partial P_1} = z + \frac{\partial W_1}{\partial P_1} \tag{23}$$

and the second term is purely periodic.

Next, suppose σ is a slow function of the time t . Then a term is added to the Hamiltonian, in the new variables, equal to

$$\frac{\partial W}{\partial t} = \frac{d\sigma}{d\,t}\,\frac{\partial W_1}{\partial \sigma} \tag{24}$$

If we are to handle the motion by perturbation methods, the addition to the Hamiltonian must be small, with smallness defined by some small parameter ε . If the variation of $\sigma(t)$ is "slow", then the time derivative on the right side of (24) is indeed of order ε , but it is equally important to note that the factor that multiplies it is "purely periodic", for if it were otherwise, the term would gradually grow until it is no longer "small".

A similar situation occurs when σ is a slow function of z . Now p_z is no longer a constant of the motion, but we can apply the transformation (16) anyway and get

$$\begin{aligned} p_z = \partial W/\partial z &= P_1 + (d\sigma/dz)\,(\partial W_1/\partial\sigma) \\ &= P_1 + (\text{purely periodic function of order } \varepsilon) \end{aligned} \tag{25}$$

This turns out to be necessary for a successful perturbation solution here.

Calculation for the Zero-order Case

In the present case the integral (17) — which essentially gives the relationship between $P_\perp$ and J — may be expressed in terms of complete elliptic integrals of the first and second kind

$$K(k) = \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}} = \frac{\pi}{2}\left(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + ..\right) \quad (26)$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 u}\; du = \frac{\pi}{2}\left(1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - ..\right) \quad (27)$$

In order to reduce the integral (17), the square root must be first transferred to the denominator. Two ways of achieving this are possible. We may integrate by parts (note that there is no contribution from the limits since integration is around a closed path)

$$\oint u\, dv = - \oint v\, du \quad (28)$$

Choosing

$$v = x \qquad u = \left\{ P_\perp^2 - (P_1 + \; x^2)^2 \right\}^{\frac{1}{2}}$$

this gives

$$J = I_1 = \oint \frac{2 P_1 \sigma x^2 + 2 \sigma^2 x^4}{u}\, dx \quad (29)$$

Alternatively, we may write

$$J = I_2 = \frac{u^2}{u}\, dx = \oint \frac{P_\perp^2 - P_1{}^2 - 2 P_1 \sigma x^2 - \sigma^2 x^4}{u}\, dx \quad (29)$$

To eliminate x^4 from the enumerator, we express J as

$$J = \frac{1}{3}(2I_2 + I_1) = \frac{2}{3}\oint \frac{P_\perp^2 - P_1{}^2 - P_1\sigma x^2}{\sqrt{P_\perp^2 - (P_1 + \sigma x^2)^2}}\, dx$$

$$= \frac{2}{3\sqrt{\sigma}}\oint \frac{P_\perp^2 - P_1{}^2 - P_1\sigma x^2}{\sqrt{(P_\perp + P_1 + \sigma x^2)(\mu^2 - x^2)}}\, dx \qquad (30)$$

where we define

$$\mu^2 = (P_\perp - P_1)/\sigma \qquad (31)$$

This will always be positive, by the following arguments. From the form of the Hamiltonian, nothing is changed if we simultaneously reverse the signs of both σ and P_1 ; we may therefore assume that σ is positive. The form of the Hamiltonian also shows that $P_\perp^2$ exceeds $P_1{}^2$, and since $P_\perp$ is defined as a positive quantity, it follows that the expression on the right side of (31) is also positive.

One now substitutes

$$\begin{aligned} x &= -\mu \cos z \\ dx &= \mu \sin z\ dz = (\mu^2 - x^2)^{\frac{1}{2}}\, dz \\ \sigma x^2 &= (P_\perp - P_1)\cos^2 z \end{aligned} \qquad (32)^*$$

Then

$$J = \frac{2}{3\sqrt{\sigma}}(P_\perp - P_1)\oint \frac{P_\perp + P_1 - P_1\cos^2 z}{\sqrt{P_\perp + P_1 + (P_\perp - P_1)\cos^2 z}}\, dz \qquad (33)$$

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* Note: z used in (32) and in the equations that follow is an auxiliary variable not related to the z-coordinate.

Let P_1 be eliminated and replaced by k, defined by

$$k^2 = \tfrac{1}{2}\left\{1 - (P_1/P_\perp)\right\} \tag{34}$$

i.e.

$$P_1 = P_\perp(1 - 2k^2) \tag{35}$$

The <u>denominator</u> in (33) then becomes

$$\left\{P_\perp + P_1 + (P_\perp - P_1)\cos^2 z\right\}^{\frac{1}{2}} = \left\{2P_\perp - (P_\perp - P_1)\sin^2 z\right\}^{\frac{1}{2}}$$
$$= \sqrt{2P_\perp}\left\{1 - k^2\sin^2 z\right\}^{\frac{1}{2}} \tag{36}$$

The <u>numerator</u> is

$$P_\perp + P_1\sin^2 z = P_\perp\left\{1 + (1 - 2k^2)\sin^2 z\right\}$$

$$= P_\perp\left\{\left(\frac{1-k^2}{k^2}\right) + \left(\frac{1-2k^2}{k^2}\right)(1 - k^2\sin^2 z)\right\} \tag{37}$$

Substituting everything and noting that the range of integration of $\oint$ is 4 times that of $K(k)$ or $E(k)$ (0 to 2π instead of 0 to $\pi/2$), we get (Sonnerup's notation):

$$J = \frac{2}{3}\sqrt{\frac{P_\perp^3}{2\sigma}}\left\{\left(\frac{1-k^2}{k^2}\right)\oint\frac{du}{\sqrt{1-k^2\sin^2 u}} - \left(\frac{1-2k^2}{k^2}\right)\oint\sqrt{1-k^2\sin^2 u}\;du\right\}$$

$$= \frac{16}{3}P_\perp^{3/2}(2\sigma)^{-\frac{1}{2}}\left\{(1-k^2)K(k) - (1-2k^2)E(k)\right\}$$

$$= \frac{16}{3}P_\perp^{3/2}(2\sigma)^{-\frac{1}{2}}f_1(k) \tag{38}$$

From general perturbation theory it follows that J is conserved adiabatically, if σ varies slowly with z and/or time. As an example demonstrating the intuitive significance of such conservation, consider a particle in the meandering mode with small P_1. Then

$$k^2 \cong \tfrac{1}{2} \qquad f_1(k) \cong \pi/4 = \text{constant} \tag{39}$$

$$J^2 = P_\perp^3/\sigma \tag{40}$$

Now by eq. (4) σ is proportional to B_0/L, which is the only non-zero component of $\nabla \underline{B}$. If σ grows slowly in time, $P_\perp$ also grows slowly, at a rate that leaves J approximately constant.

If the meandering motion carries the particle to regions of stronger gradient (i.e. with $\sigma(z)$), J is still an adiabatic invariant. However, since energy is conserved and the Hamiltonian (it can be shown) still separates as before, $P_\perp$ cannot vary and the conservation of J is maintained by the variation of k, through changes in P_1. Equation (40) may not be used in this case.

The New Hamiltonian

Since k is a function of $P_\perp$ and P_1, equation (38) defines an implicit relationship

$$P_\perp = P_\perp(J, P_1) \tag{41}$$

When this is substituted in (7), it gives the new Hamiltonian (which without time dependence equals the old one). The trouble is that inverting (38) to the form (41) is not easy: for instance, we could use the series expansions (26) and (27), but this tends to be cumbersome.

There exists a way around this difficulty, namely by redefining P_1, the new variable introduced by the canonical transformation generated by W. Instead of defining it (as before) equal to p_z, we define

$$P_1 = k = \sqrt{\tfrac{1}{2}\left[1 - (p_z/P_\perp)\right]} \qquad (42)$$

Equation (16) is now replaced by

$$W = z P_\perp (1 - 2 P_1^2) + \int \sqrt{P_\perp^2 - \left[P_\perp(1 - 2P_1^2) + \sigma x^2\right]^2}\, dx \qquad (43)$$

and (42) follows immediately from

$$p_z = \partial W/\partial z \qquad (44)$$

Also, since

$$p_x = \partial W_1/\partial x \qquad (45)$$

the derivation of J is unchanged, giving instead of (38)

$$J = (16/3)\, P_\perp^{3/2} (2\sigma)^{-\frac{1}{2}} f_1(P_1) \qquad (46)$$

from which

$$H = (1/2m)\,(3/16)^{4/3} (2\sigma)^{2/3} J^{4/3} \left\{f_1(P_1)\right\}^{-4/3} \qquad (47)$$

Since Ω grows by unity each period, the frequency ν of the oscillation is at once obtained from

$$\nu = \partial H/\partial J = (2/3m)\, P_\perp^2/J \qquad (48)$$

This choice of P_1 will be termed the <u>second choice</u>. It is useful for perturbed motions where the perturbation consists of some additional terms in the Hamiltonian (examples of this will be described later) but not where σ is a slow function of position or time. Suppose that $\sigma = \sigma(t)$; then the new Hamiltonian acquires two added terms

$$\frac{\partial W}{\partial t} = z\,(1 - 2\,P_1^{\,2})\,\frac{\partial P_1}{\partial \sigma}\,\frac{d\sigma}{dt} + \frac{\partial W_1}{\partial \sigma}\,\frac{d\sigma}{dt} \qquad (49)$$

Both terms contain a "small" factor $d\sigma/dt$. In the second term, the expression multiplying this factor is purely periodic, by arguments cited earlier, but in the first term we get an expression proportional to z , and this is liable to grow as the motion progresses. In such a case, therefore, we must use the first choice of P_1 equal to p_z and invert (38) to (41) by other means.

It is also possible for σ to depend on x , which results in a different shape of the oscillation. The problem then is rather similar, but the integrals one obtains cannot be reduced to elliptic form and must be handled by expansion, numerically or by other means.

Slow Dependence of σ on z

As an example, assume that $\sigma = \sigma(z)$, with $d\sigma/dz$ small compared, say, to the ratio between σ and the distance between two consecutive crossings of the neutral plane (in the meandering mode). The Hamiltonian again separates into two parts, and the part describing the motion orthogonal to the field (we will drop the subscript here) is

$$H = (1/2m)\left\{ p_x^2 + (\,p_z + \sigma(z)\,x^2)^2 \right\} \qquad (50)$$

Obviously p_z is no longer conserved, but we apply anyway the transformation generated by (16). By (44) and (45), with the "first choice" of P_1 and with $P_\perp$ defined as in (38)

$$p_x = \partial W/\partial x = \sqrt{P_\perp^2 - (P_1 + \sigma x^2)^2} \tag{51}$$

$$p_z = \partial W/\partial z = P_1 + (\partial W/\partial \sigma)\, d\sigma/dz \tag{52}$$

Substituting this in (50) gives as the new Hamiltonian

$$H = (1/2m)\left\{P_\perp^2 - (P_1 + \sigma x^2)^2 + \left(P_1 + \frac{\partial W}{\partial \sigma}\frac{d\sigma}{dz} + \sigma x^2\right)^2\right\}$$

$$= (1/2m)\left\{P_\perp^2 + 2\frac{\partial W}{\partial \sigma}\frac{d\sigma}{dz}(P_1 + \sigma x^2) + O(\varepsilon^2)\right\} \tag{53}$$

The zero-order part of this is a function $H^{(0)}$ implicitely defined through (38)

$$P_\perp^2/2m = H^{(0)}(J, P_1, \sigma(z)) \tag{54}$$

It is not permissible, however, to use z in the new Hamiltonian, and we therefore transform and expand, using (23)

$$\sigma(x) = \sigma(Q_1) - \frac{\partial W_1}{\partial P_1}\frac{d\sigma}{dQ_1} + O(\varepsilon^2) \tag{55}$$

Because $\sigma(z)$ differs from $\sigma(Q_1)$ only by small terms we may replace in the first-order part of (53) $d\sigma/dz$ by $d\sigma/dQ_1$. In the same expression we also must replace x^2 by a function of the new variables. By eq. (32)

$$\sigma x^2 = (P_\perp - P_1)\cos^2 z$$

Now the angle variable Ω is, with the same definition of z

$$\Omega = \frac{\partial W}{\partial J} = \frac{1}{2}\,\frac{\partial P_\perp^2}{\partial J}\int \frac{dx}{\sqrt{P_\perp^2 - (P_1 + \sigma x^2)^2}}$$

$$= \frac{1}{2\sqrt{\sigma}}\,\frac{\partial P_\perp^2}{\partial J}\int \frac{dx}{\sqrt{(P_\perp + P_1 + \sigma x^2)(\mu^2 - x^2)}}$$

$$= \frac{1}{\sqrt{8\sigma P_\perp}}\,\frac{\partial P_\perp^2}{\partial J}\int_0^z \frac{dz}{\sqrt{1 - k^2 \sin^2 z}} \qquad (56)$$

Hence (e.g. eqs. (133), (135) in section 112 of Franklin, <u>Methods of Advanced Calculus</u>)

$$\sigma x^2 = (P_\perp - P_1)\ \mathrm{cn}^2 \left\{ \frac{\Omega\sqrt{8\sigma P_\perp}}{\partial P_\perp^2/\partial J} \right\} \qquad (57)$$

The new Hamiltonian thus is

$$H = H^{(0)}(J, P_1, \sigma(Q_1)) - \frac{\partial H^{(0)}}{\partial \sigma}\,\frac{d\sigma}{dQ_1}\,\frac{\partial W_1}{\partial P_1} +$$

$$+ \frac{1}{m}\,\frac{\partial W}{\partial \sigma}\,\frac{d\sigma}{dQ_1}\left(P_1 + (P_\perp - P_1)\ \mathrm{cn}^2 \left\{ \frac{\Omega\sqrt{8\sigma P_\perp}}{\partial P_\perp^2/\partial J} \right\} \right)$$

$$+ O(\varepsilon^2) \qquad (58)$$

The $O(\varepsilon)$ term is purely periodic with zero average, since periodic factors only appear in it in odd powers, so J is indeed adiabatically conserved

$$\langle J \rangle = \text{constant} + O(\varepsilon^2) \qquad (59)$$

One could go on here and derive higher-order corrections, but there seems to exist little practical need for those.

Adding a Constant Electric Field Along z

Suppose that we add a constant electric field in the z-direction

$$\underline{E} = - \nabla \phi \qquad (60)$$

$$\phi = - E_0 z \qquad (61)$$

Then

$$H = (1/2m) \left\{ p_x^{\;2} + (p_z + \sigma x^2)^2 \right\} - e E_0 z \qquad (62)$$

If the electric field is relatively small, we can regard it as a perturbation superimposed on the type of motion previously analyzed. In beginning the calculation we disregard the perturbing term and apply the canonical transformation (43), where (P_1, J) are the new canonical momenta, $P_\perp$ is defined by (38) and the "second choice" (42) is used in defining P_1 . The main part of the Hamiltonian transforms as before and we get

$$H = P_\perp^2 /2m - e E_0 z \qquad (63)$$

To express z one must use (23)

$$Q_1 = \frac{\partial W}{\partial P_1} = z \frac{\partial P_z}{\partial P_1} + \frac{\partial W_1}{\partial P_1} \qquad (64)$$

where we have introduced an auxiliary function equal to p_z

$$P_z(J, P_1, \sigma) = P_\perp (1 - 2 P_1^{\;2}) \qquad (65)$$

Thus

$$H = \frac{1}{2m} (3/16)^{4/3} (2\sigma)^{2/3} J^{4/3} \left[f_1(P_1) \right]^{-4/3} - \frac{e E_0}{\partial P_z / \partial P_1} \left\{ Q_1 - \frac{\partial W_1}{\partial P_1} \right\}$$

$$(66)$$

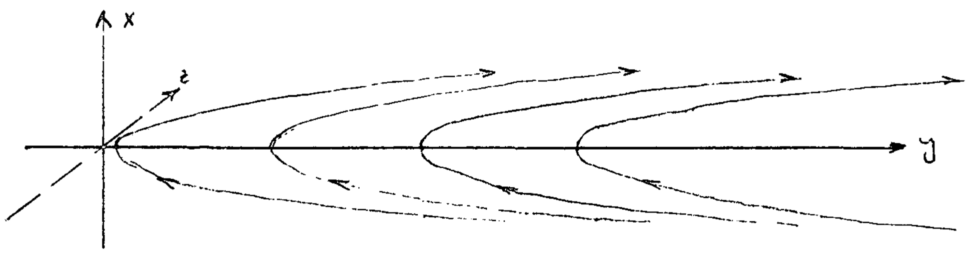
The Hamiltonian (66) contains 3 terms, only the last of which (the one proportional to $\partial W_1/\partial P_1$) involves Ω. If only that term is considered as a perturbation term, canonical transformations may be devised that push the dependence on Ω to an arbitrarily high order. There remains a one-dimensional system in (P_1, Q_1) which is readily solved, at least in principle.

Small Added Magnetic Field in x-direction

Suppose that a small magnetic component

$$\underline{b} = b\,\hat{\underline{x}} \qquad (67)$$

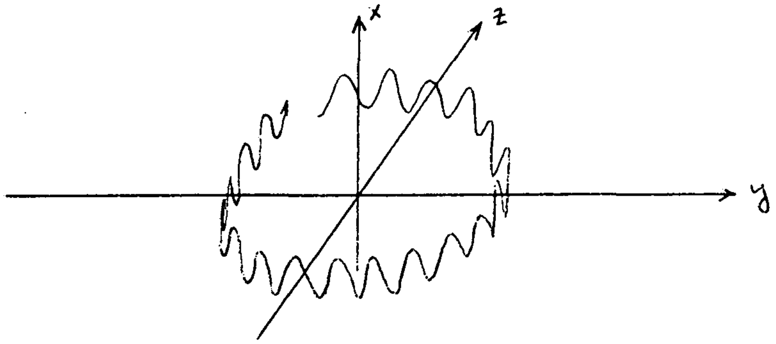
is added, so that instead of a neutral sheet we have a sheet of very low field intensity and field lines become parabola-shaped



Several approaches to this problem exist and will be described in what follows.

(1) Intuitive Treatment

It is easy to make a crude guess about what might happen here. No additional force exists in the x-direction, so the oscillatory motion orthogonal to the plane $x = 0$ should not be affected. If this motion is averaged out by use of action-angle variables (J, Ω) one is left with an average motion in that plane, presumably one affected only by $\underline{b}$. We could then get a circular motion, resembling the guiding-center motion due to $\underline{b}$ alone :



Sonnerup, who treated this case numerically, indeed obtained certain solutions which resemble this form, though for other solutions a marked y - z asymmetry is evident (see Figure 4 in his paper). We may view the situation given in the sketch above as a motion with the usual two lowest invariants, but with the ordering of their characteristic time scales reversed: the oscillation about the orbit's point of minimum field intensity is rapid, while the guiding center gyration is slow.

(2) Treatment as Coupled Oscillators

Analytically, we have

$$\underline{b} = b\,\hat{\underline{x}} = b\,(\nabla y \times \nabla z)$$
$$= \nabla \times (\,b\,y\,\hat{\underline{z}}\,) \qquad (68)$$

The y-component of the motion can no longer be separated and the Hamiltonian thus is

$$H = (1/2m)\left\{p_x^{\,2} + p_y^{\,2} + (p_z + \sigma x^2 - (eb/c)\,y\,)^2\right\} \qquad (69)$$

On the other hand, p_z is still a constant of the motion

$$p_z = P_z = \text{constant} \qquad (70)$$

Introducing new notation

$$\sigma_1 = \sigma \qquad \sigma_2 = eb/c$$

we can rearrange H to the following form

$$H = \frac{1}{2m}\left\{ p_x{}^2 + (P_z + \sigma_1 x^2)^2 \right\} + \frac{1}{2m}\left\{ p_y{}^2 + (P_z - \sigma_2 y)^2 \right\}$$

$$- \; P_z{}^2/2m \quad - \quad (\sigma_1 \sigma_2 /m)\, x^2 y \qquad (71)$$

This can be viewed as consisting of four parts. The first part is an anharmonic oscillator in the x-direction, of the type encountered in the treatment of meandering motion; the second term represents a harmonic oscillator in the y-direction ; the third term is constant and does not affect the motion, while the last term represents an interaction between the two oscillators. It is proportional to b and therefore relatively small.

To treat the motion we transform the Hamiltonian to the action-angle variables of the two oscillators and then treat the interaction term by classical perturbation theory. The problem resembles a wide class of problems in celestial mechanics and may involve resonance problems, since it involves the interaction of two oscillating systems with different frequencies. In particular, the problem of two harmonic oscillators coupled by a nonlinear term similar to the one in (71) has been extensively studied by Contopoulos (Astrophys. J. 153, 83, '68 ; Astronomical J. 73, 86, '68 ; Astronomical J. 75, 96, 108, '70 ; Astronomical J. 76, 147, '71).

(3) Treatment as Motion in a Potential

Because P_z is constant, the Hamiltonian

$$H = (1/2m) \left\{ p_x^2 + p_y^2 + (P_z + \sigma_1 x^2 - \sigma_2 y)^2 \right\} \quad (72)$$

may be viewed as describing motion in a potential

$$\begin{aligned} V &= (1/2m)\,(P_z + \sigma_1 x^2 - \sigma_2 y)^2 \\ &= (\sigma_2^2/2m) \left\{ (\sigma_1/\sigma_2)\, x^2 - (y - y_0) \right\}^2 \end{aligned} \quad (73)$$

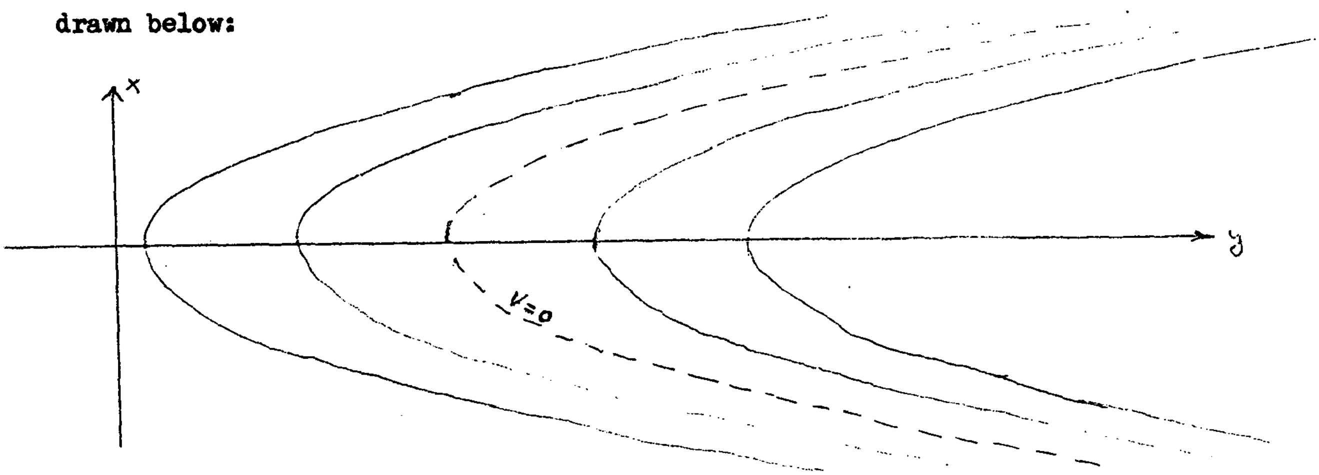
The potential V is never negative and from its second form in (73) it follows that for every value of V there exist two parabolic equipotentials of the form

$$y = y_0 + (\sigma_1/\sigma_2)\, x^2 \pm \sqrt{2\,m\,V}/\sigma_2 \quad (74)$$

These are equally spaced in relation to the equipotential

$$y = y_0 + (\sigma_1/\sigma_2)\, x^2 \quad (75)$$

which corresponds to the lowest level $V = 0$. Some typical contours are drawn below:



The topography of V is that of a parabolic channel which (when viewed in normal cross-section) narrows down with increasing y. For particles which move down the channel for a considerable distance, the usual two adiabatic invariants may be derived, and the particles will bounce back and forth across the line $y = 0$ with reflection in each of the arms of the channel.

On the other hand, consider a particle starting its motion along the y-axis. By symmetry, the particle will maintain $x = 0$ at all times, and its motion will therefore be governed by the potential

$$V = (\sigma_2^2/2m)(y - y_0)^2 \qquad (76)$$

which is a harmonic potential well. Presumably the orbits discussed in the section on "intuitive treatment" will lie close to this class. If the corresponding motion in the x-direction were also harmonic, of the same frequency, we could get circular orbits in the averaged motion in the $x = 0$ plane. In view of Sonnerup's numerical results, this is only approximately true and even that only in certain cases, while in general the averaged motion in the y-direction and in the z-direction have different properties.

(4) Generalization

Assertion: In general, if to the field of eq. (1) we add a magnetic field independent of z and with no component in the z direction

$$\left.\begin{aligned} \underline{b} &= \underline{b}(x,y) \\ b_z &= 0 \end{aligned}\right\} \qquad (77)$$

then the motion of a charged particle in the combined field can be reduced to motion in a potential in the (x, y) plane, and the field lines in this plane follow the equipotential lines.

Proof: Clearly we can choose z to be one of the Euler potentials here

$$z = \beta$$

Let $\alpha = \alpha(x, y)$ be the Euler potential conjugate to this. The Hamiltonian then has the form

$$H = (1/2m) \left\{ p_x^{\,2} + p_y^{\,2} + \left(p_z - (e/c)\,\alpha(x, y) \right)^2 \right\} \tag{78}$$

Since z is absent, p_z is a constant and our potential is

$$V = (1/2m) \left(p_z - (e/c)\,\alpha(x, y) \right)^2 \tag{79}$$

Magnetic field lines in the (x,y) plane stay in this plane by virtue of equations (77) and are characterized by constant values of $\alpha(x, y)$. By equation (79) the potential V is also constant along such lines, hence Q.E.D.

It is interesting to compare this case to Störmer's treatment of particle motion in a dipole field. There, too, the problem reduces to motion is a two-dimensional potential field (Störmer: The Polar Aurora, Oxford 1955 ; A.J.Dragt, Reviews of Geophysics 3, 255, '65) , but the equipotentials (except for $V = 0$) do not follow field lines. This may be traced to the fact that in Störmer's case the basic geometry is axisymmetric rather than independent of one of the cartesian coordinates.

X - type Null Point

If to the main field of eq. (1) we add a field in the x-direction increasing linearly with y (i.e. a field resembling the main field in its form but orthogonal to it)

$$\underline{B}' = (b/L)\, y\, \hat{\underline{x}} \tag{80}$$

then we obtain an x-type null at the origin (actually, a line of null points stretching along the z-axis). We will assume that $b \ll B_0$, so that near the null point (80) represents a small perturbation. Since (80) is a field of the general type of (77), it may be handled in the same manner as indicated there:

$$\underline{B}' = (b/L)\, y\, (\nabla y \times \nabla z) = (b/2L)\, (\nabla y^2 \times \nabla z)$$

$$= \nabla \times (b/2L)\, y^2\, \hat{\underline{z}} \tag{81}$$

Let us define

$$\left.\begin{aligned} \sigma_1 &= \sigma = e B_0/2Lc \\ \sigma_2 &= eb/2Lc \end{aligned}\right\} \tag{82}$$

then

$$H = (1/2m) \left\{ p_x{}^2 + p_y{}^2 + (p_z + \sigma_1 x^2 - \sigma_2 y^2)^2 \right\} \tag{83}$$

As in (78), p_z is again a constant P_z of the motion. As in (71), the Hamiltonian may be rearranged to the form

$$H = \frac{1}{2m} \left\{ p_x{}^2 + (P_z + \sigma_1 x^2)^2 \right\} + \frac{1}{2m} \left\{ p_y{}^2 + (P_z - \sigma_2 y^2)^2 \right\}$$

$$- P_z{}^2/2m - (\sigma_1 \sigma_2/m)\, x^2 y^2 \tag{84}$$

As in (71), this may be viewed as representing two one-dimensional oscillators -- anharmonic ones, of the "meandering" type -- coupled by a nonlinear term proportional to x^2y^2. This can be treated by perturbation methods in a manner similar to what was indicated there.

Alternatively, we can view (83) as representing motion in a two-dimensional potential

$$V = \frac{1}{2m}\left(P_z + \sigma_1 x^2 - \sigma_2 y^2\right)^2 \qquad (85)$$

Simultaneous inversion of the signs of σ_1 and σ_2 does not affect the Hamiltonian (although it also inverts the sign of P_z appropriate to any given orbit), so the question is essentially whether the signs of these two coefficients are equal or not. Suppose that both are positive:

$$\sigma_1 = 1 / s_1^2 \qquad \sigma_2 = 1 / s_2^2 \qquad (86)$$

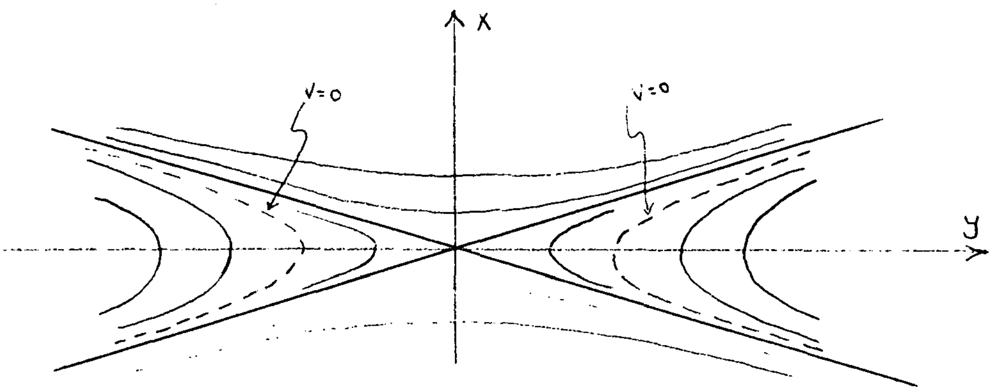
Then the equipotentials for a given value of V will be two pairs of hyperbolas

$$\frac{y^2}{s_2^2} - \frac{x^2}{s_1^2} = P_z \pm \sqrt{2\,m\,V} \qquad (87)$$

As V becomes smaller the hyperbolas converge towards the pair

$$(y^2/ s_2^2) - (x^2/ s_1^2) = P_z \qquad (88)$$

which corresponds to $V = 0$; these two hyperbolas are the bottoms of two potential valleys and are separated by a pass at which $V = P_z^2/2m$, which also is the potential on the two asymptotic straight lines. A sketch of the appropriate topography (for negative P_z) is given on the next page. If P_z is positive the valleys of $V = 0$ appear in the other family of hyperbolas, for only then can $V = 0$ be met with $x = 0$.



If σ_1 and σ_2 have opposite signs, we obtain a set of nested ellipses centered around an O-type null point. A single null point is not likely to be of this type, but if a chain of them is given (as in the next section) then O-type and X-type points alternate.

A Chain of Null Points

A practical case, possibly resembling the tail of the earth's field more than any of the preceding models, may be a chain of alternating neutral points of X-type and O-type, along the surface $x = 0$. This was suggested in theoretical work of Coppi et al. (Phys. Rev. Letters 16, 1207, 27 June '66) and by analyzing observations by Schindler and Ness (J. Geophys. Res. 77, 91, '72). The latter authors also provide a model for the field, but we will represent the situation here by a somewhat simpler model than theirs. Such a model is

$$\underline{B} = (B_0/L)\, x\, \hat{\underline{y}} + b\, \hat{\underline{x}} \sin \omega y \qquad (89)$$

This may be written

$$\underline{B} = \nabla \times \left(\frac{b}{\omega} \cos \omega y - (B_0/2L)\, x^2 \right) \hat{\underline{z}} = -\frac{c}{e} \nabla \times \left(\sigma_1 x^2 + \sigma_2 \cos \omega y \right) \hat{\underline{z}} \tag{90}$$

leading to the Hamiltonian

$$H = \frac{1}{2m} \left\{ p_x^2 + p_y^2 + (P_z + \sigma_1 x^2 + \sigma_2 \cos \omega y)^2 \right\} \tag{91}$$

This as before can be resolved into two one-dimensional coupled motions, of which one is oscillatory while the other may or may not have a steady growth , in addition to its oscillation.

Alternatively, we may view H as representing motion in a two-dimensional potential

$$V = \frac{1}{2m} (P_z + \sigma_1 x^2 + \sigma_2 \cos \omega y)^2 \tag{92}$$

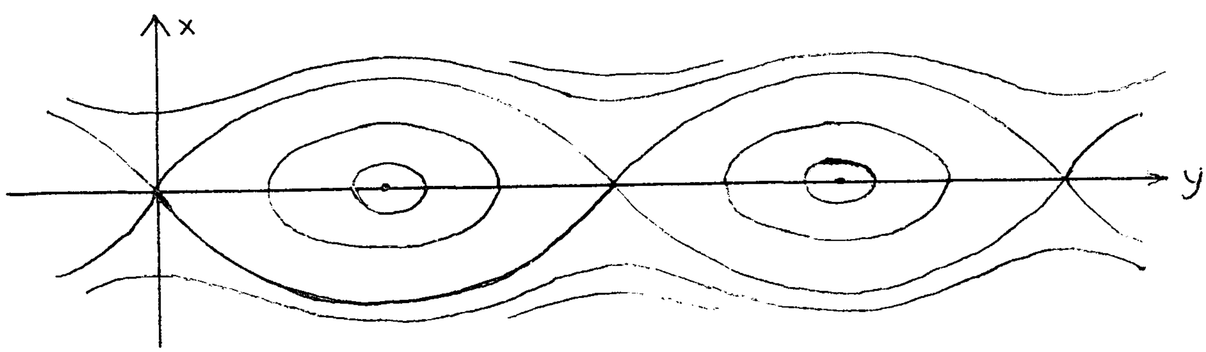
This potential is non-negative and its gradient

$$\nabla V = V_0 (2 \sigma_1 x \hat{\underline{x}} - \sigma_2 \omega \sin \omega y \hat{\underline{y}}) \tag{93}$$

vanishes if

$$x = 0 \ ; \quad \omega y = \pm \pi N \quad (N = 0, 1, 2, \ldots) \tag{94}$$

The lines of constant V are sketched below:



Since for small y

$$\sigma_2 \cos \omega y \;\cong\; \sigma_2 - \tfrac{1}{2}\sigma_2 \omega^2 y^2 \qquad (95)$$

we get hyperbola-like levels near the origin if σ_1 and σ_2 are of the same sign, i.e. the origin and all null points with even N are of X-type. At these points

$$V = \frac{1}{2m}(P_z + \sigma_2)^2 \qquad (96)$$

while for odd N we get O-type nulls and

$$V = \frac{1}{2m}(P_z - \sigma_2)^2 \qquad (97)$$

Let us assume σ_1 and σ_2 are both positive (if both are negative, the results derived below all hold with the sign of P_z everywhere reversed; if their signs differ, the whole pattern shifts phase).

On the line $x = 0$ we have

$$V = \frac{1}{2m}(P_z + \sigma_2 \cos \omega y)^2 \qquad (98)$$

If $|P_z| < \sigma_2$, (98) will pass through zero and therefore the potential will vanish on closed channels of roughly elliptical shape, each surrounding a central peak and separated from similar channels by two passes, the whole chain of passes and channels being flanked by two rising slopes.

If $|P_z| > \sigma_2$, the situation depends on the sign of P_z. If $P_z > \sigma_2$ the expression being squared in (98) is always positive, and the extra term $\sigma_1 x^2$ in (92) can only increase it. Thus V never vanishes and is smallest at the O-type nulls, where (97) holds, and these form a series of pits separated by passes, again the whole chain flanked by rising slopes. If $P_z < -\sigma_2$, the same expression is always negative. However, by (92) the expression inside V reverses sign and becomes positive for large $|x|$. We then get two wavy valleys of $V = 0$ flanking a central ridge of alternating peaks and passes, and these valleys in their turn are flanked as before by two rising slopes.