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Abstract

The diffusion equation which describes the evolution of the average one particle distribution function for an ensemble of strongly turbulent plasmas is derived. The diffusion tensor is a time integral of the autocorrelation tensor of the fluctuations as observed by particles moving along statistically distributed orbits. These orbits contain the effects of fluctuations and thus differ from those encountered in weak turbulence theory. Two statistical orbit effects quadratic in the strength of the fluctuations affect the magnitude of the diffusion: (a) modification of the ensemble average orbits by the fluctuations and (b) statistical dispersion in particle orbits about the average. The plasma trajectory equations are used to relate each to the diffusion tensor itself when the turbulence is electrostatic. The diffusion tensor is explicitly evaluated for a strongly turbulent unmagnetized plasma.

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I. INTRODUCTION

Plasma turbulence is described theoretically by statistical formalisms: one derives equations which advance in time the average properties of an ensemble of plasmas. The ensemble may be thought of as consisting of realizations which differ only in the phases of the Fourier decomposed microfields at some arbitrary instant. Rigorous specification of the ensemble is, however, generally unessential to the formalisms.

For each realization, the one particle distribution function f for each species satisfies the Vlasov Equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \underset{\sim}{F} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = 0 \tag{1}$$

with **F** the force per unit mass on the element of plasma located at the phase space point **x**, **v** at time t. Denoting the average of a quantity over the ensemble by $\langle\ \rangle$ and its deviation from the average by δ we obtain from Eq. (1)

$$\frac{\partial \langle f \rangle}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \langle f \rangle + \langle \underset{\sim}{F} \rangle \cdot \frac{\partial \langle f \rangle}{\partial \underset{\sim}{v}} = - \left\langle \delta \underset{\sim}{F} \cdot \frac{\partial \delta f}{\partial \underset{\sim}{v}} \right\rangle \tag{2}$$

By subtracting Eq. (2) from Eq. (1) the equation for δf, the fluctuating component of the distribution function in any single realization, is obtained

$$\frac{\partial \delta f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \delta f + (\langle \underset{\sim}{E} \rangle + \delta \underset{\sim}{E}) \cdot \frac{\partial \delta f}{\partial \underset{\sim}{v}} = -\delta \underset{\sim}{E} \cdot \frac{\partial \langle f \rangle}{\partial \underset{\sim}{v}} + \left\langle \delta \underset{\sim}{E} \cdot \frac{\partial \delta f}{\partial \underset{\sim}{v}} \right\rangle \tag{3}$$

In the usual theory of weak turbulence[1-9] one first discards terms quadratic in the fluctuations from Eq. (3) and then integrates the resulting linear equation for δf. Plugging this expression for δf into Eq. (2) and making standard assumptions result in a diffusion equation for $\langle f \rangle$

$$\frac{\partial \langle f \rangle}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \langle f \rangle + \langle \underset{\sim}{E} \rangle \cdot \frac{\partial \langle f \rangle}{\partial \underset{\sim}{v}} = \frac{\partial}{\partial \underset{\sim}{v}} \cdot \underset{\approx}{D} \cdot \frac{\partial \langle f \rangle}{\partial \underset{\sim}{v}} \tag{4}$$

The diffusion tensor in the weak turbulence approximation is

$$\underset{\approx}{D} = \int_0^\infty d\tau \left\langle \delta \underset{\sim}{E}(\underset{\sim}{x}, \underset{\sim}{v}, t)\, \delta \underset{\sim}{E}\left[\bar{\underset{\sim}{x}}^*(t-\tau), \bar{\underset{\sim}{v}}^*(t-\tau), t-\tau\right] \right\rangle \tag{5}$$

where $\bar{x}^*$ and $\bar{v}^*$ are the characteristic trajectories of the linearized Eq. 3, solutions to the ordinary differential equations

$$\frac{d\bar{\underset{\sim}{x}}^*}{d\tau} = \bar{\underset{\sim}{v}}^* \qquad\qquad \frac{d\bar{\underset{\sim}{v}}^*}{d\tau} = \langle \underset{\sim}{E}(\bar{\underset{\sim}{x}}^*, \bar{\underset{\sim}{v}}^*, \tau) \rangle \tag{6}$$

with the boundary conditions $\bar{x}^*(t) = x$, $\bar{v}^*(t) = v$. Note that the orbit $\bar{x}^*(t)$, $\bar{v}^*(t)$ is a non-fluctuating one.

By taking moments of Eq. (4) one obtains equations which describe the evolution of the ensemble averaged macroscopic properties of the plasma.

From Eq. (5) we see that the magnitude of the diffusion tensor element D_{ij} depends on both the strength of the fluctuations and the effective width (as a function of τ) of the correlation tensor element

$$C_{ij} = \left\langle \delta F_i (\underset{\sim}{x}, \underset{\sim}{v}, t)\, \delta F_j \left[\underset{\sim}{\bar{x}}^*(t-\tau), \underset{\sim}{\bar{v}}^*(t-\tau), t-\tau \right] \right\rangle$$

where δF_i is the i^{th} vector component. The net area under the C_{ij} vs. τ curve divided by $|\delta F_i^{rms}|\,|\delta F_j^{rms}|$ is a measure of this correlation width. Except for particles with approximately the same velocity as the typical phase velocity of the waves which comprise the turbulence, the net area under the C_{ij} vs. τ curve is small (Either C_{ij} is a rapidly damped or highly oscillating function of τ.) because the turbulence is convecting past such non-resonant particles.

The correlation width and D_{ij} are largest for resonant particles.[10] For resonant particles C_{ij} has the effective width $\tau_c = (\Delta\omega^*)^{-1}$, where $\Delta\omega^*$ is the frequency spread in the Fourier components of the turbulence as observed by the resonant particles. Thus τ_c is the correlation time of the fluctuations <u>as apparent to the resonant particles</u> moving as if the fluctuations were absent. (We shall define τ_c as such even when the fluctuating fields are large.)

In a strongly turbulent plasma the kinetic equation for $\langle f \rangle$ is a diffusion equation. It is again resonant particles which are most strongly diffused. Our treatment in this paper is concerned exclusively with them. The diffusion tensor becomes significantly different from Eq. (5) once the fluctuations reach such a magnitude that in the time interval τ_c they appreciably alter particle orbits.

In our theory of strong turbulence the effect of fluctuations on particle orbits is twofold. Consider a particle which in each realization of the ensemble is at the same phase space point at time t. As time progresses $\langle \mathbf{x}^* \rangle$, $\langle \mathbf{v}^* \rangle$ – the phase space orbit of this particle averaged over the ensemble — develops a contribution quadratic in the fluctuations analogous to the frictional drag in a Fokker-Planck description of interacting particles. [This contribution occurs because in integrating $\ddot{x}^* = F$, it is $F\ (x^*, v^*, \tau)$ which enters. Although $<\delta F\ (x, v, \tau)> = 0$, $<\delta F\ (x^*, v^*, \tau)> \neq 0$.] There will also be a statistical spread in orbits about $<x^*>$, the dispersion likewise being an effect quadratic in δF.

Suppose the motion of the particle at time t is resonant. With advancing time $<v^*>$ changes and the particle gradually becomes non-resonant. Were the particle to become non-resonant in a time interval τ_1 shorter than τ_c (We estimate τ_1 to be the time needed for the δF part of $<x^*>$ to grow to order $2\pi/\Delta k$, the characteristic scale length of the turbulence.), then the width of the auto-correlation as viewed by the particle would be τ_1 rather than τ_c. (The wave number spectrum of the turbulence has width Δk and has typical wave number k_0.) The fluctuations may similarly act to make an initially non-resonant particle resonant.

The spread in orbits also acts to modify (in the ensemble average) the auto-correlation time of the fluctuations as viewed by the particle. $P\,[\mathbf{x}^*(\tau), \mathbf{v}^*(\tau), \tau; \mathbf{x}, \mathbf{v}, t]$, the probability of finding the particle in any single realization at $\mathbf{x}^*(\tau), \mathbf{v}^*(\tau)$ at time τ given that it was at $\mathbf{x}$, $\mathbf{v}$ at time t may be maximum at $\mathbf{x}^*(\tau) = <\mathbf{x}^*(\tau)>, \mathbf{v}^*(\tau) = <\mathbf{v}^*(\tau)>$ but it has an ever increasing width as $\tau - t$ increases. If its width were to increase to $2\pi/k_0$ in the time $\tau_2 \leq \tau_c$, the phase of the wave as viewed by the particle would be effectively random and the wave-particle interaction would average to zero for $\tau - t \geq \tau_2$. In this case the width of the autocorrelation would be limited to τ_2.

Dispersion in statistically averaged orbits in strong turbulence was first considered by Dupree.[11] His work is pioneering in the field and shows that dispersion leads to a modification of the diffusion coefficient and to resonance broadening of the plasma dielectric function. To our knowledge no one has yet explicitly considered the effect of modification of the average orbit on either the diffusion tensor or the plasma dielectric tensor.

We have noted that modification of average orbits by the fluctuations and dispersion about the average orbits are both effects of order $(|\delta F|)^2$. We expect a priori that both affect resonant strong turbulence diffusion with comparable magnitude.

It is our purpose in this paper to calculate simultaneously the effect on diffusion of both average orbit modification and orbit dispersion. In a companion paper[12] we consider how these two effects non-linearly modify the plasma dielectric function and the growth (damping) rate of waves.

Our approach is via the statistical orbits perturbation technique proposed and initially developed by Dupree[11] and expanded by Weinstock[13]. As much as possible, however, we work explicitly with the dynamic trajectories of the particles in phase space rather than resorting exclusively to the operator formalism.

II. THE DIFFUSION EQUATION FOR STRONG TURBULENCE

Following Weinstock[13], let us write Eq. (3) in integral form

$$\delta f(\underset{\sim}{x},\underset{\sim}{v},t) = U_A(t,t_0)\,\delta f(\underset{\sim}{x},\underset{\sim}{v},t_0) - \int_{t_0}^{t} d\tau\, U_A(t,\tau)\,\delta \underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{v},\tau)\cdot\frac{\partial\langle f\rangle(\underset{\sim}{x},\underset{\sim}{v},\tau)}{\partial \underset{\sim}{v}} \tag{7}$$

Here $\delta f(\mathbf{x},\mathbf{v},t_0)$ is the initial value of δf and the propagator $U_A(t,t_0)$ satisfies

$$\frac{\partial U_A}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla} U_A + \left(\langle \underset{\sim}{E}\rangle + \delta\underset{\sim}{F}\right)\cdot\frac{\partial U_A}{\partial \underset{\sim}{v}} - \left\langle \delta\underset{\sim}{F}\cdot\frac{\partial U_A}{\partial \underset{\sim}{v}}\right\rangle = 0 \tag{8}$$

$$U_A(t_0,t_0) = 1$$

We next make the weak coupling approximation[11,13,14]:

$$\begin{aligned} U_A(t,t_0) &\cong \langle U(t,t_0)\rangle = \left\langle \exp\left\{\left[\underset{\sim}{x}^*(t_0)-\underset{\sim}{x}\right]\cdot\underset{\sim}{\nabla} + \left[\underset{\sim}{v}^*(t_0)-\underset{\sim}{v}\right]\cdot\frac{\partial}{\partial\underset{\sim}{v}}\right\}\right\rangle \\ &= \left\langle \exp\left\{\left[\underset{\sim}{\Pi}^*(t_0)-\underset{\sim}{\Pi}\right]\cdot\frac{\partial}{\partial\underset{\sim}{\Pi}}\right\}\right\rangle \end{aligned} \tag{9}$$

In Eq. 9 $\mathbf{x}^*$ and $\mathbf{v}^*$ are solutions of the characteristic equations

$$\frac{d\underset{\sim}{x}^*}{d\tau} = \underset{\sim}{v}^* \qquad\qquad \frac{d\underset{\sim}{v}^*}{d\tau} = \underset{\sim}{F}(\underset{\sim}{x}^*,\underset{\sim}{v}^*,\tau) \tag{10}$$

again with the boundary conditions $\mathbf{x}^*(t) = \mathbf{x}$, $\mathbf{v}^*(t) = \mathbf{v}$. The vector Π is a phase space vector with the six components $\{x_i, v_i\}$ and has been introduced for notational convenience. Note the appearance of $\mathbf{F}$ in Eqs. (10) as distinguished from $\langle \mathbf{F}\rangle$ in Eqs. 6. Successive higher order corrections to Eq. (9) are in the ratio $k_0|\delta \mathbf{F}|\tau_c^2$, considered to be a small but finite quantity in the weak coupling approximation.[13] A straightforward technique for generating higher order corrections to Eq. (9) is presented by Birmingham and Bornatici.[14]

The weak coupling approximation is a perturbation scheme for solving Eq. (3). It amounts to an iteration on the source term $\langle \delta \mathbf{F} \cdot \partial \delta f / \partial \mathbf{v} \rangle$. Elements of the convective term $\delta \mathbf{F} \cdot \partial \delta f / \partial \mathbf{v}$ are retained in lowest order, for only when one includes the influence of fluctuations on particle orbits are the effects of strong turbulence manifest. The fundamental strong turbulence solution for δf represented by Eqs. (7) and (9) thus embodies the fundamental weak turbulence solution (i.e., the solution to Eq. (3) with $\mathcal{O}(\delta^2)$ terms discarded) plus select contributions from all higher order (mode coupling) solutions of the weak turbulence analysis. In this respect the approximation made here (and also made in the work of Dupree[11] and Weinstock[13]) is similar to those made in other theories[(15-22)] of strong plasma turbulence.

Having made the approximation 9, we plug the expression for δf into Eq. (2) and obtain

$$\frac{\partial \langle f \rangle}{\partial t} + \mathbf{v} \cdot \nabla \langle f \rangle + \langle \mathbf{E} \rangle \cdot \frac{\partial \langle f \rangle}{\partial \mathbf{v}} = - \left\langle \delta \mathbf{E} \cdot \frac{\partial}{\partial \mathbf{v}} \left[\langle U(t,t_0) \rangle \, \delta f(\mathbf{x}, \mathbf{v}, t_0) \right] \right\rangle$$
$$+ \frac{\partial}{\partial \mathbf{v}} \cdot \int_{t_0}^{t} d\tau \, \langle \delta \mathbf{E}(\mathbf{x}, \mathbf{v}, t) \langle U(t,\tau) \rangle \, \delta \mathbf{E}(\mathbf{x}, \mathbf{v}, \tau) \rangle \cdot \frac{\partial \langle f(\mathbf{x}, \mathbf{v}, \tau) \rangle}{\partial \mathbf{v}} \qquad (11)$$

We now make the additional standard simplifications:

(1) We neglect the term in Eq. (11) associated with the initial value $\delta f(\mathbf{x}, \mathbf{v}, t_0)$. This term may persist for long times in the phase space

description of the plasma, and on this level its omission is unjustified. However, when velocity moments of the kinetic equation are taken (and we envisage our resultant kinetic equation to be so used), the initial value contribution phase mixes to zero in a time interval Δt. The magnitude of Δt depends on the nature of $\delta f(\mathbf{x}, \mathbf{v}, t_0)$. If u, the characteristic width of $\delta f(t_0)$ in velocity space is much greater than the average velocity v_0 associated with $\delta f(t_0)$, then $\Delta t \cong (k_0 u)^{-1}$ where k_0 is the typical wave number of the turbulent spectrum.

(2) We assume that $<f>$ changes sufficiently slowly with $\mathbf{x}$, $\mathbf{v}$, and t that on the right side of Eq. (11) $\partial <f>/\partial \mathbf{v}$ may be removed from both the $<U>$ operation and also from the time integration. Specifically the $\mathbf{x}$ and $\mathbf{v}$ scales of $<f>$ are assumed longer than the incremental change in phase space position of a resonant particle during the time τ^* characteristic of the decay of elements of the autocorrelation tensor $<\delta F(\mathbf{x}, \mathbf{v}, t)$ $<U(t, \tau)> \delta F(\mathbf{x}, \mathbf{v}, \tau)>$ and the time scale of change of $<f>$ is itself slower than τ^*. (We expect that for strong turbulence τ^* depends on the amplitude of the fluctuations in accordance with the discussion in the Introduction.) These are standard assumptions[23] made in the statistical mechanics of stochastic processes.

With these approximations Eq. (11) again becomes the diffusion equation (4). The diffusion tensor is, however, now

$$\underset{\approx}{D}(\underset{\sim}{x}, \underset{\sim}{v}, t, t-t_0) = \int_0^{t-t_0} d\tau \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{v}, t) \langle U(t, t-\tau) \rangle \delta \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{v}, t-\tau) \right\rangle$$

$$\cong \int_0^{t-t_0} d\tau \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{v}, t)\, \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau), \underset{\sim}{v}^*(t-\tau), t-\tau\right] \right\rangle \tag{12}$$

The latter form of Eq. (12) follows from the facts that (a) to lowest order in δF, $\langle U(t, t-\tau)\rangle \cong U(t, t-\tau)$ [11,13] and (b) $U(t, t-\tau)\ \delta \mathbf{F}(\mathbf{x}, \mathbf{v}, t-\tau) = \delta \mathbf{F}[\mathbf{x}^*(t-\tau), \mathbf{v}^*(t-\tau), t-\tau]$.[14]

If $t - t_0 \gg \tau^*$ (Indeed, the diffusion equation is only valid on a time scale much longer than τ^*) the upper limit of integration in Eq. (12) may be extended to infinity

$$\underset{\approx}{D}(\underset{\sim}{x}, \underset{\sim}{v}; t, \infty) = \int_0^{\infty} d\tau \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{v}, t) \langle U(t, t-\tau) \rangle \delta \underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{v}, t-\tau) \right\rangle \tag{13}$$

Note that the diffusion tensor Eq. (13) differs from the weak turbulence diffusion tensor Eq. (5) in the fact that the correlation in (13) is evaluated along the perturbed orbit of the particle. If the turbulence is approximately homogeneous and stationary, as we shall assume, the dependence of $\underset{\approx}{D}$ in Eqs. (12) and (13) on $\mathbf{x}$ and t disappears. (The dependence of $\underset{\approx}{D}$ on $t - t_0$ for $t - t_0 \le \tau^*$ remains however). We now attempt to evaluate $\langle U(t, t-\tau)\rangle$ in terms of statistical properties of the turbulence.

III. EXPLICIT EVALUATION OF $\underset{\approx}{D}$

To abbreviate this calculation, let us restrict ourselves to the case of electrostatic turbulence $\delta F = q\delta E(x, t)/m$; the extension to include a turbulent electromagnetic field is direct. We further follow Weinstock[13,24] by making a cumulant expansion of $\langle U(t, t-\tau)\rangle$, dropping cumulants higher than second order:

$$\begin{aligned}\langle U(t,t-\tau)\rangle &= \langle \exp[\underset{\sim}{x}^*(t-\tau)-\underset{\sim}{x}]\cdot\underset{\sim}{\nabla}\rangle \equiv \langle \exp \Delta\underset{\sim}{x}^*(t-\tau)\cdot\underset{\sim}{\nabla}\rangle \\ &\simeq \exp\left\{\langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle\cdot\underset{\sim}{\nabla} + \frac{1}{2}\left[\langle\Delta\underset{\sim}{x}^*(t-\tau)\Delta\underset{\sim}{x}^*(t-\tau)\rangle - \langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle\langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle\right] : \underset{\sim}{\nabla}\underset{\sim}{\nabla}\right\}\end{aligned} \tag{14}$$

The terms in Eq. (9) involving the velocity increment $(v^* - v)$ do not appear in (14) because of our assumption of electrostatic turbulence (δF independent of v).

We plug (14) into (13), expand the δF's in Fourier series, and ensemble average assuming that the initial phase of each $\mathbf{k}$-mode varies randomly from one realization to the next:

$$\begin{aligned}\underset{\approx}{D}(\underset{\sim}{v},\infty) &= \int_0^\infty d\tau \sum_{\underset{\sim}{k}} \hat{k}\hat{k} \langle|\delta F_{\underset{\sim}{k}}|^2\rangle \exp -i\omega_{\underset{\sim}{k}}\tau \\ &\exp\left\{-i\underset{\sim}{k}\cdot\langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle - \frac{\underset{\sim}{k}\underset{\sim}{k}}{2} : \left[\langle\Delta\underset{\sim}{x}^*(t-\tau)\Delta\underset{\sim}{x}^*(t-\tau)\rangle - \langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle\langle\Delta\underset{\sim}{x}^*(t-\tau)\rangle\right]\right\}\end{aligned} \tag{15}$$

$\langle|\delta F_{\mathbf{k}}|^2\rangle$ is the ensemble average square amplitude of the Fourier mode with wave vector $\mathbf{k}$, $\hat{k}$ is the unit vector along $\mathbf{k}$, and the imaginary parts of the eigenfrequencies are neglected so that the $\omega_{\mathbf{k}}$'s are real. The term $i\mathbf{k}\cdot\langle\Delta\mathbf{x}^*(t-\tau)\rangle$

contains the effect of the turbulence on the ensemble average orbit. In previous work the turbulent corrections to $\langle \Delta \mathbf{x}^* \rangle$ have been neglected so that in an unmagnetized plasma $\langle \Delta \mathbf{x}^* \rangle$ was just $-\mathbf{v}\tau$. The term $\mathbf{k}\mathbf{k} : [\langle \Delta \mathbf{x}^* \Delta \mathbf{x}^* \rangle - \langle \Delta \mathbf{x}^* \rangle \langle \Delta \mathbf{x}^* \rangle]$ represents the statistical dispersion in the trajectories of the particle passing through $\mathbf{x}$, $\mathbf{v}$ at time t.

One can relate $\Delta \mathbf{x}^*(t-\tau)$ implicitly to the fluctuations by formally integrating Eqs. (10). We here further simplify the problem by assuming that there exists no zeroth order magnetic field

$$\begin{aligned} \Delta \underset{\sim}{x}^*(t-\tau) &= -\underset{\sim}{v}\tau + \int_0^\tau d\tau' \tau' \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau+\tau'), t-\tau+\tau'\right] \\ &= -\underset{\sim}{v}\tau + \int_0^\tau d\tau' (\tau-\tau') \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau'), t-\tau'\right] \end{aligned} \tag{16}$$

It follows that

$$\begin{aligned} \langle \Delta \underset{\sim}{x}^*(t-\tau)\rangle &\cong -\underset{\sim}{v}\tau + \int_0^\tau d\tau' (\tau-\tau') \Big\langle \left[\underset{\sim}{x}^*(t-\tau') - \langle \underset{\sim}{x}^*(t-\tau')\rangle\right] \cdot \\ &\qquad \frac{\partial}{\partial \langle \underset{\sim}{x}^*(t-\tau')\rangle} \delta \underset{\sim}{F}\left[\langle \underset{\sim}{x}^*(t-\tau')\rangle, t-\tau'\right] \Big\rangle \\ &\cong -\underset{\sim}{v}\tau + \int_0^\tau d\tau_1 (\tau-\tau_1) \int_0^{\tau_1} d\tau_2 (\tau_1-\tau_2) \\ &\qquad \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_2), t-\tau_2\right] \cdot \frac{\partial}{\partial \underset{\sim}{x}^*(t-\tau_1)} \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_1), t-\tau_1\right] \right\rangle \\ &\equiv -\underset{\sim}{v}\tau + \int_0^\tau d\tau_1 (\tau-\tau_1) \int_0^{\tau_1} d\tau_2 (\tau_1-\tau_2) \underset{\sim}{T} \end{aligned} \tag{17}$$

in the approximation where terms higher than second order in δF are neglected.

In Appendix A we show that in the random phase approximation T can be transformed in such a way that

$$\langle \Delta \underset{\sim}{x}^{*}(t-\tau) \rangle \cong -\underset{\sim}{V}\tau - \frac{\partial}{\partial \underset{\sim}{V}} \int_0^{\tau} d\tau_1 \, (\tau - \tau_1) \int_0^{\tau_1} d\tau_2 \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^{*}(t-\tau_2), t-\tau_2\right] \cdot \delta \underset{\sim}{F}\left[\underset{\sim}{x}^{*}(t-\tau_1), t-\tau_1\right] \right\rangle \tag{18}$$

For stationary turbulence the correlation in Eq. (18) depends only on $|\tau_1 - \tau_2|$; hence one of the integrations can be performed

$$\begin{aligned} \langle \Delta \underset{\sim}{x}^{*}(t-\tau) \rangle &= -\underset{\sim}{V}\tau - \frac{1}{2}\frac{\partial}{\partial \underset{\sim}{v}} \int_0^{\tau} dz \, (\tau - z)^2 \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, t) \cdot \delta \underset{\sim}{F}\left[\underset{\sim}{x}^{*}(t-z), t-z\right] \right\rangle \\ &\cong -\underset{\sim}{V}\tau - \frac{\tau^2}{2}\frac{\partial}{\partial \underset{\sim}{V}} \int_0^{\tau} dz \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, t) \cdot \delta \underset{\sim}{F}\left[\underset{\sim}{x}^{*}(t-z), t-z\right] \right\rangle \end{aligned} \tag{19}$$

The latter approximation follows because the correlation is peaked at z = 0. Noting Eq. (12), we finally conclude that

$$\langle \Delta \underset{\sim}{x}^{*}(t-\tau) \rangle = -\underset{\sim}{V}\tau - \frac{\tau^2}{2}\frac{\partial}{\partial \underset{\sim}{V}} \mathrm{Tr}\left\{ \underset{\approx}{D}(\underset{\sim}{V}, \tau) \right\} \tag{20}$$

where Tr $\{\underset{\approx}{D}\}$ is the trace of the matrix of $\underset{\approx}{D}$. A result similar to Eq. (20) has been obtained in one dimension by Weinstock[25] using his operator formalism. Our result follows directly from approximating the characteristic trajectory

Eq. (16). Note from Eq. (20) that the deviation of the average orbit from a straight line may be attributed to an effective fluctuating force (per unit mass) of magnitude $\left| \frac{\partial}{\partial \underset{\sim}{v}} \mathrm{Tr} \{ \underset{\approx}{D} \} \right|$.

Next let us calculate the dispersion contribution to Eq. (15). Since $\langle \Delta x^* (t - \tau) \rangle = - v\tau + \mathcal{O}(\delta F^2)$, the dispersion tensor is

$$\underset{\approx}{I} = \langle \Delta \underset{\sim}{x}^*(t-\tau)\, \Delta \underset{\sim}{x}^*(t-\tau) \rangle - \langle \Delta \underset{\sim}{x}^*(t-\tau) \rangle \langle \Delta \underset{\sim}{x}^*(t-\tau) \rangle$$
$$\cong \int_0^\tau d\tau_1\, \tau_1 \int_0^\tau d\tau_2\, \tau_2 \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau+\tau_1), t-\tau+\tau_1\right] \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau+\tau_2), t-\tau+\tau_2\right] \right\rangle \qquad (21)$$

Using the stationarity assumption on Eq. (21), we obtain

$$\underset{\approx}{I} = \int_0^\tau d\tau_1\, \tau_1 \int_0^\tau d\tau_2\, \tau_2 \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau), \tau\right] \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau-\tau_1+\tau_2), \tau-\tau_1+\tau_2\right] \right\rangle \qquad (22)$$

In Appendix B we show that

$$\underset{\approx}{I} \cong \frac{2}{3} \tau^3 \underset{\approx}{D}(\underset{\sim}{v}, \tau) \qquad (23)$$

Thus the dispersion factor in Eq. (15) is $\exp - \mathbf{k}\,\mathbf{k} : \underset{\approx}{D}(v, t)\; \tau^3/3$. Dupree[11] and Weinstock[13] find a similar dependence for the dispersion factor although it is the asymptotic $\underset{\approx}{D}(v, \infty)$ which enters the expressions of these authors. With $\underset{\approx}{D}$ a function of τ the integral in Eq. (15) converges more rapidly than does the corresponding integral obtained by Dupree and Weinstock. Dupree's $\underset{\approx}{D}$ is also by assumption independent of v.

We now gather together the pieces, Eqs. (15), (20), and (23) to obtain

$$\underset{\approx}{D}(\underset{\sim}{v},\infty) = \int_0^\infty d\tau \sum_{\underset{\sim}{k}} \hat{k}\hat{k} \langle |\delta \underset{\sim}{E}_{\underset{\sim}{k}}|^2 \rangle \exp - i\left[(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v})\tau - \frac{\tau^2}{2}\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} Tr\{\underset{\approx}{D}(\underset{\sim}{v},\tau)\}\right] \tag{24}$$

$$\exp - \frac{\tau^3}{3}\underset{\sim}{k}\underset{\sim}{k} : \underset{\approx}{D}(\underset{\sim}{v},\tau)$$

By comparing Eqs. (12) and (13) we see that the non-asymptotic $\underset{\approx}{D}(\mathbf{v},\tau)$ is given by the right side of Eq. (24) with τ replacing ∞ as the upper limit of integration

$$\underset{\approx}{D}(\underset{\sim}{v},\tau) = \int_0^\tau d\tau' \sum_{\underset{\sim}{k}} \hat{k}\hat{k} \langle |\delta \underset{\sim}{E}_{\underset{\sim}{k}}|^2 \rangle \exp - i\left[(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v})\tau' - \frac{(\tau')^2}{2}\underset{\sim}{k}\cdot\frac{\partial}{\partial \underset{\sim}{v}} Tr\{\underset{\approx}{D}(\underset{\sim}{v},\tau')\}\right] \tag{25}$$

$$\exp - \frac{(\tau')^3}{3}\underset{\sim}{k}\underset{\sim}{k} : \underset{\approx}{D}(\underset{\sim}{v},\tau')$$

Note that Eq. (24) properly reduces to the quasi-linear result

$$\underset{\approx}{D}(\underset{\sim}{v},\infty) = \pi \sum_{\underset{\sim}{k}} \hat{k}\hat{k} \langle |\delta \underset{\sim}{E}_{\underset{\sim}{k}}|^2 \rangle \delta(\omega_{\underset{\sim}{k}} - \underset{\sim}{k}\cdot\underset{\sim}{v}) \tag{26}$$

when the $\underset{\approx}{D}$-dependent exponential terms are suppressed.

In the strongly turbulent regime, however, we expect that the width of the τ-integration in Eq. (24) is influenced by these fluctuation-dependent exponential terms. The asymptotic $\underset{\approx}{D}(\mathbf{v},\infty)$ thus requires knowledge of $\underset{\approx}{D}(\mathbf{v},\tau)$, Eq. (25), to compute it when the turbulence is strong. Fortunately $\underset{\approx}{D}(\mathbf{v},\tau)$ occurs in Eq. (24) in terms which drop off abruptly with τ. This fact allows us to use the small τ expansion of $\underset{\approx}{D}(\mathbf{v}, \tau)$ in Eq. (24). From Eq. (25) we have in the small τ limit

$$\underset{\approx}{D}(\mathbf{v},\tau) \cong \sum_{\mathbf{k}} \hat{k}\hat{k}\langle|\delta E_{\mathbf{k}}|^2\rangle\left[\tau - \frac{i(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\tau^2}{2} - \frac{(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})^2\tau^3}{6}\right]$$
$$= \sum_{\mathbf{k}} \hat{k}\hat{k}\langle|\delta E_{\mathbf{k}}|^2\rangle\left[\tau - \frac{(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})^2\tau^3}{6}\right] \quad (27)$$

the term proportional to τ^2 vanishing on $\mathbf{k}$-summation.

We now plug the result Eq. (27) into Eq. (24) and obtain

$$\underset{\approx}{D}(\mathbf{v},\omega) = \int_0^\infty d\tau \sum_{\mathbf{k}} \hat{k}\hat{k}\langle|\delta E_{\mathbf{k}}|^2\rangle \exp -i\left[(\omega_{\mathbf{k}}-\mathbf{k}\cdot\mathbf{v})\tau - \frac{\tau^5}{6}\mathbf{k}\cdot\sum_{\mathbf{k}'}\mathbf{k}'\langle|\delta E_{\mathbf{k}'}|^2\rangle(\omega_{\mathbf{k}'}-\mathbf{k}'\cdot\mathbf{v})\right] \quad (28)$$
$$\exp -\frac{\tau^4}{3}\mathbf{k}\mathbf{k}:\sum_{\mathbf{k}'}\hat{k}'\hat{k}'\langle|\delta E_{\mathbf{k}'}|^2\rangle$$

Equation (28) is the main result of this theory.

Let us now look at the time scales on which the integrand of Eq. (28) would converge if the three exponential factors occurred individually rather than in combination. When contributions from different $\mathbf{k}$'s are summed, the factor $\exp -i(\omega_{\mathbf{k}} - \mathbf{k}\cdot\mathbf{v})\tau$ produces the usual weak turbulence convergence. This convergence is due to phase mixing of waves and occurs on the time scale $\tau_c = (\Delta\omega^*)^{-1}$. If all waves have approximately the same frequency (as is the case for plasma oscillations, for example), then $\tau_c \cong (\Delta k |v|)^{-1}$, where Δk is the wave number spread in the waves propagating in the direction of $\mathbf{v}$.

The factor

$$\exp - \frac{\tau^4}{3} \underset{\sim}{k}\, \underset{\sim}{k} : \sum_{\underset{\sim}{k}'} \hat{k}' \hat{k}' \langle |\delta E_{\underset{\sim}{k}'}|^2 \rangle \equiv \exp - \frac{\tau^4}{3} \underset{\sim}{k}\, \underset{\sim}{k} : \langle \delta \underset{\sim}{E}(x,t)\, \delta \underset{\sim}{E}(x,t) \rangle$$

produces the convergence due to the statistical spread in particle orbits. This convergence occurs on the time scale $\tau_2 = (m/k_0 q\, \delta E^{rms})^{1/2}$. τ_2 is the time it takes the dispersion in the orbits of a particle to extend over one typical wavelength $2\pi/k_0$ of the fluctuations given that the orbits start at the same phase space point in all realizations. (In Dupree's notation τ_2 is τ_{tr}.)

Finally, the term

$$\exp i \left[\frac{\tau^5}{6} \underset{\sim}{k} \cdot \sum_{\underset{\sim}{k}'} \underset{\sim}{k}' \langle |\delta E_{\underset{\sim}{k}'}|^2 \rangle (\omega_{\underset{\sim}{k}'} - \underset{\sim}{k}' \cdot \underset{\sim}{v}) \right]$$

produces the convergence associated with the acceleration of the ensemble average orbit on the time scale

$$\tau_1 = \left[\frac{1}{\Delta k^2} \frac{m^2}{q^2 (\delta E^{rms})^2} \frac{1}{\Delta \omega^*} \right]^{1/5} \cong \left(\frac{k_0}{\Delta k} \right)^{2/5} (\tau_2^4 \tau_c)^{1/5}$$

In estimating τ_1 we have taken the typical wave frequency <u>seen by a resonant particle</u> to be of the order $\Delta\omega^*$, the frequency spread observed by the particle. The complicated parametric dependence of τ_1 reflects the fact that the effective fluctuating force $\partial/\partial v \cdot \underset{\approx}{D}$ behaves as

$$\tau^3 \Delta k \, \Delta\omega^* \, \frac{q^2 (\delta E^{rms})^2}{m^2}$$

Since τ_2 depends on $(\delta E^{rms})^{-1/2}$, we expect that for very large fluctuations $\tau_2 << \tau_c$. Unfortunately the weak coupling approximation[11, 13] breaks down for $\tau_2 \leq \tau_c$ and hence Eq. (28) is invalid in this regime.[26]

Equation (28) can, however, be examined for $\tau_c << \tau_2$ and the effects of average orbit distortion and orbit dispersion assessed. As previously mentioned, these orbit effects act to modify the coherence time between the fluctuations and resonant particles. Note that when $\tau_c << \tau_2$ and $\Delta k \sim k_0$, τ_1 is a shorter time interval than τ_2. Thus average orbit distortion is a more important effect than orbit dispersion in determining corrections to $\underset{\approx}{D}$ in this regime. $\Delta k \sim k_0$ implies that the spectrum is broad-band, as would result, for example, from a beam, diffuse in energy, penetrating and giving rise to instability in a background plasma.

Figure 1 illustrates the effect of average orbit distortion on the diffusion coefficient for a one dimensional plasma. The turbulence is here characterized by a spectrum

$$\langle |\delta F_k|^2 \rangle \propto \left[\exp - \frac{(k-k_0)^2}{(\Delta k)^2} + \exp - \frac{(k+k_0)^2}{(\Delta k)^2} \right]$$

wherein $k_0/\Delta k$ has been chosen to be 4. We have also assumed that the frequency of the component waves is independent of wavenumber: $\omega_k = \omega_0$ for $k > 0$ and

$\omega_k = -\omega_0$ for $k < 0$. Plotted against particle velocity (in units of ω_0/k_0) is the non-dimensionalized diffusion coefficient.

Curve I is the usual weak turbulence coefficient Eq. (26). Curve II includes the effect of average orbit distortion Eq. (28); dispersion has been neglected. (Normalization of the two curves is the same, so that they may be compared in amplitude as well as shape.) For the spectrum we have assumed,

$$\sum_{k'} k' |\delta F_{k'}|^2 (\omega_{k'} - k'v) = k_0 \omega_0 W(1 - 1.03\,\tilde{v}) \tag{29}$$

where

$$W = \sum_{k'} |\delta F_{k'}|^2 = \langle \delta F(x,t)\, \delta F(x,t) \rangle$$

is the mean square value of the fluctuating force (per unit mass). In obtaining Curve II we have had to specify a value for the parameter $\epsilon = \frac{1}{6} \frac{W \omega_0^{-2}}{\omega_0^2/k_0^2}$; we have taken ϵ to be 10^{-3}. ϵ measures (in units of the resonant energy $m \frac{\omega_0^2}{k_0^2}$) the energy that a particle would gain in one period ω_0^{-1} of the fluctuations if it were to freely accelerate in the rms electric field. Our choice of ϵ insures that for resonant particles $\tau_c \le \tau_2$.

Note that Curve II has an extremely rapid initial fall-off from its maximum value at $\tilde{v} = 0.97$. (The maxima of the two curves coincide for our choice of parameters.) This behavior is attributable to the factor $(1-1.03\,\tilde{v})$ in Eq. (29). If this factor were equal to unity — or if dispersion were added — the maximum of Curve II would be both lower and more rounded. (We feel that dispersion is important only in the immediate vicinity of the maximum.)

For $\Delta k = \frac{1}{4} k_0$, the bulk of the resonant particles lie between $\tilde{v} = .8$ and $\tilde{v} = 1.3$. For particles in this region the effect of average orbit dispersion is generally to reduce the magnitude of the diffusion from its weak-turbulence value. On the flanks of the resonance, $\tilde{v} < .8$ and $\tilde{v} > 1.3$, the diffusion coefficient of Curve II is larger than that of Curve I: the turbulence is here evidently acting to make non-resonant particles more resonant.

SUMMARY

We have derived the strong turbulence diffusion tensor $\underset{\approx}{D}$. $\underset{\approx}{D}$ includes the effects of fluctuations on both the ensemble averaged plasma orbits and the statistical dispersion in these orbits. By using the plasma trajectory equations we have calculated the contributions of these two effects explicitly.

There are two basic time scales in this problem: τ_c is the autocorrelation time of the fluctuations as seen by a resonant particle when fluctuation effects on its orbit are ignored and τ_2 is the time it takes a resonant particle to diffuse (under the influence of the turbulence) one characteristic wavelength $(2\pi/k_0)$ of the fluctuations. In the weak coupling approximation to strong plasma turbulence we consider $\tau_c \ll \tau_2$.

Effects of average orbit distortion occur on a τ_1 time scale, where $\tau_1 \cong \left(\frac{k_0}{\Delta k}\right)^{2/5} (\tau_2^4 \tau_c)^{1/5}$. Thus when $k_0 \sim \Delta k$, $\tau_1 \lesssim \tau_2$ in the weak coupling limit and the effects of average orbit distortion dominate those of orbit dispersion.

In a sample situation studied in this paper, average orbit distortion reduces the magnitude of the diffusion for the bulk of resonant particles. The diffusion coefficient for particles on the flanks of the resonance is, however, enhanced as the fluctuations here apparently act to make particles more resonant with the turbulence.

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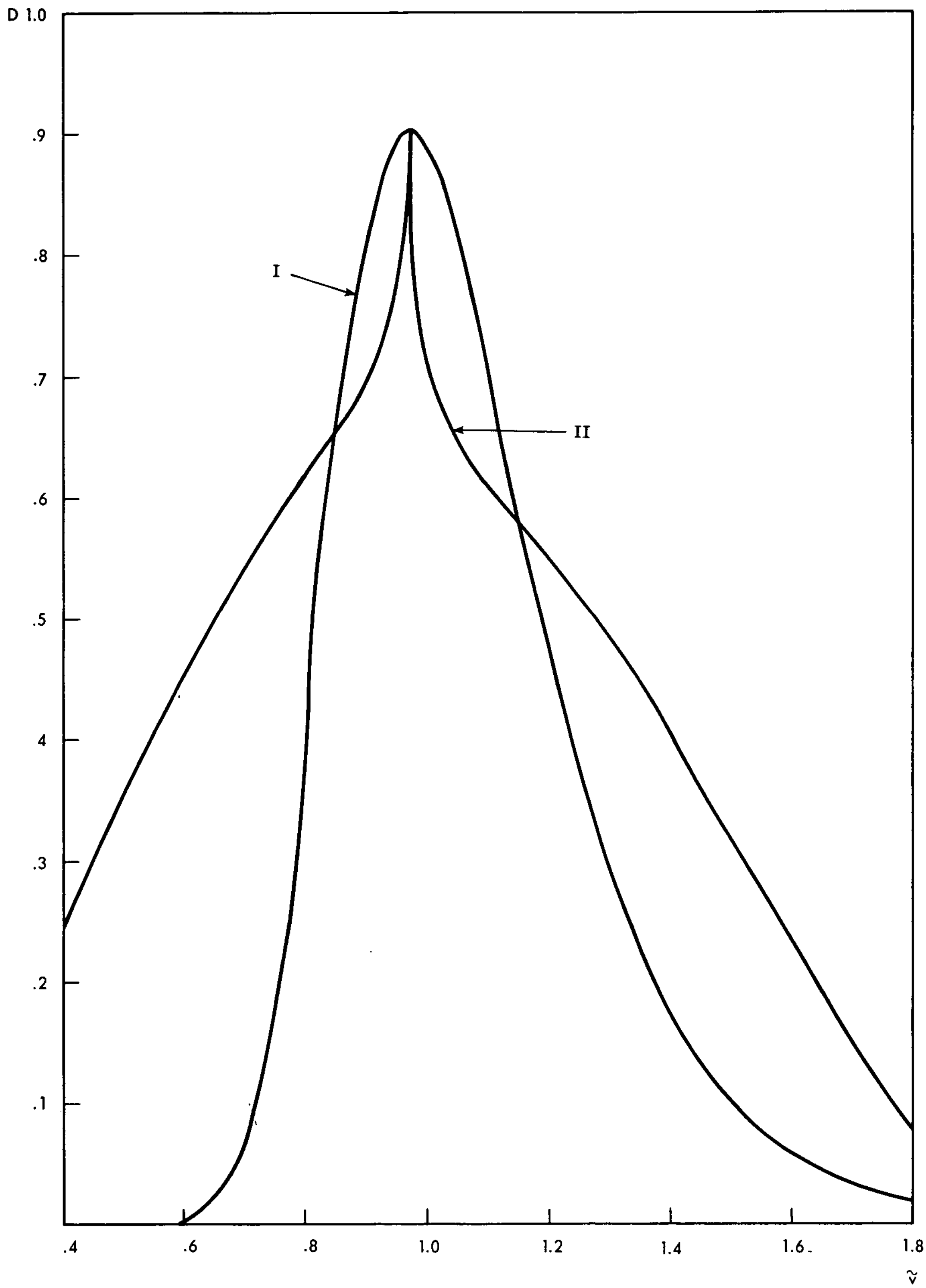
FIGURE CAPTION

Figure 1. The turbulent diffusion coefficient with (Curve II) and without (Curve I) the effects of average orbit distortion.



APPENDIX A

The Correlation T

Expanding the fluctuating force δF in a Fourier Series, we express T as

$$\underset{\sim}{T} = \Big\langle i \sum_{\underset{\sim}{k},\underset{\sim}{k}'} \underset{\sim}{k}' \cdot \hat{k}\, \hat{k}' |\delta \underset{\sim}{F}_k| |\delta \underset{\sim}{F}_{k'}| \exp i \big[\underset{\sim}{k} \cdot \underset{\sim}{x}^*(t-\tau_2) + \omega_{\underset{\sim}{k}} (t-\tau_2) + \phi_{\underset{\sim}{k}} + \underset{\sim}{k}' \cdot \underset{\sim}{x}^*(t-\tau_1) + \omega_{\underset{\sim}{k}'}(t-\tau_1) + \phi_{\underset{\sim}{k}'} \big] \Big\rangle \tag{A.1}$$

As in Eq. (15) the $\hat{k}$'s are unit vectors in the direction of electrostatic wave propagation, $|\delta \mathbf{F}_{\mathbf{k}}|$ is the rms amplitude of the mode with wave vector $\mathbf{k}$, and ϕ_k is the phase of the same wave at $\mathbf{x} = 0$ at time $t = 0$. The ϕ_k's are defined so that the $\hat{k}$'s all have positive projection on an arbitrarily chosen axis. We interpret the ensemble average as an average over the phase angles ϕ_k. If we assume that for each $\mathbf{k}$, ϕ_k varies randomly from one realization to the next, it follows that only the terms $\mathbf{k}' = -\mathbf{k}$ contribute in Eq. (A.1).

$$\underset{\sim}{T} = -i \sum_{\underset{\sim}{k}} \underset{\sim}{k} |\delta \underset{\sim}{F}_k|^2 \exp i \left\{ \underset{\sim}{k} \cdot \left[\underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1) \right] - \omega_{\underset{\sim}{k}} (\tau_2 - \tau_1) \right\} \tag{A.2}$$

Realize now that, by Eq. (16),

$$\begin{aligned} \underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1) &= \Delta \underset{\sim}{x}^*(t-\tau_2) - \Delta \underset{\sim}{x}^*(t-\tau_1) \\ &= -\underset{\sim}{v}\,(\tau_2 - \tau_1) + O(\delta F) \end{aligned} \tag{A.3}$$

so that

$$i\underset{\sim}{k} \exp i\underset{\sim}{k} \cdot \left[\underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1)\right] =$$

$$-\frac{1}{(\tau_2-\tau_1)} \frac{\partial}{\partial \underset{\sim}{v}} \exp i\underset{\sim}{k} \cdot \left[\underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1)\right] + \mathcal{O}(\delta F) \tag{A.4}$$

It thus follows that

$$\underset{\sim}{T} \simeq \frac{1}{(\tau_2-\tau_1)} \frac{\partial}{\partial \underset{\sim}{v}} \sum_{\underset{\sim}{k}} |\delta F_{\underset{\sim}{k}}|^2 \exp i\left\{\underset{\sim}{k} \cdot \left[\underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1)\right] - \omega_{\underset{\sim}{k}} (\tau_2-\tau_1)\right\} \tag{A.5}$$

But in the random phase approximation

$$\left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_2), t-\tau_2\right] \cdot \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_1), t-\tau_1\right]\right\rangle =$$

$$\sum_{\underset{\sim}{k}} |\delta F_{\underset{\sim}{k}}|^2 \exp i\left\{\underset{\sim}{k} \cdot \left[\underset{\sim}{x}^*(t-\tau_2) - \underset{\sim}{x}^*(t-\tau_1)\right] - \omega_{\underset{\sim}{k}} (\tau_2-\tau_1)\right\} \tag{A.6}$$

and

$$\underset{\sim}{T} = \frac{1}{\tau_2-\tau_1} \frac{\partial}{\partial \underset{\sim}{v}} \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_2), t-\tau_2\right] \cdot \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t-\tau_1), t-\tau_1\right]\right\rangle \tag{A.7}$$

$\langle \Delta \mathbf{x}^* (t-\tau) \rangle$ thus has the form Eq. (18).

APPENDIX B

Evaluation of $\underset{\approx}{I}$

To evaluate Eq. 22, let us first introduce the variables

$$s = \tau_1 - \tau_2 \qquad q = \tau_1 + \tau_2 \tag{B.1}$$

By the stationarity hypothesis $<\delta F[x^*(\tau),\tau)\ \delta F[x^*(\tau - \tau_1 + \tau_2), \tau - \tau_1 + \tau_2]>$ depends only on $|s|$. Since $\tau_1 \tau_2 = (q^2 - s^2)/4$, we need hence only integrate Eq. (22) over the region $s > 0$ ($\tau_1 > \tau_2$). Thus

$$\begin{aligned} \underset{\approx}{I} &= \int_0^{\tau} ds \int_s^{2\tau - s} dq\ \frac{q^2 - s^2}{4} \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau), \tau\right] \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau - s), \tau - s\right]\right\rangle \\ &= \int_0^{\tau} ds \left[\frac{2\tau^3}{3} - \tau^2 s + \frac{s^3}{3}\right] \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau), \tau\right] \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau - s), \tau - s\right]\right\rangle \end{aligned} \tag{B.2}$$

But the correlation $<\delta F[x^*(\tau),\tau]\ \delta F[x^*(\tau - s), \tau - s]>$ is strongly peaked at $s = 0$, so that

$$\begin{aligned} \underset{\approx}{I} &\approx \frac{2}{3}\tau^3 \int_0^{\tau} ds \left\langle \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau), \tau\right] \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(\tau - s), \tau - s\right]\right\rangle \\ &= \frac{2}{3}\tau^3 \int_0^{\tau} ds \left\langle \delta \underset{\sim}{F}(\underset{\sim}{x}, t)\ \delta \underset{\sim}{F}\left[\underset{\sim}{x}^*(t - s), t - s\right]\right\rangle \end{aligned} \tag{B.3}$$

the latter form of (B.3) again following from the stationarity assumption.

Compare now the integral in (B.3) with the definition of the diffusion tensor Eq. (12). They are the same and Eq. (23) follows.