NASA TECHNICAL NOTE

NASA TN D-6715

c.1

GENERALIZED REPRESENTATION OF ELECTRIC FIELDS IN INTERACTION GAPS OF KLYSTRONS AND TRAVELING WAVE TUBES WITH AXIAL SYMMETRY

by Henry G. Kosmahl

Lewis Research Center
Cleveland, Ohio 44135

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1972

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TN D-6715		
4. Title and Subtitle GENERALIZED REPRESENTATION OF ELECTRIC FIELDS IN INTERACTION GAPS OF KLYSTRONS AND TRAVELING WAVE TUBES WITH AXIAL SYMMETRY		5. Report Date March 1972 6. Performing Organization Code
7. Author(s) Henry G. Kosmahl		8. Performing Organization Report No. E-6689
9. Performing Organization Name and Address Lewis Research Center National Aeronautics and Space Administration Cleveland, Ohio 44135		10. Work Unit No. 164-21 11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Note 14. Sponsoring Agency Code

15. Supplementary Notes

16. Abstract

Analytic expressions for axial and radial electric fields in axisymmetric interaction gaps of klystrons and coupled cavity traveling wave tubes are derived. Introduction of the "field shape" parameter m results in both limiting cases of the field at the tunnel tips, that is, E equal to a constant and E approaching infinity as well as a continuous transition between these two limits. The transition represents actual, practical fields. This representation may be used to replace the somewhat arbitrary expressions being applied by various researchers to describe the fields.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Fields in gaps Klystrons TWT's	Unclassified - unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	26	$3.00

*For sale by the National Technical Information Service, Springfield, Virginia 22151

GENERALIZED REPRESENTATION OF ELECTRIC FIELDS IN INTERACTION GAPS OF KLYSTRONS AND TRAVELING WAVE TUBES WITH AXIAL SYMMETRY

by Henry G. Kosmahl

Lewis Research Center

SUMMARY

Analytic expressions for axial and radial electric fields in axisymmetric interaction gaps of klystrons and coupled cavity traveling wave tubes are derived. Introduction of the "field shape" parameter m results in both limiting cases of the field at the tunnel tips, that is, uniform field, E equal to a constant and E approaching infinity as well as a continuous transition between these two limits. The transition represents actual, practical fields. This representation may be used to replace the somewhat arbitrary expressions being applied by various researchers to describe the fields.

INTRODUCTION

Accurate computations of electron motion through tunnels of cavities in klystrons and coupled-cavity traveling wave tubes (TWT) require exact solutions for radiofrequency (rf) fringing fields between the tunnel tips. Consider figure 1 showing schematically the interaction gap of length $2l$ between the tunnel tips and the rest of the reso-

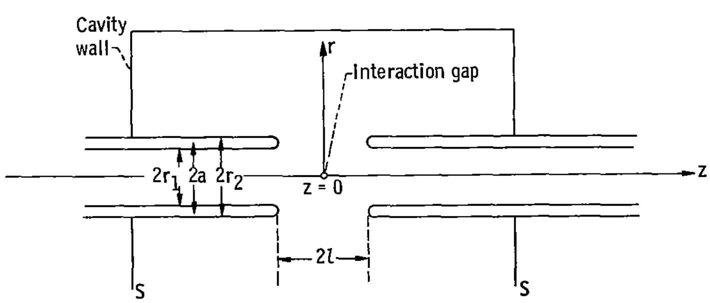


Figure 1. - Double reentrant, axisymmetric klystron tunnel in resonant cavity; z = 0 is at the center of the interaction gap for the klystron.

nator with axially symmetric fields. The tunnel has an inner radius r_1, an outer radius r_2, and an average radius a. Suppose that the shape of the electric field E is known along the path from $-l$ to l at $r = a$, for example,

$$E_z(a, z, t) = e^{i\omega t} E_0 f(z) \tag{1}$$

with E_0 specified at $z = 0$ and $r = a$. Two different cases must be considered: one for klystrons and the other for TWT's. The field of a klystron may be represented by standing waves with zero net propagation of energy (for lossless resonators). Branch (ref. 1) has shown that such a standing wave field may be represented by an infinite sum of equal and oppositely directed space harmonics that add up to zero propagating energy. Since the harmonics are continuous, they may be expressed by an integral. The field at $r = a$ is finite between $-l < z < l$ and zero elsewhere in the metal and on the metallic boundary. It must therefore be represented as a Fourier integral over the space harmonics, that is (ref. 1),

$$f(z) = \int_{-\infty}^{\infty} g(\beta) e^{-i\beta z}\, d\beta \tag{2}$$

Thus, by Fourier inversion

$$g(\beta) = \frac{1}{2\pi} \int_{-l}^{+l} f(z) e^{i\beta z}\, dz \tag{3}$$

where β designates the phase propagation constant. We shall use these expressions later.

Let us consider now the fields of traveling waves. Similar to klystron fields, traveling wave fields consist also of an infinite set of space harmonics with, however, one important difference. Since there is now a traveling wave transporting a net energy in a given direction, the space harmonics are discrete rather than continuous and are thus expressed by an infinite nonzero sum over all harmonics rather than by an integral. From Floquet's theorem for periodic systems we have (ref. 2)

$$E_z(a, z, t) = e^{i\omega t} \hat{E}(a, z) e^{-i\beta_0 z} \tag{4}$$

The field $\hat{E}(a, z) e^{-i\beta_0 z}$, where β_0 is the phase constant of the fundamental wave, may be expanded in a Fourier series of the form

$$E_0 f(z) = \hat{E}(a, z)e^{-i\beta_0 z} = \sum_{n=-\infty}^{\infty} E_n(a)e^{-i[\beta_0+(2\pi n/L)]z} = \sum_n E_n(a)e^{-i\beta_n z} \tag{5}$$

where

$$\beta_n = \beta_0 + \frac{2\pi n}{L}$$

Consider now figure 2 showing schematically a series of rf gaps of length $2l$ spaced by a distance L. Since $\hat{E}e^{-i\beta_0 z}$ is periodic in z with period L, we can determine the E_n's by Fourier analysis of equation (5). (We shall return to eq. (5) later.)

The evaluation of either $g(\beta)$ from equation (3) or $E_n(a)$ from equation (5) requires knowledge of $f(z)$. Closed-form solutions are available from the literature for the case $f(z) = \text{constant}$ in the gap and $f(z) = 0$ elsewhere. However, fields in actual gaps are not uniform but rather have the form corresponding to figure 3(c).

In figure 3 we show schematically the form of fields in gaps for three different

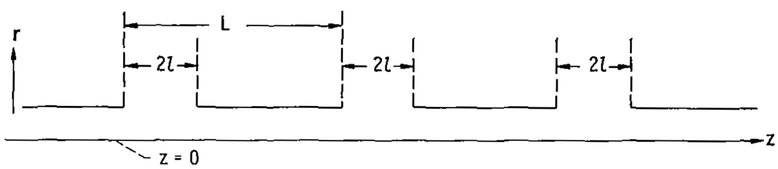
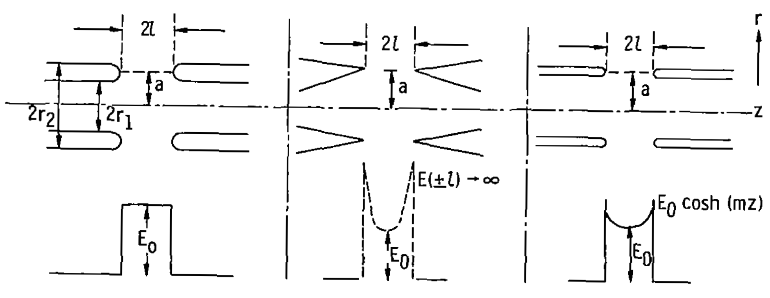


Figure 2. - Series of axisymmetric interaction gaps of length $2l$, spaced by a distance L, such as they appear in coupled cavity TWT; z = 0 is at the beginning of first cell.



(a) Constant field (E_0 = constant at r = a); blunt tips and relatively large wall thicknesses, $r_2 - r_1$, and $2l < 2a$.

(b) Knife-edge field. Field becomes infinite at $z = \pm l$.

(c) Actual field. E_z is neither constant nor infinite. (E_z at r = a can be very well approximated by $E_0 \cosh mz$.)

Figure 3. - Electric fields for three differently shaped tunnel tips.

shapes of tunnel tips. In figure 3(a) the tunnel wall is thick and the tips are blunted. If, furthermore, $2l \ll a$, the electric field has a constant value $E_0 \cdot f(z) = \text{constant} = E_0$ at $r = a$; that is, it rises instantly to its value E_0 at $z = \pm l$ and remains constant in the gap between the tips.

In figure 3(b) the tunnel tips are assumed to have zero thickness (knife edge), and the field becomes infinite at $z = \pm l$; that is, within the gap $-l \leq z \leq +l$,

$$E_z(a, z) = \frac{E_0}{\sqrt{1 - \left(\frac{z}{l}\right)^2}} \tag{6}$$

In practical resonators knife-edge tips are unacceptable for two reasons: First, because of high skin effect losses and, second, because of the danger of melting due to interception.

Actual fields (fig. 3(c)) retain some of the characteristics of the limiting cases (figs. 3(a) and (b)): The field never becomes infinite nor is it strictly uniform. The exact field description depends on the thickness of the tunnel walls, their shape, and the aspect ratio l/a in each individual case. Thus, it cannot be expressed by a single function (except an even power polynomial of infinite order). However, the general features of this function are known: It must be an even function of z, and it must approach unity (uniform field) and infinity for the two limiting cases, respectively. In addition, we wish it to be a simple and integrable function. We choose therefore the function

$$f(z, a) = \cosh(mz) \tag{7}$$

where the factor m is to be chosen such that at $z = \pm l$ the field $E_0 \cdot \cosh(mz)$ acquires the measured, actual amplitude $E_z(a, \pm l)$. It may be seen that for $m = 0$ we obtain the case of $E_0 \cdot f(z, a) = \text{constant} = E_0$ and for large m, arbitrarily large values $E_z(a, \pm l)$ may be achieved. Thus, the representation introduced in equation (7) furnishes the much needed generality for the treatment of the rf field in gaps, with both limiting cases $E_z(a, \pm l) = E_0$ or E large, following simply from a suitable choice of m.

The case $m \to \infty$ is not represented correctly by expression (7) because

$$\lim_{m \to \infty} \cosh(mz) \approx \lim_{m \to \infty} \frac{e^{mz}}{2}$$

while

$$\lim_{z \to l} \frac{1}{\sqrt{1 - \left(\frac{z}{l}\right)^2}} = \lim_{\delta \to 0} \frac{1}{\sqrt{2\delta}}$$

with $\delta = 1 - z/l$, has a pole different from that of $e^{mz}/2$. However, knife-edge tunnel tips are not realistic, and, for the finite fields E, numbers very accurately representing actual fields of figure 3(c) may be obtained. This is made evident by comparing actually measured fields with those obtained from equation (7) with the proper choice of m. In most cases $1.25 \lesssim \cosh ml \lesssim 3$.

It was the purpose of this study to derive generally valid analytical expressions for the fields in actual klystron and TWT gaps, thus replacing the somewhat arbitrary, empirical, or semianalytic expressions being used presently.

SYMBOLS

a	average tunnel radius
c	speed of light
D	denominator of integrand
E	electric field
E_0	amplitude of electric field at gap center and $r = a$
g	Fourier transform
I_0, I_1	modified Bessel functions of zero and first order, respectively
J_0, J_1	Bessel functions of zero and first order, respectively
k	free wave number equal to ω/c
L	length of period for TWT circuits
l	half length of gap
m	field shaping parameter
N	numerator of integrand
n	integer number of root of Bessel functions $J_0(\lambda_n) = 0$ where $n = 1, 2, 3, \ldots$; also, integer summation index for propagation constants β_n and γ_n for TWT equations, $-\infty < n < \infty$

p_n	$\sqrt{\lambda_n^2 - k^2a^2}$
r	radius or radial coordinate
t	time
V	voltage
α	a/l
β	axial propagation constant
γ	radial propagation constant, equal to $\sqrt{\beta^2 - k^2}$
δ	small, real number
θ	transit time parameter equal to βl
λ_n	n^{th} root of Bessel function, $J_0(\lambda_n) = 0$
ν	summation index
ξ	z/l
ρ	r/l
ω	$2\pi \times$ frequency

Subscripts:

n, ν	summation indexes
l, u	lower and upper half complex plane, respectively

Superscripts:

'	differentiation with respect to argument, $D'(X) = dD(X)/dX$
^	complex amplitude of electric field, $E(a, z)$

DERIVATION OF FIELD EQUATIONS FOR KLYSTRON GAPS

The wave equation yields the following expressions for the axial and radial electric fields with axial symmetry in klystrons for $r \leq a$.

$$E_z(r, z, t) = E_0 e^{i\omega t} \int_{-\infty}^{\infty} \frac{I_0(\gamma r)}{I_0(\gamma a)} g(\beta) e^{-i\beta z} \, d\beta \tag{7}$$

$$E_r(r, z, t) = iE_0 e^{i\omega t} \int_{-\infty}^{\infty} \frac{\beta}{\gamma} \frac{I_1(\gamma r)}{I_0(\gamma a)} g(\beta) e^{-i\beta z} \, d\beta \tag{8}$$

with $\gamma^2 = \beta^2 - (\omega^2/c^2)$ and $g(\beta)$ given by equation (3).

Branch (unpublished General Electric Company report) has evaluated the integrals (eqs. (7) and (8)) by contour integration for the case $f(z)$ = constant in the gap and zero elsewhere:

$$g(\beta) = \frac{1}{2\pi} \int_{-l}^{l} \text{constant} \cdot e^{i\beta z} \, dz = \frac{\text{constant}}{2\pi} \frac{e^{i\beta l} - e^{-i\beta l}}{i\beta} \tag{9}$$

The same method can be applied conveniently to obtain analytical expressions for the more general case $f(z) = \cosh(mz)$ in the gap and zero elsewhere. Thus,

$$g(\beta) = \frac{1}{2\pi} \int_{-l}^{l} \frac{e^{mz} + e^{-mz}}{2} e^{i\beta z} \, dz = \frac{1}{4\pi} \left[\frac{e^{l(i\beta+m)} - e^{-l(i\beta+m)}}{i\beta + m} + \frac{e^{l(i\beta-m)} - e^{-l(i\beta-m)}}{i\beta - m} \right] \tag{10}$$

Axial Electric Fields in Klystrons

Expression (10) has four terms. We evaluate first equation (7). Substitution of equation (10) into equation (7) yields

$$E_z(r, z, t) = E_0 e^{i\omega t} \left\{ \frac{e^{ml}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0(\gamma r)}{I_0(\gamma a)} \frac{e^{i\beta(l-z)}}{\beta l - iml} \, d(\beta l) - \frac{e^{-ml}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0(\gamma r)}{I_0(\gamma a)} \frac{e^{-i\beta(l+z)}}{\beta l - iml} \, d(\beta l) \right.$$

$$\left. + \frac{e^{-ml}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0(\gamma r)}{I_0(\gamma a)} \frac{e^{i\beta(l-z)}}{\beta l + iml} \, d(\beta l) - \frac{e^{ml}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0(\gamma r)}{I_0(\gamma a)} \frac{e^{-i\beta(l+z)}}{\beta l + iml} \, d(\beta l) \right\} \tag{11}$$

For compactness let $ml = \overline{m}$, and adapt the notations of Branch (ref. 3):

$$\beta l = \theta \qquad \xi = \frac{z}{l} \qquad \rho = \frac{r}{l} \qquad \alpha = \frac{a}{l} \qquad \gamma r = \rho\sqrt{\theta^2 - \theta_0^2}$$

$$\gamma a = \alpha\sqrt{\theta^2 - \theta_0^2} \qquad \theta_0 = \frac{\omega}{c}\, l = kl$$

Denoting the four integrals by $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, and $\mathcal{J}_4$, we obtain

$$\mathcal{J}_1 = \frac{e^{\overline{m}}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{i\theta(1-\xi)}}{\theta - i\overline{m}}\, d\theta \tag{12a}$$

$$\mathcal{J}_2 = -\frac{e^{-\overline{m}}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{-i\theta(1+\xi)}}{\theta - i\overline{m}}\, d\theta \tag{12b}$$

$$\mathcal{J}_3 = \frac{e^{-\overline{m}}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{i\theta(1-\xi)}}{\theta + i\overline{m}}\, d\theta \tag{12c}$$

$$\mathcal{J}_4 = -\frac{e^{\overline{m}}}{4\pi i} \int_{-\infty}^{\infty} \frac{I_0\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{-i\theta(1+\xi)}}{\theta + i\overline{m}}\, d\theta \tag{12d}$$

The quadrature may be carried out exactly by the method of residues in the complex plane. To do this we examine the denominator for poles. The poles occur at $\theta_m = \pm i\overline{m}$ and also at $\theta = \pm\theta_n$, all on the imaginary axis. The $\pm\theta_n$ are the roots of $I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)$. Following Branch, we call λ_n the real roots of the equation $J_0(\lambda_n) = 0 = I_0(i\lambda_n)$:

$$\left.\begin{aligned} \alpha\sqrt{\theta_n^2 - \theta_0^2} &= i\lambda_n \\ \lambda_n &= \alpha\sqrt{\theta_0^2 - \theta_n^2} \\ \theta_n &= \pm i\sqrt{\frac{\lambda_n^2}{\alpha^2} - \theta_0^2} \end{aligned}\right\} \tag{13}$$

We also introduce a new parameter $p_n = +\sqrt{\lambda_n^2 - k^2a^2}$, which is a real positive number. Therefore, θ_n will be $+i\frac{l}{a}p_n$ and $-i\frac{l}{a}p_n$ in the upper and lower complex half plane, respectively. It can be seen that $\lambda_n/\alpha > \theta_0$ or $ka < \lambda_n$. For the first root $ka < \lambda_1 = 2.408 \cdots$. Since in microwave tubes $ka = 0.1$ to 0.5 this condition is well satisfied. Therefore,

$$I_0\left(\alpha\sqrt{\theta_n^2 - \theta_0^2}\right) = I_0(i\lambda_n) \equiv J_0(\lambda_n) = 0$$

The integration path and the roots are shown in figure 4. The path extends along the real axis from $-\infty$ to $+\infty$, and it closes along an arc of infinite radius in either the

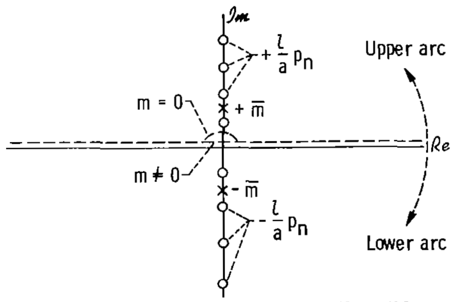


Figure 4. - Integration path for equations (12).

upper or lower half plane. In general, $\overline{m} = ml \neq \theta_n$ and all poles are single and of first order. The path of integration is to be taken such that the integral vanishes along the arc. Then,

$$\mathscr{I} = 2\pi i\sum \text{Residues} = 2\pi i\sum\frac{N(\theta)}{D'(\theta)} \tag{14}$$

where $N(\theta)$, $D(\theta)$, and $D'(\theta)$ designate the numerator, denominator, and its derivative, respectively.

For $\mathcal{J}_1$ and $\mathcal{J}_2$ the denominator and its derivative are

$$D_{1,2}(\theta) = (\theta - i\overline{m})J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) \tag{15}$$

$$D'_{1,2}(\theta) = J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) + \frac{\alpha^2\theta(\theta - i\overline{m})}{\alpha\sqrt{\theta_0^2 - \theta^2}} J_1\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) \tag{16}$$

The derivative $D'_{1,2}(\theta)$ assumes now the following: At $\theta_m = i\overline{m}$

$$D'_{1,2}(i\overline{m}) = J_0\left(\alpha\sqrt{\theta_0^2 + \overline{m}^2}\right) \tag{17}$$

and at $\theta = \theta_n$

$$D'_{1,2}(\theta_n) = \alpha^2(\theta_n - i\overline{m})\frac{\theta_n J_1(\lambda_n)}{\lambda_n} \tag{18}$$

Similarly, we get for $D_{3,4}(\theta)$ and $D'_{3,4}(\theta)$ the following values: At $\theta_m = i\overline{m}$

$$D'_{3,4}(-i\overline{m}) = J_0\left(\alpha\sqrt{\theta_0^2 + \overline{m}^2}\right) \tag{19}$$

and at $\theta = \theta_n$

$$D'_{3,4}(\theta_n) = \alpha^2\theta_n(\theta_n + i\overline{m})\frac{J_1(\lambda_n)}{\lambda_n} \tag{20}$$

The numerators $N(\theta)$ for $\mathcal{J}_1$ and $\mathcal{J}_3$ are

$$N_{1,3}(\theta) = J_0\left(\rho\sqrt{\theta_0^2 - \theta^2}\right)e^{i\theta(1-\xi)} \tag{21}$$

And, similarly, for $\mathcal{J}_2$ and $\mathcal{J}_4$

$$N_{2,4}(\theta) = J_0\left(\rho\sqrt{\theta_0^2 - \theta^2}\right)e^{-i\theta(1+\xi)} \tag{22}$$

Convergence is obtained when the real parts of the exponents $i\theta(1 - \xi)$ and $-i\theta(1 + \xi)$ go to zero as θ goes to infinity. This occurs when $\mathcal{J}_1$ and $\mathcal{J}_3$ (corresponding to $N_{1,3}$) are integrated over the upper arc if $1 - \xi > 0$ ($\xi < 1$) and over the lower arc if $1 - \xi < 0$ ($\xi > 1$), and when $\mathcal{J}_2$ and $\mathcal{J}_4$ (corresponding to $N_{2,4}$) are integrated over the upper arc if $1 + \xi < 0$ ($\xi < -1$) and over the lower arc if $1 + \xi > 0$ ($\xi > -1$).

Application of the theorem of residues (eq. (14)) and summation over the infinite number of roots λ_n, beginning with λ_1 as the first root, give the following results for the integrals $\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$, and $\mathcal{J}_4$:

$$\mathcal{J}_{1u} = \frac{e^{\overline{m}}}{2}\left[\frac{J_0\left(\frac{r}{l}\sqrt{k^2l^2 + \overline{m}^2}\right)}{J_0\left(\frac{a}{l}\sqrt{k^2l^2 + \overline{m}^2}\right)} e^{-m(l-z)} + \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{-(p_n/a)(l-z)}}{-p_n(p_n - am)J_1(\lambda_n)}\right] \tag{23}$$

for $z < l$,

$$\mathcal{J}_{1l} = -\frac{e^{\overline{m}}}{2} \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{(p_n/a)(l-z)}}{-p_n(p_n + am)J_1(\lambda_n)} \tag{24}$$

for $z > l$,

$$\mathcal{J}_{2u} = -\frac{e^{-\overline{m}}}{2}\left[\frac{J_0\left(\frac{r}{l}\sqrt{k^2l^2 + \overline{m}^2}\right)}{J_0\left(\frac{a}{l}\sqrt{k^2l^2 + \overline{m}^2}\right)} e^{m(l+z)} + \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{(p_n/a)(l+z)}}{-p_n(p_n - am)J_1(\lambda_n)}\right] \tag{25}$$

for $z < -l$,

$$\mathcal{J}_{2l} = + \frac{e^{-\overline{m}}}{2} \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{-(p_n/a)(l+z)}}{-p_n(p_n + am)J_1(\lambda_n)} \tag{26}$$

for $z > -l$,

$$\mathcal{J}_{3u} = \frac{e^{-\overline{m}}}{2} \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{-(p_n/a)(l-z)}}{-p_n(p_n + am)J_1(\lambda_n)} \tag{27}$$

for $z < l$,

$$\mathcal{J}_{3l} = - \frac{e^{-\overline{m}}}{2} \left[\frac{J_0\left(\frac{r}{l}\sqrt{k^2 l^2 + \overline{m}^2}\right)}{J_0\left(\frac{a}{l}\sqrt{k^2 l^2 + \overline{m}^2}\right)} e^{m(l-z)} + \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{(p_n/a)(l-z)}}{-p_n(p_n - am)J_1(\lambda_n)} \right] \tag{28}$$

for $z > l$,

$$\mathcal{J}_{4u} = - \frac{e^{\overline{m}}}{2} \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{(p_n/a)(l+z)}}{-p_n(p_n + am)J_1(\lambda_n)} \tag{29}$$

for $z < -l$, and

$$\mathscr{I}_{4l} = \frac{e^{\overline{m}}}{2}\left[\frac{J_0\left(\frac{r}{l}\sqrt{k^2l^2+\overline{m}^2}\right)}{J_0\left(\frac{a}{l}\sqrt{k^2l^2+\overline{m}^2}\right)} e^{-m(l+z)} + \sum_{n=1}^{\infty} \frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right) e^{-(p_n/a)(l+z)}}{-p_n(p_n - am)J_1(\lambda_n)}\right] \tag{30}$$

for $z > -l$.

The two different values for each of the integrals $\mathscr{I}_1$ to $\mathscr{I}_4$ correspond to closing the contour either in the upper or lower half plane, each value being valid in the interval indicated in equations (23) to (30).

Now, the field within the gap may be computed by adding all integrals whose validity falls into the region $-l < z < l$. We see that the appropriate sum consists of $\mathscr{I}_{1u} + \mathscr{I}_{2l} + \mathscr{I}_{3u} + \mathscr{I}_{4l}$. The result of summing up these four terms is

$$E_z(r,z,t) = E_0 e^{i\omega t}\left[\cosh(mz)\frac{J_0\left(r\sqrt{k^2+m^2}\right)}{J_0\left(a\sqrt{k^2+m^2}\right)} - \sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)}\right.$$

$$\left.\times\left(\frac{e^{ml}}{p_n - am} + \frac{e^{-ml}}{p_n + am}\right) e^{-p_n(l/a)} \cosh\left(\frac{p_n z}{a}\right)\right] \tag{31}$$

Outside the gap for $z > l$, the summation consists of $\mathscr{I}_{1l} + \mathscr{I}_{2l} + \mathscr{I}_{3l} + \mathscr{I}_{4l}$ with the results

$$E_z(r,z,t) = E_0 e^{i\omega t}\left\{\frac{e^{ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)}\, e^{-p_n z/a}\left(\frac{e^{p_n l/a}}{p_n + am} - \frac{e^{-p_n l/a}}{p_n - am}\right)\right.$$

$$\left. + \frac{e^{-ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)}\, e^{-p_n z/a}\left(\frac{e^{p_n l/a}}{p_n - am} - \frac{e^{-p_n l/a}}{p_n + am}\right)\right\} \quad (32)$$

Outside the gap but for $z < -l$, the summation consists of $\mathscr{I}_{1u} + \mathscr{I}_{2u} + \mathscr{I}_{3u} + \mathscr{I}_{4u}$:

$$E_z(r,z,t) = E_0 e^{i\omega t}\left\{\frac{e^{ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)}\, e^{p_n z/a}\left(\frac{e^{p_n l/a}}{p_n + am} - \frac{e^{-p_n l/a}}{p_n - am}\right)\right.$$

$$\left. + \frac{e^{-ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)}\, e^{p_n z/a}\left(\frac{e^{p_n l/a}}{p_n - am} - \frac{e^{-p_n l/a}}{p_n + am}\right)\right\} \quad (33)$$

Equations (32) and (33) can be combined for $|z| > l$ into one expression:

$$E_z(r,z,t) = E_0 e^{i\omega t}\left\{\frac{e^{ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)} e^{-(p_n/a)|z|}\left(\frac{e^{p_n l/a}}{p_n + am} - \frac{e^{-p_n l/a}}{p_n - am}\right)\right.$$

$$\left. + \frac{e^{-ml}}{2}\sum_{n=1}^{\infty}\frac{\lambda_n J_0\left(\frac{r}{a}\lambda_n\right)}{p_n J_1(\lambda_n)} e^{-(p_n/a)|z|}\left(\frac{e^{p_n l/a}}{p_n - am} - \frac{e^{-p_n l/a}}{p_n + am}\right)\right\} \tag{34}$$

The case $m = 0$ must be discussed separately. The integration path bypasses the origin on a semiarc as indicated in figure 4. The denominators of integrals (12a) to (12d) have a pole at $\theta = 0$ with nonzero residues. However, the signs in front of equations (12a) and (12b) as well as those in front of equations (12c) and (12d) are opposite, and the contributions due to the residues as well as of the integrals along the semiarc around the origin cancel each other.

Radial Electric Field in Klystrons

The expression for the radial field E_r is

$$E_r(r,z,t) = iE_0 e^{i\omega t}\int_{-\infty}^{\infty}\frac{\beta}{\gamma}\frac{I_1(\gamma r)}{I_0(\gamma a)} g(\beta) e^{-i\beta z}\, d\beta \tag{35}$$

Introducing the expression for $g(\beta)$ from equation (10) into equation (35) results in the following integrals:

$$E_r(r,z,t) = \frac{E_0 e^{i\omega t}}{4\pi}\left\{ e^{\bar{m}} \int_{-\infty}^{\infty} \frac{\theta}{\sqrt{\theta^2 - \theta_0^2}} \frac{I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{i\theta(1-\xi)}}{\theta - i\bar{m}}\, d\theta \right.$$

$$- e^{-\bar{m}} \int_{-\infty}^{\infty} \frac{\theta}{\sqrt{\theta^2 - \theta_0^2}} \frac{I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{-i\theta(1+\xi)}}{\theta - i\bar{m}}\, d\theta$$

$$+ e^{-\bar{m}} \int_{-\infty}^{\infty} \frac{\theta}{\sqrt{\theta^2 - \theta_0^2}} \frac{I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{i\theta(1-\xi)}}{\theta + i\bar{m}}\, d\theta$$

$$\left. - e^{\bar{m}} \int_{-\infty}^{\infty} \frac{\theta}{\sqrt{\theta^2 - \theta_0^2}} \frac{I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)}{I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right)} \frac{e^{-i\theta(1+\xi)}}{\theta + i\bar{m}}\, d\theta \right\} \tag{36}$$

or, for simplicity,

$$E_r(r,z,t) = \frac{E_0 e^{i\omega t}}{4\pi}\left(e^{\bar{m}}\mathcal{J}_1 - e^{-\bar{m}}\mathcal{J}_2 + e^{-\bar{m}}\mathcal{J}_3 - e^{\bar{m}}\mathcal{J}_4\right) \tag{37}$$

where integrals $\mathcal{J}_1$ to $\mathcal{J}_4$ stand for the four integrals in equation (36). The denominators $D(\theta)$ of integrals $\mathcal{J}_1$ to $\mathcal{J}_4$ are

$$D(\theta) = (\theta \mp i\bar{m}) \sqrt{\theta^2 - \theta_0^2}\, I_0\left(\alpha\sqrt{\theta^2 - \theta_0^2}\right) \tag{38}$$

Therefore, $\mathcal{J}_1$ to $\mathcal{J}_4$ have simple poles at

$$\theta_m = \pm i\bar{m}$$

on the imaginary axis, and at

$$\theta_n = \pm i \frac{l}{a} \sqrt{\lambda_n^2 - k^2 a^2} = \pm i \frac{l}{a} p_n$$

also on the imaginary axis.

Note that $\theta = \pm\theta_0$ does not produce poles due to cancellation, which results from the argument of $\mathscr{I}_1$ in the numerator and $\sqrt{\theta^2 - \theta_0^2}$ in the denominator. The contour of integration is shown in figure 5 together with the poles. The path of integration is similar to that shown in figure 4 with the exception of the case $\overline{m} = 0$.

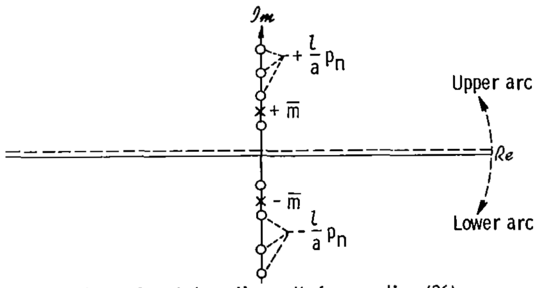


Figure 5. - Integration path for equation (36).

Examination of the numerator of the integrands yields the following conditions for convergence: Integrate

$\mathscr{I}_1$ and $\mathscr{I}_3$ along upper arc if $1 - \xi > 0$ $(\xi < 1)$
$\mathscr{I}_1$ and $\mathscr{I}_3$ along lower arc if $1 - \xi < 0$ $(\xi > 1)$
$\mathscr{I}_2$ and $\mathscr{I}_4$ along upper arc if $1 + \xi < 0$ $(\xi < -1)$
$\mathscr{I}_2$ and $\mathscr{I}_4$ along lower arc if $1 + \xi > 0$ $(\xi > -1)$.

The denominator of $\mathscr{I}_1$ and $\mathscr{I}_2$, $D_{1,2}(\theta)$, is

$$D_{1,2}(\theta) = (\theta - i\overline{m}) \sqrt{\theta^2 - \theta_0^2}\, I_0\left(\alpha \sqrt{\theta^2 - \theta_0^2}\right) \tag{39}$$

And its derivative $D'_{1,2}(\theta)$ is

$$D'_{1,2}(\theta) = \sqrt{\theta^2 - \theta_0^2}\, J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) + \frac{\theta(\theta - i\overline{m})}{\sqrt{\theta^2 - \theta_0^2}}\, J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right)$$

$$+ \alpha\theta(\theta - i\overline{m})iJ_1\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) \tag{40}$$

The numerator $N_{1,3}(\theta)$ in $\mathscr{I}_1$ is

$$N_{1,3}(\theta) = \theta I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)e^{i\theta(1-\xi)} = -i\theta J_1\left(\rho\sqrt{\theta_0^2 - \theta^2}\right)e^{i\theta(1-\xi)}$$

The examination of the quotient $N_{1,3}(\theta)/D'_{1,2}(\theta)$ yields the following residues:
At $\theta_m = i\overline{m}$

$$\frac{N_{1,3}(\theta_m)}{D'_{1,2}(\theta_m)} = \frac{\overline{m}J_1\left(\rho\sqrt{\theta_0^2 + \overline{m}^2}\right)}{i\sqrt{\theta_0^2 + \overline{m}^2}\, J_0\left(\alpha\sqrt{\theta_0^2 + \overline{m}^2}\right)}\, e^{-\overline{m}(1-\xi)} \tag{41}$$

at $\theta = \theta_n = i\frac{l}{a}p_n$

$$\frac{N_{1,3}(\theta_n)}{D'_{1,2}(\theta_n)} = -\frac{J_1\left(\frac{r}{a}\lambda_n\right)e^{-(l/a)p_n(1-\xi)}}{i(p_n - ma)J_1(\lambda_n)} \tag{42}$$

and at $\theta = \theta_n = -i\frac{l}{a}p_n$

$$\frac{N_{1,3}(\theta_n)}{D'_{1,2}(\theta_n)} = \frac{J_1\left(\frac{r}{a}\lambda_n\right)e^{(l/a)p_n(1-\xi)}}{i(p_n + ma)J_1(\lambda_n)} \tag{43}$$

Similarly, the numerator $N_{2,4}(\theta)$ in $\mathscr{I}_2$ is

$$N_{2,4}(\theta) = \theta I_1\left(\rho\sqrt{\theta^2 - \theta_0^2}\right)e^{-i\theta(1+\xi)} = -i\theta J_1\left(\rho\sqrt{\theta_0^2 - \theta^2}\right)e^{-i\theta(1+\xi)}$$

And the residues of $\mathscr{I}_2$ are as follows:

At $\theta = i\overline{m}$

$$\frac{N_{2,4}(\theta_m)}{D'_{1,2}(\theta_m)} = \frac{\overline{m} J_1\left(\rho\sqrt{\theta_0^2 + \overline{m}^2}\right) e^{\overline{m}(1+\xi)}}{i\sqrt{\theta_0^2 + \overline{m}^2}\; J_0\left(\alpha\sqrt{\theta_0^2 + \overline{m}^2}\right)} \tag{44}$$

at $\theta_n = i\frac{l}{a} p_n$

$$\frac{N_{2,4}(\theta_n)}{D'_{1,2}(\theta_n)} = -\frac{J_1\left(\frac{r}{a}\lambda_n\right) e^{(l/a)p_n(1+\xi)}}{i(p_n - ma)J_1(\lambda_n)} \tag{45}$$

and at $\theta_n = -i\frac{l}{a} p_n$

$$\frac{N_{2,4}(\theta_n)}{D'_{1,2}(\theta_n)} = \frac{J_1\left(\frac{r}{a}\lambda_n\right) e^{-(l/a)p_n(1+\xi)}}{i(p_n + ma)J_1(\lambda_n)} \tag{46}$$

The derivative of the denominator of $\mathscr{I}_3$ and $\mathscr{I}_4$, $D'_{3,4}(\theta)$ is

$$D'_{3,4}(\theta) = \sqrt{\theta^2 - \theta_0^2}\; J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) + \frac{\theta(\theta + i\overline{m})}{\sqrt{\theta^2 - \theta_0^2}}\; J_0\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right)$$

$$+\; i\alpha\theta(\theta + i\overline{m})J_1\left(\alpha\sqrt{\theta_0^2 - \theta^2}\right) \tag{47}$$

Examination of the quotient $N_{1,3}(\theta)/D'_{3,4}(\theta)$ results in the following residues of $\mathscr{I}_3$:

At $\theta_m = -i\overline{m}$

$$\frac{N_{1,3}(\theta_m)}{D'_{3,4}(\theta_m)} = \frac{-\overline{m} J_1\left(r\sqrt{k^2 + m^2}\right) e^{\overline{m}(1-\xi)}}{i\sqrt{\theta_0^2 + \overline{m}^2}\; J_0\left(a\sqrt{k^2 + m^2}\right)} \tag{48}$$

at $\theta_n = i \frac{l}{a} p_n$

$$\frac{N_{1,3}(\theta_n)}{D'_{3,4}(\theta_n)} = - \frac{J_1\left(\frac{r}{a} \lambda_n\right) e^{-(l/a)p_n(1-\xi)}}{i(p_n + ma)J_1(\lambda_n)} \tag{49}$$

and at $\theta_n = -i \frac{l}{a} p_n$

$$\frac{N_{1,3}(\theta_n)}{D'_{3,4}(\theta_n)} = \frac{J_1\left(\frac{r}{a} \lambda_n\right) e^{(l/a)p_n(1-\xi)}}{i(p_n - ma)J_1(\lambda_n)} \tag{50}$$

The residues of $\mathscr{I}_4$ are as follows:

At $\theta_m = -i\overline{m}$

$$\frac{N_{2,4}(\theta_m)}{D'_{3,4}(\theta_m)} = - \frac{\overline{m} J_1\left(r\sqrt{k^2 + m^2}\right) e^{-\overline{m}(1+\xi)}}{i\sqrt{\theta_0^2 + \overline{m}^2}\, J_0\left(a\sqrt{k^2 + m^2}\right)} \tag{51}$$

at $\theta_n = i \frac{l}{a} p_n$

$$\frac{N_{2,4}(\theta_n)}{D'_{3,4}(\theta_n)} = - \frac{J_1\left(\frac{r}{a} \lambda_n\right) e^{(l/a)p_n(1+\xi)}}{i(p_n + ma)J_1(\lambda_n)} \tag{52}$$

and at $\theta_n = -i \frac{l}{a} p_n$

$$\frac{N_{2,4}(\theta_n)}{D'_{3,4}(\theta_n)} = \frac{J_1\left(\frac{r}{a} \lambda_n\right) e^{-(l/a)p_n(1+\xi)}}{i(p_n - ma)J_1(\lambda_n)} \tag{53}$$

From equations (35) to (53) the following expressions may be derived for the resulting field $E_r = 2\pi i \sum_{\text{all}}$ residues: In the gap $-l < z < l$,

$$E_r(r,z,t) = \frac{E_0 e^{i\omega t}}{4\pi}\left\{e^{ml}\mathcal{J}_{1u} + e^{-ml}\mathcal{J}_{3u} - e^{-ml}\mathcal{J}_{2l} - e^{ml}\mathcal{J}_{4l}\right\}$$

$$= -E_0 e^{i\omega t}\left\{\sum_{n=1}^{\infty} \frac{J_1\left(\frac{r}{a}\lambda_n\right)}{J_1(\lambda_n)} e^{-p_n l/a}\left(\frac{e^{ml}}{p_n - ma} + \frac{e^{-ml}}{p_n + ma}\right)\sinh\left(\frac{p_n z}{a}\right)\right.$$

$$\left. - m\sinh(mz)\frac{J_1\left(r\sqrt{k^2+m^2}\right)}{\sqrt{k^2+m^2}\, J_0\left(a\sqrt{k^2+m^2}\right)}\right\} \tag{54}$$

Outside the gap $z > +l$,

$$E_r(r,z,t) = -E_0 e^{i\omega t}\sum_{n=1}^{\infty}\frac{J_1\left(\frac{r}{a}\lambda_n\right)e^{-p_n z/a}}{J_1(\lambda_n)}\left[\frac{\sinh\left(ml + \frac{l}{a}p_n\right)}{p_n + ma} - \frac{\sinh\left(ml - \frac{l}{a}p_n\right)}{p_n - ma}\right] \tag{55}$$

Outside the gap $z < -l$,

$$E_r(r,z,t) = -E_0 e^{i\omega t} \sum_{n=1}^{\infty} \frac{J_1\left(\frac{r}{a}\lambda_n\right) e^{-p_n|z/a|}}{J_1(\lambda_n)} \left[\frac{\sinh\left(ml - \frac{l}{a} p_n\right)}{p_n - ma} - \frac{\sinh\left(ml + \frac{l}{a} p_n\right)}{p_n + ma} \right] \tag{56}$$

Note that $E_r(r,-z,t) = -E_r(r,z,t)$, that $E_r(0,z,t) = 0$, and that $E_r(r,0,t) = 0$. Thus E_r is an odd function of z relative to the center of the gap, and E_z is an even function of z. When m is set to zero, the expressions derived here for $E_z(r,z,t)$ and $E_r(r,z,t)$ become identical with those obtained by Branch in his unpublished report.

DERIVATION OF FIELD EQUATIONS FOR COUPLED CAVITY TWT GAPS

We return now to equation (5) and evaluate from it the Fourier coefficients E_n. Let, as before in the klystron case, $E_0 f(z)$ at radius a be represented by $E_0 \cosh m(z - l)$. Multiply both sides of equation (5) by $e^{(i2\pi z\nu)/L}$, substitute $E_0 f(z)$ for $\hat{E}(a,z)e^{-i\beta_0 z}$, and integrate over the period from $z = 0$ to $z = L$. Because of the orthogonality of the function, all terms are zero except when $\nu = n$, and we obtain (with $\beta_n = \beta_0 + 2\pi n/L$)

$$E_n = \frac{E_0}{L} \int_0^{2l} \cosh m(z - l) e^{i\beta_n z} \, dz$$

$$= \frac{E_0}{L} \frac{\left[m \sinh(ml) - i\beta_n \cosh(ml)\right] e^{2i\beta_n l} + m \sinh(ml) + i\beta_n \cosh(ml)}{m^2 + \beta_n^2} \tag{57}$$

Note that $f(z) = 0$ for $2l \leq z \leq L$. For $m \to 0$, equation (57) becomes

$$\lim_{m \to 0} E_n = \frac{E_0}{L} 2l e^{i\beta_n l} \frac{\sin(\beta_n l)}{\beta_n l} \tag{58}$$

For uniform fields, $E_0 = V/2l$, therefore,

$$\lim_{m \to 0} E_n = \frac{V}{L} e^{i\beta_n l} \frac{\sin \beta_n l}{\beta_n l}$$

which is identical to that available in the literature (e.g., ref. 2) for uniform fields, that is, $V = E_0 2l$.

The appropriate expressions for the axial and radial components of E then follow:

$$E_z(r, z, t) = e^{i\omega t} \sum_{n=-\infty}^{\infty} E_n \frac{I_0(\gamma_n r)}{I_0(\gamma_n a)} e^{-i\beta_n z} = \frac{E_0}{L} \sum_{n=-\infty}^{\infty} \frac{I_0(\gamma_n r)}{I_0(\gamma_n a)} e^{i(\omega t - \beta_n z)}$$

$$\times \frac{\left[m \sinh(ml) - i\beta_n \cosh(ml)\right] e^{2i\beta_n l} + m \sinh(ml) + i\beta_n \cosh(ml)}{m^2 + \beta_n^2} \tag{59}$$

$$E_r(r, z, t) = i \frac{E_0}{L} \sum_{n=-\infty}^{\infty} \frac{\beta_n}{\gamma_n} \frac{I_1(\gamma_n r)}{I_0(\gamma_n a)} e^{i(\omega t - \beta_n z)}$$

$$\times \frac{\left[m \sinh(ml) - i\beta_n \cosh(ml)\right] e^{2i\beta_n l} + m \sinh(ml) + i\beta_n \cosh(ml)}{m^2 + \beta_n^2} \tag{60}$$

where

$$\gamma_n = \sqrt{\beta_n^2 - k^2}$$

In tube theory it is necessary to refer the field E_0 to the gap voltage V:

$$V = \int_0^{2l} E_0 \cosh m(z - l)\, dz = 2 \frac{E_0}{m} \sinh(ml) \tag{61}$$

$$E_0 = \frac{mV}{2 \sinh(ml)} \tag{62}$$

As m approaches zero

$$\lim_{m \to 0} E_0 = \lim_{m \to 0} \frac{V}{\left(2l + \frac{2}{3!} m^2 l^3 + \ldots\right)} = \frac{V}{2l}$$

which is exactly the expression for the uniform field. From equation (62) E_0 may be substituted in all equations wherever it is desirable to have the fields expressed in terms of gap voltages.

SUMMARY OF RESULTS

Radial and axial electric fields in axisymmetric interaction gaps of klystrons and coupled cavity traveling wave tubes were obtained in closed form expressions by assuming that the field at the radius of the tunnel tips $r = a$ is equal to $E_0 \cosh(mz)$, where m is a properly chosen field shaping parameter and E_0 is the amplitude of E midway between the tunnel tips.

In the case of klystron gaps the radial and axial expressions were derived by the method of contour integration of the Fourier transform of the gap times the solution of the wave equation as the sum of the residues. To obtain convergence, it was necessary to derive the expressions for the field inside and outside the gap separately.

For TWTs the radial and axial gap fields were derived by expanding the gap field at $r = a$ into discrete space harmonics whose infinite sum represents the gap field.

For both types of tube the validity of the expressions ranges from a constant field between the tunnel tips to very large nonuniform fields but excluding infinite fields. The latter are, however, of no practical interest.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, December 21, 1971,
164-21.

REFERENCES

1. Branch, Garland M., Jr.: Electron Beam Coupling in Interaction Gaps of Cylindrical Symmetry. IRE Trans. on Electron Devices, vol. ED-8, no. 3, May 1961, pp. 193-207.

2. Watkins, Dean A.: Topics in Electromagnetic Theory. John Wiley & Sons, Inc., 1958.