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CONSTRAINED CHEBYSHEV APPROXIMATIONS TO SOME ELEMENTARY FUNCTIONS SUITABLE FOR EVALUATION WITH FLOATING-POINT ARITHMETIC

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16. Abstract

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SUMMARY

Approximations which can be evaluated with precision using floating-point arithmetic are presented. The particular set of approximations thus far developed are for the function TAN and the functions of USASI FORTRAN excepting SQRT and EXPONENTIATION. These approximations are, furthermore, specialized to particular forms which are especially suited to a computer with a small memory, in all that of the approximations can share one general purpose subroutine for the evaluation of a polynomial in the square of the working argument.

INTRODUCTION

The need for approximations of known quality to the mathematical functions commonly found in the function libraries of higher level computer languages, such as FORTRAN, has existed for some time. Approximations from the recent collection in the SIAM Series in Applied Mathematics (ref. 1) fill a large part of this need. These approximations have been somewhat optimized for speed, but they generally require that their evaluations be performed with some amount of precision beyond that which is required of the result.

In situations where it is desirable, for whatever reason, to evaluate the approximations using floating-point arithmetic with the precision of the result, the approximations of reference 1 prove to be not well conditioned for the minimization of the errors inherent in floating-point arithmetic.

It is the purpose of this report to present a family of approximations which can be evaluated with good precision using floating-point arithmetic. The particular set of approximations thus far developed are for the function TAN and the functions of USASI

FORTRAN excepting SQRT and EXPONENTIATION. These approximations are, furthermore, specialized to particular forms which are thought to be especially suited to a computer with a small memory, but which has an efficient method of reference to subroutines.

GENERAL CONSIDERATIONS

In general, these approximations are designed so that when the coefficients of a selected approximation are expressed in the floating-point representation of any computer and the given algebraic form is evaluated using the floating-point arithmetic of that computer then the accuracy of the implemented approximation is limited by the given nominal value of relative error or by the precision of the floating-point arithmetic used. Hence, these approximations are designed to avoid certain important sources of error that are inherent in the use of floating-point arithmetic where recourse to an occasional step of arithmetic with greater than nominal precision is overly difficult or slow. This is usually the situation when "double precision" versions of the approximations are being implemented.

The most pervasive source of these errors is a property of floating-point multiplication and division. It can be shown that these operations cannot produce ONTO mapping in the sense of Matula (ref. 2). This has two relevant consequences. The first, and probably more important, occurs when a change of scale is used to facilitate argument reduction. This situation is illustrated for the sine function when the argument is changed to "circle measurement" by multiplying by $4/\pi$.

For every argument x and number base β such that $\pi\beta^{-n}/4 < x < \beta^{-n}$ the value of the multiplied argument lies in the interval $\beta^{-n} < y < 4\beta^{-n}/\pi$. The effect is that the exponent part of y is one unit greater than the exponent part of x and an average of $\pi\beta/4$ successive values of x are represented by a single value of y. Necessarily then, the same result is generated for each of these successive values of x. For at least one of these successive values the magnitude of the error in the result cannot be less than one-half the difference of the correct values of the sine function at the extremes of this small interval or approximately $\frac{1}{2}\cos(x)\,\mathrm{Ceil}(\pi\beta/4)$ units of the value of the least significant bit of the result even with no other sources of error. The symbol Ceil(t) denotes the smallest integer greater than t; hence, for a base sixteen computer this error is approximately 6.3 units (2π). Examples of this large an error have been observed in a case where a change in scale of the argument was used during argument reduction. For this reason, a change in scale of the argument during argument reduction should be avoided.

The second consequence of this defect occurs when a floating-point multiplication or division is used as the final step of any evaluation. Small but systematic reduction in error is achieved by writing all odd functions, the logarithm function, and the nonconstant terms of the exponential function as $y + yf(y)$ rather than $y(1 + f(y))$. Sometimes an extra step of arithmetic is added to the algorithm by this organization. If a method of argument reduction which changes the scale of the independent variable is used, the benefits of this organization will be negligible.

The approximations to be described are all some form of the Chebyshev approximation constrained to algebraic forms that terminate with an operation of addition or subtraction. It is typical of previously reported Chebyshev approximations of these elementary functions with relative error weight functions for extremes of relative error to occur at the end points of the domain of derivation and for the relative error to increase very rapidly outside this domain of derivation. This property of the previously reported approximations imposes quite severe restrictions on the choice of integer multiplier for the argument reduction. Each of the current approximations is constrained to take on the value of the function at the end point of the domain of the approximation. This has the effect of widening the valid domain somewhat beyond the nominal domain used for derivation of the coefficients; hence, the restrictions on the correct choice of integer multiplier for argument reduction are relieved. The details of the precision requirements for a reduced argument to stay well within this extended domain are discussed in the appendix.

This constraint on the approximation's value at the boundary of its nominal domain has also been imposed when no argument reduction is required. The effect of this constraint is that weak monotonicity can easily be achieved and continuity satisfactorily simulated at a point where two different approximation segments must be joined. This is realizable even for approximations whose accuracy is low compared to the nominal precision of the floating-point arithmetic in use.

A further source of errors arises from the impossibility of representing arbitrary real numbers in any finite length floating-point notation. Algebraic forms for the approximations presented here were selected so that those coefficients in which truncation could produce sizable error in the final approximation would, if unconstrained, be very nearly equal to integers or half integers. These more important coefficients are constrained to these generally representable integer or half integer values, and the remaining coefficients are calculated subject to these constraints. Specific details of these constraints as applied to each approximation are given in the DISCUSSION OF SPECIFIC APPROXIMATIONS section.

These absence of optionally rounded floating-point arithmetic or the failure of weak monotonicity or "continuity" can in some cases be compensated for by modification of the values of selected coefficients. Such "fudges" are machine, word length, and number base dependent and no attempt has been made to include any.

Given some approximation R to a function f, the relative error function for this approximation is defined by

$$ER(x) = \frac{[R(x) - f(x)]}{f(x)}$$

wherever $f(x) \neq 0$. If within the domain of validity of the approximation $f(x) = 0$, the relative error can be defined for that point by

$$ER(x) = \lim_{t \to x} \left[\frac{R(t) - f(t)}{f(t)} \right]$$

One measure of the quality of an approximation is its extremal relative error; that is the least upper bound of the magnitude of $ER(x)$ for all values x from the domain of validity of the approximation:

$$\overline{ER} = \operatorname{lub}_{x \in D} |ER(x)|$$

A term often used in describing the quality of an approximation is its precision; this is taken to be the negative of the logarithm of the extremal relative error:

$$\text{Precision} = -\log_{\beta}(\overline{ER})$$

Its value is very nearly equal to the minimum of the number of correct digits in the base β representation of the value of $R(x)$ for any argument x from the domain of validity of the approximation.

CONSEQUENT RESTRICTIONS ON FORMS USED

The current set of Chebyshev approximations was developed to avoid serious errors from the previously mentioned sources. Hence, each approximation incorporates these characteristics:

(1) The final arithmetic operation is always the addition of an exact term to an approximate term of smaller magnitude.

(2) The coefficients are jointly constrained so that the approximation takes on the value of the approximated function at the boundary points of its nominal (reduced) domain.

(3) The coefficients with most the influence on error are constrained to values that can be exactly represented in any computer's floating-point number system.

Because of a specific interest in their use in a computer which has a small memory, the forms used for these approximations are limited to those involving the use of a single polynomial in the square of an appropriately reduced argument.

It is expected that the theoretical value of extremal relative error of each approximation will be increased by observing all these constraints. Empirically this effect is small and fortuitously has not required the use of more elaborate approximations in any case that has been implemented.

CURVE FIT

The rational form used for any approximation presented is formally equivalent to one of the following: P, yP, (P + y)/(P - y), or $y \pm y^3/P$. The symbol P represents a polynomial of degree N whose independent variable y^2 is the square of the reduced argument; the symbol Q will also be used. Some of the coefficients of P (or Q) are constrained to given values; all are constrained to give the theoretically correct value for the joining point. The coefficients are computed subject to these constraints by a slightly modified version of the second algorithm of Remes (ref. 3) using especially constructed error weighting functions so that each resulting approximation is uniform throughout the nominal domain. A known restriction on the use of such rational approximations is that they be pole-free. All the approximations, as generated, turned out to be so without specific attention to the problem. The coefficients presented in this report were computed on an IBM 7094 II computer using floating-point arithmetic with 140 binary digits in the fractional part of the floating-point number. Subroutines to perform this extended precision arithmetic and to evaluate many of the elementary functions using it have been provided by C. L. Lawson (ref. 4).

DISCUSSION OF SPECIFIC APPROXIMATIONS

Logarithm

For any $x > 0$ the natural logarithm can be defined in terms of its values over a limited domain as

$$\ln(x) = n \ln(2) + \ln(y); \qquad \frac{\sqrt{2}}{2} < y < \sqrt{2} \tag{1}$$

The form of equation (1) implies the use of base two arithmetic in that the values of n and y are then obtained without error from the representation of the argument x. The rational approximation selected for ln(y) in the basic domain is

$$\ln(y) \approx 2v + \frac{v^3}{Q(v^2)} \tag{2}$$

$$v = \frac{y - 1}{y + 1}; \qquad \frac{\sqrt{2}}{2} < y < \sqrt{2} \tag{3}$$

When floating-point arithmetic is used the term $y + 1$ cannot be calculated exactly if the representation of y has a low order digit of one. The multiplier of any error in v is reduced from 2.0 to at most 0.395 by the use of the identity $2v = (y - 1) + v(1 - y)$ to convert equation (2) to the recommended form

$$\ln(y) \approx (y - 1) + v\left[1 - y + \frac{v^2}{Q(v^2)}\right] \tag{4}$$

As far as is known, further reduction in error can come only from using extended precision arithmetic.

The quantity $n \ln(2)$ should be calculated and used in two parts: The more significant part, A, is calculated using only that number of leading digits of $\ln(2)$ that give an exact product with any value of n which can occur in an implementation; the less significant part, B, is calculated using the best representation of the remainder of $\ln(2)$. The various terms of the approximation should be summed starting from the right in approximation (5):

$$\ln(x) \approx A + (y - 1) + B + v\left[(1 - y) + \frac{v^2}{Q(v^2)}\right] \tag{5}$$

Optimal use of rounding is quite difficult to achieve because of the large number of changing criteria. For most values of $n \neq 0$, the most important operation to be rounded is the left-most (final) addition of approximation (5). For $n = 0$, the second addition from the left is most important.

A change of scale of the independent variable to use logarithms of other than the natural base is not recommended because of the floating-point multiplication property unless the implementer is prepared to use somewhat extended precision arithmetic in the evaluation. In that case, an approximation from reference 1 should be applicable.

Coefficients for the approximations (2), (4), or (5) are identified according to the degree M of the polynomial $Q(v^2)$ involved as LOG($\sqrt{2}$, 0, M).

Exponential

For any argument x the exponential function can be defined as

$$e^x = 2^n e^y \tag{6}$$

in terms of its values over a base domain. Ideally, the integer n and the working argument y are selected so that

$$y = x - n \ln(2) \qquad |y| \le \frac{\ln(2)}{2} \tag{7}$$

A rational approximation

$$e^y \approx 1 + \frac{2y}{2 - y + y^2 P(y^2)} \tag{8}$$

is then used within the basic domain. The approximation described here is best implemented in base two arithmetic; the multiplication by 2^n in equation (6) can be done exactly, and the final addition of approximation (8) leaves a digit that can be used for rounding.

Because ln(2) is irrational it is not possible to guarantee computing the correct integer n, as defined by relation (7), except by completing the indicated reduction and verifying the containment $|y| \le \ln(2)/2$. The need for such care is avoided because the approximations for e^y are constrained to take on as nearly as possible the correct values at the joining points, $y = \pm\ln(2)/2$. This insures that the attainable, weaker, containment $|y| < \ln(2)/2 + \Delta$ is sufficient. (See the appendix for details.)

For negative values of the reduced argument the approximation (8) is not weakly monotonic. This is an artifact of floating-point representation in any number base β and is very similar to a situation discussed by D. W. Matula in reference 5. He pointed out the nonmonotone behavior of any floating-point implementation of $f(y) = y/(2 + y)$ for arguments y approaching 1.0 from below. The behavior is similarly nonmonotone for arguments that approach many of the positive fractions β^{-k}. In a floating-point implementation of approximation (8) the ratio $2y/\left\{\left[2 + y^2 P(y^2)\right] - y\right\}$ exhibits a similar failure of weak monotonicity for negative arguments. As the representation of y increases from some negative value to the next available value this ratio increases instead of decreasing.

This increase is sometimes sufficient to cause the sum to decrease producing a failure of weak monotonicity. The approximation can be restated in the algebraically equivalent form

$$e^y \approx 1 + y + \frac{y\left[y - y^2P(y^2)\right]}{2 - \left[y - y^2P(y^2)\right]} \tag{9}$$

The use of expression (9) is recommended whenever high accuracy is required; it avoids the previously described computational difficulty at the cost of one extra storage operation and one operation of addition.

Coefficients for the polynomial $P(y^2)$ of degree N used in approximation (8) are given the identification EXP(ln(2)/2, 0, N + 1).

Hyperbolic Sine and Hyperbolic Cosine

The formal definition

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \tag{10}$$

of the hyperbolic sine function suggests the implementation as

$$\sinh(x) = \frac{\text{sgn}(x)}{2}\left(e^t - \frac{1}{e^t}\right) \qquad t = |x| \tag{11}$$

Direct use of equation (11) is computationally unstable for small arguments because of the addition of values with opposite signs and nearly equal magnitudes.

For small arguments the rational approximation

$$\sinh(x) \approx x + \frac{x^3}{Q(x^2)} \qquad |x| < b \tag{12}$$

is used. The joining point b is selected to satisfy precision requirements of the approximation related to (11) which is used for large arguments.

A different difficulty exists for some large arguments. For any number base β direct implementation of approximation (11) is somewhat unstable whenever $\sinh(t) < \beta^n < e^t/2$ because the significance of one or more digits is lost by cancellation during the subtraction. Since $\sinh(t) = s \geq 0$ is equivalent to $t = \ln\left(s + \sqrt{s^2 + 1}\right)$ we have this instability occurring whenever

$$\ln(2\beta^n) \le t < \left(\ln\ \beta^n + \sqrt{\beta^{2n} + 1}\right) \qquad (13)$$

The most elegant known resolution of this difficulty was obtained from Mr. Hirondo Kuki in a private communication. Choose a value v large enough so that if t is any magnitude from one of the intervals (13) then, for $y = t - v$, $e^y/2$ has the same exponent part as $\sinh(t)$. From this point of view suitable values are given by

$$v \ge \ln\left(\beta^n + \sqrt{\beta^{2n} + 1}\right) - \ln(2\beta^n) = \ln\left(\frac{1 + \sqrt{1 + \beta^{-2n}}}{2}\right) \qquad (14)$$

The value of v is further selected to have a sufficient number of zero low order digits in its machine representation that no error is introduced in the subtraction $t - v$ for any magnitude t such that $\sinh(t)$ can be represented. An algebraic restatement of equation (10) leads to the approximation

$$\sinh(x) \approx \operatorname{sgn}(x)\left[e^y + \left(\frac{e^v}{2} - 1\right)e^y - \frac{e^{-v}}{2}e^{-y}\right] \qquad y = |x| - v \qquad (15)$$

In a situation where rounding is available the condition $(e^v/2) - 1 < 1/\beta$ is desirable in order that the addition provide a nearly correct rounding digit.

Another possible difficulty with the direct use of approximation (11) would occur for any magnitude t near the upper limit for which the value $\sinh(t)$ can be represented in whatever floating-point number system is used. The required value e^t fails to be representable and a machine error condition would result from attempting its calculation. The computational scheme of approximation (14) is found to prevent this whenever $v > \ln(2)$ without requiring any test except that the value $\sinh(x)$ be itself representable.

At the joining points of the approximation segments, $x = \pm b$, the rational approximations are constrained to take on the values obtained by evaluation of the formal definition (10) using high precision arithmetic. It may be necessary for an implementation that the coefficients of the rational approximation be adjusted so that its values at the joining points match the values actually produced by the approximation (14) used for large arguments. A reasonable selection of the joining point is the end of the first positive interval (13) for which the instability of a direct implementation of approximation (11) is avoided. For base two this means $n = -1$ and $b = \ln[(1 + \sqrt{5})/2]$; for any larger base use $n = 0$ and $b = \ln(1 + \sqrt{2})$.

Polynomials $Q(x^2)$ for use in the rational approximation (12) and tailored to base two arithmetic are valid in the domain $|x| < \ln[(1 + \sqrt{5})/2]$. The coefficients for the polynomial of degree M are identified as SINH$\{\ln[(1 + \sqrt{5})/2],\ 0,\ M\}$ and the value selected for v of approximation (15) must satisfy $\ln(2) \le v < \ln(3)$. Approximations using the coefficients identified as SINH$[\ln(1+\sqrt{2}),\ 0,\ M]$ are valid in the domain $|x| < \ln(1+\sqrt{2})$. These are given for use with number bases other than two; the associated value of v must satisfy $\ln(2) \le v < \ln(2.125)$.

The hyperbolic cosine function is defined as

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \tag{16}$$

A straightforward implementation would be valid for small and most large arguments. For arguments whose magnitude is near the upper limit for which $\cosh(x)$ can be represented $\cosh(x) \approx |\sinh(x)|$. The approximation

$$\cosh(x) \approx e^{y} + \left(\frac{e^{v}}{2} - 1\right)e^{y} + \frac{e^{-v}}{2}\ {}^{-y} \qquad y = |x| - v \tag{17}$$

which is similar to approximation (15) and uses the same value of v is effective for all arguments for which $\cosh(x)$ is representable.

Hyperbolic Tangent

The hyperbolic tangent function is defined as

$$\tanh(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \tag{18}$$

This equation is not suitable as the basis for an evaluating algorithm: both numerator and denominator contain exponential terms that must be approximations, neither the sum nor the difference required can be precisely calculated and finally the computation ends with a division. The form

$$\tanh(x) = \operatorname{sgn}(x)\left(1 - \frac{2}{e^{2y} + 1}\right) \qquad y = |x| \tag{19}$$

is algebraically equivalent to (18). It is sufficiently well adapted to floating-point arithmetic to be used as the basis for an approximation to $\tanh(x)$ for large arguments ($|x| > b$). The value of b is selected so that precision requirements of the approximation (19) can be satisfied. For small values of the argument x both equations (18) and (19) require the addition of values with opposite signs and nearly equal magnitudes;

hence, neither is satisfactory. The rational approximation

$$\tanh(x) \approx x - \frac{x^3}{3.0 + x^2 Q(x^2)} \tag{20}$$

is used therefore when $|x| < b$.

It is desirable to round the result of the final arithmetic operation of either approximation; hence, a rounding digit must be generated during that final operation. This is assured if the floating-point exponent of the smaller term is less than that of the result. For large arguments using equation (19) this requires

which gives

$$\left.\begin{aligned} &\frac{2}{e^{2b}+1} < \frac{1}{\beta} \\ &b > \ln\left(\frac{2\beta - 1}{2}\right) \end{aligned}\right\} \tag{21}$$

For small arguments using approximation (20) the rounding digit is generated if the floating-point exponent of $x^3/[3.0 + x^2 Q(x^2)]$ is smaller than the floating-point exponent of x for every $x \leq b$. Only for $\beta = 2$ can both requirements be satisfied; with any other number base the floating-point representation of the value of the smaller term will not extend far enough to include the needed rounding digits.

The accuracy of the rational term of approximation (20) can be marginal near the limits of its domain; hence, the constant term of the denominator is constrained to the precisely representable value 3.0 which eliminates error from one important source. An equally important source of possible error is the calculation of x^3; any available error reducing steps, such as rounding, should be used here.

When an implementation is for a number base greater than two, the floating-point representation of the value $2y$ can be in error, whether calculated as $y + y$ or as $2y$, hence the form

$$\tanh(x) = \mathrm{sgn}(x)\left[1 - \frac{2}{(e^y)^2 + 1}\right] \qquad y = |x| \tag{22}$$

should be used for equation (19) to avoid an unnecessary loss of accuracy due to the representation of $2y$.

Coefficients for the approximation (20) are identified according to the degree M of the denominator polynomial involved as TANH[ln(3)/2, 0, M].

Sine and Cosine

The sine and cosine functions can be defined by Maclaurin series as

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \tag{23}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \tag{24}$$

for all values of the argument x. Direct implementations of equations (23) and (24) are not satisfactory as approximations because the functions are periodic and have repeated zeros for large arguments.

This difficulty is overcome by limiting the nominal domain of definition of the approximations to $|x| < \pi/4$. The evaluation algorithms then become

$$x = (4n + j)\frac{\pi}{2} + y \qquad |y| \leq \frac{\pi}{4} \tag{25}$$

$$\sin(x) = \begin{cases} \sin(y) & \text{if } j = 0 \\ \cos(y) & \text{if } j = 1 \\ -\sin(y) & \text{if } j = 2 \\ -\cos(y) & \text{if } j = 3 \end{cases} \tag{26}$$

$$\cos(x) = \begin{cases} \cos(y) & \text{if } j = 0 \\ -\sin(y) & \text{if } j = 1 \\ -\cos(y) & \text{if } j = 2 \\ \sin(y) & \text{if } j = 3 \end{cases} \tag{27}$$

The polynomial approximations used for sin(y) and cos(y) are

$$\sin(y) \approx y + y^3 P(y^2) \tag{28}$$

$$\cos(y) \approx 1.0 + y^2\left[-0.5 + y^2 P_1(y^2)\right] \tag{29}$$

In approximation (28) the term $y^3P(y^2)$ has several sources of computational error: the value of y^2, the multiplication of y by y^2, and the truncated values of the coefficients. Rounding can help reduce these errors. When the implementation uses floating-point arithmetic with small number base ($\beta \leq 12$), the alinement shift prior to the final addition of approximation (28) both attenuates the effects of these computational errors in the rational term and produces a rounding digit.

Coefficients for the polynomial $P(y^2)$ of degree N - 1 used in approximation (28) are identified as SIN(π/4, N, 0). These approximations for N = 2, 3, . . ., 7 are comparable to approximations 3040, 3041, . . ., 3045 of reference 1. The loss of nominal precision of the approximations (28) caused by imposing the boundary point value constraint is less than 0.14 decimal digit in all cases.

In approximation (29) for the cosine series the term $y^2\left[-0.5 + y^2 P_1(y^2)\right]$ can have a magnitude somewhat greater than 0.25; hence, only use of base two arithmetic insures that the floating-point exponent of this term is less than that of the result. Even so, reduction in the effect of computational errors in that term may be marginal as may the accuracy of the rounding digit. The leading coefficients are constrained to precisely 1.0 and -0.5 so that no error is introduced by truncating their values for storage. The use of appropriate rounding is recommended.

Coefficients for the polynomial of degree N - 2 used as approximation (29) are identified as COS(π/4, N, 0). These approximations for N = 3, 4, . . ., 8 are comparable to approximations 3820, 3821, . . ., 3825 of reference 1. The loss of nominal precision of the approximations (29) caused by imposing the boundary point value constraint and the coefficient constraint is not overly large: in all cases it is less than 0.49 decimal digit.

Tangent and Cotangent

The tangent function can be defined in continued fraction form as

$$\tan(x) = \frac{x}{1-}\ \frac{x^2}{3-}\ \frac{x^2}{5-}\ \frac{x^2}{7-}\ \cdots \qquad (30)$$

for any value of the argument. The tangent function is periodic, but any direct implementation of equation (30) valid for the entire cycle about the origin is impractical because of the large number of terms that would be required near the poles at $\pm\pi/2$. The identity

$$\tan(x) = \frac{1}{\tan\left(\frac{\pi}{2} - x\right)} \qquad (31)$$

is used to construct an evaluation algorithm in terms of the values of the tangent from the domain $|x| \leq \pi/4$.

$$x = (2k + j)\frac{\pi}{2} + y \qquad |y| \leq \frac{\pi}{4} \tag{32}$$

$$\tan(x) = \begin{cases} \tan(y) & \text{if } j = 0 \\ \dfrac{-1}{\tan(y)} & \text{if } j = 1 \text{ and } y \neq 0 \end{cases} \tag{33}$$

The rational form used for the basic approximation is

$$\tan(y) \approx y + \frac{y^3}{3.0 + y^2 Q(y^2)} \tag{34}$$

Because the cotangent function is the reciprocal of the tangent, the same argument reduction and basic approximation can be used, with trivial modifications to equation (33), to evaluate the cotangent.

The magnitude of the rational term of approximation (34) can be almost 0.25; hence, only with the use of arithmetic of base four or less will an alinement shift occur before the final addition. When the implementation must use arithmetic of some larger number base, computational error in the rational term will not have its effect on the final result attenuated and no digit will be available for rounding. Because the accuracy of the rational term can be marginal, its constant term is constrained to the precisely representable value 3.0 so that no error is introduced by truncating that constant for storage. Another important source of error is the calculation of the numerator y^3; any possible error reducing steps, such as rounding, should be included in an implementation.

Coefficients for the approximation (34) are identified according to the degree M of the denominator polynominal involved as TAN($\pi/4$, 0, M + 1). The approximation using TAN($\pi/4$, 0, 2) is comparable to approximation 4283 of reference 1.

Inverse Tangent

For any argument x the principal value of the inverse tangent function can be defined as

$$\arctan(x) = \frac{x}{1+}\;\frac{x^2}{3+}\;\frac{4x^2}{5+}\;\cdots\;\frac{k^2x^2}{(2k+1)+}\;\cdots \tag{35}$$

This continued fraction is not an economical computational algorithm for arguments with large magnitudes because of the number of terms required in the computation. The transformation

$$\arctan(x) = \frac{\pi}{2}\,\mathrm{sgn}(x) - \arctan(y) \qquad y = \frac{1}{x} \tag{36}$$

can be used whenever $|x| > 1$ to reduce the domain for which the basic approximation used need be valid. Further reduction can be obtained by applying

$$\arctan(x) = \mathrm{sgn}(x)\left[\frac{\pi}{6} + \arctan(y)\right] \qquad y = \frac{|x|\sqrt{3} - 1}{|x| + \sqrt{3}} \tag{37}$$

whenever $\tan(\pi/12) < |x| \le 1$. The use of transformation (36) or (37) can introduce error both in calculating y and in subsequently calculating $\arctan(x)$ using the value $\arctan(y)$. For some arguments both must be used. Implementing the following elaborated scheme can avoid the cascading of these effects:

$$\arctan(x) = \begin{cases} \arctan(y) & y = x \quad \text{if } |x| < \tan\left(\frac{\pi}{12}\right) \\ \mathrm{sgn}(x)\left[\frac{\pi}{6} + \arctan(y)\right] & y = \frac{|x|\sqrt{3} - 1}{|x| + \sqrt{3}} \quad \text{if } \tan\left(\frac{\pi}{12}\right) < |x| \le 1 \\ \mathrm{sgn}(x)\left[\frac{\pi}{3} - \arctan(y)\right] & y = \frac{\sqrt{3} - |x|}{1 + |x|\sqrt{3}} \quad \text{if } 1 < |x| < \frac{1}{\tan\left(\frac{\pi}{12}\right)} \\ \frac{\pi}{2}\,\mathrm{sgn}(x) - \arctan(y) & y = \frac{1}{x} \quad \text{if } |x| > \frac{1}{\tan\left(\frac{\pi}{12}\right)} \end{cases} \tag{38}$$

The form selected for the basic approximation is

$$\arctan(y) \approx y - \frac{y^3}{Q(y^2)} \tag{39}$$

This approximation need be valid only for the domain $|y| \lesssim \tan(\pi/12)$ and is in fact quite stable there even when implemented in floating-point arithmetic of any commonly used number base.

Coefficients for the polynomial $Q(y^2)$ of degree M used by approximation (39) are identified as ATAN[tan(π/12), 0, M]. The approximation using ATAN[tan(π/12), 0, 1] is comparable to approximation 5050 of reference 1. The imposition of the boundary point value constraint causes a loss of 0.19 decimal digit of nominal precision.

Inverse Sine and Inverse Cosine

For any argument x with $|x| < 1$ the principal value of the inverse sine function is defined as

$$\arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \ldots \tag{40}$$

Various numerical problems associated with implementing this definition for arguments with magnitudes near 1.0 can be avoided by using the transformation

$$\arcsin(x) = \operatorname{sgn}(x)\left[\frac{\pi}{2} - 2\arcsin(y)\right] \qquad y = \sqrt{\frac{1 - |x|}{2}} \tag{41}$$

wherever $|x| > 0.5$. The rational approximation

$$\arcsin(y) \approx y + \frac{y^3}{Q(y^2)} \tag{42}$$

is then used in either case.

Any errors that may be introduced by the argument transformation of (41) are preserved through the approximation; hence, all possible error reducing steps should be used. Implementation in base two arithmetic eases this problem somewhat because then neither the calculation of $(1 - |x|)/2$ nor the multiplication in $2 \arcsin(y)$ can introduce error.

A suitable evaluation algorithm for the principal value of the inverse cosine function can be built around the identity

$$\arccos(x) = \frac{\pi}{2} - \arcsin(x) \tag{43}$$

transformation (41) and approximation (42).

Coefficients for the polynomial $Q(y^2)$ of degree M used in approximation (42) are identified as ARSIN(0.5, 0, M). The approximation using ARSIN(0.5, 0, 1) is comparable to approximation 4691 of reference 1; a loss of 0.19 decimal digit of precision is caused by the imposition of the boundary point value constraint.

The precision obtainable from approximation (42) increases only slowly with the degree M of the polynomial used. This may limit the utility of these approximations where high precision is required.

RESULTS

Coefficients for use in implementing any of the approximations that have been discussed are presented herein. Note that these coefficients are for the polynomial $P(y^2)$ or $Q(y^2)$ required in the description of each approximation. Any specifically constrained coefficients that may be needed were presented with that description. The coefficients are listed in order of increasing powers of the square of the appropriate variable; formally,

$$P(y^2) = P_{00} + P_{01}y^2 + P_{02}y^4 + \ldots \tag{44}$$

For each function considered the functional form and nominal interval of its approximations are presented as page headings to the lists of coefficients. Each set of coefficients is identified by an index number and the precision for which that approximation is adequate. The precision is expressed as the number of binary digits (bits) and the number of decimal digits. The coefficients are given in both binary (octal) and decimal notation; in each radix system ($\beta = 2$ or $\beta = 10$) the coefficient is expressed as (n)F where n is an integer and F is a signed fraction whose magnitude is bounded by $1/\beta$ and 1. The value of the numeral is $F*\beta^n$. Both parts of the binary numeral are, for convenience, written in the common pseudo-octal representation.

The extreme values of the relative error function ER(x) for each approximation covered by this report are given in separate lists, indexed according to the same system used for the sets of coefficients. With each value is displayed a set of points from the nominal domain at which the relative error function attains its extreme magnitude. The sign of the relative error at each point is indicated by a mark (+) or (-) attached to the point. The natural symmetries of the various relative error functions are indicated; this allows the identification of all the remaining extremal points of the approximation and the corresponding signs.

LOG(X) $\sqrt{2}/2 < X < \sqrt{2}$, $Y = (X-1)/(X+1)$, $LOG(\sqrt{2}, 0, M) = 2Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 1	PRECISION 25.0 BITS		PRECISION 7.53 DIGITS
	(1) .60000 60107 03222 63203	Q00	(1) .15000 45908 71064 92509
	(0) -.71713 02456 73527 22742	Q01	(0) -.90463 38041 61428 99733
M = 2	PRECISION 33.2 BITS		PRECISION 10.00 DIGITS
	(1) .57777 77543 30151 71753	Q00	(1) .14999 99708 26922 35389
	(0) -.71461 24554 52353 50613	Q01	(0) -.89994 27376 90583 87066
	(-3) -.66226 40472 27070 26612	Q02	(0) -.10604 28985 34924 58845
M = 3	PRECISION 41.0 BITS		PRECISION 12.35 DIGITS
	(1) .60000 00001 07254 33332	Q00	(1) .15000 00002 07617 33898
	(0) -.71463 16141 34744 31055	Q01	(0) -.90000 06629 64100 45727
	(-3) -.64502 11543 65667 77721	Q02	(0) -.10279 14103 86743 10443
	(-4) -.70346 32565 04321 70154	Q03	(-1) -.55126 98676 13972 73393
M = 4	PRECISION 48.6 BITS		PRECISION 14.64 DIGITS
	(1) .57777 77777 77360 71370	Q00	(1) .14999 99999 98458 96480
	72504 11345 41365 37661		40458 55206 78358 00268
	(0) -.71463 14621 64566 22332	Q01	(0) -.89999 99927 57963 35274
	50103 40776 61017 00366		34565 49909 43167 33935
	(-3) -.64523 53023 26453 43403	Q02	(0) -.10285 82476 25745 33080
	40246 42301 63133 02034		24971 62132 60086 81806
	(-4) -.65604 05704 67034 32315	Q03	(-1) -.52498 03914 10786 00749
	56035 30661 10535 32000		89523 51352 07795 34039
	(-4) -.44412 46161 32554 63612	Q04	(-1) -.35664 74382 63394 33715
	62776 52020 67506 00000		66549 51092 78909 07926
M = 5	PRECISION 56.1 BITS		PRECISION 16.90 DIGITS
	(1) .60000 00000 00002 03160	Q00	(1) .15000 00000 00011 65464
	35434 06433 17336 64331		63096 85635 80587 92429
	(0) -.71463 14631 53456 42473	Q01	(0) -.90000 00000 75537 53897
	37505 53067 51126 23436		93007 46572 05806 74694
	(-3) -.64523 30165 35107 17770	Q02	(0) -.10285 71265 58117 96087
	44654 00176 21560 52040		38584 40640 24884 24690
	(-4) -.65653 32474 52365 56152	Q03	(-1) -.52573 04365 88491 33689
	76714 55561 37043 10000		96979 48235 22445 22579
	(-4) -.42137 57604 42211 13455	Q04	(-1) -.33385 74644 50083 46016
	51551 12213 55076 00000		21920 69208 44438 90803
	(-5) -.64615 02576 73423 24345	Q05	(-1) -.25769 27465 06735 29956
	22774 55464 07500 00000		16388 51653 26798 42879
M = 6	PRECISION 63.6 BITS		PRECISION 19.14 DIGITS
	(1) .57777 77777 77777 76777	Q00	(1) .14999 99999 99999 91105
	20466 46254 24240 07407		36980 66750 22438 31335
	(0) -.71463 14631 46262 04776	Q01	(0) -.89999 99999 99240 51719
	26070 12605 37224 51732		82333 02557 92860 04541
	(-3) -.64523 30403 26150 75534	Q02	(0) -.10285 71430 76480 51061
	35447 26004 11714 61700		22868 69003 71271 72405
	(-4) -.65652 43326 03157 36641	Q03	(-1) -.52571 39855 82276 44815
	14272 41104 27041 20000		81158 72261 17836 70300
	(-4) -.42213 14214 03627 11651	Q04	(-1) -.33468 61224 03538 51917
	13237 74165 74360 00000		20036 71217 39398 72002
	(-5) -.60443 30330 21376 11274	Q05	(-1) -.23715 38314 83266 82035
	21777 02440 62000 00000		36668 57959 57468 93322
	(-5) -.50614 42731 51031 07731	Q06	(-1) -.19909 42545 35562 27678
	33022 22412 00000 00000		84725 88306 75557 47636

LOG(X) $\sqrt{2}/2 < X < \sqrt{2}$, $Y = (X-1)/(X+1)$, $LOG(\sqrt{2}, 0, M) = 2Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS

M = 7

PRECISION 70.9 BITS			PRECISION 21.36 DIGITS	
(1)	.60000 00000 00000 00003	Q00	(1)	.15000 00000 00000 00068
	73444 72513 31056 54354			18875 49550 21916 49472
(0)	-.71463 14631 46315 03711	Q01	(0)	-.90000 00000 00007 41309
	07376 47263 27417 34575			54371 80715 05501 57883
(-3)	-.64523 30401 34067 37506	Q02	(0)	-.10285 71428 54388 05267
	26222 02613 71412 15600			72969 81877 61855 31475
(-4)	-.65652 44342 13167 40502	Q03	(-1)	-.52571 42906 63321 82299
	10205 21621 32614 00000			34630 40944 61532 18661
(-4)	-.42212 01025 23541 50545	Q04	(-1)	-.33466 37015 39199 74780
	42005 73343 32500 00000			96430 93236 24869 12055
(-5)	-.60602 26754 20447 25421	Q05	(-1)	-.23805 96043 83019 30946
	11015 41141 50000 00000			46126 24884 07269 32876
(-5)	-.44707 50064 17000 77066	Q06	(-1)	-.18012 64438 06338 67445
	33135 35230 00000 00000			51662 60793 29354 72560
(-5)	-.40747 71256 77267 60240	Q07	(-1)	-.16090 29385 11566 82769
	12165 07100 00000 00000			67854 31151 28078 16710

M = 8

PRECISION 78.3 BITS			PRECISION 23.56 DIGITS	
(1)	.57777 77777 77777 77777	Q00	(1)	.14999 99999 99999 99999
	76042 33124 24177 00576			47612 30648 56314 23831
(0)	-.71463 14631 46314 63024	Q01	(0)	-.89999 99999 99999 92938
	77012 60340 35415 45043			34867 32037 03678 98653
(-3)	-.64523 30401 35510 10240	Q02	(0)	-.10285 71428 57175 63818
	43153 51305 00176 44600			28168 05540 31303 26867
(-4)	-.65652 44331 42702 52353	Q03	(-1)	-.52571 42856 39698 59218
	76404 43512 21326 00000			18552 21328 95235 47704
(-4)	-.42212 02562 01576 06550	Q04	(-1)	-.33466 42025 43956 87455
	25474 76737 57200 00000			39439 31786 12068 74604
(-5)	-.60577 23412 71101 57071	Q05	(-1)	-.23803 04815 28441 62770
	73551 66755 40000 00000			74473 16612 71219 33267
(-5)	-.45056 47160 13027 51756	Q06	(-1)	-.18110 85967 84934 90069
	41151 01640 00000 00000			43263 44851 10701 71337
(-6)	-.72470 54477 37474 55562	Q07	(-1)	-.14309 26191 81558 30900
	54510 36000 00000 00000			62997 74303 57543 59238
(-6)	-.67007 13545 54560 03456	Q08	(-1)	-.13431 15939 66956 16028
	34747 40000 00000 00000			47827 23900 73187 45949

M = 9

PRECISION 85.6 BITS			PRECISION 25.76 DIGITS	
(1)	.60000 00000 00000 00000	Q00	(1)	.15000 00000 00000 00000
	00007 46755 42721 13072			00402 80714 33732 77872
(0)	-.71463 14631 46314 63147	Q01	(0)	-.90000 00000 00000 00065
	12170 05447 25335 15455			91209 94427 07800 95536
(-3)	-.64523 30401 35476 63034	Q02	(0)	-.10285 71428 57142 48368
	24277 37133 17250 47400			89294 84409 75943 98853
(-4)	-.65652 44331 53144 40540	Q03	(-1)	-.52571 42857 15333 15412
	54701 26310 57574 00000			24408 00020 01834 24589
(-4)	-.42212 02541 20243 55053	Q04	(-1)	-.33466 41927 81199 15578
	12672 26506 54000 00000			94836 00681 02934 26339
(-5)	-.60577 30502 22226 01354	Q05	(-1)	-.23803 12427 08043 52534
	55636 65026 00000 00000			89594 15327 03162 14010
(-5)	-.45052 61662 72524 31151	Q06	(-1)	-.18107 20338 59276 56069
	17622 47600 00000 00000			10826 52458 24029 80602
(-6)	-.73026 50177 33650 77330	Q07	(-1)	-.14415 08717 01254 02853
	45015 60000 00000 00000			16142 50936 05080 42952
(-6)	-.60051 66434 63035 20330	Q08	(-1)	-.11738 70679 91915 36861
	53534 00000 00000 00000			49301 17713 07131 81792
(-6)	-.57033 67326 44435 73405	Q09	(-1)	-.11487 89688 26266 04733
	30400 00000 00000 00000			95477 86703 02482 31294

LOG(X) $\sqrt{2}/2 < X < \sqrt{2}$, $Y = (X-1)/(X+1)$, $LOG(\sqrt{2}, 0, M) = 2Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS
M = 10	PRECISION 92.9 BITS			PRECISION 27.95 DIGITS
(1)	.57777 77777 77777 77777 77777 74205 34271 56616	Q00	(1)	.14999 99999 99999 99999 99996 90686 66152 08173
(0)	-.71463 14631 46314 63146 31117 03721 66424 21665	Q01	(0)	-.89999 99999 99999 99999 39612 13378 20685 51923
(-3)	-.64523 30401 35476 71666 50254 55327 36450 05400	Q02	(0)	-.10285 71428 57142 86124 19451 32353 91623 68757
(-4)	-.65652 44331 53050 60713 75011 00516 40220 00000	Q03	(-1)	-.52571 42857 14271 83905 24261 70648 65962 22563
(-4)	-.42212 02541 43126 55133 26344 27466 62000 00000	Q04	(-1)	-.33466 41929 52635 38047 89577 83791 31775 63837
(-5)	-.60577 30407 33602 57316 47365 16670 00000 00000	Q05	(-1)	-.23803 12255 79813 29415 04097 91987 74818 70366
(-5)	-.45052 71141 44663 55027 47626 20000 00000 00000	Q06	(-1)	-.18107 31277 73065 53504 77854 86718 46938 89734
(-6)	-.73015 17522 50366 53507 66735 00000 00000 00000	Q07	(-1)	-.14410 61244 45174 37449 51665 88366 05890 51549
(-6)	-.60427 57073 47253 63006 06040 00000 00000 00000	Q08	(-1)	-.11852 13861 19335 92331 92458 91345 75873 80002
(-6)	-.50320 67527 77264 16335 47000 00000 00000 00000	Q09	(-2)	-.98652 21916 55058 18989 55403 12393 94560 33753
(-6)	-.51010 26155 11336 61304 50000 00000 00000 00000	Q10	(-1)	-.10013 74582 33428 10770 34834 28761 45952 82889

M = 11	PRECISION 100.2 BITS			PRECISION 30.15 DIGITS
(1)	.60000 00000 00000 00000 00000 00016 20606 07726.	Q00	(1)	.15000 00000 00000 00000 00000 02304 13512 43322
(0)	-.71463 14631 46314 63146 31465 14633 31370 34465	Q01	(0)	-.90000 00000 00000 00000 00529 41011 78343 98846
(-3)	-.64523 30401 35476 71620 33474 20723 45167 01000	Q02	(0)	-.10285 71428 57142 85710 04290 02504 35364 18644
(-4)	-.65652 44331 53051 43401 63775 11667 32600 00000	Q03	(-1)	-.52571 42857 14285 88481 51416 11171 99419 90715
(-4)	-.42212 02541 42672 67237 37412 05302 00000 00000	Q04	(-1)	-.33466 41929 49867 C7630 67857 31132 93985 16227
(-5)	-.60577 30410 47155 42433 27046 54440 00000 00000	Q05	(-1)	-.23803 12259 22959 24329 05589 19359 06454 72069
(-5)	-.45052 71001 64244 52015 20463 10000 00000 00000	Q06	(-1)	-.18107 30999 03740 49006 55085 40258 13260 89609
(-6)	-.73015 43674 02311 34472 62314 00000 00000 00000	Q07	(-1)	-.14410 76298 99721 02191 12802 45244 03720 19334
(-6)	-.60414 37501 35122 32661 11000 00000 00000 00000	Q08	(-1)	-.11846 77800 41921 68628 88432 40881 03296 99022
(-6)	-.50716 27621 15160 30070 50000 00000 00000 00000	Q09	(-2)	-.99861 00968 76734 26462 85217 23500 26513 92505
(-6)	-.42467 67566 53670 35362 00000 00000 00000 00000	Q10	(-2)	-.84494 92840 52296 98987 77241 87056 88437 51243
(-6)	-.44176 47123 72075 52060 00000 00000 00000 00000	Q11	(-2)	-.88494 35775 60089 24353 19155 81542 07996 26735

EXP(Y) $|Y| < \ln(2)/2$, $EXP(\ln(2)/2, N, 0) = 1 + 2Y/(2 - Y + Y^2 P(Y^2))$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS

N = 2

	PRECISION 30.5 BITS			PRECISION 9.19 DIGITS
(-2)	.52525 20554 06430 47441	P00	(0)	.16666 61156 57965 13756
(-10)	-.55234 61067 44521 41037	P01	(-2)	-.27652 70152 20490 35285

N = 3

	PRECISION 41.2 BITS			PRECISION 12.40 DIGITS
(-2)	.52525 25247 21467 07671	P00	(0)	.16666 66658 77010 60714
(-10)	-.55405 32224 14032 40401	P01	(-2)	-.27777 44623 70256 13492
(-15)	.42351 07317 05006 57021	P02	(-4)	.65718 27715 08735 79204

N = 4

	PRECISION 51.7 BITS			PRECISION 15.56 DIGITS
(-2)	.52525 25252 52306 31444	P00	(0)	.16666 66666 65651 06099
	14776 24730 34376 23677			77469 34173 57617 61704
(-10)	-.55405 54033 03044 30247	P01	(-2)	-.27777 77710 60931 71199
	33776 33534 43022 04300			70277 09543 79616 87108
(-15)	.42531 21327 32175 34146	P02	(-4)	.66136 09268 81204 86908
	64714 10507 50437 54000			55757 66556 02356 91215
(-23)	-.67023 10606 47073 06361	P03	(-5)	-.16402 41646 70305 09449
	40043 02172 72300 00000			06937 30159 20202 10605

N = 5

	PRECISION 62.2 BITS			PRECISION 18.71 DIGITS
(-2)	.52525 25252 52525 12352	P00	(0)	.16666 66666 66665 45923
	26233 02226 70663 75446			01319 37728 94278 11176
(-10)	-.55405 54055 36541 33223	P01	(-2)	-.27777 77777 66290 71642
	61217 06355 60461 01010			78025 35416 94085 23060
(-15)	.42531 52526 42607 15125	P02	(-4)	.66137 56233 30046 26857
	40346 13453 32232 20000			99105 97618 80329 28238
(-23)	-.67364 70603 50255 04375	P03	(-5)	-.16533 82020 85912 95473
	23436 51317 60000 00000			49479 90741 05811 75560
(-30)	.54316 70735 34507 36550	P04	(-7)	.41354 52321 95996 41995
	35201 24714 00000 00000			68233 63501 83156 98492

N = 6

	PRECISION 72.6 BITS			PRECISION 21.86 DIGITS
(-2)	.52525 25252 52525 25244	P00	(0)	.16666 66666 66666 66531
	33064 46401 56234 60566			28914 16612 37674 40640
(-10)	-.55405 54055 40552 41021	P01	(-2)	-.27777 77777 77760 31696
	36231 41761 14020 77300			00754 10376 80530 96780
(-15)	.42531 52567 74001 00430	P02	(-4)	.66137 56612 95017 81105
	44305 24443 60201 00000			99188 72335 15091 16143
(-23)	-.67365 67302 03106 27240	P03	(-5)	-.16534 38975 15594 72389
	51667 66641 37000 00000			60384 39662 32257 57352
(-30)	.54651 07206 07130 56111	P04	(-7)	.41751 47017 62701 52876
	33270 22544 00000 00000			93864 04720 03594 44883
(-35)	-.43721 54432 43311 75335	P05	(-8)	-.10451 05825 75781 09520
	16235 02000 00000 00000			97045 88539 27221 58244

EXP(Y) $|Y| < \ln(2)/2$, $\mathrm{EXP}(\ln(2)/2, N, 0) = 1 + 2Y/(2 - Y + Y^2P(Y^2))$

N = 7

BINARY COEFFICIENTS PRECISION 83.0 BITS			DECIMAL COEFFICIENTS PRECISION 24.99 DIGITS	
(-2)	.52525 25252 52525 25252	P00	(0)	.16666 66666 66666 66666
	52172 05014 10073 51705			52157 59855 38169 85653
(-10)	-.55405 54055 40554 05430	P01	(-2)	-.27777 77777 77777 75345
	72324 00436 77050 14240			15366 43305 67611 19193
(-15)	.42531 52570 00421 07274	P02	(-4)	.66137 56613 75512 67793
	65620 57445 44272 00000			18718 11873 89822 57076
(-23)	-.67365 67446 12215 04577	P03	(-5)	-.16534 39152 98975 34018
	77642 75643 60000 00000			71344 13425 37294 03655
(-30)	.54652 16721 44407 46551	P04	(-7)	.41753 50651 72805 24185
	53516 75040 00000 00000			18657 86176 04023 84655
(-35)	-.44236 67656 61366 05143	P05	(-8)	-.10567 68753 70777 42622
	11214 70000 00000 00000			81469 68199 94650 89348
(-43)	.72072 10575 53034 62337	P06	(-10)	.26426 98206 73207 02910
	64100 00000 00000 00000			93917 17232 53341 52339

N = 8

PRECISION 93.4 BITS			PRECISION 28.12 DIGITS	
(-2)	.52525 25252 52525 25252	P00	(0)	.16666 66666 66666 66666
	52525 06644 64755 43604			66651 66222 60866 15950
(-10)	-.55405 54055 40554 05540	P01	(-2)	-.27777 77777 77777 77774
	47415 05475 67105 27100			60905 45465 80175 65627
(-15)	.42531 52570 00425 31154	P02	(-4)	.66137 56613 75661 12868
	47306 73340 53530 00000			80760 47540 65509 54423
(-23)	-.67365 67446 32335 07052	P03	(-5)	-.16534 39153 43818 13963
	30716 41036 00000 00000			67085 14204 51518 66985
(-30)	.54652 17127 42346 30420	P04	(-7)	.41753 51395 39837 86328
	27305 32400 00000 00000			76035 61981 66706 50265
(-35)	-.44240 05405 72715 75547	P05	(-8)	-.10568 37738 71161 76690
	54077 00000 00000 00000			53803 78100 27858 38290
(-43)	.72664 20016 71455 77156	P06	(-10)	.26762 81459 08700 96309
	42000 00000 00000 00000			31305 68141 91869 16344
(-50)	-.57017 41717 57634 33175	P07	(-12)	-.66834 12095 26360 77066
	00000 00000 00000 00000			83655 31120 20326 70707

N = 9

PRECISION 103.8 BITS			PRECISION 31.25 DIGITS	
(-2)	.52525 25252 52525 25252	P00	(0)	.16666 66666 66666 66666
	52525 25243 14757 42424			66666 65159 31498 58437
(-10)	-.55405 54055 40554 05540	P01	(-2)	-.27777 77777 77777 77777
	55401 71733 45227 32200			77386 54717 10376 64194
(-15)	.42531 52570 00425 31525	P02	(-4)	.66137 56613 75661 37528
	41120 52547 73640 00000			33390 26775 59357 16741
(-23)	-.67365 67446 32357 00352	P03	(-5)	-.16534 39153 43915 15673
	61430 05712 00000 00000			66071 30936 21175 42212
(-30)	.54652 17127 73356 72300	P04	(-7)	.41753 51397 56822 78149
	40550 21000 00000 00000			75698 51772 50398 57503
(-35)	-.44240 05653 76626 51075	P05	(-8)	-.10568 38026 78120 61973
	41160 00000 00000 00000			63665 12669 40045 45814
(-43)	.72666 62002 67616 22773	P06	(-10)	.26765 06246 64669 42490
	50000 00000 00000 00000			50674 91072 91234 36363
(-50)	-.57546 51214 56716 26000	P07	(-12)	-.67786 45678 05886 94029
	00000 00000 00000 00000			24126 21982 22362 20566
(-55)	.46037 73171 10733 52000	P08	(-13)	.16903 08099 38185 14298
	00000 00000 00000 00000			28574 31445 48276 80538

SINH(Y) $|Y| < \ln((1+\sqrt{5})/2)$, $\mathrm{SINH}(\ln((1+\sqrt{5})/2, 0, M) = Y + Y^3/Q(Y^2)$

M	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
M = 1	PRECISION 22.9 BITS			PRECISION 6.89 DIGITS	
	(3)	.57777 11217 34261 01451	Q00	(1)	.59997 91260 38329 57553
	(-1)	-.46032 75243 52416 32765	Q01	(0)	-.29728 63482 61173 16774
M = 2	PRECISION 33.0 BITS			PRECISION 9.94 DIGITS	
	(3)	.57777 77721 67356 72537	Q00	(1)	.59999 99656 27987 80758
	(-1)	-.46314 16643 66123 12277	Q01	(0)	-.29999 13263 35020 31956
	(-7)	.77666 23672 10731 67261	Q02	(-2)	.77949 31022 65013 44420
M = 3	PRECISION 43.8 BITS			PRECISION 13.18 DIGITS	
	(3)	.57777 77777 75176 64110	Q00	(1)	.59999 99999 67958 83639
	(-1)	-.46314 63112 14732 15504	Q01	(0)	-.29999 99868 69269 90423
	(-6)	.40135 21331 17225 35003	Q02	(-2)	.78569 75676 82561 85834
	(-14)	-.43114 13114 31137 25042	Q03	(-3)	-.13408 19463 60659 96193
M = 4	PRECISION 57.0 BITS			PRECISION 17.15 DIGITS	
	(3)	.57777 77777 77777 62201	Q00	(1)	.59999 99999 99995 11608
		41762 16034 34043 73304			09179 67404 91997 36610
	(-1)	-.46314 63146 31145 02347	Q01	(0)	-.29999 99999 97070 26644
		76124 46002 34305 77734			84638 66340 85578 11207
	(-6)	.40135 47665 67405 05077	Q02	(-2)	.78571 42800 16462 04321
		00415 72527 07032 13260			50109 42384 44521 33290
	(-14)	-.43274 53223 46364 46413	Q03	(-3)	-.13492 01528 68565 91459
		53666 26131 43614 46000			94191 14046 14594 10241
	(-23)	.60473 65216 57055 02053	Q04	(-5)	.14488 95344 56593 56675
		10354 07610 42100 00000			53337 41202 49813 24478
M = 5	PRECISION 64.4 BITS			PRECISION 19.39 DIGITS	
	(3)	.60000 00000 00000 00067	Q00	(1)	.60000 00000 00000 03884
		76711 17530 25656 50471			58043 55462 15272 97226
	(-1)	-.46314 63146 31465 36127	Q01	(0)	-.30000 00000 00032 27888
		50746 22071 66540 43665			43571 62286 08335 52246
	(-6)	.40135 47671 66017 25030	Q02	(-2)	.78571 42858 03577 22224
		24002 52045 64651 47560			29701 49000 57943 75389
	(-14)	-.43274 57415 50706 63101	Q03	(-3)	-.13492 06462 78005 58812
		35245 74224 75466 40000			99876 10868 01326 39100
	(-23)	.60534 50660 42621 75005	Q04	(-5)	.14508 04888 74696 66981
		30724 57254 22300 00000			67917 91665 04235 18385
	(-34)	-.57165 30622 23242 37567	Q05	(-8)	-.27491 05451 27541 44867
		56564 67366 00000 00000			82632 64055 59002 74117
M = 6	PRECISION 73.8 BITS			PRECISION 22.22 DIGITS	
	(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00007
		06732 71722 16200 21131			51086 31781 39521 32008
	(-1)	-.46314 63146 31463 15126	Q01	(0)	-.30000 00000 00000 08212
		76026 14726 35557 14263			58978 99482 23340 49776
	(-6)	.40135 47671 62073 40020	Q02	(-2)	.78571 42857 14589 36741
		75570 54567 15770 61040			39689 40526 38552 83986
	(-14)	-.43274 57333 73033 44727	Q03	(-3)	-.13492 06349 73858 92777
		11350 64436 45144 00000			34689 33819 91230 99079
	(-23)	.60533 31024 44004 35322	Q04	(-5)	.14507 32306 84047 17765
		47641 21631 30000 00000			16233 52404 72886 83925
	(-34)	-.53225 17067 47743 03006	Q05	(-8)	-.25198 95596 43713 14936
		05244 07760 00000 00000			55658 02520 53579 12094
	(-37)	-.46711 10126 14555 76265	Q06	(-9)	-.28298 25636 83370 42599
		17602 17400 00000 00000			48101 64367 63332 68322

SINH(Y) |Y| < ln((1+√5)/2), SINH(ln((1+√5)/2, 0, M) = Y + $Y^3/Q(Y^2)$

BINARY COEFFICIENTS — DECIMAL COEFFICIENTS

M = 7 — PRECISION 84.1 BITS — PRECISION 25.31 DIGITS

Exp	Binary coefficient		Exp	Decimal coefficient
(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00000
	00003 53673 73602 06133			00780 41862 98227 74210
(-1)	-.46314 63146 31463 14631	Q01	(0)	-.30000 00000 00000 00010
	66326 75422 24576 37131			85547 03537 71061 69687
(-6)	.40135 47671 62064 53125	Q02	(-2)	.78571 42857 14286 23028
	17515 61261 02760 64140			69422 24907 49535 20244
(-14)	-.43274 57333 54066 21767	Q03	(-3)	-.13492 06349 20753 51949
	02254 64474 67467 00000			54539 78531 32705 10638
(-23)	.60533 30374 41145 57353	Q04	(-5)	.14507 31809 38256 36851
	04663 51052 14000 00000			84050 91499 36968 50352
(-34)	-.53176 55537 71742 42024	Q05	(-8)	-.25173 35117 80674 15178
	20205 26400 00000 00000			60681 38405 73631 24698
(-37)	-.47651 54323 10377 14756	Q06	(-9)	-.28981 18217 33982 20768
	25340 40000 00000 00000			91627 79620 28173 98355
(-44)	.40342 61532 76736 65633	Q07	(-11)	.73766 66483 91228 20564
	43370 00000 00000 00000			33376 50940 11111 96260

M = 8 — PRECISION 95.4 BITS — PRECISION 28.72 DIGITS

Exp	Binary coefficient		Exp	Decimal coefficient
(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00000
	00000 00072 02730 65112			00000 37511 14128 08659
(-1)	-.46314 63146 31463 14631	Q01	(0)	-.30000 00000 00000 00000
	46321 53574 33251 53013			00646 32382 01155 54757
(-6)	.40135 47671 62064 52330	Q02	(-2)	.78571 42857 14285 71466
	62223 30220 32030 60200			91024 30768 71696 79533
(-14)	-.43274 57333 54045 15102	Q03	(-3)	-.13492 06349 20635 03210
	11067 64135 07044 00000			22739 92159 08243 51947
(-23)	.60533 30373 52737 56540	Q04	(-5)	.14507 31807 87649 77954
	77611 12647 30000 00000			34722 71772 77664 01564
(-34)	-.53176 47321 04126 36577	Q05	(-8)	-.25173 23964 08916 86062
	14174 45000 00000 00000			37549 14966 51289 64450
(-37)	-.47655 04131 73372 24501	Q06	(-9)	-.28985 97308 05104 84988
	65607 00000 00000 00000			09252 24954 85857 57803
(-44)	.40733 70437 14267 30477	Q07	(-11)	.74872 92459 11306 32400
	43400 00000 00000 00000			60665 31378 34851 48385
(-53)	-.73624 34625 10371 55742	Q08	(-12)	-.10620 82708 71330 24371
	60000 00000 00000 00000			91971 99783 24581 42458

M = 9 — PRECISION 105.9 BITS — PRECISION 31.88 DIGITS

Exp	Binary coefficient		Exp	Decimal coefficient
(3)	.57777 77777 77777 77777	Q00	(1)	.59999 99999 99999 99999
	77777 77777 74702 23647			99999 99968 49111 66828
(-1)	-.46314 63146 31463 14631	Q01	(0)	-.29999 99999 99999 99999
	46314 62703 06216 17757			99999 34042 88426 42573
(-6)	.40135 47671 62064 52330	Q02	(-2)	.78571 42857 14285 71428
	40122 04170 10553 74600			52363 12182 23249 47222
(-14)	-.43274 57333 54045 14062	Q03	(-3)	-.13492 06349 20634 92046
	74257 12526 00570 00000			34410 89547 36455 43324
(-23)	.60533 30373 52675 66602	Q04	(-5)	.14507 31807 87466 14829
	13357 16133 40000 00000			01416 18382 72405 54854
(-34)	-.53176 47313 63561 24466	Q05	(-8)	-.25173 23945 85030 90132
	72554 74000 00000 00000			24055 47149 45616 71963
(-37)	-.47655 04533 15200 16022	Q06	(-9)	-.28985 98423 81444 90598
	45160 00000 00000 00000			36394 21440 10884 46424
(-44)	.40734 63643 70536 75047	Q07	(-11)	.74877 03652 17364 59467
	77000 00000 00000 00000			85123 62368 53606 41279
(-53)	-.74205 60022 51423 23164	Q08	(-12)	-.10704 54508 68381 30894
	00000 00000 00000 00000			69326 84331 31455 45193
(-62)	.64075 21101 04655 24000	Q09	(-15)	.72330 56279 28719 78243
	00000 00000 00000 00000			73933 87025 38999 60234

SINH(Y) $|Y| < \ln(1+\sqrt{2})$, $\mathrm{SINH}(\ln(1+\sqrt{2}), 0, M) = Y + Y^3/Q(Y^2)$

M	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 1	PRECISION 17.7 BITS		PRECISION 5.33 DIGITS
	(3) .57766 45071 12461 51574	Q00	(1) .59977 00163 46557 26393
	(-1) -.45200 03774 06637 43646	Q01	(0) -.29101 65768 61232 46210
M = 2	PRECISION 26.1 BITS		PRECISION 7.85 DIGITS
	(3) .57777 74527 64253 24055	Q00	(1) .59999 87302 85263 12460
	(-1) -.46306 40173 75671 33660	Q01	(0) -.29990 39259 58367 31104
	(-7) .76530 65010 62654 02624	Q02	(-2) .76505 72870 58380 35529
M = 3	PRECISION 35.1 BITS		PRECISION 10.56 DIGITS
	(3) .57777 77772 50036 54155	Q00	(1) .59999 99959 96010 75816
	(-1) -.46314 61107 43051 17006	Q01	(0) -.29999 95087 96321 87025
	(-6) .40131 54671 26456 37703	Q02	(-2) .78552 72755 15751 55082
	(-14) -.42504 30247 24242 45311	Q03	(-3) -.13211 65271 55963 39442
M = 4	PRECISION 46.3 BITS		PRECISION 13.93 DIGITS
	(3) .57777 77777 77625 26565	Q00	(1) .59999 99999 97575 17657
	12305 64112 25351 06664		62025 01713 78151 53128
	(-1) -.46314 63145 36164 61152	Q01	(0) -.29999 99995 68012 31169
	35471 73727 64230 34760		81112 44262 94321 15060
	(-6) .40135 47416 70004 21636	Q02	(-2) .78571 40370 10810 80743
	42646 42615 06106 50757		24407 30242 72627 11294
	(-14) -.43273 72201 00222 36705	Q03	(-3) -.13491 44442 89386 27731
	05327 53524 01611 21540		64555 68551 21768 27215
	(-23) .60346 03101 21016 34472	Q04	(-5) .14439 02081 03940 03484
	76234 56364 41503 04000		52057 59912 16228 30287
M = 5	PRECISION 52.3 BITS		PRECISION 15.74 DIGITS
	(3) .60000 00000 00002 21544	Q00	(1) .60000 00000 00051 76195
	65561 67270 23311 57512		54249 33793 06307 26021
	(-1) -.46314 63146 33276 36172	Q01	(0) -.30000 00000 12893 11733
	05171 41443 17512 44452		84486 26631 64777 75043
	(-6) .40135 47701 10105 23611	Q02	(-2) .78571 42964 01587 69216
	36620 51421 04153 51611		84645 18014 86803 93330
	(-14) -.43274 62733 67445 75631	Q03	(-3) -.13492 10424 15122 91066
	77365 12430 31136 73600		37281 50813 76409 66763
	(-23) .60550 72074 01130 71712	Q04	(-5) .14515 19089 84491 15644
	07511 21522 23157 40000		01959 12573 29710 11896
	(-34) -.70140 56505 01306 05101	Q05	(-8) -.32706 25787 63550 43983
	57664 34376 10000 00000		05669 53500 26375 97059
M = 6	PRECISION 59.9 BITS		PRECISION 18.04 DIGITS
	(3) .60000 00000 00000 00755	Q00	(1) .60000 00000 00000 34252
	50141 53022 72403 21554		32186 26247 17941 41493
	(-1) -.46314 63146 31473 05541	Q01	(0) -.30000 00000 00112 10805
	12414 01334 64456 10344		04598 37970 06951 35446
	(-6) .40135 47671 67435 16725	Q02	(-2) .78571 42858 38300 74413
	13311 50545 74027 65447		26179 01119 06515 19245
	(-14) -.43274 57370 22261 10362	Q03	(-3) -.13492 06414 23072 00467
	30174 02063 17122 13000		21728 23123 97097 43537
	(-23) .60533 54427 61042 60714	Q04	(-5) .14507 50047 67869 21594
	05761 77076 41000 00000		21027 82202 87050 45206
	(-34) -.53566 21502 64302 21011	Q05	(-8) -.25454 79614 65466 55925
	65640 53464 24000 00000		57911 56224 81548 07963
	(-37) -.44565 40145 72006 06376	Q06	(-9) -.26724 22725 41603 08985
	61010 43734 70000 00000		72089 64483 07916 67465

SINH(Y) $|Y| < \ln(1+\sqrt{2})$, $\mathrm{SINH}(\ln(1+\sqrt{2}), 0, M) = Y + Y^3/Q(Y^2)$

M = 7

BINARY COEFFICIENTS PRECISION 68.4 BITS			DECIMAL COEFFICIENTS PRECISION 20.59 DIGITS	
(3)	.60000 00000 00000 00001 57650 76121 34141 43436	Q00	(1)	.60000 00000 00000 00121 24635 27408 67165 23187
(-1)	-.46314 63146 31463 17044 55117 01166 71646 70717	Q01	(0)	-.30000 00000 00000 50441 66197 90985 86345 70809
(-6)	.40135 47671 62104 63306 06455 07472 64475 01556	Q02	(-2)	.78571 42857 15002 47708 94494 44374 87943 08584
(-14)	-.43274 57333 71734 32641 63136 72106 41316 22000	Q03	(-3)	-.13492 06349 69867 96827 96801 06111 27624 83063
(-23)	.60533 30544 75652 60475 51424 64030 51000 00000	Q04	(-5)	.14507 31994 91845 03689 89574 76495 96725 13288
(-34)	-.53202 20417 10261 03067 00311 27113 00000 00000	Q05	(-8)	-.25177 38070 56488 95918 15505 90900 94235 51325
(-37)	-.47607 47717 64354 35212 07212 44116 00000 00000	Q06	(-9)	-.28932 76520 36127 20711 33113 18786 31856 57815
(-45)	.76476 14357 70673 33450 32771 73100 00000 00000	Q07	(-11)	.71192 37382 01770 23894 14449 17423 43404 42979

M = 8

PRECISION 77.9 BITS			PRECISION 23.46 DIGITS	
(3)	.60000 00000 00000 00000 00141 07515 13374 25445	Q00	(1)	.60000 00000 00000 00000 20565 86315 06114 81823
(-1)	-.46314 63146 31463 14634 02474 11740 20376 50735	Q01	(0)	-.30000 00000 00000 00105 85708 93050 16835 22193
(-6)	.40135 47671 62064 55077 71311 52334 47365 10610	Q02	(-2)	.78571 42857 14287 58913 40839 96925 62173 49803
(-14)	-.43274 57333 54074 50442 02740 07130 67526 50000	Q03	(-3)	-.13492 06349 20797 60142 13555 11815 55848 72669
(-23)	.60533 30374 07475 14000 02125 55343 10540 00000	Q04	(-5)	.14507 31808 67259 80676 43134 97901 07343 97950
(-34)	-.53176 50564 42300 02170 61053 71620 00000 00000	Q05	(-8)	-.25173 26307 60334 04501 53434 76467 60138 11383
(-37)	-.47654 61234 55547 41074 45751 53720 00000 00000	Q06	(-9)	-.28985 55409 20806 99261 35738 03118 26450 04003
(-44)	.40722 14567 25553 07511 00042 57000 00000 00000	Q07	(-11)	.74829 91539 49519 11382 78228 20002 96218 98372
(-53)	-.72522 17675 33747 72523 64660 40000 00000 00000	Q08	(-12)	-.10420 22300 69056 20040 32734 96365 14992 33416

M = 9

PRECISION 86.9 BITS			PRECISION 26.17 DIGITS	
(3)	.57777 77777 77777 77777 77777 61272 64625 71673	Q00	(1)	.59999 99999 99999 99999 99951 57649 07831 02259
(-1)	-.46314 63146 31463 14631 45747 26515 60062 61630	Q01	(0)	-.29999 99999 99999 99999 69633 06618 31144 19515
(-6)	.40135 47671 62064 52323 50727 11633 01271 65730	Q02	(-2)	.78571 42857 14285 70769 44715 84835 84041 58891
(-14)	-.43274 57333 54045 05442 72304 62507 34204 00000	Q03	(-3)	-.13492 06349 20634 21233 74194 85465 52332 98464
(-23)	.60533 30373 52555 16354 70413 62440 62600 00000	Q04	(-5)	.14507 31807 87029 05459 04065 30628 47337 92840
(-34)	-.53176 47307 00042 12654 07360 67234 00000 00000	Q05	(-8)	-.25173 23929 17231 99959 84524 54794 27098 11357
(-37)	-.47655 04670 57754 35430 61276 67700 00000 00000	Q06	(-9)	-.28985 98829 48979 39181 00400 68006 59573 94829
(-44)	.40734 74645 53165 26237 23205 30000 00000 00000	Q07	(-11)	.74877 66126 47436 12784 33184 47461 68221 96870
(-53)	-.74226 13307 03740 72712 41230 00000 00000 00000	Q08	(-12)	-.10710 24447 94464 52077 62759 12036 67389 15392
(-62)	.65672 03613 30600 17021 24400 00000 00000 00000	Q09	(-15)	.74750 47854 25786 05423 34922 34938 00226 25948

SINH(Y) $|Y| < \ln(1+\sqrt{2})$, $\text{SINH}(\ln(1+\sqrt{2}), 0, M) = Y + Y^3/Q(Y^2)$

M = 10

	BINARY COEFFICIENTS PRECISION 93.7 BITS			DECIMAL COEFFICIENTS PRECISION 28.21 DIGITS
(3)	.57777 77777 77777 77777	Q00	(1)	.59999 99999 99999 99999
	77777 77656 02375 04574			99999 47033 92785 41382
(-1)	-.46314 63146 31463 14631	Q01	(0)	-.29999 99999 99999 99999
	46311 64060 27763 25540			99604 80305 07080 55565
(-6)	.40135 47671 62064 52330	Q02	(-2)	.78571 42857 14285 71418
	33260 07426 23552 51020			32470 80999 92805 59869
(-14)	-.43274 57333 54045 13765	Q03	(-3)	-.13492 06349 20634 90738
	14640 45252 33714 20000			89300 84772 15232 15383
(-23)	.60533 30373 52674 05370	Q04	(-5)	.14507 31807 87456 55406
	51512 23563 61000 00000			56607 90955 58519 48831
(-34)	-.53176 47313 53705 66356	Q05	(-8)	-.25173 23945 42557 80039
	26422 54260 00000 00000			86695 84370 91546 03471
(-37)	-.47655 04535 55462 15613	Q06	(-9)	-.28985 98434 68002 83424
	30345 35400 00000 00000			70353 93081 96730 41984
(-44)	.40734 63763 13062 60333	Q07	(-11)	.74877 04726 72253 89390
	37741 40000 00000 00000			58749 09690 59889 02453
(-53)	-.74205 10151 22511 45501	Q08	(-12)	-.10704 32916 37644 74358
	33400 00000 00000 00000			62959 57695 34993 42168
(-62)	.63427 72411 42326 67703	Q09	(-15)	.71535 42719 54820 25021
	00000 00000 00000 00000			98389 80370 68099 95241
(-70)	.42602 36045 54202 62130	Q10	(-17)	.75362 51142 21962 88768
	00000 00000 00000 00000			53373 07589 17024 83750

TANH(Y) $|Y| < \ln(3)/2$ TANH(ln(3)/2, 0, M) = $Y - Y^3/(3 + Y^2Q(Y^2))$

M		BINARY COEFFICIENTS			DECIMAL COEFFICIENTS
M = 2	PRECISION 27.6 BITS			PRECISION 8.32 DIGITS	
	(1)	.46314 44234 53547 54725	Q00	(1)	.11999 85795 99494 52967
	(-7)	-.55635 13620 31514 45077	Q01	(-2)	-.55916 74827 16129 26206
M = 3	PRECISION 36.1 BITS			PRECISION 10.86 DIGITS	
	(1)	.46314 63054 77205 53154	Q00	(1)	.11999 99893 06266 31581
	(-7)	-.56630 41260 61242 76650	Q01	(-2)	-.57126 33376 02867 10955
	(-13)	.40177 25572 12644 24753	Q02	(-3)	.24603 81338 86521 50827
M = 4	PRECISION 44.4 BITS			PRECISION 13.36 DIGITS	
	(1)	.46314 63146 00403 51675	Q00	(1)	.11999 99999 26970 15730
	(-7)	-.56637 30164 71232 12502	Q01	(-2)	-.57142 68342 81977 62645
	(-13)	.41212 00106 55612 65645	Q02	(-3)	.25382 64119 33539 28896
	(-20)	-.61120 02603 43313 17315	Q03	(-4)	-.11719 78333 83368 87540
M = 5	PRECISION 52.6 BITS			PRECISION 15.84 DIGITS	
	(1)	.46314 63146 31342 05007	Q00	(1)	.11999 99999 99538 87683
		43546 50770 23253 07602			91328 38674 45058 40958
	(-7)	-.56637 34663 06221 61567	Q01	(-2)	-.57142 85559 21155 75823
		40236 67740 22342 56042			48317 32128 30235 56011
	(-13)	.41223 31200 11443 37451	Q02	(-3)	.25396 63692 68491 99616
		33060 05014 16275 41000			45173 96182 33907 77430
	(-20)	-.63110 26056 54061 07005	Q03	(-4)	-.12193 03588 14239 34141
		06537 75475 35110 00000			90906 03028 94351 98277
	(-24)	.46215 00266 46165 43756	Q04	(-6)	.57034 79276 39625 23565
		73604 41051 72140 00000			98986 23240 21305 69162
M = 6	PRECISION 60.8 BITS			PRECISION 18.30 DIGITS	
	(1)	.46314 63146 31462 55772	Q00	(1)	.11999 99999 99997 26337
		01426 51642 50104 74017			38189 22278 63599 79133
	(-7)	-.56637 34710 17402 36732	Q01	(-2)	-.57142 85713 05487 48093
		14457 63207 50520 61160			21729 03252 05490 25482
	(-13)	.41223 41203 22370 35106	Q02	(-3)	.25396 82333 41234 85146
		62731 03720 55432 70000			21994 94120 90823 77707
	(-20)	-.63136 64166 33513 52377	Q03	(-4)	-.12203 49972 80024 49603
		15522 13231 15360 00000			62281 44905 55824 92380
	(-24)	.50104 12267 55007 24564	Q04	(-6)	.59803 02188 59822 77651
		12170 76661 64000 00000			46071 33681 43327 21188
	(-31)	-.74000 65015 31040 11200	Q05	(-7)	-.27940 43078 54559 93405
		77632 70561 00000 00000			79036 82950 18028 74625
M = 7	PRECISION 69.0 BITS			PRECISION 20.76 DIGITS	
	(1)	.46314 63146 31463 14500	Q00	(1)	.11999 99999 99999 98454
		37717 64656 21244 25702			33587 96059 70849 36613
	(-7)	-.56637 34710 32201 34046	Q01	(-2)	-.57142 85714 27677 63797
		47513 04002 56240 03040			43726 14797 29939 12356
	(-13)	.41223 41260 15653 30442	Q02	(-3)	.25396 82537 71821 63747
		47466 66665 73047 20000			92420 07417 97514 61019
	(-20)	-.63137 13174 10172 10704	Q03	(-4)	-.12203 66715 59829 89536
		36056 05446 54360 00000			14441 13566 18737 75456
	(-24)	.50134 77007 76764 17044	Q04	(-6)	.59875 26563 30262 29006
		41240 47041 40000 00000			95337 88295 25264 16977
	(-31)	-.77306 04761 36525 17705	Q05	(-7)	-.29516 81170 25780 29355
		25375 73344 00000 00000			52403 79740 96182 44051
	(-35)	.57103 77342 31202 50541	Q06	(-8)	.13717 44888 50264 70435
		00221 67000 00000 00000			02361 93333 33068 89376

TANH(Y) |Y| < ln(3)/2   TANH(ln(3)/2, 0, M) = $Y - Y^3/(3 + Y^2 Q(Y^2))$

BINARY COEFFICIENTS / DECIMAL COEFFICIENTS

M = 8

PRECISION 77.1 BITS / PRECISION 23.21 DIGITS

Binary exp.	Binary coefficient		Decimal exp.	Decimal coefficient
(1)	.46314 63146 31463 14631	Q00	(1)	.11999 99999 99999 99991
	07354 07550 21560 65763			61401 40289 38617 36341
(-7)	-.56637 34710 32251 32634	Q01	(-2)	-.57142 85714 28565 37589
	43423 47445 26530 46300			34419 60096 94033 10179
(-13)	.41223 41260 51174 42357	Q02	(-3)	.25396 82539 66570 78066
	62501 10525 06247 20000			42318 32845 25348 39084
(-20)	-.63137 13424 44156 04152	Q03	(-4)	-.12203 66932 22449 19327
	22243 04104 02300 00000			56144 43601 01556 26261
(-24)	.50135 34463 53503 13410	Q04	(-6)	.59876 61101 88486 83884
	12640 03613 20000 00000			52957 71136 89065 03889
(-31)	-.77372 05045 22461 37252	Q05	(-7)	-.29564 10686 51256 94466
	30161 75740 00000 00000			52709 83121 42530 79775
(-35)	.62120 26005 34775 30733	Q06	(-8)	.14597 58545 73020 31021
	33136 02000 00000 00000			73560 16142 64896 07605
(-41)	-.45032 45237 64412 50025	Q07	(-10)	-.67397 04972 72292 18506
	45553 60000 00000 00000			68617 19432 94533 35409

M = 9

PRECISION 85.2 BITS / PRECISION 25.66 DIGITS

Binary exp.	Binary coefficient		Decimal exp.	Decimal coefficient
(1)	.46314 63146 31463 14631	Q00	(1)	.11999 99999 99999 99999
	46171 52415 27650 43211			95598 84171 89554 57142
(-7)	-.56637 34710 32251 54107	Q01	(-2)	-.57142 85714 28571 38982
	43556 17475 41523 45200			92415 61959 04873 33086
(-13)	.41223 41260 51364 67465	Q02	(-3)	.25396 82539 68240 69817
	02422 52152 60210 40000			82091 18504 95489 83365
(-20)	-.63137 13426 20373 31325	Q03	(-4)	-.12203 66934 62870 29377
	11475 34161 34000 00000			86255 12438 77739 46003
(-24)	.50135 35025 14677 21044	Q04	(-6)	.59876 63104 89961 64847
	32401 46537 00000 00000			05124 84700 84317 04975
(-31)	-.77373 13521 41234 54260	Q05	(-7)	-.29565 10995 80076 20631
	33444 65200 00000 00000			25007 21710 76377 21388
(-35)	.62204 61207 03762 33542	Q06	(-8)	.14627 38609 16424 83720
	60706 40000 00000 00000			80390 11508 83005 02767
(-41)	-.47555 07417 56146 26766	Q07	(-10)	-.72237 74526 92537 87496
	30330 00000 00000 00000			78393 61557 94330 38998
(-46)	.72212 34211 51470 35476	Q08	(-11)	.33122 88410 30090 42978
	62600 00000 00000 00000			43927 27129 89843 30519

M = 10

PRECISION 93.4 BITS / PRECISION 28.10 DIGITS

Binary exp.	Binary coefficient		Decimal exp.	Decimal coefficient
(1)	.46314 63146 31463 14631	Q00	(1)	.11999 99999 99999 99999
	46314 30026 25351 50526			99977 53677 18891 48004
(-7)	-.56637 34710 32251 54200	Q01	(-2)	-.57142 85714 28571 42833
	30350 50406 24414 22700			47139 14551 85205 20434
(-13)	.41223 41260 51365 64263	Q02	(-3)	.25396 82539 68253 87047
	17550 60120 46354 00000			47113 79157 88734 00092
(-20)	-.63137 13426 21436 45714	Q03	(-4)	-.12203 66934 65243 37399
	24754 24112 23000 00000			72847 44885 21735 33353
(-24)	.50135 35030 03335 32171	Q04	(-6)	.59876 63130 23474 95711
	44365 16652 00000 00000			35643 21797 47805 95921
(-31)	-.77373 14661 25707 16041	Q05	(-7)	-.29565 12682 84462 86623
	62421 01000 00000 00000			04470 69605 07059 52466
(-35)	.62206 01014 70145 33604	Q06	(-8)	.14628 09451 48238 77645
	17512 00000 00000 00000			82037 77526 24438 82970
(-41)	-.47640 35573 60056 02766	Q07	(-10)	-.72420 16662 47849 72927
	37500 00000 00000 00000			22758 54077 00608 07923
(-46)	.76714 24455 12670 21105	Q08	(-11)	.35753 97943 37496 03623
	20000 00000 00000 00000			19861 05567 25127 45577
(-52)	-.55646 33751 66007 12713	Q09	(-12)	-.16280 33598 47832 25391
	00000 00000 00000 00000			15090 88434 06512 61614

TANH(Y) $|Y| < \ln(3)/2$ $\text{TANH}(\ln(3)/2, 0, M) = Y - Y^3/(3 + Y^2Q(Y^2))$

M = 11

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS
	PRECISION 101.5 BITS			PRECISION 30.54 DIGITS
(1)	.46314 63146 31463 14631	Q00	(1)	.11999 99999 99999 99999
	46314 63041 06226 14160			99999 88803 71191 86498
(-7)	-.56637 34710 32251 54200	Q01	(-2)	-.57142 85714 28571 42857
	56534 14152 36616 26200			00376 53186 50999 29293
(-13)	.41223 41260 51365 64630	Q02	(-3)	.25396 82539 68253 96757
	40601 63402 03214 00000			28041 96346 20251 62193
(-20)	-.63137 13426 21443 40651	Q03	(-4)	-.12203 66934 65264 71458
	16014 03356 74000 00000			09966 56650 53304 54787
(-24)	.50135 35030 05360 13711	Q04	(-6)	.59876 63130 51739 45359
	11362 70740 00000 00000			76793 16590 36480 82919
(-31)	-.77373 14671 75612 05774	Q05	(-7)	-.29565 12706 77863 50465
	72702 00000 00000 00000			88637 39387 46577 04003
(-35)	.62206 02411 76764 02251	Q06	(-8)	.14628 10778 72957 90633
	11020 00000 00000 00000			38605 94294 55910 82908
(-41)	-.47641 64165 63323 66532	Q07	(-10)	-.72424 96769 44759 82743
	64400 00000 00000 00000			72424 21897 75329 58876
(-46)	.77056 51307 32007 46505	Q08	(-11)	.35863 14213 42213 64605
	00000 00000 00000 00000			44842 45909 50433 26429
(-52)	-.61637 72202 76245 50620	Q09	(-12)	-.17696 89273 38728 44428
	00000 00000 00000 00000			10727 57429 35003 80594
(-56)	.44024 23013 64257 34000	Q10	(-14)	.80024 08300 09698 81946
	00000 00000 00000 00000			46548 73278 65149 97140

SIN(Y) $|Y| < \pi/4$, $SIN(\pi/4, N, 0) = Y + Y^3 P(Y^2)$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
N = 2	PRECISION 18.7 BITS		PRECISION 5.63 DIGITS
	(-2) -.52520 27557 62441 17160	P00	(0) -.16662 88040 08959 98749
	(-6) .41305 03473 37173 60520	P01	(-2) .81506 04487 86464 78757
N = 3	PRECISION 27.7 BITS		PRECISION 8.35 DIGITS
	(-2) -.52525 24143 42641 53447	P00	(0) -.16666 65308 97840 06806
	(-6) .42101 46611 32222 52235	P01	(-2) .83320 64512 22541 71326
	(-14) -.63077 33724 37067 01114	P02	(-3) -.19502 21968 59566 52365
N = 4	PRECISION 37.3 BITS		PRECISION 11.23 DIGITS
	(-2) -.52525 25251 34043 41040	P00	(0) -.16666 66663 80731 36129
	(-6) .42104 20367 60337 31400	P01	(-2) .83333 28991 38356 99561
	(-14) -.64003 57264 26127 36155	P02	(-3) -.19839 21220 80444 87767
	(-22) .55454 26661 23221 44165	P03	(-5) .27171 75168 60305 36788
N = 5	PRECISION 47.4 BITS		PRECISION 14.26 DIGITS
	(-2) -.52525 25252 52435 47116	P00	(0) -.16666 66666 66270 74116
	43506 65602 12352 35767		29525 30156 11326 70418
	(-6) .42104 21041 41502 76470	P01	(-2) .83333 33324 48792 51968
	02752 02100 65315 41012		19317 15269 78167 10240
	(-14) -.64006 37430 72416 65171	P02	(-3) -.19841 26340 43623 96882
	55335 53766 54537 53420		20446 64424 48486 35135
	(-22) .56165 63004 20257 77337	P03	(-5) .27555 27035 89941 51021
	76376 74525 33745 62000		19436 71031 95220 20689
	(-31) -.65122 66630 23132 74707	P04	(-7) -.24755 40551 16020 08202
	21517 77110 51020 00000		50538 47182 97411 26064
N = 6	PRECISION 57.8 BITS		PRECISION 17.40 DIGITS
	(-2) -.52525 25252 52525 21656	P00	(0) -.16666 66666 66666 27895
	52301 60320 65601 01506		49932 03448 60368 74392
	(-6) .42104 21042 10366 12501	P01	(-2) .83333 33333 32138 70648
	40022 14237 51421 13432		04102 24234 23306 76192
	(-14) -.64006 40063 36167 02327	P02	(-3) -.19841 26982 89844 85367
	31241 75562 16304 42620		84476 65303 32184 43492
	(-22) .56167 43246 71317 07103	P03	(-5) .27557 31340 53155 94654
	32145 63152 37365 10000		31989 68832 20509 13618
	(-31) -.65627 43157 34474 07212	P04	(-7) -.25050 71304 41265 10840
	72466 14275 46340 00000		11914 93010 12661 70306
	(-40) .53541 03442 44645 65604	P05	(-9) .15894 17006 37574 44945
	13332 21021 00000 00000		50181 39368 85205 31005
N = 7	PRECISION 68.6 BITS		PRECISION 20.66 DIGITS
	(-2) -.52525 25252 52525 25251	P00	(0) -.16666 66666 66666 66638
	27202 22652 33155 47263		40551 88761 54829 09154
	(-6) .42104 21042 10421 02471	P01	(-2) .83333 33333 33332 18598
	74141 65046 55702 56254		60773 12148 98823 48578
	(-14) -.64006 40064 00611 26441	P02	(-3) -.19841 26984 12540 55975
	27131 33773 22435 35100		47036 32604 01497 82475
	(-22) .56167 43512 30640 63057	P03	(-5) .27557 31921 36746 90955
	14642 20755 27006 40000		20661 98377 45163 38756
	(-31) -.65631 05115 76755 76207	P04	(-7) -.25052 10477 90948 31925
	75154 66036 27400 00000		01098 73486 43444 01662
	(-40) .54110 07446 53345 06010	P05	(-9) .16058 34993 36193 30562
	20701 30050 40000 00000		83530 47147 92982 06387
	(-50) -.65246 13475 77310 22433	P06	(-12) -.75778 77652 95879 24241
	24061 22400 00000 00000		82195 50034 47205 85217

SIN(Y) $|Y| < \pi/4$, $SIN(\pi/4, N, 0) = Y + Y^3 P(Y^2)$

N = 8

BINARY COEFFICIENTS PRECISION 79.8 BITS			DECIMAL COEFFICIENTS PRECISION 24.01 DIGITS	
(-2)	-.52525 25252 52525 25252	P00	(0)	-.16666 66666 66666 66666
	52475 20725 47063 70361			65073 91372 49255 88370
(-6)	.42104 21042 10421 04207	P01	(-2)	.83333 33333 33333 33250
	73177 04335 70020 51614			99082 28478 26491 52539
(-14)	-.64006 40064 00640 05120	P02	(-3)	-.19841 26984 12698 26695
	44402 00502 32364 26300			74315 45453 53840 32139
(-22)	.56167 43512 53273 53470	P03	(-5)	.27557 31922 39734 00628
	01216 12562 40561 00000			52370 48896 57994 59411
(-31)	-.65631 05317 54757 04207	P04	(-7)	-.25052 10837 95143 32751
	56062 77253 75000 00000			05607 98491 10627 67014
(-40)	.54111 05733 52167 46133	P05	(-9)	.16059 04219 62951 90032
	20074 63474 00000 00000			78722 99497 10424 84129
(-50)	-.65636 67077 30526 32751	P06	(-12)	-.76469 00143 47935 66062
	20434 20000 00000 00000			56801 98412 12862 59300
(-60)	.62170 70042 21520 42770	P07	(-14)	.27886 62968 73223 51196
	05734 00000 00000 00000			63211 46934 46447 72480

N = 9

PRECISION 91.2 BITS			PRECISION 27.45 DIGITS	
(-2)	-.52525 25252 52525 25252	P00	(0)	-.16666 66666 66666 66666
	52525 24510 61437 24077			66665 95198 69118 01941
(-6)	.42104 21042 10421 04210	P01	(-2)	.83333 33333 33333 33333
	42071 14022 26357 26141			28751 32902 12468 05397
(-14)	-.64006 40064 00640 06400	P02	(-3)	-.19841 26984 12698 41259
	25313 50045 71427 46000			71161 90127 79216 11746
(-22)	.56167 43512 53307 14173	P03	(-5)	.27557 31922 39858 79662
	53157 44142 13256 00000			36339 32702 93533 98298
(-31)	-.65631 05317 72474 72657	P04	(-7)	-.25052 10838 54349 68355
	60217 55641 70000 00000			76723 85346 32279 89154
(-40)	.54111 06047 14642 44073	P05	(-9)	.16059 04383 43159 37135
	76264 12340 00000 00000			06348 22292 68113 61866
(-50)	-.65637 63612 56317 62245	P06	(-12)	-.76471 63158 55668 19962
	56670 20000 00000 00000			03704 41844 61367 33023
(-60)	.62512 30607 52671 63371	P07	(-14)	.28113 78186 76281 91963
	03500 00000 00000 00000			39857 00023 73240 51698
(-70)	-.45504 01603 04513 46434	P08	(-17)	-.81603 27034 08918 40807
	00000 00000 00000 00000			62121 74862 72639 92345

N = 10

PRECISION 102.9 BITS			PRECISION 30.97 DIGITS	
(-2)	-.52525 25252 52525 25252	P00	(0)	-.16666 66666 66666 66666
	52525 25252 42304 71614			66666 66640 53299 54016
(-6)	.42104 21042 10421 04210	P01	(-2)	.83333 33333 33333 33333
	42104 20600 75403 12005			33331 29905 29159 63171
(-14)	-.64006 40064 00640 06400	P02	(-3)	-.19841 26984 12698 41269
	63776 00401 01364 45400			83578 08881 31667 14238
(-22)	.56167 43512 53307 14707	P03	(-5)	.27557 31922 39858 90645
	47114 16532 26760 00000			21829 60234 55422 53442
(-31)	-.65631 05317 72504 70355	P04	(-7)	-.25052 10838 54417 13133
	55346 52032 40000 00000			64010 17861 98373 53065
(-40)	.54111 06047 24150 76670	P05	(-9)	.16059 04383 68189 31317
	50264 40100 00000 00000			39496 05370 97566 83232
(-50)	-.65637 63716 23542 65270	P06	(-12)	-.76471 63731 00367 67565
	32127 00000 00000 00000			24854 15208 64203 59560
(-60)	.62513 07260 23676 55636	P07	(-14)	.28114 57095 29595 80894
	53000 00000 00000 00000			12773 15316 35118 28193
(-70)	-.45721 76171 02262 53430	P08	(-17)	-.82204 43082 57590 33326
	00000 00000 00000 00000			58798 18668 24827 85337
(-101)	.55717 56324 30012 00000	P09	(-19)	.19441 82583 40054 97532
	00000 00000 00000 00000			91781 08362 37315 80674

COS(Y) $|Y| < \pi/4$, $COS(\pi/4, N, 0) = 1 + Y^2(-.5 + Y^2P(Y^2))$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
N = 3	PRECISION 23.3 BITS			PRECISION 7.00 DIGITS	
	(-4)	.52522 07463 00117 71100	P00	(-1)	.41660 53532 40411 57057
	(-11)	-.54540 00306 20365 40161	P01	(-2)	-.13637 54633 08921 49335
N = 4	PRECISION 32.8 BITS			PRECISION 9.88 DIGITS	
	(-4)	.52525 24443 44021 14107	P00	(-1)	.41666 64390 14384 86476
	(-11)	-.55402 71656 67451 14073	P01	(-2)	-.13887 22883 10893 87275
	(-17)	.63160 06635 21233 61417	P02	(-4)	.24423 10225 40583 02461
N = 5	PRECISION 42.8 BITS			PRECISION 12.88 DIGITS	
	(-4)	.52525 25251 62704 30516	P00	(-1)	.41666 66661 59322 31130
	(-11)	-.55405 53364 54627 71040	P01	(-2)	-.13888 88319 88869 76316
	(-17)	.64004 10426 40635 26172	P02	(-4)	.24799 38184 79873 87017
	(-25)	-.44403 20551 44643 66131	P03	(-6)	-.27199 36460 83906 83604
N = 6	PRECISION 53.1 BITS			PRECISION 15.99 DIGITS	
	(-4)	.52525 25252 52453 13615	P00	(-1)	.41666 66666 65917 94443
		73622 54572 37046 76017			85143 87347 83301 00080
	(-11)	-.55405 54054 66717 37360	P01	(-2)	-.13888 88887 70069 50576
		16030 12056 64761 07006			20779 37164 92917 37632
	(-17)	.64006 37502 57572 62010	P02	(-4)	.24801 58044 43734 42288
		50622 16156 62451 56000			48727 43351 18416 82487
	(-25)	-.44767 76510 22510 77602	P03	(-6)	-.27555 47578 07623 46997
		41235 05342 57236 00000			45043 71683 35018 22276
	(-34)	.43354 76530 56620 50555	P04	(-8)	.20642 09555 82353 89232
		44317 76256 06500 00000			65302 68261 60607 69767
N = 7	PRECISION 63.8 BITS			PRECISION 19.22 DIGITS	
	(-4)	.52525 25252 52525 22407	P00	(-1)	.41666 66666 66665 88408
		03772 02541 16237 77324			71812 45750 29249 85123
	(-11)	-.55405 54055 40516 07701	P01	(-2)	-.13888 88888 88722 54526
		17742 36562 53661 53160			17140 14630 37683 04334
	(-17)	.64006 40063 42765 76300	P02	(-4)	.24801 58728 83391 08273
		22363 56124 11640 00000			11312 26216 54434 08296
	(-25)	-.44771 17371 74100 06732	P03	(-6)	-.27557 31402 80130 43649
		23141 67731 60644 00000			65052 49564 32305 33151
	(-34)	.43672 34232 05177 16024	P04	(-8)	.20875 67986 81516 77639
		27662 36267 26000 00000			87810 16328 16151 19764
	(-44)	-.61746 52166 32741 12413	P05	(-10)	-.11357 43049 11523 88965
		73307 32574 00000 00000			31209 33578 90863 38919
N = 8	PRECISION 74.9 BITS			PRECISION 22.54 DIGITS	
	(-4)	.52525 25252 52525 25251	P00	(-1)	.41666 66666 66666 66605
		42664 44371 47656 55204			83785 05079 06108 02374
	(-11)	-.55405 54055 40554 03611	P01	(-2)	-.13888 88888 88888 72237
		62047 55375 40553 66254			42987 48990 34254 17537
	(-17)	.64006 40064 00614 01700	P02	(-4)	.24801 58730 15698 91675
		74177 77536 54456 70000			51058 65223 69877 01919
	(-25)	-.44771 17556 56750 51263	P03	(-6)	-.27557 31921 47177 03559
		72347 34007 63020 00000			61688 47533 41165 94707
	(-34)	.43673 30620 21521 63040	P04	(-8)	.20876 75422 38229 91293
		60106 15412 70000 00000			27187 80025 12557 38280
	(-44)	-.62344 61163 62114 03165	P05	(-10)	-.11470 27768 69686 97229
		60534 17300 00000 00000			87721 08093 07801 20573
	(-54)	.65255 32534 24525 74253	P06	(-13)	.47374 28657 69253 93852
		71476 60000 00000 00000			20741 96320 11019 86574

COS(Y) $|Y| < \pi/4$, $COS(\pi/4, N, 0) = 1 + Y^2(-.5 + Y^2P(Y^2))$

BINARY COEFFICIENTS — DECIMAL COEFFICIENTS

N = 9 — PRECISION 86.2 BITS — PRECISION 25.96 DIGITS

	Binary			Decimal
(-4)	.52525 25252 52525 25252	P00	(-1)	.41666 66666 66666 66666
	52477 21737 33362 61427			63018 29106 66257 93155
(-11)	-.55405 54055 40554 05537	P01	(-2)	-.13888 88888 88888 88876
	76254 71366 00311 76230			40218 87309 35967 67767
(-17)	.64006 40064 00640 05220	P02	(-4)	.24801 58730 15872 85049
	01410 64624 02241 10000			27468 20764 38810 66505
(-25)	-.44771 17556 74226 74637	P03	(-6)	-.27557 31922 39744 91689
	34554 35517 12700 00000			37390 91024 41762 20536
(-34)	.43673 30737 64005 52721	P04	(-8)	.20876 75698 33079 67763
	16072 26420 00000 00000			30989 27029 76810 95660
(-44)	-.62345 64400 30737 70041	P05	(-10)	-.11470 74449 84278 92193
	16617 34000 00000 00000			71429 24117 37614 94871
(-54)	.65636 71614 43625 43117	P06	(-13)	.47793 19774 49255 62333
	33230 00000 00000 00000			35838 60538 41185 87811
(-64)	-.54523 20473 57520 05002	P07	(-15)	-.15495 45785 99266 57620
	43000 00000 00000 00000			71277 28012 63877 13620

N = 10 — PRECISION 97.8 BITS — PRECISION 29.45 DIGITS

	Binary			Decimal
(-4)	.52525 25252 52525 25252	P00	(-1)	.41666 66666 66666 66666
	52525 24522 33210 07716			66664 92869 55887 92285
(-11)	-.55405 54055 40554 05540	P01	(-2)	-.13888 88888 88888 88888
	55367 50437 32744 77540			88162 25634 52253 34788
(-17)	.64006 40064 00640 06400	P02	(-4)	.24801 58730 15873 01575
	27243 71000 30175 40000			43460 62810 67928 74573
(-25)	-.44771 17556 74237 26501	P03	(-6)	-.27557 31922 39858 80412
	67734 51250 31000 00000			35553 55971 94919 27504
(-34)	.43673 30737 74323 50431	P04	(-8)	.20876 75698 78628 48424
	66004 43740 00000 00000			72653 55488 65326 66213
(-44)	-.62345 64521 30227 40032	P05	(-10)	-.11470 74559 60467 79028
	06054 34000 00000 00000			39709 73916 88786 85239
(-54)	.65637 63616 01363 20477	P06	(-13)	.47794 76991 55716 82182
	72260 00000 00000 00000			75032 35867 75925 22700
(-64)	-.55011 20256 77561 72232	P07	(-15)	-.15618 78294 64264 54811
	70000 00000 00000 00000			09550 63341 96566 97692
(-75)	.74161 00435 71742 43100	P08	(-18)	.40807 14727 05811 77294
	00000 00000 00000 00000			11617 41514 11204 63368

N = 11 — PRECISION 109.7 BITS — PRECISION 33.02 DIGITS

	Binary			Decimal
(-4)	.52525 25252 52525 25252	P00	(-1)	.41666 66666 66666 66666
	52525 25252 42105 31427			66666 66599 36798 50191
(-11)	-.55405 54055 40554 05540	P01	(-2)	-.13888 88888 88888 88888
	55405 53527 54400 01400			88888 55154 70411 31534
(-17)	.64006 40064 00640 06400	P02	(-4)	.24801 58730 15873 01587
	63776 21110 15066 00000			29493 61242 07546 67730
(-25)	-.44771 17556 74237 27071	P03	(-6)	-.27557 31922 39858 90645
	23021 60265 60000 00000			55147 56431 32693 94788
(-34)	.43673 30737 74330 45520	P04	(-8)	.20876 75698 78680 94521
	53010 47740 00000 00000			40639 41877 47808 96525
(-44)	-.62345 64521 40170 16050	P05	(-10)	-.11470 74559 77279 07481
	32154 20000 00000 00000			81121 28478 55091 76404
(-54)	.65637 63716 23770 00530	P06	(-13)	.47794 77331 90139 87779
	33200 00000 00000 00000			71009 25725 44734 07779
(-64)	-.55011 70236 30715 35540	P07	(-15)	-.15619 20611 94840 35292
	00000 00000 00000 00000			72696 62879 51829 98406
(-75)	.74517 76367 07154 20000	P08	(-18)	.41102 24238 72858 97584
	00000 00000 00000 00000			69511 55441 34548 82007
(-105)	-.41307 25422 66445 00000	P09	(-21)	-.88380 81830 30617 65452
	00000 00000 00000 00000			58475 88127 74954 47027

TAN(Y) $|Y| < \pi/4$, $TAN(\pi/4, 0, M) = Y + Y^3/(3 + Y^2Q(Y^2))$

M	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 2	PRECISION 23.0 BITS		PRECISION 6.92 DIGITS
	(1) -.46313 55066 14434 54005	Q00	(1) -.11999 33153 01125 49259
	(-7) -.61013 11302 46375 73343	Q01	(-2) -.59841 02860 39506 38507
M = 3	PRECISION 30.4 BITS		PRECISION 9.15 DIGITS
	(1) -.46314 64261 56041 04732	Q00	(1) -.12000 01093 96830 09490
	(-7) -.56576 22330 04474 76760	Q01	(-2) -.57063 78899 57414 57668
	(-13) -.43467 57217 65225 75106	Q02	(-3) -.27167 40781 72828 22702
M = 4	PRECISION 37.6 BITS		PRECISION 11.33 DIGITS
	(1) -.46314 63135 61570 66705	Q00	(1) -.11999 99983 93864 51640
	(-7) -.56640 14752 52521 30115	Q01	(-2) -.57144 64777 45882 24994
	(-13) -.41145 32274 62715 13014	Q02	(-3) -.25328 11762 62736 99907
	(-20) -.67611 46263 64200 15045	Q03	(-4) -.13296 30577 82321 31218
M = 5	PRECISION 44.8 BITS		PRECISION 13.49 DIGITS
	(1) -.46314 63146 41023 55512	Q00	(1) -.12000 00000 21648 89123
	70146 50467 47104 00126		49134 80418 10060 98261
	(-7) -.56637 33762 22144 61620	Q01	(-2) -.57142 82293 66083 90277
	25606 37777 51526 01012		17670 95213 01954 00598
	(-13) -.41224 65441 11547 06747	Q02	(-3) -.25398 78629 02969 99415
	76437 54724 34104 61040		40064 98085 94864 68060
	(-20) -.62757 17720 73415 16542	Q03	(-4) -.12151 54703 87136 47943
	47237 56414 00631 50000		04235 57900 91982 55751
	(-24) -.54421 62474 42711 53762	Q04	(-6) -.66361 94567 60160 54299
	02111 24451 41364 40000		02968 04053 67807 55874
M = 6	PRECISION 51.9 BITS		PRECISION 15.63 DIGITS
	(1) -.46314 63146 31403 13205	Q00	(1) -.11999 99999 99727 01481
	10516 13337 45664 14551		26279 76746 54011 37397
	(-7) -.56637 34720 26776 07721	Q01	(-2) -.57142 85772 11421 89687
	12405 54364 36050 05634		94823 65683 83305 57992
	(-13) -.41223 37300 62523 66762	Q02	(-3) -.25396 77956 48178 10414
	51421 54612 46745 15000		47755 61730 71180 69445
	(-20) -.63143 00606 26225 16723	Q03	(-4) -.12205 45354 88061 93425
	23513 54643 65113 40000		58242 22135 65575 00925
	(-24) -.47737 74706 41022 21106	Q04	(-6) -.59511 37098 81351 91645
	62600 35724 43050 00000		88849 17769 62511 80089
	(-30) -.43611 37305 10342 16631	Q05	(-7) -.33312 04496 31465 49258
	24377 66426 71000 00000		24316 33755 25334 21923
M = 7	PRECISION 59.0 BITS		PRECISION 17.77 DIGITS
	(1) -.46314 63146 31463 61441	Q00	(1) -.12000 00000 00003 26541
	33633 56037 74542 21650		47510 42901 44300 96344
	(-7) -.56637 34710 22373 03212	Q01	(-2) -.57142 85713 39428 50601
	35126 53563 46603 27620		14883 03991 18496 42544
	(-13) -.41223 41305 04173 76331	Q02	(-3) -.25396 82632 53356 52147
	62230 17175 35717 04400		35540 01533 66988 26171
	(-20) -.63137 04622 56507 26643	Q03	(-4) -.12203 62018 34239 95068
	01175 14346 76075 00000		58874 34779 63798 19793
	(-24) -.50142 34653 57726 20063	Q04	(-6) -.59891 17359 81678 40743
	55233 77525 35040 00000		22292 31421 47775 78040
	(-31) -.76760 15714 72723 16005	Q05	(-7) -.29322 30671 93845 27436
	76457 02256 74000 00000		74569 87489 89484 40181
	(-35) -.71434 13162 20472 32131	Q06	(-8) -.16750 71835 16170 90874
	73441 16264 60000 00000		84465 64352 63388 69070

TAN(Y) $|Y| < \pi/4$, $TAN(\pi/4, 0, M) = Y + Y^3/(3 + Y^2 Q(Y^2))$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 8	PRECISION 66.1 BITS		PRECISION 19.90 DIGITS
	(1) -.46314 63146 31463 14301	Q00	(1) -.11999 99999 99999 96256
	63544 23277 42641 34550		60115 38230 66532 32240
	(-7) -.56637 34710 32343 20443	Q01	(-2) -.57142 85714 29849 68344
	75751 45177 76005 56020		19762 67380 38293 42161
	(-13) -.41223 41260 21563 63260	Q02	(-3) -.25396 82537 99471 99709
	26651 31721 40655 77000		22457 64738 31260 48733
	(-20) -.63137 13547 63017 13425	Q03	(-4) -.12203 67050 50353 79784
	61611 17527 14630 00000		16359 43637 37491 02932
	(-24) -.50135 22724 56551 12124	Q04	(-6) -.59876 17057 54366 32943
	54422 71167 54400 00000		84655 18718 68231 01295
	(-31) -.77407 23366 74662 40127	Q05	(-7) -.29576 13506 74539 33867
	34564 40533 60000 00000		27686 91942 61961 99827
	(-35) -.61564 46427 77450 01414	Q06	(-8) -.14472 67680 68726 02526
	17402 16744 00000 00000		08537 73367 30891 05728
	(-41) -.56250 20767 20511 30566	Q07	(-10) -.84271 31120 31317 60819
	74616 72640 00000 00000		83875 97010 61096 35648
M = 9	PRECISION 73.2 BITS		PRECISION 22.02 DIGITS
	(1) -.46314 63146 31463 14633	Q00	(1) -.12000 00000 00000 00041
	77222 06061 23100 26415		43965 50640 61400 13567
	(-7) -.56637 34710 32250 72322	Q01	(-2) -.57142 85714 28554 13688
	55470 07167 56654 30270		93052 43381 06175 38815
	(-13) -.41223 41260 51700 63036	Q02	(-3) -.25396 82539 71070 78169
	64407 22045 73122 44000		88943 36748 05974 83950
	(-20) -.63137 13424 44357 35251	Q03	(-4) -.12203 66932 23010 34397
	05300 25611 30740 00000		37269 20137 02206 22205
	(-24) -.50135 35243 23414 32166	Q04	(-6) -.59876 64367 03593 83523
	25321 41374 77000 00000		23596 92103 20153 63015
	(-31) -.77372 61115 57120 22167	Q05	(-7) -.29564 73326 53421 77400
	44777 31670 00000 00000		82061 02419 13217 70494
	(-35) -.62223 73663 30532 05075	Q06	(-8) -.14636 00663 20954 09431
	74203 76000 00000 00000		00375 32356 28238 84313
	(-41) -.47221 53737 26603 73602	Q07	(-10) -.71458 16917 30544 95844
	27761 47000 00000 00000		14458 68567 18248 90059
	(-45) -.45227 50472 53616 26216	Q08	(-11) -.42400 82641 30349 30213
	21240 50000 00000 00000		75888 89037 70047 11048
M = 10	PRECISION 80.2 BITS		PRECISION 24.15 DIGITS
	(1) -.46314 63146 31463 14631	Q00	(1) -.11999 99999 99999 99999
	44603 15010 23436 56711		55446 24582 90078 28031
	(-7) -.56637 34710 32251 54711	Q01	(-2) -.57142 85714 28571 65152
	57665 75713 41023 40750		15538 14772 48648 90469
	(-13) -.41223 41260 51362 52412	Q02	(-3) -.25396 82539 68210 10556
	21347 40633 00354 54000		44406 73367 51665 78347
	(-20) -.63137 13426 23511 73671	Q03	(-4) -.12203 66934 69872 23709
	04474 10772 40240 00000		90121 96343 20095 80066
	(-24) -.50135 35024 63004 43704	Q04	(-6) -.59876 63101 31017 38161
	32316 51622 50000 00000		72480 17038 88058 77057
	(-31) -.77373 15544 77361 40361	Q05	(-7) -.29565 13892 01552 54462
	27156 07300 00000 00000		32281 62495 31724 55859
	(-35) -.62205 37070 33520 12427	Q06	(-8) -.14627 79328 97033 08871
	52010 17000 00000 00000		60757 78438 60792 01088
	(-41) -.47661 12244 06247 67216	Q07	(-10) -.72479 48466 43567 70629
	32011 10000 00000 00000		30433 69864 24204 47248
	(-46) -.76037 62077 45270 21046	Q08	(-11) -.35278 20612 35035 28855
	11014 00000 00000 00000		81808 42291 17226 11695
	(-52) -.74031 16036 35304 61700	Q09	(-12) -.21333 78174 08640 92805
	05060 00000 00000 00000		70769 20066 26863 72069

TAN(Y) $|Y| < \pi/4$, $TAN(\pi/4,\ 0,\ M) = Y + Y^3/(3 + Y^2Q(Y^2))$

M = 11

BINARY COEFFICIENTS PRECISION 87.3 BITS		DECIMAL COEFFICIENTS PRECISION 26.27 DIGITS
(1) -.46314 63146 31463 14631	Q00	(1) -.12000 00000 00000 00000
46325 50106 20361 20267		00467 30511 68609 24801
(-7) -.56637 34710 32251 54174	Q01	(-2) -.57142 85714 28571 42581
52015 52313 34014 27540		02626 91691 15670 54899
(-13) -.41223 41260 51365 67614	Q02	(-3) -.25396 82539 68254 61313
63661 25237 15253 70000		82557 45252 18239 73834
(-20) -.63137 13426 21420 66562	Q03	(-4) -.12203 66934 65183 79880
73307 66026 03200 00000		85018 33131 95334 55860
(-24) -.50135 35030 11765 62574	Q04	(-6) -.59876 63131 14341 47767
32174 42133 60000 00000		07014 87181 33152 95053
(-31) -.77373 14656 65460 00665	Q05	(-7) -.29565 12675 89292 70749
16321 10740 00000 00000		04437 19708 79328 14605
(-35) -.62206 03562 57167 01004	Q06	(-8) -.14628 11848 62721 68610
35412 10000 00000 00000		54739 45417 55006 19778
(-41) -.47641 13010 20752 62216	Q07	(-10) -.72422 67986 11024 20895
26651 00000 00000 00000		92621 99697 16750 46426
(-46) -.77122 17773 03650 17471	Q08	(-11) -.35902 66955 87047 23765
17200 00000 00000 00000		37680 45451 90616 13853
(-52) -.61004 74557 55570 71011	Q09	(-12) -.17411 73089 16476 10869
07000 00000 00000 00000		51876 35482 07176 78230
(-56) -.60256 10243 32502 66462	Q10	(-13) -.10733 65788 05092 29020
30000 00000 00000 00000		33918 63744 29889 61420

M = 12

PRECISION 94.3 BITS		PRECISION 28.39 DIGITS
(1) -.46314 63146 31463 14631	Q00	(1) -.11999 99999 99999 99999
46314 55314 32104 74671		99995 20171 97093 03868
(-7) -.56637 34710 32251 54200	Q01	(-2) -.57142 85714 28571 42860
61735 27117 05270 22240		44725 83185 68707 30519
(-13) -.41223 41260 51365 64604	Q02	(-3) -.25396 82539 68253 95922
62553 15275 66526 00000		12966 87737 42604 08053
(-20) -.63137 13426 21443 67161	Q03	(-4) -.12203 66934 65266 23190
64704 64010 30200 00000		91257 93872 04261 55937
(-24) -.50135 35030 05315 54730	Q04	(-6) -.59876 63130 50804 76550
45223 05301 00000 00000		75664 28251 85501 78956
(-31) -.77373 14672 25551 57740	Q05	(-7) -.29565 12707 81672 94360
23506 33100 00000 00000		22696 52784 64943 78983
(-35) -.62206 02404 51021 51315	Q06	(-8) -.14628 10769 46121 80545
35266 10000 00000 00000		22529 41396 94469 36436
(-41) -.47641 67475 04307 13064	Q07	(-10) -.72425 15583 23100 71785
66670 00000 00000 00000		39982 45402 84589 22129
(-46) -.77057 76630 62427 75071	Q08	(-11) -.35864 62374 33584 82683
00400 00000 00000 00000		21045 42024 52671 84841
(-52) -.62041 25055 44264 31011	Q09	(-12) -.17786 69539 12816 61842
70000 00000 00000 00000		08694 18447 00592 34859
(-56) -.46540 71075 37003 02647	Q10	(-14) -.85907 37713 80125 38175
00000 00000 00000 00000		25888 79710 82333 04406
(-62) -.46723 26145 25445 56240	Q11	(-15) -.54002 33982 47212 55152
00000 00000 00000 00000		94379 03912 42200 24993

TAN(Y) $|Y| < \pi/4,$ $TAN(\pi/4, 0, M) = Y + Y^3/(3 + Y^2Q(Y^2))$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 13	PRECISION 101.3 BITS		PRECISION 30.51 DIGITS
	(1) -.46314 63146 31463 14631	Q00	(1) -.12000 00000 00000 00000
	46314 63204 25520 27474		00000 04836 80779 65405
	(-7) -.56637 34710 32251 54200	Q01	(-2) -.57142 85714 28571 42857
	56614 70516 63156 02400		10446 31178 10435 96961
	(-13) -.41223 41260 51365 64632	Q02	(-3) -.25396 82539 68253 96837
	32017 22750 46371 40000		53437 62920 73434 10341
	(-20) -.63137 13426 21443 43125	Q03	(-4) -.12203 66934 65264 87293
	55533 40033 51400 00000		57400 30609 31607 28936
	(-24) -.50135 35030 05373 50336	Q04	(-6) -.59876 63130 52049 69742
	73033 12730 00000 00000		61934 08268 46615 81687
	(-31) -.77373 14672 04257 02515	Q05	(-7) -.29565 12707 06365 72092
	33165 30400 00000 00000		30052 81517 21775 08607
	(-35) -.62206 02426 50263 21632	Q06	(-8) -.14628 10800 66769 22551
	31512 40000 00000 00000		50871 72522 48384 82611
	(-41) -.47641 65776 27065 40311	Q07	(-10) -.72425 06576 68148 17488
	75340 00000 00000 00000		00141 47517 02547 38014
	(-46) -.77061 47073 13445 15265	Q08	(-11) -.35866 43335 77955 90684
	54000 00000 00000 00000		39593 78423 65533 91688
	(-52) -.61775 43141 45602 32236	Q09	(-12) -.17761 86826 31385 84864
	00000 00000 00000 00000		66270 24012 63553 56557
	(-56) -.47540 26614 14011 14140	Q10	(-14) -.88125 49579 26251 17551
	00000 00000 00000 00000		04200 87114 10697 67712
	(-63) -.75037 42401 75014 63000	Q11	(-15) -.42369 99629 47304 21856
	00000 00000 00000 00000		17720 16610 29881 07390
	(-67) -.76512 61363 11666 24000	Q12	(-16) -.27168 38954 52182 85870
	00000 00000 00000 00000		30648 82622 69948 05358

ATAN(Y) $|Y| < \tan(\pi/12)$, $\mathrm{ATAN}(\tan(\pi/12), 0, M) = Y - Y^3/Q(Y^2)$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
M = 1	PRECISION 21.4 BITS			PRECISION 6.43 DIGITS	
	(2)	.60004 03245 02052 71655	Q00	(1)	.30004 94618 09009 05965
	(1)	.70730 65424 35335 27651	Q01	(1)	.17788 59653 43542 53646
M = 2	PRECISION 28.4 BITS			PRECISION 8.54 DIGITS	
	(2)	.60000 03610 31256 73645	Q00	(1)	.30000 07183 83477 47334
	(1)	.71451 34531 33213 61607	Q01	(1)	.17994 04787 69343 47166
	(-2)	-.61031 00570 03760 15475	Q02	(0)	-.19159 70724 22043 00439
M = 3	PRECISION 35.0 BITS			PRECISION 10.53 DIGITS	
	(2)	.60000 00037 41362 16551	Q00	(1)	.30000 00117 43239 03997
	(1)	.71462 74204 37163 75050	Q01	(1)	.17999 84225 04945 40601
	(-2)	-.64374 64552 22266 23257	Q02	(0)	-.20505 38906 56403 59719
	(-3)	.60137 51206 00017 43025	Q03	(-1)	.94114 85563 03785 26716
M = 4	PRECISION 41.4 BITS			PRECISION 12.45 DIGITS	
	(2)	.60000 00000 42375 75017	Q00	(1)	.30000 00002 00793 16519
	(1)	.71463 14305 06011 72673	Q01	(1)	.17999 99604 17682 81307
	(-2)	-.64520 03565 56527 64465	Q02	(0)	-.20568 89212 05988 76219
	(-3)	.65357 51000 42366 76552	Q03	(0)	.10442 97814 99643 25428
	(-4)	-.73044 15074 74577 52333	Q04	(-1)	-.57686 24302 86831 49236
M = 5	PRECISION 47.7 BITS			PRECISION 14.35 DIGITS	
	(2)	.60000 00000 00464 21114	Q00	(1)	.30000 00000 03504 60090
		30107 32033 37627 40151			71099 01652 06350 57168
	(1)	.71463 14624 37624 43630	Q01	(1)	.17999 99990 49440 24446
		41444 26056 65563 76501			06962 99517 41643 86944
	(-2)	-.64523 21231 24207 25366	Q02	(0)	-.20571 34270 00825 78607
		36724 66652 22273 51464			42588 03496 95084 23161
	(-3)	.65641 14363 50424 00227	Q03	(0)	.10510 70973 78166 27520
		56433 03542 52040 74600			01595 04438 59892 82490
	(-3)	-.41703 21156 34051 53671	Q04	(-1)	-.66174 57960 25232 56984
		20637 25337 24311 54000			86742 47553 54443 58135
	(-4)	.50373 13632 34305 24503	Q05	(-1)	.39541 59622 33395 43253
		21653 23122 61440 40000			56216 87769 27257 83352
M = 6	PRECISION 53.9 BITS			PRECISION 16.22 DIGITS	
	(2)	.60000 00000 00005 33721	Q00	(1)	.30000 00000 00061 80110
		42605 02512 02073 76710			00439 59074 42864 21914
	(1)	.71463 14631 36647 33761	Q01	(1)	.17999 99999 77959 74340
		34104 71747 23461 03065			79047 49113 89165 10631
	(-2)	-.64523 30217 21714 40355	Q02	(0)	-.20571 42591 28942 75176
		34432 15176 46556 74624			00094 70231 38174 18627
	(-3)	.65652 12702 74377 51546	Q03	(0)	.10514 13345 78546 44706
		47602 12427 40077 11000			67465 58706 98943 59320
	(-3)	-.42175 61242 61315 37510	Q04	(-1)	-.66886 02673 49397 98638
		34604 53260 60110 00000			20530 81818 55163 62367
	(-4)	.57735 04275 76577 02237	Q05	(-1)	.46808 37306 54581 06281
		65171 53245 24540 00000			65435 87999 58382 81156
	(-5)	-.73315 22112 37067 75643	Q06	(-1)	-.29004 36737 35950 99906
		21457 63400 57300 00000			19557 68436 45691 23380

ATAN(Y) $|Y| < \tan(\pi/12)$, $\mathrm{ATAN}(\tan(\pi/12),\ 0,\ M) = Y - Y^3/Q(Y^2)$

BINARY COEFFICIENTS   DECIMAL COEFFICIENTS

M = 7   PRECISION 60.0 BITS   PRECISION 18.07 DIGITS

(2)	.60000 00000 00000 06126	Q00	(1)	.30000 00000 00001 09566
	01655 51107 34047 04262			13174 57957 54888 83116
(1)	.71463 14631 46165 32736	Q01	(1)	.17999 99999 99503 30710
	62062 26420 41246 40341			18673 10828 57465 19647
(-2)	-.64523 30376 12064 47570	Q02	(0)	-.20571 42849 44780 13865
	45734 07660 44510 74270			67091 23927 79846 91303
(-3)	.65652 43352 00771 61732	Q03	(0)	.10514 27994 40705 58804
	17140 20112 77214 74000			57386 02972 39198 59670
(-3)	-.42211 32341 76515 60107	Q04	(-1)	-.66930 43955 35652 65150
	17262 60527 11703 00000			56390 24921 86618 94916
(-4)	.60540 61042 70232 16101	Q05	(-1)	.47547 84906 28131 22490
	11041 62052 13500 00000			15083 00256 59067 37949
(-4)	-.44166 13605 71502 02157	Q06	(-1)	-.35381 66765 94114 81430
	15305 64360 75000 00000			41468 37051 43826 91582
(-5)	.55463 02153 72276 45057	Q07	(-1)	.22265 46720 81135 80333
	41505 35232 40000 00000			30842 71652 60998 80676

M = 8   PRECISION 66.2 BITS   PRECISION 19.92 DIGITS

(2)	.60000 00000 00000 00070	Q00	(1)	.30000 00000 00000 01947
	11106 31515 52057 05671			84374 26244 61072 25753
(1)	.71463 14631 46312 70062	Q01	(1)	.17999 99999 99989 06646
	65620 11417 63252 12513			12618 32938 18185 60280
(-2)	-.64523 30401 27641 23747	Q02	(0)	-.20571 42856 93157 07870
	41662 10532 14514 52460			54432 49674 09824 29525
(-3)	.65652 44310 36070 16216	Q03	(0)	.10514 28551 40201 95616
	65325 03760 44755 20000			41299 55384 42246 69654
(-3)	-.42212 00705 74231 23740	Q04	(-1)	-.66932 73106 84810 73829
	50616 44676 76560 00000			73789 92658 98120 92442
(-4)	.60575 43153 13310 37353	Q05	(-1)	.47602 74914 89654 35864
	54201 70750 10400 00000			03627 96408 93393 13797
(-4)	-.45005 76444 73146 52706	Q06	(-1)	-.36144 21403 76316 52513
	34575 14172 10000 00000			99222 40843 04897 33313
(-5)	.71203 31232 63765 47637	Q07	(-1)	.27957 33962 01066 00046
	05017 07670 00000 00000			32817 52118 34836 77612
(-5)	-.44125 52222 67326 21744	Q08	(-1)	-.17659 81744 00253 43067
	32532 32554 00000 00000			79157 88388 40017 13746

M = 9   PRECISION 72.2 BITS   PRECISION 21.75 DIGITS

(2)	.60000 00000 00000 00000	Q00	(1)	.30000 00000 00000 00034
	77754 55505 02477 13033			67404 86610 46922 96625
(1)	.71463 14631 46314 60426	Q01	(1)	.17999 99999 99999 76401
	00564 41317 55514 13426			01904 42971 43796 71399
(-2)	-.64523 30401 35360 52213	Q02	(0)	-.20571 42857 13729 76892
	27452 27544 42060 25400			34038 45578 17184 78244
(-3)	.65652 44331 07331 62311	Q03	(0)	.10514 28570 77999 66255
	04541 72466 06407 60000			84479 42655 55547 43180
(-3)	-.42212 02474 13022 55527	Q04	(-1)	-.66932 83423 92628 65609
	30676 53232 26760 00000			94963 06085 39015 64526
(-4)	.60577 22346 27256 61057	Q05	(-1)	.47606 06437 71610 02949
	51155 01164 64000 00000			95063 42778 53420 00952
(-4)	-.45050 27250 23121 24311	Q06	(-1)	-.36209 80169 50393 00750
	73040 52244 00000 00000			16209 43297 59569 26720
(-5)	.72666 62044 77166 32207	Q07	(-1)	.28738 76798 51377 78986
	60645 40460 00000 00000			94047 47006 51019 04626
(-5)	-.56545 41064 21733 36444	Q08	(-1)	-.22801 89249 08286 25944
	24602 43200 00000 00000			33645 77607 09068 97686
(-6)	.72631 55522 24442 66567	Q09	(-1)	.14355 52284 70559 26050
	14311 76000 00000 00000			07582 06678 63699 64293

ATAN(Y) $|Y| < \tan(\pi/12)$, $\mathrm{ATAN}(\tan(\pi/12), 0, M) = Y - Y^3/Q(Y^2)$

M = 10

BINARY COEFFICIENTS PRECISION 78.3 BITS			DECIMAL COEFFICIENTS PRECISION 23.57 DIGITS	
(2)	.60000 00000 00000 00000 01107 20102 30177 26070	Q00	(1)	.30000 00000 00000 00000 61754 21029 25080 91544
(1)	.71463 14631 46314 63111 41376 44555 75523 10107	Q01	(1)	.17999 99999 99999 99499 07073 61636 80071 27778
(-2)	-.64523 30401 35474 72450 06320 74760 27121 12500	Q02	(0)	-.20571 42857 14271 59174 51791 81509 00755 69520
(-3)	.65652 44331 51772 44466 65254 34342 16047 40000	Q03	(0)	.10514 28571 40871 27127 62756 40718 66474 14221
(-3)	-.42212 02540 11475 47313 70554 14031 00340 00000	Q04	(-1)	-.66932 83842 76383 85770 67196 30629 33682 56248
(-4)	.60577 30171 03063 57355 43355 51753 40000 00000	Q05	(-1)	.47606 23682 80518 58411 07347 18174 87841 47127
(-4)	-.45052 57421 41322 26556 75445 06542 00000 00000	Q06	(-1)	-.36214 33777 28087 66666 78756 82114 35421 91735
(-5)	.73006 67737 64056 06214 04643 02200 00000 00000	Q07	(-1)	.28815 14932 41866 93413 41525 49459 40135 41013
(-5)	-.60250 72427 31545 53715 26460 30000 00000 00000	Q08	(-1)	-.23598 58968 41605 63310 63526 52231 73938 65186
(-5)	.47000 30120 75547 57126 50701 00000 00000 00000	Q09	(-1)	.19043 32873 42341 87943 56664 64407 42706 62833
(-6)	-.60562 35544 01547 12712 77246 00000 00000 00000	Q10	(-1)	-.11895 40099 62411 65650 80017 41646 39886 08014

M = 11

PRECISION 84.3 BITS			PRECISION 25.39 DIGITS	
(2)	.60000 00000 00000 00000 00012 30635 52545 23532	Q00	(1)	.30000 00000 00000 00000 01099 83248 41343 64062
(1)	.71463 14631 46314 63145 62733 63121 75440 42260	Q01	(1)	.17999 99999 99999 99989 51828 45848 88431 43521
(-2)	-.64523 30401 35476 66512 47101 11103 75757 21100	Q02	(0)	-.20571 42857 14285 36599 41160 00978 53981 72996
(-3)	.65652 44331 53031 15222 77734 20226 31355 00000	Q03	(0)	.10514 28571 42799 10028 44162 47559 42636 03604
(-3)	-.42212 02541 37601 05561 56465 61152 03600 00000	Q04	(-1)	-.66932 83858 43111 89607 26753 69002 26392 74854
(-4)	.60577 30402 41662 64170 22720 47567 00000 00000	Q05	(-1)	.47606 24483 04667 21700 05769 63405 80503 75098
(-4)	-.45052 70407 72410 43734 14040 75330 00000 00000	Q06	(-1)	-.36214 60543 44673 04347 23677 93191 73938 52097
(-5)	.73015 07261 53655 43341 34346 13000 00000 00000	Q07	(-1)	.28821 10099 55552 85488 88794 91016 92040 52682
(-5)	-.60404 26406 64176 74576 67647 70000 00000 00000	Q08	(-1)	-.23685 79079 67750 85084 68943 99803 65054 12213
(-5)	.50520 36322 62277 41130 10262 00000 00000 00000	Q09	(-1)	.19852 13773 98601 81322 02899 59395 49527 32824
(-5)	-.41132 12771 13554 17015 77010 00000 00000 00000	Q10	(-1)	-.16199 27565 28486 07977 47575 41581 86481 77090
(-6)	.50777 40043 76671 53704 50500 00000 00000 00000	Q11	(-1)	.10009 52773 00312 47156 14689 94583 07487 90678

ATAN(Y) $|Y| < \tan(\pi/12)$, $\text{ATAN}(\tan(\pi/12), 0, M) = Y - Y^3/Q(Y^2)$

M = 12

BINARY COEFFICIENTS PRECISION 90.4 BITS			DECIMAL COEFFICIENTS PRECISION 27.21 DIGITS	
(2)	.60000 00000 00000 00000	Q00	(1)	.30000 00000 00000 00000
	00000 13654 25074 25035			00019 58198 22407 37988
(1)	.71463 14631 46314 63146	Q01	(1)	.17999 99999 99999 99999
	30632 01773 31415 43213			78338 77930 16393 26622
(-2)	-.64523 30401 35476 71552	Q02	(0)	-.20571 42857 14285 70591
	14535 51402 16724 65200			03204 35627 11062 62742
(-3)	.65652 44331 53051 05402	Q03	(0)	.10514 28571 42855 51168
	26641 60003 01025 00000			74812 60529 40178 79752
(-3)	-.42212 02541 42610 27403	Q04	(-1)	-.66932 83858 97940 14022
	07645 52241 07600 00000			45662 58315 21721 65852
(-4)	.60577 30410 26524 51662	Q05	(-1)	.47606 24516 95409 79616
	04140 56615 40000 00000			63155 75329 00823 65629
(-4)	-.45052 70770 00026 07603	Q06	(-1)	-.36214 61940 93322 31498
	77727 71640 00000 00000			02516 89267 73008 13026
(-5)	.73015 41550 63567 41521	Q07	(-1)	.28821 49375 53372 91560
	27752 60000 00000 00000			93454 66566 93219 08575
(-5)	-.60414 21544 61145 05324	Q08	(-1)	-.23693 34841 98732 14378
	63770 00000 00000 00000			40798 32346 78347 78088
(-5)	.50667 15521 71123 76045	Q09	(-1)	.19950 11657 39839 06940
	50260 00000 00000 00000			97026 96289 49761 84759
(-5)	-.42664 03275 73650 36232	Q10	(-1)	-.17017 41473 32677 64738
	35400 00000 00000 00000			19930 35020 57257 99739
(-6)	.71214 55212 33363 41553	Q11	(-1)	.13983 11011 76868 37416
	66000 00000 00000 00000			35313 65175 07826 93028
(-6)	-.42737 27146 47116 22616	Q12	(-2)	-.85293 59105 33072 20204
	54000 00000 00000 00000			33555 34636 92587 42811

M = 13

PRECISION 96.4 BITS			PRECISION 29.02 DIGITS	
(2)	.60000 00000 00000 00000	Q00	(1)	.30000 00000 00000 00000
	00000 00153 66324 26404			00000 34848 28894 36490
(1)	.71463 14631 46314 63146	Q01	(1)	.17999 99999 99999 99999
	31452 65377 24304 43256			99557 20911 70749 81589
(-2)	-.64523 30401 35476 71617	Q02	(0)	-.20571 42857 14285 71408
	72401 02151 57137 60000			86577 06082 03588 41937
(-3)	.65652 44331 53051 42056	Q03	(0)	.10514 28571 42857 09853
	31760 42460 34406 00000			11238 03724 32205 99872
(-3)	-.42212 02541 42673 35367	Q04	(-1)	-.66932 83858 99755 34221
	34430 47641 40400 00000			04895 66545 54274 65741
(-4)	.60577 30410 45250 25557	Q05	(-1)	.47606 24518 28773 08021
	01714 45426 00000 00000			31989 75834 11979 87870
(-4)	-.45052 71003 27323 67477	Q06	(-1)	-.36214 62007 08719 39695
	55257 12000 00000 00000			19528 90685 97719 16649
(-5)	.73015 43167 57677 36174	Q07	(-1)	.28821 51654 18814 51232
	70122 20000 00000 00000			99517 68175 96958 95840
(-5)	-.60414 66551 54761 14373	Q08	(-1)	-.23693 89990 64346 19107
	06514 00000 00000 00000			53793 67942 86438 47301
(-5)	.50701 00772 44255 62345	Q09	(-1)	.19959 46451 11298 76954
	04600 00000 00000 00000			23438 97200 57446 51561
(-5)	-.43045 77133 26164 20272	Q10	(-1)	-.17126 07113 14185 12740
	66000 00000 00000 00000			31312 06173 80304 72213
(-6)	.74516 67310 61356 56616	Q11	(-1)	.14808 11381 44030 48145
	00000 00000 00000 00000			60530 99931 22489 23160
(-6)	-.62020 11214 14227 25344	Q12	(-1)	-.12214 72973 98204 44668
	00000 00000 00000 00000			56669 57339 57772 06314
(-7)	.74126 27066 44044 00560	Q13	(-2)	.73448 08826 48203 24223
	00000 00000 00000 00000			48911 19366 57991 46401

ATAN(Y) $|Y| < \tan(\pi/12)$, $\mathrm{ATAN}(\tan(\pi/12), 0, M) = Y - Y^3/Q(Y^2)$

M = 14

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS
	PRECISION 102.4 BITS			PRECISION 30.82 DIGITS
(2)	.60000 00000 00000 00000	Q00	(1)	.30000 00000 00000 00000
	00000 00001 72610 00455			00000 00619 80810 39723
(1)	.71463 14631 46314 63146	Q01	(1)	.17999 99999 99999 99999
	31463 01754 60212 67616			99991 03535 61789 95324
(-2)	-.64523 30401 35476 71620	Q02	(0)	-.20571 42857 14285 71428
	63243 76502 55705 10400			11654 25908 66329 36259
(-3)	.65652 44331 53051 42674	Q03	(0)	.10514 28571 42857 14168
	34637 03014 62060 00000			71828 27699 60558 09366
(-3)	-.42212 02541 42675 04613	Q04	(-1)	-.66932 83858 99812 67937
	51130 41622 14000 00000			20519 91746 88220 70850
(-4)	.60577 30410 45675 62722	Q05	(-1)	.47606 24518 33701 69145
	02760 44140 00000 00000			23392 89171 78640 93905
(-4)	-.45052 71003 67133 14157	Q06	(-1)	-.36214 62009 97614 02577
	73405 75000 00000 00000			74895 13796 11144 69955
(-5)	.73015 43240 54606 36724	Q07	(-1)	.28821 51773 37236 41046
	26101 00000 00000 00000			21957 98734 60819 73313
(-5)	-.60414 71043 02033 53157	Q08	(-1)	-.23693 93510 25530 65597
	64520 00000 00000 00000			02826 20673 76847 69376
(-5)	.50701 63110 51404 67445	Q09	(-1)	.19960 21184 16394 24381
	61000 00000 00000 00000			62512 20530 88294 82675
(-5)	-.43061 66423 11245 47315	Q10	(-1)	-.17137 38646 20015 72016
	10000 00000 00000 00000			08346 19273 38227 32634
(-6)	.75110 64024 70323 40274	Q11	(-1)	.14927 29813 44468 22846
	00000 00000 00000 00000			97892 50478 78306 03163
(-6)	-.65334 06212 12260 36740	Q12	(-1)	-.13044 40401 38046 81684
	00000 00000 00000 00000			99038 30615 17604 80133
(-6)	.54106 32054 30355 67400	Q13	(-1)	.10775 76046 20166 11279
	00000 00000 00000 00000			65643 06108 84854 10436
(-7)	-.64214 75300 50506 03000	Q14	(-2)	-.63812 63495 38615 73083
	00000 00000 00000 00000			86643 19435 33825 68806

ARSIN(Y) $|Y| < 0.5$, ARSIN(0.5, 0, M) = $Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
M = 1	PRECISION 16.4 BITS		PRECISION 4.92 DIGITS
	(3) .60101 33624 12324 40573	Q00	(1) .60159 75148 69820 57912
	(2) -.56013 11305 74666 32464	Q01	(1) -.28763 60676 99968 58558
M = 2	PRECISION 21.3 BITS		PRECISION 6.41 DIGITS
	(3) .57773 77305 31774 16263	Q00	(1) .59990 21093 59305 18286
	(2) -.52670 70656 46564 13311	Q01	(1) -.26788 19353 15246 86946
	(0) -.41153 25024 53777 01335	Q02	(0) -.51890 04137 06405 24248
M = 3	PRECISION 25.9 BITS		PRECISION 7.79 DIGITS
	(3) .60000 21450 42751 70200	Q00	(1) .60000 67059 29487 88279
	(2) -.53171 71166 67447 56340	Q01	(1) -.27023 79669 50386 69904
	(-1) -.56674 66426 24241 25077	Q02	(0) -.36616 29065 48671 14636
	(-1) -.46411 42271 75113 12170	Q03	(0) -.30092 67719 25491 60618
M = 4	PRECISION 30.2 BITS		PRECISION 9.10 DIGITS
	(3) .57777 76576 53467 33733	Q00	(1) .59999 95221 80392 80492
	(2) -.53144 25704 25635 04771	Q01	(1) -.26997 48770 67182 63713
	(-1) -.62573 33106 43035 23337	Q02	(0) -.39641 45512 07688 30092
	(-2) -.51561 20500 61474 16642	Q03	(0) -.16297 34786 16977 95334
	(-2) -.67600 04464 71127 11041	Q04	(0) -.21777 39862 56980 41912
M = 5	PRECISION 34.5 BITS		PRECISION 10.38 DIGITS
	(3) .60000 00056 36360 76445	Q00	(1) .60000 00346 27395 90990
	(2) -.53146 46654 42701 15452	Q01	(1) -.27000 25247 57742 46054
	(-1) -.62073 27503 45262 66747	Q02	(0) -.39153 09028 62354 80214
	(-2) -.63322 43352 20235 57074	Q03	(0) -.20082 51497 18395 83653
	(-3) -.53033 63305 15264 43371	Q04	(-1) -.84090 43462 11621 40450
	(-2) -.55055 60241 73070 11055	Q05	(0) -.17613 03325 00577 97831
M = 6	PRECISION 38.7 BITS		PRECISION 11.64 DIGITS
	(3) .57777 77774 46557 35407	Q00	(1) .59999 99974 70501 08359
	(2) -.53146 30244 02620 32536	Q01	(1) -.26999 97559 35173 39064
	(-1) -.62150 54231 03762 16201	Q02	(0) -.39222 24757 55065 46992
	(-2) -.61274 62431 02076 26672	Q03	(0) -.19284 66020 69482 43835
	(-2) -.41366 17174 22263 77030	Q04	(0) -.13078 48981 31793 06445
	(-4) -.52262 61314 35415 04367	Q05	(-1) -.41356 60527 16537 59216
	(-2) -.47007 14613 65703 36630	Q06	(0) -.15239 86784 37318 44645
M = 7	PRECISION 42.8 BITS		PRECISION 12.88 DIGITS
	(3) .60000 00000 17735 65627	Q00	(1) .60000 00001 85487 81950
	(2) -.53146 31560 46051 21441	Q01	(1) -.27000 00228 71423 36254
	(-1) -.62142 65724 53436 31762	Q02	(0) -.39213 32157 59236 09210
	(-2) -.61567 00647 77271 77007	Q03	(0) -.19426 73716 67095 18060
	(-3) -.74600 27544 43133 62004	Q04	(0) -.11865 37561 64675 10652
	(-3) -.62306 25472 73362 23427	Q05	(-1) -.98412 84841 87263 03372
	(-6) -.65421 12156 50153 36502	Q06	(-1) -.13069 70578 48487 74584
	(-2) -.43250 51273 40607 72036	Q07	(0) -.13800 54195 20651 93502

ARSIN(Y) $|Y| < 0.5$, $\mathrm{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS
M = 8	PRECISION 46.9 BITS			PRECISION 14.11 DIGITS
(3)	.57777 77777 76650 56752	Q00	(1)	.59999 99999 86374 26200
	03405 32116 73442 70216			01914 59031 31286 57232
(2)	-.53146 31455 45627 45364	Q01	(1)	-.26999 99979 10322 07187
	14764 26473 44567 13435			38736 96825 18695 69075
(-1)	-.62143 42770 36367 72732	Q02	(0)	-.39214 39612 63967 65017
	45737 02421 64027 00470			46229 53369 77955 07937
(-2)	-.61531 45524 06721 23254	Q03	(0)	-.19404 28810 44547 51459
	33336 05160 46771 77300			90092 63798 52759 71841
(-3)	-.76052 26510 10131 26027	Q04	(0)	-.12125 53167 86436 19839
	55615 55346 47513 74000			06901 99200 92104 50725
(-3)	-.51277 11526 72242 52465	Q05	(-1)	-.80807 30853 96567 08399
	57377 72405 01363 40000			62358 85012 94052 92998
(-3)	-.52105 31756 77724 06446	Q06	(-1)	-.82296 01185 28758 59680
	30565 75243 47534 00000			83709 59663 94166 48540
(-6)	.42766 63050 76772 06700	Q07	(-2)	.85405 34079 42694 57240
	07441 71160 72140 00000			70761 03447 46135 95509
(-2)	-.41036 70575 05312 75444	Q08	(0)	-.12914 18962 84259 60198
	25705 15061 57620 00000			80294 91347 35701 75071

M = 9	PRECISION 50.9 BITS			PRECISION 15.33 DIGITS
(3)	.60000 00000 00054 03151	Q00	(1)	.60000 00000 01001 58353
	77762 47127 25451 25217			80457 41554 96489 56119
(2)	-.53146 31463 54726 47571	Q01	(1)	-.27000 00001 86958 24214
	30252 56311 71351 54235			46698 94582 18804 96812
(-1)	-.62143 35661 60271 62001	Q02	(0)	-.39214 27362 39204 92588
	07725 54343 30760 63730			42927 66610 93580 56883
(-2)	-.61535 65445 67066 72244	Q03	(0)	-.19407 52952 03610 03452
	77646 34647 37060 13400			90278 64515 90092 29045
(-3)	-.75652 67515 63174 20043	Q04	(0)	-.12076 90031 50130 95291
	06574 35513 63356 60000			52287 01764 34774 78247
(-3)	-.53504 33443 17643 74303	Q05	(-1)	-.85221 98014 51068 67138
	42310 75370 73356 00000			79066 54880 23671 14516
(-4)	-.72750 15465 51632 13570	Q06	(-1)	-.57571 81658 71886 77082
	14301 13172 33440 00000			15208 57063 80544 29598
(-3)	-.46311 70064 07275 66410	Q07	(-1)	-.74988 84807 75021 96911
	73142 70316 63600 00000			10554 16995 38575 92360
(-5)	.67130 57152 27111 56561	Q08	(-1)	.26940 09553 99505 63733
	13616 04537 51000 00000			63527 71987 25714 05826
(-3)	-.77333 42005 60165 76155	Q09	(0)	-.12388 43209 28901 45983
	26130 75231 66000 00000			32886 46206 61918 63266

ARSIN(Y) $|Y| < 0.5$, $\text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)$

M = 10

BINARY COEFFICIENTS (PRECISION 54.9 BITS)			DECIMAL COEFFICIENTS (PRECISION 16.54 DIGITS)	
(3)	.57777 77777 77774 60605	Q00	(1)	.59999 99999 99926 37400
	60471 61335 64454 52211			55152 43266 06429 79890
(2)	-.53146 31463 11764 05074	Q01	(1)	-.26999 99999 83566 33982
	20324 67317 36375 64055			30545 75722 86648 49823
(-1)	-.62143 36320 63044 66353	Q02	(0)	-.39214 28699 03705 82865
	63715 26744 31550 21310			32332 55350 78730 49742
(-2)	-.61535 20775 56320 76535	Q03	(0)	-.19407 09347 19668 59545
	22332 12030 31663 04000			87352 46997 68581 40441
(-3)	-.75700 25515 24632 14032	Q04	(0)	-.12085 08998 76679 52127
	53334 06734 77507 40000			79570 82104 26724 54005
(-3)	-.53112 51421 02706 07701	Q05	(-1)	-.84269 13817 47262 47935
	65502 16453 21760 00000			68430 45212 45216 42875
(-3)	-.41070 13627 26301 53144	Q06	(-1)	-.64667 45111 97093 51401
	74554 21135 26100 00000			73570 74584 10001 93249
(-4)	-.52061 02504 25130 42671	Q07	(-1)	-.41109 16356 61913 99638
	72041 15655 01000 00000			57535 36948 45241 41593
(-3)	-.45450 66747 55001 02116	Q08	(-1)	-.73398 05081 75627 96194
	22343 13273 56000 00000			57124 22265 20421 05987
(-4)	.54775 47340 45713 32773	Q09	(-1)	.43940 76581 76581 28169
	74106 22547 44000 00000			19912 29388 09394 10163
(-3)	-.76024 56137 11566 36373	Q10	(0)	-.12117 27968 36129 05368
	45327 65271 52000 00000			67475 08003 78009 70076

M = 11

BINARY COEFFICIENTS (PRECISION 58.9 BITS)			DECIMAL COEFFICIENTS (PRECISION 17.74 DIGITS)	
(3)	.60000 00000 00000 17165	Q00	(1)	.60000 00000 00005 41057
	35153 70246 35015 47605			22799 56034 46826 97117
(2)	-.53146 31463 15026 60556	Q01	(1)	-.27000 00000 01422 91800
	55504 07050 44070 53253			32216 09519 95314 15437
(-1)	-.62143 36262 45572 65562	Q02	(0)	-.39214 28558 36863 75160
	56764 43721 40512 74770			40365 56305 42058 66908
(-2)	-.61535 25510 71675 26474	Q03	(0)	-.19407 14886 65467 15327
	74270 76436 64066 03000			05918 27524 13818 91111
(-3)	-.75675 00464 21135 76533	Q04	(0)	-.12083 82011 70378 94926
	15260 54032 00266 00000			41300 50622 74415 87265
(-3)	-.53172 55371 71422 22716	Q05	(-1)	-.84452 47936 74931 66425
	72750 06206 16420 00000			73228 88596 74698 93183
(-3)	-.40160 77346 31344 10557	Q06	(-1)	-.62931 02800 81801 07545
	37765 15532 53400 00000			39925 03742 97189 31628
(-4)	-.65235 41300 14051 61426	Q07	(-1)	-.52058 26089 88493 93996
	27733 04626 64000 00000			62830 32944 39292 19223
(-5)	-.71022 06235 42221 31561	Q08	(-1)	-.27849 29137 95544 10052
	33677 16651 00000 00000			39038 82246 09906 36830
(-3)	-.46770 33060 20023 71462	Q09	(-1)	-.76142 97236 43939 12582
	10215 13771 40000 00000			60334 33841 90749 44432
(-4)	.76050 51431 30631 26346	Q10	(-1)	.60624 40721 94381 38088
	22675 17343 00000 00000			99062 12307 07852 16317
(-3)	-.75507 03641 76343 66633	Q11	(0)	-.12038 82584 78569 75423
	31415 56563 00000 00000			47433 28549 95478 31270

ARSIN(Y) $|Y| < 0.5$, $\text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)$

	BINARY COEFFICIENTS			DECIMAL COEFFICIENTS

M = 12

	PRECISION 62.9 BITS			PRECISION 18.94 DIGITS
(3)	.57777 77777 77777 76703	Q00	(1)	.59999 99999 99999 60258
	21200 45101 75052 25562			84164 65764 67233 44180
(2)	-.53146 31463 14616 71615	Q01	(1)	-.26999 99999 99878 38638
	32036 22127 50765 16127			05384 25338 28235 53524
(-1)	-.62143 36265 53102 30166	Q02	(0)	-.39214 28572 73063 02369
	10102 45077 43446 45700			68483 40659 63351 10618
(-2)	-.61535 25050 51343 56122	Q03	(0)	-.19407 14215 50487 75059
	46251 31302 26435 62000			86696 99531 76111 74215
(-3)	-.75675 37372 36127 01734	Q04	(0)	-.12084 00421 85900 49423
	54003 74167 22202 00000			32365 32202 59939 71094
(-3)	-.53162 21270 42424 00252	Q05	(-1)	-.84420 28525 04878 55009
	62341 27762 63400 00000			42484 67785 60705 30339
(-3)	-.40323 40253 26160 74260	Q06	(-1)	-.63306 82841 91187 40483
	12004 26233 72000 00000			87492 82219 61197 59374
(-4)	-.62173 03036 56255 03346	Q07	(-1)	-.49062 82007 71770 89751
	16004 56243 40000 00000			64876 55746 29267 15711
(-4)	-.55215 63265 72515 41537	Q08	(-1)	-.44215 77916 45981 14831
	60436 72772 00000 00000			76402 56762 89698 04989
(-5)	-.40333 01655 33177 57756	Q09	(-1)	-.15833 88207 43890 71323
	22424 07350 00000 00000			42703 93931 99774 23494
(-3)	-.52240 10442 54571 34775	Q10	(-1)	-.82642 11224 14780 49342
	70366 64240 00000 00000			83106 02325 68744 28205
(-3)	.47626 20231 20406 51174	Q11	(-1)	.77721 61360 57413 26628
	11112 47540 00000 00000			40695 68836 13418 86571
(-3)	-.76020 30505 24740 23742	Q12	(0)	-.12115 62535 40790 40083
	70056 72360 00000 00000			09970 89017 06329 05341

M = 13

	PRECISION 66.9 BITS			PRECISION 20.14 DIGITS
(3)	.60000 00000 00000 00052	Q00	(1)	.60000 00000 00000 02917
	02561 07327 34307 41071			28547 74131 36139 34902
(2)	-.53146 31463 14632 40202	Q01	(1)	-.27000 00000 00010 27707
	35036 72367 02501 66374			57419 48323 66065 39814
(-1)	-.62143 36265 27372 65302	Q02	(0)	-.39214 28571 30165 14109
	37744 10047 57224 52540			54461 00297 01633 27434
(-2)	-.61535 25112 15101 61057	Q03	(0)	-.19407 14293 63715 47443
	32235 17134 62474 30000			64918 92252 12442 47239
(-3)	-.75675 33202 42525 60035	Q04	(0)	-.12083 97898 05663 68081
	03501 26324 52134 00000			90218 26921 50628 54495
(-3)	-.53163 51270 14247 70202	Q05	(-1)	-.84425 53041 88261 80943
	17303 00535 27300 00000			69423 80623 28131 63955
(-3)	-.40300 13552 50057 71203	Q06	(-1)	-.63233 11974 13687 62741
	51562 02010 64000 00000			36538 91332 23911 03093
(-4)	-.62764 50030 55237 60636	Q07	(-1)	-.49782 99284 74966 76814
	74326 21203 00000 00000			75516 07962 91786 37899
(-4)	-.50152 62236 70043 60447	Q08	(-1)	-.39266 17831 90031 58195
	76707 42350 00000 00000			01959 39150 17255 36120
(-4)	-.50504 47275 73536 02047	Q09	(-1)	-.39681 65430 31342 52713
	40065 31740 00000 00000			45919 30502 26503 14382
(-10)	-.75765 31744 27433 41034	Q10	(-2)	-.37829 16713 56429 01003
	36456 14000 00000 00000			22155 68399 95326 39776
(-3)	-.57370 71216 37461 56525	Q11	(-1)	-.92746 33397 50777 36424
	50273 75200 00000 00000			95960 89229 96181 77537
(-3)	.61025 73755 12775 63224	Q12	(-1)	.95786 80772 93746 06191
	37337 60000 00000 00000			86786 14346 13076 87328
(-3)	-.77064 53602 62425 23046	Q13	(0)	-.12324 78472 85786 69776
	52101 67400 00000 00000			32442 94610 19047 40309

ARSIN(Y) $|Y| < 0.5$, $\text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)$

M = 14

BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
PRECISION 70.8 BITS			PRECISION 21.33 DIGITS	
(3)	.57777 77777 77777 77774	Q00	(1)	.59999 99999 99999 99785
	72467 71734 65550 07361			98845 47511 62519 68744
(2)	-.53146 31463 14631 41436	Q01	(1)	-.26999 99999 99999 14011
	27046 22600 06333 32400			85782 42183 64659 92770
(-1)	-.62143 36265 31315 24433	Q02	(0)	-.39214 28571 44070 27885
	14616 12716 66221 70240			73770 52364 42492 55566
(-2)	-.61535 25106 33376 20204	Q03	(0)	-.19407 14284 84663 88368
	70145 01674 12124 70000			76116 16686 86475 12609
(-3)	-.75675 33635 76670 46124	Q04	(0)	-.12083 98228 02482 63069
	36426 76713 10220 00000			22587 81088 18241 00898
(-3)	-.53163 33711 44217 66043	Q05	(-1)	-.84424 72777 87525 43528
	65721 64124 04000 00000			00534 66362 68934 11744
(-3)	-.40303 53303 51305 13047	Q06	(-1)	-.63246 45174 27890 38088
	34305 72701 00000 00000			40835 63171 22840 53798
(-4)	-.62642 64003 25651 66453	Q07	(-1)	-.49626 94664 37586 23332
	45541 36506 00000 00000			45954 42445 01995 23959
(-4)	-.51430 75022 53444 37104	Q08	(-1)	-.40574 93914 62188 78342
	04266 10060 00000 00000			63752 31417 11846 86850
(-4)	-.40426 74047 43676 22424	Q09	(-1)	-.31782 03336 19889 96643
	35427 25400 00000 00000			78102 15047 67909 02397
(-4)	-.46557 72453 63314 53074	Q10	(-1)	-.37811 11793 37790 90284
	01652 61000 00000 00000			44779 89199 05474 73205
(-6)	.46046 56360 02072 65474	Q11	(-2)	.92958 09781 65970 86332
	57601 00000 00000 00000			83309 14344 63642 54843
(-3)	-.66442 45473 34570 41342	Q12	(0)	-.10657 72543 01983 69498
	22147 10000 00000 00000			18986 50191 57974 48060
(-3)	.73015 77107 72212 67564	Q13	(0)	.11528 77295 28556 57770
	30270 70000 00000 00000			19887 88070 71109 72998
(-2)	-.40310 02353 55133 76636	Q14	(0)	-.12652 61722 04392 97375
	62020 56000 00000 00000			47413 49590 70655 19321

ARSIN(Y) |Y| < 0.5, $ARSIN(0.5, 0, M) = Y + Y^3/Q(Y^2)$

M = 15

BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
PRECISION 74.8 BITS			PRECISION 22.52 DIGITS	
(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00015
	16361 10635 57731 17237			68945 92012 04418 79179
(2)	-.53146 31463 14631 46632	Q01	(1)	-.27000 00000 00000 07131
	26456 03226 55602 23202			55117 87444 43633 61785
(-1)	-.62143 36265 31157 73620	Q02	(0)	-.39214 28571 42743 16919
	52245 11407 44725 04540			46057 03294 81906 79659
(-2)	-.61535 25106 65707 76463	Q03	(0)	-.19407 14285 80684 66850
	53535 73220 11541 00000			14646 17067 87299 45290
(-3)	-.75675 33572 31616 52442	Q04	(0)	-.12083 98186 60494 78922
	40265 27235 00520 00000			60687 81769 12725 62339
(-3)	-.53163 35661 64070 50461	Q05	(-1)	-.84424 84422 28683 66948
	66725 32301 70000 00000			54524 69967 80600 95760
(-3)	-.40303 05455 27715 72444	Q06	(-1)	-.63244 19907 45104 30545
	66403 23452 40000 00000			79825 96296 90686 55655
(-4)	-.62663 05047 54447 24613	Q07	(-1)	-.49657 97297 75265 20302
	43537 15570 00000 00000			15570 63274 47366 56143
(-4)	-.51166 13506 53107 74422	Q08	(-1)	-.40264 47647 91516 71172
	47740 63200 00000 00000			58199 17713 53597 34452
(-4)	-.42701 33035 03623 74463	Q09	(-1)	-.34060 33089 03054 84703
	73754 61000 00000 00000			41983 68396 95343 40971
(-5)	-.64260 13623 77714 07174	Q10	(-1)	-.25558 64735 03620 52091
	55652 40000 00000 00000			76302 65042 80033 34585
(-4)	-.47254 53660 34561 00363	Q11	(-1)	-.38415 30813 65372 76047
	50323 00000 00000 00000			02488 87656 35470 31892
(-5)	.61604 54031 14646 27321	Q12	(-1)	.24296 46326 92860 20086
	40726 00000 00000 00000			98584 56208 03270 34147
(-3)	-.77560 53022 34713 62002	Q13	(0)	-.12445 32487 41516 30944
	76745 00000 00000 00000			05938 84642 40447 93084
(-2)	.42767 55733 55770 00606	Q14	(0)	.13665 55606 29094 53707
	21603 00000 00000 00000			96103 63689 34994 68779
(-2)	-.41407 33626 01425 52543	Q15	(0)	-.13091 60939 47388 86682
	56006 60000 00000 00000			09229 28580 17689 01363

ARSIN(Y) |Y| < 0.5, $ARSIN(0.5, 0, M) = Y + Y^3/Q(Y^2)$

M = 16

BINARY COEFFICIENTS		DECIMAL COEFFICIENTS
PRECISION 78.7 BITS		PRECISION 23.70 DIGITS
(3) .57777 77777 77777 77777	Q00	(1) .59999 99999 99999 99998
76741 13651 63214 60061		85054 42503 27510 07475
(2) -.53146 31463 14631 46273	Q01	(1) -.26999 99999 99999 99413
70526 47054 64006 11064		15885 66156 55828 74169
(-1) -.62143 36265 31170 54442	Q02	(0) -.39214 28571 42867 68846
42660 57346 33412 54000		24045 08276 94363 27390
(-2) -.61535 25106 63051 37734	Q03	(0) -.19407 14285 70463 65888
41523 71310 77711 60000		19437 80765 64085 42148
(-3) -.75675 33576 55532 13532	Q04	(0) -.12083 98191 62349 40874
60024 62554 22700 00000		58453 06771 42930 84687
(-3) -.53163 35447 16466 52531	Q05	(-1) -.84424 82808 93348 62098
44107 17744 66000 00000		51418 91350 40930 00039
(-3) -.40303 13472 06556 11361	Q06	(-1) -.63244 55818 45152 13411
72733 37643 00000 00000		58540 87134 49158 59213
(-4) -.62660 04443 67320 64405	Q07	(-1) -.49652 23580 74870 24888
42144 53740 00000 00000		10786 38379 14991 15878
(-4) -.51231 35747 25601 17630	Q08	(-1) -.40331 77947 50841 45135
60416 42400 00000 00000		55411 27021 30349 98662
(-4) -.42215 42227 24155 21316	Q09	(-1) -.33473 08324 39285 42021
04502 54000 00000 00000		14649 90389 57124 52508
(-5) -.74135 51315 15163 02763	Q10	(-1) -.29386 18363 13338 70001
16640 00000 00000 00000		37609 38220 06464 73970
(-5) -.50532 41671 36752 36561	Q11	(-1) -.19861 72590 06632 45197
52044 00000 00000 00000		68327 43964 76178 24874
(-4) -.52470 37541 01324 50635	Q12	(-1) -.41611 66219 40230 36023
74230 00000 00000 00000		28095 90366 70361 56714
(-4) .53104 30007 14565 35366	Q13	(-1) .42122 60288 17995 16866
21500 00000 00000 00000		45181 33697 96181 03022
(-2) -.45460 67446 33124 15472	Q14	(0) -.14685 72109 99458 32236
31530 00000 00000 00000		07437 72852 81293 55920
(-2) .51024 71626 72552 24732	Q15	(0) .16031 57275 62947 97678
12530 00000 00000 00000		16631 70681 46166 92424
(-2) -.42724 33365 40717 32610	Q16	(0) -.13638 63324 54658 67274
65374 00000 00000 00000		78660 89548 32960 86605

ARSIN(Y) $|Y| < 0.5$, $\text{ARSIN}(0.5, 0, M) = Y + Y^3/Q(Y^2)$

M = 17

BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
PRECISION 82.7 BITS			PRECISION 24.88 DIGITS	
(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00000
	00047 57413 77177 21253			08415 81302 44224 83084
(2)	-.53146 31463 14631 46316	Q01	(1)	-.27000 00000 00000 00047
	13516 77423 57275 57267			95116 27848 09402 35276
(-1)	-.62143 36265 31167 70572	Q02	(0)	-.39214 28571 42856 18029
	41555 71677 56430 04100			92040 32661 19319 31674
(-2)	-.61535 25106 63277 12562	Q03	(0)	-.19407 14285 71527 11843
	04226 36563 73763 00000			26368 85341 57970 16111
(-3)	-.75675 33576 15212 24042	Q04	(0)	-.12083 98191 03403 24913
	43535 45364 70000 00000			95057 24039 19538 31194
(-3)	-.53163 35471 53077 36350	Q05	(-1)	-.84424 83023 66831 49016
	40372 75472 60000 00000			02599 78904 23025 27707
(-3)	-.40303 12546 41070 77511	Q06	(-1)	-.63244 50375 01589 78120
	11042 70364 00000 00000			08071 93731 17051 53138
(-4)	-.62660 46012 11045 07130	Q07	(-1)	-.49653 23268 79537 18153
	25325 70100 00000 00000			22362 56432 01646 81921
(-4)	-.51222 30260 05213 54373	Q08	(-1)	-.40318 26090 54718 72442
	33142 10000 00000 00000			40152 40155 20164 61988
(-4)	-.42325 63607 12017 14157	Q09	(-1)	-.33610 93294 55481 83602
	31414 20000 00000 00000			86306 26435 62001 44017
(-5)	-.72000 56273 57351 52052	Q10	(-1)	-.28321 00341 74118 67482
	33640 00000 00000 00000			12327 96934 50625 11630
(-5)	-.65352 37612 75655 76314	Q11	(-1)	-.26102 53947 10667 54648
	55340 00000 00000 00000			06733 25567 19659 37549
(-6)	-.71540 00455 41041 50676	Q12	(-1)	-.14083 86669 23322 53573
	05000 00000 00000 00000			21616 29078 50990 12694
(-4)	-.60722 72466 42751 31006	Q13	(-1)	-.47765 57107 37561 47088
	22600 00000 00000 00000			30043 30302 21003 87167
(-3)	.40512 04021 71360 63464	Q14	(-1)	.63759 09059 98337 34473
	76400 00000 00000 00000			43661 01579 86992 31993
(-2)	-.54516 32771 37020 07130	Q15	(0)	-.17442 64349 07332 69182
	77600 00000 00000 00000			82961 90091 75714 51685
(-2)	.57630 23030 24116 75145	Q16	(0)	.18670 88136 06507 50645
	65600 00000 00000 00000			57944 12249 94332 59321
(-2)	-.44457 20314 21260 61702	Q17	(0)	-.14293 86614 52154 60051
	05000 00000 00000 00000			28580 39287 88897 82082

ARSIN(Y) $|Y| < 0.5,$ ARSIN(0.5, 0, M) $= Y + Y^3/Q(Y^2)$

M = 18

BINARY COEFFICIENTS PRECISION 86.6 BITS			DECIMAL COEFFICIENTS PRECISION 26.07 DIGITS	
(3)	.57777 77777 77777 77777	Q00	(1)	.59999 99999 99999 99999
	77775 05710 02701 14777			99384 21602 32806 41833
(2)	-.53146 31463 14631 46314	Q01	(1)	-.26999 99999 99999 99996
	54011 21212 71547 02514			10668 70292 69240 46194
(-1)	-.62143 36265 31167 75356	Q02	(0)	-.39214 28571 42857 22964
	12247 55773 25536 11700			84299 70689 18715 06062
(-2)	-.61535 25106 63257 72052	Q03	(0)	-.19407 14285 71418 68966
	51316 32534 23234 00000			75019 89758 75275 88544
(-3)	-.75675 33576 20762 20627	Q04	(0)	-.12083 98191 10139 01271
	77762 27331 16200 00000			85737 00595 37421 04807
(-3)	-.53163 35467 23353 67526	Q05	(-1)	-.84424 82996 08092 26143
	45545 45754 10000 00000			63731 84244 19482 60062
(-3)	-.40303 12652 25361 33116	Q06	(-1)	-.63244 51164 52282 15958
	34544 21006 00000 00000			00768 28196 82845 72680
(-4)	-.62660 40406 72067 34005	Q07	(-1)	-.49653 06858 69477 35387
	76477 43000 00000 00000			67001 43543 29382 80864
(-4)	-.51223 55526 55001 65565	Q08	(-1)	-.40320 80380 14814 45418
	30441 30000 00000 00000			22260 91417 08464 07859
(-4)	-.42306 10721 41300 05145	Q09	(-1)	-.33581 04554 47928 31264
	21020 00000 00000 00000			04091 93712 66676 05229
(-5)	-.72432 75574 50500 07370	Q10	(-1)	-.28590 16870 59891 75768
	16360 00000 00000 00000			29710 59686 41837 88645
(-5)	-.61506 76025 05166 52443	Q11	(-1)	-.24237 60356 27020 28231
	66500 00000 00000 00000			84553 16622 21396 30736
(-5)	-.61121 17100 11241 51614	Q12	(-1)	-.24003 25425 39282 65828
	77000 00000 00000 00000			93384 43207 55564 00576
(-7)	-.76424 64603 57067 51114	Q13	(-2)	-.76343 59439 07878 45504
	40000 00000 00000 00000			13098 52247 00012 28157
(-4)	-.72667 15411 53655 12121	Q14	(-1)	-.57478 35394 48964 80828
	20000 00000 00000 00000			59151 87302 52829 52767
(-3)	.56177 21066 53141 07311	Q15	(-1)	.90329 23619 66845 03299
	40000 00000 00000 00000			15285 19072 94348 16603
(-2)	-.65171 12335 04172 06007	Q16	(0)	-.20795 56503 02168 45043
	34000 00000 00000 00000			02476 68980 76549 63715
(-2)	.67277 55072 55506 07012	Q17	(0)	.21630 63424 43702 06217
	50000 00000 00000 00000			84403 40134 42211 51703
(-2)	-.46433 47400 17252 02614	Q18	(0)	-.15060 13274 75058 91865
	44000 00000 00000 00000			19325 94878 76176 19473

ARSIN(Y) |Y| < 0.5, ARSIN(0.5, 0, M) = $Y + Y^3/Q(Y^2)$

M = 19

BINARY COEFFICIENTS			DECIMAL COEFFICIENTS	
PRECISION 90.5 BITS			PRECISION 27.24 DIGITS	
(3)	.60000 00000 00000 00000	Q00	(1)	.60000 00000 00000 00000
	00000 15470 01053 13302			00045 02975 40853 72856
(2)	-.53146 31463 14631 46314	Q01	(1)	-.27000 00000 00000 00000
	63617 17474 43567 25423			31429 58479 29578 64100
(-1)	-.62143 36265 31167 75024	Q02	(0)	-.39214 28571 42857 13511
	13770 56771 77513 51200			72371 57646 30951 80806
(-2)	-.61535 25106 63261 33675	Q03	(0)	-.19407 14285 71429 54609
	06203 10004 01540 00000			84531 04443 50334 13975
(-3)	-.75675 33576 20436 67246	Q04	(0)	-.12083 98191 09387 97611
	01043 51774 42600 00000			77305 92592 98975 29034
(-3)	-.53163 35467 46257 17705	Q05	(-1)	-.84424 82999 51546 53584
	51217 27400 10000 00000			94287 50461 90802 12205
(-3)	-.40303 12640 70017 71332	Q06	(-1)	-.63244 51054 39383 94031
	32421 42254 00000 00000			06109 41530 56466 41491
(-4)	-.62660 41301 36352 62530	Q07	(-1)	-.49653 09434 77942 65242
	24333 52000 00000 00000			49179 83970 38282 79108
(-4)	-.51223 36405 57554 66160	Q08	(-1)	-.40320 35205 42889 16542
	32557 40000 00000 00000			48384 87429 54745 52229
(-4)	-.42311 23740 61065 12175	Q09	(-1)	-.33587 09630 36575 88829
	32770 00000 00000 00000			53105 46920 67917 19449
(-5)	-.72331 21204 61601 47446	Q10	(-1)	-.28527 51701 07311 90041
	12740 00000 00000 00000			26910 09590 61937 03951
(-5)	-.62530 16760 10716 77744	Q11	(-1)	-.24742 34952 06495 75075
	34400 00000 00000 00000			64606 18802 53745 44636
(-5)	-.52527 36305 51005 71166	Q12	(-1)	-.20835 37557 74676 35677
	30000 00000 00000 00000			13649 09431 50981 53629
(-5)	-.57161 17250 57344 74626	Q13	(-1)	-.23057 21237 61918 04552
	40000 00000 00000 00000			93444 59265 70816 70949
(-14)	.44200 67407 45562 00200	Q14	(-3)	.13828 92386 54901 19694
	00000 00000 00000 00000			99450 85812 31684 72957
(-3)	-.44522 23600 20774 43375	Q15	(-1)	-.71603 04489 84261 43291
	00000 00000 00000 00000			01616 59926 01827 38362
(-3)	.77032 27421 45373 01072	Q16	(0)	.12314 74598 85313 34360
	00000 00000 00000 00000			16787 92715 85889 00056
(-2)	-.77457 35273 11000 41204	Q17	(0)	-.24840 89571 84441 58755
	00000 00000 00000 00000			26146 38667 59186 20312
(-2)	.77716 55620 71765 25371	Q18	(0)	.24962 39880 34651 74327
	00000 00000 00000 00000			61072 05282 69894 20282
(-2)	-.50640 12473 61270 01413	Q19	(0)	-.15942 50937 39116 78788
	00000 00000 00000 00000			14240 97683 69134 32634

LOG(X) $\sqrt{2}/2 < X < \sqrt{2}$, $Y = (X-1)/(X+1)$, $LOG(\sqrt{2}, 0, M) = 2Y + Y^3/Q(Y^2)$

$ER(1) = ER(\sqrt{2}) = 0$, $ER(1/X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 1	$.29295*10^{-7}$	1.1722(-), 1.3601(+)
M = 2	$.99921*10^{-10}$	1.1255(+), 1.2770(-), 1.3846(+)
M = 3	$.44545*10^{-12}$	1.0988(-), 1.2222(+), 1.3252(-), 1.3954(+)
M = 4	$.22691*10^{-14}$	1.0815(+), 1.1846(-), 1.2767(+), 1.3518(-), 1.4012(+)
M = 5	$.12517*10^{-16}$	1.0693(-), 1.1576(+), 1.2391(-), 1.3111(+), 1.3681(-), 1.4047(+)
M = 6	$.72790*10^{-19}$	1.0603(+), 1.1373(-), 1.2098(+), 1.2766(-), 1.3342(+), 1.3787(-), 1.4069(+)
M = 7	$.43943*10^{-21}$	1.0534(-), 1.1216(+), 1.1865(-), 1.2479(+), 1.3034(-), 1.3503(+), 1.3861(-), 1.4084(+)
M = 8	$.27278*10^{-23}$	1.0479(+), 1.1090(-), 1.1676(+), 1.2240(-), 1.2765(+), 1.3232(-), 1.3621(+), 1.3913(-), 1.4095(+)
M = 9	$.17301*10^{-25}$	1.0434(-), 1.0988(+), 1.1521(-), 1.2040(+), 1.2532(-), 1.2985(+), 1.3382(-), 1.3709(+), 1.3953(-), 1.4103(+)
M = 10	$.11146*10^{-27}$	1.0397(+), 1.0903(-), 1.1392(+), 1.1871(-), 1.2332(+), 1.2764(-), 1.3157(+), 1.3498(-), 1.3776(+), 1.3983(-), 1.4110(+)
M = 11	$.70612*10^{-30}$	1.0366(-), 1.0831(+), 1.1282(-), 1.1725(+), 1.2157(-), 1.2568(+), 1.2950(-), 1.3293(+), 1.3589(-), 1.3829(+), 1.4006(-), 1.4114(+)

EXP(Y) $|Y| < \ln(2)/2$, $EXP(\ln(2)/2, N, 0) = 1 + 2Y/(2 - Y + Y^2P(Y^2))$

$ER(0) = ER(\ln(2)/2) = 0$, $ER(-X) = -ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
N = 2	$.63842 * 10^{-9}$	.18992(+), .31550(-)
N = 3	$.40152 * 10^{-12}$	.14745(+), .26087(-), .32913(+)
N = 4	$.27364 * 10^{-15}$	.12058(+), .21981(-), .29102(+), .33529(-)
N = 5	$.19349 * 10^{-18}$	.10201(+), .18908(-), .25697(+), .30741(-), .33864(+)
N = 6	$.13962 * 10^{-21}$	.08840(+), .16555(-), .22858(+), .27961(-), .31741(+), .34068(-)
N = 7	$.10202 * 10^{-24}$	.07799(+), .14705(-), .20516(+), .25461(-), .29457(+), .32399(-), .34202(+)
N = 8	$.75175 * 10^{-28}$	.06978(+), .13219(-), .18576(+), .23281(-), .27283(+), .30500(-), .32856(+), .34294(-)
N = 9	$.55720 * 10^{-31}$	.06313(+), .12001(-), .16951(+), .21392(-), .25303(+), .28611(-), .31257(+), .33183(-), .34360(+)

SINH(Y) $|Y| < \ln((1+\sqrt{5})/2)$, $\text{SINH}(\ln((1+\sqrt{5})/2, 0, M) = Y + Y^3/Q(Y^2)$

$ER(0) = ER(\ln(1+\sqrt{5})/2) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 1	$.12837*10^{-6}$	.22063(+), .42706(-)
M = 2	$.11396*10^{-9}$	.16449(+), .33988(-), .45198(+)
M = 3	$.66755*10^{-13}$	.13130(+), .27928(-), .39145(+), .46284(-)
M = 4	$.70249*10^{-17}$	.10950(+), .23645(-), .34038(+), .41925(-), .46862(+)
M = 5	$.40612*10^{-19}$	.09357(-), .20404(+), .29864(-), .37688(+), .43557(-), .47196(+)
M = 6	$.59922*10^{-22}$	.08186(-), .17956(+), .26550(-), .34007(+), .40105(-), .44630(+), .47416(-)
M = 7	$.49082*10^{-25}$	.07276(-), .16024(+), .23857(-), .30864(+), .36890(-), .41771(+), .45365(-), .47565(+)
M = 8	$.19084*10^{-28}$	.06548(-), .14464(+), .21640(-), .28194(+), .34016(-), .38979(+), .42970(-), .45890(+), .47672(-)
M = 9	$.13215*10^{-31}$	.05949(+), .13168(-), .19771(+), .25899(-), .31464(+), .36372(-), .40529(+), .43853(-), .46276(+), .47750(-)

SINH(Y) $|Y| < \ln(1+\sqrt{2})$, $\text{SINH}(\ln(1+\sqrt{2}), 0, M) = Y + Y^3/Q(Y^2)$

$ER(0) = ER(\ln(1+\sqrt{2})) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 1	$.46756*10^{-5}$	.40177(+), .78101(-)
M = 2	$.13980*10^{-7}$	.29993(+), .62098(-), .82737(+)
M = 3	$.27783*10^{-10}$	.23969(+), .51033(-), .71614(+), .84751(-)
M = 4	$.11712*10^{-13}$	.20065(+), .43324(-), .62360(+), .76799(-), .85833(+)
M = 5	$.18026*10^{-15}$	.17081(-), .37264(+), .54582(-), .68937(+), .79729(-), .86433(+)
M = 6	$.91182*10^{-18}$	.14955(-), .32813(+), .48540(-), .62205(+), .73398(-), .81715(+), .86840(-)
M = 7	$.25471*10^{-20}$	.13298(-), .29293(+), .43626(-), .56460(+), .67510(-), .76468(+), .83071(-), .87116(+)
M = 8	$.34995*10^{-23}$	.11977(-), .26456(+), .39590(-), .51591(+), .62258(-), .71359(+), .78679(-), .84041(+), .87313(-)
M = 9	$.67749*10^{-26}$	.10866(+), .24056(-), .36136(+), .47343(-), .57539(+), .66541(-), .74173(+), .80282(-), .84740(+), .87454(-)
M = 10	$.62358*10^{-28}$	.09971(+), .22110(-), .33299(+), .43790(-), .53484(+), .62240(-), .69918(+), .76389(-), .81543(+), .85290(-), .87565(+)

TANH(Y) $|Y| < \ln(3)/2$ $\text{TANH}(\ln(3)/2, 0, M) \approx Y - Y^3/(3 + Y^2 Q(Y^2))$

$ER(0) = ER(\ln(3)/2) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 2	$.48245*10^{-8}$	.33133(-), .50672(+)
M = 3	$.13686*10^{-10}$	.26542(-), .42896(+), .52491(-)
M = 4	$.43338*10^{-13}$	.22162(-), .36807(+), .47026(-), .53327(+)
M = 5	$.14459*10^{-15}$	.19028(-), .32105(+), .42007(-), .49294(+), .53790(-)
M = 6	$.49670*10^{-18}$	.16673(-), .28414(+), .37729(-), .45191(+), .50693(-), .54075(+)
M = 7	$.17375*10^{-20}$	.14837(-), .25458(+), .34137(-), .41439(+), .47310(-), .51624(+), .54264(-)
M = 8	$.61526*10^{-23}$	.13366(-), .23044(+), .31114(-), .38121(+), .44048(-), .48799(+), .52275(-), .54396(+)
M = 9	$.21976*10^{-25}$	.12160(-), .21040(+), .28551(-), .35217(+), .41044(-), .45961(+), .49888(-), .52750(+), .54492(-)
M = 10	$.79000*10^{-28}$	.11154(-), .19351(+), .26360(-), .32678(+), .38328(-), .43260(+), .47409(-), .50709(+), .53108(-), .54564(+)
M = 11	$.28537*10^{-30}$	.10302(-), .17910(+), .24468(-), .30451(+), .35891(-), .40753(+), .44984(-), .48531(+), .51344(-), .53384(+), .54620(-)

SIN(Y) $|Y| < \pi/4$, $SIN(\pi/4, N, 0) = Y + Y^3P(Y^2)$

$ER(0) = ER(\pi/4) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
N = 2	$.23205*10^{-5}$	.36559(+), .69983(-)
N = 3	$.44477*10^{-8}$	.27194(+), .55879(-), .73895(+)
N = 4	$.58471*10^{-11}$	.21661(+), .45934(-), .64141(+), .75607(-)
N = 5	$.55462*10^{-14}$	.18003(+), .38819(-), .55771(+), .68555(-), .76515(+)
N = 6	$.39562*10^{-17}$	.15403(+), .33543(-), .49007(+), .61722(-), .71204(+), .77057(-)
N = 7	$.21951*10^{-20}$	.13460(+), .29498(-), .43562(+), .55714(-), .65608(+), .72920(-), .77406(+)
N = 8	$.97348*10^{-24}$	.11952(+), .26308(-), .39134(+), .50572(-), .60376(+), .68288(-), .74096(+), .77645(-)
N = 9	$.35273*10^{-27}$	.10749(+), .23731(-), .35483(+), .46192(-), .55679(+), .63746(-), .70214(+), .74938(-), .77815(+)
N = 10	$.10634*10^{-30}$	.09765(+), .21609(-), .32427(+), .42447(-), .51527(+), .59514(-), .66262(+), .71644(-), .75562(+), .77941(-)

COS(Y) $|Y| < \pi/4$, $COS(\pi/4, N, 0) = 1 + Y^2(-.5 + Y^2P(Y^2))$

$ER(0) = ER(\pi/4) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
N = 3	$.99493*10^{-7}$	.49199(-), .73091(+)
N = 4	$.13274*10^{-9}$	.39375(-), .62490(+), .75365(-)
N = 5	$.13287*10^{-12}$	.32766(-), .53804(+), .67966(-), .76425(+)
N = 6	$.10127*10^{-15}$	.28041(-), .46957(+), .60932(-), .70960(+), .77019(-)
N = 7	$.60233*10^{-19}$	.24499(-), .41534(+), .54816(-), .65254(+), .72812(-), .77390(+)
N = 8	$.28613*10^{-22}$	.21746(-), .37174(+), .49622(-), .59945(+), .68122(-), .74048(+), .77638(-)
N = 9	$.11081*10^{-25}$	.19548(-), .33610(+), .45224(-), .55198(+), .63533(-), .70137(+), .74918(-), .77813(+)
N = 10	$.35616*10^{-29}$	.17752(-), .30651(+), .41483(-), .51014(+), .59263(-), .66158(+), .71611(-), .75556(+), .77941(-)
N = 11	$.96433*10^{-33}$	.16257(-), .28159(+), .38278(-), .47338(+), .55377(-), .62336(+), .68143(-), .72727(+), .76036(-), .78036(+)

TAN(Y) $|Y| < \pi/4$, $TAN(\pi/4, 0, M) = Y + Y^3/(3 + Y^2Q(Y^2))$

$ER(0) = ER(\pi/4) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 2	$.12158*10^{-6}$	.49264(-), .73112(+)
M = 3	$.71323*10^{-9}$	.39416(+), .62527(-), .75375(+)
M = 4	$.46965*10^{-11}$	.32799(-), .53839(+), .67987(-), .76430(+)
M = 5	$.32665*10^{-13}$	.28065(+), .46988(-), .60956(+), .70973(-), .77022(+)
M = 6	$.23421*10^{-15}$	.24517(-), .41559(+), .54840(-), .65271(+), .72821(-), .77392(+)
M = 7	$.17114*10^{-17}$	.21761(+), .37195(-), .49644(+), .59963(-), .68134(+), .74055(-), .77639(+)
M = 8	$.12665*10^{-19}$	.19559(-), .33627(+), .45243(-), .55216(+), .63547(-), .70146(+), .74923(-), .77814(+)
M = 9	$.94566*10^{-22}$	.17761(+), .30665(-), .41500(+), .51030(-), .59278(+), .66169(-), .71619(+), .75559(-), .77942(+)
M = 10	$.71075*10^{-24}$	.16265(-), .28172(+), .38293(-), .47353(+), .55392(-), .62349(+), .68151(-), .72731(+), .76039(-), .78038(+)
M = 11	$.53688*10^{-26}$	.15001(+), .26046(-), .35523(+), .44119(-), .51886(+), .58777(-), .64732(+), .69687(-), .73594(+), .76411(-), .78113(+)
M = 12	$.40712*10^{-28}$	.13919(-), .24214(+), .33112(-), .41265(+), .48734(-), .55484(+), .61465(-), .66621(+), .70904(-), .74276(+), .76706(-), .78171(+)
M = 13	$.30967*10^{-30}$	.12982(+), .22619(-), .30996(+), .38735(-), .45900(+), .52467(-), .58395(+), .63635(-), .68145(+), .71885(-), .74826(+), .76943(-), .78219(+)

ATAN(Y) $|Y| < \tan(\pi/12)$, $\mathrm{ATAN}(\tan(\pi/12)\ 0,\ M) = Y - Y^3/Q(Y^2)$

$ER(0) = ER(\tan(\pi/12)) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 1	$.36768*10^{-6}$	.12161(+), .23714(-)
M = 2	$.28901*10^{-8}$	.09071(+), .18822(-), .25135(+)
M = 3	$.29772*10^{-10}$	.07247(+), .15453(-), .21727(+), .25753(-)
M = 4	$.35092*10^{-12}$	.06038(+), .13064(-), .18857(+), .23287(-), .26080(+)
M = 5	$.44825*10^{-14}$	.05176(+), .11298(-), .16560(+), .20931(-), .24224(+), .26273(-)
M = 6	$.60388*10^{-16}$	.04530(+), .09944(-), .14720(+), .18877(-), .22289(+), .24830(-), .26397(+)
M = 7	$.84485*10^{-18}$	.04028(+), .08877(-), .13227(+), .17128(-), .20493(+), .23226(-), .25244(+), .26482(-)
M = 8	$.12157*10^{-19}$	.03627(+), .08014(-), .11998(+), .15643(-), .18889(+), .21664(-), .23900(+), .25540(-), .26542(+)
M = 9	$.17877*10^{-21}$	.03298(+), .07302(-), .10971(+), .14376(-), .17476(+), .20215(-), .22539(+), .24400(-), .25759(+), .26586(-)
M = 10	$.26747*10^{-23}$	.03024(+), .06706(-), .10102(+), .13288(-), .16234(+), .18898(-), .21236(+), .23209(-), .24781(+), .25925(-), .26620(+)
M = 11	$.40585*10^{-25}$	.02792(+), .06199(-), .09358(+), .12345(-), .15141(+), .17712(-), .20022(+), .22039(-), .23733(+), .25078(-), .26054(+), .26646(-)
M = 12	$.62304*10^{-27}$	.02593(+), .05763(-), .08714(+), .11523(-), .14176(+), .16646(-), .18905(+), .20925(-), .22681(+), .24151(-), .25314(+), .26156(-), .26667(+)
M = 13	$.96588*10^{-29}$	.02421(+), .05384(-), .08152(+), .10800(-), .13318(+), .15687(-), .17882(+), .19880(-), .21660(+), .23202(-), .24488(+), .25505(-), .26239(+), .26683(-)
M = 14	$.15099*10^{-30}$	.02270(+), .05052(-), .07657(+), .10159(-), .12554(+), .14824(-), .16949(+), .18910(-), .20688(+), .22267(-), .23631(+), .24766(-), .25661(+), .26306(-), .26696(+)

ARSIN(Y) $|Y| < 0.5$, $ARSIN(0.5, 0, M) = Y + Y^3/Q(Y^2)$

$ER(0) = ER(.5) = 0$, $ER(-X) = ER(X)$

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS
M = 1	$.11955*10^{-4}$	.24038(-), .44933(+)
M = 2	$.38635*10^{-6}$	.17844(+), .36192(-), .47235(+)
M = 3	$.16363*10^{-7}$	.14167(-), .29819(+), .41247(-), .48241(+)
M = 4	$.79250*10^{-9}$	.11739(+), .25196(-), .35967(+), .43921(-), .48778(+)
M = 5	$.41582*10^{-10}$	.10017(-), .21750(+), .31637(-), .39644(+), .45521(-), .49100(+)
M = 6	$.23011*10^{-11}$	.08735(+), .19104(-), .28125(+), .35834(-), .42033(+), .46559(-), .49308(+)
M = 7	$.13226*10^{-12}$	.07743(-), .17018(+), .25257(-), .32546(+), .38733(-), .43677(+), .47272(-), .49452(+)
M = 8	$.78206*10^{-14}$	.06953(+), .15334(-), .22889(+), .29731(-), .35748(+), .40822(-), .44857(+), .47783(-), .49555(+)
M = 9	$.47270*10^{-15}$	.06309(-), .13948(+), .20908(-), .27317(+), .33095(-), .38143(+), .42378(-), .45734(+), .48162(-), .49631(+)
M = 10	$.29077*10^{-16}$	.05774(+), .12790(-), .19232(+), .25238(-), .30749(+), .35688(-), .39981(+), .43569(-), .46404(+), .48452(-), .49689(+)
M = 11	$.18144*10^{-17}$	.05322(-), .11807(+), .17797(-), .23435(+), .28678(-), .33463(+), .37729(-), .41423(+), .44501(-), .46927(+), .48678(-), .49735(+)
M = 12	$.11457*10^{-18}$	.04936(+), .10963(-), .16556(+), .21860(-), .26843(+), .31454(-), .35644(+), .39364(-), .42575(+), .45243(-), .47344(+), .48857(-), .49771(+)
M = 13	$.73083*10^{-20}$	.04602(-), .10231(+), .15474(-), .20475(+), .25212(-), .29644(+), .33727(-), .37422(+), .40693(-), .43509(+), .45845(-), .47681(+), .49003(-), .49800(+)
M = 14	$.47018*10^{-21}$	.04311(+), .09590(-), .14523(+), .19250(-), .23756(+), .28009(-), .31972(+), .35610(-), .38891(+), .41788(-), .44278(+), .46339(-), .47958(+), .49122(-), .49824(+)
M = 15	$.30475*10^{-22}$	.04054(-), .09024(+), .13680(-), .18158(+), .22451(-), .26530(+), .30365(-), .33927(+), .37186(-), .40118(+), .42702(-), .44917(+), .46750(-), .48188(+), .49221(-), .49844(+)
M = 16	$.19881*10^{-23}$	.03826(+), .08521(-), .12928(+), .17181(-), .21275(+), .25188(-), .28894(+), .32368(-), .35583(+), .38518(-), .41153(+), .43471(-), .45456(+), .47096(-), .48381(+), .49304(-), .49860(+)
M = 17	$.13045*10^{-24}$	.03622(-), .08071(+), .12254(-), .16301(+), .20212(-), .23968(+), .27546(-), .30925(+), .34082(-), .36998(+), .39654(-), .42034(+), .44125(-), .45913(+), .47389(-), .48545(+), .49375(-), .49874(+)

ARSIN(Y) $|Y| < 0.5$, ARSIN(0.5, 0, M) = $Y + Y^3/Q(Y^2)$

ER(0) = ER(0.5) = 0, ER(-X) = ER(X)

INDEX	EXTREMAL ERROR	POINTS OF EXTREMAL RELATIVE ERROR WITH SIGNS OF THE ERRORS				
M = 18	$.86028*10^{-26}$	.03439(+),	.07666(-),	.11646(+),	.15505(-),	.19246(+),
		.22854(-),	.26308(+),	.29590(-),	.32680(+),	.35561(-),
		.38216(+),	.40630(-),	.42791(+),	.44685(-),	.46304(+),
		.47640(-),	.48685(+),	.49435(-),	.49886(+)	
M = 19	$.56991*10^{-27}$	.03273(-),	.07299(+),	.11095(-),	.14782(+),	.18366(-),
		.21834(+),	.25169(-),	.28353(+),	.31372(-),	.34207(+),
		.36845(-),	.39272(+),	.41475(-),	.43444(+),	.45169(-),
		.46642(+),	.47856(-),	.48806(+),	.49487(-),	.49897(+)

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, December 4, 1971,
132-80.

APPENDIX - STRATEGY OF ARGUMENT REDUCTION

Within the scope of this report argument reduction is required only for the exponential function and for the circular functions. No argument reduction is required for the logarithm approximation in the sense that the working argument is obtained without error from the floating-point representation of the actual argument.

For these cases, given the related transcendental constant K (either $\ln(2)$ or $\pi/2$), the reduced argument y is defined in terms of K and the given argument x by

$$y = x - nK \tag{A1}$$

where n is an integer. Because the approximations are constrained to have negligible error for $y = \pm K/2$, adequately small errors will result for a somewhat wider interval. We, therefore, require only that y lie in the interval

$$-\left(\frac{K}{2} + \Delta\right) < y < \frac{K}{2} + \Delta \tag{A2}$$

Table I given at the end of this appendix shows the value of Δ allowed by each of these approximations.

Given an upper bound N on the magnitude of the integers allowed for use in relation (A1) a value of n for which inequality (A2) is satisfied is given by

$$n = [kx] \tag{A3}$$

The symbol $[Z]$ means the nearest integer to Z and the multiplier k satisfies the inequality

$$\frac{1}{K + \dfrac{2\Delta}{2N + 1}} < k < \frac{1}{K - \dfrac{2\Delta}{2N - 1}} \tag{A4}$$

If $2\Delta/(2N + 1)$ is greater than β times the value of a one in the least significant digit of the machine precision representation of K, then the numbers $1/\left\{K + [2\Delta/(2N + 1)]\right\}$, $1/K$, $1/\left\{K - [2\Delta/(2N - 1)]\right\}$ have distinct representations. The rounded for storage representation of the value $1/K$ is then a suitable value for k.

In the case of the exponential function the bound N is typically determined by the limitations of exponent overflow or underflow on the representation of the computed result. For the circular functions which (except for poles) are defined and representable for all arguments the bound on N must be somewhat arbitrary and is related to the details of the actual evaluation of the reduced argument y.

For any of these functions the required transcendental constant, ln(2) or $\pi/2$, cannot be exactly represented. It may, however, be represented to any required precision as a sequence of constants K_1, K_2, . . . of successively decreasing magnitude whose correct sum is very nearly equal to the desired K. At least three such constants are generally required. A minimum limitation on the lengths of the constants K_1 and K_2 is that the products nK_1 and nK_2 be exactly representable in the floating-point notation of the computer of implementation.

A further requirement of any implementation is that the difference $x - nK_1$ be computed exactly. This cannot be guaranteed for an arithmetic system in which no guard digits are provided for floating point addition unless the given argument x is broken into shorter parts and the constant K_1 subject to more severe restrictions on its length. In any case, when K_1 is subjected only to the limitation that the product nK_1 be exactly representable the difference $x - nK_1$ is always exactly representable.

For any n there is always some value of x such that $x - nK_1$ equals zero. The reduced argument is then the negative of the correctly rounded sum of $nK_2 + nK_3$ which should cause a minimum of trouble.

If K_1, K_2, and K_3 are of the same sign and the sign of $x - nK_1$ is opposite to that of x, the final calculation of the reduced argument requires the correct addition of three terms of like sign. No arithmetic trouble occurs in adding these terms in the order $(nK_3 + nK_2) + (x - nK_1)$ with rounding on the final addition. If K_1, K_2, and K_3 are of the same sign and the sign of $x - nK_1$ is the same as the sign of x, which should happen in about one-half the cases, completion of the argument reduction can cause further cancellation of lead digits and result in an unrecoverable error. Greater care with regard to the details of the reduction is required to avoid unwanted loss of precision. In this situation the difficulty caused by mixed signs could be resolved by the use of a second set of constants K'_1, K'_2, . . ., where K'_1 is just larger than K_1 and the K'_2, . . . are negative; therefore, the smaller terms nK'_2, . . . have the same sign as $x - nK'_1$. The small interval for which $x - nK_1$ has the same sign as x but $x - nK'_1$ is opposite in sign remains unresolved. Assuming that this variant is implemented, difficulty with further cancellation can occur only for very small reduced arguments.

TABLE I. - VALUES OF Δ FOR VARIOUS APPROXIMATIONS

J	exp(Y) EXP[ln(2)/2, J, 0]	sin(Y) SIN(π/4, J, 0)	cos(Y) COS(π/4, J, 0)	tan(Y) TAN(π/4, 0, J)
2	0.01041	0.02881	-------	0.01780
3	.00585	.01561	0.01788	.01015
4	.00378	.00983	.01054	.00702
5	.00265	.00677	.00704	.00505
6	.00196	.00495	.00507	.00382
7	.00152	.00378	.00383	.00300
8	.00121	.00298	.00300	.00242
9	.00098	.00241	.00242	.00199
10	------	.00199	.00199	.00167
11	------	------	.00167	.00142
12	------	------	------	.00122
13	------	------	------	------

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