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CROSS-SPECTRAL FUNCTIONS BASED ON VON KÁRMÁN'S SPECTRAL EQUATION

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16. Abstract

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ABSTRACT

Cross-spectral functions for the vertical and longitudinal components of turbulence of a two-dimensional gust field are derived from the point correlation function for turbulence due to von Kármán. Closed form solutions in terms of Bessel functions of order 5/6 and 11/6 are found. An asymptotic expression for large values of the frequency agreement, and series results for small values of frequency, are also given. These results now form the base for studying the effect of spanwise variations in turbulence for a turbulence environment which is characterized by the von Kármán isotropic spectral relations. Previous studies were based mainly on the Dryden-type spectral representation.

INTRODUCTION

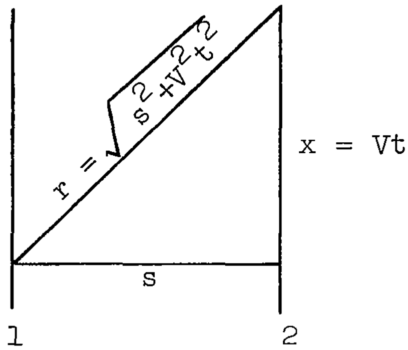
In references 1 and 2 derivations are given for the cross-spectra for a two-dimensional gust field based upon the point correlation and spectral functions which were often used in gust studies at that time; specifically, the point functions due to Dryden and which yield a $\frac{1}{\omega^2}$ behavior at high frequency for the spectrum were employed. More recent gust studies indicate that a correlation and spectral function due to von Kármán - one which indicates a $\frac{1}{\omega^{5/3}}$ behavior at high frequencies - seems to be a better model for stabilized atmospheric turbulence. Study of this function revealed that it too could be extended to the two-dimensional gust field case in closed form fashion. The purpose of this report is to present the derivation and the cross-spectral functions that apply for this case.

SYMBOLS

$I_n(\ )$	modified Bessel function of the first kind
$K_n(\ )$	modified Bessel function of the second kind
L	integral scale of turbulence
r	correlation distance $\sqrt{s^2 + V^2t^2}$
$R(x)$	point correlation function
$R_{12}(\ )$	cross-correlation function for w component
$R_{12_u}(\ )$	cross-correlation function for u component
s	separation distance between paths 1 and 2
t	time
u	nondimensional correlation distance $\frac{x}{1.339L}$ or $\frac{\sigma}{1.339}\sqrt{1 + \left(\frac{Vt}{s}\right)^2}$
V	velocity
x	correlation distance Vt
z	general variable
ν	reduced frequency $\frac{\omega L}{V}$
σ	nondimensional separation distance $\frac{s}{L}$
σ_w	rms value of w turbulence component
$\phi(\ )$	power spectra
$\phi_{12}(\ )$	cross-spectra
ω	angular frequency

CROSS-SPECTRA FOR TWO-DIMENSIONAL GUST STRUCTURE BASED ON THE VON KÁRMÁN SPECTRAL FUNCTION

As in reference 1, consider the gust field to be homogenous and isotropic and momentarily frozen (Taylor hypothesis). As such, the correlation function should be independent of any flight traverse path taken. With reference to the following sketch,



correlation between gust velocities along paths 1 and 2 should thus be associated with the correlation distance $r = \sqrt{s^2+V^2t^2}$ in place of the correlation distance x of the point correlation function.

The von Kármán correlation function for vertical gust velocities is given by

$$R(x) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \left[u^{1/3} K_{1/3}(u) - \frac{1}{2} u^{4/3} K_{2/3}(u)\right] \tag{1}$$

where $u = \frac{x}{1.339L}$, and where L is the integral scale length. To derive the cross-spectra between the vertical velocities along path 1 and the vertical velocities along path 2, we replace x by r and hence define the cross-correlation function as

$$R_{12}(t) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \left[u^{1/3} K_{1/3}(u) - \frac{1}{2} u^{4/3} K_{2/3}(u)\right] \tag{2}$$

where $u = \frac{\sigma}{1.339} \sqrt{1 + \left(\frac{Vt}{s}\right)^2}$

$\sigma = \frac{s}{L}$

By means of

$$K_{n-1}(z) - K_{n+1}(z) = - \frac{2n}{z} K_n(z) \tag{3}$$

equation (2) is rewritten so as to appear

$$R_{12}(t) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \left[\frac{4}{3} u^{1/3} K_{1/3}(u) - \frac{1}{2} u^{4/3} K_{4/3}(u)\right] \tag{4}$$

For later use, the fact is also noted that

$$K_n(z) = K_{-n}(z) \tag{5}$$

The cross-spectral function follows from equation (4) as

$$\phi_{12}(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{12}(t)\, e^{-i\omega t} dt \tag{6}$$

Fortunately, this equation, with R_{12} represented by equation (4) has a closed form integration (transform 941 in reference 3). After some manipulation and use of equation (5), the result for ϕ_{12} for the case under consideration is found to be

$$\phi_{12}(\omega) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \frac{1}{\sqrt{2\pi}} \frac{L}{V} \left(\frac{1}{1.339}\right)^{8/3} \left[\frac{8}{3}(1.339)^2 \frac{\sigma^{5/3}}{z^{5/6}} K_{5/6}(z) - \frac{\sigma^{1/3}}{z^{11/6}} K_{11/6}(z)\right] \tag{7}$$

where $z = \frac{\sigma}{1.339}\sqrt{1 + (1.339\nu)^2}$

$\nu = \frac{\omega L}{V}$

$\sigma = \frac{s}{L}$

Because

$$\phi_{12}(\nu) = \frac{V}{L}\,\phi_{12}(\omega)$$

equation (7) may also be written in terms of the frequency argument ν as

$$\phi_{12}(\nu) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1.339}\right)^{8/3} \left[\frac{8}{3}\,(1.339)^2\,\frac{\sigma^{5/3}}{z^{5/6}}\,K_{5/6}(z) - \frac{\sigma^{11/3}}{z^{11/6}}\,K_{11/6}(z)\right] \tag{8}$$

This is the basic result of the present paper. Its evaluation is encumbered because of the uncommon fractional orders for the Bessel functions. This evaluation and associated curves are discussed in a subsequent section.

As a check, equation (8) should reduce to the one-dimensional spectral result when $\sigma = 0$. This reduction is as follows. For z small

$$K_n(z) = 2^{n-1}(n-1)!\; z^{-n} \quad ; \quad n \neq 0$$

With this result, equation (8) becomes

$$\phi_{12}(\nu) = \sigma_w^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1.339}\right)^{8/3} \left[\frac{\frac{8}{3}\, 2^{-\frac{1}{6}} \left(-\frac{1}{6}\right)! \, (1.339)^{11/3}}{\left[1 + (1.339\nu)^2\right]^{5/6}} - \frac{2^{5/6} \left(\frac{5}{6}\right)! \, (1.339)^{11/3}}{\left[1 + (1.339\nu)^2\right]^{11/6}} \right] \tag{9}$$

From the known relations

$$n! = \Gamma(n + 1) = n\Gamma(n)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{6}\right)} = 1.339$$

this equation reduces, as it should, to the following well known result for the one-dimensional case

$$\phi(\nu) = \frac{\sigma_w^2}{\pi} \frac{1 + \frac{8}{3}\,(1.339\nu)^2}{\left[1 + (1.339\nu)^2\right]^{11/6}} \tag{10}$$

EVALUATION OF EQUATION (8)

Since no tables were found for the Bessel functions of order $\frac{5}{6}$ and $\frac{11}{6}$, evaluation was made by considering the analytically continuous property of Bessel functions with respect to their order. The known closed form half-order functions were introduced as follows:

$$y_{1/2}(z) = \sqrt{\frac{2z}{\pi}}\, e^{z}\, K_{1/2}(z) = 1$$

$$y_{3/2}(z) = \sqrt{\frac{2z}{\pi}}\, e^{z}\, K_{3/2}(z) = 1 + \frac{1}{z}$$

$$y_{5/2}(z) = \sqrt{\frac{2z}{\pi}}\, e^{z}\, K_{5/2}(z) = 1 + \frac{3}{z} + \frac{3}{z^{2}}$$

Similar expressions were written for the functions of order 1 and 2. Then for a given value of z, a plot of $y_n(z)$ versus n was made, and from the resulting curve values of $y_n(z)$ at $n = \frac{5}{6}$ and $n = \frac{11}{6}$ were read by interpolation. The values of the Bessel function needed in equation (8) then followed from the relations

$$\frac{1}{z^{5/6}} K_{5/6}(z) = \frac{1}{z^{5/6}} \sqrt{\frac{\pi}{2z}}\, e^{-z} y_{5/6}(z)$$

$$\frac{1}{z^{11/6}} K_{11/6}(z) = \frac{1}{z^{11/6}} \sqrt{\frac{\pi}{2z}}\, e^{-z} y_{11/6}(z)$$

Specific numerical results found for the cross spectra are shown in figure 1. These may be used directly in studies where variation in gust velocities across the span is being treated. The interpolation technique as used here for evaluation was aided by considering the asymptotic expression for equation (8) and the results at very low z, as discussed in the next section.

<u>Asymptotic result and expression for low z.</u>- For large z

$$K_n(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

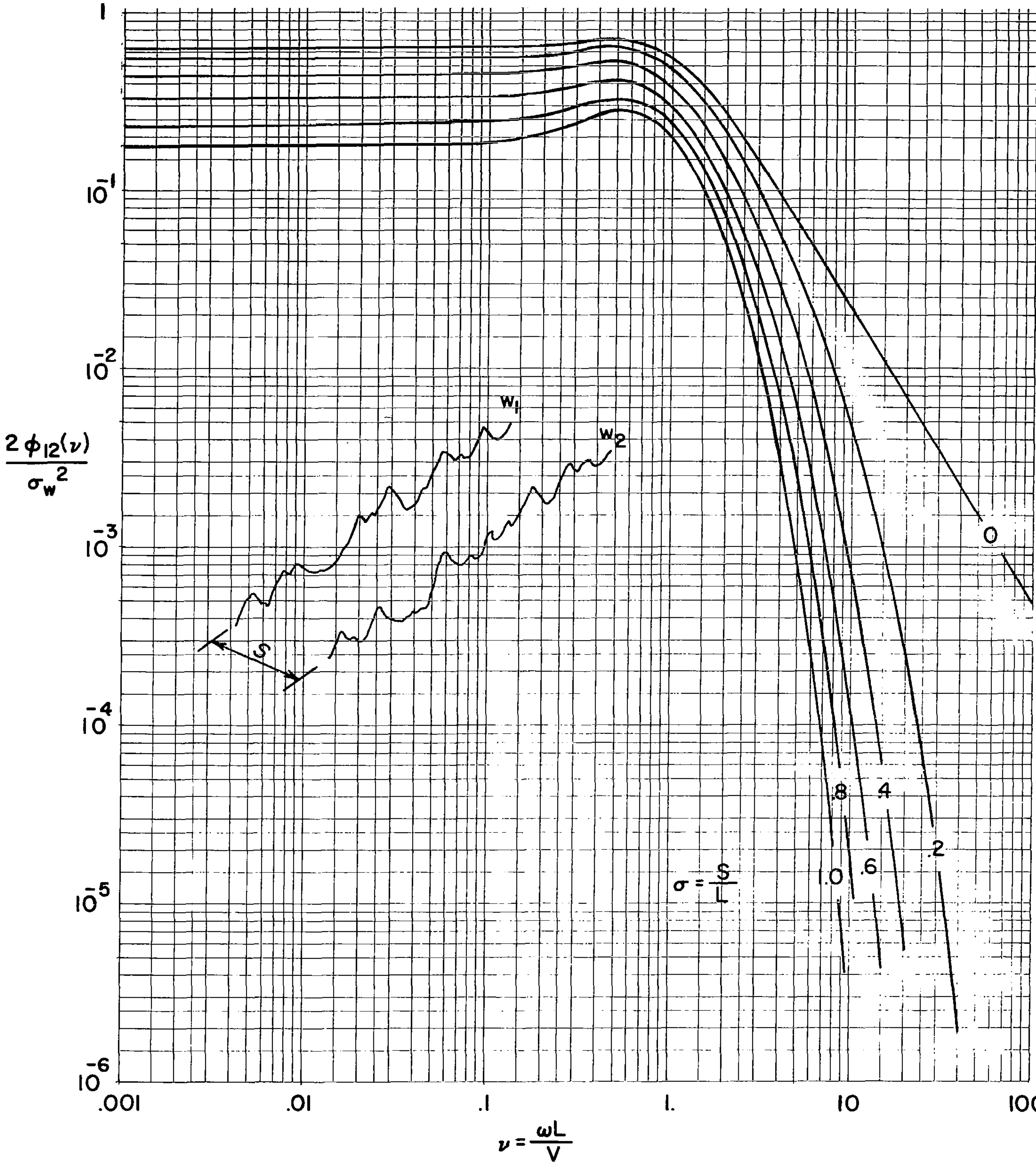


Figure 1. Cross-Spectra for Treatment of Nonuniform Spanwise Gusts

With this result, equation (8) reduces to the following asymptotic result, applicable at large z

$$\phi_{12}(\nu) = \sigma_w^2 \frac{1}{2^{1/3}\Gamma\left(\frac{1}{3}\right)}\left(\frac{1}{1.339}\right)^{8/3}\left[\frac{8}{3}\,(1.339)^2\sigma^{5/3}z^{-4/3} - \sigma^{11/3}z^{-7/3}\right]e^{-z} \tag{11}$$

Figure 2 gives a comparison of results as obtained from this equation and from equation (8). It is seen that the asymptotic result serves quite well for $\nu > 1$.

A fact of interesting physical significance can be brought out by considering the behavior of the cross-spectral function at large frequency argument in a particular nondimensional manner. In response studies it is often convenient or instructive to express results in terms of the ratio of the cross spectra to the point spectra, that is, equation (8) divided by equation (10). With the use of equation (11) instead of equation (8), it can be shown that for high frequencies this ratio is simply

$$\left.\frac{\phi_{12}(\nu)}{\phi(\nu)}\right|_{\nu\ \text{large}} = \frac{1.339\pi}{2^{1/3}\Gamma\left(\frac{1}{3}\right)}\left(\frac{\omega s}{V}\right)^{1/3} e^{-\frac{\omega s}{V}}$$

It is noted that the ratio is a function of the frequency argument $\frac{\omega s}{V}$ only; the scale L no longer appears. Because of this fact, it can be reasoned that certain response phenomena may be expected to peak at a certain $\frac{\omega s}{V}$ regardless of the scale of turbulence. For example, in the case of rolling response due to spanwise variations in vertical gust velocities, maximum rolling power may be expected, as expressed in the spectral ratio form, when $\frac{\omega s}{V}$ is near π, where s is taken as the wing span.

A useful expression for small z is also readily found through use of the following definitions.

$$K_n(z) = \frac{\pi}{2} \frac{I_{-n}(z) - I_n(z)}{\sin n\pi} \tag{12}$$

$$I_n(z) = \sum_{k=o}^{\infty} \frac{1}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n} \tag{13}$$

Through means of these expressions, equation (8) may be expanded into a series involving σ and z, with the following result.

$$\begin{aligned}
\frac{\phi_{12}(\nu)}{\sigma_w^2} = {} & .522\left(\frac{\sigma}{z}\right)^{5/3}\left(1 + \frac{3}{2} z^2 + \frac{9}{56} z^4 + \frac{9}{1456} z^6 + \frac{27}{221312} z^8 + \ldots\right) \\
& + .1815\left(\frac{\sigma}{z}\right)^{11/3}\left(-1 + \frac{3}{10} z^2 + \frac{9}{40} z^4 + \frac{9}{560} z^6 + \frac{27}{58240} z^8 + \ldots\right) \\
& - .977\sigma^{5/3}\left(1 + \frac{3}{22} z^2 + \frac{9}{1456} z^4 + \frac{9}{68816} z^6 + \ldots\right) \\
& - .0556\sigma^{11/3}\left(1 + \frac{3}{34} z^2 + \frac{9}{3128} z^4 + \frac{9}{181424} z^6 + \ldots\right)
\end{aligned} \tag{14}$$

This equation was used to check the evaluation of the cross-spectral functions at low values of z .

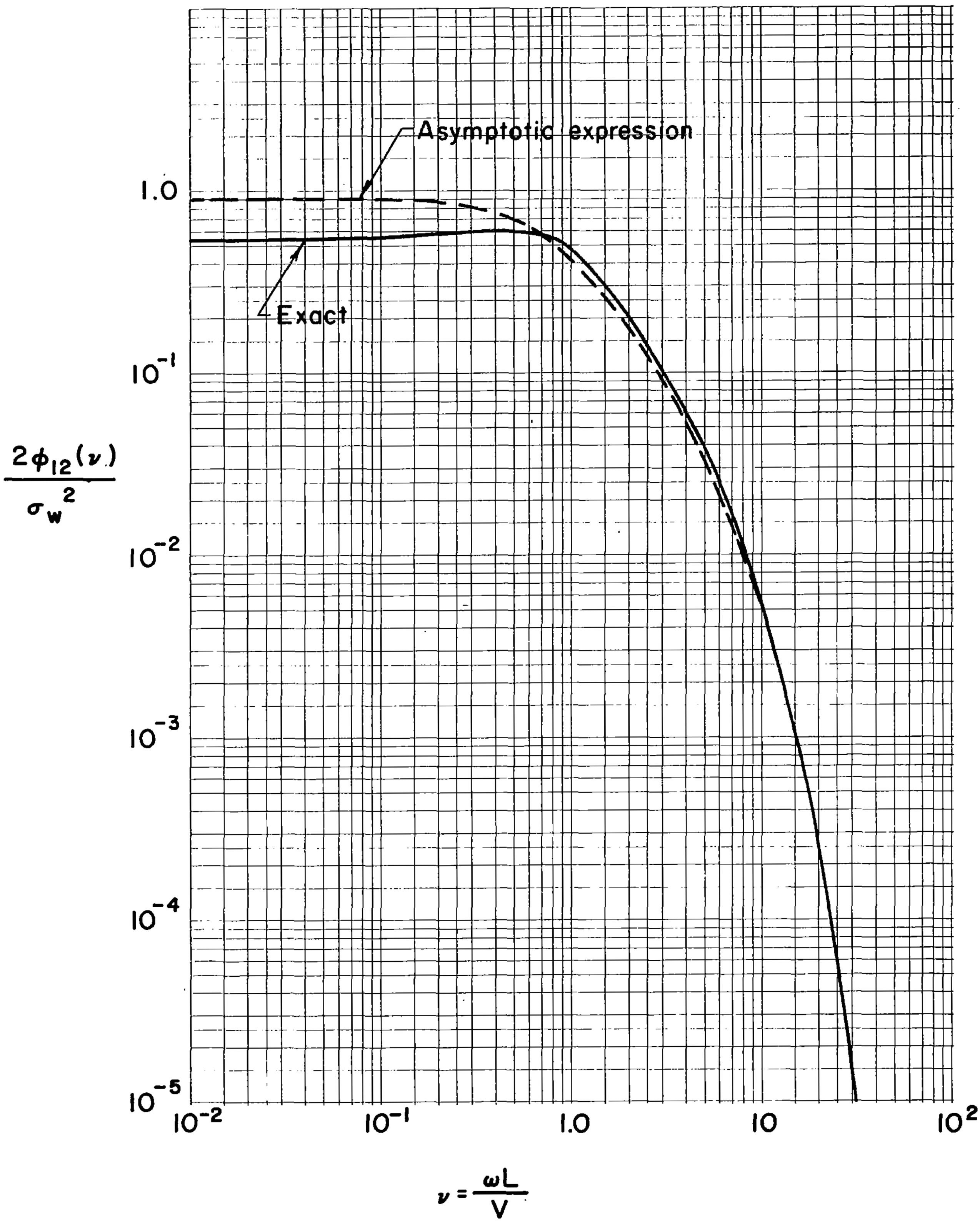


Figure 2a. Cross-Spectra for $\sigma = 0.2$.

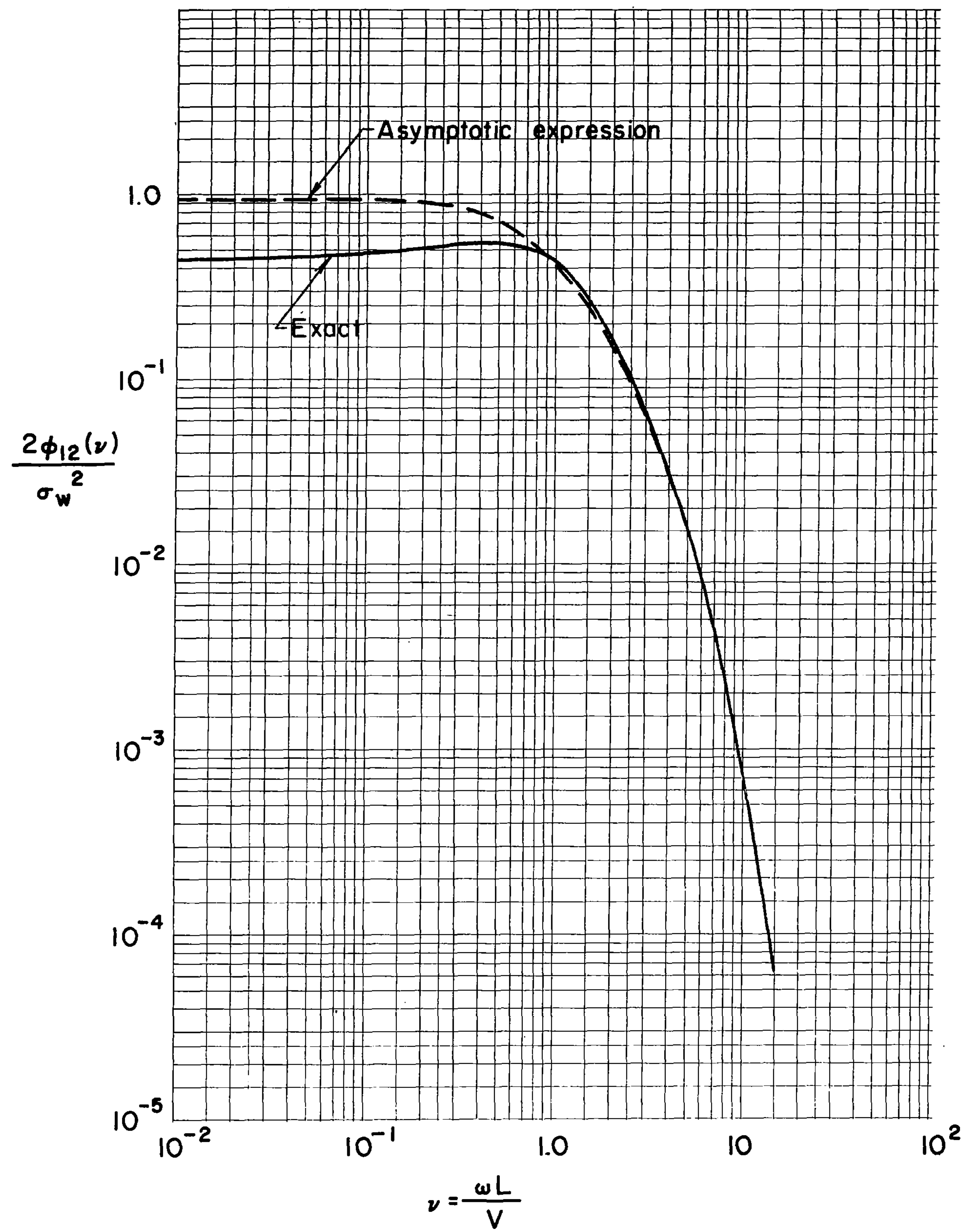


Figure 2b. Cross-Spectra for $\sigma = 0.4$.

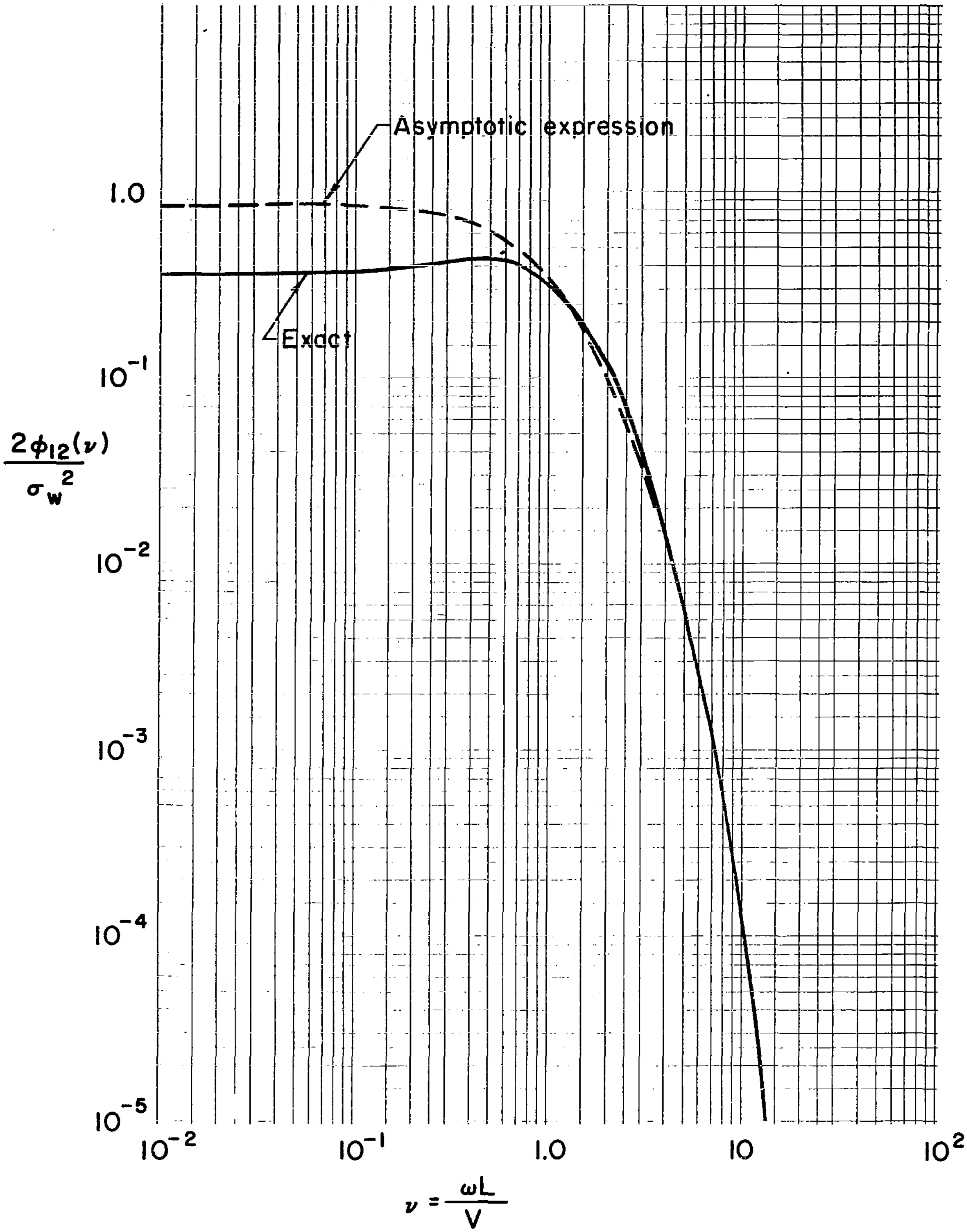


Figure 2c. Cross-Spectra for $\sigma = 0.6$.

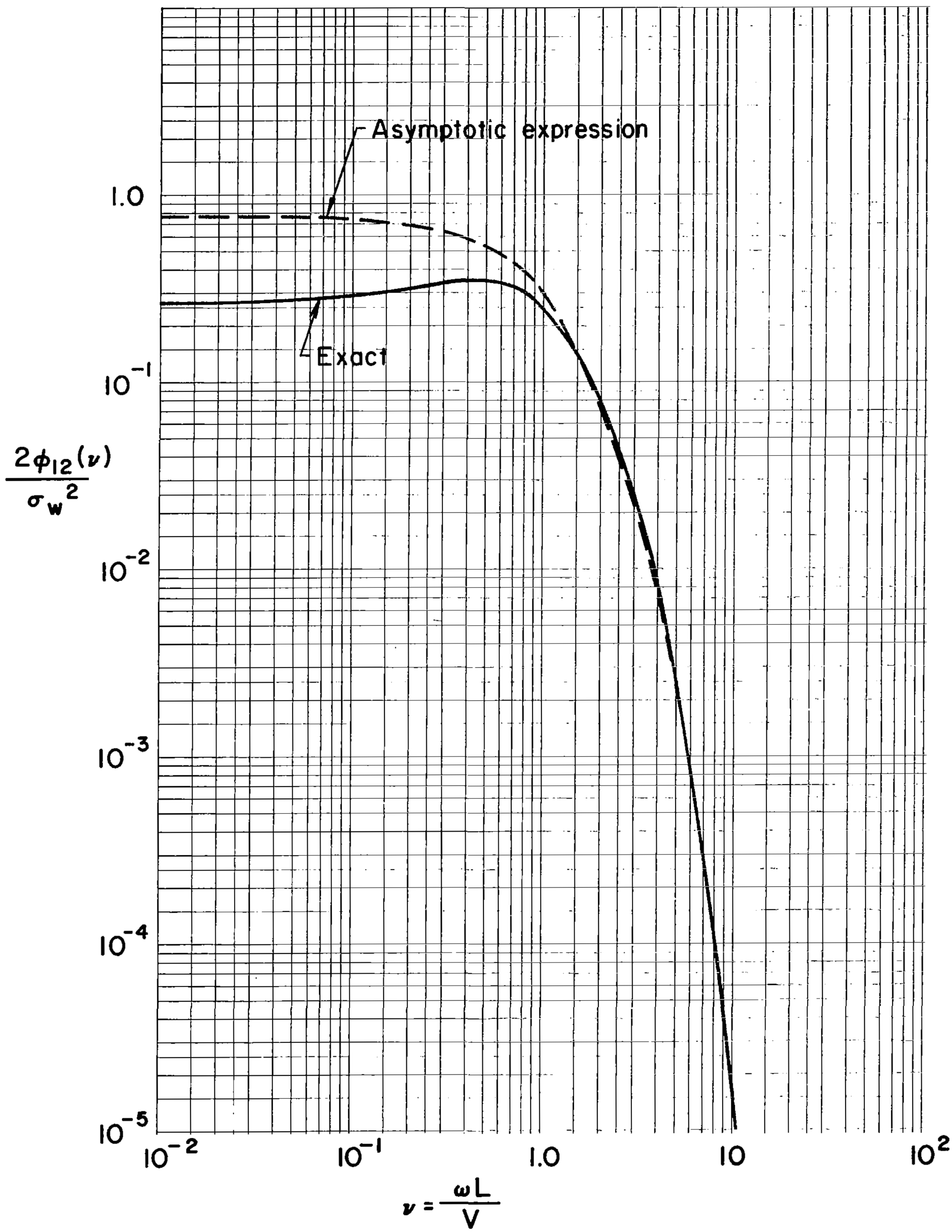


Figure 2d. Cross-Spectra for $\sigma = 0.8$.

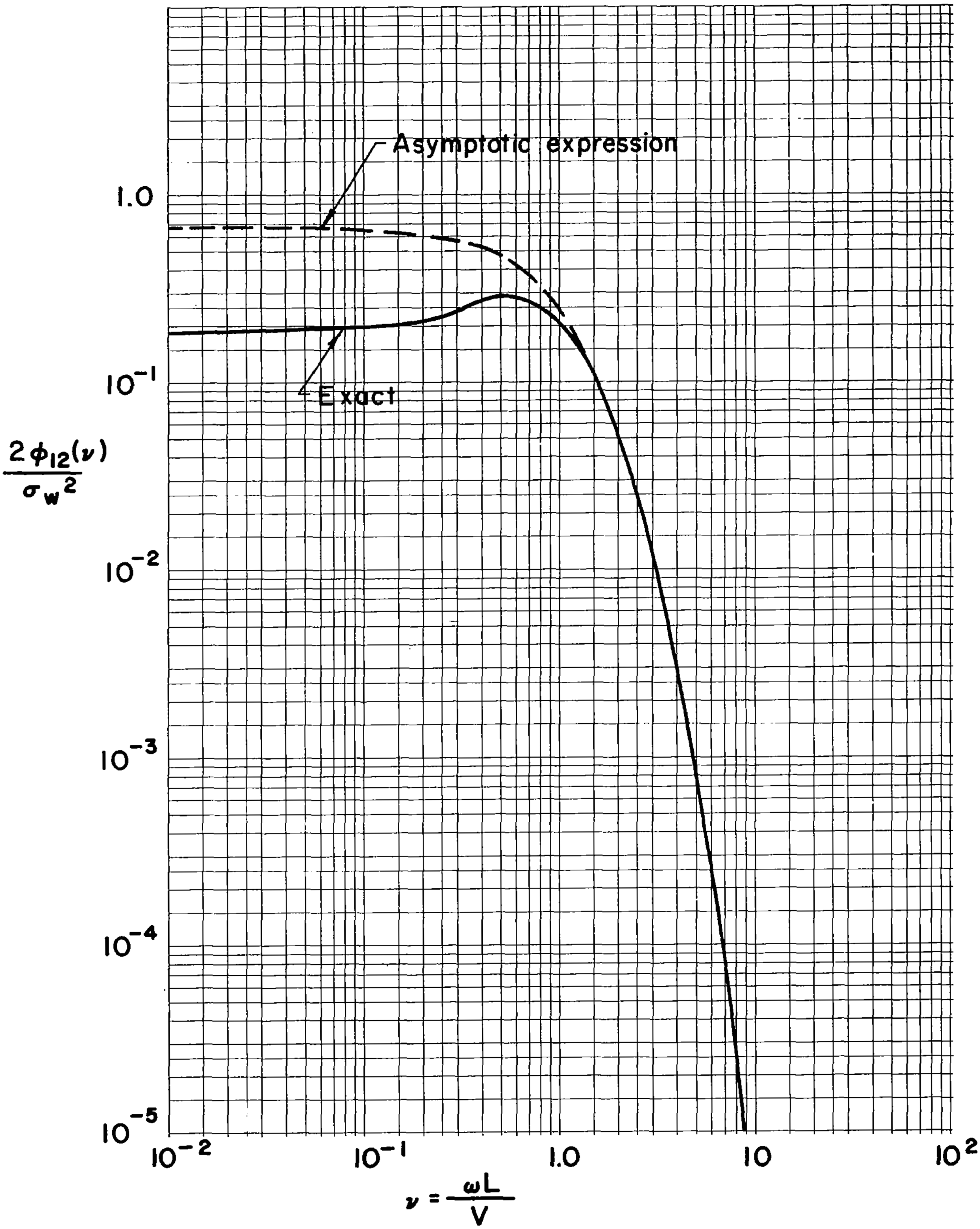


Figure 2e. Cross-Spectra for $\sigma = 1.0$.

CROSS SPECTRA FOR LONGITUDINAL TURBULENCE

The point correlation function for the longitudinal component of turbulence that corresponds to equation (1) is given by

$$R_u(x) = \sigma_u^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} u^{1/3} K_{1/3}(u) \tag{15}$$

where $u = \frac{x}{1.339L}$. If x is replaced by $r = \sqrt{s^2 + V^2t^2}$, as done with equation (2), the following cross-correlation function applicable to longitudinal turbulence is obtained.

$$R_{12_u}(t) = \sigma_u^2 \frac{2^{2/3}}{\Gamma\left(\frac{1}{3}\right)} u^{1/3} K_{1/3}(u) \tag{16}$$

where $u = \frac{\sigma}{1.339} \sqrt{1 + \left(\frac{Vt}{s}\right)^2}$, and $\sigma = \frac{s}{L}$. From this function the cross-spectral function for longitudinal turbulence is found to be

$$\phi_{12_u}(\nu) = \sigma_u^2 \frac{2^{5/3}}{\Gamma\left(\frac{1}{3}\right)} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1.339}\right)^{2/3} \frac{\sigma^{5/3}}{z^{5/6}} K_{5/6}(z) \tag{17}$$

CONCLUDING REMARKS

This paper has presented closed form solutions for the cross-spectral functions based on von Kármán's spectral equation. These relations and associated curves thus form a new and more appropriate base for studying the effects of spanwise variations in turbulence on aircraft gust response.

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