ON THE CALCULATION OF

THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER FLOWS

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NOMENCLATURE

a = cylinder radius

$f = \int_0^\eta F(\xi,\lambda,\zeta)\,d\lambda$

$F = u/U$

$g = \int_0^\eta G(\xi,\lambda,\zeta)\,d\lambda$

$G = w/W$

p = pressure

U = x-component of inviscid flow

W = z-component of inviscid flow

U_∞ = velocity upstream of the flat plate

u,v,w = velocity components in the x,y,z directions

$\bar{x}$ distance from cylinder axis to the leading edge of the flat plate (=45.7 cm)

x,y,z = Cartesian coordinates

ξ,η,ζ = transformed coordinates

λ = dummy variable of integration

ψ,ϕ = streamfunctions

ρ = density

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$$\beta = 2\xi/U \; U_\zeta$$

$$\beta^* = 2\xi/U \; W_\zeta$$

$$K = 2\xi/U^2 U_\zeta$$

$$K^* = 2\xi/W \; W_\xi$$

τ_{xy}, τ_{zy} = shear stress components

μ = dynamic viscosity

ν = kinematic viscosity

Subscripts

$x, y, z, n, \ldots$ = differentiation (except on τ)

INTRODUCTION

In a recent paper, Wang[1] introduces an approximation which reduces the computation of three-dimensional, laminar, compressible, boundary-layer equations to the problem of solving two-dimensional type boundary layer equations. Assuming familiarity with Wang's work, it is to be noted that the test case chosen by Wang had been previously studied by Fannelop[2] who had introduced a small cross-flow perturbation method. The perturbation method itself reduced the computation of the three-dimensional boundary layer equations to that of two-dimensional type boundary layer equations. While on the one hand Wang's results compare favorably with Fannelop's, it would seem that a more stringent test of Wang's method would be the comparison with a fully three-dimensional boundary layer calculation. In this note we provide such a comparison with the results from our three-dimensional calculations. In addition we consider another aspect of Wang's approximation for the calculation of three-dimensional flows.

The test case which we have chosen is incompressible, laminar flow past a flat plate with attached cylinder (see Fig. 1). The boundary layer of interest is formed on the plate upstream of the cylinder. This problem has been studied by Sowerby[3] with a Blasius-type series solution and by Dwyer[4] with a finite difference method. Our method differs from Dwyer's in that we introduce two stream functions such that the continuity equation is satisfied.

ANALYSIS

The governing boundary layer equations, written in Cartesian coordinates are

$$u_x + v_y + w_z = 0 \tag{1}$$

$$uu_x + vu_y + wu_z = -1/\rho \, p_x + \nu u_{yy} \tag{2}$$

$$uw_x + vw_y + ww_z = -1/\rho \, p_z + \nu w_{yy} \tag{3}$$

subject to boundary conditions

$$y = 0: u = v = w = 0, \ y = \infty : u = U, w = W \tag{4}$$

where $-1/\rho \, p_x = UU_x + WU_z$, $-1/\rho \, p_z = UW_x + WW_z$ $\tag{5}$

and

$$U = U_\infty \left\{ 1 + \frac{z^2 - (x-\bar{x})^2}{(1/a^2)[(x-\bar{x})^2 + z^2]^2} \right\} \ ; \ W = - \frac{2U_\infty z(x-\bar{x})}{(1/a^2)[(x-\bar{x})^2 + z^2]^2} \tag{6}$$

for the present problem under investigation.

Introducing a Blasius-type transformation, $\eta = (U/2\nu x)^{\frac{1}{2}} y$, $\xi = x$ and $\zeta = z$ as well as stream functions

$$u = \psi_y, \ v = -\psi_x - \phi_z, \ w = \phi_y \tag{7}$$

where

$$\psi(x,y,z) \equiv (2\nu x U)^{\frac{1}{2}} f(\xi,\eta,\zeta); \quad \phi(x,y,z) \equiv W\left(\frac{2\nu x}{U}\right)^{\frac{1}{2}} g(\xi,\eta,\zeta) \quad (8)$$

Eqs. (2) and (3) become

$$F_{\eta\eta} + (1+\beta/2)fF_\eta + \beta(1-F^2) + K(1-FG) + (\beta^*-K/2)gF_\eta$$

$$+ 2\xi(f_\xi F_\eta - FF_\xi) + 2\xi\frac{W}{U}(g_\zeta F_\eta - GF_\zeta) = 0 \quad (9)$$

$$G_{\eta\eta} + (1+\beta/2)fG_\eta - \beta(1-G^2) + K^*(1-GF) + (\beta^*-K/2)gG_\eta$$

$$+ 2\xi(f_\xi G_\eta - FG_\xi) + 2\xi\frac{W}{U}(g_\zeta G_\eta - GG_\zeta) = 0 \quad (10)$$

with the boundary conditions at $\eta = 0$: $f = F = G = g = 0$ and at
$\eta = \infty$: $F = 1$, $G = 1$.

To integrate Eqs. (9) and (10) two initial conditions are needed, and
these are deduced by considering two limiting cases of Eqs. (9) and (10):
(a) $\xi = 0$, $\zeta \neq 0$; (b) $\xi \neq 0$, $\zeta = 0$. Eqs. (9) and (10), for condition (a),
reduce to Blasius equations. Referring to Fig. (1) condition (b) is the
"line of symmetry" condition. Along this line $w = 0$ as well as W. In
other words $w/W = G$ is indeterminate. On the other hand, w_z and W_z are
not zero along $z = 0$ so that by applying L'Hopital's rule to w/W,
$G = w_\zeta/W_\zeta$ along $\zeta = 0$. Consequently the equations for condition (b) are
determined from Eqs. (9) and (10) by simply letting ζ go to zero.

To solve Eqs. (9) and (10) as well as the equations resulting from
the initial conditions, an implicit finite difference procedure of
Crank-Nicholson type is applied. The resulting non-linear equations are
replaced initially by linear difference equations. An iteration procedure
is used until the non-linear difference equations corresponding to the
grid points employed have been solved. Calculations proceeded in the
downstream direction, for a given z-station, until flow reversal appeared

in the u-component of velocity. Step sizes used in the calculations are: $\Delta\xi = \Delta\zeta = .61$, $\Delta\eta = .20$.

In order to check Wang's approximation against the full three-dimensional calculation, we must simplify Eqs. (9) and (10) accordingly. We replace derivatives in the crossflow direction, i.e., the ζ derivatives, by their values at the edge of the inviscid flow, thereby reducing Eqs. (9) and (10) to quasi-two dimensional equations. Therefore, the following relations will be used:

$$GF_\zeta = G(1/U\, u_\zeta - 1/U\, U_\zeta F) \approx G(1-F)\, 1/U\, U_\zeta, \quad GG_\zeta = G(\frac{1}{W}\, w_\zeta - \frac{1}{W}\, W_\zeta G)$$

$$\approx G(1-G)\, \frac{1}{W}\, W_\zeta, \quad g_\zeta = 0 \tag{11}$$

Insofar as shear stress data and flow reversal are of importance, Tables 1 and 2 are a comparison between the full and approximate three-dimensional calculations of $F_\eta (=f_{\eta\eta})$ and $G_\eta (=g_{\eta\eta})$ at $\eta = 0$. These values are proportional to the shear stress at the surface since

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U\, (\frac{U}{2\nu x})^{\frac{1}{2}} f_{\eta\eta}(\xi,0,\zeta)\ ,\quad \tau_{zy} = \mu \frac{\partial w}{\partial y}\bigg|_{y=0}$$

$$= \mu W\, (\frac{U}{2\nu x})^{\frac{1}{2}} g_{\eta\eta}(\xi,0,\zeta) \tag{12}$$

Fig. (1) contains the flow reversal line as determined by both methods.

Up to a z or ζ value of 6.10, the full and approximate three-dimensional calculations predict flow reversal in the u-component, the approximate method predicting flow reversal further downstream from the leading edge than the full three-dimensional calculations. At $\zeta = 7.32$, the full three-dimensional calculations predict flow reversal while the approximate method does not. By the approximate method, $F_\eta(0)$ (or $f_{\eta\eta}(0)$) reaches a

minimum and then increases with x or ξ, the minimum point shifting to smaller ξ values as ζ increases. The calculations of $G_\eta(0)$ by both methods, and for all ζ values, even in the region where the approximate method fails to predict flow reversal, show good agreement.

While in two-dimensional, incompressible flow, the necessary condition for flow reversal is an adverse pressure gradient, we find that for three-dimensional flows an adverse pressure gradient is not necessarily a necessary condition for flow reversal. As a matter of fact, for the present problem at $\zeta = 9.05$ and flow reversal, the pressure gradient is favorable. As Dwyer[4] noted, the flow reversal of the u-component of velocity is caused by crossflow effects near the bottom of the boundary layer. That is, flow reversal is driven by the convective term, wu_z, which is positive and overcomes the negative p_x term. Consequently wu_z is an important term, and in particular, u_z, insofar as flow reversal is concerned so that by approximating u_z in certain regions of the flow field, wu_z can take on values which do not balance properly with the pressure gradient term. This is the case at $\zeta = 7.32$ - the quasi-two dimensional approximation breaks down as flow reversal is approached, although for a large range of ξ values, the approximate values of $F_\eta(0)$ are in good agreement with the full three-dimensional calculations. This is also true at $\zeta = 9.15$.

At most, then, what one may conclude from these results, is that Wang's approximation may not be valid in some applications and, more likely, in some particular region of a flow field, for example, as discussed above. Where the method does not break down, agreement is most satisfactory with the full calculations.

To explore the possible extension of Wang's method for further research, we considered the following calculation: adopting the approximate method, Eq. (11), along the line $\zeta = 3.05$ = constant, and using the subsequent results as initial conditions for the full three-dimensional calculations, we find that at the next station, $\zeta = 3.66$, flow reversal is predicted at the same ξ value as that obtained by starting the calculations along the line of symmetry, $\zeta = 0$. In other words, the approximate method may be of interest in calculating three-dimensional flows where a line of symmetry is not present. The major difficulty in such flows are the initial conditions with respect to the crossflow derivatives and their calculation. The idea is to apply the approximate method along some initial line and then revert to the full three-dimensional calculations. This idea is being applied to other problems as well.

REFERENCES

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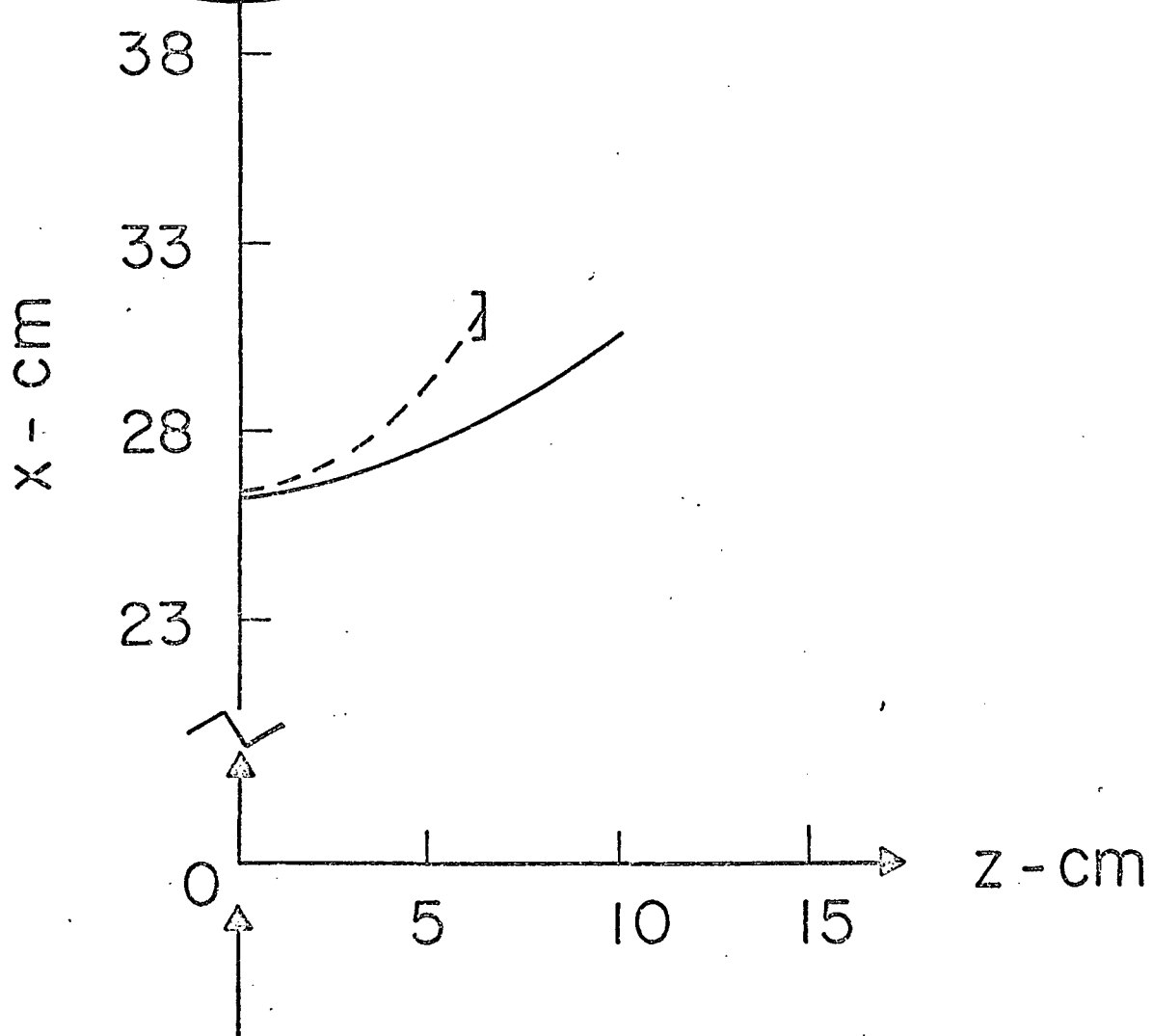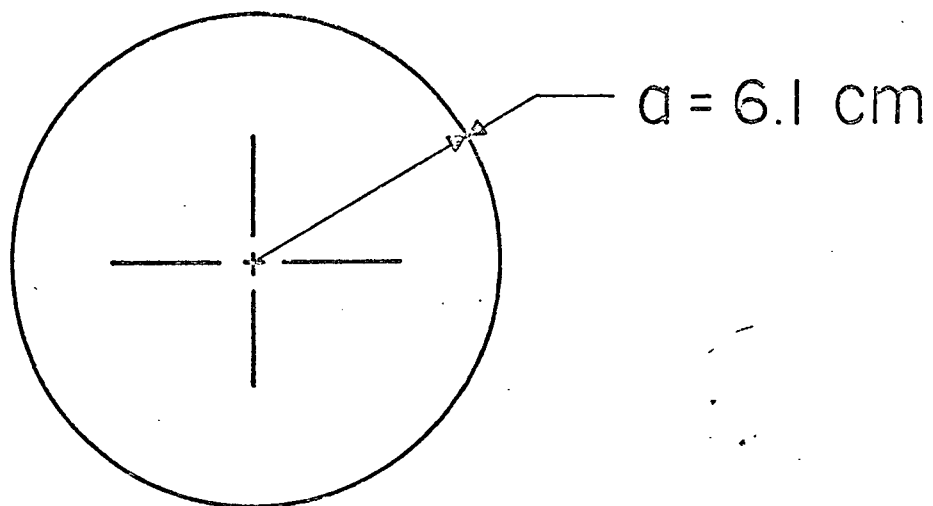
TABLE 1

COMPARISON BETWEEN FULL AND APPROXIMATE
THREE-DIMENSIONAL BOUNDARY LAYER CALCULATIONS OF $F_\eta(0)$

ξ-cm.	$\zeta = 3.05$ cm. Full	Approx.	$\zeta = 6.10$ cm. Full	Approx.	$\zeta = 7.32$ cm. Full	Approx.	$\zeta = 9.15$ cm. Full	Approx.
0	.4696	.4696	.4696	.4696	.4696	.4696	.4696	.4696
.61	.4677	.4677	.4677	.4677	.4678	.4678	.4678	.4679
3.66	.4631	.4631	.4637	.4637	.4640	.4640	.4646	.4646
7.93	.4508	.4508	.4528	.4529	.4539	.4540	.4558	.4560
10.37	.4408	.4408	.4442	.4444	.4461	.4464	.4493	.4496
13.42	.4230	.4231	.4296	.4300	.4331	.4337	.4389	.4397
17.69	.3814	.3818	.3974	.3989	.4056	.4075	.4188	.4213
20.13	.3417	.3428	.3691	.3722	.3827	.3865	.4039	.4087
23.79	.2340	.2393	.3006	.3120	.3315	.3444	.3766	.3906
24.40	.2051	.2126	.2839	.2987	.3198	.3361	.3715	.3885
25.01	.1701	.1811	.2646	.2843	.3067	.3275	.3661	.3869
25.62	.1250	.1431	.2714	.2685	.2916	.3187	.3605	.3861
26.23		.0946	.2126	.2512	.2735	.3097	.3543	.3863
26.84			.1685	.2322	.2495	.3008	.3471	.3876
27.45			.0874	.2113	.2061	.2919	.3378	.3906
28.06				.1880	.1041	.2835	.3209	.3955
28.67				.1617		.2759	.2547	.4028
29.28				.1310		.2694		.4130
29.89				.0926		.2646		.4269
30.50						.2623		.4452
31.11						.2632		.4687

TABLE 2

COMPARISON BETWEEN FULL AND APPROXIMATE
THREE-DIMENSIONAL BOUNDARY LAYER CALCULATIONS OF $G_\eta(0)$

ξ-cm.	$\zeta = 3.05$ cm.		$\zeta = 6.10$ cm.		$\zeta = 7.32$ cm.		$\zeta = 9.15$ cm.	
	Full	Approx.	Full	Approx.	Full	Approx.	Full	Approx.
0	.4696	.4696	.4696	.4696	.4696	.4696	.4696	.4696
.61	.5309	.5307	.5298	.5295	.5292	.5289	.5280	.5277
3.66	.8567	.8555	.8488	.8477	.8443	.8432	.8363	.8353
7.93	1.355	1.351	1.333	1.330	1.321	1.318	1.300	1.297
10.37	1.669	1.664	1.637	1.632	1.618	1.614	1.586	1.582
13.42	2.102	2.094	2.050	2.044	2.021	2.015	1.970	1.965
17.69	2.809	2.798	2.713	2.703	2.661	2.650	2.569	2.559
20.13	3.288	3.278	3.152	3.139	3.077	3.064	2.949	2.935
23.79	4.173	4.172	3.928	3.909	3.797	3.774	3.580	3.552
24.40	4.348	4.351	4.075	4.055	3.932	3.904	3.693	3.660
25.01	4.534	4.543	4.230	4.207	4.071	4.039	3.809	3.770
25.62	4.729	4.750	4.392	4.366	4.216	4.178	3.927	3.⁣⁣1
26.23		4.973	4.561	4.532	4.366	4.321	4.048	3.⁣
26.84			4.723	4.707	4.520	4.469	4.171	4.⁣
27.45			4.892	4.891	4.647	4.621	4.296	4.⁣
28.06				5.086	4.754	4.777	4.408	4.⁣⁣
28.67				5.293		4.938	4.411	4.4⁣8
29.28				5.513		5.102		4.545
29.89				5.747		5.270		4.648
30.50						5.441		4.746
31.11						5.613		4.837

a = 6.1 cm

U_∞ = 3050 cm / sec

Fig. 1 Flow geometry and the prediction of flow reversal: --- quasi-
two-dimensional (approximate); _____ fully three-dimensional.