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UNIVERSITY OF CINCINNATI
DEPARTMENT OF AEROSPACE ENGINEERING

DYNAMIC CHARACTERISTICS OF A VARIABLE-MASS FLEXIBLE MISSILE

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ABSTRACT

This investigation is concerned with the general motion of a variable-mass flexible missile with internal flow and aerodynamic forces. The present formulation extends previous work under the NASA Research Grant NGR 36-004-030 by including the aerodynamic forces. The resulting formulation comprises six ordinary differential equations for the rigid body motion and three partial differential equations for the elastic motion. The simultaneous differential equations are nonlinear and possess time-dependent coefficients. Since no closed-form solution can be expected, the differential equations are being solved by a semi-analytical method leading to a set of purely ordinary differential equations which are then solved numerically. A computer program has been developed for the numerical solution and results for a given set of initial conditions are presented.

1. Introduction

Investigations of the behavior of a rocket in flight can be divided for the most part into two major classes according to the mathematical models: the first is concerned with a rigid missile of variable mass and the second with a flexible missile of constant mass.

The treatment of the missile as a rigid-body of time-dependent mass has been adequately covered by many researchers, including Grubin[1*], Dryer[2], and Leitmann[3]. The ballistic trajectories of spin- and fin-stabilized rigid bodies are treated in the book by Davis, Follin, and Blitzer[4].

A considerable amount of effort has been devoted to the analysis of an elastic body subjected to longitudinal acceleration. For example, Seide[5] has treated the effect of both a compressive and a tensile force on the frequencies and mode shapes of transverse vibration of a continuous slender body. Others, such as Beal[6] have been concerned with the problem of buckling instability of a uniform bar subjected to an end thrust as well as with the change in the body natural frequencies as a result of that thrust. These investigations regard the mass of the body as constant in time.

A series of reports by Miles, Young, and Fowler[7] offers a comprehensive treatment of a wide range of subjects associated with the dynamics of missiles, including fuel sloshing. The report by Keith et al[8] also covers a wide range of subjects associated with the dynamics of

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* See References listed at the end of this work.

missiles. Again the mass variation is not accounted for.

Attempts have been made to consider simultaneously the mass variation and missile flexural elasticity by investigators such as Birnbaum[9] and Edelen[10]. Both were concerned with solid-fuel rockets and neither of them included the axial elasticity of the missile. On the other hand, Price[11] investigated the internal flow in a solid-fuel rocket and ignored entirely the vehicle motion. An attempt to synthesize the problem of rocket dynamics has been made by Meirovitch and Wesley[12]. This latter work accounts for the mass variation, rigid-body translation and rotation, and axial and transverse deformations but it assumes the motion to be planar, which exludes spinning rockets. A later work by Meirovitch[13,14] does away with the restriction of planar motion and considers the general motion of a variable-mass flexible missile.

This investigation represents an extension of References 13, 14 and it includes the aerodynamic effects on the missile during its flight through the atmosphere.

2. Equations of Motion for a General Variable-Mass System†

By a variable-mass system we understand a system of changing composition. To examine this concept more closely we envision a control volume in space and assume that the amount of matter within the control volume may change with time, although the shape of the control volume is fixed. Since the system composition changes, it is not proper to equate the time-derivative of the sum of momenta associated with the particles to the sum of the time derivatives, because the summation involves different sets of particles at different times. In this case, the proper procedure for obtaining the equations of motion is to write the force equation in the form $\underset{\sim}{F} = \dot{\underset{\sim}{p}}^*$, where the rate of change of the momentum, $\dot{\underset{\sim}{p}}$, is derived by a limiting process consisting of calculating $\underset{\sim}{p}$ at two different instants, a time interval Δt apart, dividing the difference of the two values by Δt, and letting $\Delta t \to 0$. In so doing, we ensure that the same total mass is involved, although at one time it is entirely inside the control volume and at the other time part of the mass is outside.

Let us consider a certain volume in space at time t, namely the control volume enclosed by the control surface shown in solid line in Figure 1. If the control volume is fixed in an inertial space,

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† The material presented in this and in several of the following sections parallels closely that of References 13 and 14.

*A wavy line under the symbol denotes a vector quantity or operation.

then it is shown in Reference 15 (p. 96) that the force equation has the form

$$\underset{\sim}{F} = \underset{\sim}{F}_S + \underset{\sim}{F}_B = \frac{\partial}{\partial t}\int_{CV} \underset{\sim}{v}\,\rho\, d\tau + \int_{CS} \underset{\sim}{v}(\rho \underset{\sim}{v} \cdot d\underset{\sim}{A}) \qquad (1)$$

where $\underset{\sim}{F}_S$ and $\underset{\sim}{F}_B$ are the resultants of the surface and body forces, respectively, acting upon the system.

More pertinent to the present discussion is a control volume translating and rotating relative to an inertial space. Further it will be convenient to assume that part of the matter is fixed in the control volume while part of it moves relative to it. Denoting by x,y,z a system of axes fixed in the control volume, so that they can be regarded as a set of body axes translating and rotating in space, the force equation can be written

$$\underset{\sim}{F}_S + \underset{\sim}{F}_B = \int_M \underset{\sim}{a}\, dM = \int_M [\underset{\sim}{a}_0 + \dot{\underset{\sim}{v}} + 2\underset{\sim}{\omega} \times \underset{\sim}{v} + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})]\, dM \qquad (2)$$

in which $\underset{\sim}{a}$ is the absolute acceleration of the mass element dM, $\underset{\sim}{a}_0$ is the acceleration of the origin 0 of the system x,y,z, $\underset{\sim}{\omega}$ is the angular velocity vector of axes x,y,z, and $\underset{\sim}{r}$ is the position of dM relative to these axes. If we recognize that if the body axes were fixed in the inertial space, the only term to survive in Eq. (2) would be $\int_{M_f} \dot{\underset{\sim}{v}}\, dM$, where M_f is the mass moving relative to the control volume, then from Eqs. (1) and (2) we conclude that

$$\underline{F}_S + \underline{F}_B = \frac{\partial}{\partial t}\int_{cv} \underline{v}\,\rho\, d\tau + \int_{cs} \underline{v}(\rho \underline{v} \cdot d\underline{A}) + \int_M [\underline{a}_o + 2\underline{\omega} \times \underline{v} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})]\, dM \quad (3)$$

where the partial derivative $\partial/\partial t$ is to be calculated by regarding axes x,y,z as fixed. It is convenient to introduce the following equivalent forces

$$\underline{F}_C = -2\,\underline{\omega} \times \int_{M_f} \underline{v}\, dM$$
$$\underline{F}_U = -\frac{\partial}{\partial t}\int_{M_f} \underline{v}\, dM \quad (4)$$
$$\underline{F}_R = -\int_A \underline{v}(\rho \underline{v} \cdot d\underline{A})$$

where $\underline{F}_C$ is recognized as the Coriolis force, $\underline{F}_U$ is a force due to the unsteadiness of the relative motion, and $\underline{F}_R$ is referred to as a reactive force. With this notation, Eq. (3) becomes

$$\underline{F}_S + \underline{F}_B + \underline{F}_C + \underline{F}_U + \underline{F}_R = \int_M [\underline{a}_o + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})]\, dM \quad (5)$$

The terms on the right side of Eq. (5) may be regarded as pertaining to a rigid body of instantaneous mass M.

In a similar manner, the torque equation about the origin 0 can be written

$$\underset{\sim}{N}_S+\underset{\sim}{N}_B+\underset{\sim}{N}_C+\underset{\sim}{N}_U+\underset{\sim}{N}_R=\int_M \underset{\sim}{r}\times\left[\underset{\sim}{a}_o+\dot{\underset{\sim}{\omega}}\times\underset{\sim}{r}+\underset{\sim}{\omega}\times(\underset{\sim}{\omega}\times\underset{\sim}{r})\right]dM \qquad (6)$$

where

$$\underset{\sim}{N}_C=-2\int_{M_f}\underset{\sim}{r}\times(\underset{\sim}{\omega}\times\underset{\sim}{v})\,dM$$

$$\underset{\sim}{N}_U=-\frac{\partial}{\partial t}\int_{M_f}\underset{\sim}{r}\times\underset{\sim}{v}\,dM \qquad (7)$$

$$\underset{\sim}{N}_R=-\int_A(\underset{\sim}{r}\times\underset{\sim}{v})(\rho\,\underset{\sim}{v}\cdot d\underset{\sim}{A})$$

The significance of the various torques is self-evident. Moreover, the expression for $\underset{\sim}{N}_U$ can be easily explained by recalling that $\partial/\partial t$ implies a time rate of change with axes x,y,z regarded as fixed.

The above equations must be supplemented by the continuity equation

$$\int_{cs}\rho\,\underset{\sim}{v}\cdot d\underset{\sim}{A}=-\frac{\partial}{\partial t}\int_{cv}dM \qquad (8)$$

which expresses the fact that the net efflux rate of mass across the control surface must equal the rate of mass decrease inside the control volume.

Equations (5) and (6) can be given an interesting physical interpretation by recalling that the system comprises one part solid and another part of changing composition, and observing that the right sides of these equations represent the motion of the system as if it were rigid in its entirety. Equations (5) and (6) can be regarded as

the equations of motion of a fictitious rigid body of instantaneous mass M, provided that the actual surface and body forces acting upon the system are supplemented by three equivalent forces, namely the Coriolis force, the force due to the unsteadiness of the relative motion, and the reactive force. This statement is sometimes referred to as the "principle of solidification for a system of changing composition" (Reference 16, p. 13).

3. The Rigid-Body Motion of a Rocket

The formulation of Section 2 is ideally suited for treating problems associated with the motion of a rocket. Although a rocket is in general flexible, a first-approximation solution for its dynamic behavior can be obtained by regarding it as rigid. The mathematical model of the rocket is assumed to comprise a long cylindrical shell open at the aft end and closed at the fore end. The inner part of the rocket consists of the propellant which surrounds a cylindrical cavity whose axis coincides with the rocket's longitudinal axis, namely axis x in Figure 2. The cavity plays the role of the combustion chamber as it contains the burned gas which flows relative to the shell until expelled through a nozzle at the aft end. This mathematical model is more representative of a solid-fuel rather than a liquid-fuel rocket. We consider first the case in which the rocket shell is rigid.

Let us assume that the missile has rotational symmetry, so that $\int_M y dM = \int_M z dM = 0$, and since

$$\underset{\sim}{r} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j} + z\,\underset{\sim}{k} \quad , \quad \underset{\sim}{\omega} = \omega_x\,\underset{\sim}{i} + \omega_y\,\underset{\sim}{j} + \omega_z\,\underset{\sim}{k} \tag{9}$$

Eq. (5) can be written as

$$\underset{\sim}{F}_S + \underset{\sim}{F}_B + \underset{\sim}{F}_C + \underset{\sim}{F}_U + \underset{\sim}{F}_R = M\underset{\sim}{a}_o + \dot{\underset{\sim}{\omega}}\,\underset{\sim}{x}\int_M x\,\underset{\sim}{i}\,dM + \underset{\sim}{\omega}\,\underset{\sim}{x}\left(\underset{\sim}{\omega}\,\underset{\sim}{x}\int_M x\,\underset{\sim}{i}\,dM\right) \tag{10}$$

or

$$\underset{\sim}{F}_S + \underset{\sim}{F}_B + \underset{\sim}{F}_C + \underset{\sim}{F}_U + \underset{\sim}{F}_R = M\underset{\sim}{a}_o - \left[(\omega_y^2+\omega_z^2)\underset{\sim}{i} - (\dot{\omega}_z+\omega_x\omega_y)\underset{\sim}{j} + (\dot{\omega}_y-\omega_x\omega_z)\underset{\sim}{k}\right]\int_M x\,dM \tag{11}$$

while Eq. (6) can be written as

$$\underline{N}_S + \underline{N}_B + \underline{N}_C + \underline{N}_U + \underline{N}_R = -\underline{a}_0 \times \int_M \underline{x}\, \underline{i}\, dM + \int_M \underline{r} \times [\dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})]\, dM$$
$$= -\underline{a}_0 \times \int_M \underline{x}\, \underline{i}\, dM + \dot{\underline{L}}' + \underline{\omega} \times \underline{L} \qquad (12)$$

where

$$\underline{L} = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\underline{i} + (-I_{xy}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\underline{j}$$
$$+ (-I_{xz}\omega_x - I_{yz}\omega_y + I_{zz}\omega_z)\underline{k} \qquad (13)$$

is the angular momentum of the vehicle about the origin 0 and $\dot{\underline{L}}'$ is the rate of change of $\underline{L}$ due to the change in the body angular velocity relative to the body axes. It is obtained by replacing the components of $\underline{\omega}$ by the components of $\dot{\underline{\omega}}$ in Eq. (13). The quantities

$$I_{xx} = \int_M (y^2+z^2)\,dM\,,\quad I_{yy} = \int_M (x^2+z^2)\,dM\,,\quad I_{zz} = \int_M (x^2+y^2)\,dM$$
$$I_{xy} = \int_M xy\,dM\,,\quad I_{xz} = \int_M xz\,dM\,,\quad I_{yz} = \int_M yz\,dM \qquad (14)$$

are the instantaneous moments and products of inertia of the vehicle about the body axes. We note that in our case the moments of inertia are time-dependent because of the mass variation.

There remains to derive explicit expressions for the actual and equivalent forces and torques. The surface force consists of the aerodynamic force on the vehicle wetted area and the pressure force

across the exit area. Denoting by $\underset{\sim}{f}_A^*$ the aerodynamic force per unit of the wetted area A_w, by p_e the pressure across the exit area A_e, and by p_a the atmospheric pressure, the surface force takes the form

$$\underset{\sim}{F}_S = \int_{A_w} \underset{\sim}{f}_A^* \, dA_w + (p_e - p_a) A_e \underset{\sim}{i} \tag{15}$$

Assuming that the gravitational field is uniform, the body force is simply

$$\underset{\sim}{F}_B = \int_L m \underset{\sim}{g} \, dx = M \underset{\sim}{g} \tag{16}$$

where L is the length of the rocket, m the distributed mass, and $\underset{\sim}{g}$ the acceleration due to gravity. Assuming that the flow everywhere is along the x-axis, with the possible exception of the exit point, we have

$$\underset{\sim}{v} = -v(x,y,z,t)\underset{\sim}{i} = -v(x,t)\underset{\sim}{i} \tag{17}$$

Moreover, assuming that the flow across the cross-sectional area is uniform, the Coriolis force can be written

$$\begin{aligned} \underset{\sim}{F}_C &= -2\underset{\sim}{\omega} \times \int_L \underset{\sim}{v} m_f \, dx = 2(\omega_z \underset{\sim}{j} - \omega_y \underset{\sim}{k}) \int_L v m_f \, dx \\ &= -2(\omega_z \underset{\sim}{j} - \omega_y \underset{\sim}{k}) \int_L \left(\int_x^b \dot{m} \, d\xi \right) dx \end{aligned} \tag{18}$$

where use has been made of the continuity equation, namely

$$v m_f = -\int_x^b \dot{m} \, d\xi \tag{19}$$

Equation (19) results from the continuity equation, Eq. (8), by considering a control volume from a point x to the closed end of the vehicle. In Eq. (19), m_f denotes fluid mass per unit length at point x, b is the distance from the vehicle mass center to the closed end, $\dot{m}$ is the mass rate of change per unit length, and ξ is a dummy variable of integration. Similarly, the force due to the flow unsteadiness takes the form

$$\underset{\sim}{F}_U = -\frac{\partial}{\partial t}\int_L \underset{\sim}{v} m_f \, dx = -\left[\frac{\partial}{\partial t}\int_L \left(\int_x^b \dot{m} \, d\xi\right) dx\right] \underset{\sim}{i} \tag{20}$$

Finally, the reactive force can be written as

$$\underset{\sim}{F}_R = -\int_L \left[\frac{\partial}{\partial x}(v \underset{\sim}{v} m_f) + \Delta(v \underset{\sim}{v} m_f)\,\delta(x)\right] dx = v \underset{\sim}{v} m_f \Big|_{x_e} \tag{21}$$

where the symbol x_e indicates that the quantity $v\underset{\sim}{v}m_f$ is to be evaluated at the exit point. The integrand in Eq. (21) can be easily derived by assuming one-dimensional flow along the x-axis. It will be noticed that the expression makes allowance for possible abrupt changes in the flow pattern, as would occur if the rocket engine were to be gimbaled at a certain angle with respect to the x-direction. This is reflected by the second term in the integrand, in which $\delta(x)$ is a spatial Dirac's delta function. Letting the flow direction at the exit be defined with respect to axes x,y,z by the direction cosines ℓ_{xR}, ℓ_{yR}, ℓ_{zR}, respectively, and using the continuity equation, Eq. (19), the reactive force becomes

$$\underset{\sim}{F}_R = -\dot{M}\, v(x_e,t)\,(\ell_{xR}\underset{\sim}{i} + \ell_{yR}\underset{\sim}{j} + \ell_{zR}\underset{\sim}{k}) \tag{22}$$

where $\dot{M}$ represents the total mass rate of change which, of course, is a negative quantity.

The forces $\underset{\sim}{F}_S$ and $\underset{\sim}{F}_R$ can be written in the form

$$\underset{\sim}{F}_S + \underset{\sim}{F}_R = \underset{\sim}{F}_A + \underset{\sim}{F}_T \tag{23}$$

where $\underset{\sim}{F}_A$ denotes the aerodynamic force

$$\underset{\sim}{F}_A = \int_{A_w} \underset{\sim}{f}_A^* \, dA_w \tag{24}$$

and $\underset{\sim}{F}_T$ is the "engine thrust"

$$\underset{\sim}{F}_T = (p_e - p_a)A_e \underset{\sim}{i} + |\dot{M}| v(x_e,t)(\ell_{xR}\underset{\sim}{i} + \ell_{yR}\underset{\sim}{j} + \ell_{zR}\underset{\sim}{k}) \tag{25}$$

In an analogous manner, the torques are obtained as

$$\begin{aligned}
\underset{\sim}{N}_A &= \int_{A_w} \underset{\sim}{r}_S \times \underset{\sim}{f}_A^* \, dA_w \\
\underset{\sim}{N}_T &= -a|\dot{M}| v(x_e,t)(\ell_{zR}\underset{\sim}{j} - \ell_{yR}\underset{\sim}{k}) \\
\underset{\sim}{N}_B &= \underset{\sim}{0} \\
\underset{\sim}{N}_C &= -2(\omega_y \underset{\sim}{j} + \omega_z \underset{\sim}{k}) \int_L x \left(\int_x^b \dot{m} \, d\xi \right) dx \\
\underset{\sim}{N}_U &= \underset{\sim}{0}
\end{aligned} \tag{26}$$

in which $\underset{\sim}{r}_S$ is the radius vector to a point on the rocket surface and a is the distance from the origin 0 to the exit point.

Using Eqs. (16), (18), (20), (24), (25) and (26), Eqs. (11) and (12) become

$$M\frac{d\dot{\underset{\sim}{R}}_o}{dt} - [(\omega_y^2+\omega_z^2)\underset{\sim}{i} - (\dot{\omega}_z+\omega_x\omega_y)\underset{\sim}{j} + (\dot{\omega}_y-\omega_x\omega_z)\underset{\sim}{k}]\int_M \underset{\sim}{x}\, dM =$$
$$\int_{A_w} \underset{\sim}{f}_A^*\, dA_w + (p_e-p_a)A_e\underset{\sim}{i} + |\dot{M}|v(x_e,t)(\ell_{xR}\underset{\sim}{i} + \ell_{yR}\underset{\sim}{j} + \ell_{zR}\underset{\sim}{k})$$
$$+ M\underset{\sim}{g} - 2(\omega_z\underset{\sim}{j} - \omega_y\underset{\sim}{k})\int_L\left(\int_x^b \dot{m}\, d\xi\right) dx - \left[\frac{\partial}{\partial t}\int_L\left(\int_x^b \dot{m}\, d\xi\right) dx\right]\underset{\sim}{i} \quad (27)$$

$$\dot{\underset{\sim}{L}}' + \underset{\sim}{\omega}\times\underset{\sim}{L} - \frac{d\dot{\underset{\sim}{R}}_o}{dt}\times\int_M \underset{\sim}{x}\,\underset{\sim}{i}\, dM = \int_{A_w} \underset{\sim}{r}_s\times\underset{\sim}{f}_A^*\, dA_w - a|\dot{M}|v(x_e,t)(\ell_{zR}\underset{\sim}{j} - \ell_{yR}\underset{\sim}{k})$$
$$- 2(\omega_y\underset{\sim}{j} + \omega_z\underset{\sim}{k})\int_L \underset{\sim}{x}\left(\int_x^b \dot{m}\, d\xi\right) dx \quad (28)$$

Next let us introduce the notation

$$\dot{\underset{\sim}{R}}_o = U\underset{\sim}{i} + V\underset{\sim}{j} + W\underset{\sim}{k} \quad (29)$$

for the velocity of the center of mass of the missile in terms of components along the moving axes, use the fact that the missile is symmetric, and assume that x,y, and z are principal axes, such that the products of inertia vanish and $I_{yy} = I_{zz}$. Under these circumstances Eqs. (27) and (28) can be shown to have the components

$$M[\dot{U}+W\omega_y - V\omega_z] - (\omega_y^2+\omega_z^2)\int_M x\, dM = F_{Ax} + (p_e-p_a)A_e + |\dot{M}|\, v(x_e,t)\,\ell_{xR}$$
$$+ M\underset{\sim}{g}\cdot\underset{\sim}{i} - \frac{\partial}{\partial t}\int_L\left(\int_x^b \dot{m}\, d\xi\right) dx \quad (30a)$$

$$M[\dot{V}+U\omega_z - W\omega_x] + (\dot{\omega}_z+\omega_x\omega_y)\int_M x\, dM = F_{Ay} + |\dot{M}|\, v(x_e,t)\,\ell_{yR} + M\underset{\sim}{g}\cdot\underset{\sim}{j}$$
$$- 2\omega_z\int_L\left(\int_x^b \dot{m}\, d\xi\right) dx \quad (30b)$$

$$M[\dot{W} + V\omega_x - U\omega_y] - (\dot{\omega}_y - \omega_x\omega_z)\int_M x\,dM = F_{Az} + |\dot{M}|\,v(x_e,t)\,\ell_{zR} + M\underset{\sim}{g}\cdot\underset{\sim}{k}$$
$$+ 2\omega_y\int_L \left(\int_x^b \dot{m}\,d\xi\right) dx \qquad (30c)$$

and

$$I_{xx}\,\dot{\omega}_x = N_{Ax} \qquad (31a)$$

$$I_{yy}\,\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{yy}) - (\dot{W} + V\omega_x - U\omega_y)\int_M x\,dM = N_{Ay}$$
$$- a|\dot{M}|\,v(x_e,t)\,\ell_{zR} - 2\omega_y\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx \qquad (31b)$$

$$I_{yy}\,\dot{\omega}_z - \omega_x\omega_y(I_{xx} - I_{yy}) + (\dot{V} + U\omega_z - W\omega_x)\int_M x\,dM = N_{Az}$$
$$+ a|\dot{M}|\,v(x_e,t)\,\ell_{yR} - 2\omega_z\int_l x\left(\int_x^b \dot{m}\,d\xi\right)dx \qquad (31c)$$

where the definitions

$$\underset{\sim}{i}\cdot\int_{A_w}\underset{\sim}{f}_A^*\,dA_w = F_{Ax} \quad , \quad \underset{\sim}{i}\cdot\int_{A_w}\underset{\sim}{r}_s \times \underset{\sim}{f}_A^*\,dA_w = N_{Ax}$$
$$\underset{\sim}{j}\cdot\int_{A_w}\underset{\sim}{f}_A^*\,dA_w = F_{Ay} \quad , \quad \underset{\sim}{j}\cdot\int_{A_w}\underset{\sim}{r}_s \times \underset{\sim}{f}_A^*\,dA_w = N_{Ay}$$
$$\underset{\sim}{k}\cdot\int_{A_w}\underset{\sim}{f}_A^*\,dA_w = F_{Az} \quad , \quad \underset{\sim}{k}\cdot\int_{A_w}\underset{\sim}{r}_s \times \underset{\sim}{f}_A^*\,dA_w = N_{Az} \qquad (32)$$

were used.

In order to conform to the more traditional notation we define a set of axes, as shown in Figure 3, where XYZ are a set of inertial

axes with Z pointing downward. Next we consider a rotation ψ about axis Z to obtain the set $x_1y_1z_1$ (yaw), a rotation θ about the y_1 axis to obtain the set $x_2y_2z_2$ (pitch), and a rotation ϕ about the x_2 axis to obtain the set xyz (roll). The relationships between the inertial and the moving coordinate systems are*

$$\begin{aligned}
\underset{\sim}{i} &= c\theta\, c\psi\, \underset{\sim}{i}' + c\theta\, s\psi\, \underset{\sim}{j}' - s\theta\, \underset{\sim}{k}' \\
\underset{\sim}{j} &= (s\phi\, s\theta\, c\psi - c\phi\, s\psi)\underset{\sim}{i}' + (s\phi\, s\theta\, s\psi + c\phi\, c\psi)\underset{\sim}{j}' + s\phi\, c\theta\, \underset{\sim}{k}' \\
\underset{\sim}{k} &= (c\phi\, s\theta\, c\psi + s\phi\, s\psi)\underset{\sim}{i}' + (c\phi\, s\theta\, s\psi - s\phi\, c\psi)\underset{\sim}{j}' + c\phi\, c\theta\, \underset{\sim}{k}'
\end{aligned} \tag{33}$$

Moreover, the angular velocities, in terms of the rate of change of ϕ, θ, ψ are

$$\begin{aligned}
\omega_x &= \dot{\phi} - \dot{\psi}\, s\theta \\
\omega_y &= \dot{\theta}\, c\phi + \dot{\psi}\, c\theta\, s\phi \\
\omega_z &= -\dot{\theta}\, s\phi + \dot{\psi}\, c\theta\, c\phi
\end{aligned} \tag{34}$$

while the velocity of the origin 0 has the following components along the inertial axes

$$\begin{aligned}
\dot{R}_{ox} &= U\, c\theta\, c\psi + V(s\phi\, s\theta\, c\psi - c\phi\, s\psi) + W(c\phi\, s\theta\, c\psi + s\phi\, s\psi) \\
\dot{R}_{oy} &= U\, c\theta\, s\psi + V(s\phi\, s\theta\, s\psi + c\phi\, c\psi) + W(c\phi\, s\theta\, s\psi - s\phi\, c\psi) \\
\dot{R}_{oz} &= -U\, s\theta + V\, s\phi\, c\theta + W\, c\phi\, c\theta
\end{aligned} \tag{35}$$

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* The abbreviations $\cos\phi = c\phi$, $\sin\phi = s\phi$, etc., are used throughout this investigation.

Equations (30), (31), (34) and (35) are sufficient to define the position and orientation of the missile as a function of time.

Under certain circumstances Eqs. (30) and (31) can be simplified appreciably. First, let us assume that the flow is steady and that the missile is not controlled, which implies that $\ell_{xR} = 1$, $\ell_{yR} = \ell_{zR} = 0$, so that Eqs. (30) and (31) reduce to

$$M[\dot{U}+W\omega_y-V\omega_z] + (\omega_y^2+\omega_z^2)\int_M x\,dM = F_{Ax} + (P_e-P_a)A_e + |M|v(x_e,t) + M\underset{\sim}{g}\cdot\underset{\sim}{i}$$

$$M[\dot{V}+U\omega_z-W\omega_x] + (\dot{\omega}_z-\omega_x\omega_y)\int_M x\,dM = F_{Ay} + M\underset{\sim}{g}\cdot\underset{\sim}{j} - 2\omega_z\int_L\left(\int_x^b \dot{m}\,d\xi\right)dx$$

$$M[\dot{W}+V\omega_x-U\omega_y] - (\dot{\omega}_y-\omega_x\omega_z)\int_M x\,dM = F_{Az} + M\underset{\sim}{g}\cdot\underset{\sim}{k} + 2\omega_y\int_L\left(\int_x^b \dot{m}\,d\xi\right)dx \tag{36}$$

and

$$I_{xx}\dot{\omega}_x = N_{Ax}$$

$$I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx}-I_{yy}) - (\dot{W}+V\omega_x-U\omega_y)\int_M x\,dM = N_{Ay} - 2\omega_y\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx$$

$$I_{yy}\dot{\omega}_z - \omega_x\omega_y(I_{xx}-I_{yy}) + (\dot{V}+U\omega_z-W\omega_x)\int_M x\,dM = N_{Az} - 2\omega_z\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx \tag{37}$$

Next assume that the origin of the coordinate system is at the center of mass, located at x = 0, and that the origin does not shift due to the mass variation (see, for example, Reference 16, p. 15). This is particularily true when the fuel rate of burning is proportional to the mass at any point of the rocket. Hence, from the definition of the center of mass

$$\int_M x \, dM = 0 \tag{38}$$

and using

$$\underset{\sim}{g} = g \underset{\sim}{k}'$$

Eqs. (36) and (37) become

$$\begin{aligned}
M[\dot{U} + W\omega_y - V\omega_z] &= F_{Ax} + (p_e - p_a)A_e + |\dot{M}|v(x_e,t) - Mg\,s\Theta \\
M[\dot{V} + U\omega_z - W\omega_x] &= F_{Ay} + Mg\,s\phi\,c\Theta - 2\omega_z\int_L\left(\int_x^b \dot{m}\,d\xi\right)dx \\
M[\dot{W} + V\omega_x - U\omega_y] &= F_{Az} + Mg\,c\phi\,c\Theta + 2\omega_y\int_L\left(\int_x^b \dot{m}\,d\xi\right)dx
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
I_{xx}\,\dot{\omega}_x &= N_{Ax} \\
I_{yy}\,\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{yy}) &= N_{Ay} - 2\omega_y\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx \\
I_{yy}\,\dot{\omega}_z - \omega_x\omega_y(I_{xx} - I_{yy}) &= N_{Az} - 2\omega_z\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx
\end{aligned} \tag{40}$$

The last two of Eqs. (40) can be rearranged as

$$\begin{aligned}
I_{yy}\left[\dot{\omega}_y + \omega_x\omega_z\left(\frac{I_{xx}}{I_{yy}} - 1\right)\right] &= N_{Ay} - 2\omega_y\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx \\
I_{yy}\left[\dot{\omega}_z - \omega_x\omega_y\left(\frac{I_{xx}}{I_{yy}} - 1\right)\right] &= N_{Az} - 2\omega_z\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx
\end{aligned} \tag{41}$$

and, because $I_{yy} >> I_{xx}$, Eqs. (41) can be reduced to

$$\begin{aligned}
I_{yy}(\dot{\omega}_y - \omega_x\omega_z) &= N_{Ay} - 2\omega_y\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx \\
I_{yy}(\dot{\omega}_z + \omega_x\omega_y) &= N_{Az} - 2\omega_z\int_L x\left(\int_x^b \dot{m}\,d\xi\right)dx
\end{aligned} \tag{42}$$

4. The Equations of Motion of a Flexible Rocket

When the rocket casing can undergo elastic deformations the problem requires further attention. To this end, consider a rocket translating and rotating relative to the inertial space X,Y,Z, as shown in Figure 2. As the control volume, we consider the volume occupied by a rocket element of unit length when the vehicle is at rest relative to the body axes x,y,z. Figure 4 shows the corresponding element. Because the rocket shell is elastic, the entire mass associated with the control volume in question can move relative to that volume. The rocket case and unburned fuel are assumed to move together and their motion is different from the motion of the burned fuel, so that it will prove convenient to denote the motions and mass associated with the case element by the subscript c and the ones related to the burned fuel element by the subscript f. In analogy with Eq. (2), and considering the rocket element shown in Figure 4, we can write the force equation of motion in the form

$$\underset{\sim}{f}_S + \underset{\sim}{f}_B = \int_{m_c} \left[\underset{\sim}{a}_o + \dot{\underset{\sim}{v}}_c + 2\,\underset{\sim}{\omega} \times \underset{\sim}{v}_c + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_c + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_c)\right] dm$$
$$+ \int_{m_f} \left[\underset{\sim}{a}_o + \dot{\underset{\sim}{v}}_f + 2\,\underset{\sim}{\omega} \times \underset{\sim}{v}_f + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_f + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_f)\right] dm \quad (43)$$

where $\underset{\sim}{f}_S$ and $\underset{\sim}{f}_B$ are distributed surface and body forces, respectively, $\underset{\sim}{v}_c$ is the elastic motion of a point inside the case element, and $\underset{\sim}{v}_f$ is the fluid velocity relative to the body axes. It will be assumed that the elastic motion is the same for the entire case element, and a similar statement can be made concerning the velocity of the fluid element. Introducing the notation

$$\underset{\sim}{v}_c = \dot{\underset{\sim}{u}} \quad , \qquad \dot{\underset{\sim}{v}}_f = \dot{\underset{\sim}{u}} + \dot{\underset{\sim}{v}} \tag{44}$$

where $\underset{\sim}{u}$ represents the elastic displacement vector, and $\underset{\sim}{v}$ is the velocity of the fluid relative to the case, we can rewrite Eq. (43) as

$$\underset{\sim}{f}_s + \underset{\sim}{f}_B = \int_m [\underset{\sim}{a}_o + \ddot{\underset{\sim}{u}} + 2\underset{\sim}{\omega} \times \dot{\underset{\sim}{u}} + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})]\, dm$$
$$+ \int_{m_f} (\dot{\underset{\sim}{v}} + 2\underset{\sim}{\omega} \times \underset{\sim}{v})\, dm = (\underset{\sim}{a}_o + \ddot{\underset{\sim}{u}} + 2\underset{\sim}{\omega} \times \dot{\underset{\sim}{u}})m + \dot{\underset{\sim}{\omega}} \times \int_m \underset{\sim}{r}\, dm$$
$$+ \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \int_m \underset{\sim}{r}\, dm) + (\dot{\underset{\sim}{v}} + 2\underset{\sim}{\omega} \times \underset{\sim}{v})m_f \tag{45}$$

where $m = m_c + m_f$ is the mass of the rocket per unit length. Moreover, the radius vector $\underset{\sim}{r}$ has the expression

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} + \underset{\sim}{u} = (x+u_x)\underset{\sim}{i} + (y+u_y)\underset{\sim}{j} + (z+u_z)\underset{\sim}{k} \tag{46}$$

in which u_x, u_y, and u_z are the elastic displacements of the case element in the x,y, and z directions, respectively.

Invoking the analogy with Eq. (5), we can rewrite Eq. (45) to read

$$\underset{\sim}{f}_S + \underset{\sim}{f}_B + \underset{\sim}{f}_C + \underset{\sim}{f}_U + \underset{\sim}{f}_R = [\underset{\sim}{a}_o + \ddot{\underset{\sim}{u}} + 2\underset{\sim}{\omega} \times \dot{\underset{\sim}{u}}]m + \dot{\underset{\sim}{\omega}} \times \int_m \underset{\sim}{r}\, dm + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \int_m \underset{\sim}{r}\, dm)$$
$$= (\underset{\sim}{a}_o + \bar{\underset{\sim}{a}})m = \underset{\sim}{a}m \tag{47}$$

where $\underset{\sim}{a}$ is the absolute acceleration consisting of the acceleration $\underset{\sim}{a}_0$ of the origin 0 and the acceleration $\bar{\underset{\sim}{a}}$ of the case element relative to the body axes. Moreover

$$\underline{f}_C = -2\underline{\omega} \times \underline{v} m_f$$

$$\underline{f}_U = -\frac{\partial}{\partial t}(\underline{v} m_f) \qquad (48)$$

$$\underline{f}_R = -\frac{\partial}{\partial x}(v \underline{v} m_f) - \Delta(v \underline{v} m_f)\delta(x+a)$$

are the Coriolis force, the force due to the unsteadiness of the fluid flow relative to the case, and the reactive force, respectively, all per unit length of rocket.

If we express $\underline{R}_0$ in terms of components along axes x,y,z, then the position of the case element at any time is given by

$$\underline{R} = \underline{R}_o + \underline{r}$$

$$= (R_{ox} + x + u_x)\underline{i} + (R_{oy} + y + u_y)\underline{j} + (R_{oz} + z + u_z)\underline{k} \qquad (49)$$

Recalling that the unit vectors $\underline{i}$, $\underline{j}$, and $\underline{k}$ rotate with angular velocity $\underline{\omega}$, the absolute acceleration of the case element can be written in the form

$$\underline{a} = \underline{a}_o + \bar{\underline{a}} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k} \qquad (50)$$

where

$$a_x = \dot{U} + \ddot{u}_x + \omega_y(W + 2\dot{u}_z) - \omega_z(V + 2\dot{u}_y) + (\dot{\omega}_y + \omega_x\omega_z)(z + u_z)$$
$$-(\dot{\omega}_z - \omega_x\omega_y)(y + u_y) - (\omega_y^2 + \omega_z^2)(x + u_x)$$

$$a_y = \dot{V} + \ddot{u}_y + \omega_z(U + 2\dot{u}_x) - \omega_x(W + 2\dot{u}_z) + (\dot{\omega}_z + \omega_x\omega_y)(x + u_x)$$
$$-(\dot{\omega}_x - \omega_y\omega_z)(z + u_z) - (\omega_x^2 + \omega_z^2)(y + u_y) \qquad (51)$$

$$a_z = \dot{W} + \ddot{u}_z + \omega_x(V + 2\dot{u}_y) - \omega_y(U + 2\dot{u}_x) + (\dot{\omega}_x + \omega_y\omega_z)(y + u_y)$$
$$-(\dot{\omega}_y - \omega_x\omega_z)(x + u_x) - (\omega_x^2 + \omega_y^2)(z + u_z)$$

In the above expressions the y and z coordinates may be considered as an offset such as may result from the missile not being perfectly symmetrical about the x axis.

Similarly, using Eq. (45), the torque equation about point 0 for the rocket element in question takes the form

$$\underline{n}_s + \underline{n}_B = \int_m \underline{r} \times [\underline{a}_o + \ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})]dm$$
$$+ \int_{m_f} \underline{r} \times (\dot{\underline{v}} + 2\underline{\omega} \times \underline{v})\, dm = \int_m \underline{r} \times (\underline{a}_o + \ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}})dm$$
$$+ \dot{\underline{l}} + \underline{\omega} \times \underline{l} + \int_{m_f} \underline{r} \times (\dot{\underline{v}} + 2\underline{\omega} \times \underline{v})\, dm \qquad (52)$$

where

$$\underline{l} = (i_{xx}\omega_x - i_{xy}\omega_y - i_{xz}\omega_z)\underline{i} + (-i_{xy}\omega_x + i_{yy}\omega_y - i_{yz}\omega_z)\underline{j}$$
$$+ (-i_{xz}\omega_x - i_{yz}\omega_y + i_{zz}\omega_z)\underline{k} \qquad (53)$$

is the angular momentum of the mass element m about the body axes x,y,z in which

$$i_{xx} = \int_m [(y+u_y)^2+(z+u_z)^2]dm\ , \quad i_{yy} = \int_m [(x+u_x)^2+(z+u_z)^2]dm$$
$$i_{zz} = \int_m [(x+u_x)^2+(y+u_y)^2]dm\ , \quad i_{xy} = \int_m (x+u_x)(y+u_y)dm$$
$$i_{xz} = \int_m (x+u_x)(z+u_z)dm\ , \quad i_{yz} = \int_m (y+u_y)(z+u_z)dm \tag{54}$$

are recognized as the associated moments and products of inertia. Moreover, $\dot{\underline{\underline{i}}}'$ is obtained from Eq. (53) by replacing ω_x, ω_y, ω_z by $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$, respectively. Equation (52) can be rewritten as

$$\underline{n}_S + \underline{n}_B + \underline{n}_C + \underline{n}_U + \underline{n}_R = \int_m \underline{r} \times (\underline{a}_o + \ddot{\underline{u}} + 2\underline{\omega} \times \dot{\underline{u}})dm + \dot{\underline{\ell}}' + \underline{\omega} \times \underline{\ell} \tag{55}$$

where the torques

$$\underline{n}_C = -2\int_{m_f} \underline{r} \times (\underline{\omega} \times \underline{v})\, dm$$
$$\underline{n}_U = -\int_{m_f} \underline{r} \times \frac{\partial}{\partial t}(\underline{v} m_f)\, dm \tag{56}$$
$$\underline{n}_R = -\int_{m_f} \underline{r} \times \left[\frac{\partial}{\partial x}(v \underline{v} m_f) + \Delta(v \underline{v} m_f)\, \delta(x+o)\right] dm$$

follow directly from Eq. (48).

Equations (47) and (55) must be supplemented by the continuity equation, Eq. (19).

5. The Equations for the Axial and Transverse Vibrations of a Rocket

Let us consider the rocket of the preceding section in which u_x is the axial elastic displacement and u_y and u_z are the elastic transverse displacements in the y and z directions, respectively. Assuming axial symmetry and uniform mass distribution and, moreover, that the elastic displacements u_x, u_y, u_z and the angular velocity components ω_y, ω_z, as well as their time derivatives, are small quantities, we can integrate Eqs. (47) and (55) and obtain

$$\underline{F}_S+\underline{F}_B+\underline{F}_C+\underline{F}_U+\underline{F}_R=\int_L[\underline{a}_o+\ddot{\underline{u}}+2\underline{\omega}\underline{x}\dot{\underline{u}}]m\,dx+\dot{\underline{\omega}}\,\underline{x}\iint_m\underline{r}\,dm\,dx+\underline{\omega}\underline{x}(\underline{\omega}\underline{x}\iint_m\underline{r}\,dm\,dx)$$

$$\cong M\underline{a}_o+\int_L(\ddot{\underline{u}}+2\underline{\omega}\underline{x}\dot{\underline{u}})m\,dx-\dot{\omega}_x\int_L(u_z\underline{j}-u_y\underline{k})m\,dx \tag{57}$$

$$\underline{N}_S+\underline{N}_B+\underline{N}_C+\underline{N}_U+\underline{N}_R=\int_L\int_m\underline{r}\,\underline{x}\,(\underline{a}_o+\ddot{\underline{u}}+2\underline{\omega}\underline{x}\dot{\underline{u}})dm\,dx+\int_L(\dot{\underline{\ell}}'+\underline{\omega}\underline{x}\underline{\ell})\,dx$$

$$=\dot{\underline{L}}'+\underline{\omega}\underline{x}\underline{L}-\int_L x\,[(\ddot{u}_z+2\omega_x\dot{u}_y+\dot{\omega}_x u_y)\underline{j}-(\ddot{u}_y-2\omega_x\dot{u}_z-\dot{\omega}_x u_z)\underline{k}]m\,dx \tag{58}$$

Comparing Eqs. (11) and (57) on the one hand, and Eqs. (12) and (58) on the other hand, we conclude that the elastic motion does not affect the rigid body motion $\underline{a}_0$ and $\underline{\omega}$, provided the following relations are satisfied

$$\int_L\underline{u}\,m\,dx=\int_L\dot{\underline{u}}\,m\,dx=\int_L\ddot{\underline{u}}\,m\,dx=\underline{0}$$

$$\int_L x\,u_y m\,dx=\int_L x\,\dot{u}_y m\,dx=\int_L x\,\ddot{u}_y m\,dx=0 \tag{59}$$

$$\int_L x\,u_z m\,dx=\int_L x\,\dot{u}_z m\,dx=\int_L x\,\ddot{u}_z m\,dx=0$$

We shall assume that this is the case, and indeed Eqs. (59) imply that the elastic modes of deformation are orthogonal (with respect to mass) to the rigid body modes of displacement, namely the translation and rotation of the vehicle as a whole. In view of the above arguments, the problem can be solved in two stages. In the first stage we solve for the rigid body motion $\underset{\sim}{a}_0$ and $\underset{\sim}{\omega}$ from Eqs. (11) and (12) and then, considering $\underset{\sim}{a}_0$ and $\underset{\sim}{\omega}$ as known, turn to Eqs. (47) through (56) for the elastic motion $\underset{\sim}{u}$.

Equations (47) and (48), representing the equations of motion for the three components u_x, u_y, u_z of the elastic displacement $\underset{\sim}{u}$, are of a general form and, before we can attempt their solution, we must specify the nature of the surface force $\underset{\sim}{f}_S$. This force depends not only on the external aerodynamic forces but also on the internal stresses in the shell and fluid pressure. Moreover, we must also know the fluid flow characteristics, as can be concluded from Eqs. (48).

As far as the elastic motion is concerned, the vehicle shell is assumed to behave like a bar in axial and flexural vibration. Under these circumstances, the distributed surface force can be written in the form

$$\begin{aligned}\underset{\sim}{f}_S = &\left[\frac{\partial}{\partial x}\left(EA_c\frac{\partial u_x}{\partial x}\right)\right]\underset{\sim}{i} + \left[-\frac{\partial^2}{\partial x^2}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(P\frac{\partial u_y}{\partial x}\right)\right]\underset{\sim}{j}\\ &+\left[-\frac{\partial^2}{\partial x^2}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(P\frac{\partial u_z}{\partial x}\right)\right]\underset{\sim}{k} - \left[\frac{\partial}{\partial x}(pA_f)\right]\underset{\sim}{i}\\ &+f_{Ax}\underset{\sim}{i} + f_{Ay}\underset{\sim}{j} + f_{Az}\underset{\sim}{k}\end{aligned} \tag{60}$$

where the first three terms represent the force components due to internal stresses caused by the axial and flexural vibrations (see, for example Reference 17, Sections 5-7 and 10-3), the fourth term is due to internal fluid pressure differential, and the remaining terms are due to aerodynamic effects. The term P denotes the axial force on the vehicle due to internal stresses and has the expression

$$P = E A_c \frac{\partial u_x}{\partial x} \tag{61}$$

It should be pointed out that the longitudinal stiffness EA_c, where E is the modulus of elasticity and A_c the cross-sectional area of the casing, and the flexural stiffnesses EI_{cy} and EI_{cz}, where I_{cy} and I_{cz} are area moments of inertia of the case about axes y and z through the cross-sectional center, take into account the casing material only. This implies that the unburned propellant possesses inertia properties but no structural stiffness.

Using the above definitions for the real and equivalent forces the differential equation for the axial vibration takes the form

$$\frac{\partial}{\partial x}(EA_c\frac{\partial u_x}{\partial x}) - \frac{\partial}{\partial x}(pA_f) + f_{Ax} + m\underline{g} \cdot \underline{i} - \frac{\partial}{\partial t}(vm_f) - \frac{\partial}{\partial x}(v^2 m_f) =$$
$$m\left[\dot{U} + \ddot{u}_x + \omega_y(W + 2\dot{u}_z) - \omega_z(V + 2\dot{u}_y) + (\dot{\omega}_y + \omega_x\omega_z)u_z\right.$$
$$\left. -(\dot{\omega}_z - \omega_x\omega_y)u_y - (\omega_y^2 + \omega_z^2)(x + u_x)\right] \tag{62}$$

which is subject to the boundary conditions

$$EA_c \frac{\partial u_x}{\partial x} = P_{x1} \quad \text{at} \quad x = b$$
$$EA_c \frac{\partial u_x}{\partial x} = P_{x2} \quad \text{at} \quad x = -a \tag{63}$$

where the functions P_{x1} and P_{x2} are axial forces exerted by the gases on the case at the ends x = - a,b.

In a similar manner, the differential equation for the flexural vibration in the xy-plane is

$$-\frac{\partial^2}{\partial x^2}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + \frac{\partial}{\partial x}\left(P\frac{\partial u_y}{\partial x}\right) + f_{Ay} + mg\underline{\cdot}\underline{j} + 2\omega_z v m_f$$
$$= m\left[\dot{V} + \ddot{u}_y + \omega_z(U+2\dot{u}_x) - \omega_x(W+2\dot{u}_z) + (\dot{\omega}_z + \omega_x\omega_y)(x+u_x)\right.$$
$$\left. -(\dot{\omega}_x - \omega_y\omega_z)u_z - (\omega_x^2 + \omega_z^2)u_y\right] \tag{64}$$

with the boundary conditions

$$EI_{cz}\frac{\partial^2 u_y}{\partial x^2} = 0 \qquad \text{at } x = -a, b$$
$$-\frac{\partial}{\partial x}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + P_{x1}\frac{\partial u_y}{\partial x} = P_{y1} \quad \text{at } x = b \tag{65}$$
$$-\frac{\partial}{\partial x}\left(EI_{cz}\frac{\partial^2 u_y}{\partial x^2}\right) + P_{x2}\frac{\partial u_y}{\partial x} = P_{y2} \quad \text{at } x = -a$$

The first of boundary conditions (65) indicates that there are no bending moments at the ends $x = - a,b$, the second and third express the fact that there may be forces in the y-direction at the ends $x = - a,b$ resulting from aerodynamic effects as well as due to a possible change in the flow direction at the end $x = - a$.

Moreover, the differential equation for the flexural vibration in the xz-plane is

$$
\begin{aligned}
&-\frac{\partial^2}{\partial x^2}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(P\frac{\partial u_z}{\partial x}\right)+f_{Az}+m\,\underline{g}\cdot\underline{k}-2\omega_y v m_f \\
&=m\Big\{\dot{W}+\ddot{u}_z+\omega_x(V+2\dot{u}_y)-\omega_y(U+2\dot{u}_x)+(\dot{\omega}_x+\omega_y\omega_z)u_y \\
&\quad-(\dot{\omega}_y-\omega_x\omega_z)(x+u_x)-(\omega_x^2+\omega_y^2)u_z\Big\}
\end{aligned}
\tag{66}
$$

with the boundary conditions

$$
\begin{aligned}
&EI_{cy}\frac{\partial^2 u_z}{\partial x^2}=0 && \text{at } x=-a,\,b \\
&-\frac{\partial}{\partial x}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right)+P_{x_1}\frac{\partial u_z}{\partial x}=P_{z1} && \text{at } x=b \\
&-\frac{\partial}{\partial x}\left(EI_{cy}\frac{\partial^2 u_z}{\partial x^2}\right)+P_{x_2}\frac{\partial u_z}{\partial x}=P_{z2} && \text{at } x=-a\ .
\end{aligned}
\tag{67}
$$

At this point a discussion of some additional assumptions implied by Eqs. (62) through (67) is in order. First we note that the aerodynamic forces are treated as distributed forces causing no torques on the case element. Such torques, if they exist, are assumed to affect only the rocket rigid body rotations. Although the nozzle has finite

length, it was assumed, for simplicity, to be of negligible length. In fact the term P_{x2} represents the axial force on the nozzle wall from the gas flow between the two end points of the nozzle. In a more exact treatment of the gas flow, this assumption may have to be relaxed by considering the pressure distribution along the finite-length nozzle (see Appendix).

The flow has been treated as if it possessed no viscosity. As a result, any reactions between the gases and the unburned fuel are assumed to be normal to the flow. This is implied by the fact that the velocity is uniform over the entire cross-sectional area which implies, in turn, perfect burning in the sense that no gas-dynamic eccentricity is present. The lack of gas-dynamic eccentricity is ensured by any type of radially symmetric flow, of which the uniform flow is a special case. Any torques due to gas flow may result from engine thrust missalignment, if at all. Moreover, the velocity of the flow relative to the body is assumed to have only one component, namely along the x-axis. Although, due to the transverse elastic displacements u_y and u_z, there are velocity components $v\partial u_y/\partial x$ and $v\partial u_z/\partial x$ in the y- and z-directions, respectively, the terms involved are assumed to be small and, therefore, ignored.

6. Distributed Aerodynamic Forces

Before a solution for the motion of the missile can be attempted, we must determine the distribution of the aerodynamic forces along the missile. To obtain the transverse forces, we use the method of virtual masses, whereas the axial forces are obtained by semi-empirical means. The latter forces are assumed to act at several discrete stations of the missile.

The method of virtual, or apparent mass can be traced to Lamb[18]. The method was extended by Munk[19] and Jones[20] and applied to missiles by Bryson[21]. The present derivation represents an extension of the method and reduces to the results of References 22, 23, and 24 if suitable simplifications and assumptions are made.

Consider a missile moving through an infinite expanse of fluid which is stationary at infinity. With the coordinate system shown in Figure 5, consider a set of axes $x_1y_1z_1$, displaced relative to xyz by

$$\underset{\sim}{r} = (x+u_x)\underset{\sim}{i} + u_y\underset{\sim}{j} + u_z\underset{\sim}{k} \tag{68}$$

where $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ are unit vectors along axes xyz. The $x_1y_1z_1$ axes are such that the $x_1 = 0$ plane is a plane at rest with respect to the fluid far away from the body and such that the x_1-axis is parallel to the x-axis at the instant under consideration.

Next consider the element of unit length shown in Figure 5, and define the translational velocity of this element, expressed in terms

of component. ong the coordinate system with origin at 0, by

$$\underset{\sim}{V}_1 = u_1 \underset{\sim}{i} + v_1 \underset{\sim}{j} + w_1 \underset{\sim}{k} \tag{69}$$

Then the linear momentum of the element expressed in terms of the same set of axes can be written as

$$\begin{aligned} \underset{\sim}{p} &= p_1 \underset{\sim}{i} + p_2 \underset{\sim}{j} + p_3 \underset{\sim}{k} \\ &= m_v (u_1 \underset{\sim}{i} + v_1 \underset{\sim}{j} + w_1 \underset{\sim}{k}) \end{aligned} \tag{70}$$

in which

$$\begin{aligned} u_1 &= U + \dot{u}_x + \omega_y u_z - \omega_z u_y \\ v_1 &= V + \dot{u}_y + \omega_z (x + u_x) - \omega_x u_z \\ w_1 &= W + \dot{u}_z + \omega_x u_y - \omega_y (x + u_x) \end{aligned} \tag{71}$$

and

$$m_v = \rho S(x) \tag{72}$$

where ρ is the free stream density and S is the cross-sectional area. The distributed force acting on the missile is then

$$\underset{\sim}{f}_A = f_{Ax} \underset{\sim}{i} + f_{Ay} \underset{\sim}{j} + f_{Az} \underset{\sim}{k} = - \frac{d \underset{\sim}{p}}{dt} \tag{73}$$

As the axial component for the distributed forces is derived by a different method, we only consider the derivation for the transverse components. Considering Eqs. (70) and (71), we can write the compo-

nents for the linear momentum in the functional form

$$p_i = p_i(x,y,z,U,V,W,u_x,u_y,u_z,\dot{u}_x,\dot{u}_y,\dot{u}_z,\omega_x,\omega_y,\omega_z,m_v), \quad i=2,3 \tag{74}$$

The total time derivative of Eq. (74) is then

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial x}\frac{dx}{dt} + \frac{\partial p_i}{\partial y}\frac{dy}{dt} + \cdots + \frac{\partial p_i}{\partial m_v}\frac{dm_v}{dt}, \quad i=2,3 \tag{75}$$

Introducing Eq. (71) into Eq. (70), using Eqs. (75), and recalling that the unit vectors $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ are rotating, the transverse components in Eq. (73) become

$$\begin{aligned} f_{Ay} = {} & -m_v a_y - \dot{m}_v\left[V+\dot{u}_y+\omega_z(x+u_x)-\omega_x u_z\right] - m_v\left[U+\dot{u}_x\right. \\ & \left.+\omega_y u_z-\omega_z u_y\right]\left[\frac{\partial \dot{u}_y}{\partial x}+\omega_z+\omega_z\frac{\partial u_x}{\partial x}-\omega_x\frac{\partial u_z}{\partial x}\right]-\rho\left[U+\dot{u}_x\right. \\ & \left.+\omega_y u_z-\omega_z u_y\right]\left[V+\dot{u}_y+\omega_z(x+u_x)-\omega_x u_z\right]\frac{dS}{dx} \end{aligned} \tag{76}$$

$$\begin{aligned} f_{Az} = {} & -m_v a_z - \dot{m}_v\left[W+\dot{u}_z+\omega_x u_y-\omega_y(x+u_x)\right] - m_v\left[U+\dot{u}_x\right. \\ & \left.+\omega_y u_z-\omega_z u_y\right]\left[\frac{\partial \dot{u}_z}{\partial x}+\omega_x\frac{\partial u_y}{\partial x}-\omega_y-\omega_y\frac{\partial u_x}{\partial x}\right]-\rho\left[U+\dot{u}_x\right. \\ & \left.+\omega_y u_z-\omega_z u_y\right]\left[W+\dot{u}_z+\omega_x u_y-\omega_y(x+u_x)\right]\frac{dS}{dx} \end{aligned} \tag{77}$$

where a_y and a_z are given by Eqs. (51).

In the above expressions S(x) represents an area in a plane perpendicular to the elastic axis. For a circular segment this area is

$$S(x) = \pi r^2(x) \tag{78}$$

in which r(x) is the radius. For a segment that has fins the equivalent area is represented by the expression

$$S_{eq}(x) = \pi s^2 \left(1 - \frac{r^2}{s^2} + \frac{r^4}{s^4}\right) \tag{79}$$

in which s is the distance from the elastic line of the missile to the tip of the fins in the cross-flow plane.

The axial aerodynamic force per unit length, f_{Ax}, is defined as

$$f_{Ax} = -q S_r c_x \tag{80}$$

in which q is the free stream dynamic pressure

$$q = \frac{1}{2} \rho \dot{\underline{R}}_o \cdot \dot{\underline{R}}_o \tag{81}$$

and c_x is the axial coefficient, which in general depends on the local angle of attack, local sideslip angle and local Mach number. However, we shall assume that it is only a function of Mach number and that it acts at discrete stations along the missile. These stations are generally located at points where there are changes in the cross-sectional area, such as at the forward end and the aft end of the missile, where the fins are located. Base pressure also acts at the aft end of the missile. Viscous forces due to friction are neglected. Hence, we can write Eq. (80) as

$$f_{Ax} = -q S_r c_x(M, x_i) \delta(x - x_i) \tag{82}$$

where $\delta(x-x_i)$ is a spatial Dirac delta function.

7. Axially Symmetric, Spinning Rocket

As indicated in Section 5, the rigid body motion is independent of the elastic motion provided Eqs. (59) are satisfied. As a first approximation we regard the body as rigid, so that Eqs. (59) are identically satisfied. The elastic motion is regarded as a perturbation on the rigid body motion.

Let us consider a slender missile, as shown in Figure 6, where the missile is subject to the following assumptions : (1) the nose and fins are short in comparison to the total length of the missile, so that the transverse aerodynamic forces associated with the nose and fins can be regarded as acting at the ends of the missile, (2) the axial aerodynamic forces act only on nose and fins, where the nose has the shape of a cone, (3) the missile is unguided and the thrust is directed along the x-axis at all times, and (4) the internal flow is steady.

As a result of the first two assumptions, the effect of aerodynamic forces on the nose and fins of the missile can be expressed in the form of boundary conditions. From the third assumption it follows that the direction cosines have the values $\ell_{xR} = 1$, $\ell_{yR} = \ell_{zR} = 0$. As a result of the fourth assumption, we conclude from Reference 13 that the internal flow satisfies the equation

$$-\frac{\partial}{\partial x}(\rho A_f) - \frac{\partial}{\partial x}(v^2 m_f) \cong 0 \ . \tag{83}$$

Since the nose and fins are assumed to be short, the missile can be regarded as being uniform, so that it proves convenient to choose the

origin of the moving coordinate system xyz at the center of the missile, from which it follows that a = b = L/2.

To solve for the rigid body motion it is necessary to calculate the aerodynamic forces, which can be obtained from Eqs. (76), (77), and (82). Since as a first approximation the body is regarded as rigid, Eqs. (71) reduce to

$$u_1 = U, \quad v_1 = V + x\omega_z, \quad w_1 = W - x\omega_y \tag{84}$$

so that Eqs. (76) and (77) become

$$f_{Ay} = -m_v(\dot{V} + x\dot{\omega}_z) - \dot{m}_v(V + x\omega_z) - m_v U\omega_z - m_v U\omega_z + m_v\omega_x(W - x\omega_y) - U\rho(V + x\omega_z)\frac{dS}{dx} \tag{85}$$

$$f_{Az} = -m_v(\dot{W} - x\dot{\omega}_y) - \dot{m}_v(W - x\omega_y) + m_v U\omega_y - m_v\omega_x(V + x\omega_z) + m_v\omega_y U - U\rho(W - x\omega_y)\frac{dS}{dx} \tag{86}$$

Integrate Eqs. (82), (85), and (86) along the missile length, use the fact that the forward end is pointed such that S(L/2) = 0, and obtain

$$F_{Ax} = \int_{-L/2}^{L/2} f_{Ax}\,dx = -q_\infty S_r \sum_{s=1}^{n} c_x(M, x_s) = -q_\infty S_r[c_x(M, L/2) + c_x(M, -L/2)] \tag{87}$$

$$F_{Ay} = \int_{-L/2}^{L/2} f_{Ay}\,dx = -M_v[\dot{V} + U\omega_z - W\omega_x] - M_v\omega_z U - \dot{M}_v V + UV_\rho S_{-L/2} - U\rho\omega_z L/2\, S_1 \tag{88}$$

$$F_{Az} = \int_{-L/2}^{L/2} f_{Az}\,dx = -M_v[\dot{W} + V\omega_x - U\omega_y] + M_v\omega_y U - \dot{M}_v W + UW_\rho S_{-L/2} + U\rho\omega_y L/2\, S_1 \tag{89}$$

in which

$$M_v = m_v L \tag{90}$$

$$S_{-L/2} = S(-L/2) \tag{91}$$

$$S_1 = S(-L/2) - \frac{2}{L}\int_{-L/2}^{L/2} S(x)\,dx \tag{92}$$

and n is the number of stations at which axial forces are assumed to act.

The rigid body aerodynamic torques are found from the first of Eqs. (26) where

$$\underline{r}_S = x\,\underline{i} \tag{93}$$

and the resulting expressions are

$$\begin{aligned}
N_{Ax} &= 0 \\
N_{Ay} &= -\frac{m_v L^3}{12}(\dot{\omega}_y - \omega_x\omega_z) - \frac{\dot{m}_v L^3}{12}\omega_y + UW\rho L/2 S_1 + U\rho\omega_y\frac{L^2}{4}S_2 \\
N_{AZ} &= -\frac{m_v L^3}{12}(\dot{\omega}_z + \omega_x\omega_y) - \frac{\dot{m}_v L^3}{12}\omega_z - UV\rho L/2 S_1 + U\rho\omega_z\frac{L^2}{4}S_2
\end{aligned} \tag{94}$$

in which

$$S_2 = S(-L/2) + \frac{8}{L^2}\int_{-L/2}^{L/2} x\,S(x)\,dx \tag{95}$$

It may be noted that there is no torque produced about the longitudinal axis of the missile by the aerodynamic forces and this needs further clarification. Physically it may be assumed that there are control systems to maintain the missile under a steady rolling velocity and therefore cancel any aerodynamic forces that are produced about the x-axis. Mathematically the torque vanishes because the missile was assumed to have negligible width.

With the above expressions for the aerodynamic forces and torques, the rigid body equations of motion, Eqs. (39), (40), and (42) become

$$m\left[\dot{U}+W\omega_y-V\omega_z\right]=-q_\infty S_r\left[C_x(M,\tfrac{1}{2})+C_x(M,-\tfrac{1}{2})\right]+(p_e-p_a)A_e+|\dot{M}|v(x_e,t)-Mg\sin\theta$$

$$M^*\left[\dot{V}+U\omega_z-W\omega_x\right]=-M_v\omega_z U-\dot{M}_v V+UV\rho S_{-1/2}+Mg\sin\phi\cos\theta-U\omega_z\rho\tfrac{1}{2}S_1$$

$$M^*\left[\dot{W}+V\omega_x-U\omega_y\right]=+M_v\omega_y U-\dot{M}_v W+UW\rho S_{-1/2}+Mg\cos\phi\cos\theta+U\omega_y\rho\tfrac{1}{2}S_1$$

(96)

and

$$I_{xx}\dot{\omega}_x=0$$

$$I^*_{yy}\left(\dot{\omega}_y-\omega_x\omega_z\right)=-\frac{\dot{m}_v L^2}{12}\omega_y+UW\rho\frac{L}{2}S_1+U\rho\omega_y\frac{L^2}{4}S_2$$

$$I^*_{yy}\left(\dot{\omega}_z+\omega_x\omega_y\right)=-\frac{\dot{m}_v L^3}{12}\omega_z-UV\rho\frac{L}{2}S_1+U\rho\omega_z\frac{L^2}{4}S_2$$

(97)

in which we introduced the notation

$$M^* = M + M_v$$
$$I^*_{yy} = I_{yy} + \frac{m_v L^2}{12} \tag{98}$$

The differential equations (96) and (97), together with Eqs. (34) and (35), must be solved simultaneously to obtain the position and orientation of the missile as a function of time.

Before turning to the elastic motion of the missile, a few mathematical preliminaries must be considered, the first of which deals with the transformation of a nonhomogeneous differential equation with nonhomogeneous boundary conditions into a nonhomogeneous differential equation with homogeneous boundary conditions. For a discussion of this type of transformation, see the text by L. Meirovitch (Reference 17, Section 7-14). To accomplish our aim we introduce the transformation

$$u_x(x,t) = w_1(x,t) + P_{x1}(t)\, g_1(x) + P_{x2}(t)\, g_2(x) \tag{99}$$

where g_1 and g_2 are so chosen as to render the boundary conditions, Eqs. (63), in terms of $w_1(x,t)$ homogeneous. Introducing Eq. (99) into Eqs. (63), and insisting that the boundary conditions in terms of w_1 have the form

$$EA_c \frac{\partial w_1}{\partial x}\bigg|_{x=L/2} = EA_c \frac{\partial w_1}{\partial x}\bigg|_{x=-L/2} = 0 \tag{100}$$

we are led to the following relations

$$EA_c \frac{dg_1}{dx}\Big|_{x=L/2} = 1 \quad , \qquad EA_c \frac{dg_1}{dx}\Big|_{x=-L/2} = 0$$
$$EA_c \frac{dg_2}{dx}\Big|_{x=L/2} = 0 \quad , \qquad EA_c \frac{dg_2}{dx}\Big|_{x=-L/2} = 0 \tag{101}$$

It is not difficult to verify that expressions

$$EA_c \frac{dg_1}{dx} = h(x-L/2) \quad , \quad EA_c \frac{dg_2}{dx} = 1-h(x+L/2) \tag{102}$$

where $h(x-x_0)$ is a unit step function applied at $x = x_0$, satisfy conditions (101). Integrating Eqs. (102), and substituting the result into Eq. (99), we obtain

$$u_x(x,t) = w_1(x,t) + \frac{P_{x1}}{EA_c}(x-L/2)h(x-L/2) + \frac{P_{x2}}{EA_c}(x+L/2)[1-h(x+L/2)] \tag{103}$$

Similarily we make the transformations

$$u_y(x,t) = w_2(x,t) + P_{y_1}(t)\, g_3(x) + P_{y_2}(t)\, g_4(x)$$
$$u_z(x,t) = w_3(x,t) + P_{z_1}(t)\, g_5(x) + P_{z_2}(t)\, g_6(x) \tag{104}$$

and require the boundary conditions, Eqs. (65) and (67), to have the form

$$
\begin{aligned}
EI_{cz}\frac{\partial^2 w_2}{\partial x^2} &= 0 && \text{at } x=-L/2, L/2\\
-EI_{cz}\frac{\partial^3 w_2}{\partial x^3}+P_{x1}\frac{\partial w_2}{\partial x} &= 0 && \text{at } x=L/2\\
-EI_{cz}\frac{\partial^3 w_2}{\partial x^3}+P_{x2}\frac{\partial w_2}{\partial x} &= 0 && \text{at } x=-L/2
\end{aligned}
\tag{105}
$$

and

$$
\begin{aligned}
EI_{cy}\frac{\partial^2 w_3}{\partial x^2} &= 0 && \text{at } x=-L/2, L/2\\
-EI_{cy}\frac{\partial^3 w_3}{\partial x^3}+P_{x1}\frac{\partial w_3}{\partial x} &= 0 && \text{at } x=L/2\\
-EI_{cy}\frac{\partial^3 w_3}{\partial x^3}+P_{x2}\frac{\partial w_3}{\partial x} &= 0 && \text{at } x=-L/2
\end{aligned}
\tag{106}
$$

It is not difficult to show that

$$
\begin{aligned}
g_3(x) &= -\frac{1}{6EI_{cy}}(x-L/2)^3\,h(x-L/2)\\
g_4(x) &= -\frac{1}{6EI_{cy}}(x+L/2)^3\,[1-h(x+L/2)]\\
g_5(x) &= -\frac{1}{6EI_{cz}}(x-L/2)^3\,h(x-L/2)\\
g_6(x) &= -\frac{1}{6EI_{cz}}(x+L/2)^3\,[1-h(x+L/2)]
\end{aligned}
\tag{107}
$$

satisfy the necessary requirements so that Eqs. (104) become

$$u_y(x,t) = w_2(x,t) - \frac{P_{y1}}{6EI_{cy}}(x-L/2)^3 h(x-L/2) - \frac{P_{y2}}{6EI_{cy}}(x+L/2)^3 [1-h(x+L/2)]$$

$$u_z(x,t) = w_3(x,t) - \frac{P_{z1}}{6EI_{cz}}(x-L/2)^3 h(x-L/2) - \frac{P_{z2}}{6EI_{cz}}(x+L/2)^3 [1-h(x+L/2)] \tag{108}$$

These transformations, Eqs. (103) and (108), will be used subsequently to solve the boundary value problem associated with the elastic displacements of the missile.

One more mathematical preliminary is in order and this deals with the solution of the boundary value problems. The solution is possible by means of modal analysis, provided the mass m is constant. This, of course, is not the case but let us assume for the moment that it is. The modal analysis amounts to solving the eigenvalue problem associated with the constant mass system, obtain the so-called normal modes, and express the system response as a superposition of the normal modes multiplied by corresponding generalized coordinates; such a solution is referred to as normal-mode vibration. Because the actual boundary-value problem possesses time-dependent coefficients, however, no normal-mode vibration is possible. Nevertheless, by virtue of the uniform-burning assumption, it turns out that a procedure based on the normal-mode approach can be used here to obtain sets of ordinary differential equations which are far simpler to solve than partial differential equations. But, because the normal modes imply a physical behavior which the actual system does not possess, we shall regard the

solution as a superposition of eigenfunctions associated with the constant-mass system, rather than superposition of normal modes. To this end we will assume that

$$w_1(x,t) = \sum_{r=1}^{\infty} \mu_r(x)\, q_r(t)$$
$$w_2(x,t) = \sum_{r=1}^{\infty} \nu_r(x)\, \eta_r(t) \qquad (109)$$
$$w_3(x,t) = \sum_{r=1}^{\infty} \nu_r(x)\, \kappa_r(t)$$

where q_r, η_r, κ_r are generalized coordinates and μ_r and ν_r are certain functions representing the normal modes. To obtain μ_r we consider the eigenvalue problem consisting of the differential equation

$$EA_c\mu'' + \Omega^2 m_o \mu = 0 \qquad (110)$$

over the domain $-L/2 < x < L/2$ and the boundary conditions

$$\mu'(L/2) = \mu'(-L/2) = 0 \qquad (111)$$

where primes denote differentiation with respect to x. The eigenvalue problem, Eqs. (110) and (111), corresponds to the axial vibration of a uniform, constant mass bar with both ends unconstrained. The solution of the problem can be shown to consist of the denumberably infinite set of eigenfunctions (see, for example, Reference 17, pp. 151-154)

$$\mu_r = \sqrt{2/m_0 L}\cos r\pi(x/L - 1/2) \qquad r = 1,2,3\ \text{---} \qquad (112)$$

and the eigenvalues

$$\Omega_r = r\pi\sqrt{EA_c/m_0 L^2} \qquad (113)$$

The eigenfunctions are orthogonal to one another and, in addition, are normalized so as to satisfy the relation

$$\int_{-L/2}^{L/2} m_0\,\mu_r(x)\mu_s(x)\,dx = \delta_{rs} \qquad r,s = 1,2,3,\text{---} \qquad (114)$$

where δ_{rs} is the Kronecker delta. The eigenfunction corresponding to $r = 0$ represents the rigid body mode $\mu_0 = \sqrt{1/m_0 L}$ and the associated eigenvalue is zero, $\Omega_0 = 0$, as is to be expected for a semidefinite system. It is easy to see also that μ_0 is orthogonal to the eigenfunctions μ_s ($s = 1,2,3,\text{---}$).

Similarily to obtain ν_r we consider the eigenvalue problem for transverse vibration of a uniform beam comprising the differential equation

$$EI_c\,\nu'''' = \Lambda^2 m_0\,\nu \qquad (115)$$

and the boundary conditions

$$z'' = z''' = 0 \quad \text{at} \quad x = -L/2, L/2. \tag{116}$$

The solution to this problem (also given in Reference 17, Sections 5-10 and 10-5) consist of the denumberably infinite set of eigenfunctions. They can be shown to have the expressions

$$z_r = \begin{cases} \dfrac{1}{\sqrt{m_o L}} \left(\dfrac{\cos \beta_r x}{\cos \beta_r L/2} + \dfrac{\cosh \beta_r x}{\cosh \beta_r L/2} \right) & r = 1,3,5,--- \\ \dfrac{1}{\sqrt{m_o L}} \left(\dfrac{\sin \beta_r x}{\sin \beta_r L/2} + \dfrac{\sinh \beta_r x}{\sinh \beta_r L/2} \right) & r = 2,4,6,--- \end{cases} \tag{117}$$

where the eigenvalues are found by solving the equation $\cos \beta_r L \cdot \cosh \beta_r L = 1$, or equivalently

$$\begin{aligned} \tan \frac{\beta_r L}{2} + \tanh \frac{\beta_r L}{2} = 0 \qquad & r = 1,3,5,--- \\ \tan \frac{\beta_r L}{2} - \tanh \frac{\beta_r L}{2} = 0 \qquad & r = 2,4,6,--- \end{aligned} \tag{118}$$

in which

$$\beta_r^4 = \frac{\Lambda_r^2 m_o}{E I_c} \tag{119}$$

The eigenfunctions are orthogonal to one another and are also normalized so as to satisfy

$$\int_{-L/2}^{L/2} m_0 \, v_r(x) \, v_s(x) \, dx = \delta_{rs} \qquad r,s = 1,2,3,\text{---} \tag{120}$$

It may be noted that two rigid body modes exist and it is not difficult to show that they are orthogonal to the remaining eigenfunctions.

a) Axial vibration of a rocket.

Using the above assumptions and the aerodynamic forces of the previous section we may write Eq. (62) as

$$EA_c \frac{\partial^2 u_x}{\partial x^2} - q_\infty S_r c_x(M,L/2)\delta(x-L/2) - q_\infty S_r c_x(M,-L/2)\delta(x+L/2) + m\underline{g}\cdot\underline{i} =$$
$$m[\dot{U} + \ddot{u}_x + \omega_y(W+2\dot{u}_z) - \omega_z(V+2\dot{u}_y) + (\dot{\omega}_y + \omega_x\omega_z)u_z - (\dot{\omega}_z - \omega_x\omega_y)u_y$$
$$-(x+u_x)(\omega_y^2 + \omega_z^2)\} \tag{121}$$

while the boundary conditions, Eqs. (63), retain their form

$$EA_c \frac{\partial u_x}{\partial x} = P_{x1} \qquad \text{at } x = L/2$$
$$EA_c \frac{\partial u_x}{\partial x} = P_{x2} \qquad \text{at } x = -L/2 \tag{122}$$

Using the transformations defined by Eqs. (103) and (108), Eq. (121) becomes

$$-EA_c\frac{\partial^2 w_1}{\partial x}+m\ddot{w}_1-m(\omega_y^2+\omega_z^2)w_1=P_{x1}\delta(x-L/2)-P_{x2}\delta(x+L/2)$$
$$-q_\infty S_r c_x(M,L/2)\delta(x-L/2)-q_\infty S_r c_x(M,-L/2)\delta(x+L/2)$$
$$+m\underline{g}\cdot\underline{i}-m\left[\dot{U}+\omega_y W-\omega_z V\right]-m\left[2\omega_y\dot{w}_3\right.$$
$$\left.-2\omega_z\dot{w}_2+(\dot{\omega}_y+\omega_x\omega_z)w_3-(\dot{\omega}_z-\omega_x\omega_y)w_2-x(\omega_y^2+\omega_z^2)\right] \qquad (123)$$

while the boundary conditions, Eqs. (122) become

$$EA_c\frac{\partial w_1}{\partial x}=0 \quad \text{at} \quad x=-L/2,\ L/2 \qquad (124)$$

We may now use the expressions given by (109) in Eq. (123) with the result

$$\sum_r -EA_c\mu_r''q_{pr}+m\mu_r\ddot{q}_{pr}-m(\omega_y^2+\omega_z^2)\mu_r q_{pr}=P_{x1}\delta(x-L/2)-P_{x2}\delta(x+L/2)$$
$$-q_\infty S_r c_x(M,L/2)\delta(x-L/2)-q_\infty S_r c_x(M,-L/2)\delta(x+L/2)+m\underline{g}\cdot\underline{i}$$
$$-m\left[\dot{U}+\omega_y W-\omega_z V\right]-m\left\{2\omega_y\sum_r v_r\dot{\zeta}_r-2\omega_z\sum_r v_r\dot{\eta}_r\right.$$
$$+(\dot{\omega}_y+\omega_x\omega_z)\sum_r v_r\zeta_r-(\dot{\omega}_z-\omega_x\omega_y)\sum_r v_r\eta_r$$
$$\left.-x(\omega_y^2+\omega_z^2)\right\} \qquad (125)$$

Use Eq. (110), multiply Eq. (125) by μ_s, integrate along the missile, use the orthogonality conditions and obtain

$$\frac{m}{m_o}\left[\ddot{q}_{pr}-(\omega_y^2+\omega_z^2)q_{pr}\right]+\Omega_r^2 q_{pr}=P_{x1}\mu_r(L/2)-P_{x2}\mu_r(-L/2)$$
$$-q_\infty S_r\left[c_x(M,L/2)\mu_r(L/2)+c_x(M,-L/2)\mu_r(-L/2)\right]-\frac{m}{m_o}\sum_s\left[2\omega_y\dot{\zeta}_s-2\omega_z\dot{\eta}_s+(\dot{\omega}_y\right.$$
$$\left.-\omega_x\omega_z)\zeta_s-(\dot{\omega}_z-\omega_x\omega_y)\eta_s\right]\int_{-L/2}^{L/2}m_o v_s(x)\mu_r(x)dx+\frac{m}{m_o}(\omega_y^2+\omega_z^2)\int_{-L/2}^{L/2}m_o x\mu_r(x)\,dx$$
$$r=1,2,3,\cdots \qquad (126)$$

which are subject to the initial conditions

$$q_r(0)=\int_{-L/2}^{L/2}m_o w_1(x,0)\mu_r(x)\,dx, \qquad \dot{q}_{pr}(0)=\int_{-L/2}^{L/2}m_o\frac{\partial w_1(x,0)}{\partial t}\mu_r(x)dx$$
$$\eta_r(0)=\int_{-L/2}^{L/2}m_o w_2(x,0)\,v_r(x)\,dx, \qquad \dot{\eta}_r(0)=\int_{-L/2}^{L/2}m_o\frac{\partial w_2(x,0)}{\partial t}v_r(x)dx$$
$$\zeta_r(0)=\int_{-L/2}^{L/2}m_o w_3(x,0)\,v_r(x)\,dx, \qquad \dot{\zeta}_r(0)=\int_{-L/2}^{L/2}m_o\frac{\partial w_3(x,0)}{\partial t}v_r(x)\,dx$$
(127)

The functions P_{x1} and P_{x2} are forces due to internal fluid flow and thrust. Their form is similar to that of Reference 13 which is

$$P_{x1}=pA_f(L/2), \qquad P_{x2}=pA_f(L/2)-(p_a-p_e)A_e-v(x_e,t)M_o\beta \qquad (128)$$

in which M_0 is the total mass $M_0=\int_{-L/2}^{L/2}m_0\,dx$ and β is the burning rate. Eqs. (128) are obtained by considering Eq. (83) and the thrust force.

b. Transverse vibration of a rocket.

Consider the differential equation for vibration in the xy-plane. Assuming constant stiffness, $I_{cy} = I_{cz} = I_c$, neglecting Coriolis forces (see Reference 16, page 14), and using Eq.(76), we write Eq.(62) as

$$-EI_c \frac{\partial^4 u_y}{\partial x^4} + \frac{\partial}{\partial x}\left(P \frac{\partial u_y}{\partial x}\right) + mg\underline{i}\cdot\underline{j} - m_v\left[U + \dot{u}_x + \omega_y u_z - \omega_z u_y\right]\left[\frac{\partial \dot{u}_y}{\partial x} + \omega_z + \omega_z \frac{\partial u_x}{\partial x} - \omega_x \frac{\partial u_z}{\partial x}\right] - \rho S\left[V + \dot{u}_y + \omega_z (x + u_x) - \omega_x u_z\right] = m^*\left[\dot{V} + \ddot{u}_y + \omega_z (U + 2\dot{u}_x) - \omega_x (W + 2\dot{u}_z) + \omega_z (x + u_x) + \omega_x \omega_y (x + u_x) - \omega_y \omega_z u_z - (\omega_x^2 + \omega_z^2) u_y\right] \tag{129}$$

in which we introduced the notation

$$m^* = m + m_v = m_c + m_f + m_v . \tag{130}$$

Introducing the transformations, Eqs.(103) and (108) into Eq. (129), using expressions (109) as well as Eqs. (115), and (61), multiplying the resulting expression by ν_s, integrating along the missile, and using the orthogonality conditions, we obtain

$$\frac{m^*}{m_o}\ddot{\eta}_r + \frac{\dot{\rho} S}{m_o}\dot{\eta}_r + \left[\Lambda_r^2 - \frac{m_v}{m_o}\omega_z^2 - \frac{m^*}{m_o}(\omega_x^2 + \omega_z^2)\right]\eta_r - \sum_s \sum_t \eta_s q_t E A_c \int_{-L/2}^{L/2} \mu_t' \nu_s' \nu_r' dx + \frac{m_v}{m_o} U \omega_z \sum_s q_s \int_{-L/2}^{L/2} m_o \mu_s' \nu_r dx + \frac{m_v}{m_o} \sum_s \sum_t \dot{q}_s (\dot{\eta}_t - \omega_x \eta_t) \int_{-L/2}^{L/2} m_o \mu_s \nu_t' \nu_r dx + \sum_s \left[\frac{m^*}{m_o}(\dot{\omega}_z q_s + \omega_x \omega_y q_s + 2\omega_z \dot{q}_s) + \frac{m_v}{m_o}\omega_z \dot{q}_s + \frac{\dot{\rho} S}{m_o}\omega_z q_s\right] \int_{-L/2}^{L/2} m_o \mu_s \nu_r dx$$

$$+\frac{m_v}{m_0}\sum_s\sum_t(\omega_y\kappa_s-\omega_z\eta_s)(\dot\eta_t-\omega_x\kappa_t)\int_{-L/2}^{L/2}m_0\,v_s\,v_t'\,v_r\,dx$$

$$+\frac{m_v}{m_0}\omega_z\sum_s\sum_t\dot q_s q_t\int_{-L/2}^{L/2}m_0\mu_s\mu_t'v_r\,dx+\frac{m_v}{m_0}\sum_s\sum_t(\omega_y\kappa_s-\omega_z\eta_s)\omega_z q_t\int_{-L/2}^{L/2}m_0\,v_s\mu_t'v_r\,dx$$

$$-2\frac{m^*}{m_0}\omega_x\dot\kappa_r-\left[\frac{m^*}{m_0}\omega_y\omega_z-\frac{m_v}{m_0}\omega_y\omega_z+\frac{\dot S}{m_0}\omega_x\right]\kappa_r-P_{y1}\,v_r(L/2)$$

$$-P_{y2}\,v_r(-L/2)=0\qquad r=1,2,3,\cdots\qquad(131)$$

which are subject to the initial conditions, Eqs. (127).

The transverse boundary forces P_{y1} and P_{y2} arise from aerodynamical effects and can be obtained from Eq. (76). They are simply the definite integrals of the last term in Eq. (76) with proper integration limits. The forward portion of the missile is assumed to consist of a cone starting at x = L/2 and ending at $x = x_n$ at which point $r(x_n) = r^*$. Hence

$$S(x)=\pi r^2(x)=\pi\left[(L/2-x)\frac{r^*}{(L/2-x_n)}\right]^2,\quad L/2\le x\le x_n\qquad(132)$$

from which

$$\frac{dS}{dx}=-\frac{2\pi r^{*2}}{(L/2-x_n)^2}(L/2-x)\qquad(133)$$

so that

$$P_{y1}=+\int_{L/2-x_n}^{L/2}\frac{2\pi r^{*2}}{(L/2-x_n)^2}\rho\left[U+\dot u_x+\omega_y u_z-\omega_z u_y\right]\left[V+\dot u_y+\omega_z(x+u_x)-\omega_x u_z\right](L/2-x)\,dx$$

$$\cong+\pi\rho r^{*2}\left\{U+\sum_s\dot q_s\mu_s(L/2)+\sum_r\left[\omega_y\kappa_r-\omega_z\eta_s\right]v_r(L/2)\right\}\left\{V+\frac{\omega_z L}{2}\right.$$

$$\left.+\sum_s\omega_z q_s\mu_s(L/2)+\sum_r\left[\dot\eta_r-\omega_x\kappa_r\right]v_r(L/2)\right\}\qquad(134)$$

The aft force, P_{y2}, is found in a similar manner. Because the equivalent area for the finned region is

$$S = \pi s^2 \left(1 - \frac{r^2}{s^2} + \frac{r^4}{s^4} \right), \qquad x_r \le x \le -L/2 \tag{135}$$

and since $r = r^*$ is constant, whereas s is the variable, we obtain

$$\frac{dS}{dx} = 2\pi \left(s - \frac{r^4}{s^3} \right) \frac{ds}{dx} . \tag{136}$$

Let s increase linearly from $s = r^*$ to $s = s^*$, where s^* is the distance from the center line of the missile to the tip of the fin at its aft end. Then

$$s = \left(\frac{s^* - r^*}{x_r - L/2} \right) x + \frac{x_r s^* - r^* L/2}{x_r - L/2} \tag{137}$$

in which x_r is the position from the origin along the missile axis to the point where the fin begins. Hence

$$\frac{ds}{dx} = \frac{s^* - r^*}{x_r - L/2} \tag{138}$$

and Eq. (136) becomes

$$\frac{dS}{dx} = 2\pi \left\{ \left(\frac{s^* - r^*}{x_r - L/2} \right) x + \frac{x_r s^* - r^* L/2}{x_r - L/2} - r^{*4} \left[\left(\frac{s^* - r^*}{x_r - L/2} \right) x + \frac{x_r s^* - r^* L/2}{x_r - L/2} \right]^{-3} \right\} \left(\frac{s^* - r^*}{x_r - L/2} \right) \tag{139}$$

Using Eq. (76), we write

$$\begin{aligned} P_{y2} = -\rho \int_{-L/2}^{-L/2 + x_r} & 2\pi \left(\frac{s^* - r^*}{x_r - L/2} \right) (U + \dot{u}_x + \omega_y u_z - \omega_z u_y) \left[V + \dot{u}_y + \omega_z (x + u_x) - \omega_x u_z \right] \left\{ \left(\frac{s^* - r^*}{x_r - L/2} \right) x \right. \\ & \left. + \frac{x_r s^* - r^* L/2}{x_r - L/2} - r^{*4} \left[\left(\frac{s^* - r^*}{x_r - L/2} \right) x + \frac{x_r s^* - r^* L/2}{x_r - L/2} \right]^{-3} \right\} dx \\ \cong -\rho \pi & \left\{ (s^* - r^*)(2s^* - r^*) + r^{*4} \left[\frac{1}{(2s^* - r^*)^2} - \frac{1}{s^{*2}} \right] \right\} \left\{ U + \sum_s \dot{q}_{ps} \mu_s (-L/2) \right. \\ & \left. + \sum_r (\omega_y \zeta_r - \omega_z \eta_r) \nu_r (-L/2) \right\} \left\{ V + \frac{\omega_z L}{2} + \sum_s \omega_z q_{ps} \mu_s (-L/2) + \sum_r (\dot{\eta}_r - \omega_x \zeta_r) \nu_r (-L/2) \right\} \end{aligned} \tag{140}$$

For vibration in the xz-plane we use the same technique as above and obtain the equation for κ_r in the form

$$\frac{m^*}{m_o}\ddot{\kappa}_r + \frac{\dot{\rho} S}{m_o}\dot{\kappa}_r + \left[\Lambda_r^2 - \frac{m_v}{m_o}\omega_y^2 - \frac{m^*}{m_o}(\omega_x^2+\omega_y^2)\right]\kappa_r$$
$$- \sum_s\sum_t \kappa_s q_{pt} E A_c \int_{-L/2}^{L/2} \mu_t' z_s' z_r' dx - \frac{m_v}{m_o} U\omega_y \sum_s q_{ps}\int_{-L/2}^{L/2} m_o \mu_s' z_r dx$$
$$+ \frac{m_v}{m_o}\sum_s\sum_t \dot{q}_{ps}(\dot{\kappa}_t + \omega_x \eta_t)\int_{-L/2}^{L/2} m_o \mu_s z_t' z_r dx + \sum_s\left[\frac{m^*}{m_o}(2\omega_y \dot{q}_{ps}\right.$$
$$\left. + \dot{\omega}_y q_{ps} - \omega_x\omega_z q_{ps}) - \frac{m_v}{m_o}\omega_y \dot{q}_{ps} - \frac{\dot{\rho} S}{m_o}\omega_y q_{ps}\right]\int_{-L/2}^{L/2} m_o \mu_s z_r dx$$
$$+ \frac{m_v}{m_o}\sum_s\sum_t(\omega_y\kappa_s - \omega_z\eta_s)(\dot{\kappa}_t + \omega_x\eta_t)\int_{-L/2}^{L/2} m_o z_s z_t' z_r dx - \frac{m_v}{m_o}\omega_y\sum_s\sum_t \dot{q}_{ps} q_{pt}\int_{-L/2}^{L/2} m_o\mu_s\mu_t' z_r dx$$
$$- \frac{m_v}{m_o}\sum_s\sum_t(\omega_y^2\kappa_s - \omega_y\omega_z\eta_s) q_{pt}\int_{-L/2}^{L/2} m_o z_s \mu_t' z_r dx + 2\frac{m^*}{m_o}\omega_x\dot{\eta}_r - \left[\frac{m^*}{m_o}\omega_y\omega_z\right.$$
$$\left. - \frac{m_v}{m_o}\omega_y\omega_z - \frac{\dot{\rho} S}{m_o}\omega_x\right]\eta_r - P_{z1} z_r(L/2) - P_{z2} z_r(-L/2) = 0 \quad r = 1,2,3,\cdots \tag{141}$$

where the initial conditions, Eqs. (127), apply and

$$P_{z1} \cong \pi\rho r^{*2}\left[U + \sum_s \dot{q}_{ps}\mu_s(L/2) + \sum_r(\omega_y\kappa_r - \omega_z\eta_r) z_r(L/2)\right]\left\{W - \frac{\omega_y L}{2}\right.$$
$$\left. - \omega_y\sum_s q_{ps}\mu_s(L/2) + \sum_r(\dot{\kappa}_r + \omega_x\eta_r) z_r(L/2)\right\} \tag{142}$$

$$P_{z2} \cong -\rho\pi\left\{(s^*-r^*)(2s^*-r^*) + r^{*4}\left[\frac{1}{(2s^*-r^*)^2} - \frac{1}{s^{*2}}\right]\right\}\left\{U + \sum_s \dot{q}_{ps}\mu_s(-L/2) + \sum_r(\omega_y\kappa_r\right.$$
$$\left. - \omega_z\eta_r) z_r(-L/2)\right\}\left\{W - \frac{\omega_y L}{2} - \omega_y\sum_s q_{ps}\mu_s(-L/2) + \sum_r(\dot{\kappa}_r + \omega_x\eta_r) z_r(-L/2)\right\} \tag{143}$$

This completes the development of the differential equations for the elastic displacements.

8. Results

The equations of motion were solved numerically on an IBM 360/65 computer. The reference configuration used was as follows:

$E = 30\times10^{6}$ psi , $p(L/2) = 1000$ psi , $L = 100$ in.

$A_c = 7.53$ in^2 , $m_0 g = 4.25$ lbs/in , $\dot{m}_c g = 0.5$ lbs/in/sec

$I_c = 93$ in^4 , $v(x_e,t) = 1000$ ft/sec , $\theta(0) = 90$ deg

$\omega_x = 0$ rad/sec , $\omega_y(0) = \omega_z(0) = 0$ rad/sec, $\psi(0) = \phi(0) = 0$ rad,

$U(0) = V(0) = W(0) = 0$ ft/sec , $X(0) = Y(0) = Z(0) = 0$ ft.

$u_x(x,0) = u_z(x,0) = 0$ ft.

$$u_y(x,0) = 10^{-6}\left(\cos\frac{\pi x}{L} - \frac{2}{\pi}\right) + 0.5\times10^{-6}\left(\sin\frac{2\pi x}{L} - \frac{6x}{\pi L}\right), \text{ ft.}$$

The results obtained by using this configuration are shown in Figures 7 and 8. In Figure 7 the rigid body translation is shown as a function of time and in Figure 8 the elastic motion of the missile is shown during a time interval approaching $t = 0.001$ sec. It may be noticed from Figure 8 that the axial deflection has the appearance of waves traveling along the length of the missile.

A comparison with results obtained for a different configuration at a given time are shown in Figure 9. Figure 9a presents results obtained by using the angular velocity ω_x as a parameter. There is no change in axial deflection due to a change in ω_x. Figure 9b shows the effect of varying the interval pressure, $p(L/2)$. Figure 9c shows the results for different values of the exit velocity $v(x_e,t)$ whereas Figure 9d shows the effect of the time rate of change of mass. In all above cases the variable u_x, u_y or u_z are not shown when no significant

changes occur due to changes in the parameters. Sometimes, due to the scale chosen, small changes in the variables cannot be readily plotted, so that several cases are represented by one curve.

The effect of aerodynamic forces was also investigated. For the burning times considered their influence is primarily felt in the transverse deflection whose magnitude is slightly smaller with aerodynamic forces than without. The effect could be dramatized by using a greatly exaggerated scale, but this would serve no purpose as the shape of the curves is the same. The aerodynamic forces begin to be felt at larger times than those investigated.

APPENDIX - Calculation of the Engine Thrust.

The purpose of a nozzle is to convert the enthalpy of the flowing gas into kinetic energy in an efficient manner while, at the same time, restricting the escape of the gas to a rate suitable for the propellant reaction inside the combustion chamber. We shall assume that the nozzle under consideration is convergent-divergent, designed to allow an isentropic expansion to an ambient pressure less than critical. In the convergent portion of the nozzle, before the throat, the flow is subsonic, reaching sonic level at the throat section, at which point the flow properties are referred to as critical, and becoming supersonic in the divergent portion after the throat. Although losses may occur in the nozzle, they are assumed to be small so that the analysis is based on the equations for one-dimensional isentropic steady flow of a compressible perfect gas.

Let us consider the one-dimensional isentropic flow of Figure A1 and assume that the stagnation conditions, denoted by the subscript 0, are known. Under these circumstances, we may write the equations governing the flow as follows:

First the flow must satisfy the first law thermodynamics. Considering the control volume shown in Figure A1, and denoting the enthalpy per unit mass by h, this law can be stated

$$h_0 = h_1 + \frac{1}{2} v_1^2 = h_2 + \frac{1}{2} v_2^2 \qquad \text{(A1)}$$

Assuming that there is no friction or heat transfer present, the second law of thermodynamics becomes simply

$$s = s_0 = \text{const} \tag{A2}$$

or the entropy s is constant, as implied by the name of the type of flow under consideration.

The flow must also satisfy the countinuity equation. Since there is no mass addition within the nozzle, we must have

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 = \text{const} \tag{A3}$$

where the flow properties at stations 1 and 2 are denoted by the corresponding subscripts.

Similarly the flow must satisfy the momentum equation. Denoting the force exerted by the nozzle wall on the gas by F_T, this equation can be written

$$F_T = p_1 A_1 - p_2 A_2 = \rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2 \tag{A4}$$

Equations (A1) through (A4) must be supplemented by the equation of state which for a perfect gas has the form

$$p = \rho R T \tag{A5}$$

in which R is the universal gas constant and T the temperature.

The above relations can be used to derive expressions for the pressure, density, etc., at any point along the nozzle. For a perfect gas the speed of sound is given by

$$c = (kRT)^{1/2} \tag{A6}$$

where

$$k = c_p / c_v \tag{A7}$$

in which c_p and c_v are the specific heats. Then the following relations can be shown to hold true.*

$$\frac{T_o}{T} = 1 + \frac{(k-1)}{2} M^2 \tag{A8}$$

$$\frac{p_o}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{k}{k-1}} \tag{A9}$$

$$\frac{\rho_o}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{\frac{1}{k-1}} \tag{A10}$$

where M = v/c is the Mach number. Moreover, the cross-sectional area A at any point is related to the cross-sectional area A_* at the throat by

$$\frac{A}{A_*} = \frac{G_*}{G} = \frac{1}{M}\left[\frac{2}{k+1}\left(1 + \frac{k-1}{2} M^2\right)\right]^{\frac{k+1}{2(k-1)}} \tag{A11}$$

---

* See Reference 15, Section 13-5.

where

$$G = \rho v \tag{A12}$$

is the mass flow per unit area at any point and

$$G_* = \left(\frac{k p_0^2}{R T_0}\right)^{1/2} \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} \tag{A13}$$

is the mass flow per unit area at the throat.

Equations (A8) through (A13) are sufficient to determine the isentropic flow in the nozzle provided the stagnation conditions are known. We are interested primarily in the flow conditions at the nozzle exit. For a given rocket design the cross-sectional areas A_e and A_* may be regarded as known. Since k is also a known quantity, we can use Eq. (A11) and obtain the Mach number M_e at the exit. Introducing this value into Eq. (A9) we can determine the exit pressure p_e, which enables us to write the expression for rocket thrust

$$F_T = p_e A_e + \rho_e A_e v_e^2 = p_e A_e (1 + k M_e^2) \tag{A14}$$

for flight in vacuum. If the rocket operates in the lower fringes of the atmosphere, then the term $p_a A_e$, where p_a is the atmospheric pressure, must be subtracted from the right side of Eq. (A14).

$$F_T = (p_e - p_a) A_e + \rho_e A_e v_e^2 \tag{A15}$$

In the above analysis, we have assumed that the stagnation conditions are known. This assumption necessitates further scrutiny. The stagnation conditions are determined by events occurring upstream of the nozzle. The flow in the combustion chamber may be regarded as a steady, adiabatic flow in a channel of uniform cross-sectional area with mass addition at constant enthalpy, and at negligible kinetic energy. The flow is not isentropic and the stagnation conditions are not constant but decreasing as the nozzle is approached. This problem is discussed in detail in Reference 11. The conclusion that can be reached is that for a Mach number less than 0.4 in the combustion chamber the drop in the stagnation pressure may not be significant. Hence, we shall assume that the stagnation pressure as well as the remaining stagnation conditions occurring at the fore end of the combustion chamber are equally applicable to the nozzle. In a more refined analysis of the gas flow this assumption may have to be revised.

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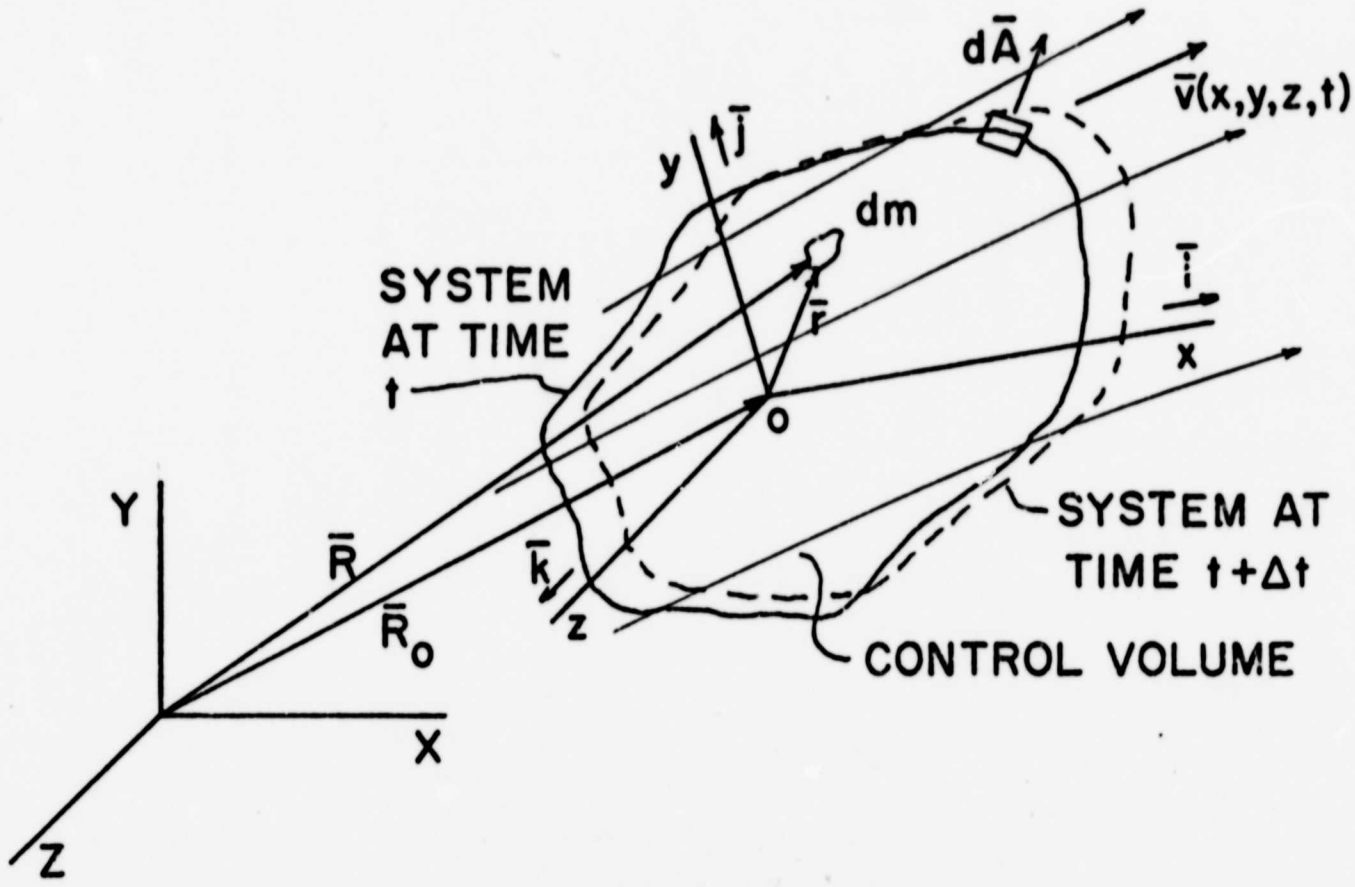


Figure I – The Control Volume

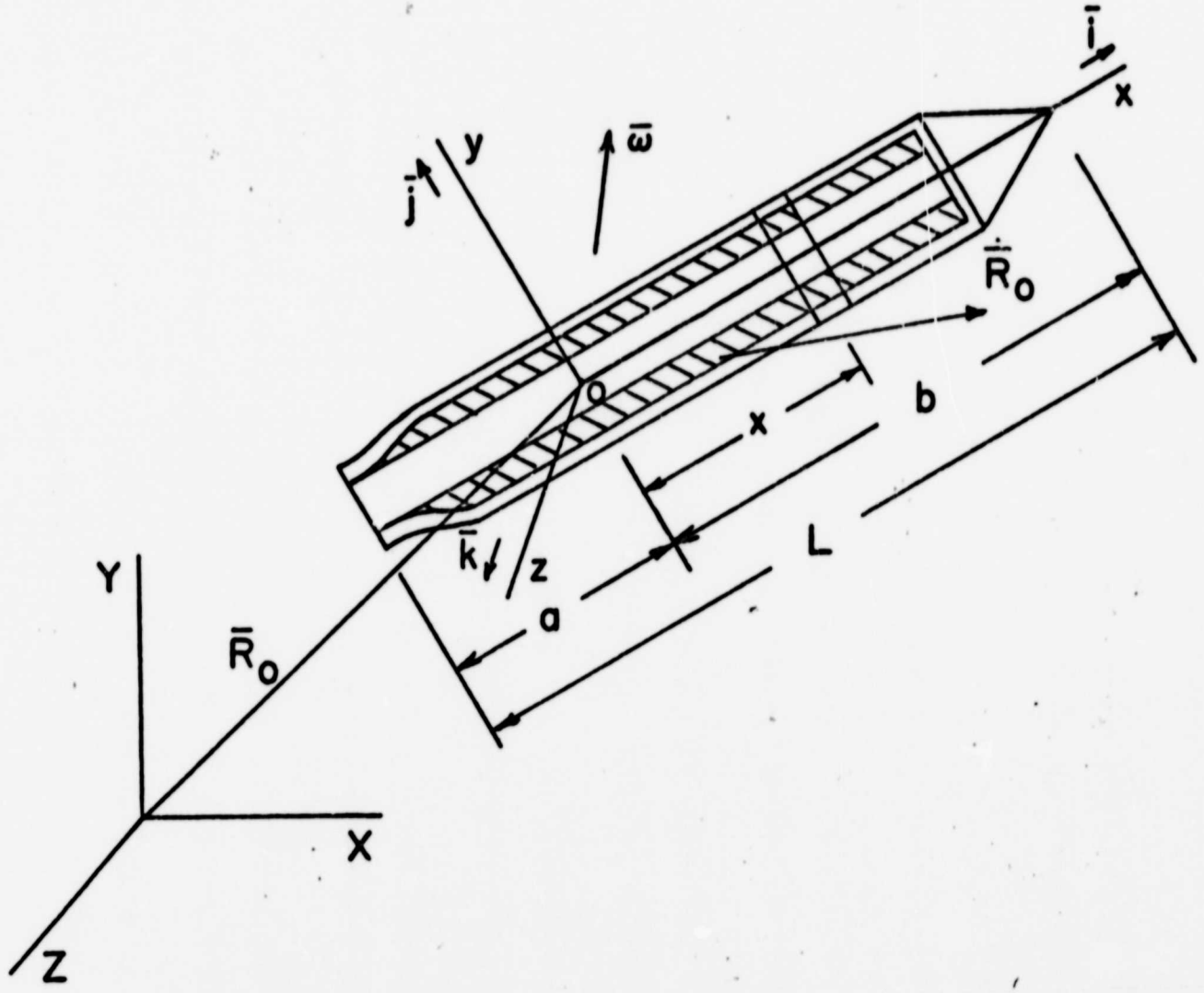


Figure 2 – Coordinate System for the Rocket in Flight.

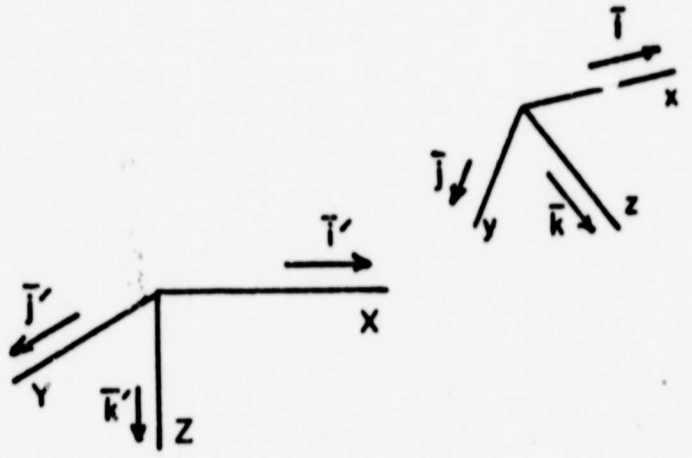


Figure 3 - Inertial and Moving Coordinate Systems

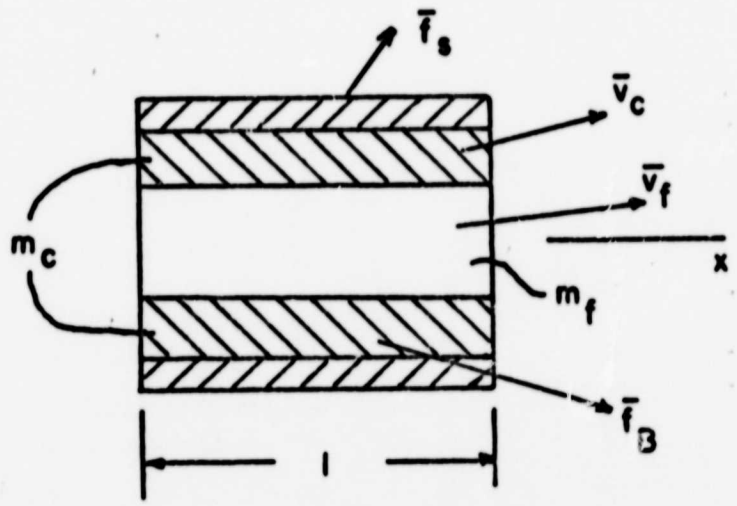


Figure 4 - The Rocket Element of Unit Length

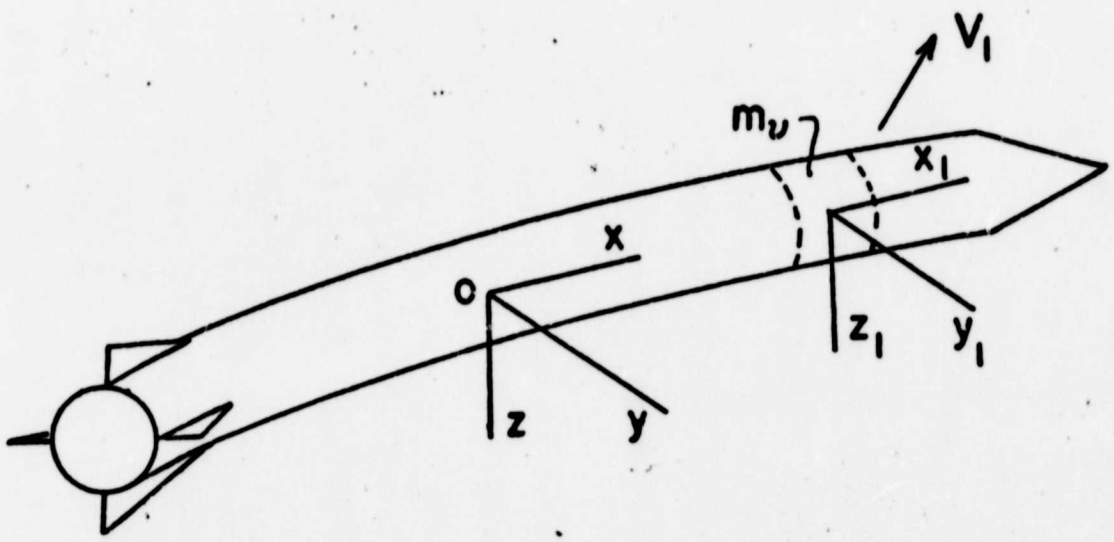


Figure 5 – Coordinate Systems for the Rocket

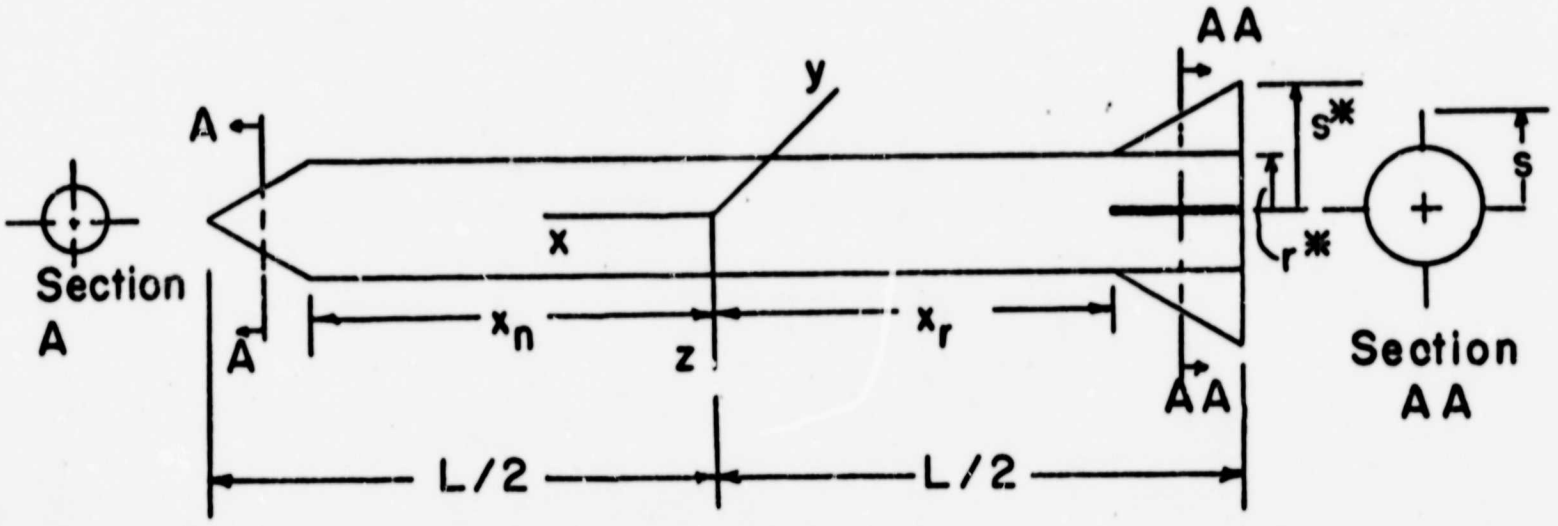


Figure 6 – Rocket Characteristics

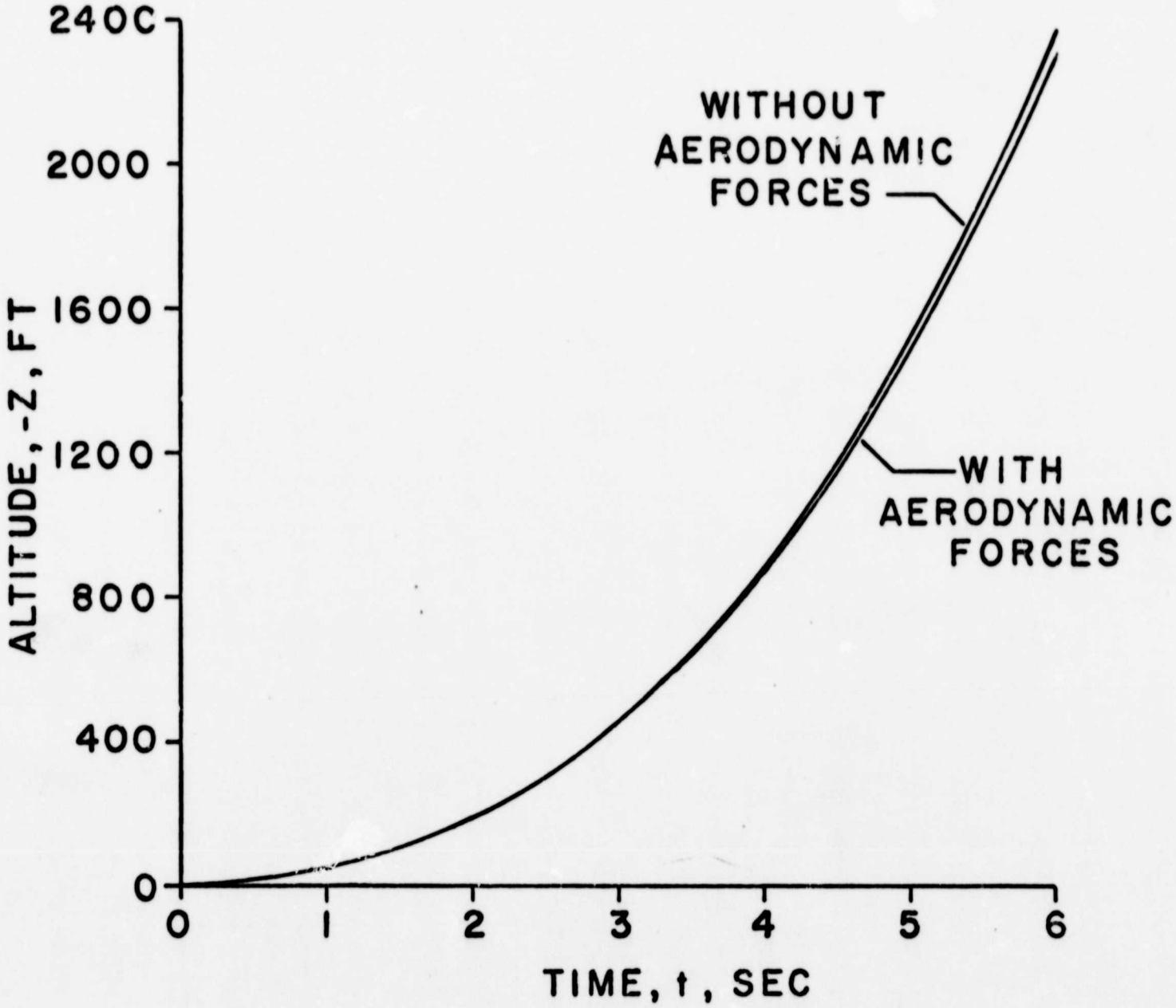


FIGURE 7 - ALTITUDE vs TIME

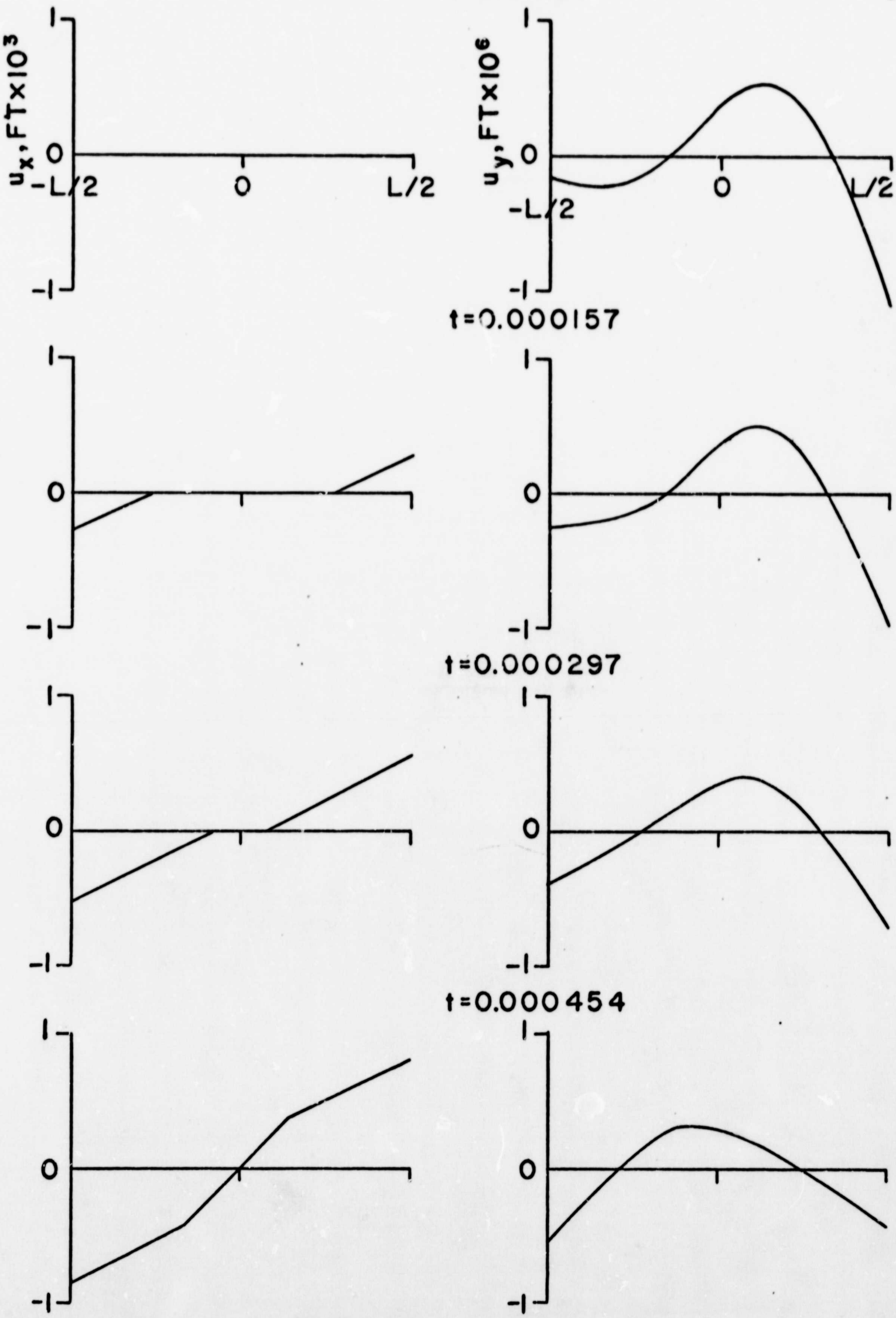


FIGURE 8 - ELASTIC DISPLACEMENTS THROUGH APPROXIMATELY t = 0.0001 SEC

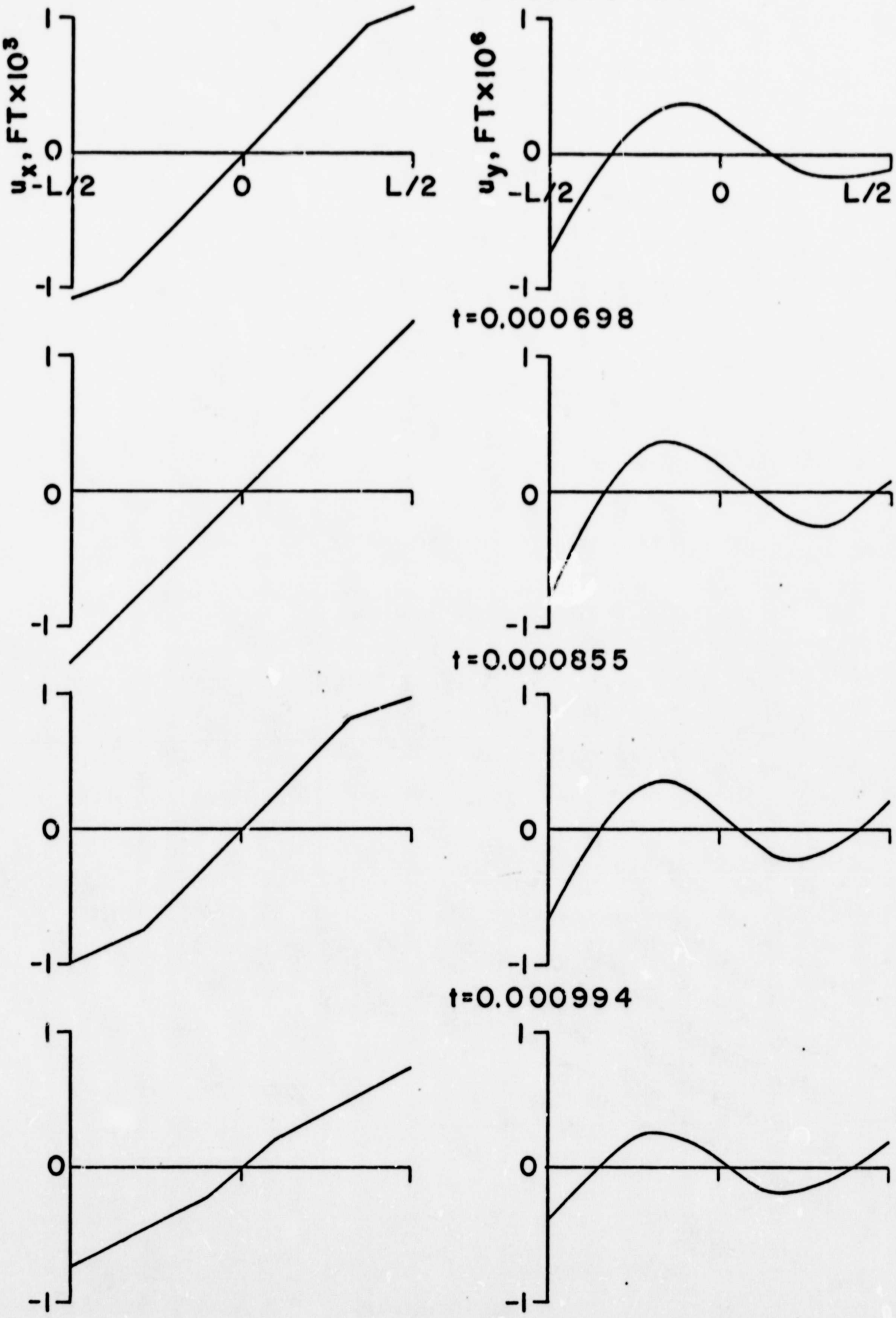
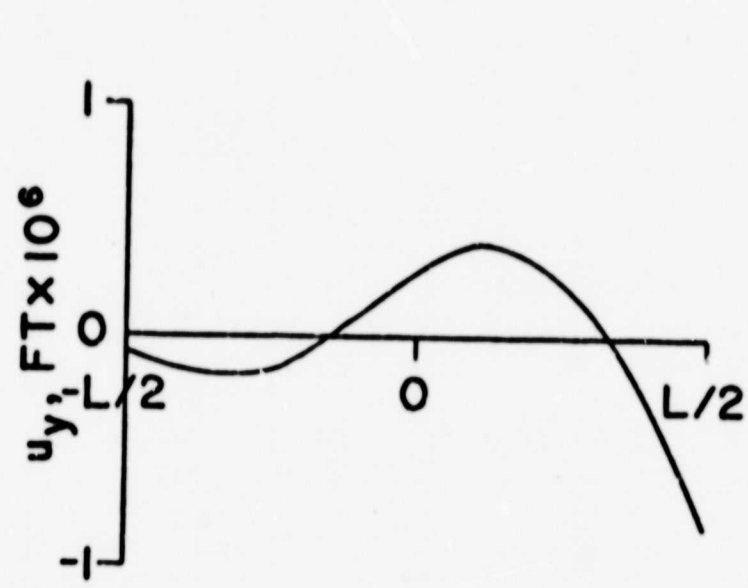
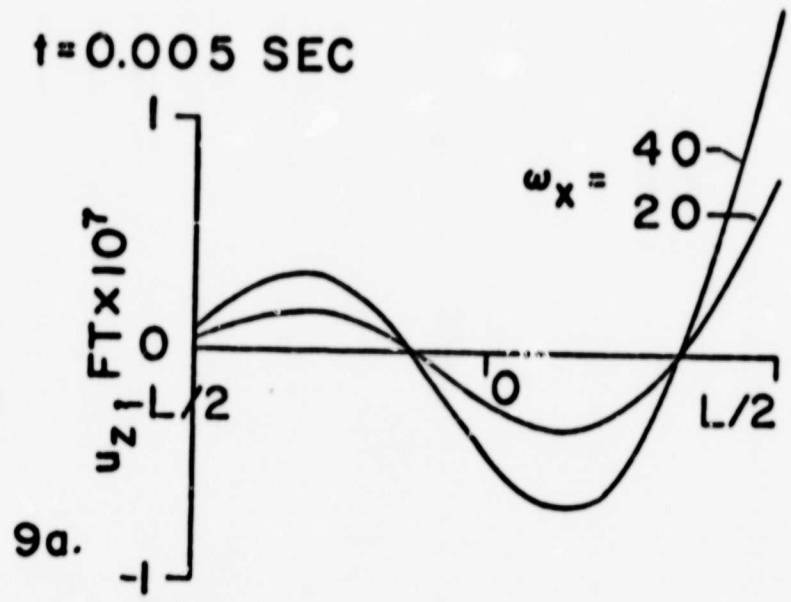
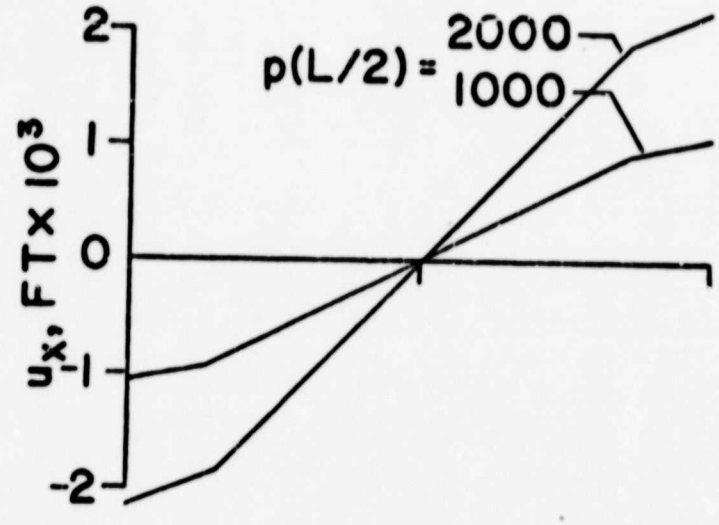


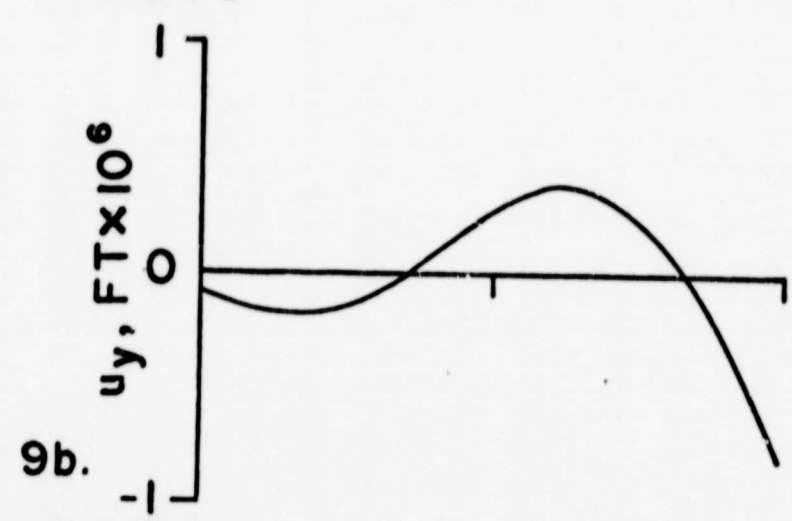
FIGURE 8 (Cont.) - ELASTIC DISPLACEMENTS THROUGH APPROXIMATELY t = 0.0001 SEC



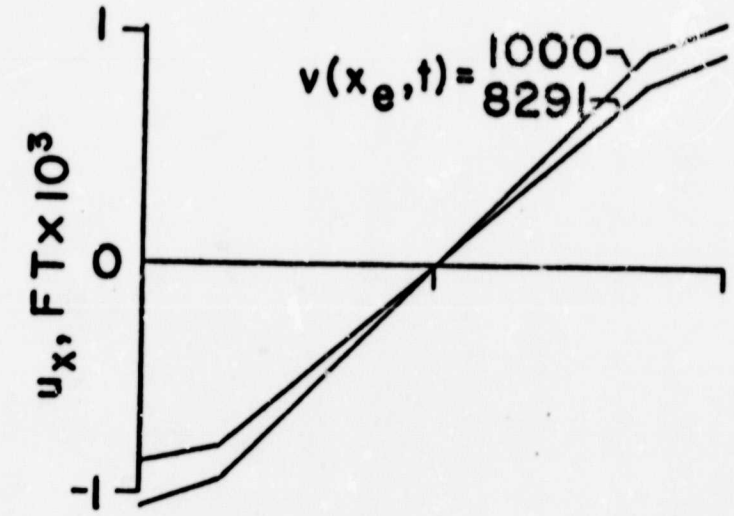


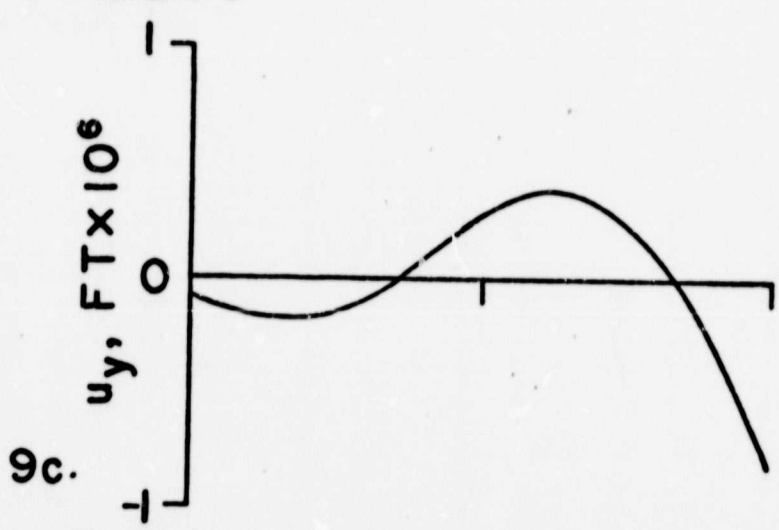
9a.



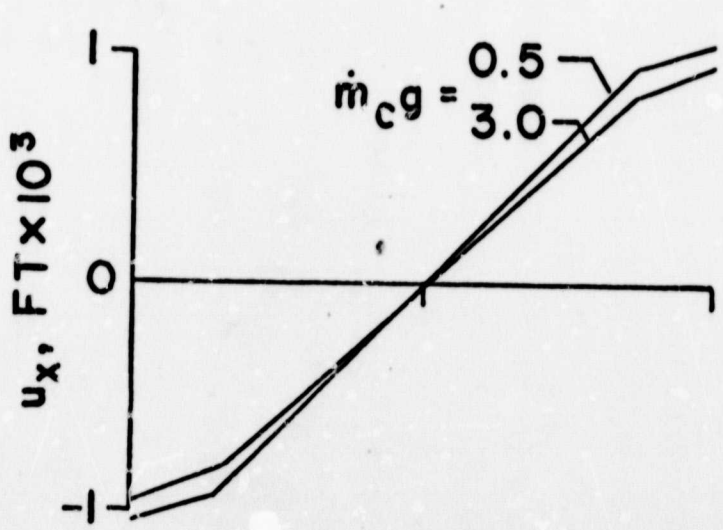


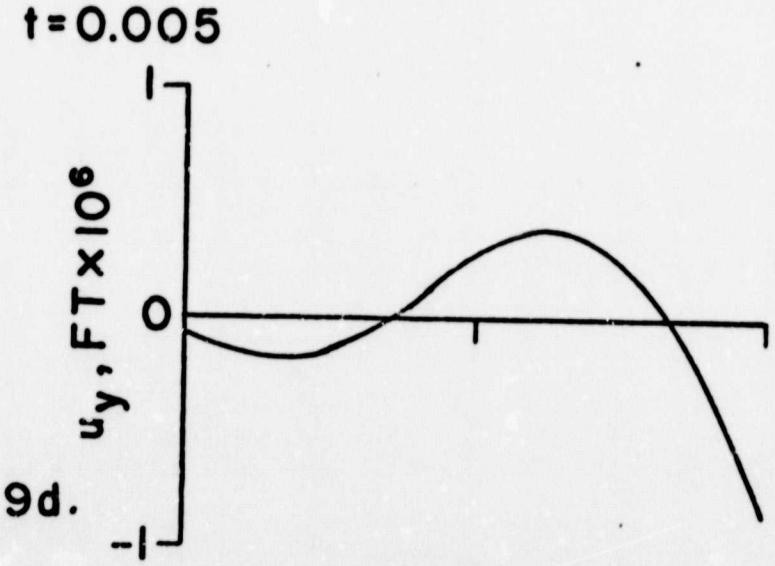
9b.





9c.





9d.

FIGURE 9 - VARIATION IN ELASTIC DISPLACEMENTS AT A FIXED TIME DUE TO PARAMETER VARIATION

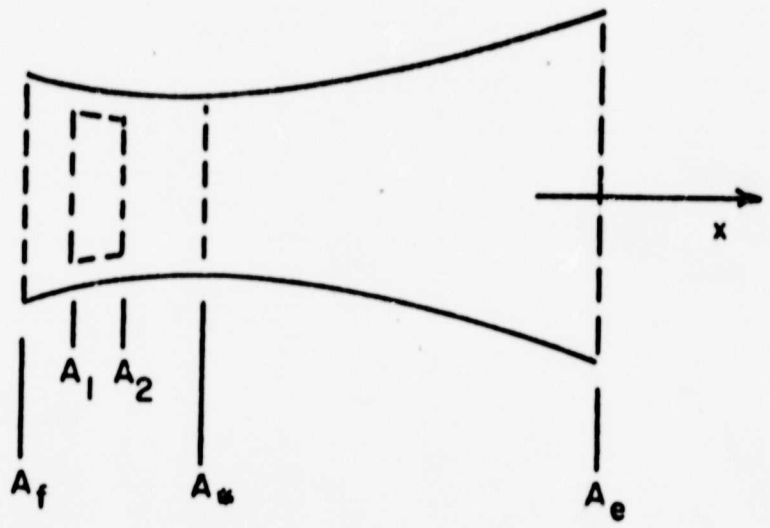


Figure A1- The Nozzle