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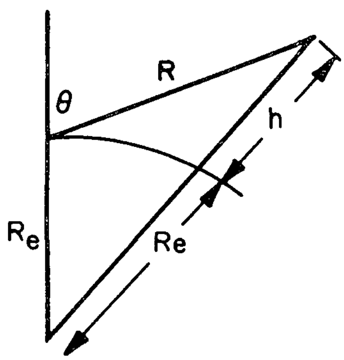
1. INTRODUCTION

The following paper is an analysis of the trade-offs and requirements for a laser cube-corner reflector array to be placed aboard the GEOS-C satellite. The objective of the study is to develop an improved array design which will give stronger returns at lower elevation angles, and reduce pulse spread in the return.

2. ORBIT

For purpose of the analysis we will assume a 500 nautical mile (927 km) circular orbit.

2.1 Range Vs Zenith Angle – Using the following figure



$$R = \sqrt{(R_e + h)^2 - R_e^2 \sin^2 \theta} - R_e \cos \theta$$

where

$h =$ 927 km (500 n. mi.)

$R_e =$ 6370 km (3440 n. mi.)

Therefore

$$R = \sqrt{(7.297 \times 10^3)^2 - (6.37 \times 20^3)^2 \sin^2 \theta} - 6.37 \times 10^3 \cos \theta$$

$$= \sqrt{53.0 \times 10^6 - 40.5 \times 10^6 \sin^2 \theta} - 6.37 \times 10^3 \cos \theta$$

$$= [\sqrt{53.0 - 40.5 \sin^2 \theta} - 6.37 \cos G] \times 10^3 \text{ km}$$

This equation is evaluated in the following table and plotted in Figure 1.

θ (Deg)	R (km)	R^4 (m.[14])	(Deg)	R (km)	R^4 (m.[24])
0	927	0.738	50	1310	2.94
10	930	0.748	60	1570	6.05
20			70	1980	15.5
30	1040	1.17	80	2620	47.8
40			90	3540	88.0

2.2 Velocity Abberration Vs. Zenith Angle – We will first calculate the orbital velocity of the satellite

$$\frac{K\, m\, m_e}{(R_e + h)^2} = \frac{m\, V_T^2}{(R_e + h)} \quad \text{but} \quad \frac{K\, m\, m_e}{R_e^2} = mg$$

Therefore

$$\frac{R_e^2\, g}{(R_e + h)^2} = \frac{V_T^2}{(R_e + h)}$$

$$V_T = \sqrt{\frac{(R_e + h)\, R_e^2\, g}{(R_e + h)^2}} = \sqrt{\frac{R_e^2\, g}{R_e + h}}$$

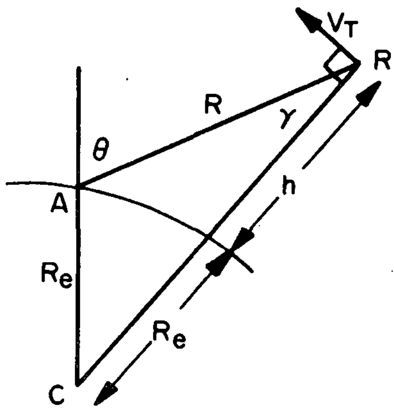
$$= \sqrt{\frac{(6.37 \times 10^6)^2 \times 10}{7.297 \times 10^6}} = 7.45 \times 10^3 \text{ meters/sec}$$

At zenith velocity abberration is

$$\phi_z = \frac{2\,V}{C} = \frac{2 \times 7.45 \times 10^3}{3 \times 10^8} = 43.8 \times 10^{-6} \text{ radians}$$

$$= 10.3 \text{ arc sec}$$

However, the velocity abberration is caused only by that component of the velocity normal to the line of site between the transmitter and cube-corner array. Denoting this component by V_N we will try to calculate V_N and ϕ_z vs θ. For a zenith pass, the situation is a rather simple planar problem. Referring to the following figure



$$\sin \gamma = \frac{R_e \sin \theta}{(R_e + h)}$$

Therefore

$$V_N = V_T \cos \gamma = V_T \sqrt{1 - \left(\frac{R_e}{R_e + h}\right)^2 \sin^2 \theta}$$

We will further define V_T^* as the tangential satellite velocity component in the plane defined by the laser tracking station (A) satellite (B) and center of the earth (C). For all passes of interest (those passing at reasonably high elevation angles) $V_T^* \approx V_T$ in magnitude (except where very close to zenith). Therefore

$$\phi = \frac{2\,V_n}{c} \approx 49.8 \times 10^{-6} \cos\gamma$$

$$= 49.8\sqrt{1 - \left(\frac{6.37 \times 10^6}{7.297 \times 10^6}\right)^2 \sin^2\theta} \times 10^{-6} = 49.8 \times 10^{-6} \sqrt{1 - .75 \sin^2\theta}$$

or

$$\phi/\phi_z = \sqrt{1 - 0.75 \sin^2\theta}$$

which is evaluated in the following table and shown in Figure 2.

θ (Deg)	ϕ/ϕ_z	θ (Deg)	ϕ/ϕ_z
0	1.00	50	
10		60	0.66
20		70	
30	0.90	80	
40		90	0.5

Since γ is also of interest we will give the following relationship

$$\sin\gamma = \left[\frac{R_e}{R_e + h}\right] \sin\theta = \left[\frac{6.37 \times 10^6}{7.297 \times 10^6}\right] \sin\theta$$

$$= 0.875 \sin\theta$$

which means that $V_{max} = \sin^{-1} 0.875 = 61.0$ deg. Figure 3 shows γ vs θ

3. SIGNAL STRENGTH VS ZENITH ANGLE FOR PERFECT ARRAY

Now that we have the orbital parameters of importance let us calculate the way the signal strength of the return from the array should vary with zenith angle. In order to simplify the evaluation we will normalize by dividing by the signal strength at zenith and denote this normalized power by W_N. We will also assume for the present, that there is no velocity abberration, and that therefore the cube-corners have been made diffraction limited. The perfect array is approximated reasonably well by a sphere coated with cube corners to provide a retrodirector which has a constant specific intensity independent of incidence angle. Under these idealized conditions the factors which affect us are the range of the target (R^4)* and the variation of atmospheric transmission with zenith angle.

In Figure 4 the space loss due to the R^4 dependence of signal is presented as a function of θ. The ordinate is R^4 at θ divided by the R^4 at $\theta = 0$ (zenith).

In addition to space loss, a certain amount of power is lost due to scatter and absorption in the atmosphere. Assuming that at zenith the absorption on a one way pass is 0.70, the transmission will be a power function of sec θ

$$T = 0.70^{\mathrm{Sec}\,\theta}$$

For a two way pass it is

$$T^2 = [0.70^{\sec\theta}]^2 = 0.70^{2\sec\theta}$$

which is plotted in Figure 5.

θ (Deg)	T^2	θ (Deg)	T^2
0	0.49	50	0.33
10		60	0.24
20	0.47	70	0.12
30		80	0.016
40	0.40	90	0

Taking the product of the R^4 and atmospheric absorption loss we get the total loss as a function of zenith angle.

*See Section 8

θ (Deg)	$R_z^4T^2/R^4T_z^2$	θ (Deg)	$R_z^4T^2/R^4T_z^2$
0	1.00	50	0.167
10	0.97	60	0.059
20	0.797	70	0.0125
30	0.58	80	0.0009
40	0.344	90	0

$$W_w = W/W_z = \frac{R_z^4\ T^2}{R^4\ T_z^2}$$

It should be noted that the values for T^2 are not accurate beyond 80°, overestimating the strength of absorption.

4. SPECIFIC INTENSITIES OF CUBE CORNER ARRAYS

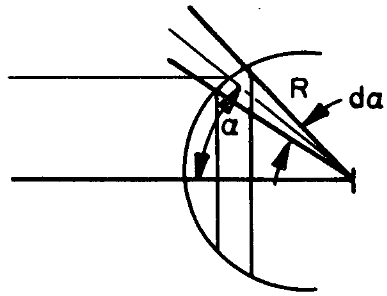
Let us define the energy per unit solid angle (intensity) returned by an individual cube corner for a unit of incident power as the specific intensity (SI). For a diffraction limited cube corner SI is a function of diameter squared

$$SI = K_1\ D^2$$

We will not consider wavelength dependence here, because wavelength is considered fixed at 6943A, nor will efficiency of optical coatings etc. be considered, but will be assumed to be part of the constant K_1. When a cube corner is used off axis the entrance pupil becomes a function of the incidence angle α. Therefore the SI must be adjusted as a function of α. Because the entrance pupil area tends to shrink along the axis in the plane of incidence with only a small decrease at right angles to this direction the area of the entrance pupil may be approximated by the on axis area multiplied by a correction factor of $[1 - K_2\ \alpha]$. In order to take care a diffraction a second $[1 - K_2\alpha]$ must be used to correct for the increased diffraction of the smaller entrance pupil. Therefore

$$SI\ (\alpha) = K_1\ D^2\ [1 - K_2\ \alpha]^2$$

where α is the angle of evidence with respect to the axis (in radians) and $K_2 \approx 1$ for fused silica cube corners of the type used on previous GEOS satellites. This relationship has been checked experimentally and found to give reasonably accurate prediction of the SI. If the cube corners are analyzed as a sphere, the specific intensity will be



$$S\ I = K_1\ D^2 \int_0^1 [2\pi R \sin \alpha] \cdot [1 - K_2\ \alpha]^2\ R\ d\alpha$$

$$= 2\pi R^2 K_1 D^2 \int_0^1 \sin\alpha\ [1 - \alpha]^2\ d\alpha$$

$$S\ I = 2\pi K_1 R^2 D^2 \int_0^1 \sin\alpha\ [1 - 2\alpha + \alpha^2] \cdot d\alpha$$

$$= 2\pi K_1 R^2 D^2 \left[\int_0^1 \sin\alpha\ d\alpha - 2\int_0^1 \alpha \sin\alpha\ d\alpha + \int_0^1 \alpha^2 \sin\alpha\ d\alpha\right]$$

$$= 2\pi K_1 R^2 D^2\ [\{-\cos\alpha\}\ \Big|_0^1 - 2\ \{\sin\alpha - \alpha\cos\alpha\}\ \Big|_0^1$$

$$+ \{2\alpha\sin\alpha + 2\cos\alpha - \alpha^2\cos\alpha\}\ \Big|_0^1]$$

$$= 2\pi K_1 R^2 D^2 [\{1 - 0.54\} - 2\{0.840 - 0 - 0.54 + 0\}$$

$$+ \{2 \times .840 - 0 + 2 \times 0.54 - 2 - 0.54 + 0\}]$$

$$= 2\pi K_1 R^2 D^2 [0.46 - 2(.30) + (1.68 + 1.08 - 2.54)]$$

$$= 2\pi K_1 R^2 D^2 [0.08] = \pi K_1 R^2 D^2 [0.16]$$

For a flat array with a circular area of radius R

$$S I = \pi K_1 R^2 D^2 [1 - \alpha]^2$$

If the same type of cube corner is used, the flat array is

$$1/0.16 = 6.25$$

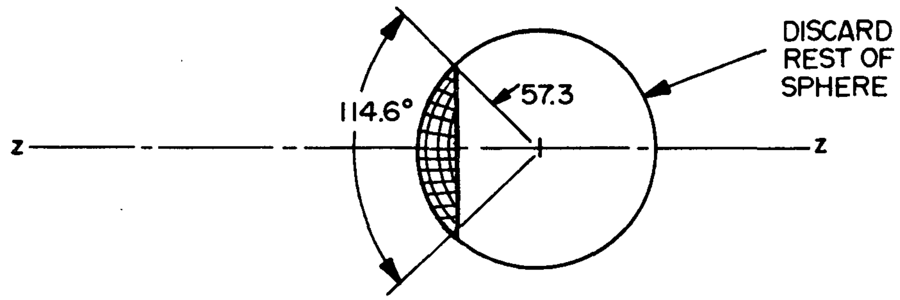
times more effective on axis. However, at

$$[1 - \alpha]^2 = 0.16$$

$$[1 - \alpha] = 0.40$$

$$\alpha = 0.60 \text{ rad} = 34.3 \text{ deg}$$

the spherical array becomes better than the flat. Considerable reduction in the number of cube corners in a spherical array can be made if only a section of the sphere can be used. Since only the cube corners out to $\approx 57.3°$ either side of the line joining the source and center of the spherical array are actually contributing to the return, the others can be eliminated



If this array is attached to the GEOS-C satellite with its axis coincident and parallel to the spacecraft axis, it will give a good return when at zenith, but will degrade when off axis. Since from Figure 3 the max γ is 61° the total included angle would have to be 114.6 + 2 (61) = 236.6°. This would make the array appear like a sphere for all possible parts of the orbit. However, it is not necessary to go to $\theta \geq 80°$ therefore the max γ required is only 59° reducing the required part of the sphere to 232.6° a slight improvement. Since the cube corners where $\alpha > 1$ contribute very little to the array the array can further be reduced with little loss in signal.

Returning to the equation for SI for the spherical array

$$SI = 2\pi K_1 R^2 D^2 [\{-\cos\alpha\} |_0^{0.5} - 2\{\sin\alpha - \alpha\cos\alpha\}|_0^{0.5}$$

$$+ \{2\alpha\sin\alpha + 2\cos\alpha - \alpha^2\cos\alpha\} |_0^{0.5}]$$

$$= 2\pi K_1 R^2 D^2 [(1 - 0.879) - 2(0.479 - 0.439 - 0 + 0)$$

$$+ (0.479 + 1.759 - 0.220 - 0 - 2 + 0)]$$

$$= 2\pi K_1 R^2 D^2 [0.121 - 0.080 + 0.018] = 2\pi K_1 R^2 D^2 [0.059]$$

$$= \pi K_1 R^2 D^2 [0.118]$$

This array has only about 1/4 the area but has

$$\frac{0.118}{0.160} = 74\%$$

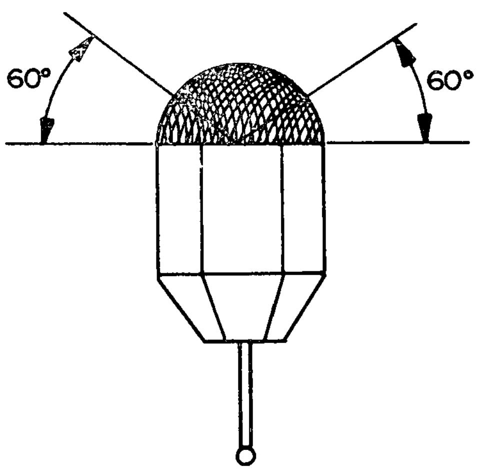
of the SI. Therefore at only slight reduction in performance we get a large reduction in the number of cube corners. Total included angle would now be only

$$[0.5 + 0.5]\ 57.3 + 2\ [59°] = 165.3°$$

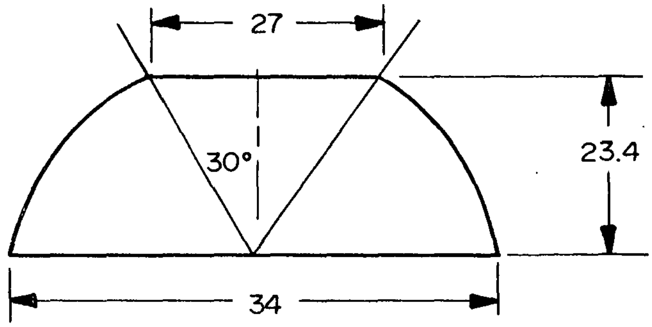
If we assume that this array is placed on the bottom of the GEOS-C and we call the radius of the satellite R_S then

$$R_S = R \sin \left[\frac{165.3}{2}\right] = R \sin (82.9) = 0.99\ R \approx R$$

Let us assume we make the array a hemisphere



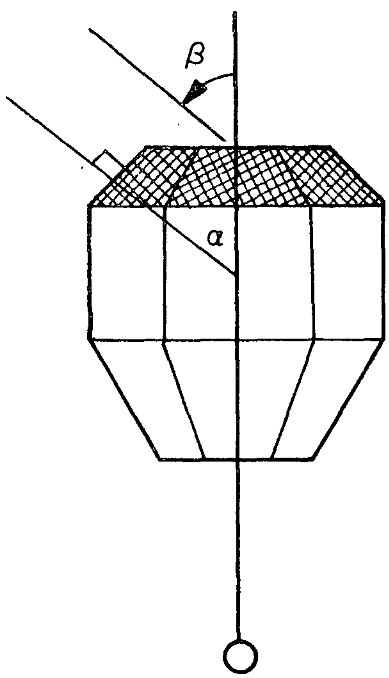
Assume we cut out the central 60° cone from hemisphere array to allow antennas etc. to be placed on the earth facing side of the satellite. Then the height of the array would be R sin 60 = 0.867R or for a 54" diameter satellite 0.867 × 27 = 23.4".

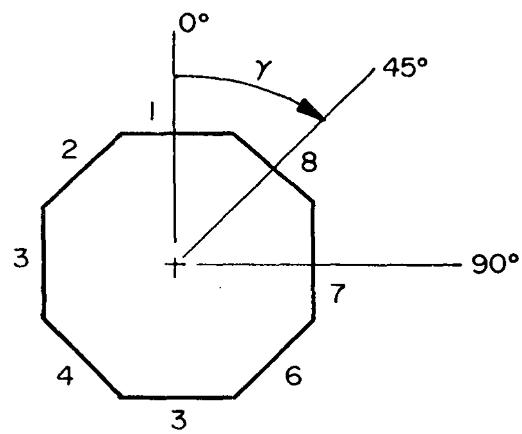


This is obviously much too large! We will therefore now consider other types of arrays.

5. THE TRUNCATED PYRAMID ARRAY

From considerations of the satellite shape (An octagonal cross section) the possibility of placing the cube curves on eight sloped panels as shown in the following diagram appears feasible





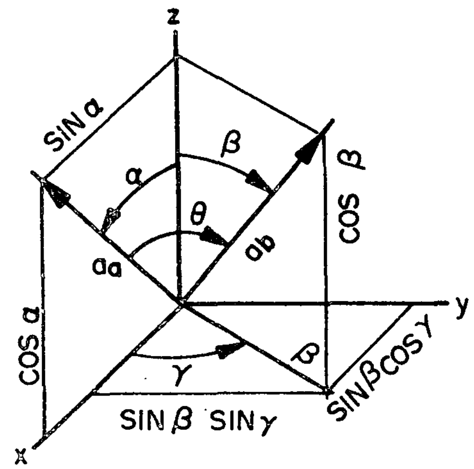
α = slope of panel with respect to satellite axis

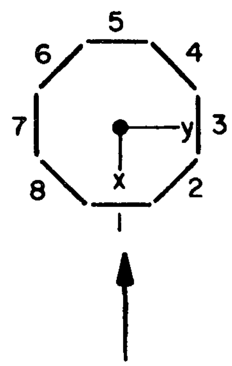
β = slope of observation axis with respect to satellite axis

γ = rotation about satellite axis with 0° the plane containing the line of sight and the satellite axis

This array is very similar to the earlier array used on BE-C but lacks the bottom panel because space must be left for other experiments.

Using the next figure we can calculate the incidence angle as a function of β, α, and γ.





$$\hat{a}_a = \sin\alpha\,\hat{a}_x + \cos\alpha\,\hat{a}_z$$

$$\hat{a}_b = \sin\beta\cos\gamma\,\hat{a}_x + \sin\beta\sin\gamma\,\hat{a}_y + \cos\beta\,\hat{a}_z$$

$$\cos\theta = \hat{a}_a\cdot\hat{a}_b = \sin\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta$$

We have assigned numbers to each of the eight panels, and assume then to have equal numbers of cube corners. Calculations of the effective area are shown in the following tables, and Figures 7 and 8.

$\beta = \pi/3 \quad \sin\beta = \sqrt{3}/2 \quad \cos\beta = 1/2$

Panel	cos V	cos θ
1	1	$\frac{1}{2}\left(\sqrt{3}\sin\alpha + \cos\alpha\right)$
2	$\sqrt{2}/2$	$\frac{1}{2}\left(\frac{\sqrt{6}}{2}\sin\alpha + \cos\alpha\right)$
3	0	$\frac{1}{2}(\cos\alpha)$
4	$-\sqrt{2}/2$	$\frac{1}{2}\left(-\frac{\sqrt{6}}{2}\sin\alpha + \cos\alpha\right)$
5	-1	$\frac{1}{2}(-\sqrt{3}\sin\alpha + \cos\alpha)$
6	$-\sqrt{2}/2$	$\frac{1}{2}\left(-\frac{\sqrt{6}}{2}\sin\alpha + \cos\alpha\right)$
7	0	$\frac{1}{2}(\cos\alpha)$
8	$\sqrt{2}/2$	$\frac{1}{2}\left(\frac{\sqrt{6}}{2}\sin\alpha + \cos\alpha\right)$

cos θ

	$\alpha \to 0$	15	30	45	60	75	90
1	1/2			0.965	1.00	0.966	0.866
2 & 8	1/2			0.795	0.780	0.720	0.613
3 & 7	1/2			0.354	0.250	0.129	0
4 & 6	1/2			–	-0.28	–	-0.613
5	1/2			–			-0.866

θ (Radians)

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	> 1			0.262	0	0.262	0.525
2 & 8	> 1			0.655	0.672	0.766	0.91
3 & 7	> 1			>1	>1	>1	>1
4 & 6	> 1			>1			>1
5	> 1			>1			>1

$(1 - \theta)$ Radians (Neg values = 0)

	0	15	30	45	60	75	90
1	0			0.738	1.00	0.738	0.475
2	0			0.245	0.328	0.234	0.09
3	0			0	0	0	0
4	0			0	0	0	0
5	0			0	0	0	0
6	0			0	0	0	0
7	0			0	0	0	0
8	0			0.245	0.328	0.234	0.09
Total	0			1.228	1.656	1.206	0.655

$\beta = \pi/4$

Panel	γ	$\sin\gamma$	$\cos\gamma$	$\cos\gamma$
1	0	0	1	$\frac{\sqrt{2}}{2}[\sin\alpha + \cos\alpha]$
2	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{\sqrt{2}}{2}\left[\frac{\sqrt{2}}{2}\sin\alpha + \cos\alpha\right]$
3	$\pi/3$	1	0	$\frac{\sqrt{2}}{2}[\cos\alpha]$
4	$3\pi/4$	$2/2$	$-\sqrt{2}/2$	$\frac{\sqrt{2}}{2}\left[-\frac{\sqrt{2}}{2}\sin\alpha + \cos\alpha\right]$
5	π	0	-1	$\frac{\sqrt{2}}{2}[-\sin\alpha + \cos\alpha]$
6	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\frac{\sqrt{2}}{2}[-\sin\alpha + \cos\alpha]$
7	$3\pi/2$	-1	0	$\frac{\sqrt{2}}{2}[\cos\alpha]$
8	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{\sqrt{2}}{2}\left[\frac{\sqrt{2}}{2}\sin\alpha + \cos\alpha\right]$

$\cos\theta$

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	$\sqrt{2}/2$	0.866	$(\sqrt{6}+\sqrt{2})/4$	1	$(\sqrt{6}+\sqrt{2})/4$	0.866	$\sqrt{2}/2$
2 & 8	$\sqrt{2}/2$	0.815		$(2+\sqrt{2})/4$	$(\sqrt{3}+\sqrt{2})/4$	0.668	1/2
3 & 7	$\sqrt{2}/2$	0.680		1/2	$\sqrt{2}/4$		0
4 & 6	$\sqrt{2}/2$	0.554		$(2-\sqrt{2})/4$			-1/2
5	$\sqrt{2}/2$			0			$-\sqrt{2}/2$

radians

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	0.785	0.525	0.28	0	0.28	0.525	0.785
2 & 8	0.785	0.620	0.53	0.55	0.67	0.840	1.05
3 & 7	0.785	0.870	0.91	1.05	1.2	>1	1.57
4 & 6	0.785	0.99	>1	1.42	>1	>1	2.10
5	0.785	>1	>1	1.57	>1	>1	2.36

$(1 - \theta)$ radians (neg. values = 0)

α	0	15	30	45	60	75	90
1	0.215	0.475	0.72	1	0.72	0.475	0.215
2	0.215	0.380	0.47	0.45	0.33	0.16	0
3	0.215	0.180	0.09	0	0	0	0
4	0.215	0.01	0	0	0	0	0
5	0.215	0	0	0	0	0	0
6	0.215	0.01	0	0	0	0	0
7	0.215	0.18	0.09	0	0	0	0
8	0.215	0.38	0.47	0.45	0.33	0.16	0
Total	1.72	1.615	1.84	1.90	1.38	0.795	0.215

Total $\times \cos^4 \alpha$

$\alpha \rightarrow$	0	15	30	45	60	75	90
$\cos^4 \alpha$	1	0.87	0.560	0.25	0.063	0.005	0
	1.72	1.41	1.03	0.475	0.087	0.004	0

$8 (1 - \alpha) \cos^4 \alpha$

$\alpha \rightarrow$	0	15	30	45	60	75	90
	8	5.15	2.12	0.420	0	0	0

$\beta = \pi/6 \ \sin\beta = 1/2 \ \cos\beta = \sqrt{3}/2$

Panel		$\sin\gamma$	$\cos\gamma$	$\cos\theta$
1	0	0	1	$\frac{1}{2}(\sin\alpha + \sqrt{3}\cos\alpha)$
2	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \sqrt{3}\cos\alpha\right)$
3	$\pi/2$	1	0	$\frac{1}{2}(\sqrt{3}\cos\alpha)$
4	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\sin\alpha + \sqrt{3}\cos\alpha\right)$
5	π	0	-1	$\frac{1}{2}(-\sin\alpha + \sqrt{3}\cos\alpha)$
6	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$\frac{1}{2}\left(-\frac{\sqrt{2}}{2}\sin\alpha + \sqrt{3}\cos\alpha\right)$
7	$3\pi/2$	-1	0	$\frac{1}{2}(\sqrt{3}\cos\alpha)$
8	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{1}{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \sqrt{3}\cos\alpha\right)$

$\cos\theta$

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	$\sqrt{3}/2$	–	1	0.966	0.866	0.707	1/2
2 & 8	$\sqrt{3}/2$	–	0.926	0.866	0.739	0	0.354
3 & 7	$\sqrt{3}/2$	–	0.750	0.612	0	0	0
4 & 6	$\sqrt{3}/2$	–	0.574	0.364	0	0	-0.354
5	$\sqrt{3}/2$	–	0.500	0.266	0	0	-1/2

θ (radians)

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	0.525	–	0	0.262	0.525	0.785	1.05
2 & 8	0.525	–	0.384	0.525	0.742	0	1
3 & 7	0.525	–	0.726	0.913	0	0	1.57
4 & 6	0.525	–	0.960	>1	0	0	0
5	0.525	–	>1	>1	0	0	0

$(1 - \theta)$ radians (neg. values = 0)

	0	15	30	45	60	75	90
1	0.475	–	1.0	0.738	0.475	0.215	0
2	0.475	–	0.616	0.473	0.258	0	0
3	0.475	–	0.274	0.087	0	0	0
4	0.475	–	0.040	0	0	0	0
5	0.475	–	0.000	0	0	0	0
6	0.475	–	0.040	0	0	0	0
7	0.475	–	0.274	0.087	0	0	0
8	0.475	–	0.616	0.475	0.991	0	0
Total	3.80		2.840	1.862	0.991	0.215	0

$(1 - \theta)^2$

$\beta = 0 \; (\theta = \alpha)$

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	1	0.541	0.225	0.046	0	0	0
2	1	0.541	0.225	0.046	0	0	0
3	1	0.541	0.225	0.046	0	0	0
4	1	0.541	0.225	0.046	0	0	0
5	1	0.541	0.225	0.046	0	0	0
6	1	0.541	0.225	0.046	0	0	0
7	1	0.541	0.225	0.046	0	0	0
8	1	0.541	0.225	0.046	0	0	0
	8	4.33	1.80	0.368	0.00	0	0

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	0.225		1.0	0.541	0.225		
2	0.225		0.380	0.225	0.066		
3	0.225		0.075	0.008	0		
4	0.225		0.002	0	0		
5	0.225		0.00	0	0		
6	0.225		0.002	0	0		
7	0.225		0.075	0.008	0		
8	0.225		0.380	0.225	0.066		
	1.80		1.914	1.007	0.357		

$(1 - \theta)^2$

$\alpha \rightarrow$	0	15	30	45	60	75	90
1	0.046	0.225	0.52	1	0.52		
2	0.046	0.144	0.22	0.202	0.11		
3	0.046	0.032	0.008	0			
4	0.046	0		0			
5	0.046	0		0			
6	0.046	0		0			
7	0.046	0.032	0.008	0			
8	0.046	0.144	0.22	0.202	0.11		
	0.370	0.571	0.976	1.404	0.74		

6. MAXIMUM ALLOWABLE SIZE FOR DIFFRACTION LIMITED CUBE CORNER

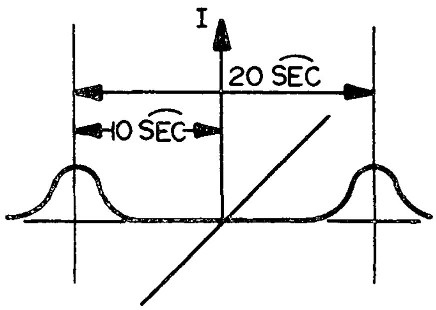
If we set the diameter of the cube corner so that the first null of the Airy disk equals the velocity abberation at zenith, we get an upper bound for the size of an individual cube corner:

$$\alpha_{BE} \leq \frac{1.22\,\lambda}{D} = 49.8 \times 10^{-6}$$

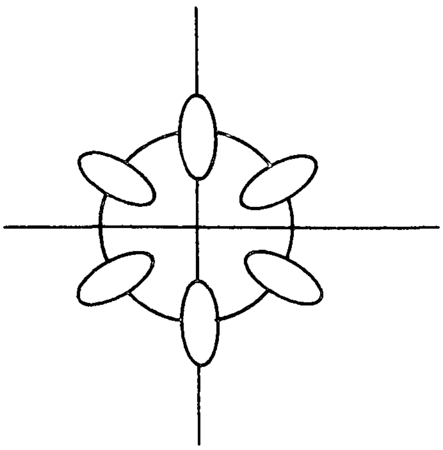
Then

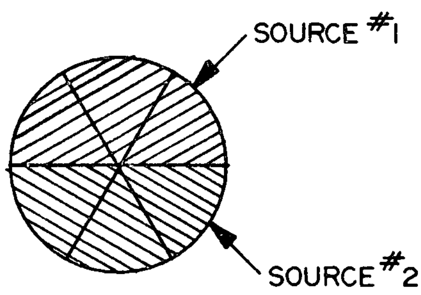
$$D \leq \frac{1.22\,\lambda}{\alpha_{BE}} = \frac{1.22\,\lambda}{49.8 \times 10^{-6}} = \frac{1.22 \times 10^{-6}}{49.8 \times 10^{-6}} = 1.71 \text{ cm} = 0.675 \text{ in. } D \approx 5/8 \text{ inch}$$

But if we build into the return of the cube corner an error equal to the Bradley Effect or about 49.8 μrad. by building an error into one of the angles of the cube corner of $49.8/5 \approx 10$ μrad. (2 arc sec) we can compensate for the velocity abberration. The effect is to split the return as shown in the following diagram. See also Figures 9 and 10.



One of these two intensity peaks will fall on the receiver if the direction of motion is in the plane determined by the vector of satellite velocity and the line joining transmitter to satellite. However since the satellite is not spin stabilized we will orient the cube corner errors in the rotational positions 120° apart to create the toroidal ring effect shown in the next figures in plan view





Now in order for there to be no empty spots in the toroid we will set the requirements that the Airy Disk radius must be 1/6 the circumference of the velocity abberration so that the intensity patterns overlap at about one half the distance to the null. Therefore

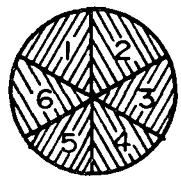
$$\frac{1.22\,\lambda}{D} = \frac{\pi\,(2\,\alpha_{BE})}{6} = \frac{\pi}{3}\,\alpha_{BE} = 1.05\,\alpha_{BE}$$

Therefore

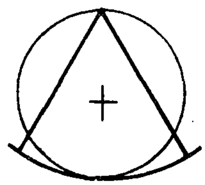
$$D = 0.642'' \text{ still} \approx 5/8 \text{ inch}$$

It should be noted that the spots are no longer circular, but tend towards ellipses with 2 to 1 ratio of major to minor axis. The major axis is aligned radially.

From a cost standpoint we would like to make the cube corners as large as possible. Therefore if we make all the angles of the cube corner offset from 90° by 2 sec will also get six spots in a circle 20 sec in diameter, but the spots will be larger. Looking at the face of the cube corner



Each of the pie shaped sections acts as a separate source and since each is a smaller fraction of the cube corner diameter the diffraction will be larger



Approximating the sector by a circle 1/2 the diameter of the cube corner, we can then set the size of the cube corner to

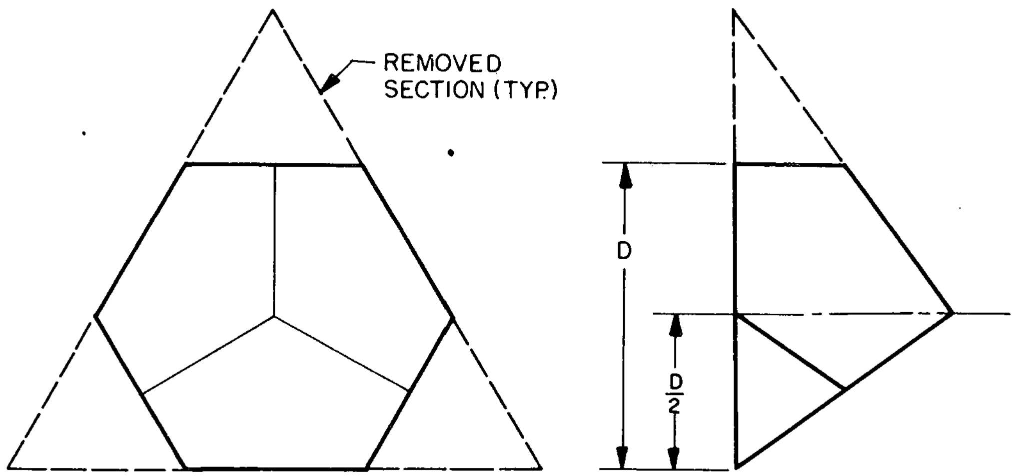
$$D = 1.71 \times 2 = 3.42 \text{ cm} = 1.35 \text{ inches} \approx 1\frac{3''}{8}$$

Thermal distortion places a practical limit on the size of the cube corner, however the distortion can be tolerated near zenith where signals tend to be strong. For large zenith angles the effective aperture of the cube corner is reduced to such a small dimension that diffraction effects are so strong that the thermal distortion is completely masked.

$$V = \frac{1}{3} BT = \frac{1}{3} \left[\frac{\sqrt{3}}{3} S^2\right] \left[\frac{\sqrt{2}}{3} S\right] = \frac{\sqrt{6}}{27} S^3$$

$$= 0.0908\ S^3$$

In order to save weight, and to allow the cube corners to be packed closer together, the corners are cut off at a point 1/2 the distance between the point of the triangle and the axis of symmetry of the prism. This does not affect the optical performance of the cube corner on axis and has only minor effects off axis. The process of cutting off the edges (called optimization) leaves the cube corner as shown below.



The resulting cube corner is an equilateral hexagonal entrance pupil (base) with a width across flats (D) equal to 2/3 S.

When the corners are cut off at 2/3 for optimization each corner has a base

$$B_c = \frac{\sqrt{3}}{3}\left[\frac{1}{3}S\right]^2 = \frac{\sqrt{3}}{27}S^2$$

and a depth

$$T_c = \left[\frac{1}{3}S\right]\left[\frac{\sqrt{6}}{2\sqrt{3}}\right] = \frac{\sqrt{2}}{6}S$$

Therefore the volume of each of the removed sections is

$$V_c = \frac{1}{3}B_c T_c = \frac{1}{3}\left[\frac{\sqrt{3}}{27}S^2\right]\left[\frac{\sqrt{2}}{6}S\right] = \frac{\sqrt{6}}{486}S^3$$

The volume of the optimized cube corner is

$$V_0 = V - 3V_c = \left[\frac{\sqrt{6}}{27} - \frac{\sqrt{6}}{486}\right]S^3$$

$$= 0.0755\ S^3$$

Converting to terms of D (S = 3/2 D)

$$V_0 = 0.0755\left[\frac{3}{2}D\right]^3 = 0.255D^3$$

For fused silica with a density ρ of 2.8×10^3 kg/m.^{3} the weight is

$$W_0 = 0.255D^3 \times 2.8 \times 10^3 = 715D^3 \text{ kg.} \quad \text{(D in m.)}$$

or

$$W_0 = 0.715D^3 \text{ grams (D in cm.)}$$

8. CALCULATION OF EXPECTED RETURN AT ZENITH FOR A FLAT ARRAY

$$W_R = \frac{P_T\ P_T}{\pi\,(\theta_t\ R)^2} \cdot \tau_A^2 \cdot \frac{A_{cc}\ P_{cc}}{\pi\,(\theta_{cc}\ R)^2} \cdot \tau_R\ \eta\ A_R$$

W_R = Return signal in photoelectrons

P_T = Power transmitted = 1 joule = 3.4×10^{18} photons

θ_T = Transmitter divergence (half angle) = 5×10^{-4} rad.

τ_A = Atmospheric transmission = 70%

ρ_T = Optical efficiency of transmitter = 80%

ρ_{cc} = Optical efficiency of cube corner = 80%

θ_{cc} = Cube corner divergence (half angle) = 5×10^{-5} rad.

R = Range of Satellite = 9.27×10^5 meters.

τ_R = Transmission of receiver = 0.25

η = Quantum efficiency of receiver 0.02

A_R = Area of receiver = 0.12 meters^2

$$W_R = \frac{3.4 \times 10^{18} \times 0.80 \times 0.49}{3.14\ (5 \times 10^{-4} \times 9.27 \times 10^5)^2} \cdot \frac{A_{cc} \times 0.80 \times 0.25 \times 0.02}{3.14\ (5 \times 10^{-5} \times 9.27 \times 10^5)^2} \times 0.12$$

$$= \frac{0.425 \times 10^{18}}{2150 \times 10^2} \cdot \frac{A_{cc} \times 4 \times 10^{-3} \times 0.12}{3.14\ (2150)} = \frac{0.54 \times 10^{15}\ A_{cc} \times 0.12}{4.60 \times 10^8}$$

$$= 0.117 \times 10^7\ A_{cc} \times 0.12 = 1.4 \times 10^5\ A_{cc}$$

If we make the array with an area of 0.28 meters^2.

$$W_R = 1.4 \times 10^5 \times 0.2 = 28{,}000 \text{ photoelectrons}$$

9. CALCULATION OF MAXIMUM WORKING ZENITH ANGLE

Assuming we can tolerate a signal of 40 photoelectrons we can tolerate a degradation of 10^3 from the zenith case which occurs (from Figure 6) at about 80°, for a spherical array, (or a flat array kept pointed toward the transmitter). Since a spherical array is only about 16% as efficient as the flat array our allowable degradation factor is only about 160, therefore θ from Figure 6 for the spherical array is about 75°.

For the truncated pyramid array from the array graphs for $\beta \approx 45°$ the 10^3 degradation point also comes at about 75°.

A truncated shaped surface of revolution with a cone half angle of 45° would be a better surface than the pyramid because it would be rotationally symmetric, and is nearly as simple to construct.

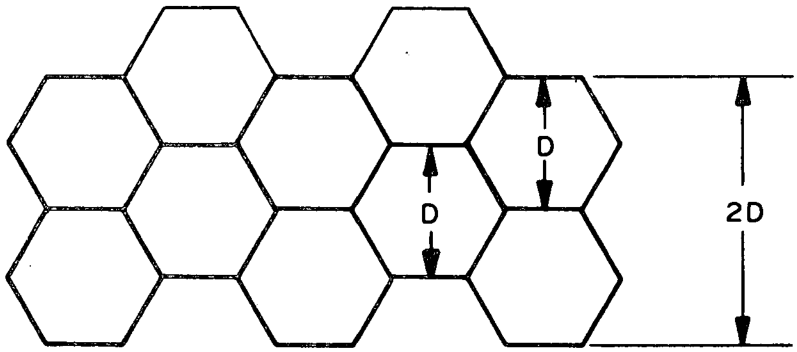
10. CONCLUSIONS

A zenith angle of 75° can be obtained with an array with an area of 0.28 meters^2, if the array is shaped on the surface of a 45° cone whose axis is coincident to the satellite axis. Since the area of the proposed array is 0.28 meters^2 and the diameter of the satellite is about 1.37 cm (54 in.) the width of the array must be approximately

$$0.28 = \pi D W = 4.30 W$$

$$W = \frac{0.28}{4.30} = 0.065 \text{ m} = 2.56''$$

Since the cube corners will be $\approx$ 3.5 cm across flats make the array look as follows. (See also Figure 11.)



Also $5D/2 = 3.5 \times 2.5 = 8.75$ cm slightly larger than our area based estimate.

In this configuration each cube corner takes up a circumferential length of $D \tan 30^\circ + D/\cos 30 = 1.732\ D = 6.14$ cm therefore we should be able to get

$$\frac{4 \times 137\,\pi}{6.14} = 280$$

in the ring. Since the area of each is $\approx 10.5\ \mathrm{cm}^2$ this gives us a total array area of

$$280 \times 10.5 = 2940\ \mathrm{cm}^2 = 0.294\ \mathrm{m}^2$$

Because of the space between cube corners, this will have to be reduced slightly. Assuming tha: 270 cube corners which will provide the desired area of $0.28\ \mathrm{m}^2$ can be fitted in the total weight of the cube corners will be

$$270 \times 31.4 = 8460\ \mathrm{grams} = 8.46\ \mathrm{kg} = 18.6\ \mathrm{lbs.}$$

an increase in weight over the GEOS II array of 14.0 lbs.

However, the panels should not be any heavier than before so the overall weight of the array has pr9bably not gone up more than 50% of the GEOS II array, while the efficiency compared to the GEOS II array of small cubes has gone up 275%.

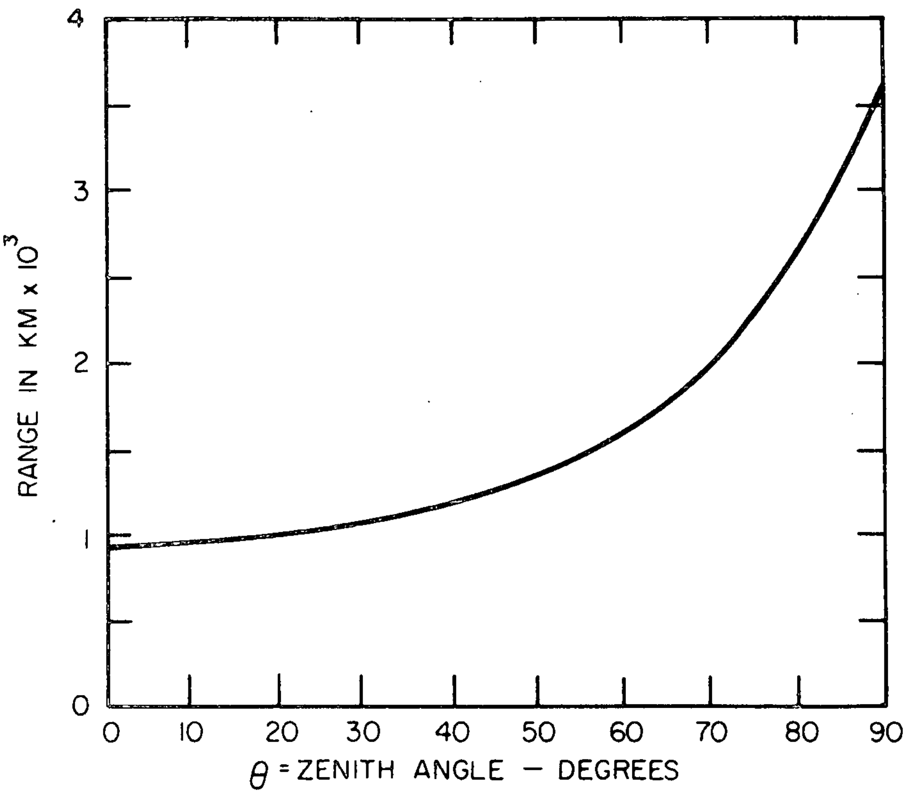


Figure 1. Range of GEOS-C Vs. Zenith Angle

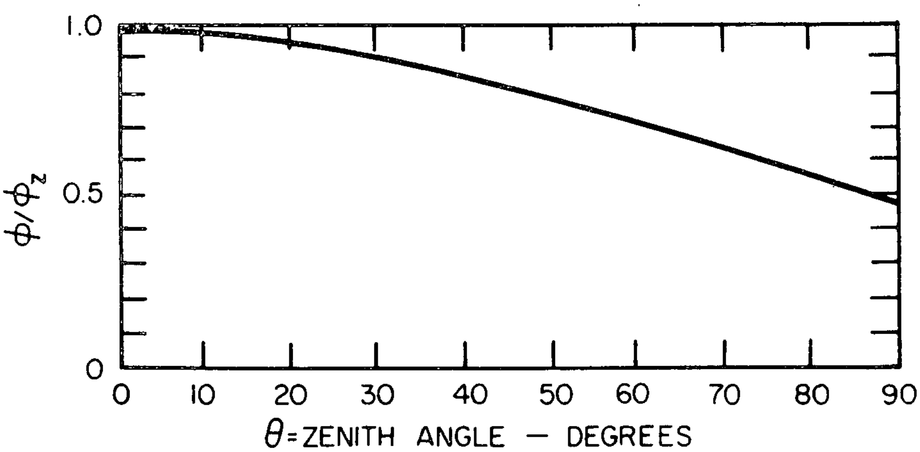


Figure 2. Velocity Abberation Vs. Zenith Angle

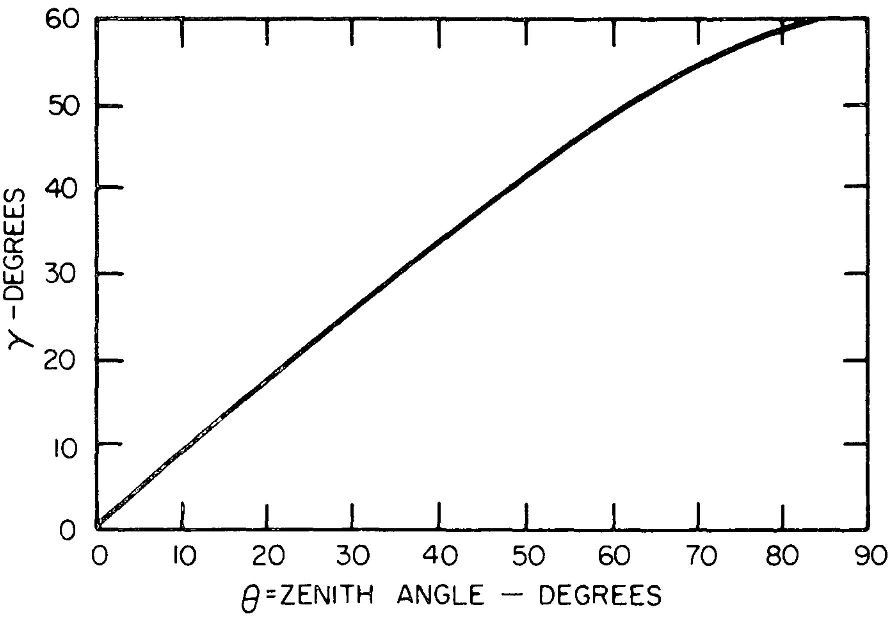


Figure 3. Incidence Angle Vs. Zenith Angle

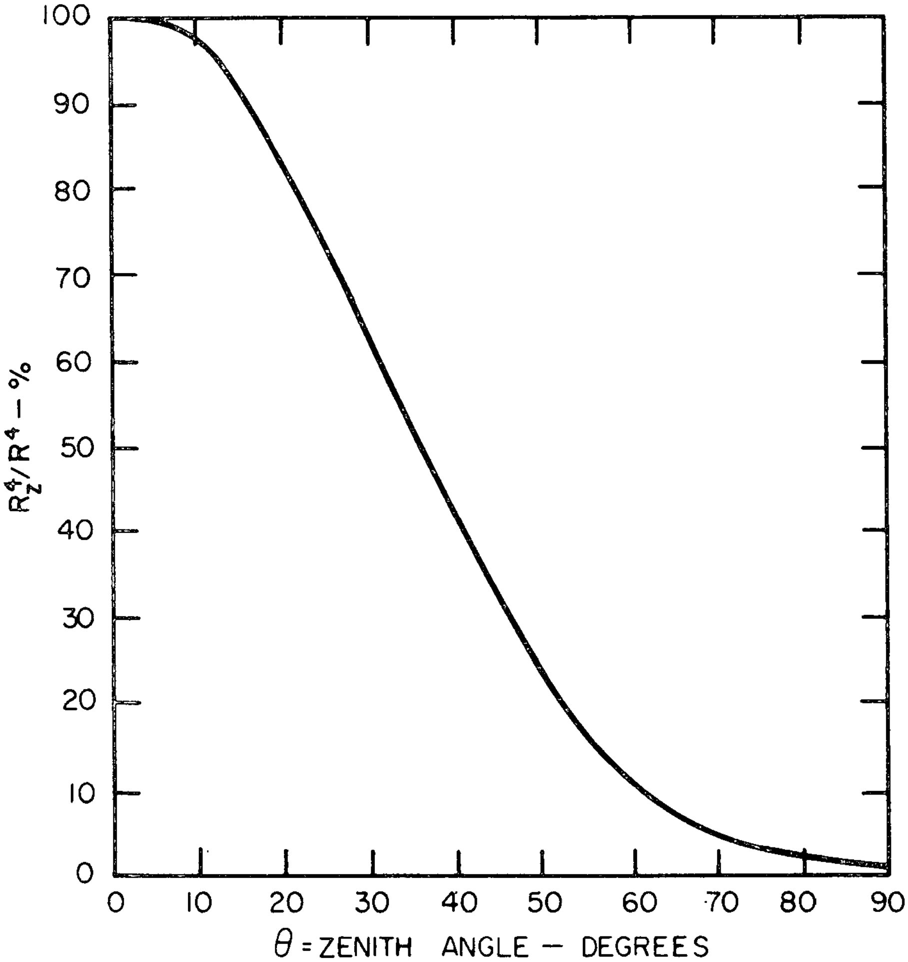


Figure 4. Normalized Space Loss

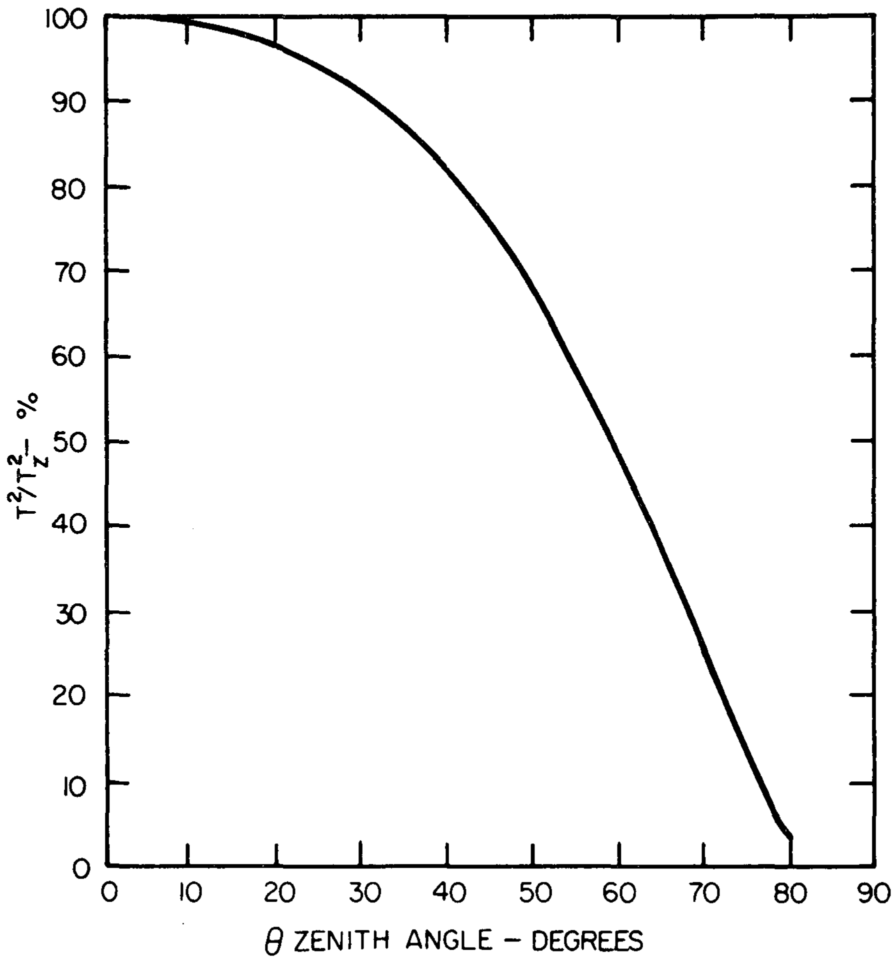


Figure 5. Atmospheric Transmission

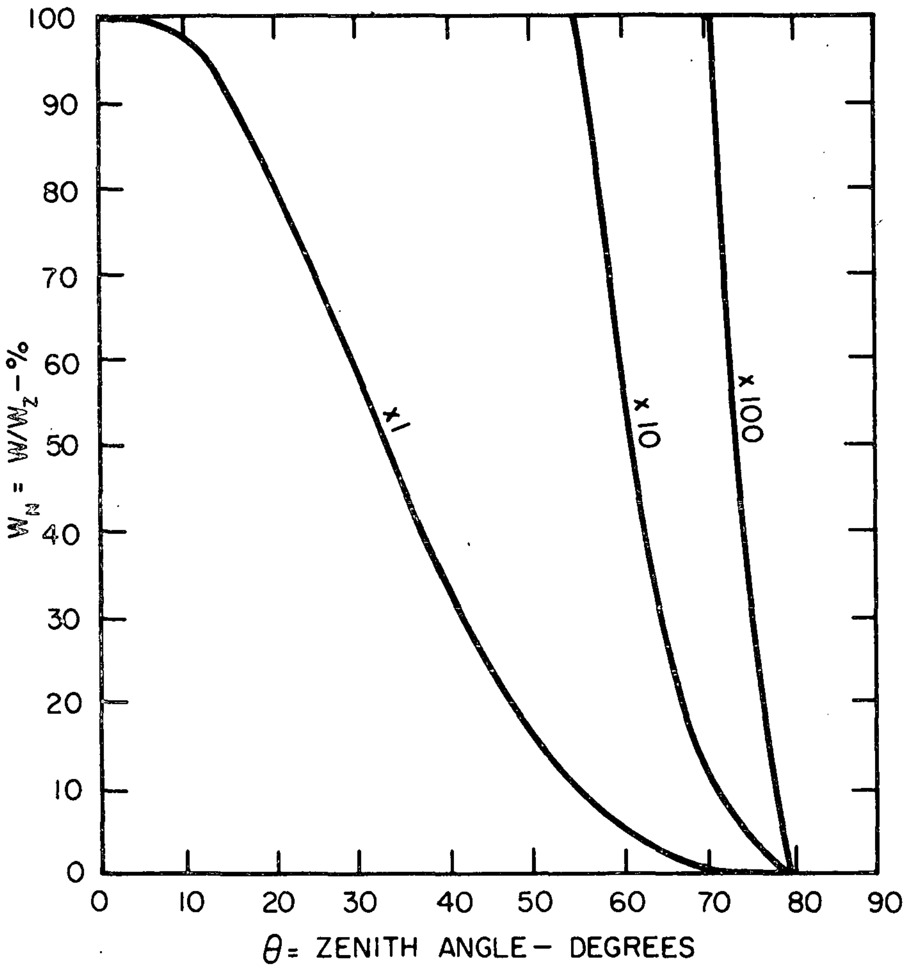


Figure 6. Normalized Returns

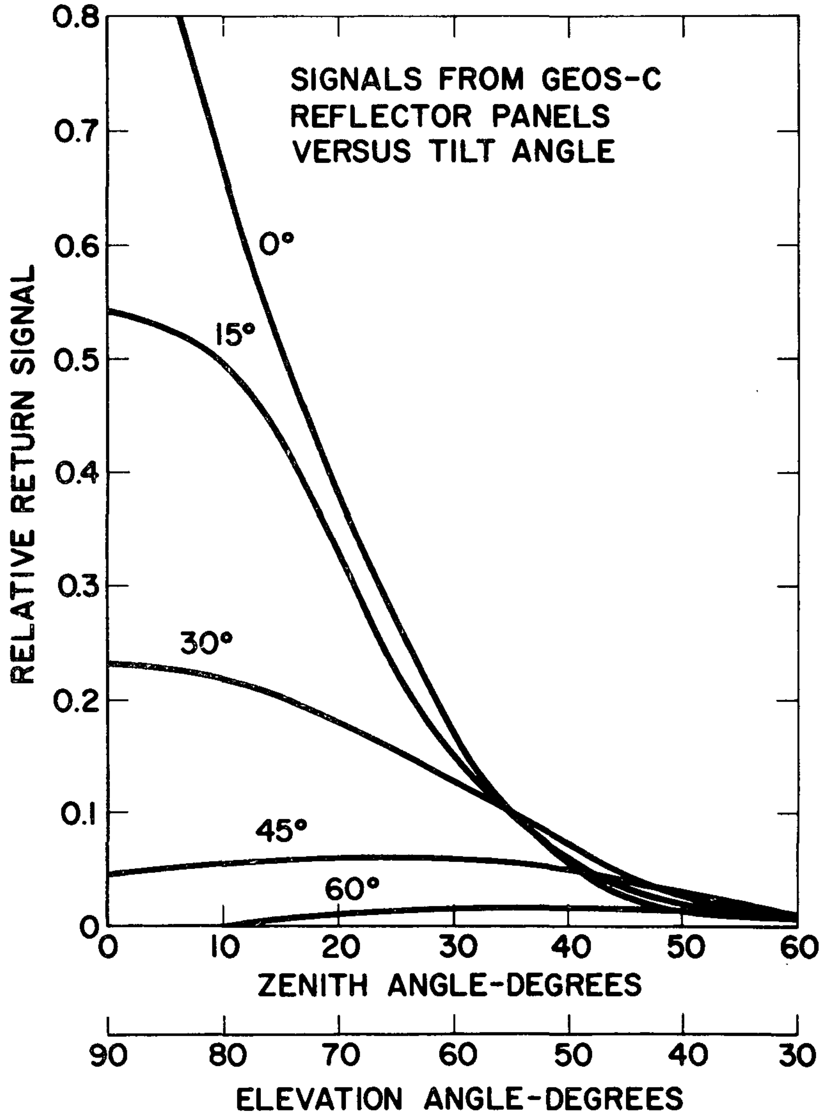


Figure 7. Signals from GEOS-C Reflector Panels Vs. Tilt Angle

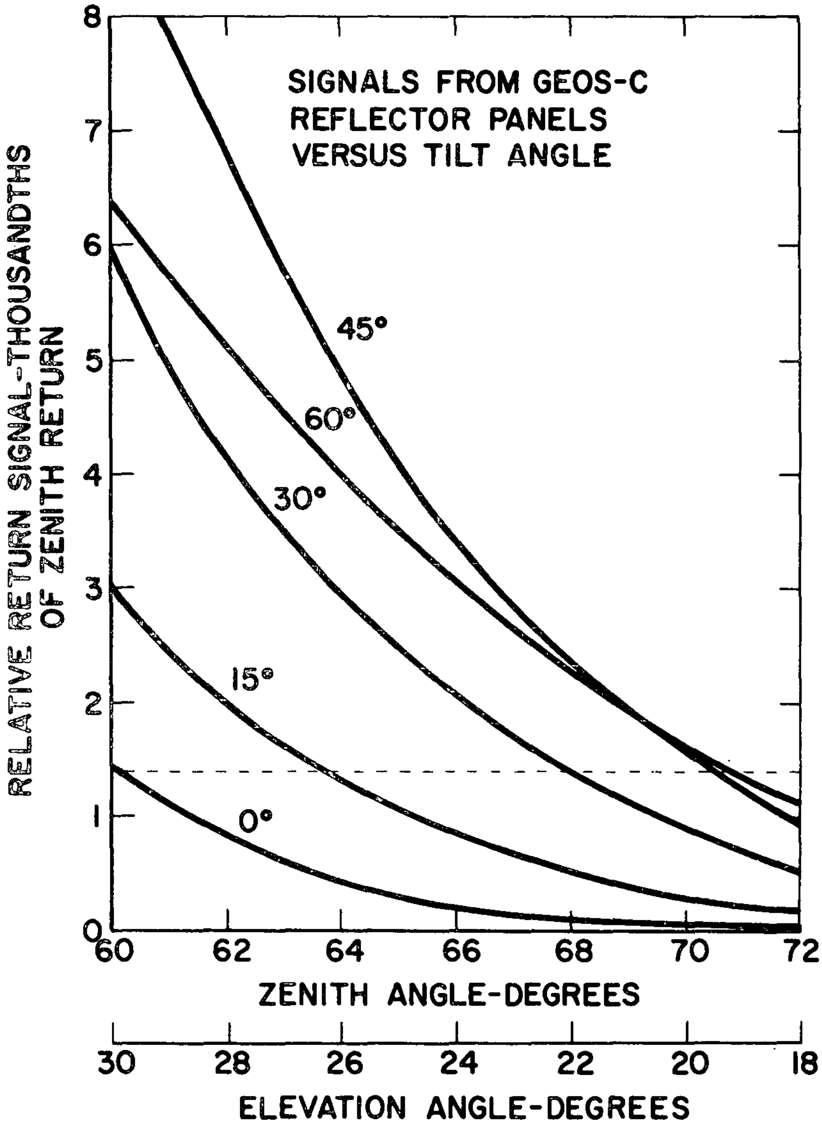


Figure 8. Signals from GEOS-C Reflector Panels Vs. Tilt Angle

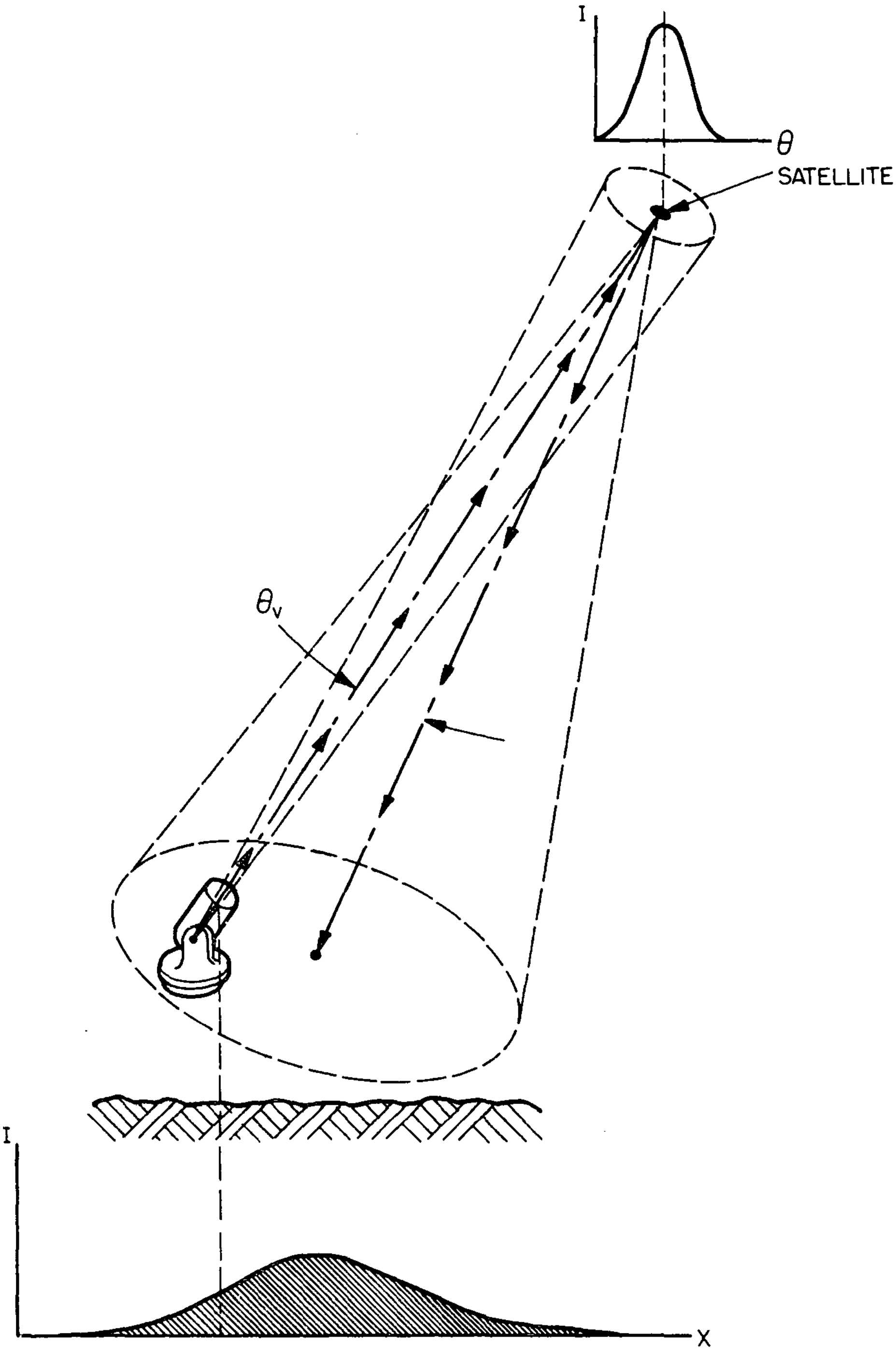


Figure 9. Velocity Abberration Compensation by Spreading of Cube Corner Return

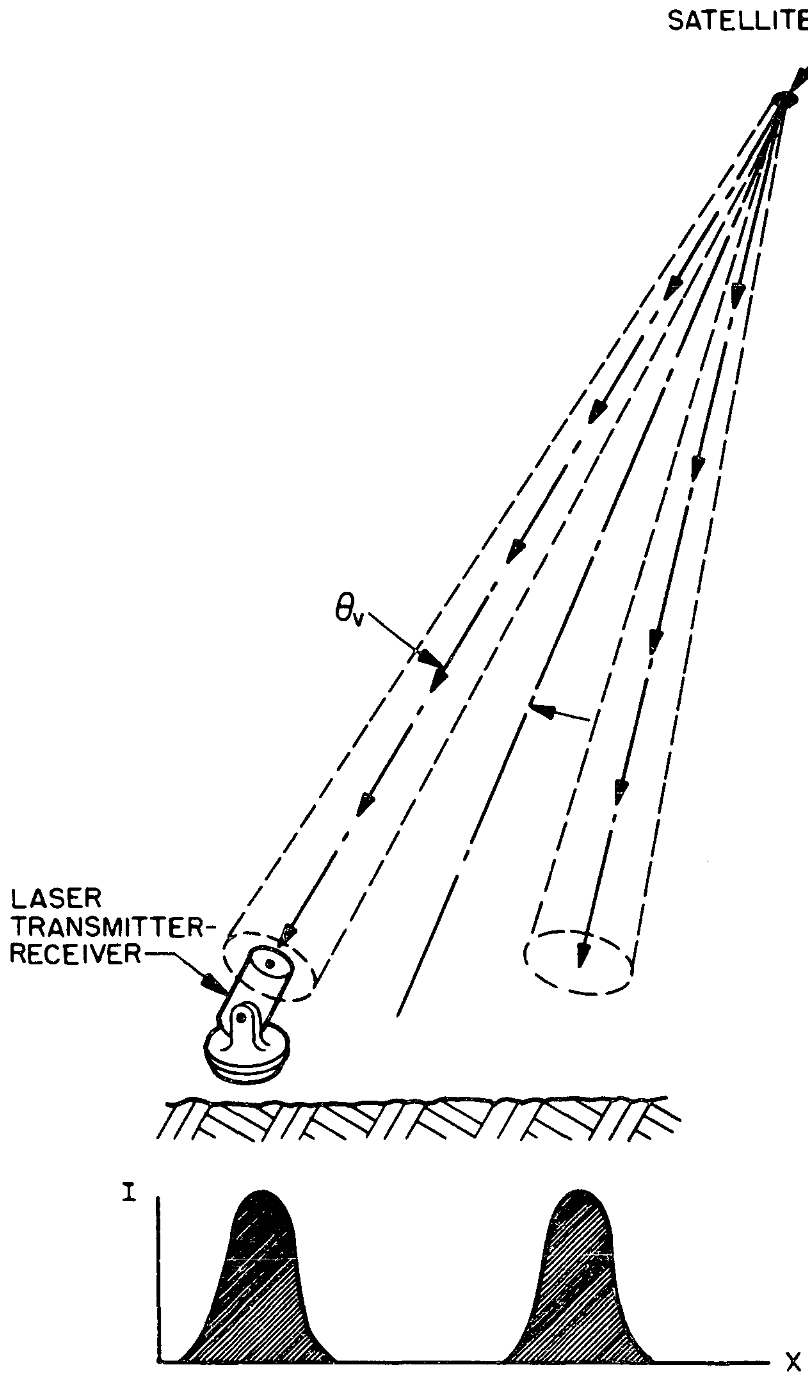


Figure 10. Velocity Abberration Compensation
by Splitting of Cube Corner Return

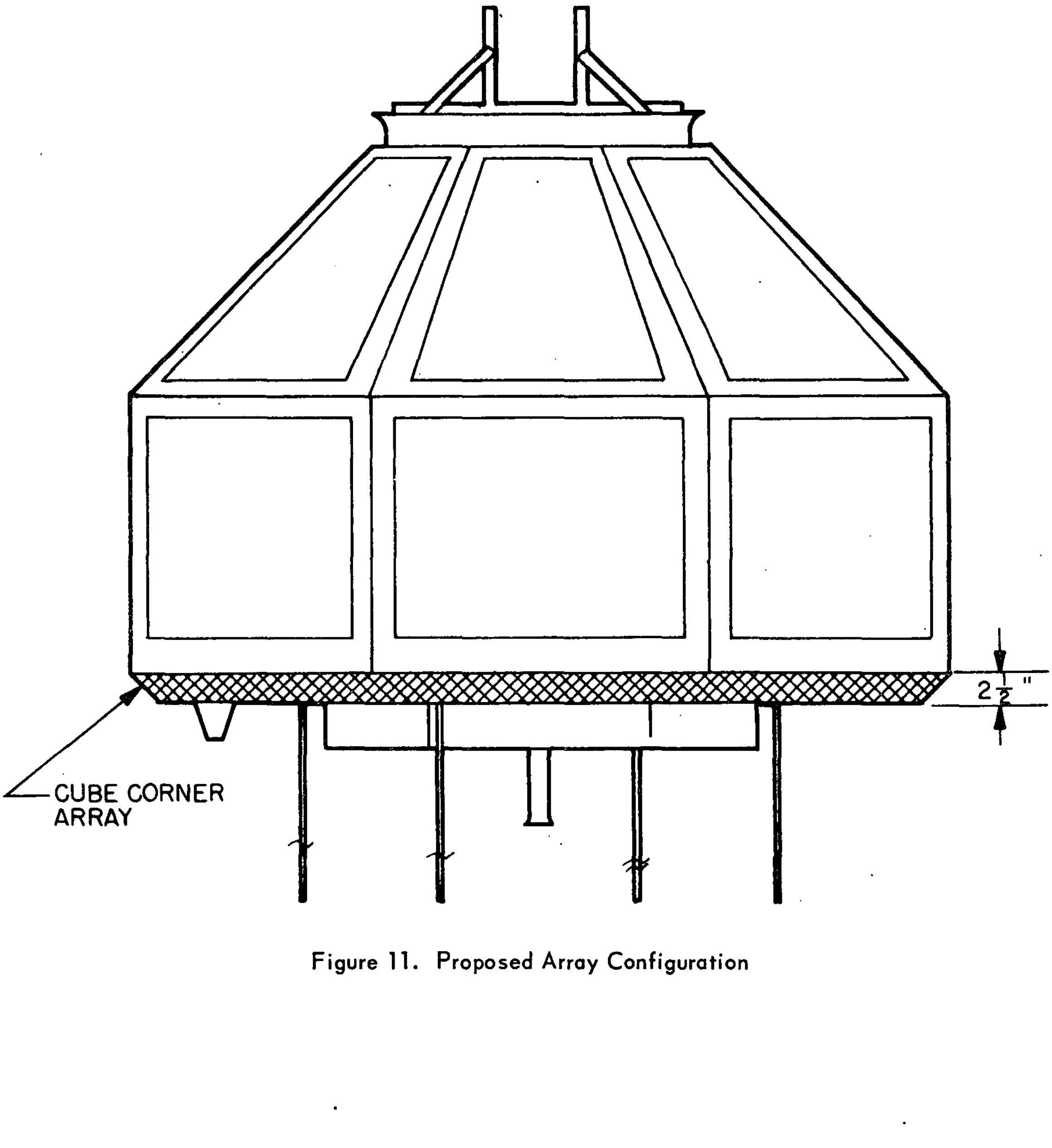


Figure 11. Proposed Array Configuration

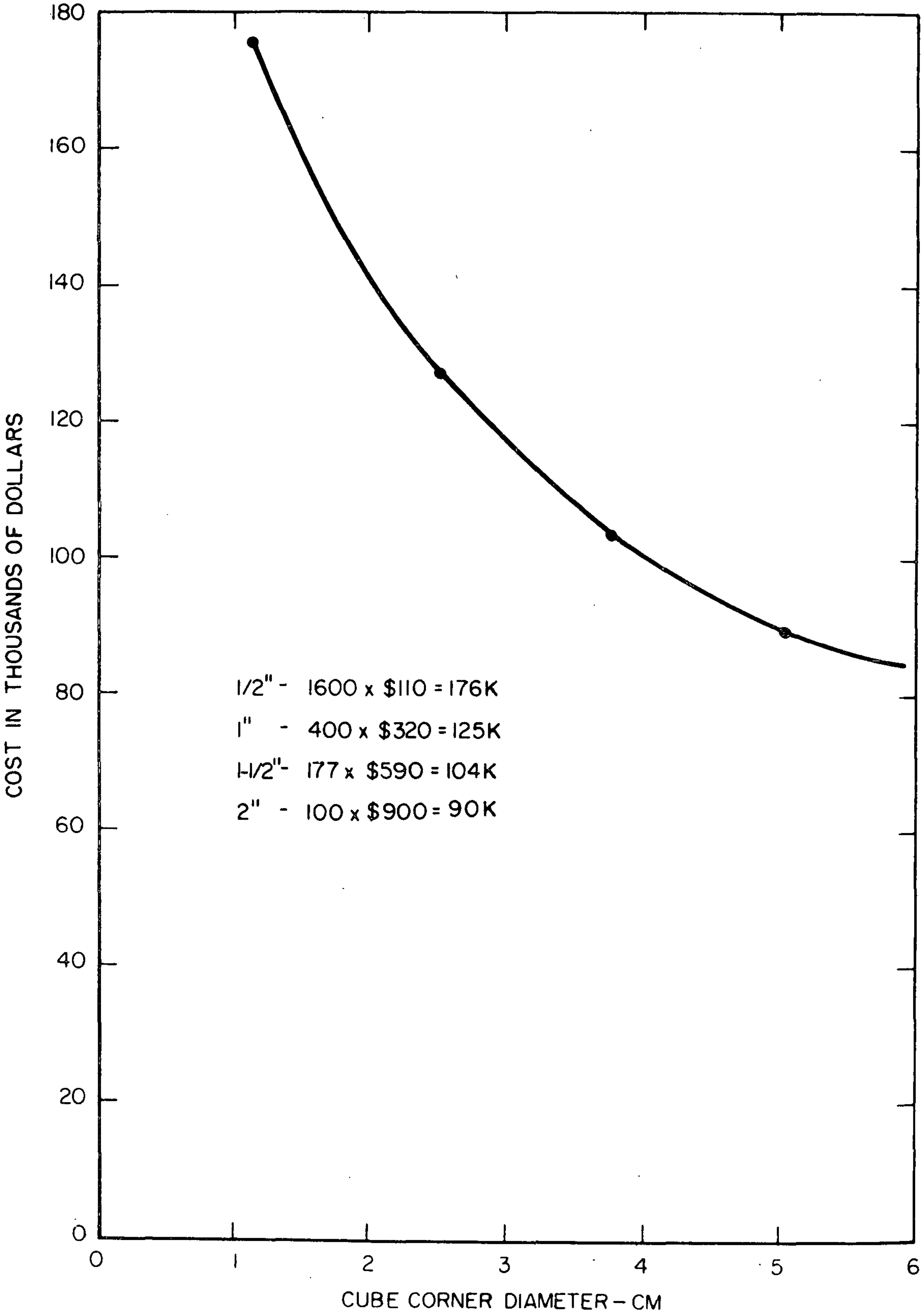
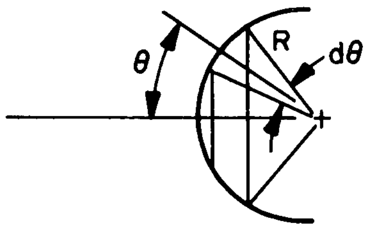


Figure 12. Costs for Equal Area Cube Corner Arrays

APPENDIX A

NORMALIZED INTENSITIES OF A CANNONBALL TYPE RETROFLECTOR ARRAY

This calculation determines the effectiveness of a cannonball type retroreflector compared to the same number (or area) of reflectors mounted on a flat surface



Normalized Intensity of Array

$$I_N = \frac{\int_0^1 2\pi R^2 (1-\theta)^2 \sin\theta \, d\theta}{4\pi R^2} = \frac{1}{2} \int_0^1 (1-\theta)^2 \sin\theta \, d\theta$$

$$= \frac{1}{2}\int_0^1 (1-2\theta+\theta^2)\sin\theta \, d\theta = \frac{1}{2}\int_0^1 \sin\theta \, d\theta - \frac{1}{2}\int_0^1 2\theta \sin\theta \, d\theta + \frac{1}{2}\int_0^1 \theta^2 \sin\theta \, d\theta$$

$$= \frac{\sin 1}{2} - \int_0^1 \theta \sin\theta \, d\theta + \frac{1}{2}\int_0^1 \theta^2 \sin\theta \, d\theta$$

$$= \frac{\sin 1}{2} - [-\theta\cos\theta + \sin\theta]_0^1 + \frac{1}{2}[-\theta^2\cos\theta + 2\theta\sin\theta + 2\cos\theta]_0^1$$

$$= \frac{\sin 1}{2} - [-\cos 1 + \sin 1 - 0] + \frac{1}{2}[-\cos 1 + 2\sin 1 + 2\cos 1 - 2\cos 0]$$

$$= \frac{\sin 1}{2} + \cos 1 - \sin 1 - \frac{\cos 1}{2} + \sin 1 + \cos 1 - 1$$

$$= \frac{\sin 1}{2} + 2 \cos 1 - \frac{\cos 1}{2} - 1 = \frac{\sin 1}{2} + \frac{3}{2} \cos 1 - 1$$

$$J_N = 0.422 + 1.5\,(.540) - 1 = 0.422 + 0.810 - 1 = 1.232 - 1 = 0.232$$

$$\theta = 4S^\circ = 0.785 \text{ rod} \quad \theta^2 = 0.615$$

$$J_N = \frac{\sin 4S}{2} - [-\theta \cos\theta + \sin\theta]_0^{4S} + \frac{1}{2}[-\theta^2 \cos\theta + 2\theta \sin\theta + 2\cos\theta]_0^{4S}$$

$$= \frac{\sqrt{2}}{4} - \left[\frac{-.78S\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0\right] + \frac{1}{2}\left[-\frac{0.615\sqrt{2}}{2} + 0.785\sqrt{2} + \frac{2\sqrt{2}}{2} + 0 - 0 - 2\right]$$

$$= .2S\sqrt{2} - \left[+ .285\frac{\sqrt{2}}{2}\right] + \frac{1}{2}[-0.307\sqrt{2} + 0.785\sqrt{2} + \sqrt{2} - 2]$$

$$= .2S\sqrt{2} + .143\sqrt{2} - .155\sqrt{2} + .392\sqrt{2} + .500\sqrt{2} - 1$$

$$= 1.142\sqrt{2} - .298\sqrt{2} - 1 = .844\sqrt{2} - 1 = 1.195 - 1 = 0.135$$

$$\theta = 28.6 = 0.500 \text{ rod} \;\; \theta^2 = .250 \;\; \sin\theta = .479 \;\; \cos\theta = .878$$

$$I_N = \frac{.479}{2} - [-.500\,(.878) + .479 - 0] + \frac{1}{2}[-.2S\,(.878) + 979 + 2\,(.878) - 2]$$

$$= .24 - [-.439 + .479] + \frac{1}{2}[-.219 + .479 + 1.755 - 2]$$

$$= .29 - (.040) + \frac{1}{2}(.015) = .24 - .04 + .007 = .297 - .04$$

$$I_N = .207$$

APPENDIX B

REFLECTIVE AREA OF A CUBE CORNER

This calculation is to determine the area of a cube corner as a function of angle of incidence of the light. Figure B-1 is a sketch of the dimensions of the cube under analysis. This cube has been optimized for maximum on-axis area to weight ratio by removing the corners of the cube so that the entrance and exit pupils are equilateral hexagons.

The hexagonal forward perimeter of the cube corner forms the entrance pupil, and its image by reflection in the cube forms the exit pupil. When observed from the front at an angle off axis from the line of symmetry it appears as though it were two hexagonal stops whose axes are coincident with the line of symmetry of the prism. (See Figure No. B-2)

To the observer it appears as shown in Figure B-3. (Assuming that the cube corner and observer are immersed in a fluid with an optical index equal to the index of the cube corner.)

The area of the clear area is by inspection

$$A = (w \tan 30°) S + 2 \frac{HS}{2} = S (w \tan 30° + H)$$

$$S = (w - 2D \tan I) \cos I = \left(w - 2 \frac{w}{\sqrt{2}} \tan I\right) \cos I$$

$$= w (1 - \sqrt{2} \tan I) \cos I$$

also

$$H = \frac{w}{2} \tan 30° \frac{S}{w \cos I} = \frac{w}{2} \tan 30° \frac{w (1 - \sqrt{2} \tan I) \cos I}{w \cos I}$$

$$= \frac{w}{2} \tan 30° (1 - \sqrt{2} \tan I)$$

Therefore

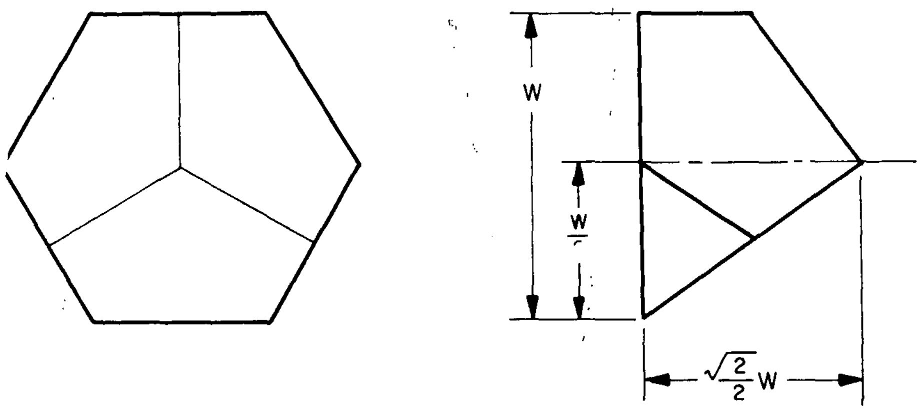


Figure B-1. Geometry of Cube Corner

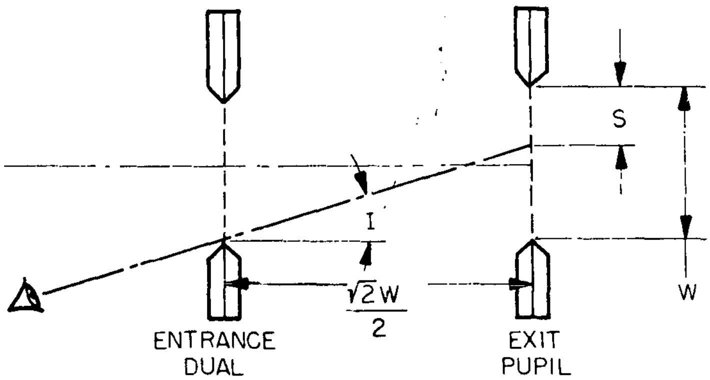


Figure B-2. Entrance-Exit Pupil Geometry

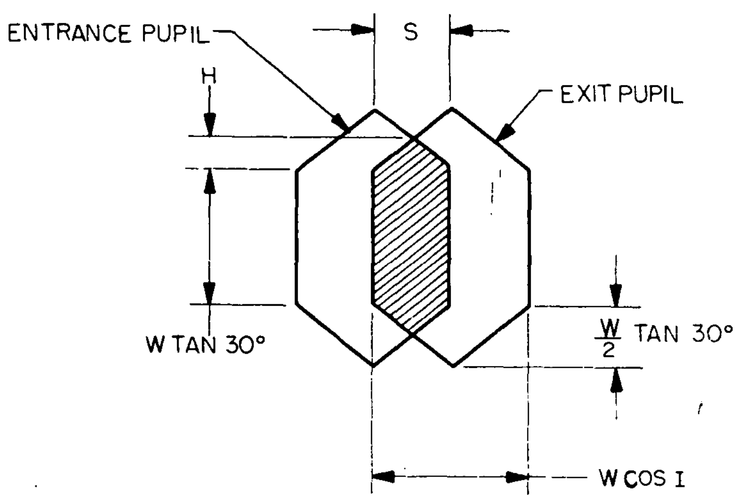


Figure B-3. Entrance-Exit Pupil Areas

$$A = w(1 - \sqrt{2}\tan I)\, w \tan 30° + \frac{w}{2} \tan 30° (1 - \sqrt{2}\tan I) \cos I$$

$$= w^2 \cos I \tan 30° (1 - \sqrt{2}\tan I) \left(\frac{3}{2} - \frac{\tan I}{\sqrt{2}}\right)$$

If the cube corner is made of glass, and is not immersed in a refractive fluid, account of the refraction at the glass air surface must be taken. To do this, I is changed to I' in all parts of the equation except cos I. Therefore

$$A = w^2 \cos I \tan 30° (1 - \sqrt{2}\tan I') \left(\frac{3}{2} - \frac{\sqrt{2}}{2} \tan I'\right)$$

$$= \frac{\sqrt{3}}{6} w^2 \cos I (1 - \sqrt{2}\tan I')(3 - \sqrt{2}\tan I')$$

where

$$I' = \sin^{-1}\left(\frac{\sin I}{N'}\right)$$

At zero angle of incidence this reduces to

$$A = \frac{3}{2} w^2 \tan 30° = \frac{\sqrt{3}}{2} \omega^2$$

This area reaches zero when

$$(1 - \sqrt{2}\tan I') = 0$$

$$\sqrt{2}\tan I' = 1$$

$$\tan I' = \frac{1}{\sqrt{2}}$$

$$I' = 35.2°$$

For mirror type cube corners this is the limit of retrodirectance. For fused silica cube corners ($N' = 1.46$) the angle is extended in

$$I = \sin^{-1}(N' \sin I') = \sin^{-1}(1.46 \times \sin 35.2°)$$

$$= \sin^{-1}(0.841) = 57.3°$$