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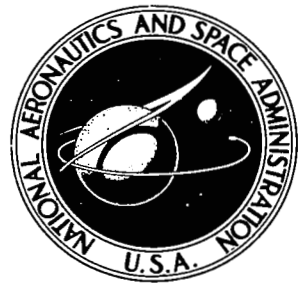

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DISTRIBUTION OF ERROR IN LEAST-SQUARES SOLUTION OF AN OVERDETERMINED SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

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16. Abstract

Probability density functions are derived for the errors in the evaluation of unknowns by the least-squares method in an overdetermined system of nonhomogeneous linear equations. Coefficients of the unknowns are assumed correct and computational precision is assumed. A vector space is used, with a number of dimensions equal to the number of equations. An error vector is defined and assumed to have uniform distribution of orientation throughout the vector space. The density functions are shown to be insensitive to the biasing effects of the source of the system of equations.

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SUMMARY

Probability density functions are derived for errors in the evaluation of unknowns by the least-squares method for a system of n nonhomogeneous linear equations in p unknowns, for n greater than p. The coefficients of the unknowns are assumed to be correct and computational precision is assumed. The treatment deals only with errors due to inaccurate constant terms or due to the existence of unknowns within the physical system that generates the system of equations, which affect the values of the constant terms, and which are omitted from the equations. Columns of the coefficient matrix and the column of constant terms are viewed as vectors in an n-dimensional space. The method involves definition of an error vector and an assumption that the error vector will be randomly oriented, with uniform distribution throughout the n-dimensional space.

A probability density function that gives due regard to the effect of any known or assumed biasing effects associated with the source of the system of equations is shown to be insensitive to those biasing effects. This fact justifies a substantial degree of utility for an approximate density function that is derived without regard to any biasing effect associated with the source.

Possible applications are mentioned for use of the density functions to enhance the protection provided by a warning system in which critical values of unknowns represent the hazards for which warning must be provided.

INTRODUCTION

A least-squares method (ref. 1) is generally used for finding a compromise solution for values of p unknowns from a system of n nonhomogeneous linear equations in which n is greater than p. An analysis of the least-squares method that may have novel aspects will be described here. The analysis involves, in effect, a method of

inversion of a matrix by use of vectors in n-dimensional space. The vector method of matrix inversion serves as an essential groundwork for the method of statistical analysis of probable error that will be presented.

The method of estimation of probable error begins with an assumption that the coefficients of the unknowns are dependable. It is assumed that the constant terms may be more or less undependable either because of inaccuracies in their measurement or because of other effects on the constant terms than the values of the unknowns and their coefficients. The method involves division of the n-dimensional vector space into a p-dimensional and an (n - p)-dimensional subspace. The p-dimensional space is that spanned by the column vectors of the coefficient matrix. The (n - p)-dimensional space is its orthogonal complement. An error vector is defined, and it is shown that the component of the error vector within the (n - p)-dimensional subspace can be evaluated directly. Statistical relations may then be developed between this component of the error and the component within the p-dimensional subspace.

The analysis is performed both without and with regard to the distribution of magnitude of the error to be expected because of characteristics of the source of the system of simultaneous equations. The two results are critically compared.

This analysis was performed for application to a problem in infrared spectroscopy. The problem concerned monitoring of a gas-filled space for detection of traces of specific toxic gases by measurement of the absorption of infrared radiation of various wavelengths. Because of the impossibility of excluding the effects of many unknown gases that might be present in admixture with those to be monitored, a mathematical method was desired for predicting the probable error in the estimation of concentrations of the specific toxic gases, produced by the presence of the unknown gases. The mathematical concepts developed here are part of the method that is under development for that purpose.

SYMBOLS

$[A]$	coefficient matrix of n rows and p columns
$\vec{A}_i$	vector defined by eq. (8)
$\vec{A}'_i$	vector extending in same direction as $\vec{A}'_{ui}$, but with magnitude such as to yield unit dot product with vector $\vec{A}_i$
$\vec{A}_s$	any specific unit vector
$\vec{A}_{ui}$	unit vector directed same as vector $\vec{A}_i$
$\vec{A}'_{ui}$	unit vector extending in same direction as component of $\vec{A}_{ui}$ orthogonal to each $\vec{A}_{uk}$ vector for $k \neq i$

$\vec{A}^*_{ui}$	an intermediate vector used in Gram-Schmidt orthogonalization
a_{ji}	coefficient within j^{th} equation for i^{th} unknown
[B]	matrix defined by eq. (4)
[C]	matrix defined by eq. (6)
$\vec{E}$	error vector defined by eq. (18)
E[ ]	expected value of any argument within brackets
$\vec{E}_i$	component of error vector $\vec{E}$ lying outside subspace S_o but within subspace S_{o+i}
$\vec{E}_o$	component of error vector $\vec{E}$ lying within subspace S_o
$\vec{E}_{o+i}$	component of error vector $\vec{E}$ lying within subspace S_{o+i}
$\vec{E}_p$	component of error vector $\vec{E}$ lying within subspace S_p
F( )	distribution function of random variable appearing within parentheses
f( )	density function for random variable appearing within parentheses
f(x|y)	density function of any random variable x when given y
g( )	density function for random variable appearing within parentheses (used instead of f( ) only when desirable to avoid confusion)
g(x|y)	density function of random variable x when given y (used instead of f(x|y) only when desirable to avoid confusion)
j	(subscript) order number of an equation within overdetermined system
$\vec{M}$	measured vector, defined by eq. (9)
[M]	column matrix of constant (measured) terms
$\vec{M}_o$	component of vector $\vec{M}$ within subspace S_o
$\vec{M}_p$	component of vector $\vec{M}$ within subspace S_p
m	(subscripted) constant (or measured) term in equation designated by subscript
n	number of simultaneous equations in overdetermined system
$\vec{O}_k$	k^{th} member of a set of (n-p) orthogonal unit vectors spanning subspace S_o
$\vec{P}_k$	k^{th} member of a set of p orthogonal unit vectors spanning subspace S_p
p	number of unknowns in overdetermined system of equations
$\vec{R}$	any unit vector randomly oriented with uniform distribution throughout n-dimensional space

r	ratio of $\lvert\vec{E}_i\rvert$ to $\lvert\vec{E}_o\rvert$
S_n	total n-dimensional space
S_o	(n - p)-dimensional subspace orthogonal to S_p
S_{o+i}	(n - p + 1)-dimensional subspace consisting of subspace S_o plus the direction of $\vec{A}_i'$ vector
S_p	p-dimensional subspace spanned by $\vec{A}_i$ vectors (i = 1 to p)
$\vec{u}_j$	j^{th} member of orthogonal set of n unit vectors spanning n-dimensional space
$[X]$	column matrix of unknowns
$[X_c]$	column matrix of unknowns to be calculated by least-squares method
X_j	magnitude of component of vector $\vec{R}$ in direction of $\vec{u}_j$
x_{ci}	value of i^{th} unknown as calculated by least-squares method
x_i	i^{th} unknown in overdetermined system of equations
x_{ti}	postulated true value of i^{th} unknown
α_i	angle between vectors $\vec{A}_i'$ and $\vec{A}_i$
β	absolute value of dot product of a specific unit vector $\vec{A}_s$ and a unit vector $\vec{R}$ randomly oriented with uniform distribution
ϵ_j	component of error vector so defined as to make j^{th} equation exact as illustrated in eq. (17)
η_{ai}	algebraic error in calculation of unknown designated by subscript i by least-squares method, defined by eq. (19)
η_i	fractional error defined by eq. (51)
θ_{o+i}	angle between vectors $\vec{E}_{o+i}$ and $\vec{A}_i'$
ξ_i	limit of integration defined by eq. (C6)
σ_{η_i}	standard deviation of η_i

DESCRIPTION OF METHODS OF SOLUTION

The least-squares method of solution of an overdetermined system in its usual matrix form will first be described. Then the vector method will be described and will be used as a foundation to describe the method of estimation of probable error.

Matrix Form of Least-Squares Method

We assume the existence of a system of equations, derived from whatever sources, as follows:

$$\sum_{i=1}^{p} a_{ji}x_i = m_j \qquad \begin{pmatrix} j = 1 \text{ to } n \\ n > p \end{pmatrix} \tag{1}$$

In equations (1), p is the number of unknowns x_i for which values are to be estimated, j identifies a particular equation within the system of n equations, the a_{ji} values are known constant coefficients, and the m_j values are known constant terms.

Equations (1) rewritten in matrix form are

$$[A]\,[X] = [M] \tag{2}$$

Here [A] is a coefficient matrix including only the a_{ji} values in the same column and row arrangement as in equations (1). [X] is a column matrix including only the x_i variables. [M] is a column matrix containing the values of m_j. The column vector $\vec{M}$ or the matrix [M] will be referred to hereafter as the measured vector or the measured matrix.

In the matrix form of the least-squares method of solving equations (1), or equation (2), the following steps are now set forth in detail in order to define certain symbolisms and intermediate relations that will be used later.

(1) Equation (2) is converted to

$$[X_c] = [B]^{-1}[A]^T[M] \tag{3}$$

where

$$[B] = [A]^T[A] \tag{4}$$

$[X_c]$ now represents values x_{ci} calculated by the least-squares method, with the subscript c to distinguish the calculated values from postulated true values x_{ti} that will later be introduced.

(2) Equation (3) is converted to

$$[X_c] = [C]\,[M] \tag{5}$$

where

$$[C] = [B]^{-1}[A]^T \tag{6}$$

Vector Form of Least-Squares Method

In the vector form of the least-squares method an orthogonal set of unit coordinate vectors $\vec{u}_j$ is assumed, spanning an n-dimensional space. Each of the n equations (1) is interpreted as a vector equation confined to a single dimension in the direction of the $\vec{u}_j$ coordinate. That is, the j^{th} equation (1) may be represented by

$$a_{j1}x_1\vec{u}_j + a_{j2}x_2\vec{u}_j + a_{j3}x_3\vec{u}_j + \cdots + a_{jp}x_p\vec{u}_j = m_j\vec{u}_j \tag{7}$$

Thus the left sides, or the right sides, of equations (1) become an orthogonal set of vectors within the n-dimensional space.

Vectors are now defined as follows:

$$\vec{A}_i = \sum_{j=1}^{n} a_{ji}\vec{u}_j \tag{8}$$

$$\vec{M} = \sum_{j=1}^{n} m_j\vec{u}_j \tag{9}$$

The vectors $\vec{A}_i$(i = 1 to p) must be linearly independent to allow a least-squares solution by the vector method. Nonsingularity of matrix [B] (eq. (4)) implies linear independence of the vectors $\vec{A}_i$.

From equations (7) and the defining equations (8) and (9), we see that

$$\sum_{i=1}^{p} x_i\vec{A}_i = \vec{M} \tag{10}$$

Normalization of the $\vec{A}_i$ vectors to unity gives

$$\vec{A}_{ui} = \frac{\vec{A}_i}{|\vec{A}_i|} = \frac{\sum_{j=1}^{n} a_{ji}\vec{u}_j}{\left(\sum_{j=1}^{n} a_{ji}^2\right)^{1/2}} \tag{11}$$

We now develop a set of unit vectors $\vec{A}'_{ui}$ each of which extends in the same direc-

tion as the component of $\vec{A}_{ui}$ that is orthogonal to each $\vec{A}_{uk}$ for $k \neq i$. Each $\vec{A}'_{ui}$ vector individually is the end result of a Gram-Schmidt orthogonalization (ref. 2), though collectively they are not the set of mutually orthogonal vectors obtained from a single Gram-Schmidt orthogonalization. A different Gram-Schmidt orthogonalization must be performed for the determination of each $\vec{A}'_{ui}$ vector, though to varying extents parts of the orthogonalizations may be performed in common. For determination of a particular $\vec{A}'_{ui}$ vector, the Gram-Schmidt orthogonalization may be performed in any order, with only one exception; the first $p - 1$ mutually orthogonal vectors obtained must span the same $(p - 1)$-dimensional subspace to which are confined all the $\vec{A}_{uk}$ vectors for $k \neq i$. The final (p^{th}) member of the mutually orthogonal set must then be the component of the $\vec{A}_{ui}$ vector that is orthogonal to the $(p - 1)$-dimensional subspace, normalized to unity. That p^{th} member of the set, normalized, will be the $\vec{A}'_{ui}$ vector.

An example for the procedure of finding the vector $\vec{A}'_{ui}$, with $n = 4$, $p = 3$, and $i = 1$, is given in appendix A.

We now define a set of vectors

$$\vec{A}'_i = \frac{\vec{A}'_{ui}}{\vec{A}_i \cdot \vec{A}'_{ui}} \qquad (i = 1 \text{ to } p) \tag{12}$$

From equation (10),

$$\vec{A}'_i \cdot \vec{M} = \sum_{k=1}^{p} x_i \vec{A}'_i \cdot \vec{A}_k \tag{13}$$

But since, for $k \neq i$, the vector $\vec{A}'_i$ is orthogonal to the vector $\vec{A}_k$, equation (13) may be rewritten as

$$\vec{A}'_i \cdot \vec{M} = x_i \vec{A}'_i \cdot \vec{A}_i \tag{14}$$

From equation (12) we see that

$$\vec{A}'_i \cdot \vec{A}_i = 1 \tag{15}$$

by definition, so that equation (14) is equivalent to

$$x_{ci} = \vec{A}'_i \cdot \vec{M} \tag{16}$$

The subscript ci is used in equation (16) to indicate that the equation yields values of x_i calculated by the least-squares method. It can be shown that the $\vec{A}_i'$ vectors are always identical with the row vectors of matrix [C] in equation (5).

A sample solution of a system of equations by both matrix and vector methods appears in appendix B. Included (eqs. (B3) and (B8)) is an example of the fact the $\vec{A}_i'$ vectors are identical with the row vectors of matrix [C].

STATISTICAL STUDY OF ERROR

We assume that the source of equations (1) involved effects due to the existence of a true value of each x_i, which we denote with the symbol x_{ti}. We assume each a_{ji} in equations (1) to be a precisely known constant. The m_j values (often results of measurements in a physical system) we assume to be unreliable for one or the other or both of two reasons: (1) The physical system that generates equations (1) may actually involve the effects of more than p unknowns. (2) The entities that are measured to obtain the values m_j may not be measured with great accuracy. As an example of the first reason, the p values x_{ti} might be the concentrations of p gases such as CO, CO_2, H_2O, and so on, within a mixture of gases. But additional gases could be present within the same mixture and could affect the entities that are measured to obtain the values m_j. The presence of those other gases within the mixture might be unknown or might be deliberately neglected. We may rewrite equations (1) as

$$\sum_{i=1}^{p} a_{ji}x_{ti} + \epsilon_j = m_j \tag{17}$$

where unknowns ϵ_j are included and are now defined as having such values as may be necessary in order to make the equations exact. It is, of course, to be expected that the ϵ_j values will usually be nonzero. For, if replacement of all x_i's in equations (1) by true values x_{ti} did not result in inequalities, there would be no point in a least-squares solution of the overdetermined system. That is, if inequalities were not created, one could as well accept p of the n equations at random and reject the rest. We now define an error vector as

$$\vec{E} = \sum_{j=1}^{n} \epsilon_j \vec{u}_j \tag{18}$$

We now wish to examine the statistical effect of the error vector $\vec{E}$ on the interrelation

between the x_{ci} and the x_{ti} values, that is, the influence of $\vec{E}$ upon an algebraic error η_{ai} defined as

$$\eta_{ai} = x_{ci} - x_{ti} \tag{19}$$

With use of equations (17) instead of (1), equations (13) and (14) become

$$\vec{A}_i' \cdot \vec{M} = \sum_{k=1}^{p} x_{ti} \vec{A}_i' \cdot \vec{A}_k + \vec{A}_i' \cdot \vec{E} = x_{ti} \vec{A}_i' \cdot \vec{A}_i + \vec{A}_i' \cdot \vec{E} \tag{20}$$

Equation (16) properly becomes

$$x_{ti} = \vec{A}_i' \cdot \vec{M} - \vec{A}_i' \cdot \vec{E} = x_{ci} - \vec{A}_i' \cdot \vec{E} \tag{21}$$

From equation (21) and the defining equation (19),

$$\eta_{ai} = \vec{A}_i' \cdot \vec{E} \tag{22}$$

Statistical distributions can be derived for the algebraic error η_{ai}, under a basic assumption that the vector $\vec{E}$ is randomly oriented, with uniform distribution throughout all the possible orientations within the n-dimensional space. These derivations are possible because a component of $\vec{E}$ is uniquely determined by equations (17).

Determination of a Component of Error Vector

The full n-dimensional space, which will be referred to hereafter as S_n, may be resolved into two mutually orthogonal subspaces S_p and S_o. The subspace S_p is p-dimensional, and is so defined that the vector $\vec{A}_i$ and, hence, $\vec{A}_i'$ lie exclusively within it for any value of i. The subspace S_o is (n - p)-dimensional. Let the component of $\vec{E}$ lying within subspace S_o be denoted by $\vec{E}_o$. As will be shown, this component is uniquely determined both as to magnitude and direction by equations (17). Neither magnitude nor direction of $\vec{E}_p$, the component of $\vec{E}$ within the subspace S_p, can be determined. However, with $|\vec{E}_o|$ known, useful statistical relations can be developed.

For any value of i, we will also consider an (n - p + 1)-dimensional subspace S_{o+i} that will include the entire subspace S_o and, in addition, the direction of the $\vec{A}_i'$ vector. We will consider the interrelations of $\vec{E}_o$, $\vec{E}_i$ (the component of $\vec{E}$ lying outside subspace S_o but within subspace S_{o+i}, i.e., in the direction of $\vec{A}_i'$), and $\vec{E}_{o+i}$ (the compo-

nent of $\vec{E}$ lying within subspace S_{o+i}). Thus,

$$\left|\vec{E}_o\right|^2 = \left|\vec{E}_{o+i}\right|^2 - \left|\vec{E}_i\right|^2 \tag{23}$$

With $|\vec{E}_o|$ known exactly, we will show that a statistical distribution of η_{ai} may be derived.

From equations (8), (9), (17), and (18)

$$\sum_{i=1}^{p} \vec{A}_i x_{ti} + \vec{E} = \vec{M} \tag{24}$$

or

$$\sum_{i=1}^{p} \vec{A}_i x_{ti} + \vec{E}_p + \vec{E}_o = \vec{M}_p + \vec{M}_o \tag{25}$$

where $\vec{E}_p$ and $\vec{M}_p$ are respectively the components of $\vec{E}$ and $\vec{M}$ within the subspace S_p, and $\vec{M}_o$ is the component of $\vec{M}$ within the subspace S_o.

Because of the mutual orthogonality of S_p and S_o, we may separate equation (25) into two equations,

$$\sum_{i=1}^{p} \vec{A}_i x_{ti} + \vec{E}_p = \vec{M}_p \tag{26}$$

and

$$\vec{M}_o = \vec{E}_o \tag{27}$$

We see from equation (27) that with the vector $\vec{M}$ and a set of mutually orthogonal unit vectors spanning the subspace S_o we could determine the vector component $\vec{E}_o$ directly. That is, if we denote that orthogonal set of unit vectors as $\vec{O}_k (k = 1 \text{ to } n - p)$,

$$\vec{E}_o = \vec{M}_o = \sum_{k=1}^{n-p} \vec{M} \cdot \vec{O}_k \vec{O}_k \tag{28}$$

In many cases, much saving of computation time may be effected with use of an indirect method suggested in personal communication from Lynn U. Albers. This method uses a set of mutually orthogonal unit vectors spanning the S_p subspace, which we denote by

$\vec{P}_k$(k = 1 to p). The vector component $\vec{M}_p$ is determined and subtracted from the vector $\vec{M}$ to give $\vec{M}_o$ or $\vec{E}_o$. That is,

$$\vec{E}_o = \vec{M}_o = \vec{M} - \vec{M}_p = \vec{M} - \sum_{k=1}^{p} \vec{M} \cdot \vec{P}_k \vec{P}_k \tag{29}$$

The $\vec{P}_k$ vectors may be obtained by a Gram-Schmidt orthogonalization of the $\vec{A}_i$ vectors. In the example presented in appendix A, with n = 4 and p = 3, the $\vec{P}_k$ vectors might be

$$\left.\begin{aligned} \vec{P}_1 &= \vec{A}'_{u1} \quad \text{(from eq. (A4))} \\ \vec{P}_2 &= \vec{A}^*_{u2} \quad \text{(from eq. (A2))} \\ \vec{P}_3 &= \vec{A}_{u3} \quad \text{(from eq. (A1))} \end{aligned}\right\} \tag{30}$$

The $\vec{O}_k$ vectors may be found by the following procedure:

(1) Select a set of n - p unit vectors $\vec{U}_k$ randomly oriented within the space S_n. Alternatively, use $\vec{U}_1 = \vec{u}_1$, $\vec{U}_2 = \vec{u}_2$, and so on.

(2) Continue the Gram-Schmidt orthogonalization by which the $\vec{P}_k$ vectors were obtained to include the $\vec{U}_k$ vectors for k = 1 to n - p.

(3) Denote the n - p additional unit vectors resulting from the continuation of the Gram-Schmidt orthogonalization by $\vec{O}_k$(k = 1 to n - p).

Use of the $\vec{P}_k$ vectors, with equation (29), would involve much less computation in any case where a single set of simultaneous equations is to be solved by the least-squares method. In cases where many sets of overdetermined simultaneous equations are to be solved, always with the same $\vec{A}_i$ vectors but with different $\vec{M}$ vectors, and with n - p smaller than p, the method using the $\vec{O}_k$ vectors and equation (28) might involve less computation.

Probability Density Function for Ratio of Magnitudes of Two Components of Error Vector

We designate θ_{o+i} as the angle between $\vec{E}_{o+i}$ and $\vec{A}'_i$. We see that

$$\left|\vec{E}_i\right| = \left|\vec{A}'_{ui} \cdot \vec{E}\right| = \left|\vec{E}_{o+i}\right| \cos \theta_{o+i} \tag{31}$$

and that

$$|\vec{E}_o| = |\vec{E}_{o+i}| \sin \theta_{o+i} \tag{32}$$

Thus

$$\frac{|\vec{E}_i|}{|\vec{E}_o|} = \frac{\cos \theta_{o+i}}{\sin \theta_{o+i}} \tag{33}$$

Because the ratio $|\vec{E}_i| / |\vec{E}_o|$ will be used repeatedly, we introduce here the symbol

$$r = \frac{|\vec{E}_i|}{|\vec{E}_o|} \tag{34}$$

Later we shall also need the following relation, from equations (23) and (34):

$$|\vec{E}_{o+i}| = \sqrt{|\vec{E}_o|^2 + |\vec{E}_i|^2} = |\vec{E}_o| \sqrt{1 + r^2} \tag{35}$$

It can be shown easily that the basic assumption, that the orientation of $\vec{E}$ is uniformly distributed within the space S_n, implies that the orientation of $\vec{E}_{o+i}$ is uniformly distributed within the S_{o+i} subspace. Under that assumption, we now wish to derive a probability density function for r of equation (34). We will do so disregarding for the moment the fact that $|\vec{E}_o|$ is known. We will make the further assumption that the magnitude of $\vec{E}$ and the orientation of $\vec{E}$ are independently distributed and, hence, that the magnitude of $\vec{E}$ and the value of r are independently distributed.

We will derive the density function for r indirectly, using the density function for the absolute value of the dot product of a fixed or specific unit vector $\vec{A}_s$ with another unit vector $\vec{R}$ that is randomly oriented, with uniform distribution, within an n-dimensional space. For convenience, we will use the symbol

$$\beta = |\vec{A}_s \cdot \vec{R}| \tag{36}$$

With such notation, appendix C presents a derivation of the following equation for the expected value $E[\beta]$:

$$E[\beta] = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \left(\frac{2}{\pi}\right)^s \prod_{k=1}^{n-1} k^{(-1)^{s+k+1}} \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \end{pmatrix} \tag{C12}$$

The density function for β, also derived in appendix C, is

$$f(\beta) = \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-2} k^{(-1)^{s+k+1}}\right] \left(1 - \beta^2\right)^{(n-3)/2} \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \end{pmatrix} \tag{C16}$$

Other functions of β derived in appendix C, but which will not be needed here, are the distribution function $F(\beta)$ (eq. (C18)) and, for the special case of $n = 100$, an approximation of $f(\beta)$ as one side of a normal distribution (eq. (C19)). All these equations for functions of β will also apply if $\vec{A}_s$ is a randomly oriented unit vector according to any type of distribution, so long as $\vec{R}$ has the uniform distribution.

Equation (C16) gives the density function for $\cos \theta_{o+i}$ upon substitution of $n - p + 1$ for n, and substitution of $\cos \theta_{o+i}$ for β. From the density function for $\cos \theta_{o+i}$ we can deduce the density function for r. For that purpose, and for several later applications, we will need the standard formula for change of variable in the probability density function,

$$g(y) = f(x) \left| \frac{dx}{dy} \right| \tag{37}$$

where y is a monotone increasing or decreasing function of x, $g(y)$ is the probability density function for y, and $f(x)$ is the density function for x. If it is wished to eliminate x from the expression obtained for $g(y)$, x may, of course, be replaced by its equivalent in terms of y. (See ref. 3 or other text on mathematical statistics for derivation of eq. (37).)

From equations (33), (34), (C16), and (37), the probability density function of r is

$$g(r) = \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-p-1} k^{(-1)^{s+k+1}}\right] \left(1 - \cos^2\theta_{o+i}\right)^{(n-p-2)/2} \left|\frac{d\cos\theta_{o+i}}{dr}\right|$$

$$= \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-p-1} k^{(-1)^{s+k+1}}\right] \sin^{n-p+1}\theta_{o+i}$$

$$= \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-p-1} k^{(-1)^{s+k+1}}\right] \left(1 + r^2\right)^{(p-n-1)/2} \qquad \begin{pmatrix} s = 1 \text{ for } n - p + 1 \text{ even} \\ s = 0 \text{ for } n - p + 1 \text{ odd} \end{pmatrix} \tag{38}$$

Values of $g(r)$ according to equation (38), for $n = 100$ and $p = 10$, are plotted in figure 1. The plotted points fit, within the plotting accuracy, the following probability density function:

$$g(r) = 7.5484 \exp\left[-\frac{1}{2}\left(\frac{r}{0.1057}\right)^2\right] \tag{39}$$

Note that the standard deviation 0.1057, as it appears in equation (39), is approximately $(n - p + 1)^{-1/2}$. This condition should not be expected for small $n - p$.

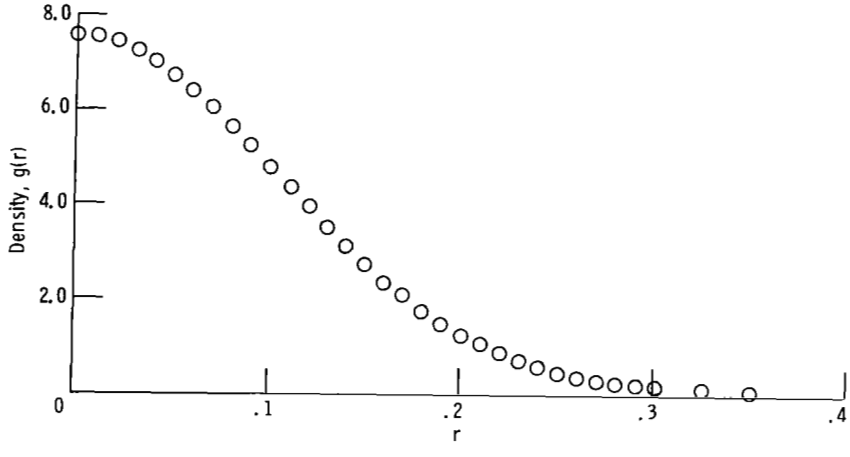


Figure 1. - Density function for r with prior density function of $|\vec{E}_{o+1}|$ disregarded for $n = 100$ and $p = 10$.

Probability Density Function for Algebraic Error

An approximate expression for the probability density function of η_{ai} given $|\vec{E}_o|$ may now be obtained with use of equation (22) rewritten with use of equation (34) as

$$\eta_{ai} = |\vec{A}_i'|\ \vec{A}'_{ui} \cdot \vec{E} = \pm|\vec{A}_i'|\ |\vec{E}_i| = \pm r|\vec{A}_i'|\ |\vec{E}_o| \tag{40}$$

and with use of equations (37) and (38) or (39), with due allowance for the fact the density function of η_{ai} must be normalized for both plus and minus values rather than for plus only as with equations (38) and (39). An assumption will be made that the density function of r given $|\vec{E}_o|$ approximates the density function of r. That is,

$$f\left(r \middle| |\vec{E}_o|\right) \cong g(r) \tag{41}$$

The accuracy of this approximation will be examined later. With use of the relations indicated, and with normalization for both positive and negative values of η_{ai},

$$f\left(\eta_{ai} \middle| |\vec{E}_o|\right) = \frac{1}{2} f\left(r \middle| |\vec{E}_o|\right) \left| \left(\frac{\partial r}{\partial \eta_{ai}}\right)_{|\vec{E}_o|} \right|$$

$$\cong \frac{1}{2} g(r) \frac{1}{|\vec{A}_i'|\ |\vec{E}_o|}$$

$$\cong \frac{1}{2|\vec{A}_i'|\ |\vec{E}_o|} \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-p-1} k^{(-1)^{s+k+1}}\right] \left(1 + \frac{\eta_{ai}^2}{|\vec{A}_i'|^2 |\vec{E}_o|^2}\right)^{(p-n-1)/2}$$

$$\begin{pmatrix} s = 1 \text{ for } n - p + 1 \text{ even} \\ s = 0 \text{ for } n - p + 1 \text{ odd} \\ -\infty < \eta_{ai} < \infty \\ 0 \le r < \infty \end{pmatrix} \tag{42}$$

or, according to equation (39), for $n = 100$ and $p = 10$,

$$f\left(\eta_{ai} \,\middle|\, |\vec{E}_o|\right) \cong \frac{3.7742}{|\vec{A}_i'|\ |\vec{E}_o|} \exp\left[-\frac{1}{2}\left(\frac{\eta_{ai}}{0.1057\ |\vec{A}_i'|\ |\vec{E}_o|}\right)^2\right] \tag{43}$$

Here also note that the standard deviation approximates $|\vec{A}_i'|\ |\vec{E}_o|\ (n - p + 1)^{-1/2}$ for large $n - p$.

In order to derive an exact equation corresponding to the approximate equation (42), a prior distribution represented by a density function $f\left(|\vec{E}_{o+i}|\right)$ must be considered. This prior distribution depends on the source of equations (1). It relates to the probability of existence of any given value of $|\vec{E}_{o+i}|$ if the values of r and $|\vec{E}_o|$ are unknown or disregarded. An equation for $f\left(r \,\middle|\, |\vec{E}_o|\right)$ and an exact equation corresponding to the approximate equation (42) are derived in appendix D with due regard to the prior density function $f\left(|\vec{E}_{o+i}|\right)$. They are as follows:

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \frac{\left(1 + r^2\right)^{(p-n)/2} f\left(|\vec{E}_{o+i}|\right)\Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}}}{\int_0^\infty \left(1 + r^2\right)^{(p-n)/2} f\left(|\vec{E}_{o+i}|\right)\Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}} dr} \tag{D7}$$

$$f\left(\eta_{ai} \,\middle|\, |\vec{E}_o|\right) = \frac{1}{2|\vec{A}_i'|\ |\vec{E}_o|} f\left(r \,\middle|\, |\vec{E}_o|\right)\Big|_{r = |\eta_{ai}| / \left(|\vec{A}_i'|\ |\vec{E}_o|\right)} \tag{D8}$$

The density function $g(r)$ according to equation (38) and the density function $f\left(r \,\middle|\, |\vec{E}_o|\right)$ according to equation (D7) are identical if the density function of $\ln |\vec{E}_{o+i}|$ is made equal to a constant. Jeffreys (ref. 4) argues that, in the absence of any information to the contrary, the unbiased distribution for a random variable constrained to non-negative values should be that giving uniform distribution to its logarithm. The identity of the right-hand sides of equations (38) and (D7) when the density function is uniform for $\ln |\vec{E}_{o+i}|$ is in harmony with that opinion.

With a given physical system that generates the same system of equations (1) many times, but with different m_j values from one time to another, a histogram could be

constructed for the frequencies of $|\vec{E}_o|$ values determined from time to time with use of equation (28) or (29). From the histogram, an expression for the density function $f\left(|\vec{E}_o|\right)$ could be found empirically. From that density function $f\left(|\vec{E}_o|\right)$, a density function for $f\left(|\vec{E}_{o+i}|\right)$ could be derived for use in equations (D7) and (D8).

Density Function of r as an Approximation for Density Function of r Given $|\vec{E}_o|$

We will now use equation (D7) to obtain an estimate of the degree of accuracy of equation (41). In order to do so, we will assume that $|\vec{E}_{o+i}|$ is distributed as the positive side of a standard normal distribution curve. That is, the following density function is assumed:

$$f\left(|\vec{E}_{o+i}|\right) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}|\vec{E}_{o+i}|^2\right) \tag{44}$$

Under that assumption, equation (D7) becomes

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \frac{\left(1+r^2\right)^{(p-n)/2} \exp\left[-\frac{1}{2}\left(|\vec{E}_o|\sqrt{1+r^2}\right)^2\right]}{\int_0^\infty \left(1+r^2\right)^{(p-n)/2} \exp\left[-\frac{1}{2}\left(|\vec{E}_o|\sqrt{1+r^2}\right)^2\right] dr} \tag{45}$$

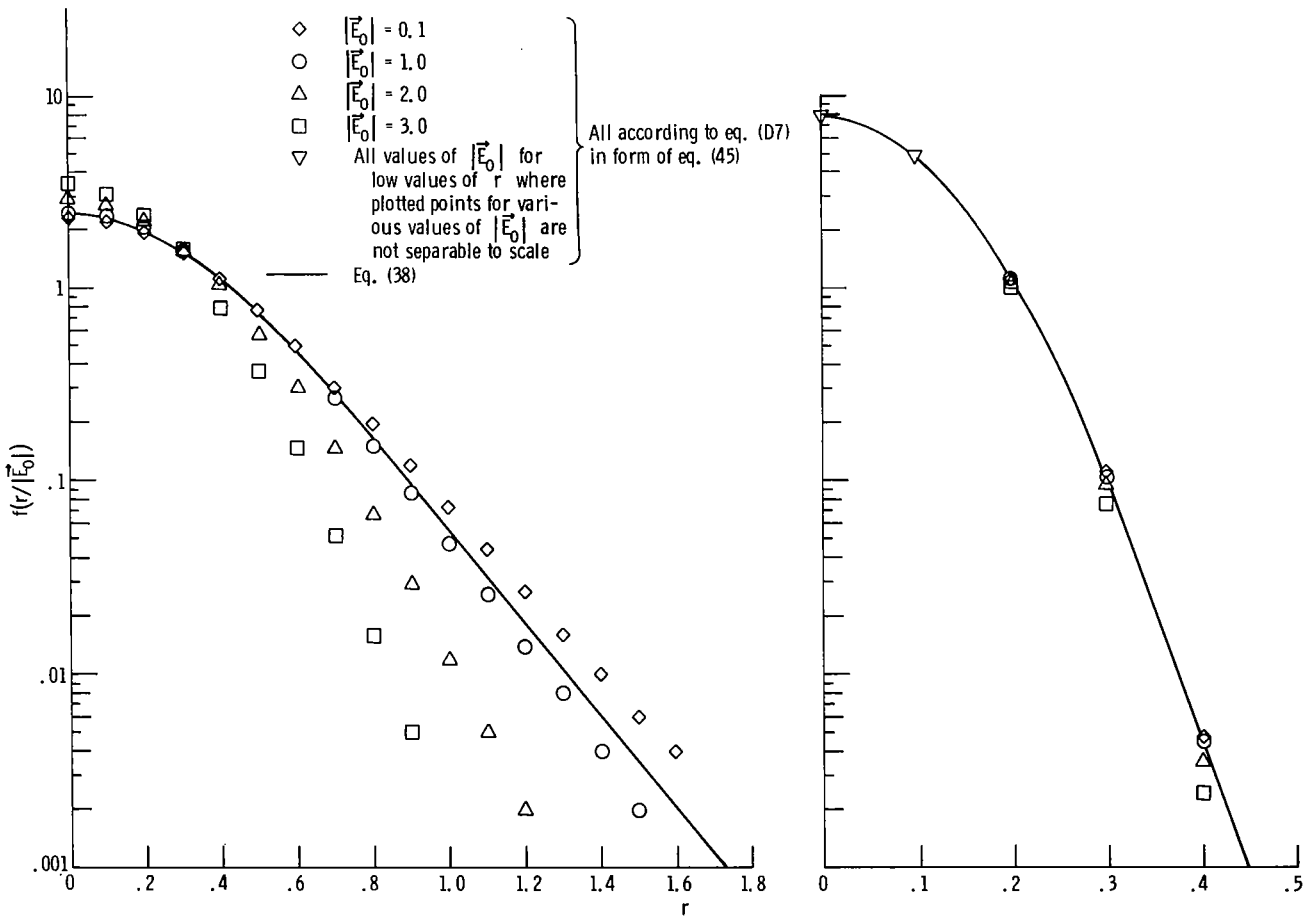
The solid curve in figure 2(a) represents values of density function plotted on a logarithmic scale against values of r according to equation (38) for n - p = 10. Also shown for the same value of n - p, as the plotted points, are probability densities according to equation (45) for values of $|\vec{E}_o|$ equal to 0.1, 1.0, 2.0, and 3.0. These values of $|\vec{E}_o|$, of course, can be considered as multiples of the standard deviation of $|\vec{E}_{o+i}|$, which by equation (44) has been assumed to be one.

From the figure we see that, for n - p = 10, in the use of g(r) according to equation (38) as a substitute for $g\left(r \,\middle|\, |\vec{E}_o|\right)$ according to equation (D7):

(1) Equation (38) is accurate within a range from 30-percent underestimate to 100-percent overestimate of the true value of the density function, for all values $0.1 \leq |\vec{E}_o| < 3.0$, for values of $r < 0.5$.

(2) Equation (38) is even more accurate for all values of $r < 1.0$ and values $0.1 \leq |\vec{E}_o| < 1.0$.

(3) For large $|\vec{E}_o|$ and large r, where equation (38) is not very accurate, it is at least conservative in the sense that it overestimates the density function.



(a) n - p = 10.

(b) n - p = 100.

Figure 2. - Comparison of results according to approximate equation (38) with results according to exact equation (D7).

Figure 2(b) is a similar plot for $n - p = 100$. It shows that equation (38) is a good approximation except for the combination of conditions $|\vec{E}_o| > 2.0$ and $r \geq 0.3$. The fact that this combination of conditions is relatively rare may be seen from the ordinates in the figure. It is clear from comparison of the two parts of figure 2 that equation (38) is increasingly exact for use as if it were an expression for $g\left(r \middle| |E_o|\right)$ instead of an expression for $g(r)$ with increasing value of $n - p$.

A numerical example, with use of equations (38) and (42), for $n = 6$ and $p = 3$, is included in appendix E. Also reported in that appendix are the statistical results of least-squares solutions for 10 000 similar systems with random values of a_{ji}, ϵ_j, and x_{ti}. The results of the 10 000 solutions are plotted and compared there with a curve representing equation (38). The results from the random problems agree with equation (38) within the plotting accuracy. The results show, therefore, that with the density function $f\left(|\vec{E}_{o+i}|\right)$ that existed for the method of random selection of values of ϵ_j, equation (38) gives on the average quite accurate values of $g\left(r \middle| |\vec{E}_o|\right)$ even with $n - p + 1$ as small as four.

To the same extent that $g(r)$ may be used as an approximation for $g\left(r \middle| |\vec{E}_o|\right)$, equation (42) may be used as an approximation for equation (D8). This condition is true because equation (41) was used in derivation of equation (42).

Statistical Considerations Regarding an Anticipated System of Equations

Relative to guidance of design for a physical system that is expected to generate an overdetermined system of simultaneous equations, we now wish to consider some statistical relations that may be expected within the system of equations before it has been generated.

We postulate that the following details only are known regarding the system of equations (17) that are expected to be generated:

(1) The values of n and p

(2) All of the a_{ji} values

(3) Possible or probable x_{ti} values

(4) Possible or probable values of $|\vec{E}|$ (eq. (18)), which may be inferred from known or postulated facts regarding the potential sources of error

In particular, no values of m_j as yet exist and, in fact, no overdetermined system of simultaneous equations yet exists.

We wish to consider the magnitudes of the algebraic errors η_{ai} that may be anticipated when the simultaneous equations have been generated and solved, in relation to the anticipated magnitudes of x_{ti} and $\vec{E}$.

From equations (11), (12), (15), and (22),

$$|\eta_{ai}| = |\vec{A}'_i \cdot \vec{E}| = |\vec{A}'_i|\,|\vec{E}| \left| \vec{A}'_{ui} \cdot \frac{\vec{E}}{|\vec{E}|} \right| = \frac{|\vec{E}| \left| \vec{A}'_{ui} \cdot \frac{\vec{E}}{|\vec{E}|} \right|}{|\vec{A}_i|\,|\vec{A}'_{ui} \cdot \vec{A}_{ui}} \tag{46}$$

The denominator in the final form of equation (46) is necessarily positive (eq. (15)). Note that this equation applies in general without need for specific known values of m_j. The expected absolute value of the dot product of unit vectors in the numerator in the final form of equation (46) should have the value given for $E[\beta]$ in equation (C12). Hence, the expected absolute value of the algebraic error is

$$E\left[|\eta_{ai}|\right] = \frac{E[\beta] \dfrac{|\vec{E}|}{|\vec{A}_i|}}{(\vec{A}'_{ui} \cdot \vec{A}_{ui})} \tag{47}$$

or

$$E\left[|\eta_{ai}|\right] = \frac{E[\beta]}{\cos \alpha_i} \frac{|\vec{E}|}{|\vec{A}_i|} \tag{48}$$

where α_i is the angle between the vectors $\vec{A}'_i$ and $\vec{A}_i$.

We now wish to show that $|\eta_{ai}|$ generally tends to be smaller with larger values of n, for a given number of unknowns p. In equation (48), the ratio $|\vec{E}|/|\vec{A}_i|$ should not be systematically affected by the value of n. But $E[\beta]$ will be smaller with large n according to equation (C12). So if we can show that $\cos \alpha_i$ tends to be greater with larger n it will necessarily follow that $E\left[|\eta_{ai}|\right]$ tends to be smaller with larger n. This fact may be shown as follows.

In the absence of contrary information, $\vec{A}_i$ may be regarded as randomly oriented with uniform distribution within the S_n space. Consider a (p - 1)-dimensional subspace S_{p-1}, spanned by the $\vec{A}_{uk(k\neq i)}$ vectors, and its orthogonal complement S_{n-p+1}. The vector $\vec{A}_i$ has components that we now denote by $\vec{A}_{p-1}$ and $\vec{A}_{n-p+1}$ within subspaces S_{p-1} and S_{n-p+1}, respectively. $\vec{A}_{n-p+1}$ extends in the same direction as $\vec{A}'_i$. Hence,

$$\cot \alpha_i = \frac{|\vec{A}_{n-p+1}|}{|\vec{A}_{p-1}|} \tag{49}$$

We may now infer whether the expected value of $\cot \alpha_i$ is an increasing function of n with fixed p. To do so, we may use equation (38), recognizing that S_{p-1} is analogous with S_o, S_{n-p+1} with S_p, $\vec{A}_i$ with $\vec{E}$, $\vec{A}_{n-p+1}$ with $\vec{E}_p$, and $\vec{A}_{p-1}$ with $\vec{E}_o$. If we can infer from equation (38) that the expected value of $|\vec{E}_p|/|\vec{E}_o|$ increases when n and p are increased by the same integral value, it follows that the expected value of $\cot \alpha_i$ of equation (49) increases when n is increased with p unchanged.

Now consider a special case in which the $\vec{A}_{ui}$ vectors, and hence the $\vec{A}'_{ui}$ vectors also, are an orthogonal set. Then

$$\frac{|\vec{E}_p|}{|\vec{E}_o|} = \left(\frac{\sum_{i=1}^{p} |\vec{E}_i|^2}{|\vec{E}_o|^2} \right)^{1/2} = \sqrt{\sum_{i=1}^{p} r_1^2} \tag{50}$$

where r_i is the same as r of equation (34). But by inspection of equation (38) we see that the expected value of each r_i will be unchanged when n and p are increased by the same integral value. So increase in the number of summed terms under the radical in equation (50), with unchanged expected values of the individual terms, means that the expected value of $|\vec{E}_p|/|\vec{E}_o|$ increases. It follows that the expected value of $\cot \alpha_i$ of equation (49) increases with increase of n without change of p, $E[\cos \alpha_i]$ increases, and $E\left[|\eta_{ai}|\right]$ decreases.

An objection might be made that the values of r_i are not independent. But now consider, in turn, the expected values of r_{iA}, r_{iB}, r_{kA}, and r_{kB} $(k \neq i)$, for two systems A and B with the same value of $n_A - p_A$ and $n_B - p_B$ but with p_B larger than p_A. By equation (38), with no value given for any r_{kA} or r_{kB}, the expected values of r_{iA} and r_{iB} must be the same. Then, given the same values of r_{iA} and r_{iB}, the values $|\vec{E}_{(o+i)A}| = \sqrt{|\vec{E}_{oA}|^2 + |\vec{E}_{iA}|^2}$ and $|\vec{E}_{(o+i)B}|$, similarly defined for B, must be the same. Now if we designate r'_{kA} as $|\vec{E}_{kA}|/|\vec{E}_{(o+i)A}|$ and r'_{kB} as $|\vec{E}_{kB}|/|\vec{E}_{(o+i)B}|$, we see that the expected value of either r'_{kA} or r'_{kB} is given by equation (38), with n - p increased by one. It follows that the expected values of r'_{kA} and r'_{kB} are the same, hence the expected values of $|\vec{E}_{kA}|$ and $|\vec{E}_{kB}|$ are the same, and finally the expected values of r_{kA} given r_{iA} and r_{kB} given r_{iB} are the same. This procedure could be continued for expected values of each r_{jA} $(j \neq i \text{ or } k)$ and the corresponding expected values of r_{jB}. It is clear, therefore, that the interdependence of the values of r_{iA} (i = 1 to p_A) and of the values of r_{iB} (i = 1 to p_B) does not vitiate the foregoing argument.

In the derivation of a density function for the error in the least-squares value of an

unknown, it will prove convenient to use the absolute value of η_{ai} expressed as a fraction of the absolute value of x_{ti}. That is, we will use an absolute fractional error

$$\eta_i = \left| \frac{\eta_{ai}}{x_{ti}} \right| \tag{51}$$

As an example, for the specific case of n = 100, we may use the normal approximation for $f(\beta)$ given by the equation

$$f(\beta) = 7.876 \exp\left[-\frac{1}{2}\left(\frac{\beta}{0.1013}\right)^2 \right] \tag{C19}$$

derived in appendix C. With equations (C19) and (46), the probability density function of η_i may readily be found. With use of the relation expressed by equation (37), the probability density function for η_i is

$$g(\eta_i) = \frac{|x_{ti}\vec{A}_i| \cos \alpha_i}{|\vec{E}|} f(\beta) \Bigg|_{\beta=\left(\eta_i |x_{ti}\vec{A}_i| \cos \alpha_i\right)/|\vec{E}|} \tag{52}$$

Equation (52) is the ordinary normal distribution equation, normalized relative to positive values only, for the special case of n = 100. That is,

$$g(\eta_i) = \frac{\sqrt{2}}{\sqrt{\pi}\,\sigma_{\eta_i}} \exp\left[-\frac{1}{2}\left(\frac{\eta_i}{\sigma_{\eta_i}}\right)^2 \right] \tag{53}$$

where

$$\sigma_{\eta_i} = \frac{0.1013|\vec{E}|}{\cos \alpha_i |x_{ti}\vec{A}_i|} \tag{54}$$

Equations (52), (53), and (54) depend on the value of $|\vec{E}|$. They are useful for judging the accuracy to be expected in the least-squares solution of an overdetermined system of simultaneous equations that will be generated by a given physical system with n = 100 when the values of $|x_{ti}\vec{A}_i|$ and $|\vec{E}|$ that are likely to be encountered are known or postulated. Analogous expressions might be found for getting comparable results with other dimensionalities.

APPLICATIONS

Some types of physical systems provide overdetermined systems of linear simultaneous equations on a continuing basis. The coefficients a_{ji} may be unchanging and highly dependable. The m_j values may change from one time to another. Moreover, the m_j values may be more or less undependable either because of inaccuracies in their measurement or because of other effects on the m_j values than the effects of the a_{ji} and x_{ti} values. In such systems, the methods of error analysis that have been presented may be useful in (1) the design stage and (2) the use stage.

In the design stage, the engineer would want to use the analysis to help him judge the degree of dependability that might be expected from each of several proposed systems. He may know, or be able to estimate, the magnitudes of extraneous effects upon the m_j values. He will want to be able to tell the user of the equipment how dependable the solutions for the unknowns may be under the condition of those anticipated extraneous effects being present.

Equation (48), for example, would be useful in the design stage. That equation yields the expected fractional error due to the anticipated extraneous effects, that is, the magnitude of the $\vec{E}$ vector. Equation (48), then, would allow an estimate of the accuracy to be expected for a given value of n, or could be used with a series of values of n to determine the necessary value of n to achieve a desired accuracy. The engineer, of course, would have the responsibility of judging whether the basic assumption of a uniformly distributed orientation of the error vector $\vec{E}$ should reasonably be applied to the specific system concerned.

After a physical system has been designed and constructed and is in use, for example, for monitoring concentrations of toxic gases, the equations that have been developed would allow warnings of several types. Suppose a decision were made that a critical concentration for gas i would be $x_{i(crit)}$. Then equation (16) could be used to sound a warning whenever the value x_{ci} were as great as $x_{i(crit)}$. That warning could be accompanied by an auxiliary signal if, at the same time, an effective standard deviation as great as x_{ci} from equation (16) were indicated. (Such an effective standard deviation would be the product $0.1057|\vec{A}_i'||\vec{E}_o|$ in eq. (43), or some similar parameter in an equation analogous to eq. (43) for some other system than those to which eq. (43) applies.) In that case, the probability would be about 0.16 (positive values of error only) that the value x_{ci} could be caused entirely by interferents. That is, in about 16 percent of cases the value of x_{ci}, equal to x_{ti} plus the error η_{ai}, could equal or exceed $x_{i(crit)}$ even with $x_{ti} = 0$. With this probability, the user of the warning equipment would, of course, be less disturbed by the warning than if the indicated standard deviation of η_{ai} were much smaller.

A most serious and urgent warning might be given at any time when x_{ci} minus the

effective standard deviation were in excess of $x_{i(crit)}$. Regardless of the value of x_{ci}, a warning might be given that the system could not successfully monitor the concentration of gas i whenever an effective standard deviation greater than $x_{i(crit)}$ was shown.

These possibilities of continuous error monitoring should add greatly to the protection provided by a monitoring system.

CONCLUDING REMARKS

Expressions have been derived for the probability density function of the ratio of absolute value of error in determination of an unknown to a uniquely determinable component of the error function that exists within the system of equations. These expressions are for both known and unknown prior distributions of error. It has been shown that the effect of the prior distribution is usually small. The analytical results have been well confirmed by Monte Carlo results.

A method has been presented by which an analytical analysis can be made of the error to be expected in least-squares solutions of overdetermined systems of linear simultaneous equations that are expected to be generated by a given physical system. It has been shown that, other things being equal, including a constant number of unknowns, the expected error diminishes with increasing number of simultaneous equations. A method has also been shown by which an estimate can be made of the expected value of error as a function of the number of simultaneous equations and the magnitudes of error sources expressed as an error vector. It has been shown that a normal distribution is approximated for the error for the special case of 100 simultaneous equations with 10 unknowns.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, January 24, 1972,
111-05.

APPENDIX A

EXAMPLE FOR METHOD OF DETERMINING $\vec{A}'_{ui}$ VECTOR

A specific example will be solved for determination of the $\vec{A}'_{u1}$ vector, with given $\vec{A}_{ui}$ vectors (i = 1 to 3), in four-dimensional space. A Gram-Schmidt orthogonalization will first be performed with use of the two $\vec{A}_{uk}$ vectors, for $k \neq 1$. This orthogonalization could be performed in either of the two possible ways. Then the orthogonalization will be continued to find the third member of the mutually orthogonal Gram-Schmidt set, which will be the $\vec{A}'_{u1}$ vector.

The three $\vec{A}_{ui}$ vectors assumed are

$$\left.\begin{aligned} \vec{A}_{u1} &= 0.258\,\vec{u}_1 + 0.516\,\vec{u}_2 - 0.258\,\vec{u}_3 - 0.775\,\vec{u}_4 \\ \vec{A}_{u2} &= -0.365\,\vec{u}_1 + 0.548\,\vec{u}_2 - 0.183\,\vec{u}_3 + 0.730\,\vec{u}_4 \\ \vec{A}_{u3} &= 0.547\,\vec{u}_1 + 0.730\,\vec{u}_2 + 0.365\,\vec{u}_3 + 0.183\,\vec{u}_4 \end{aligned}\right\} \quad \text{(A1)}$$

As a first step we find a vector $\vec{A}^*_{u2}$. That vector is the component of $\vec{A}_{u2}$ that is orthogonal to the one-dimensional subspace to which the vector $\vec{A}_{u3}$ is confined, normalized to unity. That is,

$$\vec{A}^*_{u2} = \frac{\vec{A}_{u2} - \vec{A}_{u2} \cdot \vec{A}_{u3}\vec{A}_{u3}}{\left|\vec{A}_{u2} - \vec{A}_{u2} \cdot \vec{A}_{u3}\vec{A}_{u3}\right|}$$

$$= -0.531\,\vec{u}_1 + 0.366\,\vec{u}_2 - 0.291\,\vec{u}_3 + 0.707\,\vec{u}_4 \quad \text{(A2)}$$

We next find a vector $\vec{A}^*_{u1}$. That vector is the component of $\vec{A}_{u1}$ that is orthogonal to the two-dimensional subspace to which the vectors $\vec{A}_{u2}$ and $\vec{A}_{u3}$ are confined, normalized to unity. That is,

$$\vec{A}^*_{u1} = \frac{\vec{A}_{u1} - \vec{A}_{u1} \cdot \vec{A}^*_{u2}\vec{A}^*_{u2} - \vec{A}_{u1} \cdot \vec{A}_{u3}\vec{A}_{u3}}{\left|\vec{A}_{u1} - \vec{A}_{u1} \cdot \vec{A}^*_{u2}\vec{A}^*_{u2} - \vec{A}_{u1} \cdot \vec{A}_{u3}\vec{A}_{u3}\right|}$$

$$= -0.138\,\vec{u}_1 + 0.538\,\vec{u}_2 - 0.561\,\vec{u}_3 - 0.614\,\vec{u}_4 \quad \text{(A3)}$$

For $p > 3$ this procedure would be continued in the same manner. For $p = 3$, the

procedure is terminated with

$$\vec{A}'_{u1} = \vec{A}^*_{u1} \tag{A4}$$

APPENDIX B

NUMERICAL EXAMPLE OF SOLUTION OF AN OVERDETERMINED SYSTEM OF SIMULTANEOUS EQUATIONS BY BOTH VECTOR AND MATRIX FORMS OF LEAST-SQUARES METHOD

A vector solution will be described in detail for a system of six simultaneous equations with three unknowns. Then the matrix solution for the same problem will be shown. The a_{ji} and m_j values were selected by a random procedure, details of which are immaterial at this point.

In general, for the vector solution, a series of values of any vector $\vec{V}_i (i = 1 \text{ to } k)$ will be represented by a $6 \times k$ matrix that will be designated [V] and the $\vec{u}_i$ vectors will be omitted. A series of scalars like $v_i (i = 1 \text{ to } k)$ will be represented by a column matrix designated as [v].

The a_{ji} values are

$$[A] = \begin{bmatrix} -0.3809 & 0.8993 & 0.2657 \\ -0.3644 & -0.1383 & -0.3259 \\ -0.5705 & -0.0159 & -0.1912 \\ -0.3020 & -0.6430 & -0.6428 \\ -0.1861 & -0.7559 & 0.2378 \\ -0.8513 & 0.7626 & -0.9064 \end{bmatrix} \tag{B1}$$

The m_j values are

$$[M] = \begin{bmatrix} 1.3737 \\ -0.0505 \\ 0.2161 \\ -0.9771 \\ -0.3574 \\ 0.3180 \end{bmatrix} \tag{B2}$$

The following $\vec{A}_i'$ vectors were determined with use of equations (11), (12), and (B1), by the method described in appendix A:

$$[A'] = \begin{bmatrix} -0.6036 & 0.2697 & 0.6276 \\ -0.2753 & -0.1293 & -0.0395 \\ -0.7092 & -0.1587 & 0.3490 \\ 0.0025 & -0.3009 & -0.4630 \\ -0.7948 & -0.4671 & 0.6588 \\ -0.1386 & 0.2499 & -0.4775 \end{bmatrix} \tag{B3}$$

By equations (16), (B2), and (B3), the calculated values x_{ci} are

$$[x_c] = \begin{bmatrix} -0.7309 \\ 0.8832 \\ 1.0047 \end{bmatrix} \tag{B4}$$

Solution of the same problem by the matrix method will now be described. Matrices [A] and [M] in equation (2) are, of course, the same as the matrices in equations (B1). The matrix [X] in equation (2) is simply

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{B5}$$

The matrix [B] of equation (4) will therefore be

$$[B] = [A]^T[A] = \begin{bmatrix} 1.4539 & -0.5974 & 1.0482 \\ -0.5975 & 2.3944 & -0.1706 \\ 1.0482 & -0.1706 & 1.5046 \end{bmatrix} \tag{B6}$$

The inversion of matrix [B] in equation (B6) is

$$[B]^{-1} = \begin{bmatrix} 1.5941 & 0.3212 & -1.0741 \\ 0.3212 & 0.4858 & -0.1687 \\ -1.0741 & -0.1687 & 1.3937 \end{bmatrix} \tag{B7}$$

Finally, matrix [C] of equation (6) is

$$[C] = [B]^{-1}[A]^{T} = \begin{bmatrix} -0.6036 & -0.2753 & -0.7092 & 0.0025 & -0.7948 & -0.1386 \\ 0.2697 & -0.1293 & -0.1587 & -0.3009 & -0.4671 & 0.2499 \\ 0.6276 & -0.0395 & 0.3490 & -0.4630 & 0.6588 & -0.4775 \end{bmatrix} \tag{B8}$$

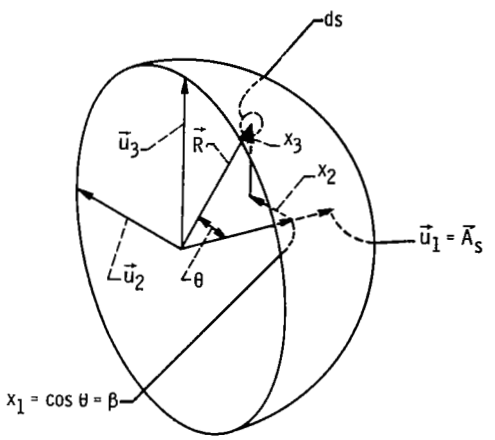
Note that the row vectors in matrix [C] of equation (B8) are the same as the $\vec{A}_i'$ (column) vectors in equation (B3). Hence, the values x_{ci} determined by the matrix method (eq. (5)) will be the same dot products as by the vector method and equation (B4) will represent the solution by the matrix method as well as by the vector method.

APPENDIX C

STATISTICAL DISTRIBUTION OF ABSOLUTE VALUES OF DOT PRODUCTS OF RANDOMLY ORIENTED VECTORS

The expected absolute value of a dot product, as well as the statistical distribution, will be sought for two unit vectors $\vec{A}_s$ and $\vec{R}$ within n-dimensional space. The results sought will be for a special case of a more general treatment by Lord (ref. 5). The unit vector $\vec{A}_s$ will be treated as fixed, though the result would be the same if it were randomly oriented with any distribution of orientation. The unit vector $\vec{R}$ will be assumed to have a uniformly distributed orientation.

The method of finding the expected absolute value of the dot product, β according to equation (36) will be discussed with reference to sketch (a), which, of course, applies



(a)

only to three-dimensional space. The method, however, will be extended to n-dimensional space. Integrations will be applied to the surface of a hemisphere that is centered about an axis in the direction of $\vec{u}_1$. The specific unit vector $\vec{A}_s$ will be taken as the vector $\vec{u}_1$.

The random unit vector $\vec{R}$ is shown, extending in one of the infinite number of possible directions. The magnitudes of its components in the directions of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are X_1, X_2, and X_3. A surface element dS is shown, which contains the terminus of $\vec{R}$. The basic assumption of uniformly distributed orientation of $\vec{R}$ may now be more explicitly stated as an assumption that all surface elements of equal area on the hemisphere will have equal probabilities of enclosing the terminus of $\vec{R}$.

Now,

$$\beta = X_1 \tag{C1}$$

so the expected value of β is

$$E[\beta] = \frac{\int_S X_1 \, dS}{\int_S dS} \tag{C2}$$

where $\int_S$ indicates integration over the entire surface of the hemisphere.

A standard formula for determining the area of the X_1 centered hemispherical surface of unit radius in rectangular coordinates, in three dimensions, is

$$\int_S dS = \int_{-1}^{1} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_2^2}} \frac{dX_3 \, dX_2}{\sqrt{1 - X_2^2 - X_3^2}} \tag{C3}$$

in which the denominator of the integrand is equal to the numerical value of X_1. So, from equations (C2) and (C3),

$$E[\beta] = \frac{\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_2^2}} dX_3 \, dX_2}{\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-X_2^2}}^{\sqrt{1-X_2^2}} \frac{dX_3 \, dX_2}{\sqrt{1 - X_2^2 - X_3^2}}} \tag{C4}$$

Extension of this formula to n dimensions gives

$$E[\beta] = \frac{\int_{-\xi_2}^{\xi_2}\int_{-\xi_3}^{\xi_3}\int_{-\xi_4}^{\xi_4}\cdots\int_{-\xi_{n-1}}^{\xi_{n-1}}\int_{-\xi_n}^{\xi_n}\prod_{i=2}^{n} dX_{n-i+2}}{\int_{-\xi_2}^{\xi_2}\int_{-\xi_3}^{\xi_3}\int_{-\xi_4}^{\xi_4}\cdots\int_{-\xi_{n-1}}^{\xi_{n-1}}\int_{-\xi_n}^{\xi_n}\frac{\prod_{i=2}^{n} dX_{n-1+2}}{\xi_{n+1}}} \qquad \text{(C5)}$$

where

$$\xi_i = \sqrt{1 - \sum_{j=2}^{i-1} X_j^2} \qquad \left(\sum_{j=2}^{i-1} X_j^2 = 0 \text{ for } i=2\right) \qquad \text{(C6)}$$

If integrations in numerator and denominator are performed concurrently, the overall task of integration becomes quite simple. Constants can be cancelled at every step of the process after the first. Thus,

$$\left(\int_{-\xi_n}^{\xi_n} \frac{dX_n}{\sqrt{1 - \sum_{k=2}^{n} X_k^2}}\right)^{-1} = \frac{1}{\pi} \qquad \text{(C7)}$$

$$\frac{1}{\pi}\frac{\int_{-\xi_n}^{\xi_n} dX_n}{\int_{-\xi_{n-1}}^{\xi_{n-1}} dX_{n-1}} = \frac{1}{\pi}\frac{2\xi_n}{2\xi_{n-1}} \qquad \text{(C8)}$$

$$\frac{\frac{1}{\pi}\int_{-\xi_{n-1}}^{\xi_{n-1}} \xi_n \, dX_{n-1}}{\frac{1}{\pi}\int_{-\xi_{n-2}}^{\xi_{n-2}} \xi_{n-1} \, dX_{n-2}} = \frac{\frac{1}{2}\pi\xi_{n-1}^2}{\frac{1}{2}\pi\xi_{n-2}^2} \tag{C9}$$

$$\frac{\frac{1}{\pi}\int_{-\xi_{n-2}}^{\xi_{n-2}} \xi_{n-1}^2 \, dX_{n-2}}{\frac{1}{\pi}\int_{-\xi_{n-3}}^{\xi_{n-3}} \xi_{n-2}^2 \, dX_{n-3}} = \frac{\frac{1}{\pi}\xi_{n-2}^3}{\frac{1}{\pi}\xi_{n-3}^3} \tag{C10}$$

and so on until we have

$$E[\beta] = \frac{1}{\pi}\int_{-\xi_2}^{\xi_2} \xi_3^{n-2} \, dX_2 \tag{C11}$$

or

$$E[\beta] = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \left(\frac{2}{\pi}\right)^s \prod_{k=1}^{n-1} k^{(-1)^{s+k+1}} \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \end{pmatrix} \tag{C12}$$

Now,

$$E[\beta] = \int_0^1 \beta f(\beta) \, d\beta$$

$$= \int_0^{\pi/2} f(\beta) \cos\theta \sin\theta \, d\theta \tag{C13}$$

where $f(\beta)$ is the probability density function of β.

If any of the expressions for $E[\beta]$ in equation (C12) contained X_1, that is $\cos\theta$, that expression could be equated to the right side of equation (C13) and the result could be solved for $f(\beta)$. Unfortunately, such is not the case. But an expression may be found for $f(\beta)$ as follows.

Equation (C5) can be regarded as applying to the entire hypersphere if both numerator and denominator are considered to be multiplied by 2. Then, for the expected value of β, an integration relative to any X_i is as valid as an integration relative to X_1. Hence, instead of multiplying the innermost integrand of the numerator by X_1, we could have multiplied the integrand by X_2 in the position just after the first integral sign in the numerator of equation (C5). Then the integration shown in equation (C7) would have been omitted. Equations (C8) to (C10) would have had identical numerators and denominators. We would have arrived at

$$E[\beta] = \frac{2\int_0^1 \xi_3^{n-3} X_2 \, dX_2}{\int_{-1}^1 \xi_3^{n-3} \, dX_2} \tag{C14}$$

or

$$E[\beta] = E[\cos\theta] = \frac{\int_0^{\pi/2} \cos\theta \sin^{n-2}\theta \, d\theta}{\int_0^{\pi/2} \sin^{n-2}\theta \, d\theta} \tag{C15}$$

in which it is justified to use $\theta = \cos^{-1} X_2$ as if it were $\theta = \cos^{-1} X_1$. By comparison of equations (C12), (C13), and (C15), it may be seen that

$$f(\beta) = \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-2} k^{(-1)^{s+k+1}}\right] \left(1 - \beta^2\right)^{(n-3)/2} \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \end{pmatrix} \tag{C16}$$

And the distribution function is

$$F(\beta) = \left(\frac{2}{\pi}\right)^s \left[\prod_{k=1}^{n-2} k^{(-1)^{s+k+1}}\right] \int_\theta^{\pi/2} \sin^{n-2}\varphi \, d\varphi \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \\ \theta = \cos^{-1}\beta \end{pmatrix} \tag{C17}$$

or

$$F(\beta) = s + \left(\frac{2}{\pi}\right)^{s}\left\{\beta \sin^{s}\theta\left[1 + \sum_{r=1}^{(n-s-3)/2} \sin^{2r}\theta \prod_{k=1}^{2r+s} k^{(-1)^{s+k+1}}\right] - s\theta\right\} \qquad \begin{pmatrix} s = 1 \text{ for } n \text{ even} \\ s = 0 \text{ for } n \text{ odd} \\ \theta = \cos^{-1}\beta \end{pmatrix} \tag{C18}$$

We may disregard any question regarding the Π factors or summations in equations (C12), (C16), and (C18) at certain very low values of n because those values of n are all below the values of interest in this report.

For n taken arbitrarily as 100, the solid curve in figure 3 represents equation (C16). The plotted points in the same figure are calculated values obtained with the following probability density function:

$$f(\beta) = 7.876 \exp\left[-\frac{1}{2}\left(\frac{\beta}{0.1013}\right)^{2}\right] \tag{C19}$$

The agreement seems to be sufficiently precise that, for $n = 100$, equation (C19) may be used for practical purposes in lieu of equation (C16). Note that the standard deviation 0.1013 in equation (C19) is approximately $(n - 3)^{-1/2}$.

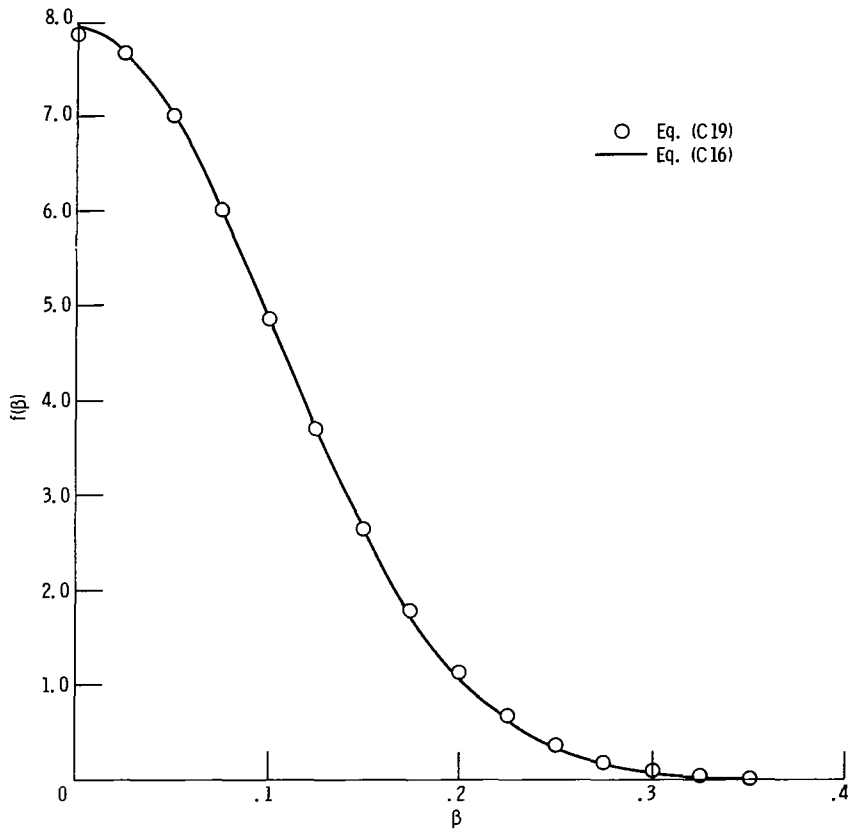


Figure 3. - Density function of absolute value of dot product of a specific unit vector and a randomly oriented unit vector in 100-dimensional space ($f(\beta)$).

APPENDIX D

EFFECT OF PRIOR DISTRIBUTION OF MAGNITUDE OF ERROR VECTOR

Equations (38) and (42) were derived for the case where it was assumed that nothing was known about the error vector or its components except its uniform distribution in space. Now we wish to find a density function for r for the situation where $|\vec{E}_o|$ is known exactly. We will show that a prior density function of the magnitude of the component of the error vector $|\vec{E}_{o+i}|$ must be taken into account. By prior density function we mean a density function that is inherent in the method of generation of the system of equations (1). This prior density function would depend on the source of equations (1), and might differ for any two different physical systems that could give rise to the same system of equations.

We shall develop here equations analogous to (38) and (42) that will include consideration of the prior density function $f\left(|\vec{E}_{o+i}|\right)$. That is, we wish to derive the probability density function of r given $|\vec{E}_o|$, where r is the ratio $|\vec{E}_i|/|\vec{E}_o|$, $|\vec{E}_i|$ is the magnitude of the component of the error vector $\vec{E}$ lying in an arbitrary direction (labeled i) within the subspace S_p, and $|\vec{E}_o|$ is the known magnitude of the component of the error vector $\vec{E}_o$ lying in subspace S_o. Again the basic assumption that is to be employed here is the uniform distribution over space of the direction of the error vector $\vec{E}$, in particular, that portion $\vec{E}_{o+i}$ of $\vec{E}$ lying in the $(n - p + 1)$-dimensional space consisting of the addition of the i^{th} direction within the space S_p to the totality of the space S_o. This $(n - p + 1)$-dimensional subspace is denoted S_{o+i}.

By the uniform distribution hypothesis, the direction of $\vec{E}_{o+i}$ is uniformly distributed over S_{o+i} so that the ratio $|\vec{E}_i|/|\vec{E}_{o+i}|$ can be identified with β and the probability density function of this ratio is identical with that given by equation (C16) where n takes on the value $n - p + 1$. This condition of uniform distribution of direction of $\vec{E}_{o+i}$ was used to derive equation (38) for the density function $g(r)$.

We assume that the prior distribution represented by the density function $f\left(|\vec{E}_{o+i}|\right)$ is independent of the direction of $\vec{E}_{o+i}$, which implies that it is independent of r. Hence,

$$f\left(r \,\middle|\, |\vec{E}_{o+i}|\right) = g(r) \tag{D1}$$

where $g(r)$ is as given by equation (38).

The quantity that we wish to determine is $f\left(r \,\middle|\, |\vec{E}_o|\right)$ where $|\vec{E}_o|$ (and not $|\vec{E}_{o+i}|$) is known. Using equation (37) once again, with equation (35), we can write

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \left|\left(\frac{\partial |\vec{E}_{o+i}|}{\partial r}\right)_{|\vec{E}_o|}\right| f\left(|\vec{E}_{o+i}| \,\middle|\, |\vec{E}_o|\right)\Bigg|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}}$$

$$= |\vec{E}_o| \frac{r}{\sqrt{1+r^2}} f\left(|\vec{E}_{o+i}| \,\middle|\, |\vec{E}_o|\right)\Bigg|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}} \qquad \text{(D2)}$$

To arrive at an expression for $f\left(|\vec{E}_{o+i}| \,\middle|\, |\vec{E}_o|\right)$, which appears in equation (D2), we we apply the rule of Bayes, namely,

$$f\left(|\vec{E}_{o+i}| \,\middle|\, |\vec{E}_o|\right) = \frac{f\left(|\vec{E}_{o+i}|\right) f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right)}{\int_{|\vec{E}_o|}^{\infty} f\left(|\vec{E}_{o+i}|\right) f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right) d|\vec{E}_{o+i}|} \qquad |\vec{E}_o| \leq |\vec{E}_{o+i}| \qquad \text{(D3)}$$

We now need an expression for $f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right)$, which appears in equation (D3). As that equation will be used for substitution into equation (D2), the expression for $f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right)$ must be for given r, namely, the specific value of r in the left side of equation (D2). So, with $|\vec{E}_{o+i}|$ fixed and with use of equations (37) and (35) under the assumption that $|\vec{E}_{o+i}|$ and r are independently distributed, we get

$$f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right) = g(r)\left|\frac{dr}{d|\vec{E}_o|}\right| = \frac{1+r^2}{r|\vec{E}_o|} g(r) \qquad \text{(D4)}$$

where $g(r)$ is as given by equation (38).

Now, substituting $f\left(|\vec{E}_o| \,\middle|\, |\vec{E}_{o+i}|\right)$ from equation (D4) into equation (D3) and then substituting $f\left(|\vec{E}_{o+i}| \,\middle|\, |\vec{E}_o|\right)$ from equation (D3) into equation (D2), we get

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \frac{|\vec{E}_o| \dfrac{r}{\sqrt{1+r^2}} f\left(|\vec{E}_{o+i}|\right) \Big|_{|E_{o+i}| = |E_o|\sqrt{1+r^2}} \dfrac{1+r^2}{r|\vec{E}_o|} g(r)}{\displaystyle\int_{|\vec{E}_o|}^{\infty} f\left(|\vec{E}_{o+i}|\right) \Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}} \dfrac{1+r^2}{r|\vec{E}_o|} g(r)\, d|\vec{E}_{o+i}|} \tag{D5}$$

or, with use of equation (35),

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \frac{g(r)\sqrt{1+r^2}\, f\left(|\vec{E}_{o+i}|\right) \Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}}}{\displaystyle\int_0^{\infty} g(r)\sqrt{1+r^2}\, f\left(|\vec{E}_{o+i}|\right) \Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}} dr} \tag{D6}$$

If we now substitute from equation (38) and simplify,

$$f\left(r \,\middle|\, |\vec{E}_o|\right) = \frac{\left(1+r^2\right)^{(p-n)/2} f\left(|\vec{E}_{o+i}|\right) \Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}}}{\displaystyle\int_0^{\infty} \left(1+r^2\right)^{(p-n)/2} f\left(|\vec{E}_{o+i}|\right) \Big|_{|\vec{E}_{o+i}| = |\vec{E}_o|\sqrt{1+r^2}} dr} \tag{D7}$$

Equation (D7) may be replaced by equation (38) even for use with given $|\vec{E}_o|$ if we have reason to assume that $f\left(r \,\middle|\, |\vec{E}_o|\right)$ is only negligibly sensitive to $f\left(|\vec{E}_{o+i}|\right)$.

With the use of equation (D7), an exact equation corresponding to the approximate equation (42) may be obtained in the same way as before. The result is

$$f\left(\eta_{ai} \,\middle|\, |\vec{E}_o|\right) = \frac{1}{2|\vec{A}_i'|\ |\vec{E}_o|} f\left(r \,\middle|\, |\vec{E}_o|\right)\Bigg|_{r=|\eta_{ai}|/\left(|\vec{A}_i'|\,|\vec{E}_o|\right)} \tag{D8}$$

APPENDIX E

STATISTICAL RESULT OF ERROR DETERMINATIONS IN SOLUTIONS OF 10 000 SYSTEMS OF SIMULTANEOUS EQUATIONS

As a Monte Carlo verification of equation (38), least-squares solutions were executed for 10 000 systems of simultaneous equations. Each system included six equations and three unknowns. Each of the 10 000 systems involved use of 27 outputs from a random number generator, 270 000 random numbers in all. Each random number was within the range from minus one to plus one, with rectangular distribution within that range.

Least-squares solutions in both vector and matrix forms were shown for an overdetermined system in appendix B. That system was one of the 10 000 used in the Monte Carlo analysis now to be described. As an example, discussion of that specific problem will now be continued.

For this example, 18 of the 27 outputs from the random number generator were used for the $\vec{A}_i$ vectors shown in equation (B1) of appendix B, which is as follows:

$$[A] = \begin{bmatrix} -0.3809 & 0.8993 & 0.2657 \\ -0.3644 & -0.1383 & -0.3259 \\ -0.5705 & -0.0159 & -0.1912 \\ -0.3020 & -0.6430 & -0.6428 \\ -0.1861 & -0.7559 & 0.2378 \\ -0.8513 & 0.7626 & -0.9064 \end{bmatrix} \tag{B1}$$

Three of the 27 outputs were used as

$$[x_t] = \begin{bmatrix} -0.7850 \\ 0.9955 \\ 0.8661 \end{bmatrix} \tag{E1}$$

The remaining six from the 27 outputs were used as

$$[E] = \begin{bmatrix} -0.0506 \\ 0.0834 \\ -0.0504 \\ -0.0173 \\ 0.0430 \\ -0.3244 \end{bmatrix} \tag{E2}$$

With use of the 27 random values shown in equations (B1), (E1), and (E2), the vector $\vec{M}$ was determined by using equation (17), with the result shown by equation (B2) of appendix B, which is as follows:

$$[M] = \begin{bmatrix} 1.3737 \\ -0.0505 \\ 0.2161 \\ -0.9771 \\ -0.3574 \\ 0.3180 \end{bmatrix} \tag{B2}$$

From this point on, the values in equation (E2) were not used in any way. Thus, the true values x_{ti} from which the problem arose were known, but could not be recovered from the $\vec{A}_i$ and $\vec{M}$ vectors because the same $\vec{M}$ vector could have resulted from any one of an infinity of possible combinations of x_{ti} values. So the true errors relative to the combination of x_{ti} values that actually gave rise to the problem could be determined and compared with the calculated error distribution.

The Gram-Schmidt orthonormal set, spanning the S_p subspace, was determined with use of the $\vec{A}_i$ vectors of equation (B1) as follows:

$$[P] = \begin{bmatrix} 0.13699 & 0.58117 & 0.53166 \\ 0.34924 & -0.08939 & -0.03348 \\ 0.50292 & -0.01025 & 0.29560 \\ 0.40480 & -0.41554 & -0.39221 \\ 0.32800 & -0.48848 & 0.55803 \\ 0.57870 & 0.49281 & -0.40448 \end{bmatrix} \tag{E3}$$

Continuation of the Gram-Schmidt orthogonalization, with use of $\vec{U}_i = \vec{u}_i$ (i = 1 to 3) gives the vectors $\vec{O}_i$ (i = 1 to 3) spanning the subspace S_o as follows:

$$[O] = \begin{bmatrix} 0.60067 & 0.00000 & 0.00000 \\ 0.03648 & 0.93144 & 0.00000 \\ -0.36642 & -0.16457 & 0.70586 \\ 0.65687 & -0.23148 & 0.15683 \\ -0.09609 & -0.14604 & -0.55841 \\ -0.25078 & -0.17441 & -0.40662 \end{bmatrix} \tag{E4}$$

Either with use of equations (28), (B2), and (E4), or with use of equations (29), (B2), and (E3), the vector component $\vec{E}_o$ was determined as

$$[E_o] = \begin{bmatrix} 0.03421 \\ 0.13275 \\ 0.00517 \\ 0.01585 \\ -0.06482 \\ -0.06705 \end{bmatrix} \tag{E5}$$

From equation (E5),

$$|\vec{E}_o| = 0.16664 \tag{E6}$$

The values η_{ai}, now, are simply the values x_{ci} of equation (B4) from appendix B, reproduced here,

$$[x_c] = \begin{bmatrix} -0.7309 \\ 0.8832 \\ 1.0047 \end{bmatrix} \tag{B4}$$

minus the values x_{ti} of equation (E1). The η_{ai} values are

$$[\eta_a] = \begin{bmatrix} 0.05409 \\ -0.11236 \\ 0.13806 \end{bmatrix} \tag{E7}$$

The $\vec{A}_i'$ vectors were given in equation (B3) of appendix B, which is reproduced here:

$$[A'] = \begin{bmatrix} -0.6036 & 0.2697 & 0.6276 \\ -0.2753 & -0.1293 & -0.0395 \\ -0.7092 & -0.1587 & 0.3490 \\ 0.0025 & -0.3009 & -0.4630 \\ -0.7948 & -0.4671 & 0.6588 \\ -0.1386 & 0.2499 & -0.4775 \end{bmatrix} \tag{B3}$$

From equation (B3)

$$\left.\begin{aligned} |A_1'| &= 1.26255 \\ |A_2'| &= 0.69698 \\ |A_3'| &= 1.18054 \end{aligned}\right\} \tag{E8}$$

Finally, from equations (E6), (E7), and (E8),

$$\left.\begin{aligned} \frac{\eta_{a1}}{|\vec{A}_1'|\,|\vec{E}_o|} &= 0.25710 \\ \frac{\eta_{a2}}{|\vec{A}_2'|\,|\vec{E}_o|} &= -0.96743 \\ \frac{\eta_{a3}}{|\vec{A}_3'|\,|\vec{E}_o|} &= 0.70456 \end{aligned}\right\} \qquad \text{(E9)}$$

Absolute values of the results shown in equations (E9) are equivalent to Monte Carlo trial values of r as used in equation (38). Accordingly, from the 10 000 random problems solved, 30 000 Monte Carlo trial values were obtained for r.

The 30 000 trial results were counted by class, with class marks at 0.1, 0.3, 0.5, and so on. The normalized counts, or discrete values of approximate probability density f(r) are plotted as circular symbols in figure 4. The solid curve in the same figure represents equation (38) for n - p + 1 = 4. The abscissa values and corresponding probability densities (ordinates) for the absolute values of the results in equation (E9) are marked by the three vertical lines in the figure. They contributed to the counts at class marks 0.3, 0.7, and 0.9.

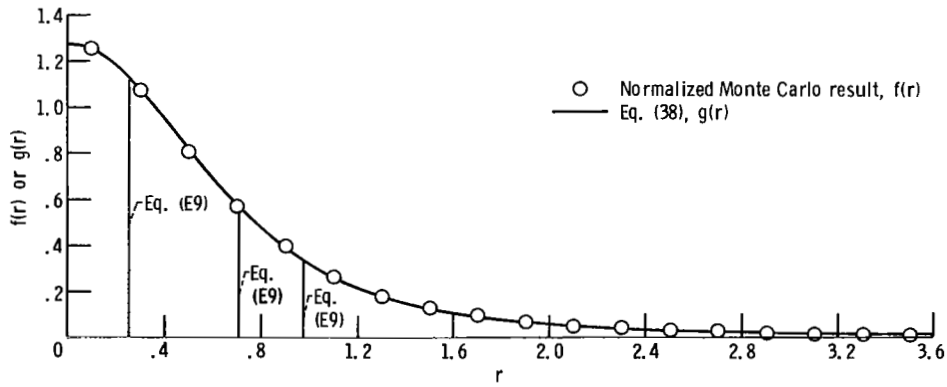


Figure 4. - Probability densities for r for n = 6 and p = 3.

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