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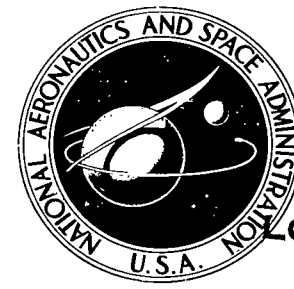

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VLASOV EQUATION EIGENVALUES AND EIGENVECTORS FOR FOURIER-HERMITE DISPERSION MATRICES OF ORDER GREATER THAN 1000

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SUMMARY

The connection between the Van Kampen and Landau representations of the Vlasov equations has been extended to Fourier-Hermite expansions containing more than 1000 terms by taking advantage of the properties of tridiagonal matrices. These numerical results are regarded as conclusive indications of the nonuniformly convergent behavior of the approximation curve in the limit of an infinite number of terms and represent an extension of work begun by Grant (1967) and by Grant and Feix (1967).

INTRODUCTION

In a fully ionized plasma such as the solar corona, which has a large number of electrons in a Debye cube (the edge of which measures the screening distance about an introduced charge), the effect of collisions is negligible and the Vlasov equations describe the space-time evolution of the single-particle distribution function.

The nonlinear Vlasov equations have been studied only for special cases, a common one of which is assumed herein: a single space and velocity variable, periodic spatial boundary conditions, a uniform steady ion background against which the electrons move, and an initial Maxwellian electronic distribution function.

Because of their formidable nature, the Vlasov equations have been treated mostly by numerical techniques. However, when ordinary x-v-t space is used, an insuperable difficulty arises in the velocity space. Increasingly steep oscillatory gradients with ever-shorter wavelengths develop in the distribution function. To circumvent the difficulty, transform methods have been introduced, and among these is the Fourier expansion on space and the Hermite expansion on velocity which lead, in the nonlinear case, to a set of coupled ordinary differential equations for the elements of a matrix, the Fourier-Hermite expansion coefficients of the Fourier-Hermite representation. (Each transformed set of equations is called a representation of the original x-v-t equations.)

When the nonlinear Fourier-Hermite equations are linearized and an eigenvalue-eigenvector solution is sought, another matrix appears, the Fourier-Hermite dispersion matrix. At this point contact is made with two classical linearized representations: that of Landau (ref. 1) and that of Van Kampen (ref. 2). Landau uses a Fourier-Laplace transform on space and time, while Van Kampen performs essentially a Fourier transform on both space and time. As it happens, the continuum of eigenvalues in Van Kampen's representation are all undamped (real) while the discrete poles of Landau are all damped (complex). In particular, a pair of highly excited poles, called Landau poles, dominates the motion for long times when the initial conditions are stable. The Fourier-Hermite representation has discrete, real eigenvalues, and hence it seems clearly akin to Van Kampen's representation.

When numerical integration of the Fourier-Hermite equations is performed, a numerical instability appears at the highest order Hermite index and spreads toward the lower. If a Fokker-Planck (weak-collision or distant-encounter) term is introduced into the Fourier-Hermite representation, it damps most strongly the coefficients of highest Hermite order. Macroscopic physical entities such as the electric field depend on coefficients of the lowest orders, and thus the selective character of the Fokker-Planck term makes it possible, with many Fourier-Hermite coefficients, to quench the instability with few collisions and little degradation of the results. At this point the fact that the Fourier-Hermite representation bridges the gap between Landau and Van Kampen representations was discovered (ref. 3).

In reference 3 the transition between Van Kampen and Landau representations of the Vlasov equations is shown in terms of a delicate limiting process that requires an indefinite increase in the number of terms kept in the approximate Fourier-Hermite distribution function and, simultaneously, an indefinite decrease in the strength of the introduced Fokker-Planck collision term. A full account of the Fourier-Hermite representation may be found in reference 4. The full matrix techniques used in reference 3 allowed only about 60 terms to be kept because of computer storage limitations. When recursion techniques are used in the tridiagonal dispersion matrices, the storage required is proportional to the matrix order rather than to the order squared. The economy of recursive methods has allowed the calculation of the eigenvalues and eigenvectors for dispersion matrices larger than 1000×1000.

The ability to treat Fourier-Hermite matrices of orders as great as 1000 carries the number of Fourier-Hermite coefficients far enough to make available the appropriate value of Fokker-Planck damping for best plasma simulation over the entire practical range of expansion coefficient order (several hundred) for the nonlinear problem.

In the general, nonlinear problem, as in the linear problem, addition of a small damping term of proper strength extends the time of good simulation at the cost of slightly degraded results.

SYMBOLS

$a_{\alpha\beta}$	Fourier-Hermite expansion coefficient
B	strength parameter of Fokker-Planck collision term
C(f)	collision term
D_n	determinant of Fourier-Hermite dispersion matrix
E	electric field
e	magnitude of electronic charge
F	Maxwellian velocity distribution, $\frac{\exp(-v^2/2)}{\sqrt{2\pi}}$
$F' = \frac{dF}{dv}$	
$\mathcal{F}\{\}$	Fourier-transform operator
f	distribution function of electrons
f_o	Fourier transform of initial perturbation in f
f_t	Fokker-Planck collision term
$\overline{H}_\beta$	Hermite polynomial of order β
$i = \sqrt{-1}$	
k	wave number, $\frac{2\pi}{\text{Wavelength}}$; Boltzmann constant
$\mathcal{L}\{\}$	Laplace-transform operator
m	electronic mass
N	order of highest term kept in Fourier-Hermite expansion of f
n	electron density; Fourier-Laplace transform of electron density

n_o	mean electron density
s	complex argument of Laplace transform
T	absolute temperature
t	time
v	electron velocity
$v_{j,0}$	first component of normalized eigenvector corresponding to jth eigenvelocity w_j
w	phase velocity, ω/k
w_n	nth eigenvelocity
x	spatial coordinate
α,β	Fourier and Hermite indices, respectively
$-\gamma$	imaginary part of ω (γ is negative for stability)
ΔB	arbitrary small increment in B
ΔD_n	change in D_n when ω becomes $\omega + \Delta\omega$
$\Delta\omega$	arbitrary small change in ω
δ	small positive quantity
$\delta_{\alpha\beta}$	Kronecker delta
ϵ	plasma dielectric constant
ϵ_0	$\epsilon(k,\omega=0)$; permittivity of vacuum
λ	eigenvelocity
μ	complex argument, $\omega/k\sqrt{2}$

$\varphi = \frac{\omega}{ik^2} n(k,s)$

ψ Fourier transform of time variation of n

$\psi_{j,j+1}$ ordinate of histogram between w_j and w_{j+1}

ω argument of D_n; complex eigenvalue (root, pole, eigenfrequency); s/i

ω_n nth order approximation to eigenvalue

$\langle\ \rangle$ average over velocity space

Subscripts:

i imaginary part

r real part

Superscript:

* complex conjugate

ANALYSIS

Fourier-Hermite Representation

Consider the ordinary one-dimensional Boltzmann equation (in natural units) for the single-particle distribution function f of the electrons:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = C(f) \tag{1}$$

where the right-hand side denotes a collision term that is a functional of f. The natural units of equation (1) are

$\sqrt{\frac{\epsilon_0 kT}{n_o e^2}}$ Debye length

$\sqrt{\frac{kT}{m}}$ thermal speed

$\sqrt{\frac{\epsilon_0 m}{n_0 e^2}}$ inverse plasma frequency

The Debye length is a measure of the distance at which an introduced charge is screened, and the plasma frequency measures the rapidity of oscillations in electron density. A neutralizing uniform positive background is assumed. The external field E acting on the electrons is the self-field of the electrons themselves given by Poisson's equation:

$$-\frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f \, dv - 1 \tag{2}$$

If the number of particles in a Debye cube is large, as in a thermonuclear plasma, the collision term C may be neglected. In the case of identically vanishing C, equations (1) and (2) are known as Vlasov equations.

The Fourier-Hermite expansion of f is

$$\sqrt{2\pi} f = \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=0}^{\infty} a_{\alpha\beta} \exp(i\alpha kx) \overline{H}_\beta(v) \exp(-v^2/2) \tag{3}$$

where

$$\overline{H}_\beta(v) = (-1)^\beta \exp(v^2/2) \left(\frac{d}{dv}\right)^\beta \exp(-v^2/2)$$

is Rodrigues' formula and

$$\int_{-\infty}^{\infty} \overline{H}_\alpha(v) \overline{H}_\beta(v) \exp(-v^2/2) dv = \delta_{\alpha\beta} \sqrt{2\pi} \, n!$$

Substitution of equation (3) into equations (1) and (2) yields, after multiplication by $\exp(-i\alpha kx)\overline{H}_\beta(v)$ and integration over x-v space, an infinite set of ordinary differential equations for the coefficients $a_{\alpha\beta}$ of the Fourier-Hermite expansion. This set of equations for the matrix elements $a_{\alpha\beta}$ is the Fourier-Hermite representation of the Vlasov equations (1) and (2). Truncation after the second row yields the Fourier-Hermite linearization. Further truncation at $\beta = N$ yields the finite approximation to the linearized case. Finally, assumption of an $\exp(i\omega t)$ variation of the elements $a_{\alpha\beta}$ leads to a dispersion relation of order $(N + 1)$.

With Maxwell's distribution as background, the Fourier-Hermite dispersion relation (from ref. 5 with a sign change on B) for cutoff of the expansion at order N is

$$\det\begin{bmatrix} \omega & (1+k^2)^{1/2} & 0 & 0 & 0 \\ (1+k^2)^{1/2} & \omega + iB & k\sqrt{2} & 0 & 0 \\ 0 & k\sqrt{2} & \omega + 2iB & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & kN^{1/2} \\ 0 & 0 & 0 & kN^{1/2} & \omega + NiB \end{bmatrix} = 0 \qquad (4)$$

In equation (4), ω is an eigenfrequency (eigenvalue) and k is the wave number of an assumed small $\cos(kx)$ disturbance in electron density. If ω is a root of equation (4), then so is $-\omega^*$. The parameter $B \geq 0$ is a measure of the strength of the Fokker-Planck collision term (ref. 6),

$$f_t = B\left[\frac{\partial}{\partial v}(vf) + \langle v^2 \rangle \frac{\partial^2 f}{\partial v^2}\right]$$

which modifies the one-dimensional Vlasov equations (1) and (2) into

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E\frac{\partial f}{\partial v} = f_t$$

$$-\frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f\, dv - 1$$

The Vlasov case occurs at $B = 0$. When $B = 0$ the eigenvalues of equation (4) are all real, which suggests the equivalence of the Fourier-Hermite representation (ref. 5) to that of Van Kampen (ref. 2) rather than to that of Landau (ref. 1). However, as B increases from zero, the surprising behavior first reported in reference 3 and shown in figure 1 occurs. All but one of the poles (eigenvalues) move rapidly into the $\gamma\omega_r$-plane where $\omega = \omega_r - i\gamma$ and $f \propto \exp(i\omega t)$. The anomalous pole is detained in the region of the Landau pole. Detailed investigation of the behavior of the anomalous pole for Fourier-Hermite expansions up to order 63 suggested that the equivalence of Van Kampen and Landau treatments was being observed within the Fourier-Hermite representation. A more powerful numerical technique has verified the behavior to orders over 1000, which is sufficiently near infinity to establish the nonuniformity convergent behavior of the approximation curve as $N \to \infty$ and $B \to 0$.

Numerical Algorithm

The improved technique is based on recursion (ref. 7). If the determinant of order $(N + 1)$ in equation (4) is denoted as D_{N+1}, expansion on the last column yields

$$D_{N+1} = (\omega + iNB)D_N - Nk^2D_{N-1} \tag{5}$$

In particular,

$$D_1 = \omega$$

$$D_2 = \omega(\omega + iB) - \left(i + k^2\right)$$

In principle, the value of D_{N+1} for arbitrary large N is obtained in about N steps, each of simple multiplication and addition. In contrast, the full matrix treatment of reference 3 requires N^2 words just to record the elements of the matrix.

To apply recursion equation (5) to the eigenvalue problem, the Fourier-Hermite root (eigenvalue) nearest in value to the real part of the Landau pole is identified for $B = 0$. A small positive increment in B, ΔB, and a small increment in ω, $\Delta\omega$, are introduced and the approximate derivative $\Delta D_{N+1}/\Delta\omega$ is computed. Use of the approximate derivative in the Newton-Raphson method gives a first approximation to the shifted eigenvalue. In general,

$$\left.\begin{aligned} \omega_{n+1} &= \omega_n - D_{N+1}\left(B+\Delta B, \omega_n\right)\frac{\Delta\omega}{\Delta D_{N+1}} \\ \Delta D_{N+1} &= D_{N+1}\left(B+\Delta B, \omega_n+\Delta\omega\right) - D_{N+1}\left(B+\Delta B, \omega_n\right) \end{aligned}\right\} \tag{6}$$

After the first step, iteration on the quasi-Newton-Raphson equations (6) yields a sharp value of $\omega(\Delta B)$. A second increment of ΔB leads on to $\omega(2\Delta B)$, and in like manner the progress of $\omega(B) = \omega_r - i\gamma$ on the $\gamma\omega_r$-plane is tracked. The chief difficulties are found in scaling the large numbers which arise for large N and in the necessity of iteration to sharpen the values of $\omega(B)$. A sample FORTRAN program is shown in appendix A.

Use of the recursion equation (5) simplifies the task of finding eigenvalues for the $B = 0$ (collisionless) case of equation (4). Easy evaluation of $D_{N+1}(\omega)$ at any N permits a movement from $\omega = 0$ in steps of $\Delta\omega > 0$. Sign changes of $D_{N+1}(\omega)$ localize each root within $\Delta\omega$, and iteration sharpens the root. The tridiagonal form of the matrix in equation (4) makes trivial (for $k = 0.5$) the calculation of the corresponding eigenvectors. In reference 5, histograms corresponding to an initial $\cos(kx)$ spatial disturbance of plasma density were constructed from the eigenvectors of the matrix of equation (4). Full matrix methods were barely adequate to yield histograms for the case of truncation

at $N = 99$, and then only because $B = 0$ produces real eigenvalues and eigenvectors. With the recursion methods, histograms for $N > 10^3$ have now been constructed.

The oscillations in density which follow an initial $\cos(kx)$ spatial density perturbation may be represented exactly by the following Fourier integral over a real phase velocity w:

$$n(k,t) = \int_{-\infty}^{\infty} \psi(w)\exp(ikwt)dw \tag{7}$$

where $kw = \omega$, a real frequency. The reality of the Fourier-Hermite eigenfrequencies for $B = 0$ permits the construction of a histogram analogous to $\psi(w)$ in equation (7). In the limit of infinite N the histogram approaches $\psi(w)$. In appendix B the derivation of ψ is given. As shown in reference 5, the ordinate of the histogram between the jth and $(j + 1)$th eigenvelocities may be assigned as

$$\psi_{j,j+1} = \frac{\frac{1}{2}\left(v_{j,0}^2 + v_{j+1,0}^2\right)}{w_{j+1} - w_j} \qquad \left(w_{j+1} > w_j > 0; \quad j = 0,1,2, \ldots, N\right)$$

where $v_{n,0}$ is the first component of the normalized eigenvector corresponding to the nth eigenvelocity w_n found by solving

$$\det\begin{bmatrix} w & (1 + k^{-2})^{1/2} & 0 & 0 & 0 \\ (1 + k^{-2})^{1/2} & w & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & w & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & N^{1/2} \\ 0 & 0 & 0 & N^{1/2} & w \end{bmatrix} = 0 \tag{8}$$

Equation (8) is derived by similarity transformations of the matrix of equation (4) and by the introduction of the phase velocity $w = \omega/k$.

RESULTS AND DISCUSSION

The behavior of the Fourier-Hermite poles with vanishing collisions is indicated in figures 2 and 3 for approximations up to $N = 1023$. In figure 2 the behavior of the real part of the eigenfrequency is shown. A scale to accommodate all the values of N has been used in figure 2(a). The extreme deviations by which the curves reach Fourier-

Hermite values at $N = 255$ and 511 as B goes to zero are quite striking. The curve for $N = 1023$ lies so nearly on a straight line to the Landau value that it would not show in figure 2(a). Therefore, the behavior for the three highest values of N is shown at larger scale in figure 2(b). For $N = 1023$ it can be seen that the chance closeness of the Fourier-Hermite root to the Landau value causes a nearly straight approach to the Landau value. Actually, a deviation from a straight approach starts at about $B = 0.0006$.

In figure 3 the behavior of the damping is shown for vanishing collisions. Again the two parts of the figure are small-scale and large-scale graphs. The curves for the higher N can be clearly seen only in the large-scale graph of figure 3(b). In every case the sharp swerve toward $\gamma = 0$ from a straight approach to the Landau value is striking.

It is clear from figures 2 and 3 that, numerically, the connection between the Van Kampen and Landau representations has been demonstrated within the Fourier-Hermite representation. Moreover, the pathological character of the curves $\omega_r(B)$ and $\gamma(B)$ for large values of N as B goes to zero clearly indicates the nonuniformly convergent behavior of the approximation curve in the limit of infinite N at vanishing B.

In figure 4 a histogram for $N = 1023$ is shown along with one for $N = 99$ that was presented in reference 5. The curve is symmetrical about $\lambda = 0$, so only the positive half is shown. Although a better representation of the curve for $N = \infty$ is obtained with $N = 1023$ than with $N = 99$, the peak is still rather crudely modeled despite a tenfold increase in the number of terms in the Fourier-Hermite expansion. The root separation diminishes so slowly with increasing N that most roots can make only negligible contributions to the peak.

The advantage of a collision term barely large enough to move a root very close to the Landau pole is suggested by figure 4. With a small collision term, a single Fourier-Hermite root can well represent the Landau damping, which is the most prominent feature of the behavior of the plasma at long times for weak initial disturbances of the plasma.

CONCLUDING REMARK

In addition to yielding conclusive indications of the nonuniformly convergent behavior of the Fourier-Hermite expansion as order increases to infinity and collisions decrease, the results have provided a guide to the choice of the size of collision term that best simulates the secular behavior of the plasma.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., March 23, 1972.

APPENDIX A

CALCULATION OF EIGENVALUES

The FORTRAN IV computer program used to compute the curves for N = 1023 in figures 2 and 3 follows. Although the dimensions 100 and 1024 have been used respectively for arrays APP (corresponding to successive root approximations) and DET (corresponding to the successive determinants in the recursive process), these locations are a luxury for diagnostic purposes. Several locations would suffice if the successive root approximations and higher order determinants were written over the earlier values. For the case shown, central processor time on the CDC 6600 computer was less than 30 seconds.

```
      PROGRAM CHASE(INPUT,OUTPUT),
*     TRACKS F-H ROOTS INTO COMPLEX PLANE AS B INCREASES FROM ZERO,
      COMPLEX D,OMEGA,DERD,DELTOM,DELMEG,APP,DET,OMEGA0,
      COMPLEX DERN1,DERN2,
      REAL K,KSQR,
      COMMON KSQR,B,IS,SCALE,
      DIMENSION APP(100),DET(1024),
      K=5.E-01,
*VALUE OF N AS USED BELOW CORRESPONDS TO (N-1) HERMITE FUNCTIONS,
      N=1024,
*LANDAU POLE FOR K=0.5 IS AT (1.41566,.15336),
*FOURIER-HERMITE EIGENVALUE NEAREST REAL PART OF LANDAU POLE,
      OMEGA0=(1.41424,0.),
      PRINT 900,N,K,
*MAXIMUM NUMBER OF ITERATIONS ALLOWED TO FIX OMEGA,
      NOIT=100,
*SHARPNESS CRITERION FOR OMEGA,
      CVLIM=2.E-09,
*STEP SIZE IN APPROXIMATE DERIVATIVE,
      DELTOM=(0.,1.E-06),
*B-RANGE AND STEP SIZE,
      BMIN=.0002,
      DELTB=.00002,
      BMAX=.0013,
      KSQR=K*K,
      OMEGA=OMEGA0,
      B=BMIN-DELTB,
*LOOP ON B,
    1 B=B+DELTB,
*NEWTON-RAPHSON LOOP,
      DO2I=1,NOIT,
*W2,
      DERN2=D(N,OMEGA+DELTOM,DET),
*NUMBER OF SCALINGS,
      IS2=IS,
*W1,
      DERN1=D(N,OMEGA,DET),
      IS1=IS,
```

```
*CHECK TO INSURE SAME SCALING ON W2,W1,
      IF(IS2.EQ.IS1) GO TO 4,
      IF(IS2-IS1.GT.0) GO TO 6,
      DERN2=DERN2*SCALE,
      GO TO 4,
    6 DERN1=DERN1*SCALE,
*APPROXIMATE DERIVATIVE (W2-W1)/DELTOM,
    4 DERD=(DERN2-DERN1)/DELTOM,
*NEWTON-RAPHSON ROOT INCREMENT,
      DELMEG=DERN1/DERD,
      OMEGA =OMEGA-DELMEG,
      APP(I)=OMEGA,
*CHECK ON IMPROVEMENT OF OMEGA,
    2 IF(CABS(DELMEG).LT.CVLIM) GO TO 3,
    3 PRINT 100,B,I,OMEGA,
      PRINT 101,IS,
  100 FORMAT(  3H B=F10.6,5X11HITERATIONS=I3,5X6HOMEGA=F11.8,F11.8),
  101 FORMAT(1X3HIS=I3),
  900 FORMAT(//3H N=I4/3H K=F6.3//),
      IF(B.LT.BMAX-DELTB/2.),
     2GO TO 1,
      STOP,
      END,
*SUBPROGRAM FOR DISPERSION DETERMINANT,
      COMPLEX FUNCTION D(IORD,OMEGA,DET),
      COMPLEX OMEGA,DET,
      REAL KSQR,
      COMMON KSQR,B,IS,SCALE,
      DIMENSION DET(IORD),
*FIRST DETERMINANTS,
      DET(1)=OMEGA,
      DET(2)=OMEGA*(OMEGA+CMPLX(0.,1.0)*B)-(1.+KSQR),
      SCALE=1.E-18,
      IS=0,
*RECURSION LOOP,
      DO11I=3,IORD,
      DET(I)=DET(I-1)*(OMEGA+FLOAT(I-1)*CMPLX(0.,1.)*B)-FLOAT(I-1)*,
     2KSQR*DET(I-2),
*CHECK ON SIZE OF D,
      IF(CABS(DET(I)).LT.1.E18) GO TO 1,
*SCALING OF D,
    2 DET(I)=DET(I)*SCALE,
      DET(I-1)=DET(I-1)*SCALE,
*SCALING COUNT,
      IS=IS+1,
    1 CONTINUE,
      D=DET(IORD),
      RETURN,
      END,
```

APPENDIX B

DERIVATION OF $\psi(\lambda)$

The linearized one-dimensional Vlasov equations, in natural units, for the electronic perturbation f on a uniform ionic background distribution are

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - EF' = 0 \tag{B1a}$$

$$-\frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f \, dx \tag{B1b}$$

where F is the unperturbed velocity distribution of the electrons. The Fourier-Laplace transform of f(x,v,t) may be written as

$$f(k,s) = \frac{1}{2\pi} \int_0^{\infty} dt \int_{-\infty}^{\infty} dx \, e^{-st} e^{+ikx} f(x,v,t)$$

where the same symbol f is used for both function and transform. The corresponding inverse is

$$f(x,v,t) = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{+i\infty+\delta} ds \int_{-\infty}^{\infty} dk \, e^{+st} e^{-ikx} f(k,s)$$

where the integral on s is known as Bromwich's integral. On application of the Fourier-Laplace transform to equations (B1), the result is

$$s f(k,s) - ikv f(k,s) - F' E(k,s) = f_0(k,v) \tag{B2a}$$

$$+ik E(k,s) = \int_{-\infty}^{\infty} f(k,s) dv = n(k,s) \tag{B2b}$$

To obtain equations (B2) the usual relations

$$\mathcal{L}\left\{\frac{\partial f}{\partial t}\right\} = s\mathcal{L}\{f\} - f(x,v,0)$$

$$\mathcal{F}\left\{\frac{\partial f}{\partial x}\right\} = -ik\mathcal{F}\{f\}$$

are employed. The notation $f_o(k,v)$ refers to $\mathcal{F}\{f(x,v,0)\}$, the Fourier transform of the initial conditions. In equations (B2), $f(k,s)$ has an implicit dependence on velocity v, but $n(k,s)$ and $E(k,s)$ do not, since only in equation (B2b) has velocity been integrated upon.

If one first solves for $f(k,s)$ in equation (B2a), then integrates over v to find $n(k,s)$ from equation (B2b), and, finally, substitutes $s = i\omega$, the result is

$$n(k,s) = \frac{I(k,s)}{\epsilon(k,s)} = \frac{\frac{1}{i}\int_{-\infty}^{\infty} \frac{f_o(k,v)dv}{\omega - kv}}{1 + \frac{1}{k}\int_{-\infty}^{\infty} \frac{F'dv}{\omega - kv}} \qquad (s = i\omega)$$

where the denominator $\epsilon(k,s)$ is the so-called plasma dielectric constant. Under an initial $\cos(kx)$ density perturbation the assumed Maxwellian electronic velocity distribution

$$F = \frac{1}{\sqrt{2\pi}} \exp(-v^2/2)$$

is untouched. Thus

$$f_o(k,v) = F = -\frac{F'}{v}$$

and in terms of F',

$$n(k,s) = \frac{\frac{1}{i}\int_{-\infty}^{\infty} \frac{1}{v}\frac{-F'}{\omega - kv}dv}{1 + \frac{1}{k}\int_{-\infty}^{\infty} \frac{F'dv}{\omega - kv}}$$

In the integral of the numerator of $n(k,s)$ the coefficient of $-F'$ may be written as

$$\frac{1}{v}\frac{1}{\omega - kv} = \frac{k}{\omega}\left(\frac{-1}{0 - kv} + \frac{1}{\omega - kv}\right)$$

so that $n(k,s)$ may be written in terms of $\epsilon(k,\omega)$ and $\epsilon(k,0)$:

$$n(k,s) = \frac{ik^2}{\omega}\frac{\epsilon(k,\omega) - \epsilon(k,0)}{\epsilon(k,\omega)} = \frac{ik^2}{\omega}\frac{\epsilon - \epsilon_0}{\epsilon}$$

By means of a change in variable $w = v/\sqrt{2}$, and the substitution

$$\frac{w}{\mu - w} = \frac{\mu}{\mu - w} - 1$$

the dielectric constant $\epsilon(k,\omega)$ may be expressed in terms of the complex function

$$Z(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - \mu} \qquad \left(\mathrm{Im}(\mu) < 0; \quad \mu = \frac{\omega}{k\sqrt{2}} = x + iy\right)$$

as

$$k^2 \epsilon(k,\mu) = 1 + k^2 + \mu Z(\mu)$$

In terms of $Z = Z_r + iZ_i$,

$$k^2 \epsilon_r = 1 + k^2 + xZ_r - yZ_i$$

$$k^2 \epsilon_i = yZ_r + xZ_i$$

The function Z is related to the complementary error function by

$$Z(z) = \frac{\sqrt{\pi}}{i} e^{-z^2} \mathrm{erfc}(-iz)$$

On the real axis of the μ-plane, $y = 0$; so for real $\lambda = \omega/k$,

$$k^2 \epsilon_r(k,\omega) = 1 + k^2 + \frac{\lambda}{\sqrt{2}} Z_r\left(\frac{\lambda}{\sqrt{2}}\right) \tag{B3a}$$

$$k^2 \epsilon_i(k,\omega) = \frac{\lambda}{\sqrt{2}} Z_i\left(\frac{\lambda}{\sqrt{2}}\right) \tag{B3b}$$

$$k^2 \epsilon_r(k,0) = 1 + k^2 \tag{B3c}$$

$$k^2 \epsilon_i(k,0) = 0 \tag{B3d}$$

Inversion by Bromwich's integral with the contour along the imaginary axis of the s-plane ($\delta = 0$) yields

$$n(k,t) = \frac{ik^2}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon - \epsilon_0}{\omega\epsilon} e^{i\omega t}d\omega = \frac{ik^2}{2\pi}\int_{-\infty}^{\infty} \frac{\varphi_r + i\varphi_i}{\omega} e^{i\omega t}d\omega \qquad \text{(B4)}$$

At all $t < 0$, $n(k,t) = 0$; thus

$$0 = \int_{-\infty}^{\infty} \frac{\varphi_r + i\varphi_i}{\omega} e^{-i\omega t}d\omega \qquad (t > 0)$$

Taking the complex conjugate yields

$$0 = \int_{-\infty}^{\infty} \frac{\varphi_r - i\varphi_i}{\omega} e^{i\omega t}d\omega \qquad (t > 0)$$

Subtraction of $ik^2/2\pi$ times this null member from equation (B4) for $n(k,t)$ yields

$$n(k,t) = -\frac{k^2}{\pi}\int_{-\infty}^{\infty} \frac{\varphi_i(\omega)}{\omega} e^{i\omega t}d\omega = \int_{-\infty}^{\infty} \psi(\omega)e^{i\omega t}d\omega$$

Writing φ_i in terms of the values of ϵ_r and ϵ_i given in equation (B3) yields

$$\psi(\omega) = \frac{1 + k^{-2}}{\pi\sqrt{2}} \frac{Z_i^+\left(\frac{\omega}{k\sqrt{2}}\right)}{\epsilon_r^2(k,\omega) + \epsilon_i^2(k,\omega)} \frac{1}{k}$$

where the function $Z^+ = -Z^*$ is one available on the computer library tape at Langley Research Center. The substitution $\omega = k\lambda$ (using the same symbol for ψ) yields

$$n(k,t) = \int_{-\infty}^{\infty} \psi(\lambda)e^{ik\lambda t}d\lambda$$

where

$$\psi(\lambda) = \frac{1 + k^{-2}}{\pi\sqrt{2}} \frac{Z_i^+\left(\frac{\lambda}{\sqrt{2}}\right)}{\epsilon_r^2 + \epsilon_i^2}$$

is the function corresponding to infinite N that is plotted in figure 4.

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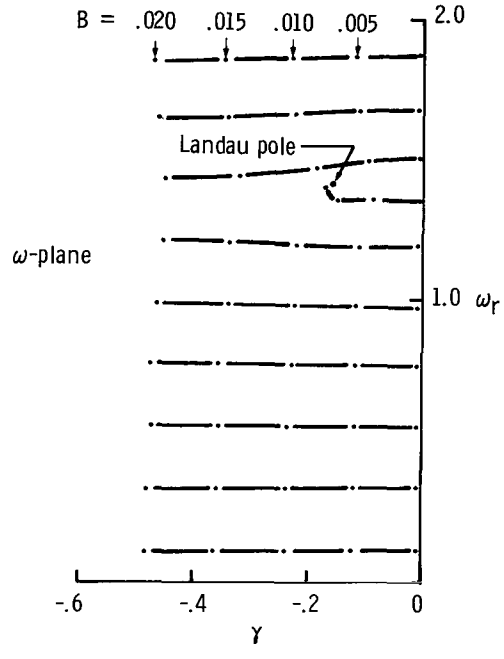
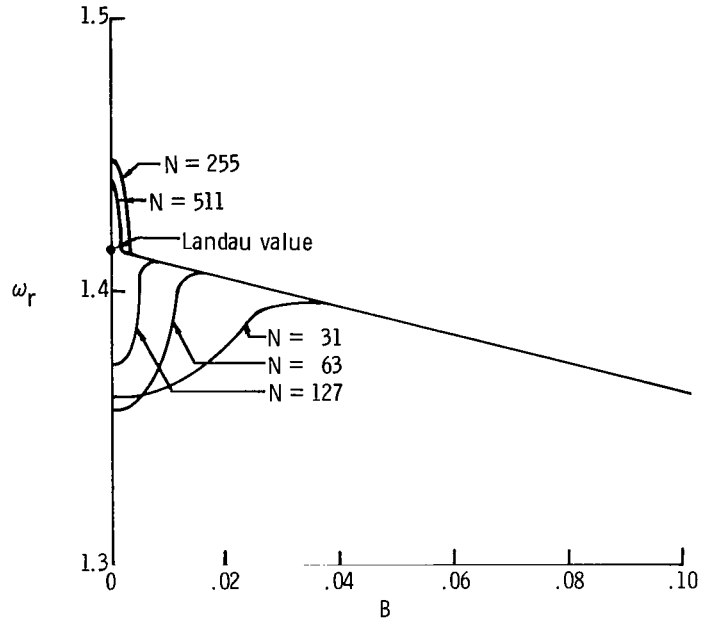
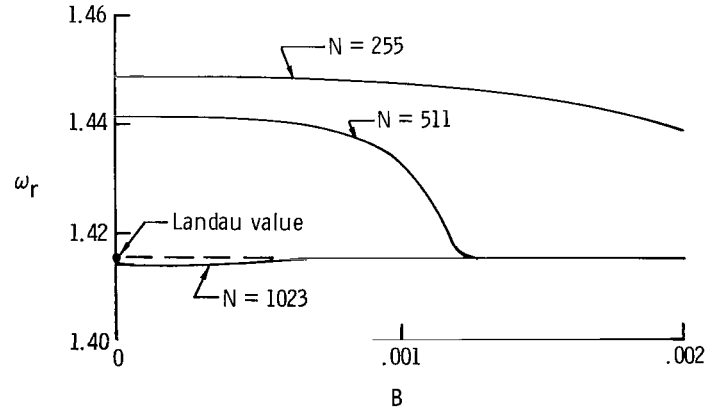


Figure 1.- Movement of Fourier-Hermite poles into complex frequency plane as collisions increase. N = 63; k = 0.5.

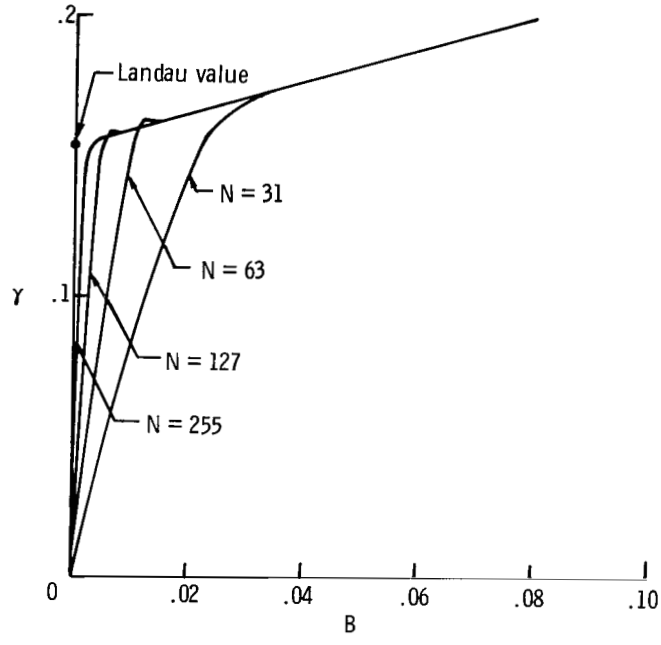


(a) Wide range of N.

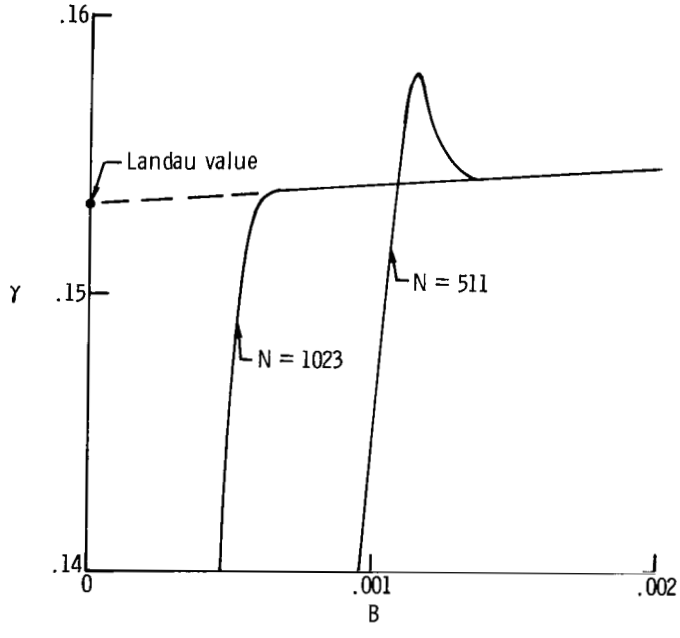


(b) Higher values of N; expanded scale.

Figure 2.- Real part of complex frequency as a function of collision parameter B. k = 0.5.

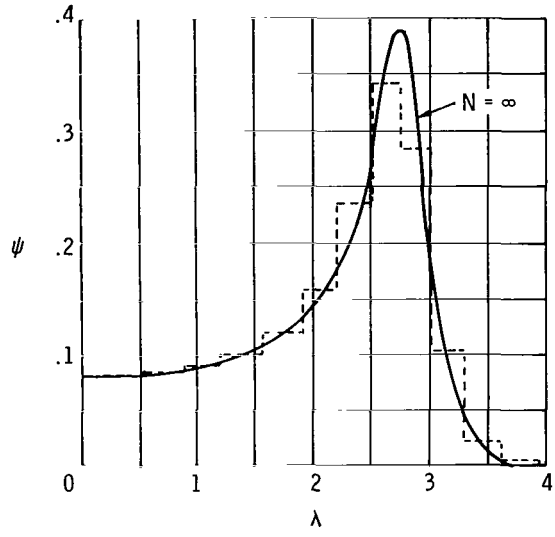


(a) Lower values of N.

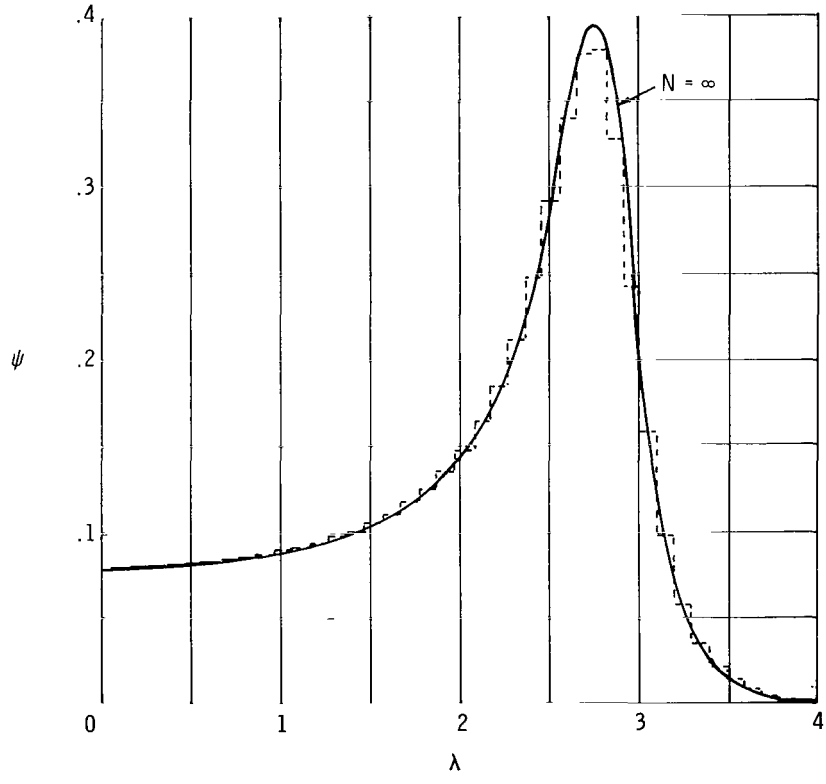


(b) Higher values of N; expanded scale.

Figure 3.- Damping part of complex frequency as a function of collision parameter B. k = 0.5.



(a) N = 99.



(b) N = 1023.

Figure 4.- Exact Fourier transform compared with corresponding Fourier-Hermite histograms for initial cos(kx) density disturbance. k = 0.5.

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