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NONLINEAR AXISYMMETRIC FLEXURAL VIBRATION OF SPHERICAL SHELLS

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16. Abstract

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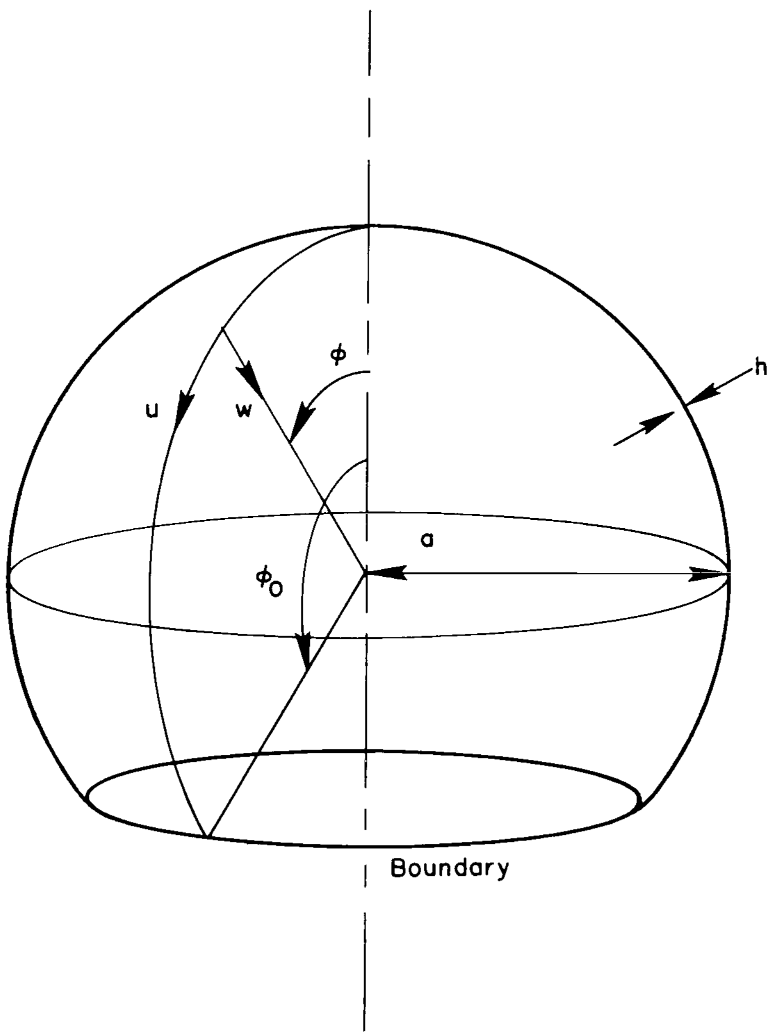
NOTATION

a	radius of spherical shell
E	modulus of elasticity
g	acceleration of gravity
$H_0(\ )$	differential operator defined as $H_0(\ ) = \left(\dfrac{\partial^2}{\partial\phi^2} + \cot\phi\,\dfrac{\partial}{\partial\phi}\right)(\ )$
$H_1(\ )$	differential operator defined as $H_1(\ ) = \{H_0 + (1 - \nu)\}(\ )$
$H_2(\ )$	differential operator defined as $H_2(\ ) = \{H_0 + 2\}(\ )$
h	shell thickness
k	$\dfrac{Eh^3}{12(1 - \nu^2)}$
m	weight of shell per unit surface area
$n!!$	$\begin{cases} n(n-2)\ .\ .\ .\ 3\times1 & (n\text{: odd}) \\ n(n-2)\ .\ .\ .\ 4\times2 & (n\text{: even}) \\ 0!! \equiv (-1)!! \equiv 1 \end{cases}$
P	external force per unit surface area
$P_i(\cos\phi)$, P_i	Legendre polynomial of ith order, first kind
$P_i^n(\cos\phi)$, P_i^n	associated Legendre polynomial of ith order
u	in-plane shell displacement
w	deflection of shell
α	$12(1 - \nu^2)\,\dfrac{a^2}{h^2}$
γ	viscous damping coefficient
ε	$\dfrac{h}{a}$
Ψ	stress function defined in equations (2)
ν	Poisson ratio

$(\ ^{\circ})$	$\frac{\partial}{\partial \phi}$
$(\dot{\ })$	$\frac{\partial}{\partial t}$
$(\ )'$	$\frac{\partial}{\partial \tau}$
$\left[\frac{\mu}{2}\right]$	Gaussian symbol, the maximum integer not more than $\frac{\mu}{2}$



Sketch (a).- Geometry of spherical shell.

NONLINEAR AXISYMMETRIC FLEXURAL VIBRATION OF SPHERICAL SHELLS*

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SUMMARY

This report presents axisymmetric responses of a nonshallow thin-walled spherical shell on the basis of nonlinear bending theory.

An ordinary differential equation with nonlinearity of quadratic as well as cubic terms associated with variable time is derived. The derivation is based on the assumption that the deflection mode is the sum of four Legendre polynomials, and the Galerkin procedure is applied. The equation is solved by asymptotic expansion, and a first approximate solution is adopted. Unstable regions of this solution are discussed.

INTRODUCTION

The analysis of dynamic behavior is among the most important current research bearing on engineering reliability for thin-walled shells. This report is concerned with the forced and free axisymmetric nonlinear vibrations of nonshallow thin spherical shells. It may be expected that for thin-walled spherical shells, especially nonshallow ones, analysis on the basis of nonlinear theory will reveal such phenomena of response that cannot be provided by linear theory.

The few investigations that have been made concerning this problem of spherical shells are limited to the shallow shells (refs. 1 and 2). Reference 3 has provided the solutions that apply to generalized variational equations of motion with the substitution of spatial mode of the deflection derived by linear theory. Since the deflection mode of linear vibration appears generally as the sum of Legendre functions, with real fractional and complex conjugate orders for nonshallow spherical shells, equations cannot be integrated easily in variational procedure (ref. 3).

In this report the deflection is set as the sum of four Legendre polynomials to satisfy the ordinary boundary conditions, and with the Galerkin procedure an ordinary nonlinear equation with time as the independent variable is derived.

Although the assumed deflection may not fit the response mode rigorously, it appears to be sufficient by comparison with the expansion of the sinusoidal function or of the Legendre function with fractional order with Legendre polynomials. Of course, if the mode of exciting force is the same as that of assumed deflection then a rigorous response mode results.

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The fundamental nonlinear equations and relations for spherical shells used here are derived from reference 4 with the assumption of small rotations and the neglect of longitudinal inertia terms and nonlinear terms in the equilibrium equations of circumferential direction.

The ordinary differential equation that is finally derived is analogous to a nonlinear, single-degree-of-freedom mass spring oscillator. This equation, however, has a nonlinearity of quadratic as well as cubic terms of dependent variable, $X(t)$, whereas only a cubic term appears for cylindrical shells (refs. 5 and 6). Because of the quadratic term, the solution for $X(t)$ can exhibit the nonlinear property of both softening and hardening alternately as $X(t)$ changes sign. The range of $X(t)$ over which this behavior occurs depends on the deflection mode assumed, on the open angle of the shell, and on the thickness-radius ratio. Consequently, it is found that an increase or decrease in the amplitude of vibration with an increase in frequency does not necessarily reflect the effect of a purely softening or hardening spring.

The solutions of this nonlinear differential equation use the asymptotic expansion of this equation, with the expansions in power of thickness-radius ratio. The region of the frequency for unstable response is discussed numerically as are the effects of thickness-radius ratio, exciting force, boundary conditions, and viscous damping.

GOVERNING EQUATIONS

The following assumptions may be applicable in the derivation of governing equations for thin-walled nonshallow spherical shells.

1. Large deflection and small strain
2. Extremely large radius-thickness ratio in comparison with unity
3. Small rotations
4. Negligible rotary inertia and transverse shear deformations

Then, if the angle of rotation is approximated as

$$\Psi_\phi = (u + w^{o})/a \simeq w^{o}/a$$

and the longitudinal inertia terms and nonlinear term in equilibrium equations in circumferential directions are neglected, the following equation can be derived from reference 4 as a governing equation for axisymmetric vibration of spherical shells.

$$H_2H_2(w) - \frac{a}{k} H_2(\Psi) - \frac{1}{k}\Big[w^{oo}H_2(\Psi) - \frac{k}{a} H_2(w)H_0(w) + \Psi H_0(w) + \Psi^{oo}H_0(\Psi) - 2\Psi^{oo}w^{oo}$$
$$- \frac{w^{o}}{a} k\{H_2(w)\}^{o}\Big] + \frac{m}{g}\frac{a^4}{k}\ddot{w} + \gamma\frac{a^4}{k}\dot{w} - \frac{a^4}{k}P = 0 \quad (1)$$

The stress resultants N_θ and N_ϕ are related to w and the Airy-type stress function Ψ as follows:

$$\left.\begin{aligned} N_\theta &= \frac{1}{a^2}\left[\Psi^{oo} + \Psi - \frac{k}{a} H_2(w)\right] \\ N_\phi &= \frac{1}{a^2}\left[(\cot\phi)\Psi^{o} + \Psi - \frac{k}{a} H_2(w)\right] \end{aligned}\right\} \tag{2}$$

The compatibility condition (in terms of w and Ψ) is

$$H_2H_1(\Psi) - (1 - \nu)\frac{k}{a} H_2H_2(w) + aEhH_2(w) + Eh\left[H_0(w)w^{oo} - \left(w^{oo}\right)^2 - \left(w^{o}\right)^2\right] = 0 \tag{3}$$

Equation (3) is derived from the middle surface strain-displacement relations

$$\left.\begin{aligned} e_\phi &= \frac{1}{a}\left[u^{o} - w + \frac{1}{2a}\left(w^{o}\right)^2\right] \\ e_\theta &= \frac{1}{a}\left[(\cot\phi)(u) - w\right] \end{aligned}\right\} \tag{4}$$

in conjunction with equations (2).

SOLUTIONS

In the axisymmetric linear problem, the purely torsional motion of shells can be solved (ref. 3); then the torsionless mode shape of vibration is constituted by three fundamental mode shapes represented as Legendre functions with small real fractional order and complex conjugate orders, or two Legendre functions with real fractional orders of small and extremely large value and a Kegelfunction (ref. 3). The mode shape represented by Legendre functions with real (usually fractional) order determines the mode shape of the overall shell vibration as shown in reference 3. In low frequency vibration, the mode shape corresponding to small order is dominant; whereas at higher frequencies, the mode shape corresponding to large order becomes dominant. These modes can be expanded in the series of Legendre polynomials.

Each shell discussed here has a closed apex and one edge (see sketch (a)). For convenience of calculation and to satisfy ordinary boundary conditions, the deflection of the following form will be adopted:

$$\begin{aligned} w &= W(\phi)\cdot X(t) \\ &= \left(A_0P_{\mu_0} + A_1P_{\mu_1} + A_2P_{\mu_2} + A_3P_{\mu_3}\right)h\cdot X(t) = \sum_{i=0}^{3} A_iP_{\mu_i}h\cdot X(t) \end{aligned} \tag{5}$$

where

$$P_{\mu_i} = P_{\mu_i}(\cos\phi)$$

A_0 a given constant

A_i $(i \neq 0)$ unknown constant that can be determined with the consideration of boundary conditions

μ_i arbitrary integers that specify the particular modes selected

Stress function Ψ can be obtained by substitution of equation (5) into equation (3) as follows.

$$\left.\begin{aligned} \Psi &= \Psi_1 + \Psi_2 \\ \Psi_1 &= \frac{k}{a} \sum_{i=0}^{3} B_{\mu_i} A_i P_{\mu_i} hX(t) \\ \Psi_2 &= Eh \sum_{j=0}^{2\mu} b_j a_j P_j h^2 X^2(t) \qquad (j \neq 1) \end{aligned}\right\} \tag{6}$$

where

$$B_i = \frac{(1 - \nu)\{2 - \mu_i(\mu_i + 1)\} - \alpha}{(1 - \nu) - \mu_i(\mu_i + 1)}$$

$$b_j = \frac{1}{\{2 - j(j + 1)\}\{j(j + 1) - (1 - \nu)\}}$$

μ maximum of μ_i

a_j see appendix A, equation (A5)

Longitudinal displacement u can be obtained from the following equation deduced from equations (2), (4), and (6).

$$u^o = \frac{1}{Eha}\left\{H_1(\Psi) - \frac{(1 - \nu)k}{a} H_2(w) - Ehaw\right\} - \frac{1}{2a}(w^o)^2 - \frac{(1 + \nu)}{Eha}\Psi^{oo}$$

$$= \frac{1}{Eha} H_1(\Psi_2) - \frac{1}{2a}(w^o)^2 - \frac{(1 + \nu)}{Eha}(\Psi_1 + \Psi_2)^{oo}$$

Then

$$u = \frac{(1+\nu)}{Eha}\frac{k}{a}\sum_{i=0}^{3} B_i A_i P_{\mu_i}^1 hX(t) - \frac{1}{2a} d_o \phi h^2 X^2(t)$$

$$+ \frac{1}{a}\sum_{j=2}^{2\mu} \{(1+\nu)b_j a_j + d_j\} P_j{}^1 h^2 X^2(t) \qquad (7)$$

where

$$d_m = \frac{1}{m^2}\left[\frac{a_m}{m(m+1)-2} - \frac{c_m}{2}\right], \qquad d_o = (a_o + c_o)$$

c_m see appendix A, equation (A2)

Equations (5), (6), and (7) allow the satisfaction of any ordinary boundary conditions. After the boundary conditions are satisfied, the only unknown value in the above relations is $X(t)$. Galerkin's procedure applied to equation (1) yields an ordinary nonlinear differential equation in $X(t)$. The substitution of equations (5) and (6) into equation (1) and integration of equation (1) for overall shell surface after multiplication of $w \sin\phi/[hX(t)]$ will result in the equation

$$X''(\tau) + 2\kappa X'(\tau) + X(\tau) + \varepsilon\beta_2 X^2(\tau) + \varepsilon^2\beta_3 X^3(\tau) = \bar{P} \qquad (8)$$

where the dimensionless independent variable τ is defined as $\tau = \omega_o t$, and the prime (') means differentiation with τ. Other constants are as follows.

$$\beta_1 = \sum_{i,j=0}^{3} \frac{\mu_i(\mu_i+1)-2}{\mu_i(\mu_i+1)-(1-\nu)}\left\{1 + \frac{\varepsilon^2}{12(1-\nu^2)}\mu_i(\mu_i+1)[\mu_i(\mu_i+1)-2]\right\}$$

$$\times A_i A_j E(\mu_i,\mu_j;\phi_o)$$

$$\beta_2 = -\frac{1}{\beta_1}\left(\sum_{j=0}^{2\mu}\sum_{i=0}^{3} b_j a_j A_i E(j,\mu_i;\phi_o)\right.$$

$$+ \varepsilon \frac{1}{Eh^5}\left\{ww^o \sin\phi\left[\Psi_1{}^o \cot\phi + \Psi_1 - \frac{k}{a} H_2(w)\right]\right\}\Bigg|_{\phi=\phi_o}$$

$$+ \frac{\varepsilon^2}{12(1-\nu^2)}\sum_{j=0}^{2\mu}\sum_{i=0}^{3} c_j A_i \left\{[(\mu_i - 1)B_i + 2 - \mu_i(\mu_i + 1)]\ E(j,\mu_i;\phi_o)\right.$$

$$\left.\left. + \sum_{q=1}^{[\mu_i/2]} [2(\mu_i - 2q) + 1]B_i E(j,\mu_i - 2q;\phi_o)\right\}\right)$$

$$\beta_3 = -\frac{1}{\beta_1}\left(\frac{1}{12(1-\nu^2)Eh^5}\left[ww^o \sin\phi(\Psi_2{}^o \cot\phi + \Psi_2)\right]\Big|_{\phi=\phi_o}\right.$$

$$\left. + \sum_{j=0}^{2\mu}\sum_{k=0}^{2\mu} c_k b_j a_j \left\{(j-1)E(k,j;\phi_o) + \sum_{q=1}^{[j/2]} [2(j-2q)+1]E(k,j-2q;\phi_o)\right\}\right)$$

$$\beta_0 = \sum_{i,j=0}^{3} A_i A_j E(\mu_i,\mu_j;\phi_o)$$

$$\omega_0{}^2 = \frac{\beta_1}{\beta_0}\,\frac{Ehg}{ma^2}$$

$$2\kappa = \sqrt{\frac{\gamma^2 g a^2}{Ehm}\,\frac{\beta_0}{\beta_1}}$$

$$\bar{P} = \frac{1}{\beta_1}\,\frac{a^2}{Eh^2}\int_o^{\phi_o} P \sum_{A_i=0}^{3} A_i P_{\mu_i} \sin\phi\ d\phi$$

and $E(i,j;\phi_o)$ is given in appendix B.

Equation (8) contains the nonlinearity of both quadratic and cubic terms, but as the coefficients of these nonlinear terms are expected to be very small compared to the coefficients of linear terms in thin shells, the solutions utilize the expansion of this equation in power of the nonlinearity parameter $\varepsilon = k/a$. Note that β_2, β_3, and the sign and magnitude of $X(t)$ determine whether the structure acts like a hardening or softening spring.

Applied surface load is assumed to be harmonic in time and is fixed in a mode spatially as in the form

$$\bar{P} = p \cos \omega t = p \cos \Omega\tau \tag{9}$$

$$\Omega = \omega/\omega_o$$

Then a first approximate solution of equation (8) is obtained

$$X(\tau) = \varepsilon\Phi_0 + \Phi_1 \cos \xi + \varepsilon\Phi_2 \cos 2\xi + \varepsilon^2\Phi_3 \cos 3\xi \tag{10}$$

where

$$\xi = \Omega\tau + \theta$$

$$\Phi_0 = -\left(\frac{1}{2} \bar{\beta}_2 A^2 + \frac{\varepsilon}{3} \bar{\beta}_2{}^2 A^3\right)$$

$$\Phi_1 = A + \frac{\varepsilon}{3} \bar{\beta}_2 A^2 - \frac{2}{9} \varepsilon\bar{\beta}_2\bar{p}A + \frac{29}{144} \varepsilon^2\bar{\beta}_2{}^2 A^3 - \frac{\varepsilon^2}{32} \bar{\beta}_3 A^3$$

$$\Phi_2 = \frac{1}{6} \bar{\beta}_2 A^2 + \frac{2}{9} \bar{\beta}_2\bar{p}A + \frac{\varepsilon}{9} \bar{\beta}_2{}^2 A^3$$

$$\Phi_3 = \frac{1}{8}\left(\frac{1}{6} \bar{\beta}_2{}^2 + \frac{1}{4} \bar{\beta}_3\right)A^3$$

$$\bar{\beta}_2 = \frac{\beta_2}{1 - \kappa^2}, \quad \bar{\beta}_3 = \frac{\beta_3}{1 - \kappa^2}, \quad \bar{p} = \frac{p}{1 - \kappa^2}$$

and amplitude parameter A included in those coefficients can be determined from

$$\varepsilon^2\left(\frac{3}{4} \bar{\beta}_3 - \frac{5}{6} \bar{\beta}_2{}^2\right)A^3 + \left(1 + \frac{\varepsilon}{3} \bar{\beta}_2\bar{p} - \frac{\Omega^2}{1 - \kappa^2}\right)A - \bar{p} = 0 \tag{11}$$

The following relations give the angle of phase difference that is due to the viscous damping term κ.

$$p \cos \theta = -\Phi_1\Omega^2 + \Phi_1 + \varepsilon^2\beta_2\Phi_1(2\Phi_0 + \Phi_2) + \varepsilon^2\beta_3\Phi_1{}^3 \frac{3}{4}$$

$$p \sin \theta = -2\kappa\Phi_1\Omega$$

Note that the solutions given by equation (10) are not symmetrical; that is, nonlinear vibration of spherical shells is physically nonsymmetrical with regard to the middle surface. Therefore, the amplitude parameter A tends to increase or decrease with frequency depending on the value of β_2 and β_3; it does not necessarily mean a hardening or softening of this structure. Neither does the increase or decrease of amplitude of $X(t)$ with frequency necessarily mean a hardening or softening structure.

Equations (8) and (10) indicate that nonlinearity of softening and hardening appears alternatively according to the change of sign of $X(t)$ (in some region of that value) in relation to the values of β_2 and β_3, when the shell is considered a mass.

STABILITY OF THE SOLUTIONS IN STEADY STATE

If small disturbances are added to A and ξ in such forms as

$$\bar{A} = A + \eta e^{\lambda\tau}$$

and

$$\bar{\xi} = \xi + \zeta e^{\lambda\tau} = \xi + \xi_1$$

the solutions of equation (8) can be given as

$$X(\tau) = \varepsilon(\Phi_0 + \bar{\Phi}_0) + (\Phi_1 + \bar{\Phi}_1)\cos\bar{\xi} + \varepsilon(\Phi_2 + \bar{\Phi}_2)\cos 2\bar{\xi} + \varepsilon^2(\Phi_3 + \bar{\Phi}_3)\cos 3\bar{\xi} \tag{12}$$

Since the values of η and ζ can be assumed to be very small and λ is considerably smaller, the high order terms of η and ζ can be negligible against η and ζ, and the next approximation will be applicable.

$$\cos 3\xi_1 \simeq 1 \quad , \qquad \sin 3\xi_1 \simeq 3\xi_1 \qquad (A'' \simeq 0,\ \xi_1'' = 0)$$

Then equation (12) can be rewritten

$$X(\tau) = X_0(\tau) + \frac{1}{\Omega} X_0'(\tau)\xi_1 + X_1(\tau) \tag{12a}$$

where $X_0(\tau)$ satisfies equation (10) and $X_1(\tau)$ is of the form

$$X_1(\tau) = \varepsilon\bar{\Phi}_0 + \bar{\Phi}_1 \cos\xi + \varepsilon\bar{\Phi}_2 \cos 2\xi + \varepsilon^2\bar{\Phi}_3 \cos 3\xi$$

Substituting equation (12a) into equation (8), and neglecting the high order terms of X_1 and ξ_1 gives the following equation

$$X_1'' + 2\kappa X_1'' + X_1 + 2\varepsilon\beta_2 X_0 X_1 + 3\varepsilon^2\beta_3 X_0^2\, X_1 - p(\sin\,\Omega\tau)\,\xi_1 + \frac{2}{\Omega}\,(X_0'' + \kappa X_0')\xi_1' = 0$$

From this equation the following two relations can be deduced.

$$-\Omega^2\bar{\Phi}_1 + 2\kappa\lambda\bar{\Phi}_1 + \bar{\Phi}_1 + 2\varepsilon^2\beta_2\left(\Phi_0\bar{\Phi}_1 + \bar{\Phi}_0\Phi_1 + \bar{\Phi}_1\Phi_2\,\frac{1}{2} + \frac{1}{2}\,\bar{\Phi}_2\Phi_1\right)$$

$$+ 3\varepsilon^2\beta_3\left(\frac{3}{4}\,\Phi_1{}^2 + \frac{1}{2}\,\varepsilon^2\Phi_2{}^2\right)\bar{\Phi}_1 + p\,\sin\,\Phi\,\xi_1 - 2\lambda\Omega\Phi_1\xi_1 = 0$$

$$2\lambda\bar{\Phi}_1\Omega + 2\kappa\bar{\Phi}_1\Omega + p(\cos\,\Phi)\xi_1 + 2\kappa\Phi_1\lambda\xi_1 = 0$$

where

$$\bar{\Phi}_i = \frac{d\Phi_i}{dA}\,\eta e^{\lambda\tau} = \bar{\bar{\Phi}}_i(\eta e^{\lambda\tau})$$

To determine η and ζ from these two equations, the determinant of the coefficients of η and ζ must be zero, and λ is given as the root of this determinant in the following form:

$$X\lambda^2 + Y\lambda + Z = 0$$

where

$$X = 4(\kappa^2 + \Omega^2)$$

$$Y = 4\kappa\left[1 + \Omega^2 + \beta_2\varepsilon^2\left(2\Phi_0 + \Phi_2 + \delta\bar{\bar{\Phi}}_0 + \frac{1}{2}\,\delta\bar{\bar{\Phi}}_2\right) + \frac{3}{2}\,\beta_3\varepsilon^2\left(\Phi_1{}^2 + \frac{1}{2}\,\varepsilon\Phi_2{}^2\right)\right]$$

$$Z = \left[1 - \Omega^2 + 2\varepsilon^2\beta_2\left(\Phi_0 + \bar{\bar{\Phi}}_0\delta + \frac{1}{2}\,\Phi_2 + \frac{1}{2}\,\bar{\bar{\Phi}}_2\delta\right) + 3\varepsilon^2\beta_3\left(\frac{3}{4}\,\Phi_1{}^2 + \frac{1}{2}\,\varepsilon^2\Phi_2{}^2\right)\right]\left[1 - \Omega^2 + \varepsilon^2\beta_2\left(2\Phi_0 + \Phi_2\right) + \frac{3}{4}\,\varepsilon^2\beta_3\Phi_1{}^3\right] + 4\kappa^2\Omega^2$$

$$\delta = \frac{\Phi_1}{\bar{\bar{\Phi}}_1}$$

If a root λ of this equation has a real part of positive value, the oscillation is divergent. This condition occurs if

$$\left.\begin{array}{ll} & Y \geq 0 \quad \text{and} \quad Z < 0 \\ \text{or} & \\ & Y < 0 \end{array}\right\} \qquad (13)$$

Then it is concluded that the oscillation concerning an amplitude parameter A that satisfies conditions (13) with associated Ω is unstable.

NUMERICAL RESULTS AND CONCLUSION

A few numerical calculations will be shown for hinged ($w = M\phi = u = 0$) and clamped boundary conditions.

Since a part of plane displacement u contains quadratic terms of A_i, coefficients A_i are determined as solutions of three simultaneous nonlinear equations associated with boundary conditions. These quadratic terms with the multiplying factor ε can, however, be assumed to be smaller than linear ones. Therefore, only the linear part of u will be considered here, and A_i can be given approximately from linear equations. Numerical results have shown the propriety of this assumption.

Shell With Hinged Edge

Deflection mode shape depends not only on the selection of orders of Legendre polynomials but also on the open angle of shells and boundary conditions. Deflection shapes will be shown for various sets of μ_i with the parameter of open angle ϕ_0 in figure 1.

$$\left.\begin{array}{ll} \text{Case (a)} & (0,1,2,3) \\ \text{Case (b)} & (0,1,2,4) \\ \text{Case (c)} & (0,2,3,4) \end{array}\right\} \qquad \text{for } (\mu_0, \mu_1, \mu_2, \mu_3)$$

The case of $\phi_0 = 120°$ will be discussed here. Figure 2 with regard to $W(\phi)$ and $U(\phi)$ shows that the inertia term of u may not be negligible compared with that of w in low order mode shape, despite its neglect in this analysis. Figure 3 gives relations between amplitude parameter $|A|$ and modified frequency Ω.[1] The case $p = 0$ means free vibration. Tendency of increase or decrease of $|A|$ with increasing Ω depends only on the sign of the term $[(3/4)\bar{\beta}_3] - [(5/6)\bar{\beta}_2^2]$ (see eq. (11)) and this does not indicate the stiffness of hardening or softening of this system as easily recognized from equation (8). Relations between nonlinearity and the inverse of multiplying parameter $1/\varepsilon = a/h$

[1]Note that $\Omega = \omega/\omega_o$ depends on the ratio β_1/β_0. Values for β_i (and A_i) are given in table 1.

will be given in figure 4 in free vibration. The linear state is $\varepsilon = 0$, and nonlinearity increases rapidly with increasing ε. Figure 5 also shows $|A| \sim \Omega$ relations of $p = 0.1$ with the parameter of viscous damping κ. Large values of $|A|$ decrease with increasing κ. When three kinds of $|A|$ exist, the smallest $|A|$ increases with increasing κ. Simultaneously, the stability zone will be indicated on figure 5 for the case of $\kappa = 0$. Oscillation for positive A is unstable in the domain bounded by +UL and +LL. Similarly, oscillation for negative A is unstable in the domain bounded by -UL and -LL. Figure 5(a) reveals that in the range of Ω shown in this figure the following cases appear in turn with increasing Ω.

(2S,1U) → (1S,2U) → (2S,1U) → (1S)

S = stable oscillation

U = unstable oscillation

Variation of the instability zone with κ will be given in figure 6. The existence of κ makes the unstable region of $|A|$ small. Figure 7 shows the angle of phase difference according to κ in relation with Ω. Finally, time-dependent function $X(\tau)$ will be shown in figure 8 for some fixed Ω. Note that the maximum $|A|$ of an Ω does not necessarily correspond to the one of three $X(\tau)$ that has maximum amplitude, and the term $\Phi_1 \cos \xi$ is dominant in $X(\tau)$. Also figure 8 indicates that shell surface does not vibrate symmetrically with respect to midsurface. There is a slight difference in $|A|$ of $p = 0.1$ and $p = -0.1$ in figure 3, but figure 8 shows that in the final results for $X(\tau)$ only a difference of phase angle appears.

All β_3 are positive, in these examples, $\beta_2 > 0$ in cases (a) and (b), and $\beta_2 < 0$ in (c). In all cases $X(\tau)$ satisfies the condition

$$|X(\tau)| < \left|\frac{\beta_2}{\beta_3 \varepsilon}\right|$$

Then it can be concluded from equation (8) that the effect of softening and hardening appears alternately according to the change of sign of $X(\tau)$.

		Cases (a) and (b)	*Case (c)*
$X(\tau)$	positive	hardening	softening
$X(\tau)$	negative	softening	hardening

The deflection shape $W(\phi)$ in case (a) is similar to that in case (b) but has a different magnitude (figs. 2(a) and 2(b)). However, the final results for $w(\phi,t)$ differ only slightly.

Similar figures are shown in figures 9 through 14 for the clamped edge condition.

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APPENDIX A

DETERMINATION OF a_i AND C_i

The expressions for the stress function ψ and the longitudinal displacement u contain the terms a_i and C_i (see eqs. (6) and (7)). These terms are evaluated as follows. From equation (5),

$$w = \sum_{i=0}^{3} A_i P_{\mu_i} hX(t) \tag{A1}$$

$$(w^o)^2 = \sum_{i=0}^{3} \sum_{j=0}^{3} A_i A_j P'_{\mu_i} P'_{\mu_j} h^2 X^2(t) \tag{A2}$$

If equation (A2) is set as

$$(w^o)^2 = \sum_{k=0}^{2\mu} C_k P_k h^2 X^2(t) \tag{A3}$$

the coefficients of this finite series are determined with Legendre polynomials. From the expansion (ref. 7),

$$P_n P_m = \sum_{r=|n-m|}^{n+m} {}_{n,m}C_r P_r$$

it follows by differentiation with respect to ϕ that

$$P_n{}^1 P_m{}^1 = \sum_{r=|n-m|}^{n+m} {}_{n,m}D_r P_r$$

where

$${}_{i,j}D_k = \frac{1}{2}\{i(i + 1) + j(j + 1) - k(k + 1)\}\ {}_{i,j}C_k$$

$$ {}_{i,j}C_k = \frac{2k + 1}{i + j + k + 1} \; \frac{R_{\frac{i-j+k}{2}} R_{\frac{j-i+k}{2}} R_{\frac{i+j-k}{2}}}{R_{\frac{i+j+k}{2}}} $$

$$ R_s = \frac{(2s - 1)!!}{s!} $$

for integer values of s,

$$ (-1)!! = 0!! = 1 $$

and if $|i - j|$ is even, then $\quad {}_{i,j}D_k \equiv 0 \quad (k \text{ is odd})$

$|i - j|$ is odd, then $\quad {}_{i,j}D_k \equiv 0 \quad (k \text{ is even})$

Then the coefficients C_k of equation (A3) can be determined as

$$ C_k = \sum_{i=0}^{3} \sum_{j=0}^{3} A_i A_j \; {}_{\mu_i,\mu_j}D_k \tag{A4} $$

At $\cos \phi = 1$,

$$ P_m{}' = 0, \; P_m = 1 \qquad (m \text{ is arbitrary integer}) $$

These coefficients then satisfy the relation

$$ \sum_{k=0}^{2\mu} C_k = \sum_{k=0,2,4,\ldots}^{2\mu} C_k = \sum_{k=1,3,5,\ldots}^{2\mu} C_k = 0 \tag{A5} $$

The nonlinear terms of compatibility of equation (3) can be expanded in a series of Legendre polynomials.

$$ H_0(w)w^{oo} - (w^{oo})^2 - (w^o)^2 = \sum_{i=0}^{2\mu} a_i P_i \tag{A6} $$

The left term of equation (A6) can be represented by substitution of equation (A3)

$$- \sum_{k=0}^{2\mu} C_k P_k - \frac{\cos \phi}{2 \sin \phi} \sum_{r=0}^{2\mu} C_r P_r{}^1$$

Then

$$a_i = -C_i - \frac{2i + 1}{4} \int_0^{\pi} \sum_{r=0}^{2\mu} C_r P_r{}^1 P_i \cos \phi \, d\phi$$

$$= \frac{i - 1}{2} C_i + \frac{2i + 1}{4} \sum_{r=i}^{2\mu} C_r \{(-1)^{i+r+1} - 1\} \qquad \text{(A7)}$$

because

$$\int_0^{\pi} P_r{}^1 P_i \cos \phi \, d\phi = \begin{cases} 0 & r + 1 \leq i \\ \dfrac{2i}{2i + 1} & r = i \\ 1 - (-1)^{i+r+1} & r > i \end{cases}$$

From equations (A5) and (A7)

$$a_1 \equiv 0$$

APPENDIX B

DETERMINATION OF $E(i,j;\phi_0)$

The coefficients appearing in equation (8) depend on the quantity $E(i,j;\phi_0)$ that is evaluated below.

$$E(i,j;\phi_0) = \int_0^{\phi_0} P_i P_j \sin\phi \, d\phi$$

$$= \frac{1}{i(i+1) - j(j+1)} \left[\sin\phi (P_i{}^1 P_j - P_j{}^1 P_i)\right]_0^{\phi_0}$$

$$= \frac{1}{i(i+1) - j(j+1)} \left[\sin\phi (P_i{}^1 P_j - P_j{}^1 P_i)\right]\Big|_{\phi=\phi_0} \qquad (i \neq j)$$

$$E(i,i;\phi_0) = \int_0^{\phi_0} P_i P_i \sin\phi \, d\phi$$

$$= \frac{1}{i + (1/2)} \left\{ \int_0^{\phi_0} P_i \frac{d}{d\phi} P_{i-1} \, d\phi - \frac{1}{2}\left[P_i P_i \cos\phi\right]_0^{\phi_0} \right\}$$

$$= \frac{1}{i + (1/2)} \left[\frac{1}{2} - \frac{1}{2}\{P_i P_i \cos\phi\}\Big|_{\phi=\phi_0} - \sum_{p=1}^{[i/2]*} [2(i - 2p) + 1] E(i,i-2p;\phi_0) \right]$$

by using the following expansion

$$\frac{d}{d\phi} P_{i-1} = (-1) \sum_{p=1}^{[i/2]} \{2(i - 2p) + 1\} P_{i-2p} \sin\phi$$

*[i/2]: gaussian symbol

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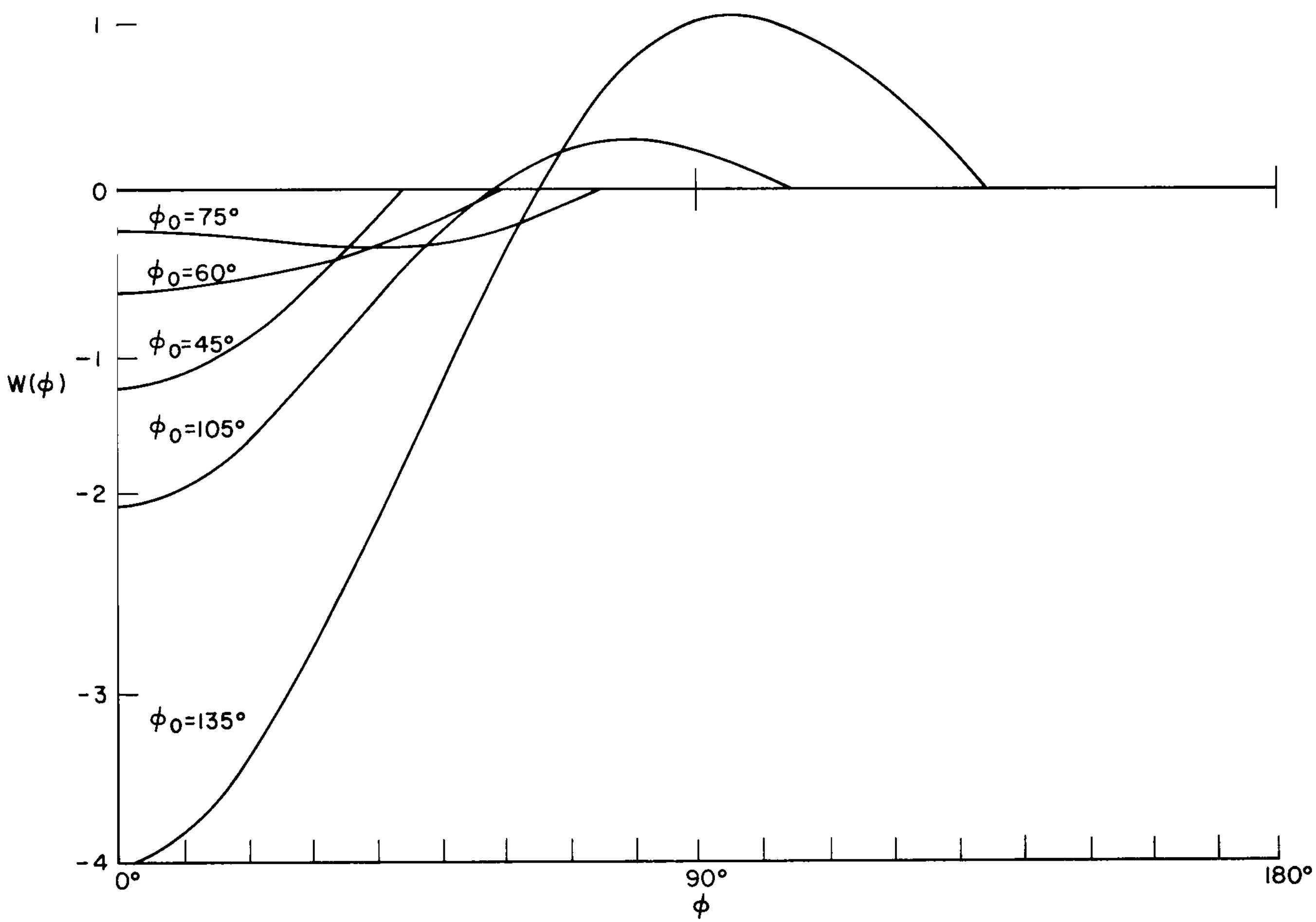
TABLE 1.- NUMERICAL VALUES FOR CONSTANTS USED IN

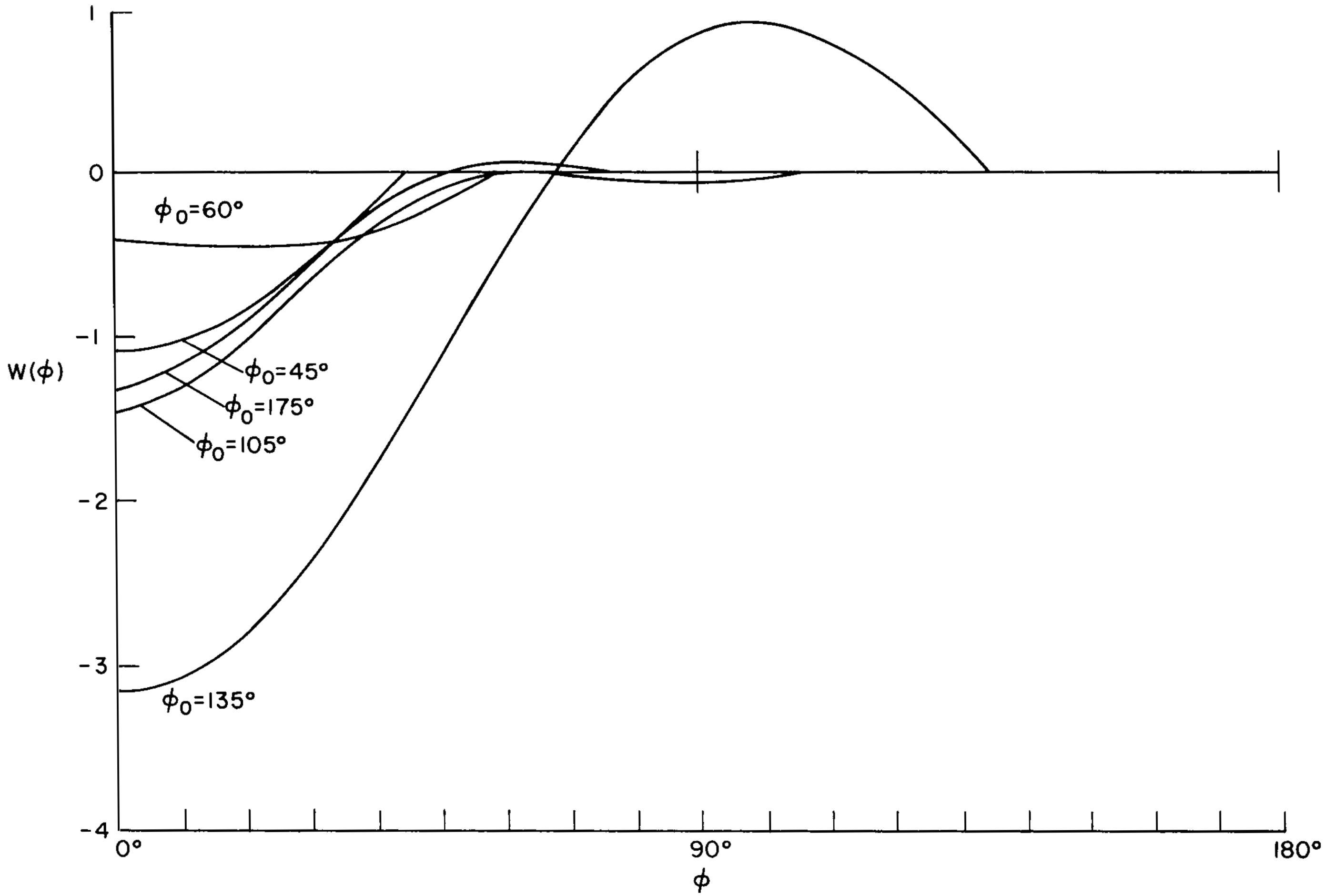
EXAMPLE CALCULATIONS

$(\mu_0, \mu_1, \mu_2, \mu_3)$	Hinged edge			Clamped edge
	Case (a) (0,1,2,3)	Case (b) (0,1,2,4)	Case (c) (0,2,3,4)	(0,1,2,3)
A_0	-0.2	-0.2	-0.2	-0.2
A_1	-.42379	-.35341	.73030	.035121
A_2	-1.19505	-.87531	1.85114	.15909
A_3	-.36864	.29794	1.79404	.54274
β_0	.74510	.43495	1.93892	.00310
β_1	.72470	.78662	.94502	36.67234
β_2	4.35677	2.78319	-6.08482	2.40445
β_3	4.23661	2.25442	54.60953	1.35567



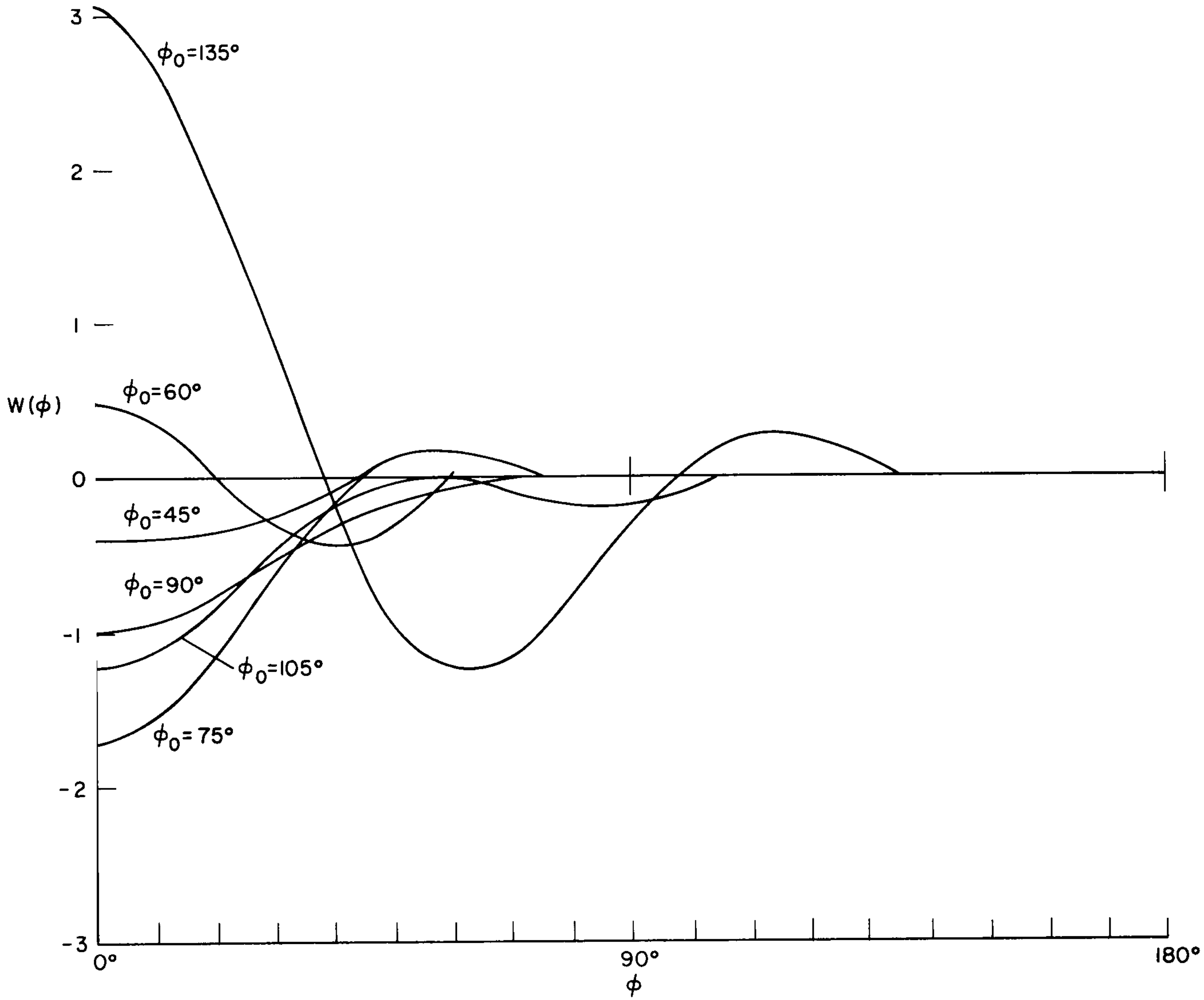
(a) μ_i = (0,1,2,3)

Figure 1.- Deflection shapes for various open angles; a/h = 100, hinged edge.



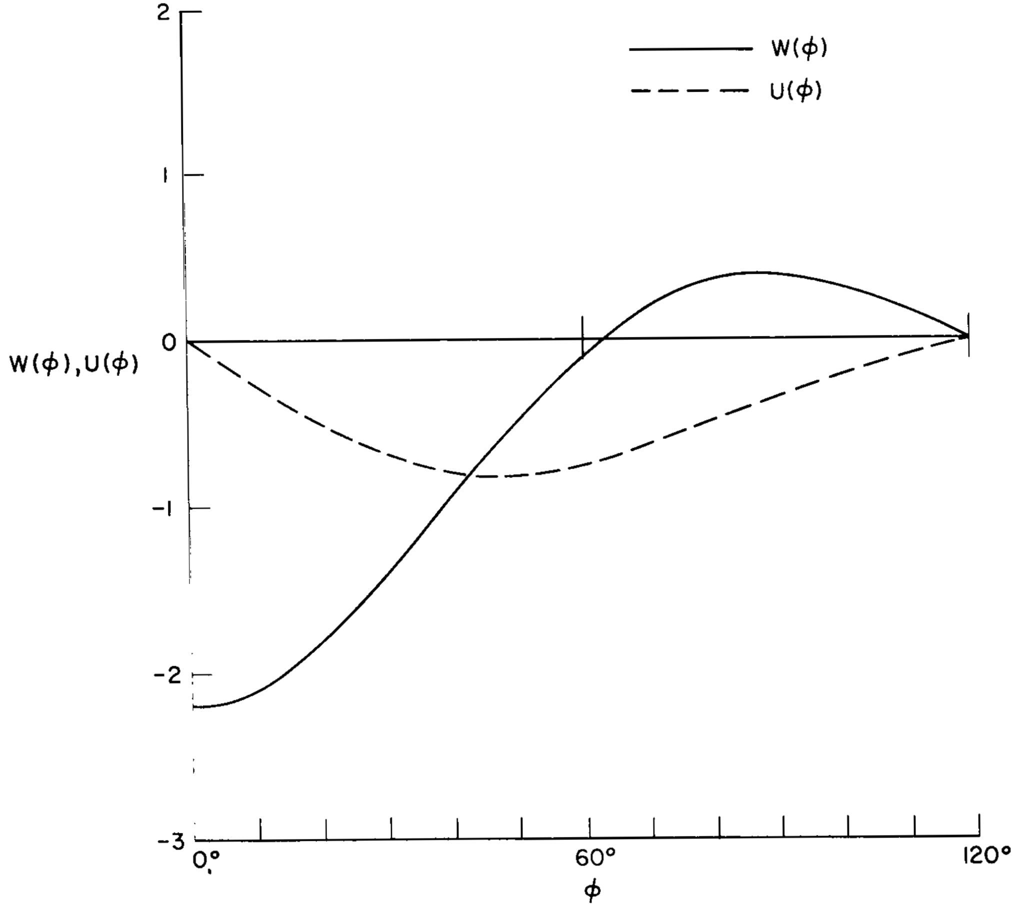
(b) $\mu_i = (0,1,2,4)$

Figure 1.- Continued.



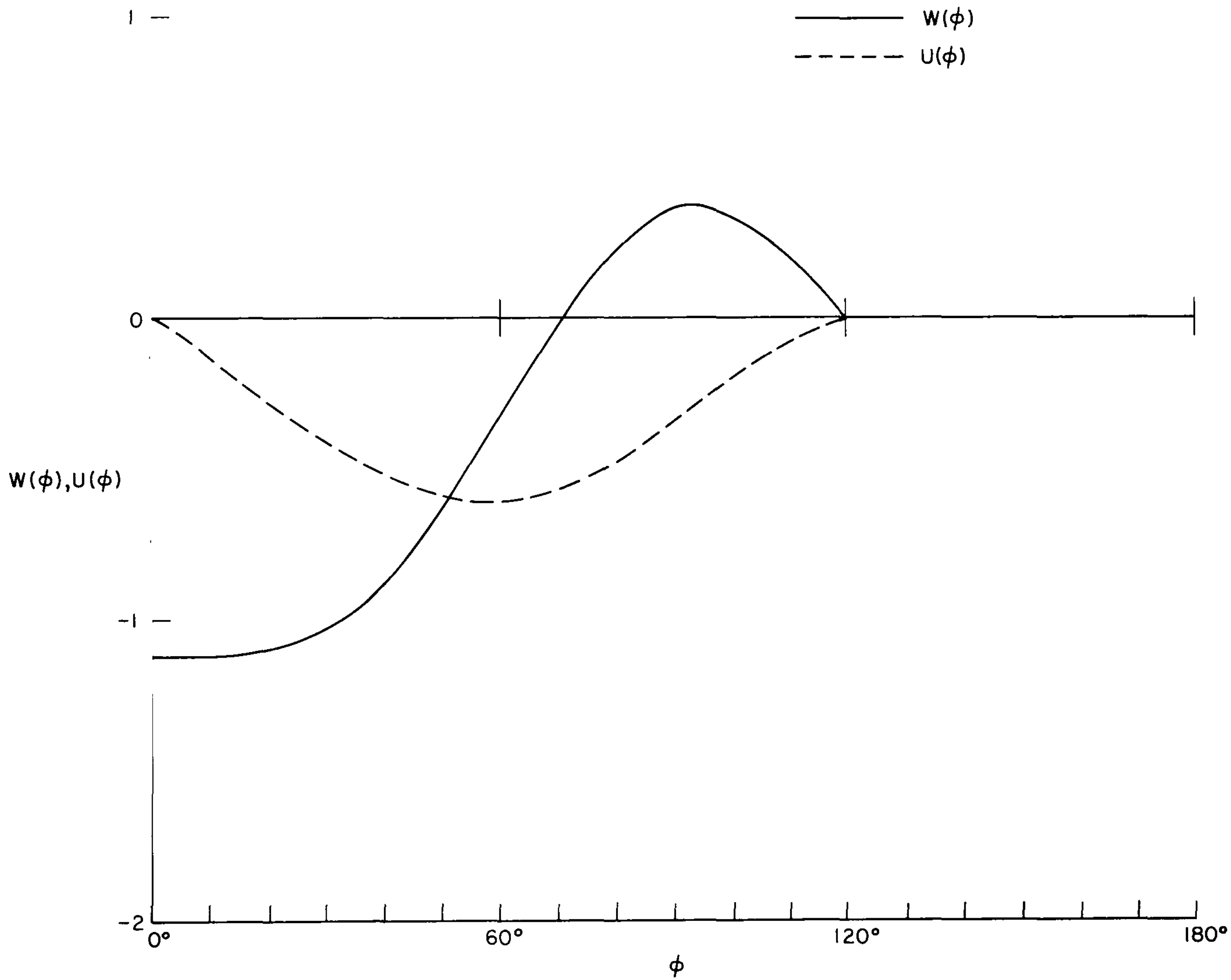
(c) μ_i = (0,2,3,4)

Figure 1.- Concluded.



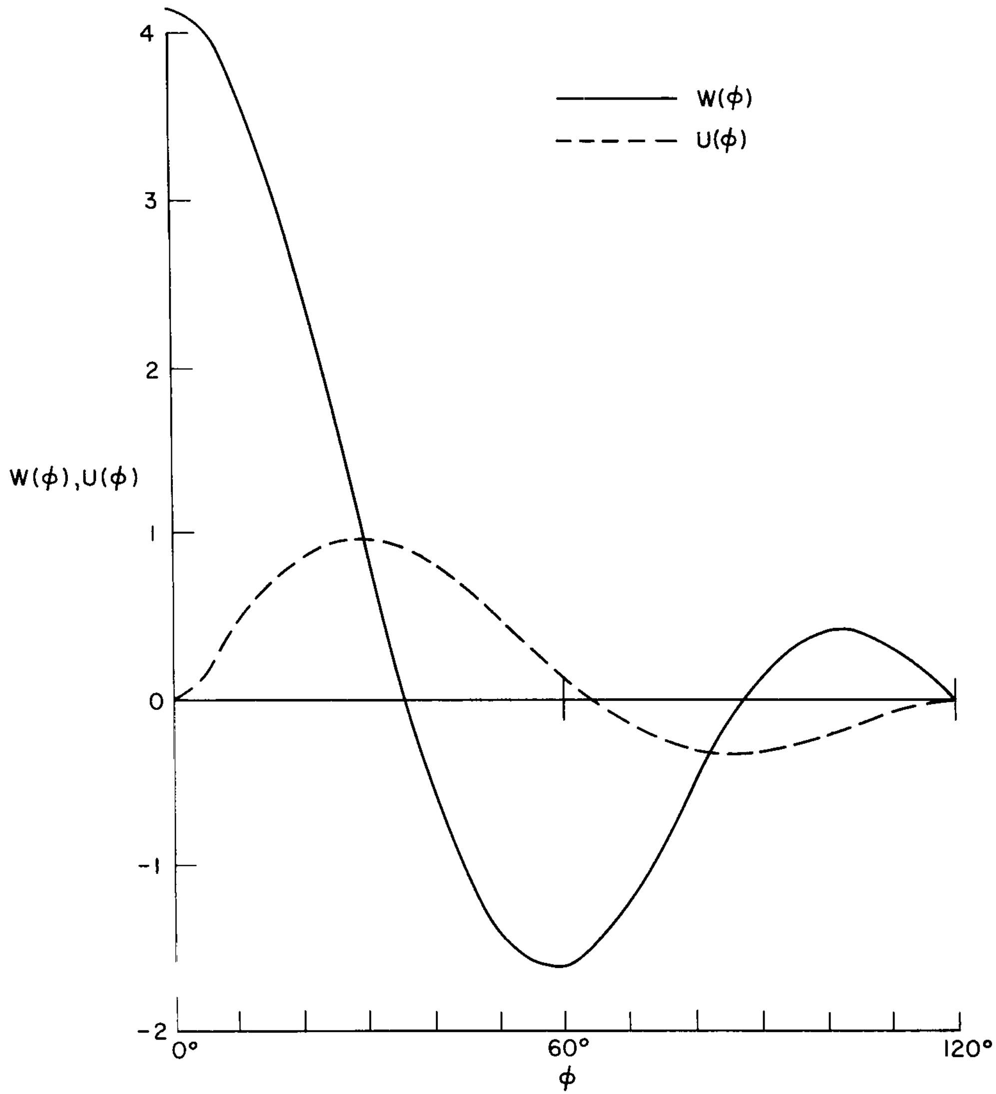
(a) $\mu_i = (0,1,2,3)$

Figure 2.- Deflection and in-plane displacement shapes for a shell of open angle $\phi_0 = 120°$; a/h = 100, hinged edge.



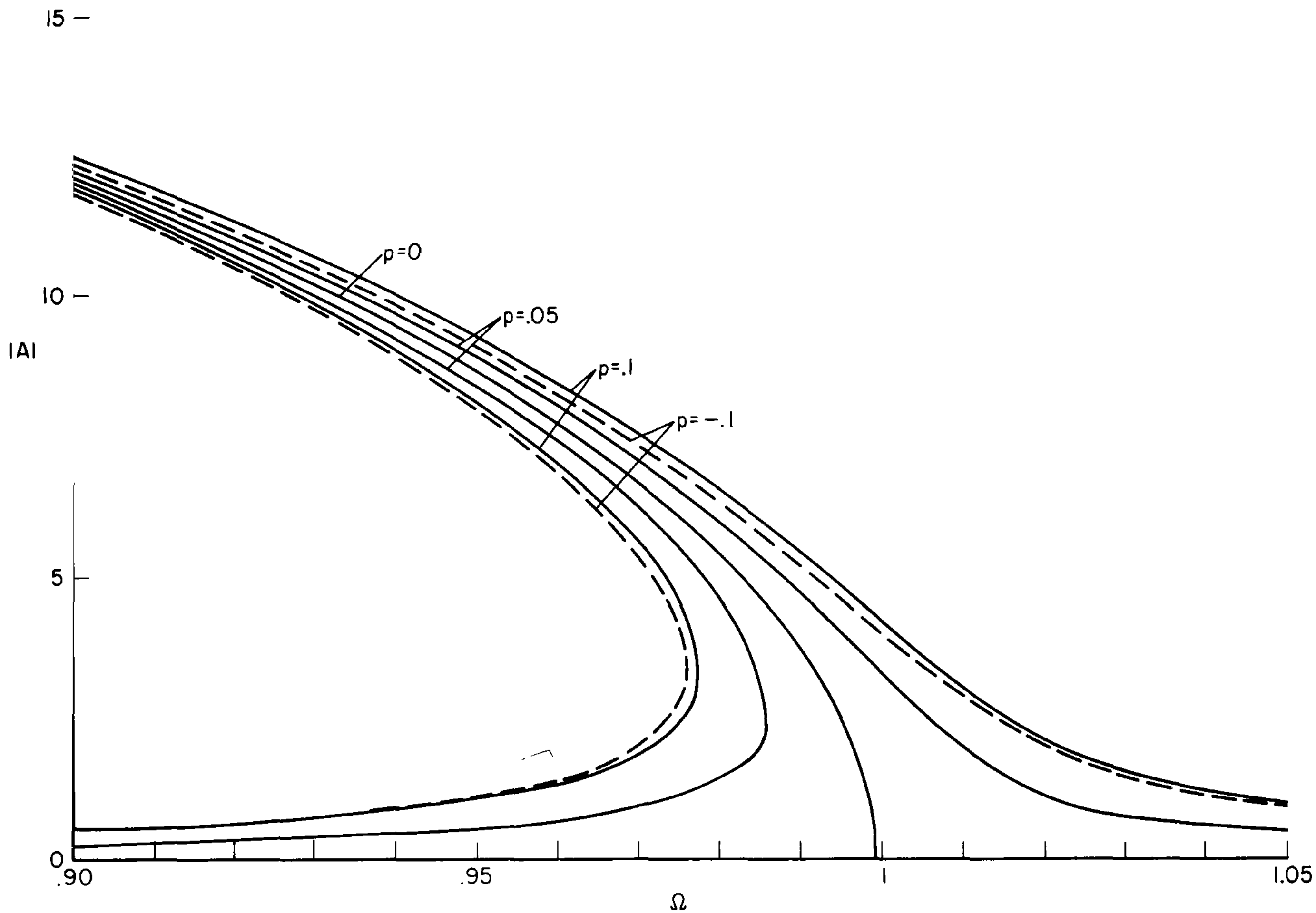
(b) $\mu_i = (0,1,2,4)$

Figure 2.- Continued.



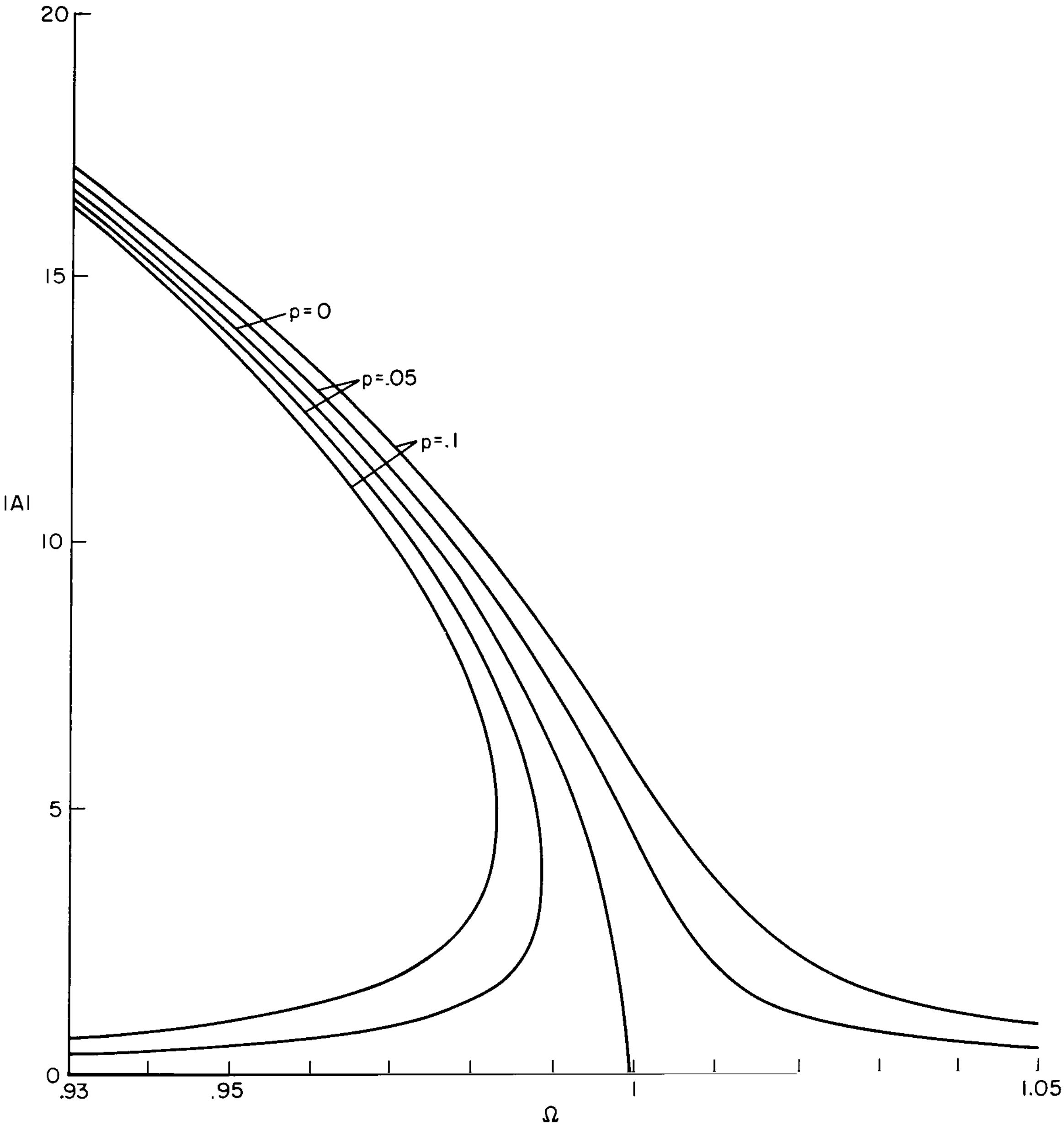
(c) $\mu_i = (0,2,3,4)$

Figure 2.- Concluded.



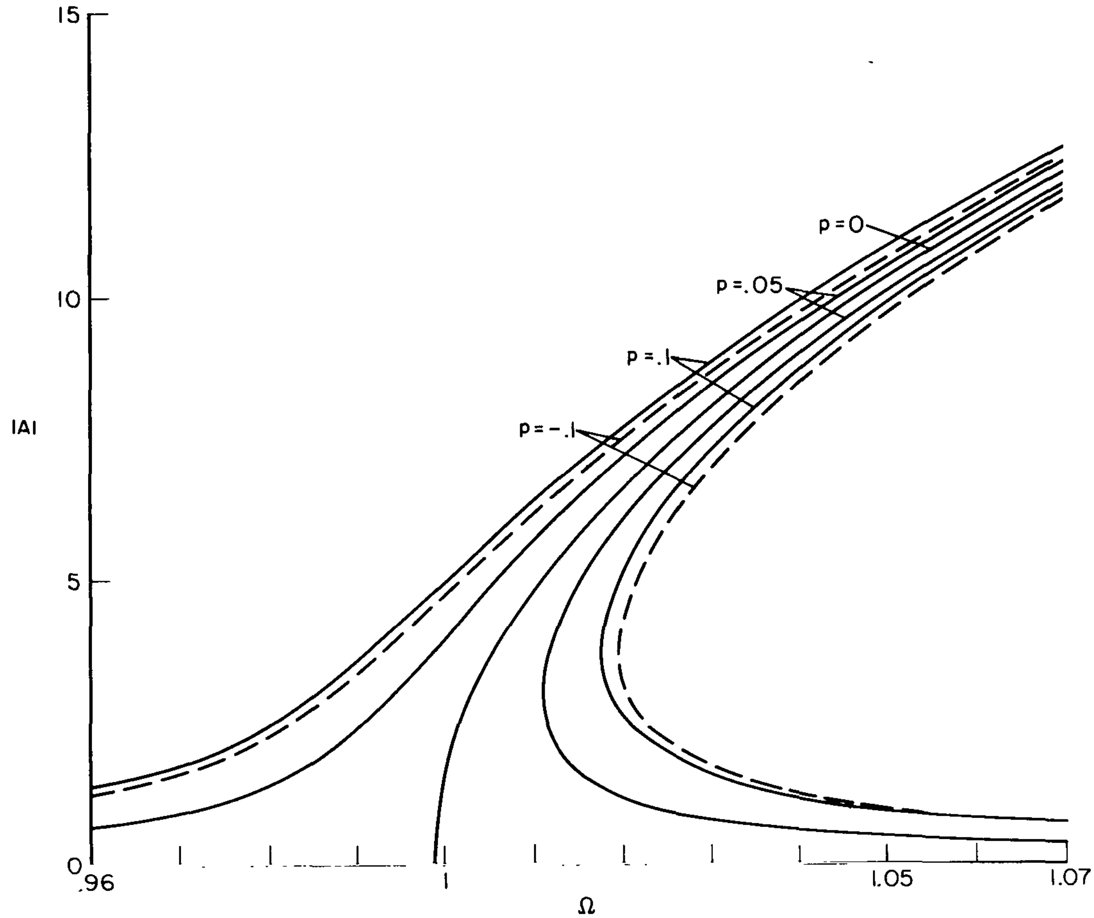
(a) $\mu_i = (0,1,2,3)$

Figure 3.- Relation between amplitude parameter $|A|$ and modified angular frequency Ω with various magnitudes of external forces p; $\phi_0 = 120°$, $a/h = 100$, $\kappa = 0.05$, hinged edge.



(b) $\mu_i = (0,1,2,4)$

Figure 3.- Continued.



(c) $\mu_i = (0,2,3,4)$

Figure 3.- Concluded.

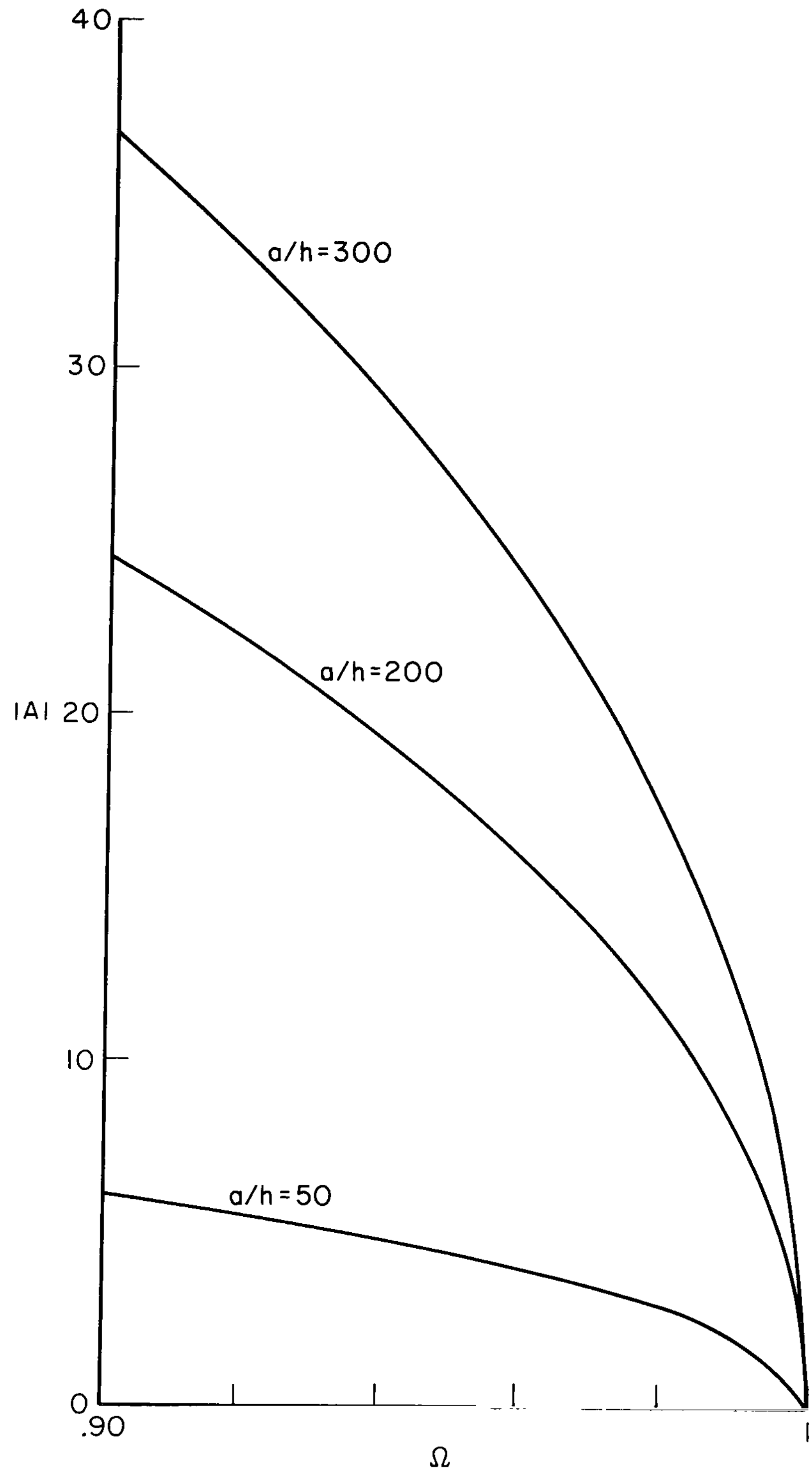
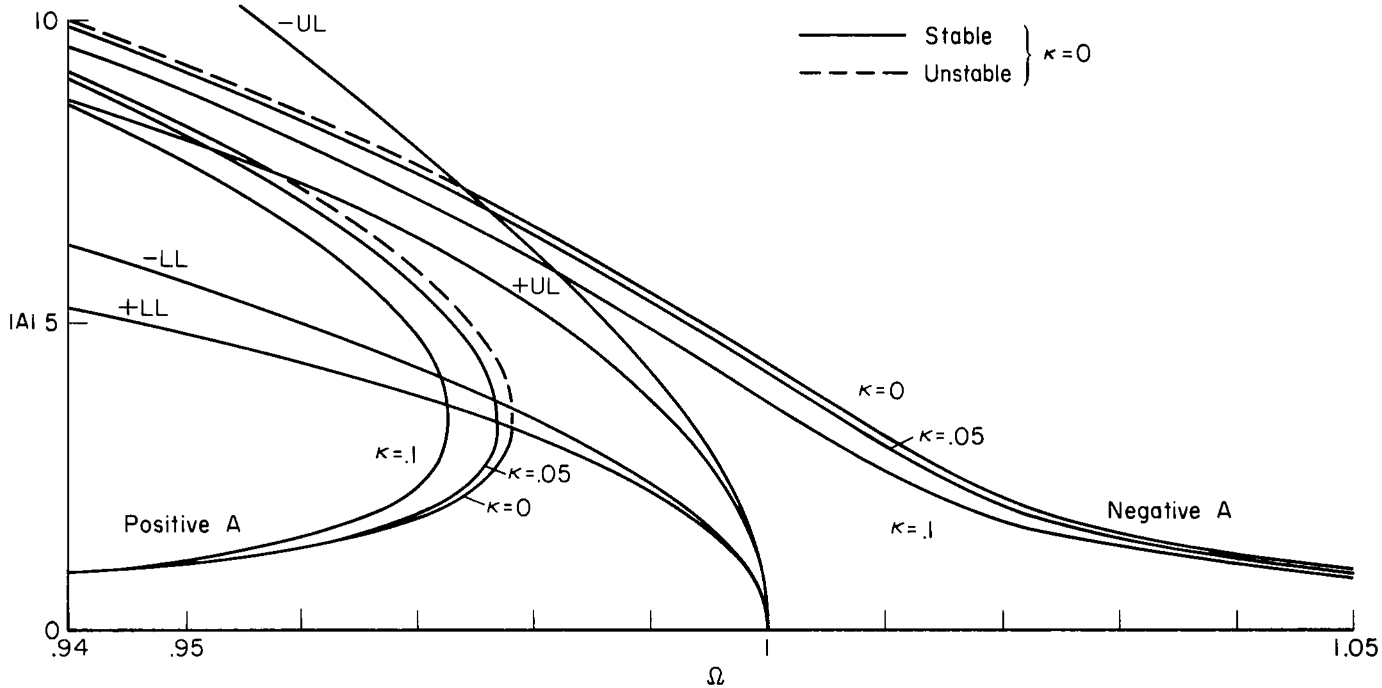
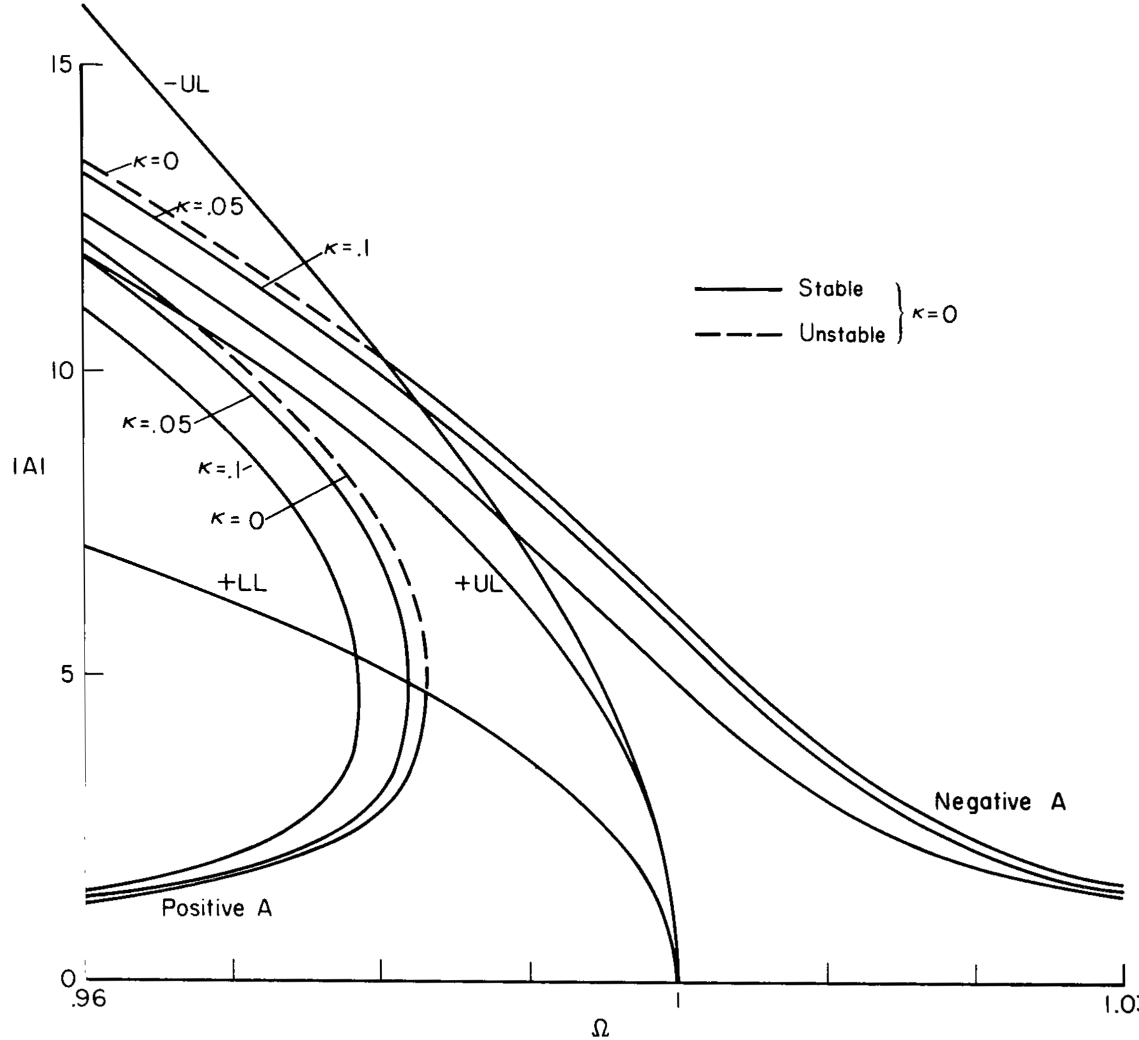


Figure 4.- Variation of $|A|$ - Ω relation with respect to radius-thickness ratio in undamped free vibration; $p = 0$, $\kappa = 0$, $\phi_0 = 120°$, hinged edge, $\mu_i = (0,1,2,3)$.



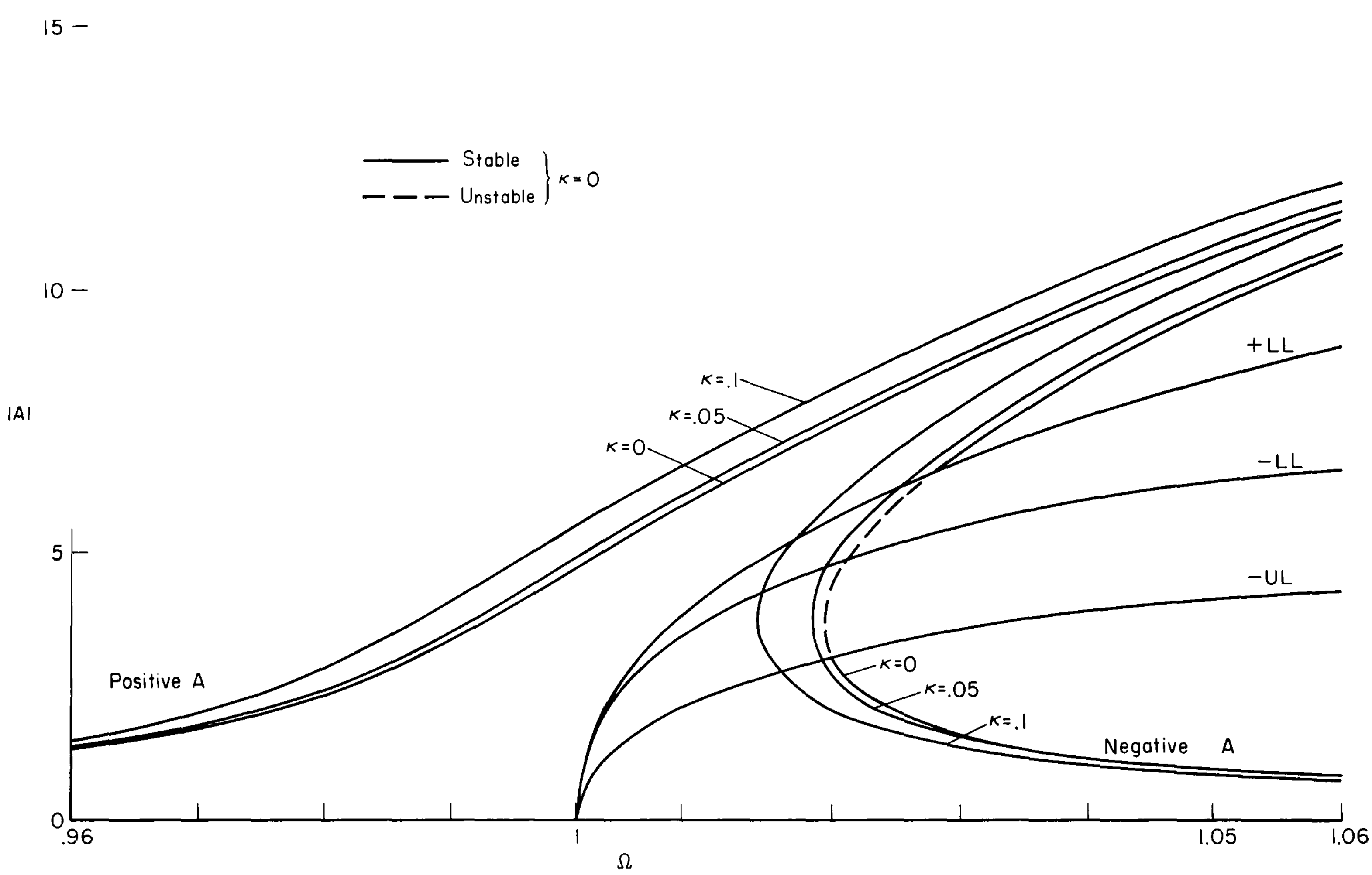
(a) $\mu_i = (0,1,2,3)$

Figure 5.- Variation of $|A|$ - Ω relation with respect to viscous damping κ and instability zone in nondamping state ($\kappa = 0$). Domains limited by +UL and +LL, and -UL and -LL are unstable regions; $p = 0.1$, $\phi_0 = 120°$, $a/h = 100$, hinged edge.



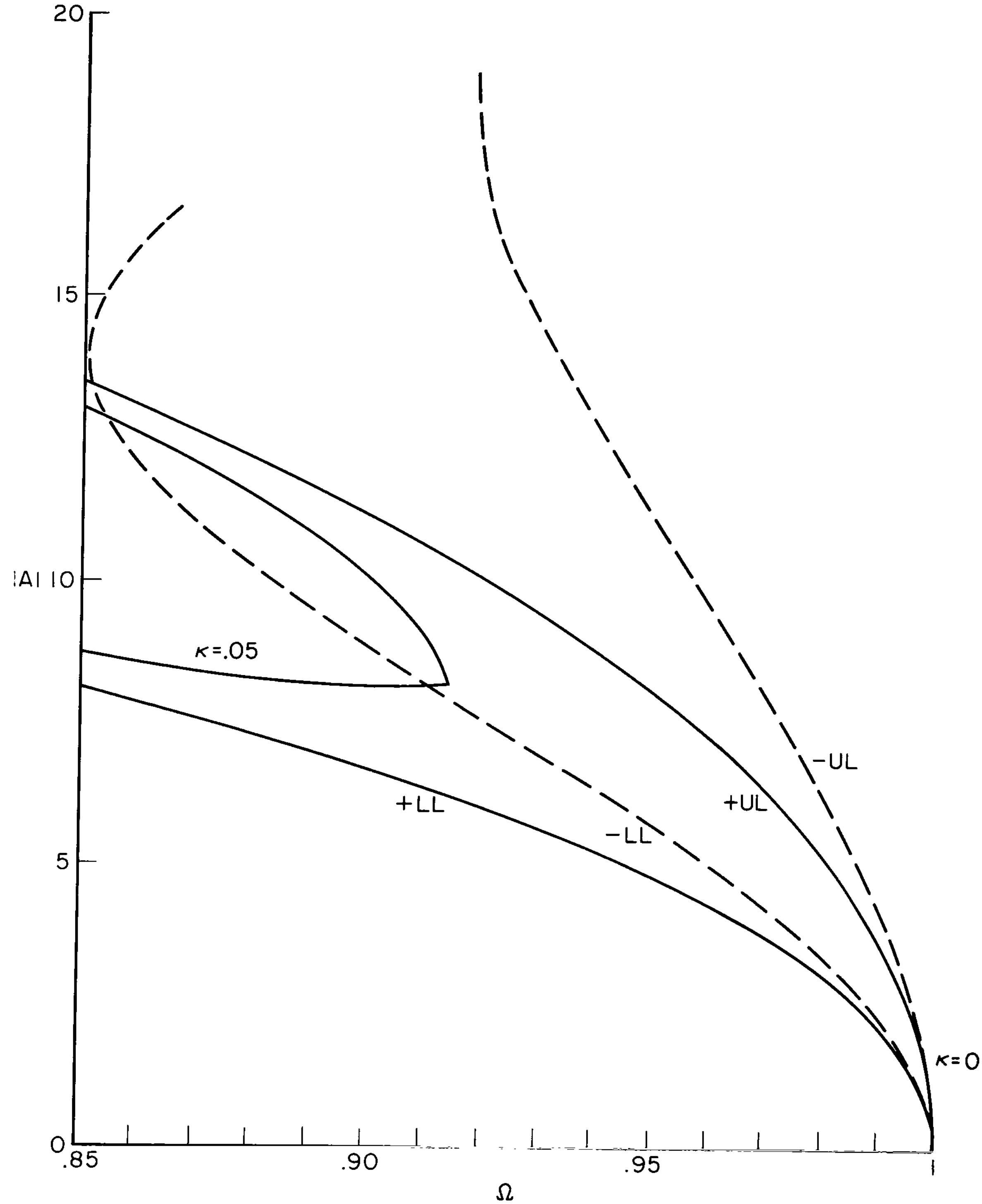
(b) $\mu_i = (0,1,2,4)$

Figure 5.- Continued.



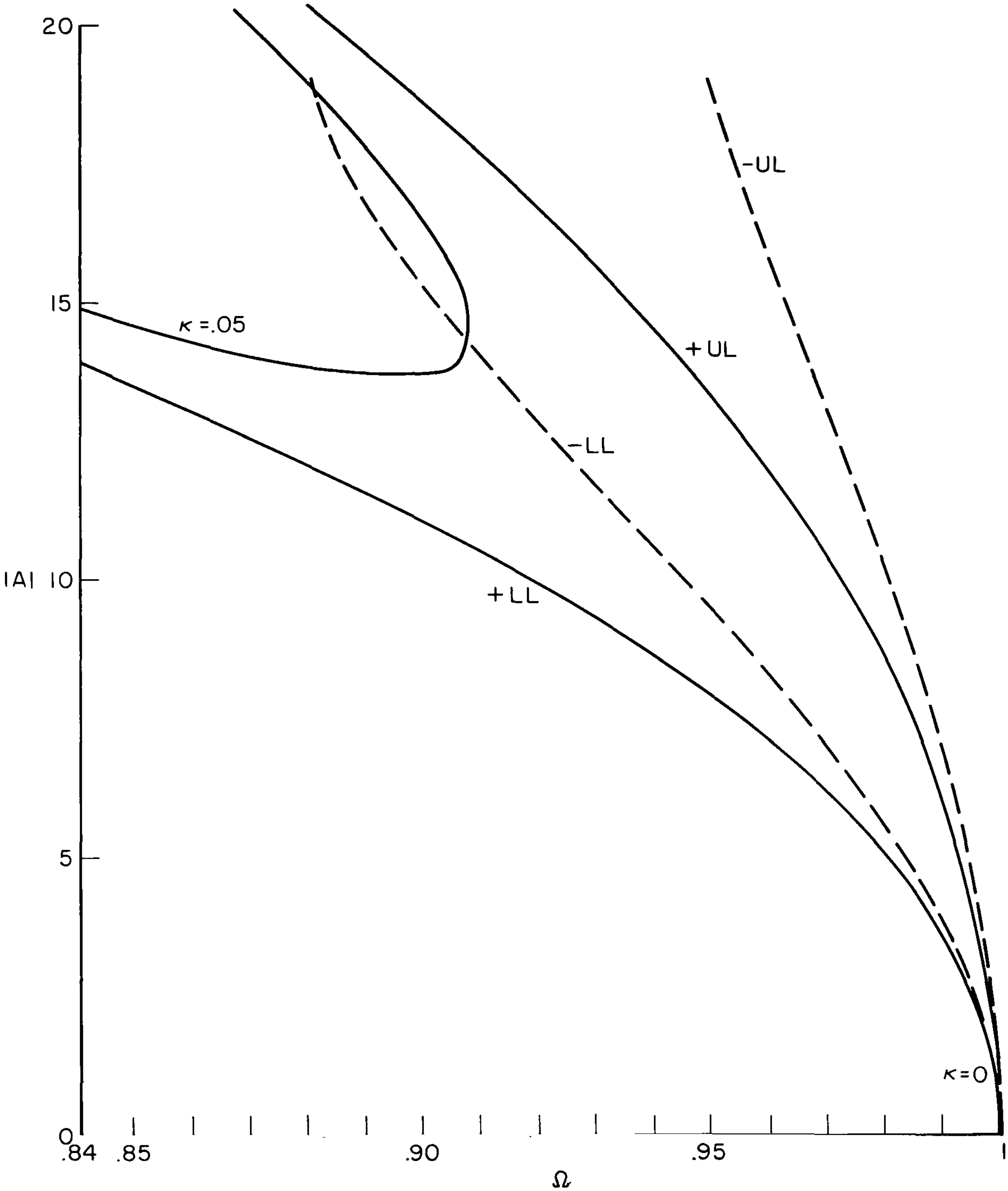
(c) $\mu_i = (0,2,3,4)$

Figure 5.- Concluded.



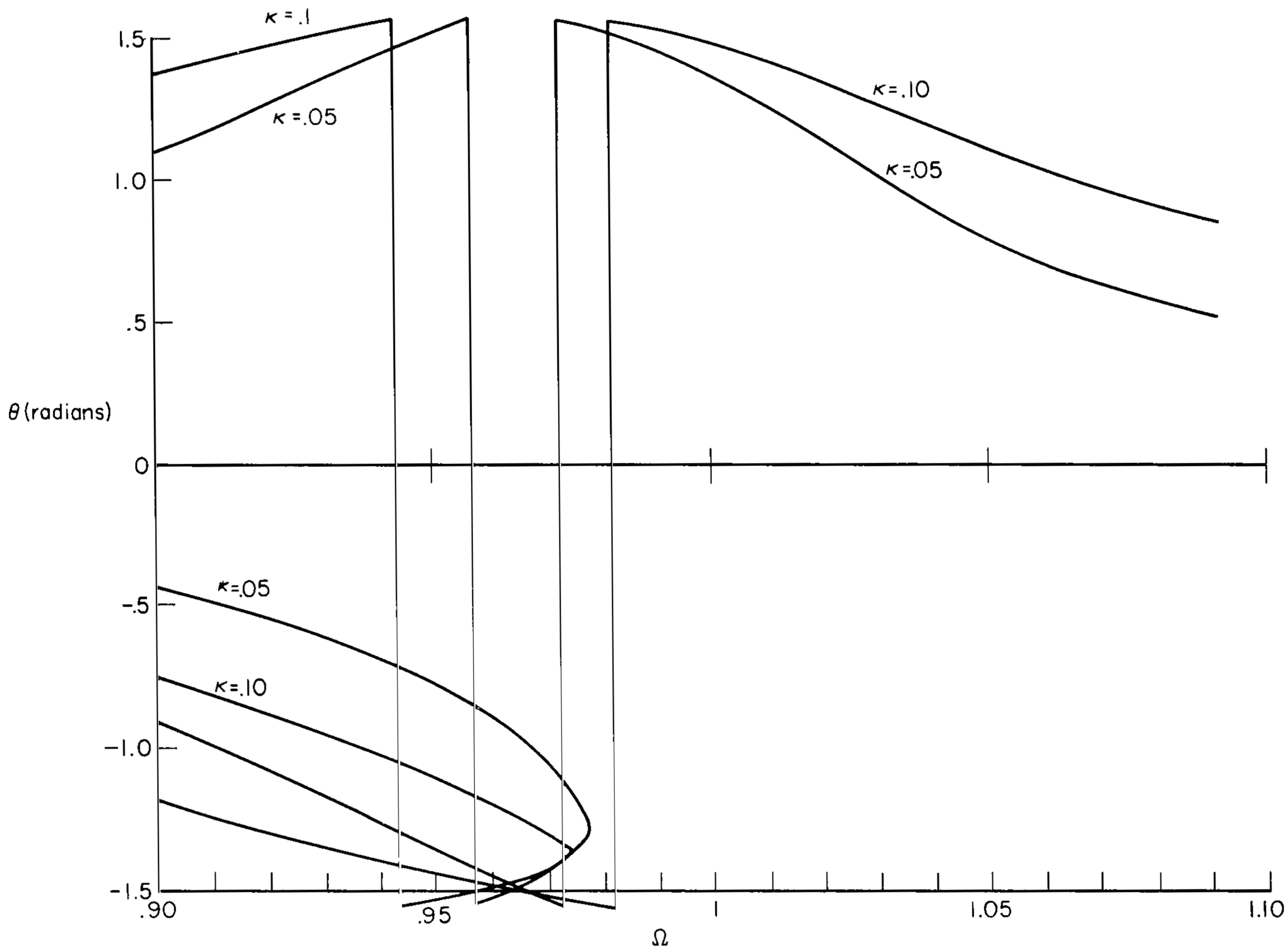
(a) μ_i = (0,1,2,3)

Figure 6.- Variation of instability zone in relation with viscous damping; p = 0.05, ϕ_0 = 120°, a/h = 100, hinged edge.



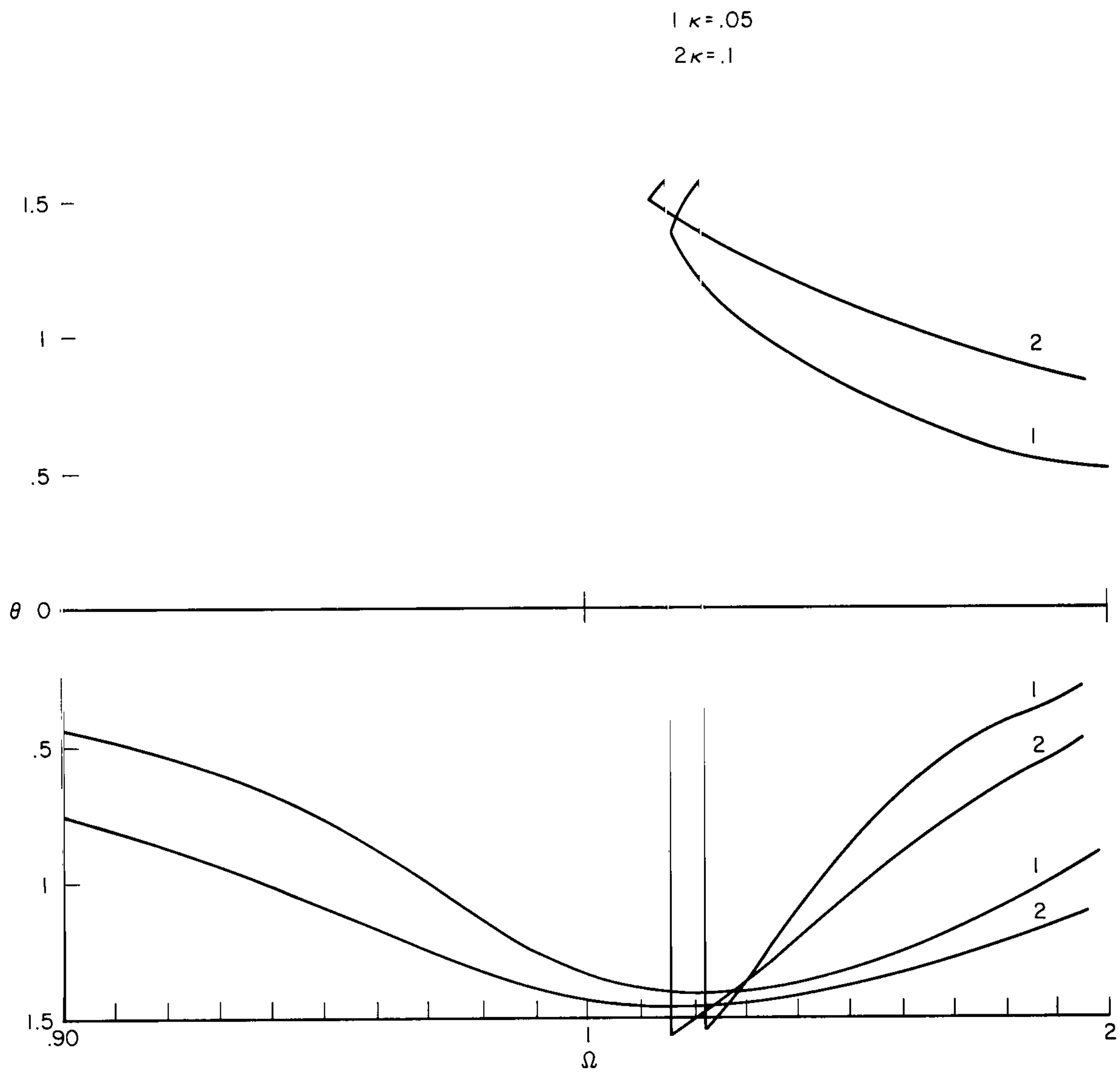
(b) $\mu_i = (0,1,2,4)$

Figure 6.- Concluded.



(a) $\mu_i = (0,1,2,3)$

Figure 7.- Relation between phase-difference and modified angular frequency; $p = 0.1$, $\phi_0 = 120°$, $a/h = 100$, hinged edge.

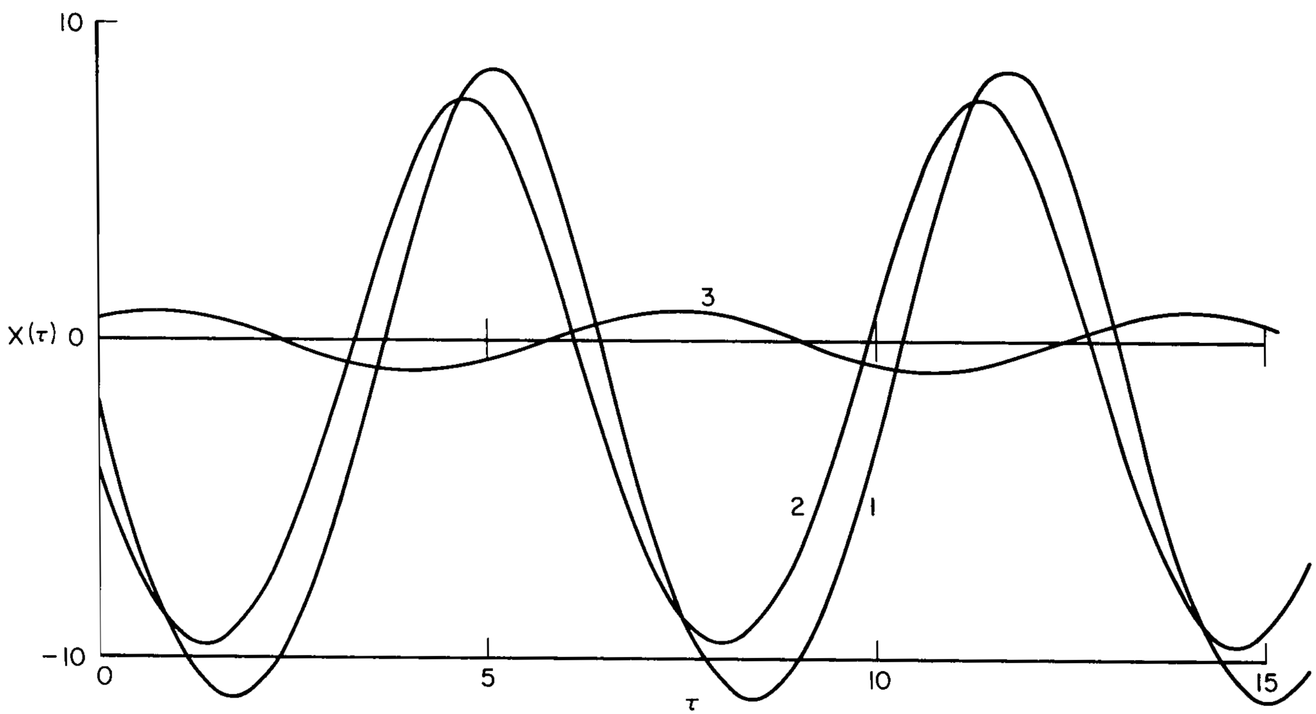


(b) $\mu_i = (0,2,3,4)$

Figure 7.- Concluded.

	1	2	3
Φ_0	− 203.2796	−144.513	− 2.0445
Φ_1	9.96177	− 8.5325	.967036
Φ_2	68.599	47.2388	.77434
Φ_3	544.335	−745.7630	.737443
A	8.6254	− 9.5798	.954408

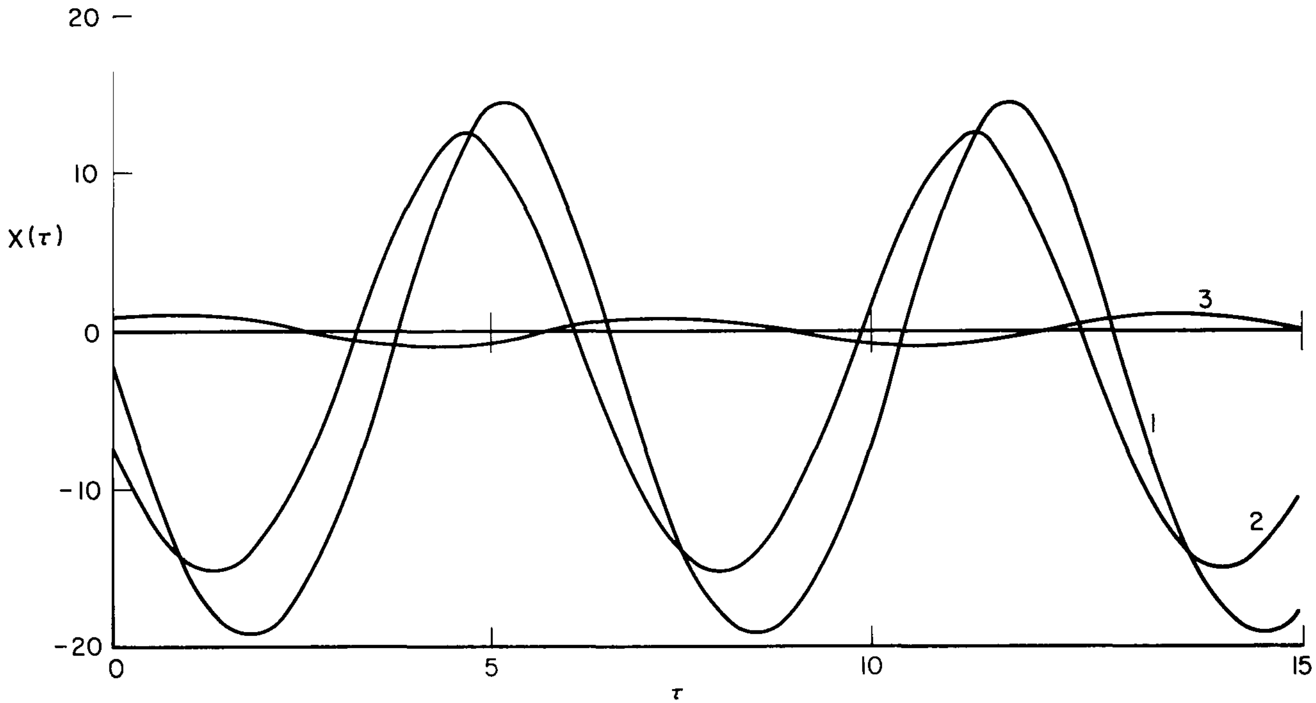
$p = .1$
$\kappa = .05$



(a1) $\mu_i = (0,1,2,3)$; $\Omega = 0.945$ (three real roots to eq. (11)).

Figure 8.- Time-dependent variable; $a/h = 100$, $\phi_0 = 120°$, hinged edge.

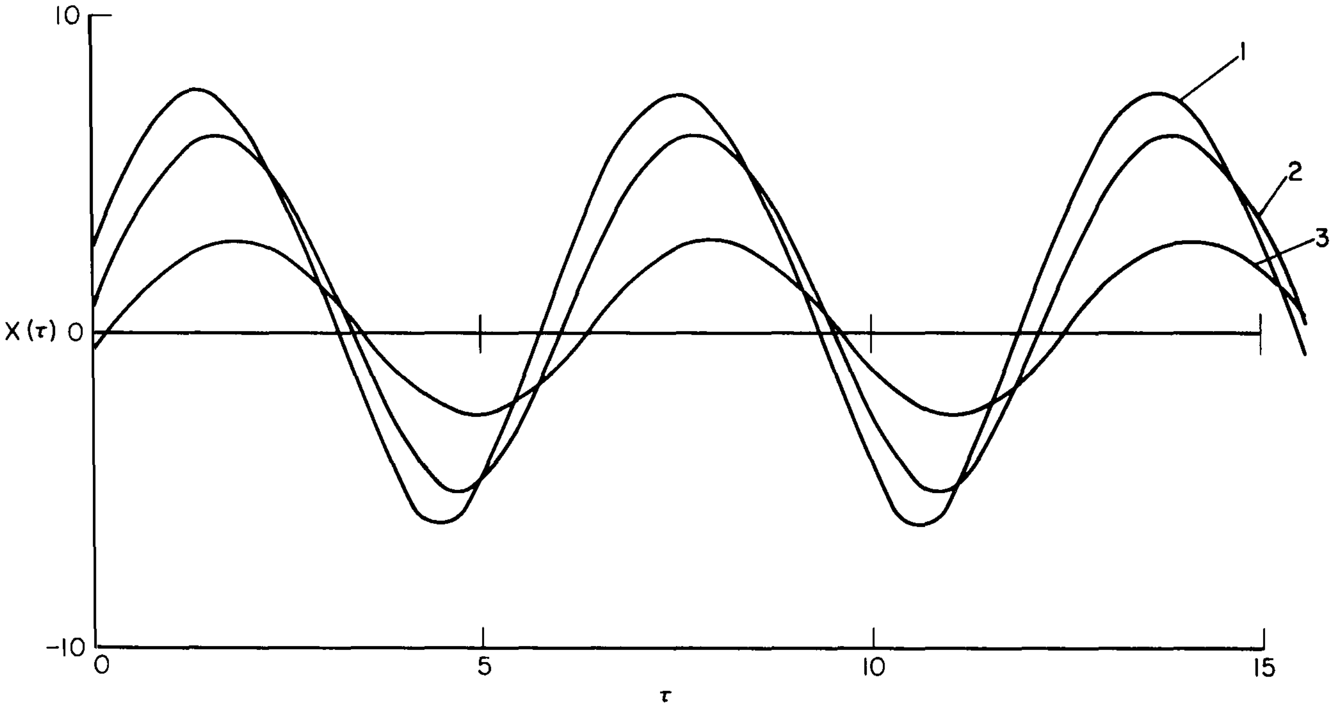
	1	2	3	
Φ_0	− 363.940	−233.6883	−1.28847	
Φ_1	16.65689	− 13.6013	.96059	p=.1
Φ_2	122.2054	76.9449	.48870	κ.05
Φ_3	1100.660	−1334.7495	.32203	
A	14.3496	− 15.3022	.952626	



(b1) μ_i = (0,1,2,4); Ω = 0.945 (three real roots to eq. (11)).

Figure 8.- Continued.

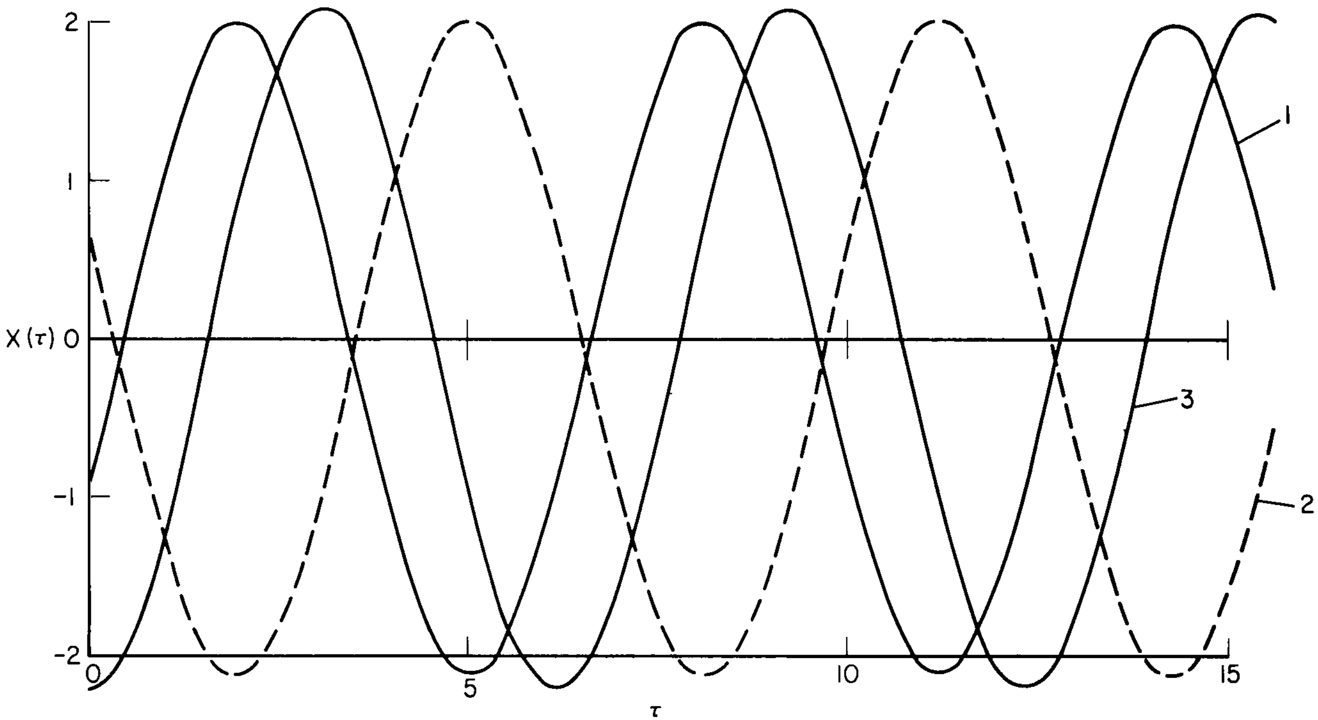
Φ_0	121.98933	90.8875	-23.4022	$p = .1$
Φ_1	6.6325	-5.5417	-2.784997	$\kappa = .05$
Φ_2	-41.6974	-29.6193	-7.4431	
Φ_3	1753.550	-490.7163	-72.5795	
A	7.61078	-4.9781	-2.632659	



(c1) μ_i = (0,2,3,4); Ω = 1.020 (three real roots to eq. (11)).

Figure 8.- Continued.

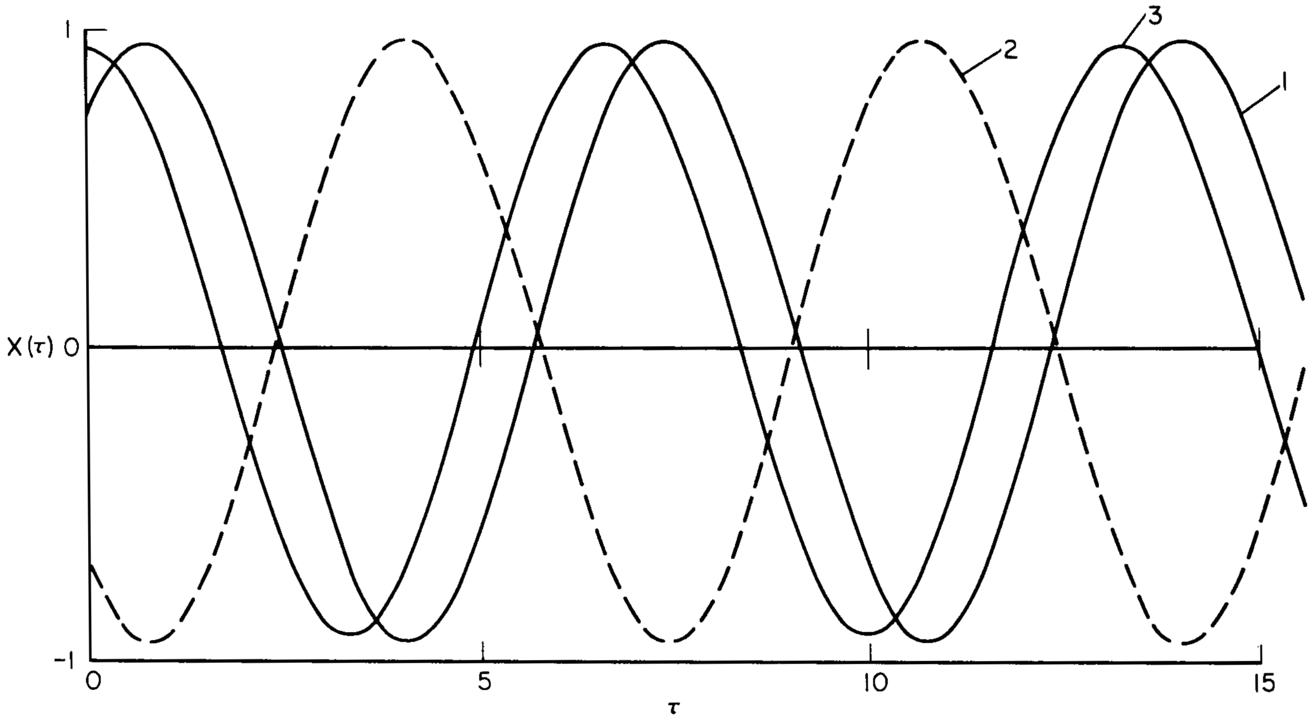
	1	2	3
Φ_0	−9.21742	−9.427106	−9.999247
Φ_1	−2.05665	−2.08328	−2.145181
Φ_2	2.86609	2.94589	3.11864
Φ_3	−8.09346	6.98300	−9.17676
A	−2.120977	2.019167	−2.21490
ρ	.1	−.1	.1
κ	.05	.05	0



(a2) μ_i = (0,1,2,3); Ω = 1.020 (one real root to eq. (11)).

Figure 8.- Continued.

	1	2	3
Φ_0	2.741602	2.74548	2.611421
Φ_1	.94998	- .94800	.92828
Φ_2	-1.04532	-1.04172	- .99822
Φ_3	3.60017	-3.21280	3.3431
A	.96730	- .93129	.94475
p	.1	-.1	.1
κ	.05	.05	0



(c2) μ_i = (0,2,3,4); Ω = 0.945 (one real root to eq. (11))

Figure 8.- Concluded.

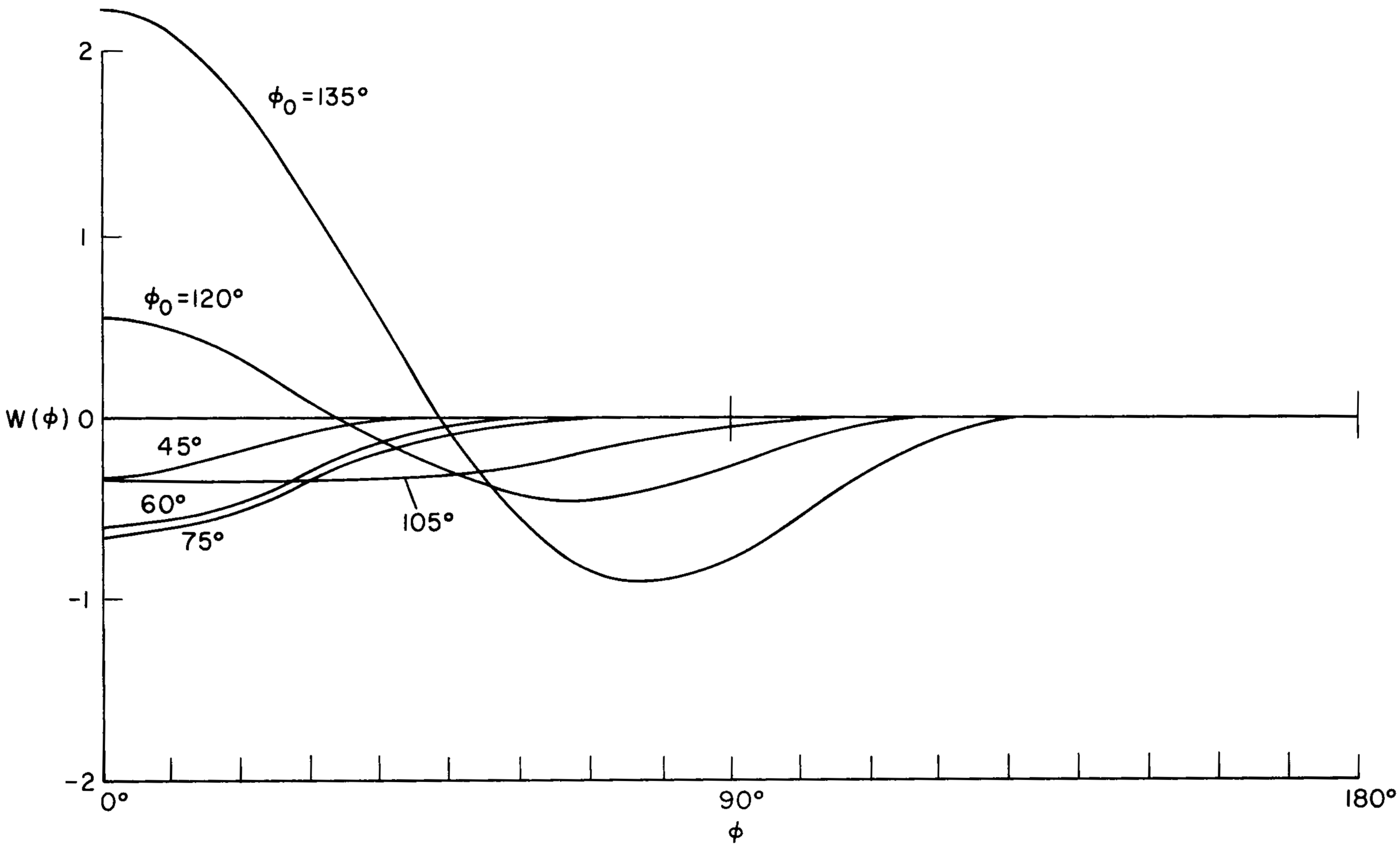


Figure 9.- Deflection shapes for various open angles; a/h = 100, clamped edge, μ_i = (0,1,2,3).

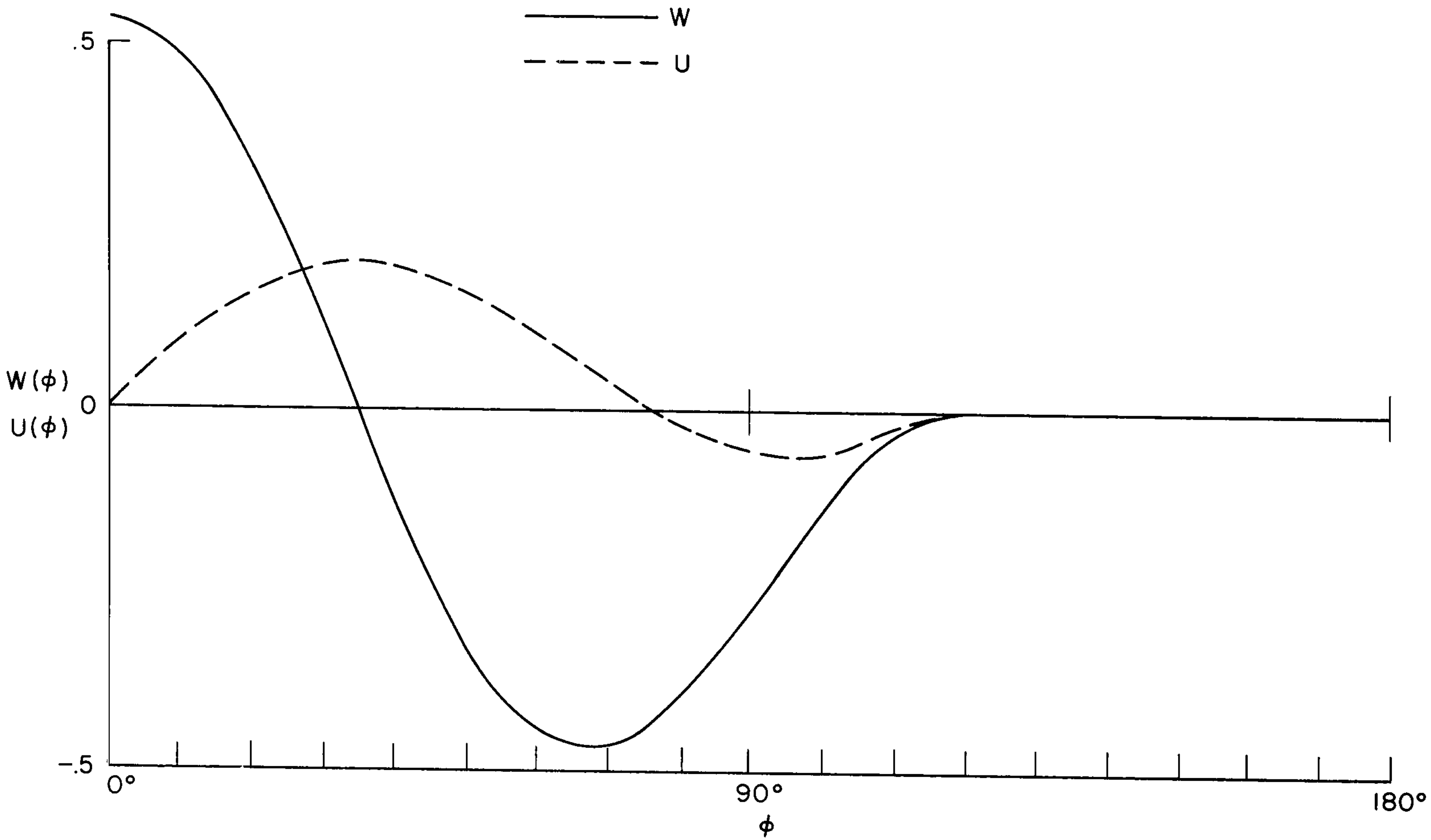


Figure 10.- Deflection and in-plane displacement shapes for a shell of open angle $\phi_0 = 120°$, a/h = 100, clamped edge, $\mu_i = (0,1,2,3)$.

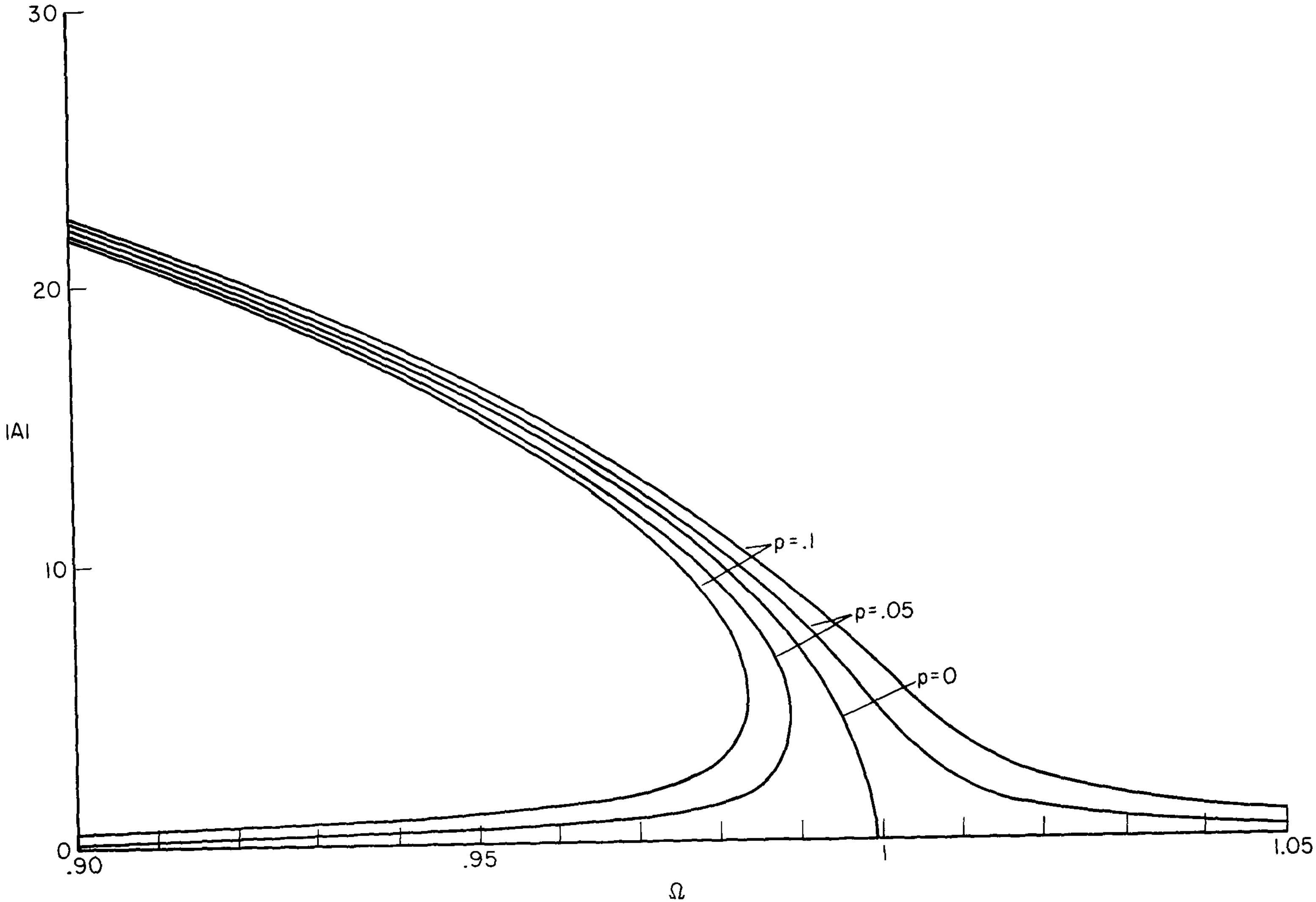


Figure 11.- Relation between amplitude parameter $|A|$ and modified angular frequency Ω with various magnitudes of external forces p; $\phi_0 = 120°$, $a/h = 100$, $\kappa = 0.05$, clamped edge, $\mu_i = (0,1,2,3)$.

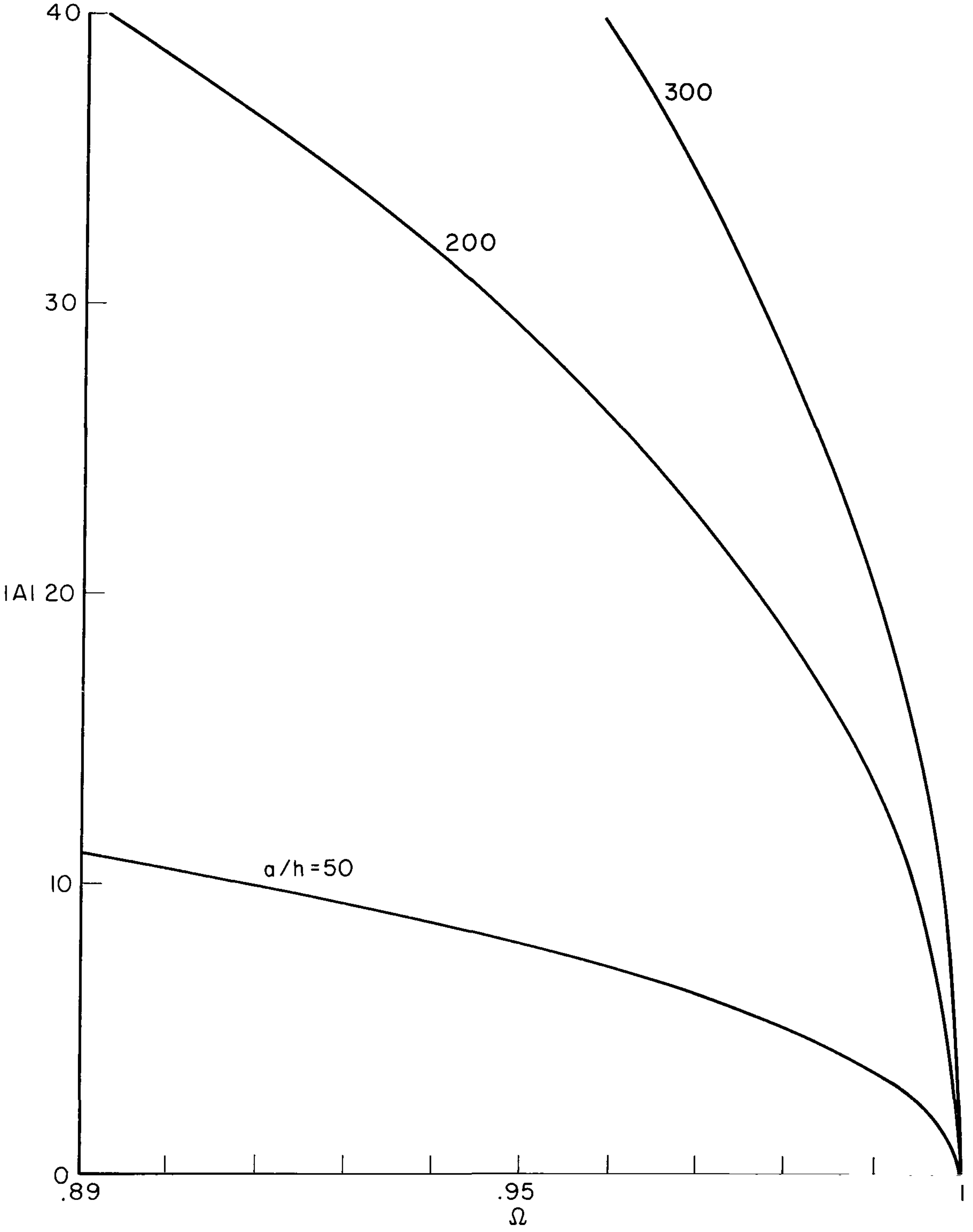


Figure 12.- Variation of $|A|$ - Ω relation with respect to radius-thickness ratio in undamped free vibration; $p = 0$, $\kappa = 0$, $\phi_0 = 120°$, clamped edge, $\mu_i = (0,1,2,3)$.

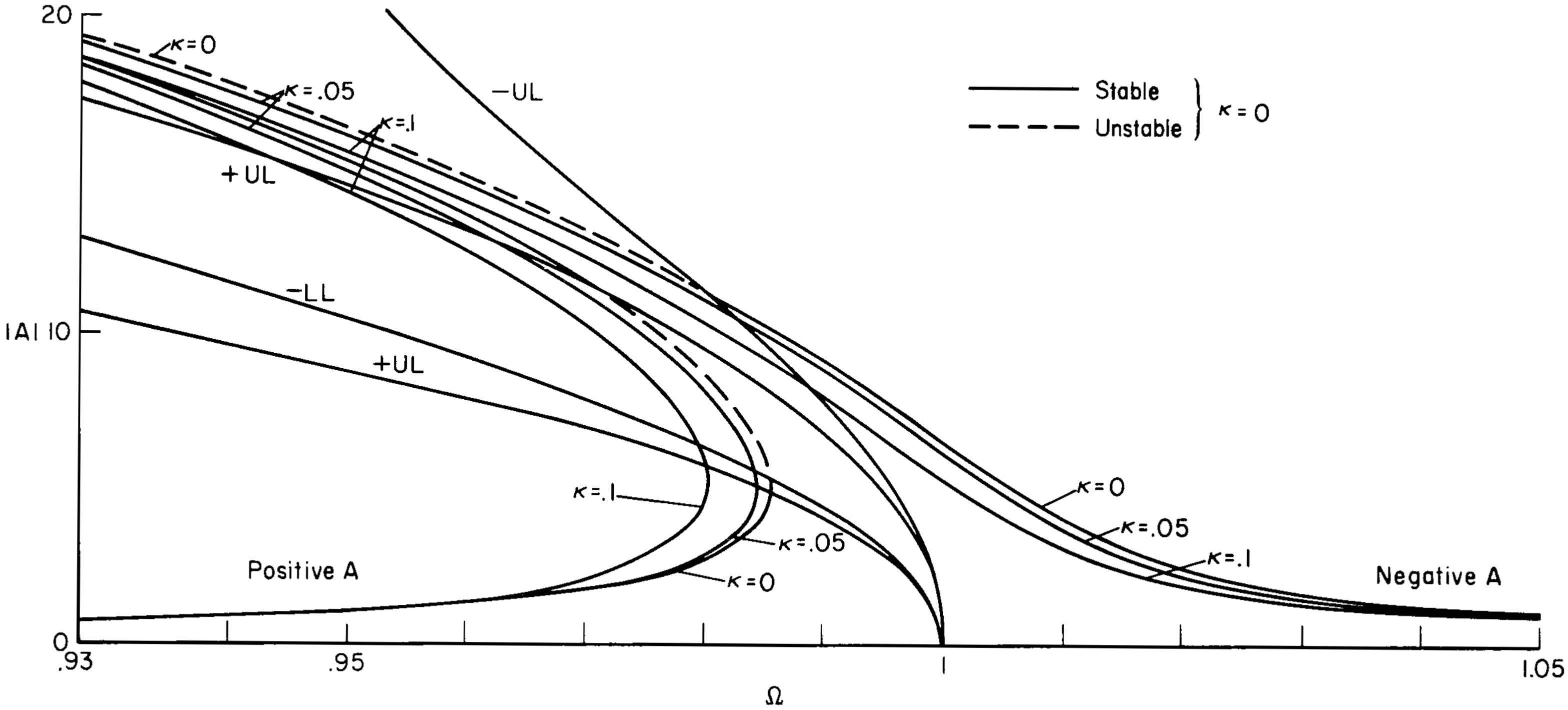
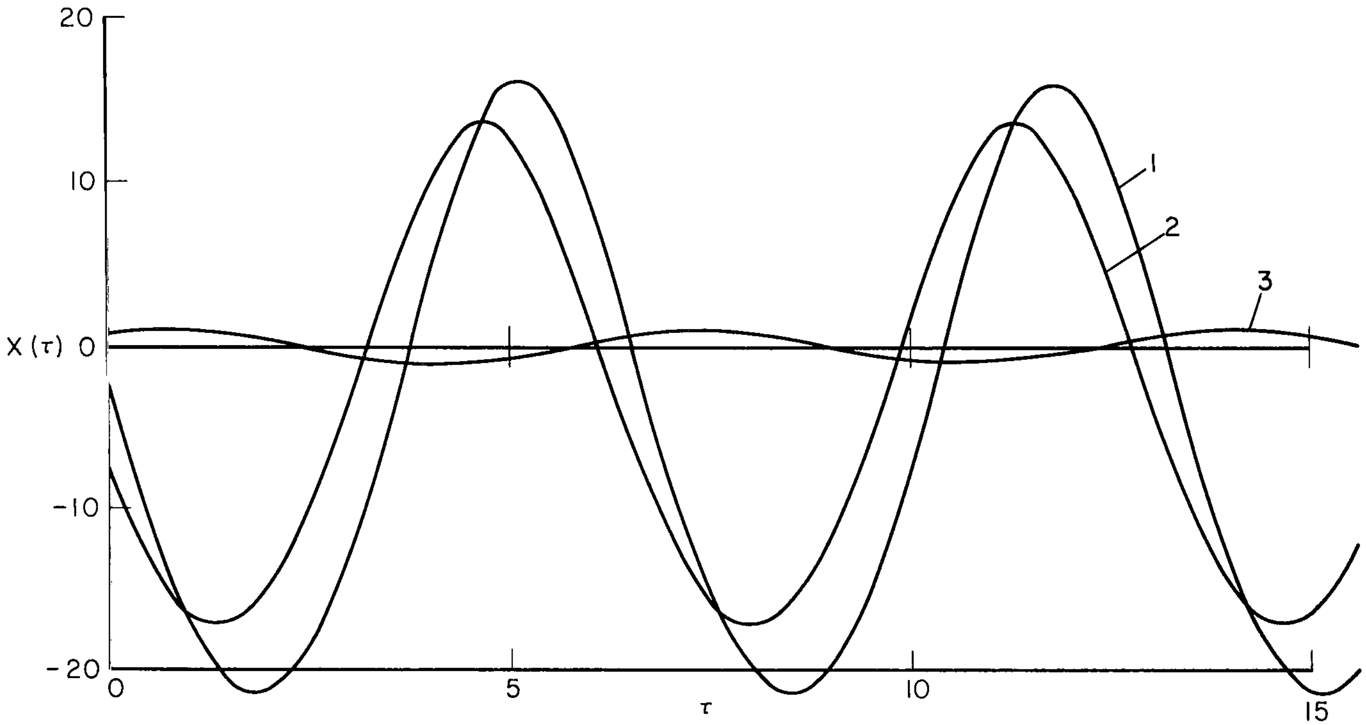


Figure 13.- Variation of $|A|$ - Ω relation with respect to viscous damping κ and instability zone in nondamping state (κ = 0); p = 0.1, ϕ_0 = 120°, a/h = 100, clamped edge, μ_i = (0,1,2,3).

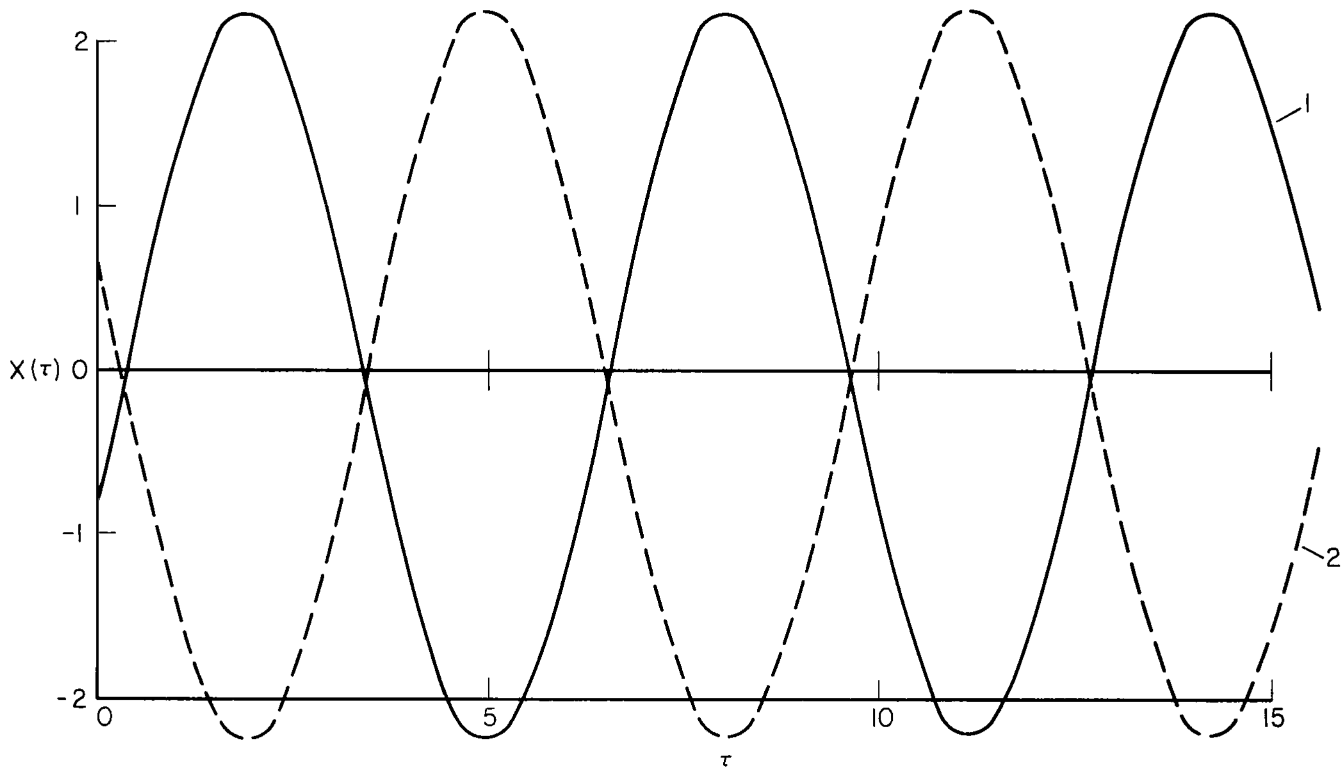
	1	2	3	
Φ_0	−394.2677	−254.9041	−1.1113	
Φ_1	18.6293	−15.2334	.95985	p = .1
Φ_2	132.2882	84.05127	.421615	κ = .05
Φ_3	1095.7618	.42165	.22643	
A	16.11912	−17.072125	.952979	



(a) μ_i = (0,1,2,3); Ω = 0.945 (three real roots to eq. (11)).

Figure 14.- Time-dependent variable; a/h = 100, ϕ_0 = 120°, clamped edge.

	1	2
ϕ_0	-5.98189	-6.01607
ϕ_1	-2.22820	2.23679
ϕ_2	1.87208	1.88744
ϕ_3	-3.05875	2.76989
A	-2.26960	2.19578
p	.1	-.1
κ	.05	.05



(b) μ_i = (0,1,2,3); Ω = 1.020 (one real root to eq. (11)).

Figure 14.- Concluded.

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