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SUMMARY

In many optimal control problems, it is reasonable to neglect certain of the terms in the system equations which are thought to have small effects in order to make the solution of the problem more tractable. When this results in a decrease in system order, however, the resulting approximation will not in general be able to satisfy all of the system boundary conditions and thus the approximation will not be valid, at least in a small region. In this case, the system is said to be singularly perturbed. Unfortunately most of the results of singular perturbation theory to date have been concerned with initial value problems whereas optimal control problems are of two-point boundary value type. The portions of this theory applicable to the open loop state regulator problem are reviewed in this paper.

For obtaining approximate solutions to the state regulator problem the method of matched asymptotic expansions is employed. This method has been developed in connection with certain fluid mechanics problems and is applicable to nonlinear as well as linear problems. It has been found in the past to be advantageous not to formulate this method generally but to apply it to each individual problem and this approach is adopted here. A general recipe for the method is given and its application is illustrated by using the method to obtain an approximate solution to a simple, specific state regulator problem.

I. INTRODUCTION

In the state regulator problem, it is desired to keep the state of a controlled, linear, dynamic system near zero without using excessive control expenditure. It is well known that application of the necessary conditions for optimal control to the state regulator problem results in a two point boundary value problem for a system of linear ordinary differential equations. This paper is concerned with obtaining approximate solutions to such systems of equations by using singular perturbation methods. Although the feedback solution of the state regulator problem is known[1], we do not take advantage of this fact since we desire a method which is applicable to nonlinear as well as linear systems.

Since efficient algorithms are available for the solution of systems of linear equations, the question arises as to why methods of obtaining approximate solutions need be studied. Approximate methods may be desirable or even necessary when computational speed or storage capacity is at a premium; for example in preliminary design in which a large number of trial systems must be evaluated or for use in a system on board a flight vehicle. Another important reason for studying linear systems is to gain insight into techniques which may be later applied to nonlinear systems. The primary factor determining computational requirements for solving state regulator problems is system dimensionality. Thus approximate solutions based on reduced order systems are often employed in practice.

A standard technique of obtaining approximate solutions of mathematical problems is to introduce perturbations about a nominal solution. This technique is particularly useful in problems in which there is a "small parameter" present because in this case the nominal solution and the method of introducing the perturbations are suggested in an obvious way. When

the nominal solution is obtained from a reduced order system, it will not in general meet all of the boundary conditions of the complete system and thus the nominal solution will not approximate the solution of the complete system uniformly. This loss of uniform convergence characterizes singular perturbation theory. This theory is reviewed in references 2 and 3.
The purpose of this paper is to briefly review the singular perturbation theory of ordinary differential equations as it applies to linear systems with two point boundary values and to illustrate application of this theory to optimal state regulator problems by solving a simple example. In this example, the theory is implemented by using the technique of matched asymptotic expansions[4,5] which has been used to solve boundary layer problems in fluid mechanics. Use of singular perturbations and matched asymptotic expansions in optimal control problems was suggested by Kelly[6,7], and has been studied most extensively by Hadlock[8].

In the recent literature, there have been two distinct methods (other than that under consideration here) proposed to obtain approximate solutions based on reduced order models of linear systems. In the first of these, the existence of a small parameter is assumed and the control is expanded in a Taylor series in the small parameter and the first two terms retained[9,10]. The zero order term is the optimal control of the reduced order system (obtained by setting the small parameter equal to zero) and the first order term is used to account for the initial conditions. In the second method, the characteristics of the model, including its order, are adjusted to give the best agreement (integral norm) between the output of the model and the output of the system using the model optimum control[11]. Either or both of these methods suffer from the following deficiencies:
(1) inadequate treatment of control variables, (2) lack of proof of uniform

convergence to exact solution, (3) failure to deal directly with loss of boundary conditions, and (4) inability to generate a sequence of successively better approximations. The method of singular perturbations, using matched asymptotic expansions, does not suffer from these shortcomings. Further, as opposed to the methods of references 9, 10 and 11, it is applicable to nonlinear systems.

II. STATE REGULATOR PROBLEM

In this section the solution of the conventional state regulator problem is first presented. The analysis follows that of reference 1, but with the nomenclature and sign conventions of reference 12, and is only summarized in this paper. Next, the problem is reformulated, in a form suitable for application of singular perturbation theory, for systems in which there is a small parameter present.

Conventional Problem

The most general problem to be considered is as follows, It is desired to minimize the cost functional

$$J = \frac{1}{2}\int_0^T [< \underline{x}\ (t), \underline{\underline{Q}}\ (t)\ \underline{x}\ (t) > + < \underline{\sigma}\ (t), \underline{\underline{R}}\ (t)\ \underline{\sigma}\ (t) >]\ dt \tag{2.1}$$

for the following system

$$\dot{\underline{x}}\ (t) = \underline{\underline{A}}\ (t)\ \underline{x}\ (t) + \underline{\underline{B}}\ (t)\ \underline{\sigma}\ (t)\ ;\ \underline{x}\ (0) = \underline{x}_0 \tag{2.2}$$

where $\underline{x}$ is an n dimensional state vector, $\underline{\sigma}$ is an r dimensional control vector, and $<\cdot, \cdot>$ is the scalar product on R^n. It is assumed throughout that (a) T is fixed, (b) $\underline{\underline{Q}}$ is positive semi-definite on $0 \leq t \leq T$, (c) $\underline{\underline{R}}$ is positive definite on $0 \leq t \leq T$, (d) optimal control exists, and (e) $\underline{\sigma}(t)$ is unconstrained. To formulate the necessary conditions for optimal control, the $\mathcal{H}$ function[12] is formed as*

$$\mathcal{H} = -\frac{1}{2} < \underline{x}, \underline{\underline{Q}}\ \underline{x} > - \frac{1}{2} < \underline{\sigma}, \underline{\underline{R}}\ \underline{\sigma} > + < \underline{\underline{A}}\ \underline{x}, \underline{\lambda} > + < \underline{\underline{B}}\ \underline{\sigma}, \underline{\lambda} > \tag{2.3}$$

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*Hereafter functional dependence will be omitted when this does not result in confusion.

where λ_o has been taken as -1. The adjoint vector $\underline{\lambda}$ is defined as the solution of

$$\underline{\dot{\lambda}} = - \frac{\partial \mathcal{H}}{\partial \underline{x}} \; ; \; \underline{\lambda} \, (T) = \underline{0} \tag{2.4}$$

Since $\underline{\underline{Q}}' = \underline{\underline{Q}}$, this gives

$$\underline{\dot{\lambda}} = \underline{\underline{Q}} \; \underline{x} - \underline{\underline{A}}' \; \underline{\lambda} \tag{2.5}$$

The optimal control $\underline{\sigma}$ is that which maximizes $\mathcal{H}$*: since $\underline{\underline{R}}$ is positive definite ($\underline{\underline{R}}' = \underline{\underline{R}}$, $|\underline{\underline{R}}| \neq 0$) this gives

$$\frac{\partial \mathcal{H}}{\partial \underline{\sigma}} = - \frac{1}{2} \underline{\underline{R}} \; \underline{\sigma} - \frac{1}{2} \underline{\underline{R}}' \underline{\sigma} + \underline{\underline{B}}' \; \underline{\lambda} = \underline{0}$$

$$\underline{\sigma} = \underline{\underline{R}}^{-1} \; \underline{\underline{B}}' \underline{\lambda} \tag{2.6}$$

That (2.6) maximizes $\mathcal{H}$ follows from $\frac{\partial^2 \mathcal{H}}{\partial \underline{\sigma} \partial \underline{\sigma}} = - \underline{\underline{R}}$ and the positive definiteness of $\underline{\underline{R}}$. Putting (2.6) in (2.2) and adding (2.5) gives a 2n dimensional linear homogeneous system with evenly split boundary values

$$\underline{\dot{x}} \, (t) = \underline{\underline{A}} \, (t) \; \underline{x} \, (t) + \underline{\underline{S}} \, (t) \; \underline{\lambda} \, (t) \; ; \; \underline{x} \, (0) = \underline{x}_0$$

$$\underline{\dot{\lambda}} \, (t) = \underline{\underline{Q}} \, (t) \; \underline{x} \, (t) - \underline{\underline{A}}' \, (t) \; \underline{\lambda} \, (t) \; ; \; \underline{\lambda} \, (T) = \underline{0} \tag{2.7}$$

where

$$\underline{\underline{S}} = \underline{\underline{B}} \; \underline{\underline{R}}^{-1} \; \underline{\underline{B}}' \tag{2.8}$$

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*The adjoint vector used here is the negative of the costate vector used in reference 1; thus $\mathcal{H}$ is maximized to minimize J, whereas it is minimized in reference 1. A further difference from reference 1 is in the handling of the integrand of the cost functional.

Integration of this system gives the (open loop) solution of the problem.

It can be established for the system (2.7) that there exists a symmetric matrix $\underline{\underline{K}}(t)$ such that

$$\underline{\lambda} = \underline{\underline{K}}\ \underline{x} \tag{2.9}$$

and this relation may be used to eliminate $\underline{\lambda}$ in the first of (2.7):

$$\dot{\underline{x}} = (\underline{\underline{A}} + \underline{\underline{S}}\ \underline{\underline{K}})\ \underline{x}\ ;\ \underline{x}\ (0) = \underline{x}_0 \tag{2.10}$$

The second of (2.7) is used to obtain a relation for $\underline{\underline{K}}$; differentiating (2.9) and using (2.5) and (2.10),

$$\dot{\underline{\lambda}} = \dot{\underline{\underline{K}}}\ \underline{x} + \underline{\underline{K}}\ \dot{\underline{x}}$$

$$\underline{\underline{Q}}\ \underline{x} - \underline{\underline{A}}'\ \underline{\underline{K}}\ \underline{x} = \dot{\underline{\underline{K}}}\ \underline{x} + \underline{\underline{K}}\ (\underline{\underline{A}} + \underline{\underline{S}}\ \underline{\underline{K}})\ \underline{x}$$

Since $\underline{x} \neq \underline{0}$*, this implies

$$\dot{\underline{\underline{K}}} = \underline{\underline{Q}} - \underline{\underline{A}}'\ \underline{\underline{K}} - \underline{\underline{K}}\ \underline{\underline{A}} - \underline{\underline{K}}\ \underline{\underline{S}}\ \underline{\underline{K}}\ ;\ \underline{\underline{K}}\ (T) = \underline{\underline{0}} \tag{2.11}$$

where the boundary condition follows from (2.9) and $\underline{\lambda}\ (T) = \underline{0}$ provided $\underline{x}\ (T) \neq \underline{0}$. Putting (2.9) in (2.6) gives the closed loop (feedback) control as

$$\underline{\sigma} = \underline{\underline{R}}^{-1}\ \underline{\underline{B}}'\ \underline{\underline{K}}\ \underline{x} \tag{2.12}$$

Thus solution of the nonlinear (Riccati) system (2.11) and then integration of (2.10) gives the solution to the original problem; this is an alternate procedure to solving (2.7) and has the advantage of giving closed loop control.

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*The trivial case $\underline{x}_0 = \underline{0}$ is excluded.

Problem With a Small Parameter

In many systems, there is present a parameter which may be identified as "small" based on physical reasoning. For example, in viscous fluid flow, the viscous effects are small in many problems, at least in large regions of the flow field. Thus the viscosity is often treated as a small parameter in the Navier-Stokes equations[4,5]. In other problems, however, no such parameter appears from physical grounds and a parameter may be artificially inserted to suppress terms in the equations which are expected to have relatively small effects. There is nothing new in this procedure, since this is in effect what the analyst does everytime he formulates a problem in dynamics, i.e., decides which effects to account for and which ones to ignore. In a flight mechanics problem for a manned vehicle, for example, a complete set of equations of motion would consist of the coupled system of the six equations of rigid body motion of the vehicle as a whole, the equations describing the dynamics of the control systems, the equations describing the dynamics of the pilot's arm and foot, etc. It is obvious that many of these effects can be neglected if, say, the vehicle trajectory is the only thing of interest. In what follows, it is assumed that the small parameter in the system has either been identified or inserted in a suitable manner.

Consider the problem of minimizing

$$J = \frac{1}{2} \int_0^T [< \underline{x} (\varepsilon, t), \underline{\underline{Q}} (\varepsilon, t) \underline{x} (\varepsilon, t)> + <\underline{\sigma} (\varepsilon, t), \underline{\underline{R}} (\varepsilon, t) \underline{\sigma} (\varepsilon, t)>] dt \tag{2.13}$$

where the system is given by

$$\underline{\underline{\varepsilon}} (\varepsilon) \dot{\underline{x}} (\varepsilon, t) = \underline{\underline{A}} (\varepsilon, t) \underline{x} (\varepsilon, t) + \underline{\underline{B}} (\varepsilon, t) \underline{\sigma} (\varepsilon, t) \; ; \; \underline{x} (0) = \underline{x}_0 \tag{2.14}$$

with the same assumptions as before, and where ε is a scalar parameter.

Without loss of generality it may be assumed that $\underline{\underline{\varepsilon}}$ is a diagonal matrix, $|\underline{\underline{\varepsilon}}| \neq 0$, and that all matrices in (2.13) and (2.14) are of order zero or higher in the parameter ε, that is, for example,

$$\lim_{\varepsilon \to 0^+} \varepsilon \underline{\underline{\varepsilon}} = \underline{\underline{0}}$$

We call this the singularly perturbed state regulator problem. The most common form of $\underline{\underline{\varepsilon}}$ in applications is a matrix with the first m elements equal to unity and the last n-m equal to ε, i.e.

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \varepsilon & \\ 0 & & & & \ddots \; \varepsilon \end{pmatrix} \tag{2.15}$$

Multiplying (2.14) by $\underline{\underline{\varepsilon}}^{-1}$ results in

$$\underline{\dot{x}} = \underline{\underline{\varepsilon}}^{-1} \underline{\underline{A}} \, \underline{x} + \underline{\underline{\varepsilon}}^{-1} \underline{\underline{B}} \, \underline{\sigma} \tag{2.16}$$

to which the results from the conventional system may be applied. Noting that, for example, $(\underline{\underline{\varepsilon}}^{-1} \underline{\underline{A}})' = \underline{\underline{A}}' \underline{\underline{\varepsilon}}^{-1}$, (2.6), (2.7), (2.8) give

$$\begin{aligned} \underline{\dot{x}} &= \underline{\underline{\varepsilon}}^{-1} \underline{\underline{A}} \, \underline{x} + \underline{\underline{\varepsilon}}^{-1} \underline{\underline{S}} \, \underline{\underline{\varepsilon}}^{-1} \underline{\tilde{\lambda}} \; ; \; \underline{x}(0) = \underline{x}_0 \\ \dot{\underline{\tilde{\lambda}}} &= \underline{\underline{Q}} \, \underline{x} - \underline{\underline{A}}' \underline{\underline{\varepsilon}}^{-1} \underline{\tilde{\lambda}} \; ; \; \underline{\tilde{\lambda}}(T) = \underline{0} \end{aligned} \tag{2.17}$$

with

$$\underline{\sigma} = \underline{\underline{R}}^{-1} \underline{\underline{B}}' \underline{\underline{\varepsilon}}^{-1} \underline{\tilde{\lambda}} \tag{2.18}$$

where the adjoint vector is denoted by $\underline{\tilde{\lambda}}$. Making the change of variable

$$\underline{\tilde{\lambda}} = \underline{\underline{\varepsilon}} \, \underline{\lambda} \tag{2.19}$$

gives

$$\begin{aligned} \underline{\underline{\varepsilon}}(\varepsilon) \, \underline{\dot{x}}(\varepsilon, t) &= \underline{\underline{A}}(\varepsilon, t) \, \underline{x}(\varepsilon, t) + \underline{\underline{S}}(\varepsilon, t) \, \underline{\lambda}(\varepsilon, t) \; ; \; \underline{x}(0) = \underline{x}_0 \\ \underline{\underline{\varepsilon}}(\varepsilon) \, \underline{\dot{\lambda}}(\varepsilon, t) &= \underline{\underline{Q}}(\varepsilon, t) \, \underline{x}(\varepsilon, t) - \underline{\underline{A}}'(\varepsilon, t) \, \underline{\lambda}(\varepsilon, t) \; ; \; \underline{\lambda}(T) = \underline{0} \end{aligned} \tag{2.20}$$

with $\underline{\sigma}$ and $\underline{\underline{S}}$ given by (2.6) and (2.8). Equations (2.20) may be called a singularly perturbed linear system; integration of this system gives the open loop solution to the problem.

The closed loop solution may be obtained in the same manner as for the conventional system. Multiplying (2.9) by $\underline{\underline{\varepsilon}}$, differentiating, and using (2.20) gives

$$\underline{\underline{\varepsilon}}\,\dot{\underline{\underline{K}}} = \underline{\underline{Q}} - \underline{\underline{A}}'\,\underline{\underline{K}} - \underline{\underline{\varepsilon}}\,\underline{\underline{K}}\,\underline{\underline{\varepsilon}}^{-1}\,\underline{\underline{A}} - \underline{\underline{\varepsilon}}\,\underline{\underline{K}}\,\underline{\underline{\varepsilon}}^{-1}\,\underline{\underline{S}}\,\underline{\underline{K}}\ ;\ \underline{\underline{K}}\,(\varepsilon,\,T) = \underline{\underline{0}} \tag{2.21}$$

which may be termed a singularly perturbed Riccati differential equation. After (2.21) has been solved for $\underline{\underline{K}}$, the state history may be found from

$$\underline{\underline{\varepsilon}}\,\dot{\underline{x}} = (\underline{\underline{A}} + \underline{\underline{S}}\,\underline{\underline{K}}\,)\,\underline{x}\ ;\ \underline{x}\,(0) = \underline{x}_0 \tag{2.22}$$

and the feedback control from (2.12). An important special class of systems are those for which (a) the system is controllable, (b) $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{Q}}$, $\underline{\underline{R}}$ are time-invariant, and (c) $T \to \infty$. In this case, $\underline{\underline{K}}$ is not a function of time[1] and (2.21) reduces to an algebraic equation

$$\underline{\underline{0}} = \underline{\underline{Q}} - \underline{\underline{A}}'\,\underline{\underline{K}} - \underline{\underline{\varepsilon}}\,\underline{\underline{K}}\,\underline{\underline{\varepsilon}}^{-1}\,\underline{\underline{A}} - \underline{\underline{\varepsilon}}\,\underline{\underline{K}}\,\underline{\underline{\varepsilon}}^{-1}\,\underline{\underline{S}}\,\underline{\underline{K}} \tag{2.23}$$

The analyses of references 9, 10 and 11 utilize the closed loop solution to the problem as a starting point. In this paper, we use the system (2.20) as a starting point since we desire a general technique; closed loop solutions are not generally available for nonlinear systems.

III. SINGULAR PERTURBATION THEORY AND THE METHOD OF MATCHED ASYMPTOTIC EXPANSION

It is desired to review results of singular perturbation theory as it applies to linear, two-point boundary value problems and to develop the method of matched asymptotic expansions for the approximate solution of such systems. Specifically, the system (2.20) is of interest; to formulate this system more concisely, let

$$\underline{w} = \begin{bmatrix} \underline{x} \\ \underline{\lambda} \end{bmatrix}$$
$$\underline{\underline{E}} = \begin{bmatrix} \underline{\underline{\varepsilon}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{\varepsilon}} \end{bmatrix} \tag{3.1}$$
$$\underline{\underline{C}} = \begin{bmatrix} \underline{\underline{A}} & \underline{\underline{S}} \\ \underline{\underline{Q}} & -\underline{\underline{A}}' \end{bmatrix}$$

The resulting problem is called P_E:

$$P_E\text{:}\ \underline{\underline{E}}\ (\varepsilon)\ \dot{\underline{w}}\ (\varepsilon,\ t) = \underline{\underline{C}}\ (\varepsilon,\ t)\ \underline{w}\ (\varepsilon,\ t)\ ;\ \underline{\underline{R}}\ (t)\ \underline{w}\ (t,\ 0) + \underline{\underline{S}}\ (\varepsilon)\ \underline{w}\ (\varepsilon,\ T) = \underline{k}(\varepsilon) \tag{3.2}$$

where the boundary conditions have been generalized by allowing "mixing" and dependence on ε. The <u>reduced</u> (or degenerate) problem associated with (3.2), P_o, is obtained by setting $\varepsilon = 0$:

$$P_o\ :\ \underline{\underline{E}}\ (0)\ \dot{\underline{w}}\ (0,\ t) = \underline{\underline{C}}\ (0,\ t)\ \underline{w}\ (0,\ t)\ ;\ \underline{\underline{R}}\ (0)\ \underline{w}\ (0,\ 0) + \underline{\underline{S}}\ (0)\ \underline{w}\ (0,\ T) = \underline{k}\ (0) \tag{3.3}$$

Of particular importance is the case when $\underline{\underline{\varepsilon}}$ has the form given by (2.15). Let

$$\underline{x} = \begin{bmatrix} \underline{y} \\ \underline{z} \end{bmatrix} \; ; \; \underline{\lambda} = \begin{bmatrix} \underline{p} \\ \underline{q} \end{bmatrix} \tag{3.4}$$

$$\underline{\underline{A}} = \begin{bmatrix} \underline{\underline{A}}_1 & \underline{\underline{A}}_2 \\ \underline{\underline{A}}_3 & \underline{\underline{A}}_4 \end{bmatrix} \quad ; \; \underline{\underline{S}} = \begin{bmatrix} \underline{\underline{S}}_1 & \underline{\underline{S}}_2 \\ \underline{\underline{S}}_3 & \underline{\underline{S}}_4 \end{bmatrix} \quad ; \; \underline{\underline{Q}} = \begin{bmatrix} \underline{\underline{Q}}_1 & \underline{\underline{Q}}_2 \\ \underline{\underline{Q}}_3 & \underline{\underline{Q}}_4 \end{bmatrix} \tag{3.5}$$

where $\underline{y}$ and $\underline{p}$ have dimension m, $\underline{z}$ and $\underline{q}$ have dimension n-m, matrices subscripted 1 are mxm, and those subscripted 4 are (n-m)x(n-m). Then the system (2.20) assumes the following form, called the P_ε problem,

$$P_\varepsilon : \quad \begin{aligned} \dot{\underline{y}} &= \underline{\underline{A}}_1 \underline{y} + \underline{\underline{A}}_2 \underline{z} + \underline{\underline{S}}_1 \underline{p} + \underline{\underline{S}}_2 \underline{q} \; ; \; \underline{y}(0) = \underline{y}_0 \\ \varepsilon\dot{\underline{z}} &= \underline{\underline{A}}_3 \underline{y} + \underline{\underline{A}}_4 \underline{z} + \underline{\underline{S}}_3 \underline{p} + \underline{\underline{S}}_4 \underline{q} \; ; \; \underline{z}(0) = \underline{z}_0 \\ \dot{\underline{p}} &= \underline{\underline{Q}}_1 \underline{y} + \underline{\underline{Q}}_2 \underline{z} - \underline{\underline{A}}_1' \underline{p} - \underline{\underline{A}}_3' \underline{q} \; ; \; \underline{p}(T) = \underline{0} \\ \varepsilon\dot{\underline{q}} &= \underline{\underline{Q}}_3 \underline{y} + \underline{\underline{Q}}_4 \underline{z} - \underline{\underline{A}}_2' \underline{p} - \underline{\underline{A}}_4' \underline{q} \; ; \; \underline{q}(T) = \underline{0} \end{aligned} \tag{3.6}$$

As before, the reduced problem associated with (3.6) is denoted P_0:

$$P_0 : \quad \begin{aligned} \dot{\underline{y}} &= \underline{\underline{A}}_1 \underline{y} + \underline{\underline{A}}_2 \underline{z} + \underline{\underline{S}}_1 \underline{p} + \underline{\underline{S}}_2 \underline{q} \; ; \; \underline{y}(0) = \underline{y}_0 \\ \underline{0} &= \underline{\underline{A}}_3 \underline{y} + \underline{\underline{A}}_4 \underline{z} + \underline{\underline{S}}_3 \underline{p} + \underline{\underline{S}}_4 \underline{q} \\ \dot{\underline{p}} &= \underline{\underline{Q}}_1 \underline{y} + \underline{\underline{Q}}_2 \underline{z} - \underline{\underline{A}}_1' \underline{p} - \underline{\underline{A}}_3' \underline{p} \; ; \; \underline{p}(T) = \underline{0} \\ \underline{0} &= \underline{\underline{Q}}_3 \underline{y} + \underline{\underline{Q}}_4 \underline{z} - \underline{\underline{A}}_2' \underline{p} - \underline{\underline{A}}_4' \underline{q} \end{aligned} \tag{3.7}$$

where the matrices are evaluated at $\varepsilon = 0$.

An even more concise statement of this system is obtained as follows; let

$$\underline{u} = \begin{bmatrix} \underline{y} \\ \underline{p} \end{bmatrix} \quad ; \underline{v} = \begin{bmatrix} \underline{z} \\ \underline{q} \end{bmatrix} \tag{3.8}$$

$$\begin{aligned} \underline{\underline{F}} &= \begin{bmatrix} \underline{\underline{A}}_1 & \underline{\underline{S}}_1 \\ \underline{\underline{Q}}_1 & -\underline{\underline{A}}_1' \end{bmatrix} \quad ; \underline{\underline{D}} = \begin{bmatrix} \underline{\underline{A}}_2 & \underline{\underline{S}}_2 \\ \underline{\underline{Q}}_2 & -\underline{\underline{A}}_3' \end{bmatrix} \\ \underline{\underline{H}} &= \begin{bmatrix} \underline{\underline{A}}_3 & \underline{\underline{S}}_3 \\ \underline{\underline{Q}}_3 & -\underline{\underline{A}}_2' \end{bmatrix} \quad ; \underline{\underline{G}} = \begin{bmatrix} \underline{\underline{A}}_4 & \underline{\underline{S}}_4 \\ \underline{\underline{Q}}_4 & -\underline{\underline{A}}_4' \end{bmatrix} \end{aligned} \tag{3.9}$$

where $\underline{u}$ and $\underline{v}$ are 2m and 2n-2m dimensionaly, respectively; then (3.6) may be written:

$$P_\varepsilon : \quad \begin{aligned} &\frac{d\,\underline{u}\,(\varepsilon, t)}{dt} = \underline{\underline{F}}\,(\varepsilon, t)\,\underline{u}\,(\varepsilon, t) + \underline{\underline{D}}\,(\varepsilon, t)\,\underline{v}\,(\varepsilon, t) \\ &u_i\,(\varepsilon, 0) = x_{o_i} \; ; \; i = 1, \cdot\cdot, m \\ &u_i\,(\varepsilon, T) = 0 \; ; \; i = m + 1, \cdot\cdot, 2m \\ &\varepsilon\,\frac{d\,\underline{v}\,(\varepsilon, t)}{dt} = \underline{\underline{H}}\,(\varepsilon, t)\,\underline{u}\,(\varepsilon, t) + \underline{\underline{G}}\,(\varepsilon, t)\,\underline{v}\,(\varepsilon, t) \\ &v_i\,(\varepsilon, 0) = x_{o_{i+m}} \; ; \; i = 1, \cdot\cdot, n\text{-}m \\ &v_i\,(\varepsilon, T) = 0 \; ; \; i = n - m + 1, \cdot\cdot, 2n\text{-}2m \end{aligned} \tag{3.10}$$

with associated reduced problem

$$P_o : \quad \begin{aligned} &\frac{d\,\underline{u}\,(0, t)}{dt} = \underline{\underline{F}}\,(0, t)\,\underline{u}\,(0, t) + \underline{\underline{D}}\,(0, t)\,\underline{v}\,(0, t) \\ &\underline{0} = \underline{\underline{H}}\,(0, t)\,\underline{u}\,(0, t) + \underline{\underline{G}}\,(0, t)\,\underline{v}\,(0, t) \\ &u_i\,(0, 0) = x_{o_i} \; ; \; i = 1, \cdot\cdot, m \\ &u_i\,(0, T) = 0 \; ; \; i = m + 1, \cdot\cdot, 2m \end{aligned} \tag{3.11}$$

Relation of Solution of Full Problem to That of Reduced Problem

If ε is a small parameter, it is natural to attempt to approximate the solution to P_E or to P_ε by solving the associated P_0*. However, a fundamental difficulty is encountered at once, since the P_0 solution will in general satisfy only 2m of the 2n boundary conditions of P_E or P_ε, and thus convergence of the solutions of the reduced problems to the solutions of P_E and P_ε will be non-uniform at best.

Unfortunately, almost all of the theoretical work in singular perturbation theory has dealt with either boundary value problems for finite order scalar differential equations or with initial value problems for first order systems, either linear or nonlinear; this work is reviewed in refs. 2 and 3. Two authors have, however, considered two point boundary value problems for first order systems[8,13], and the following result gives the relation of the solution of P_0 to that of P_ε.

Theorem 1: For P_ε, suppose that there exists an ε_0 such that on the domain $D = \{(\varepsilon, t) \mid 0 \leq \varepsilon \leq \varepsilon_0, 0 \leq t \leq T\}$ we have 1) $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{Q}}$, $\underline{\underline{R}}$ are continuous in both arguments, 2) $\underline{\underline{G}}$ is nonsingular, and 3) n-m of the eigenvalues of $\underline{\underline{G}}$ have negative real parts. Then, 1) a unique solution to P_ε exists, denoted $\underline{u}(\varepsilon, t)$, $\underline{v}(\varepsilon, t)$, 2) a unique solution to P_0 exists, denoted $\underline{u}(0, t)$, $\underline{v}(0, t)$, 3) $\lim_{\varepsilon \to 0^+} \underline{u}(\varepsilon, t) = \underline{u}(0, t)$ uniformly on $0 \leq t \leq T$, and 4) $\lim_{\varepsilon \to 0^+} \underline{v}(\varepsilon, t) = \underline{v}(0, t)$ uniformly on any closed subinterval of $0 < t < T$. Hypotheses 1 and 2 are sufficient to arrive at conclusions 1 and 2; 3 is a stability condition and its significance will become apparent subsequently.

The theorem may be generalized in several important ways. Harris[13] has considered systems of the type P_E except that his form of $\underline{\underline{E}}$ is somewhat

*In what follows, the context will make it clear whether (3.3) or (3.11) are to be used for P_0.

more restricted than that in P_E. It is found that several complications occur. First, with boundary conditions of the mixed type as in P_E it is not immediately clear which boundary conditions should be cancelled for the reduced problem P_0. Thus a "cancellation law"[3,13] must be formulated and an additional hypothesis added to the theorem to the effect that this law is satisfied unambiguously. Also, hypothesis 2 of Theorem 1 must be reformulated in such a way that all matrices which must be inverted to solve P_0 are nonsingular. Another important generalization is to the time independent case discussed previously for which $T \to \infty$. This case has been studied by Hoppensteadt[14] for the initial value problem and a result similar to Theorem 1 has been obtained.

For nonlinear problems, extensive modification of the theorem is required. In this case, it must be assumed that there exists an isolated root of the reduced system (3.11b), that this root is asymptotically stable in the sense of Liapunov, and that the boundary conditions are in the domain of influence of this root. For nonlinear initial value problems this theory was developed by Tihonov and is reviewed in ref. 2. An interesting geometrical interpretation of Tihonov's theory is contained in ref. 15 where it is seen to include the phenomenon of relaxation oscillation. A preliminary extension of these results to the two-point boundary value problem is contained in ref. 8.

For the feedback solution to the state regulator problem, there is a result analogous to Theorem 1; under suitable hypotheses it can be shown that the limiting form of the gain matrix $\underline{\underline{K}}(\varepsilon, t)$ obtained by setting $\varepsilon = 0$ in (2.21) is the same as the gain matrix of the reduced system.[16]

Boundary Layer Equations

Theorem 1 indicates that $\underline{v}$ (ε,t) will change very rapidly from its boundary values at t = 0 and t = T given by (3.10) to values approximating $\underline{v}$ (0, 0) and $\underline{v}$ (0, T), respectively. These rapid changes are called "boundary layers," a term which comes from fluid mechanics in connection with certain problems of viscous flow past a solid body. In these problems, the viscosity is a small parameter multiplying the highest derivatives in the Navier-Stokes equations. If this parameter is set to zero, the hydrodynamic system of equations results (reduced system); the solution of this system violates the no-slip boundary condition at the body surface. Thus in a thin layer of fluid near the surface of the body, the boundary layer, the velocity varies rapidly from zero on the surface of the body to the value given by the hydrodynamic solution. The boundary layer is characterized by high velocity gradients and high viscous forces. It is interesting to note that if the hydrodynamic solution is used without accounting for the boundary layer, the erroneous result that there is no streamwise force (D'Alembert's paradox) is obtained, even though the flow field is accurately described in all but a small region. This is indicative of the caution which must be exercised when drawing conclusions form reduced order systems.

The phenomenon of boundary layers occurs in all singular perturbation problems. For the system (3.10) there will be boundary layers at both t = 0 and t = T; in these layers $\underline{v}$ (ε, t) will change rapidly from its "inviscid" value $\underline{v}$ (0, t) to its boundary values $\underline{v}$ (ε, 0) and $\underline{v}$ (ε, T) in a time of order ε. To study this rapid change, we "stretch" the time scale by introducing the transformations

$$\tau_o = \frac{t}{\varepsilon}$$
$$\tau_T = \frac{T - t}{\varepsilon} \tag{3.12}$$

Using (3.12) in (3.6) the boundary layer equations associated with P_ε at the initial point are found to be

$$P_{BL_1} : \begin{array}{l} \dot{\underline{y}}_1 = \varepsilon \, \tilde{\underline{\underline{A}}}_1 \, \underline{y}_1 + \varepsilon \, \tilde{\underline{\underline{A}}}_2 \, \underline{z}_1 + \varepsilon \, \tilde{\underline{\underline{S}}}_1 \, \underline{p}_1 + \varepsilon \, \tilde{\underline{\underline{S}}}_2 \, \underline{q}_1 \; ; \; \underline{y}_1 \, (0) = \underline{y}_o \\ \dot{\underline{z}}_1 = \tilde{\underline{\underline{A}}}_3 \, \underline{y}_1 + \tilde{\underline{\underline{A}}}_4 \, \underline{z}_1 + \tilde{\underline{\underline{S}}}_3 \, \underline{p}_1 + \tilde{\underline{\underline{S}}}_4 \, \underline{q}_1 \; ; \; \underline{z}_1 \, (0) = \underline{z}_o \\ \dot{\underline{p}}_1 = \varepsilon \, \tilde{\underline{\underline{Q}}}_1 \, \underline{y}_1 + \varepsilon \, \tilde{\underline{\underline{Q}}}_2 \, \underline{z}_1 - \varepsilon \, \tilde{\underline{\underline{A}}}_1' \, \underline{p}_1 - \varepsilon \, \tilde{\underline{\underline{A}}}_3' \, \underline{q}_1 \\ \dot{\underline{q}}_1 = \tilde{\underline{\underline{Q}}}_3 \, \underline{y}_1 + \tilde{\underline{\underline{Q}}}_4 \, \underline{z}_1 - \tilde{\underline{\underline{A}}}_2' \, \underline{p}_1 - \tilde{\underline{\underline{A}}}_4' \, \underline{q}_1 \end{array} \tag{3.13}$$

In these equations, the independent variable is τ_o, for example $\dot{\underline{y}}_1 = \frac{d \, \underline{y}_1}{d \tau_o}$ and $\tilde{\underline{\underline{A}}}_1 \, (\varepsilon, \tau_o) = \underline{\underline{A}}_1 \, (\varepsilon, \varepsilon\tau_o)$. The boundary layer system at the final point, P_{BL_2}, is the same as (3.13) except that the boundary conditions are replaced by $\underline{p}_2(0) = \underline{0}$, $\underline{q}_2 \, (0) = \underline{0}$ and the independent variable is τ_T. The abbreviated forms of these equations are

$$P_{BL_1} : \begin{array}{l} \dfrac{d \, \underline{u}_1 \, (\varepsilon, \tau_o)}{d\tau_o} = \varepsilon \, \underline{\underline{F}}_1 \, (\varepsilon, \tau_o) \, \underline{u}_1 \, (\varepsilon, \tau_o) + \varepsilon \, \underline{\underline{D}}_1 \, (\varepsilon, \tau_o) \, \underline{v}_1 \, (\varepsilon, \tau_o) \\ \qquad u_{1_i} \, (\varepsilon, 0) = x_{o_i} \; ; \; i = 1, \cdot\cdot, m \\ \dfrac{d \, \underline{v}_1 \, (\varepsilon, \tau_o)}{d\tau_o} = \underline{\underline{H}}_1 \, (\varepsilon, \tau_o) \, \underline{u}_1 \, (\varepsilon, \tau_o) + \underline{\underline{G}}_1 \, (\varepsilon, \tau_o) \, \underline{v}_1 \, (\varepsilon, \tau_o) \\ \qquad v_{1_i} \, (\varepsilon, 0) = x_{o_{i+m}} \; ; \; i = 1, \cdot\cdot, n\text{-}m \end{array} \tag{3.14}$$

$$P_{BL_2}: \quad \begin{aligned} &\frac{d\,\underline{u}_2(\varepsilon,\tau_T)}{d\tau_T} = \varepsilon\,\underline{\underline{F}}_2(\varepsilon,\tau_T)\,\underline{u}_2(\varepsilon,\tau_T) + \varepsilon\,\underline{\underline{D}}_2(\varepsilon,\tau_T)\,\underline{v}_2(\varepsilon,\tau_T) \\ &u_{2_i}(\varepsilon, 0) = 0\ ;\ i = m+1,\cdots, 2m \\ &\frac{d\,\underline{v}_2(\varepsilon,\tau_T)}{d\tau_T} = \underline{\underline{H}}_2(\varepsilon,\tau_T)\,\underline{u}_2(\varepsilon,\tau_T) + \underline{\underline{G}}_2(\varepsilon,\tau_T)\,\underline{v}_2(\varepsilon,\tau_T) \\ &v_{2_i}(\varepsilon, 0) = 0\ ;\ i = n-m+1,\cdots, 2n-2m \end{aligned} \tag{3.15}$$

where, for example, $\underline{\underline{F}}_1(\varepsilon, \tau_o) = \underline{\underline{F}}(\varepsilon, \varepsilon\tau_o)$ and $\underline{\underline{F}}_2(\varepsilon, \tau_T) = \underline{\underline{F}}(\varepsilon, T - \varepsilon\tau_T)$.

Matched Asymptotic Expansions

It is natural to attempt the solution of (3.10) by asymptotic expansion in the small parameter ε. Asymptotic methods for both regular and singular problems are reviewed in refs. 4 and 17. Before proceeding, some definitions are needed. Given functions $f(\varepsilon)$ and $g(\varepsilon)$, we write $f(\varepsilon) = O(g(\varepsilon))$ if $f(\varepsilon)/g(\varepsilon)$ is bounded as $\varepsilon \to 0^+$ and $f(\varepsilon) = o(g(\varepsilon))$ if $\frac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to 0^+$. The sequence $f_n(\varepsilon)$ is an asymptotic sequence if $f_{i+1}(\varepsilon) = o(f_i(\varepsilon))$ as $\varepsilon \to 0^+$; $f_i(\varepsilon)$ is an asymptotic expansion of the function $F(\varepsilon)$ if it is an asymptotic sequence and if

$$\left| F(\varepsilon) - \sum_{i=0}^{n} a_i f_i(\varepsilon) \right| = o(f_n(\varepsilon))$$

which implies

$$\left| F(\varepsilon) - \sum_{i=0}^{n} a_i f_i(\varepsilon) \right| = O(f_{n+1}(\varepsilon))$$

We call $\sum_{i=0}^{n} a_i f_i(\varepsilon)$ the nth order approximation of $F(\varepsilon)$. It should be noted that different asymptotic sequences may lead to different asymptotic expansions for the same function but that the expansion in terms of a given sequence is unique. If the sequence is infinite, we write

$$F(\varepsilon) \sim \sum_{i=0}^{\infty} a_i f_i(\varepsilon)$$

and say that the series $a_i f_i(\varepsilon)$ is asymptotically convergent to $F(\varepsilon)$. The series need not be convergent in the ordinary sense. The only asymptotic sequence which will be used in this paper is the power series, ε^i.

There is at present no theoretically justifiable technique of obtaining an asymptotic solution to (3.10). The most general result available is due to Vasileva[18] and deals with the Cauchy problem for nonlinear first order systems. In her method a solution is sought in the form of a sum of two asymptotic expansions, one which approximates the solution "inside" the boundary layer and one "outside," and an explicit method for obtaining the terms in these expansions is obtained. This method is not used in the present paper, but the following result of ref. 18 is of interest in motivating the use of asymptotic methods in singular perturbation problems.

Theorem 2: Consider the system

$$\frac{dx}{dt} = f(x,y,\varepsilon,t) \qquad ; x(0) = x_0$$

$$\varepsilon \frac{dy}{dt} = g(x,y,\varepsilon,t) \qquad ; y(0) = y_0$$

with associated reduced system,

$$\frac{dx}{dt} = f(x,y,0,t) \qquad ; x(0) = x_0$$

$$0 = g(x,y,0,t)$$

and associated boundary layer system ($t = \varepsilon\tau$),

$$\frac{dx}{dt} = \varepsilon f(x,y,\varepsilon,\varepsilon\tau) \qquad ; \; x(0) = x_0$$

$$\frac{dy}{dt} = g(x,y,\varepsilon,\varepsilon\tau) \qquad ; \; y(0) = y_0$$

and suppose that: 1) f and g are n + 1 times continuously differentiable in x, y, and ε in some domain Ω and continuous in t for $0 \leq t \leq T$, 2) the full system and the reduced system both have unique solutions in Ω, 3) there exists in isolated root $y = \phi\,(x,t)$ of the reduced problem $0 = g(x,y,0,t)$ in Ω, and 4) the root $y = \phi\,(x,t)$ is an asymptotically stable equilibrium point of the boundary layer system. Let the asymptotic sequences $x_i(t)\varepsilon^i$, $\overline{x}_i(t)\varepsilon^i$, $y_i(t)\varepsilon^i$, $\overline{y}_i(t)\varepsilon^i$ be those discussed in ref. 18. Then these sequences are asymptotic expansions of the solution the full system, $x(\varepsilon,t)$ and $y(\varepsilon,t)$, that is

$$|x(\varepsilon,t) - \sum_{i=1}^{n} x_i(t)\varepsilon^i - \sum_{i=1}^{n} \overline{x}_i(t)\varepsilon^i| = o(\varepsilon^n)$$

$$|y(\varepsilon,t) - \sum_{i=1}^{n} y_i(t)\varepsilon^i - \sum_{i=1}^{n} \overline{y}_i(t)\varepsilon^i| = o(\varepsilon^n)$$

in Ω for $0 \leq t \leq T$.

Hadlock[8] formally applies Vasileva's method to two-point boundary value problems and it appears that the method is applicable to this class of problems as well. The method to be used in this paper was originally developed in fluid mechanics and is called the method of matched asymptotic expansions (or the method of inner and outer expansions)[4,5]. The method was first conceived by Prandtl and has been formalized by Kaplun, Lagerstrom and others. In this method it has been found to be advantageous not to give an explicit general formulation of the required expansions but to formulate only a general recipe and then treat each new problem individually, and this approach will be used here. This is due to the large diversity of behavior which may be encountered in even the most elementary problems as well to the great algebraic complexity generally involved. Theoretical justification of the method is not of great importance since it becomes readily apparent in the course of a solution whether or not the method is working. However, it seems likely that the hypotheses of Theorem 1 plus increased restrictions on the smoothness of the system matrices in the domain of interest are required. Specifically, hypothesis 1) will be replaced by: $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{Q}}$, $\underline{\underline{R}}$ are of class C^∞ in t and have asymptotic expansions in ε in domain D.

In the method of matched asymptotic expansions we proceed as follows: 1) solve the "outer" system (3.10) asymptotically, leaving the boundary conditions free, 2) solve the "inner" or boundary layer systems (3.14) and (3.15) asymptotically subject to the appropriate boundary conditions, and 3) determine the remaining constants of integration by invoking a "matching principle."

To solve the outer system set

$$\begin{aligned}
\underline{u}(\varepsilon,t) &\sim \sum_{i=0}^{\infty} \underline{u}^{(i)}(t)\varepsilon^i \quad ; \quad \underline{v}(\varepsilon,t) \sim \sum_{i=0}^{\infty} \underline{v}^{(i)}(t)\varepsilon^i \\
\underline{\underline{F}}(\varepsilon,t) &\sim \sum_{i=0}^{\infty} \underline{\underline{F}}^{(i)}(t)^{\varepsilon i} \quad ; \quad \underline{\underline{D}}(\varepsilon,t) \sim \sum_{i=0}^{\infty} \underline{\underline{D}}^{(i)}(t)\varepsilon^i \qquad (3.16) \\
\underline{\underline{H}}(\varepsilon,t) &\sim \sum_{i=0}^{\infty} \underline{\underline{H}}^{(i)}(t)\varepsilon^i \quad ; \quad \underline{\underline{G}}(\varepsilon,t) \sim \sum_{i=0}^{\infty} \underline{\underline{G}}^{(i)}(t)\varepsilon^i
\end{aligned}$$

Putting (3.16) in (3.10) gives

$$\sum_{i=0}^{\infty} \frac{d\underline{u}^{(i)}}{dt} \varepsilon^{i} \sim \left(\sum_{i=0}^{\infty} \underline{\underline{F}}^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{u}^{(i)}\varepsilon^{i}\right) + \left(\sum_{i=0}^{\infty} \underline{\underline{D}}^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{v}^{(i)}\varepsilon^{i}\right)$$

$$\sum_{i=0}^{\infty} \frac{d\underline{v}^{(i)}}{dt} \varepsilon^{i+1} \sim \left(\sum_{i=0}^{\infty} \underline{\underline{H}}^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{u}^{(i)}\varepsilon^{i}\right) + \left(\sum_{i=0}^{\infty} \underline{\underline{G}}^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{v}^{(i)}\varepsilon^{i}\right) \quad (3.17)$$

Equating the coefficients of successive orders of ε of these series in the usual way results in a sequence of linear problems, each of which is of order 2m and hence 2m arbitrary constants will appear at each order. If it is assumed that $\underline{\underline{G}}^{(o)}$ is nonsingular these problems will have unique solutions. Note that the zeroth order problem is homogeneous while the higher order ones are inhomogeneous.

To solve P_{BL_1}, set

$$\underline{u}_1(\varepsilon,\tau_o) \sim \sum_{i=0}^{\infty} \underline{u}_1^{(i)}(\tau_o)\varepsilon^{i} \quad ; \quad \underline{v}_1(\varepsilon,\tau_o) \sim \sum_{i=0}^{\infty} \underline{v}_1^{(i)}(\tau_o)\varepsilon^{i} \qquad (3.18)$$

in (3.14) to get

$$\sum_{i=0}^{\infty} \frac{d\underline{u}_1^{(i)}}{d\tau_o} \varepsilon^{i} \sim \left(\sum_{i=0}^{\infty} \underline{\underline{F}}_1^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{u}_1^{(i)}\varepsilon^{i+1}\right)$$

$$+ \left(\sum_{i=0}^{\infty} \underline{\underline{D}}_1^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{v}^{(i)}\varepsilon^{i+1}\right)$$

$$\sum_{i=0}^{\infty} \frac{d\underline{v}_1^{(i)}}{d\tau_o} \varepsilon^{i} \sim \left(\sum_{i=0}^{\infty} \underline{\underline{H}}_1^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{u}_1^{(i)}\varepsilon^{i}\right)$$

$$+ \left(\sum_{i=0}^{\infty} \underline{\underline{G}}_1^{(i)}\varepsilon^{i}\right) \left(\sum_{i=0}^{\infty} \underline{v}_1^{(i)}\varepsilon^{i}\right) \qquad (3.19)$$

where, for example,

$$\underline{\underline{F}}_1\ (\varepsilon,\tau_0) \sim \sum_{i=0}^{\infty} \underline{\underline{F}}_1{}^{(i)}(\tau_0)\varepsilon^i$$

The zeroth order terms in these expansions must satisfy the initial conditions since these conditions are of zeroth order in ε and the higher order terms must vanish at $t = 0$, that is

$$\begin{aligned} u_{1_j}{}^{(i)} &= \begin{Bmatrix} x_{o_j} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{Bmatrix} \quad ; \; j = 1, \ldots, m \\ v_{1_j}{}^{(i)} &= \begin{Bmatrix} x_{o_{j+m}} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{Bmatrix} ; \; j = 1, \ldots, n\text{-}m \end{aligned} \qquad (3.20)$$

The system (3.19) is of ordinary perturbation type and has 2n constants of integration at each order; imposing the n conditions (3.20) leaves n arbitrary constants. Note that 2m of the equations (3.19) are simple quadratures. Similarly for P_{BL_2} we set

$$\underline{u}_2(\varepsilon,\tau_T) \sim \sum_{i=0}^{\infty} \underline{u}_2{}^{(i)}(\tau_T)\varepsilon^i \quad ; \quad \underline{v}_2(\varepsilon,\tau_T) \sim \sum_{i=0}^{\infty} \underline{v}_2{}^{(i)}(\tau_T)\varepsilon^i \qquad (3.21)$$

so that

$$\sum_{i=0}^{\infty} \frac{d\underline{u}_2{}^{(i)}}{d\tau_T}\,\varepsilon^i \sim \left(\sum_{i=0}^{\infty} \underline{\underline{F}}_2{}^{(i)}\varepsilon^i\right)\left(\sum_{i=0}^{\infty} \underline{u}_2{}^{(i)}\varepsilon^{i+1}\right) + \left(\sum_{i=0}^{\infty} \underline{\underline{D}}_2{}^{(i)}\varepsilon^i\right)\left(\sum_{i=0}^{\infty} \underline{v}_2{}^{(i)}\varepsilon^{i+1}\right)$$

$$\sum_{i=0}^{\infty} \frac{d\underline{v}_2^{(i)}}{d\tau_T} \varepsilon^i \sim \left(\sum_{i=0}^{\infty} \underline{\underline{H}}_2^{(i)} \varepsilon^i\right) \left(\sum_{i=0}^{\infty} \underline{u}_2^{(i)} \varepsilon^i\right)$$

$$+ \left(\sum_{i=0}^{\infty} \underline{\underline{G}}_2^{(i)} \varepsilon^i\right) \left(\sum_{i=0}^{\infty} \underline{v}_2^{(i)} \varepsilon^i\right) \tag{3.22}$$

subject to

$$u_{2_j}^{(i)}(0) = 0 \ ; \quad i \geq 0, \quad j = m+1, \ .., \ 2m$$

$$v_{2_j}^{(i)}(0) = 0 \ ; \quad i \geq 0, \quad j = n-m+1, \ .., \ 2n-2m \tag{3.23}$$

where, for example,

$$F_2(\varepsilon,\tau_T) \sim \sum_{i=0}^{\infty} F_2^{(i)}(\tau_T)\varepsilon^i$$

The unknown constants in the outer and boundary layer expansions are determined by a matching principle. This principle assumes that there exist two overlap regions, one near t=0 and one near t=T such that in the first of these both the outer and the P_{BL_1} expansions are valid and agree, and in the second of these both the outer and P_{BL_2} expansions are valid and agree. In the limit, this leads to the matching conditions:

$$\lim_{t\to 0} \underline{u}^{(i)}(t) = \lim_{\tau_o\to\infty} \underline{u}_1^{(i)}(\tau_o)$$

$$\lim_{t\to 0} \underline{v}^{(i)}(t) = \lim_{\tau_o\to\infty} \underline{v}_1^{(i)}(\tau_o)$$

$$\lim_{t\to T} \underline{u}^{(i)}(t) = \lim_{\tau_T\to\infty} \underline{u}_2^{(i)}(\tau_T)$$

$$\lim_{t\to T} \underline{v}^{(i)}(t) = \lim_{\tau_T\to\infty} \underline{v}_2^{(i)}(\tau_T) \tag{3.24}$$

These conditions are to hold for all orders i of ε. It is evident that asymptotic stability of the boundary layer solutions is required. If it is assumed that n-m of the eigenvalues of $\underline{\underline{G}}^{(o)}$ are negative this stability is insured since n-m of the n constants appearing at each order of the boundary layer solutions may be chosen to suppress the unstable solutions, leaving m undetermined constants. The m constants in P_{BL_1}, the m in P_{BL_2} and the 2m of P_ε are determined by the matching conditions. It will be shown later that the matching conditions associated with $\underline{z}$ and $\underline{q}$ are redundant.

The representation of the solution obtained by the method outlined above has the undesirable feature that different representations are required for different values of the independent variable. It is desirable to obtain a single expansion valid uniformly for $0 \leq t \leq T$ and this may be accomplished by adding the two boundary layer expansions to the outer expansion and subtracting out the common parts, i.e., the terms which cancel out in the matching. This expansion is

$$\underline{u}(\varepsilon,t) \sim \sum_{i=0}^{\infty} \underline{u}^{(i)}(t)\varepsilon^i + \sum_{i=0}^{\infty} \underline{u}_1{}^{(i)}(\tau_o)\varepsilon^i + \sum_{i=0}^{\infty} \underline{u}_2{}^{(i)}(\tau_T)\varepsilon^i$$

$$- \lim_{\tau_o \to \infty} \sum_{i=0}^{\infty} \underline{u}_1{}^{(i)}(\tau_o)\varepsilon^i - \lim_{\tau_T \to \infty} \sum_{i=0}^{\infty} \underline{u}_2{}^{(i)}(\tau_T)\varepsilon^i$$

$$\underline{v}(\varepsilon,t) \sim \sum_{i=0}^{\infty} \underline{v}^{(i)}(t)\varepsilon^i + \sum_{i=0}^{\infty} \underline{v}_1{}^{(i)}(\tau_o)\varepsilon^i + \sum_{i=0}^{\infty} \underline{v}_2{}^{(i)}(\tau_T)\varepsilon^i$$

$$- \lim_{\tau_o \to \infty} \sum_{i=0}^{\infty} \underline{v}_1{}^{(i)}(\tau_o)\varepsilon^i - \lim_{\tau_T \to \infty} \sum_{i=0}^{\infty} \underline{v}_2{}^{(i)}(\tau_T)\varepsilon^i \qquad (3.25)$$

To summarize the results of this section, we state the following:

Theorem 3: Consider the system P_ε as given by (3.6) or (3.10) and suppose that there exists an ε_0 such that on $D = \{(\varepsilon,t) | 0 \leq \varepsilon \leq \varepsilon_0,\ 0 \leq t \leq T\}$ we have 1) $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{Q}}$, $\underline{\underline{R}}$ are of class C^∞ in t and have asymptotic expansions in ε, 2) $\underline{\underline{G}}^{(o)}$ is non-singular, and 3) n-m of the eigenvalues of $\underline{\underline{G}}^{(o)}$ are negative. Then the method of matched asymptotic expansions leads to a unique asymptotic representation of the solution of P_ε.

It should be noted that the theorem does not state that the resulting asymptotic sequences converge to the solution of P_ε. The theorem will be proved explicitly for the zeroth order in the following section (extension to higher orders is similar) where it is also shown that the zeroth order outer problem is precisely the reduced problem P_0.

Zeroth Order Approximation

To illustrate the method of matched asymptotic expansions, the zeroth order approximation will be worked out in detail. Retention of only zeroth order terms in the expansions gives a sufficiently good approximation for many applications. From (3.17)

$$\frac{d\underline{u}^{(o)}}{dt} = \underline{\underline{F}}^{(o)}\ \underline{u}^{(o)} + \underline{\underline{D}}^{(o)}\ \underline{v}^{(o)}$$

$$\underline{0} = \underline{\underline{H}}^{(o)}\ \underline{u}^{(o)} + \underline{\underline{G}}^{(o)}\ \underline{v}^{(o)} \qquad (3.26)$$

The solution of (3.26) is, referring to (3.8),

$$\begin{bmatrix} \underline{y}^{(o)} \\ \underline{p}^{(o)} \end{bmatrix} = \underline{u}^{(o)} = e^{(\underline{\underline{F}}^{(o)} - \underline{\underline{D}}\,\underline{\underline{G}}^{(o)^{-1}}\,\underline{\underline{H}}^{(o)})t}\,\underline{C}$$

$$\begin{bmatrix} \underline{z}^{(o)} \\ \underline{q}^{(o)} \end{bmatrix} = \underline{v}^{(o)} = -\,\underline{\underline{G}}^{(o)-1}\,\underline{\underline{H}}^{(o)} \begin{bmatrix} \underline{y}^{(o)} \\ \underline{p}^{(o)} \end{bmatrix} \qquad (3.27)$$

where $\underline{C}$ is a vector of 2m constants of integration.

From (3.19) the zeroth order term in the P_{BL_1} expansion satisfies

$$\frac{d\underline{u}_1^{(o)}}{d\tau_o} = \underline{0}\ ; \quad u_{1_i}^{(o)}(0) = x_{o_i} \quad ; \quad i=1, .., m$$

$$\frac{d\underline{v}_1^{(o)}}{d\tau_o} = \underline{\underline{H}}^{(o)}\underline{u}_1^{(o)} + \underline{\underline{G}}^{(o)}\underline{v}_1^{(o)}\ ; \quad v_{1_i}^{(o)}(0) = x_{o_{i+m}} \quad ; \quad i=1, .., n-m \qquad (3.28)$$

The solution to this system may be written

$$\begin{bmatrix} \underline{y}_1^{(o)} \\ \underline{p}_1^{(o)} \end{bmatrix} = \underline{u}_1^{(o)} = \begin{pmatrix} \underline{y}_o \\ \underline{C}_1' \end{pmatrix}$$

$$\begin{bmatrix} \underline{z}_1^{(o)} \\ \underline{q}_1^{(o)} \end{bmatrix} = \underline{v}_1^{(o)} = \underline{\underline{M}}_G^{(o)}\,\underline{h}_1 - \underline{\underline{G}}^{(o)^{-1}}\,\underline{\underline{H}}^{(o)} \begin{pmatrix} \underline{y}_o \\ \underline{C}_1' \end{pmatrix} \qquad (3.29)$$

where $\underline{\underline{M}}_G^{(o)}$ is the modal matrix of $\underline{\underline{G}}^{(o)}$ and

$$h_{1_i} = C_{1_i}\, e^{S_i \tau_o}$$

where the C_{1_i} are the n-m constants of integration associated with the eigenvalues S_i of $\underline{\underline{G}}^{(o)}$ which have negative real parts. The solutions associated with eigenvalues which have positive or zero real parts have been suppressed by setting their associated constants of integration to zero. The constants $\underline{C}_1'$ and $\underline{C}_1$ are related by

$$\begin{bmatrix} \underline{z}_o \\ \underline{g}_1^{(o)}(0) \end{bmatrix} = \underline{\underline{M}}_G^{(o)} \underline{C}_1 - \underline{\underline{G}}^{(o)^{-1}} \underline{\underline{H}}^{(o)} \begin{pmatrix} \underline{y}_o \\ \underline{C}_1' \end{pmatrix} \tag{3.30}$$

so that only m of their components are independent. The zeroth order term in P_{BL_2} satisfies

$$\begin{aligned} &\frac{d\underline{u}_2^{(o)}}{d\tau_T} = \underline{0} \quad ; \quad u_{2_i}^{(o)}(0) = 0 \quad ; \quad i = m+1, \ldots, 2m \\ &\frac{d\underline{v}_2^{(o)}}{d\tau_T} = \underline{\underline{H}}^{(o)} \underline{u}_2^{(o)} + \underline{\underline{G}}^{(o)} \underline{v}_2^{(o)} \quad ; \quad v_{2_i}^{(o)}(0) = 0 \quad ; \quad i = n-m+1, \ldots, 2n-2m \end{aligned} \tag{3.31}$$

with solution

$$\begin{bmatrix} \underline{y}_2^{(o)} \\ \underline{p}_2^{(o)} \end{bmatrix} = \underline{u}_2^{(o)} = \begin{pmatrix} \underline{C}_2' \\ \underline{0} \end{pmatrix}$$

$$\begin{bmatrix} \underline{z}_2^{(o)} \\ \underline{g}_2^{(o)} \end{bmatrix} = \underline{v}_2^{(o)} = \underline{\underline{M}}_G^{(o)} \underline{h}_2 - \underline{\underline{G}}^{(o)^{-1}} \underline{\underline{H}}^{(o)} \begin{pmatrix} \underline{C}_2' \\ \underline{0} \end{pmatrix} \tag{3.32}$$

where

$$h_{2_i} = C_{2_i} e^{S_i \tau_T}$$

with

$$\begin{bmatrix} \underline{z}_2^{(o)}(0) \\ \underline{0} \end{bmatrix} = \underline{\underline{M}}_G^{(o)}\, \underline{C}_2 - \underline{\underline{G}}^{(o)^{-1}}\, \underline{\underline{H}}^{(o)} \begin{pmatrix} \underline{C}_2' \\ \underline{0} \end{pmatrix} \qquad (3.33)$$

First consider the matching of $\underline{y}^{(o)}$ at the initial point. From (3.24)

$$\lim_{t \to o} \underline{y}^{(o)} = \lim_{\tau_o \to \infty} \underline{y}_1^{(o)}$$

$$\underline{y}^{(o)}(0) = \underline{y}_o$$

Similarly, the matching of $\underline{p}^{(o)}$ at the final condition gives

$$\underline{p}^{(o)}(T) = \underline{0}$$

These conditions together with (3.26) imply that the zeroth order outer solution is just the solution to the reduced problem (3.7). The constants $\underline{C}$ have now been determined. The matching of $\underline{p}^{(o)}$ at t = 0 and $\underline{y}^{(o)}$ at t = T gives the constants $\underline{C}_1'$ and $\underline{C}_2'$:

$$\begin{pmatrix} \underline{y}_o \\ \underline{C}_1' \end{pmatrix} = \underline{C}$$

$$\begin{pmatrix} \underline{C}_2' \\ \underline{0} \end{pmatrix} = e^{(\underline{\underline{F}}^{(o)} - \underline{\underline{D}}\, \underline{\underline{G}}^{(o)^{-1}}\, \underline{\underline{H}}^{(o)})T}\, \underline{C}$$

The matching conditions for $\underline{z}^{(o)}$ and $\underline{q}^{(o)}$ at the initial and final points are then satisfied identically as may be seen from (3.27b), (3.29b), and (3.30b).

An Alternate Procedure

In the preceding development, the necessary conditions were applied to the full system and the resulting two point boundary value problem was solved by the method of matched asymptotic expansions. It is natural to ask whether or not the reverse procedure is valid, that is, is it permissible to first expand the state equations asymptotically and then to apply the necessary conditions to each order of the expansions? This latter procedure is attractive since it may be expected to result in less algebraic manipulation for a given problem. To pursue this question, we partition (2.14) in the manner of (3.6)

$$\dot{\underline{y}} = \underline{\underline{A}}_1 \, \underline{y} + \underline{\underline{A}}_2 \, \underline{z} + \underline{\underline{B}}_1 \, \underline{\sigma}$$

$$\varepsilon\dot{\underline{z}} = \underline{\underline{A}}_3 \, \underline{y} + \underline{\underline{A}}_4 \, \underline{z} + \underline{\underline{B}}_2 \, \underline{\sigma}$$

where $\underline{\underline{B}}_1$ is m x r, $\underline{\underline{B}}_2$ is (n-m) x r and the other quantities are as defined by (3.4) and (3.5). The following result gives the conditions for equivalence of procedures for the zeroth order.

Theorem 4: Let the first two hypotheses of Theorem 1 apply and further suppose that the matrix $\left[\underline{\underline{A}}_4 \; \underline{\underline{B}}_2\right]$ has maximum rank (i.e., rank n-m) in a neighborhood of the extremal. Then the reduced problem obtained by setting $\varepsilon = o$ in the necessary conditions of the full problem is the same as the problem obtained by setting $\varepsilon = 0$ in the state equations and applying the necessary conditions to the result.

The significance of the additional hypothesis is clear; in either procedure, if it is satisfied n-m of the components of $(\underline{z}, \underline{\sigma})$ may be eliminated from the problem. When ε is set to zero in the state equations, $\underline{z}$ becomes effectively a control variable and the two control variables $\underline{z}$ and $\underline{\sigma}$ are constrained by a set of state dependent algebraic equations. The adjoint variables associated with the state variables whose derivatives are neglected correspond formally to the ordinary Lagrange multipliers arising in the maximization of the $\mathcal{H}$ function in the second procedure. The proof of the theorem follows from section 3.6 of ref. 12. A more restrictive equivalence of procedures result than Theorem 4 is given in ref. 8 where it is shown that a sufficient condition for equivalence is the non-singularity of $\underline{\underline{A}}_4$. If one takes the viewpoint that a control variable is a variable which may be changed instantaneously*, it may be concluded that, for problems in which a small parameter is to be inserted, the parameter should be inserted in such a way as to make the relatively "fast" variables behave as control variables in low order approximations.

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*I.e., controls may be piecewise continuous or even just measurable.

IV. AN EXAMPLE

Problem Formulation

Consider a forced two degree of freedom spring-mass-damper constant coefficient system with specified initial conditions.

$$a\ddot{y} + b\dot{y} + cy = \sigma \;; \quad y(0) = y_0, \; \dot{y}(0) = \dot{y}_0$$

It is desired to select the control $\sigma(t)$ to minimize

$$J = \frac{1}{2}\int_0^\infty (q_{11}y^2 + q_{22}\dot{y}^2 + 2q_{12}y\dot{y} + R\sigma^2)dt$$

In state variable form (let $x_1 = y$, $x_2 = \dot{y}$, $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$)

$$\dot{x}_1 = x_2 \qquad ; \quad x_1(0) = x_{10}$$

$$a\dot{x}_2 = - bx_2 - cx_1 + \sigma \quad ; \quad x_2(0) = x_{20}$$

$$J = \frac{1}{2}\int_0^\infty \left[< \underline{x}, \underline{\underline{Q}}\underline{x} > + R\sigma^2\right]dt$$

If b or c is "small" this is a regular perturbation problem and if a is "small" it is a singular perturbation problem. Suppose a is small and $q_{11} = q_{12} = 0$. Without loss of generality, let $q_{22} = q_2$, $R = 1$ and $a = \varepsilon$ so that

$$\dot{x}_1 = x_2 \qquad ; \quad x_1(0) = x_{10}$$

$$\varepsilon\dot{x}_2 = - bx_2 - cx_1 + \sigma \quad ; \quad x_2(0) = x_{20}$$

$$J = \frac{1}{2}\int_0^\infty (q_2x_2^2 + \sigma^2)dt$$

The system matrices are

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{\underline{Q}} = \begin{pmatrix} 0 & 0 \\ 0 & q_2 \end{pmatrix}$$

$$\underline{\underline{R}} = \begin{pmatrix} 1 \end{pmatrix}$$

From (2.8)

$$\underline{\underline{S}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

From (3.5) and (3.9)

$$\underline{\underline{A}}_1 = (0) \;;\; \underline{\underline{A}}_2 = (1) \;;\; \underline{\underline{A}}_3 = (-c) \;;\; \underline{\underline{A}}_4 = (-b)$$

$$\underline{\underline{S}}_1 = (0) \;;\; \underline{\underline{S}}_2 = (0) \;;\; \underline{\underline{S}}_3 = (0) \;;\; \underline{\underline{S}}_4 = (1)$$

$$\underline{\underline{Q}}_1 = (0) \;;\; \underline{\underline{Q}}_2 = (0) \;;\; \underline{\underline{Q}}_3 = (0) \;;\; \underline{\underline{Q}}_4 = (q_2)$$

$$\underline{\underline{F}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \;;\; \underline{\underline{D}} = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$$

$$\underline{\underline{H}} = \begin{pmatrix} -c & 0 \\ 0 & -1 \end{pmatrix} \;;\; \underline{\underline{G}} = \begin{pmatrix} -b & 1 \\ q_2 & b \end{pmatrix}$$

Since $\underline{\underline{G}}$ is nonsingular with eigenvalues $+\sqrt{q_2{}^2+b^2}$ and $-\sqrt{q_2{}^2+b^2}$, Theorems 1 and 3 hold ($\underline{\underline{G}} = \underline{\underline{G}}^{(o)}$), and it may be expected that the method of matched asymptotic expansions will work on this problem. Further, $[\underline{\underline{A}}_4, \underline{\underline{B}}_2] = (-b\ 1)$ has maximum rank (in fact $\underline{\underline{A}}_4$ is nonsingular) so that Theorem 4 holds and the alternate procedure may be used, that is we may set $\varepsilon = 0$ in the state equations, treat x_2 and σ as control variables, and apply necessary conditions to obtain the reduced problem. This equivalence may be verified by direct computation for this simple example.

In the following, both the exact and zeroth order asymptotic solutions for the case of no damping (b = 0) will be obtained. Consideration of this case exhibits the essential elements of the method while minimizing algebraic complexity. Note that now

$$\underline{\underline{A}}_4 = (0)$$

$$\underline{\underline{G}} = \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix}$$

and $[\underline{A}_4\ \underline{B}_2] = (0\ 1)$ so that Theorems 1, 3 and 4 still hold. However the alternate procedure will not be used in the following.

Exact Solution

The system to be solved is (see 2.20)

$$\dot{x}_1 = x_2 \qquad ; \quad x_1(0) = x_{10}$$

$$\varepsilon\dot{x}_2 = -c x_1 + \lambda_2 \qquad ; \quad x_2(0) = x_{20}$$

$$\dot{\lambda}_1 = c\lambda_2 \qquad ; \quad \lambda_1(\infty) = 0$$

$$\varepsilon\dot{\lambda}_2 = q_2 x_2 - \lambda_1 \qquad ; \quad \lambda_2(\infty) = 0$$

Set

$$x_1 = A_1 e^{St},\ x_2 = A_2 e^{St},\ \lambda_1 = A_3 e^{St},\ \lambda_2 = A_4 e^{St}$$

to get

$$A_1 S = A_2$$

$$\varepsilon A_2 S = -cA_1 + A_4$$

$$A_3 S = cA_4$$

$$\varepsilon A_4 S = q_2 A_2 - A_3$$

Let

$$\mu_2 = A_2/A_1 = S$$

$$\mu_3 = A_3/A_1 = \frac{c}{S}\,(\varepsilon S^2 + c)$$

$$\mu_4 = A_4/A_1 = \varepsilon S^2 + c$$

The characteristic equation is

$$\varepsilon^2 S^4 + (2\varepsilon c - q_2)S^2 + c^2 = 0$$

with solution

$$S_{1,2,3,4} = \pm\sqrt{\frac{q_2}{2\varepsilon^2} - \frac{c}{\varepsilon} \pm\sqrt{\frac{q_2^{\,2}}{4\varepsilon^4} - \frac{cq_2}{\varepsilon^3}}}$$

The exact solution is represented as

$$x_1 = A_1^{(1)}e^{S_1t} + A_1^{(2)}e^{S_2t} + A_1^{(3)}e^{S_3t} + A_1^{(4)}e^{S_4t}$$

$$x_2 = \mu_2^{(1)}A_1^{(1)}e^{S_1t} + \mu_2^{(2)}A_1^{(2)}e^{S_2t} + \mu_2^{(3)}A_1^{(3)}e^{S_3t} + \mu_2^{(4)}A_1^{(4)}e^{S_4t}$$

$$\dot{\lambda}_1 = \mu_3^{(1)}A_1^{(1)}e^{S_1t} + \mu_3^{(2)}A_1^{(2)}e^{S_2t} + \mu_3^{(3)}A_1^{(3)}e^{S_3t} + \mu_3^{(4)}A_1^{(4)}e^{S_4t}$$

$$\lambda_2 = \mu_4^{(1)}A_1^{(1)}e^{S_1t} + \mu_4^{(2)}A_1^{(2)}e^{S_2t} + \mu_4^{(3)}A_1^{(3)}e^{S_3t} + \mu_4^{(4)}A_1^{(4)}e^{S_4t}$$

The $A_1^{(i)}$ are determined by the boundary conditions; this leads to

$$A_1^{(1)} = 0$$

$$A_1^{(2)} = 0$$

$$A_1^{(3)} = \frac{x_{20} - \mu_2^{(4)}\,x_{10}}{\mu_2^{(3)} - \mu_2^{(4)}}$$

$$A_1^{(4)} = - \frac{x_{20} - \mu_2^{(3)} x_{10}}{\mu_2^{(3)} - \mu_2^{(4)}}$$

where

$$S_{\substack{3\\4}} = -\sqrt{\frac{q_2}{2\varepsilon^2} - \frac{c}{\varepsilon} \pm \sqrt{\frac{q_2^2}{4\varepsilon^4} - \frac{cq_2}{\varepsilon^3}}}$$

For future purposes, we expand the solution for small ε and retain only the first terms. For S_3:

$$S_3 = -\sqrt{\frac{q_2}{2\varepsilon^2} - \frac{c}{\varepsilon} + \sqrt{\frac{q_2^2}{4\varepsilon^4} - \frac{cq_2}{\varepsilon^3}}}$$

$$= - \frac{1}{\varepsilon} \sqrt{q_2 - 2c\varepsilon - \frac{c^2}{q_2} \varepsilon^2 + \ldots}$$

$$S_3 \approx - \frac{\sqrt{q_2}}{\varepsilon}$$

$$\mu_2^{(3)} = S_3 \approx - \frac{\sqrt{q_2}}{\varepsilon}$$

$$\mu_3^{(3)} = \frac{c}{S_3} (\varepsilon S_3^2 + c) \approx - c \sqrt{q_2}$$

$$\mu_4^{(3)} = \varepsilon S_3^2 + c \approx \frac{q_2}{\varepsilon}$$

For S_4:

$$S_4 = -\sqrt{\frac{q_2}{2\varepsilon^2} - \frac{c}{\varepsilon} - \sqrt{\frac{q_2^2}{4\varepsilon^4} - \frac{cq_2}{\varepsilon^3}}}$$

$$= - \frac{1}{\varepsilon} \sqrt{\frac{c^2}{q_2} \varepsilon^2 + \ldots}$$

$$S_4 \approx -\frac{c}{\sqrt{q_2}}$$

$$\mu_2^{(4)} \approx -\frac{c}{\sqrt{q_2}}$$

$$\mu_3^{(4)} \approx -c\sqrt{q_2}$$

$$\mu_4^{(4)} \approx c$$

Thus

$$A_1^{(3)} \approx -\frac{\varepsilon}{\sqrt{q_2}}\left(\frac{c}{\sqrt{q_2}}x_{10} + x_{20}\right)$$

$$A_1^{(4)} \approx x_{10}$$

so that the zeroth order approximation is

$$x_1 \approx x_{10}\, e^{-\frac{c}{\sqrt{q_2}}t}$$

$$x_2 \approx \left(\frac{c}{\sqrt{q_2}}x_{10} + x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon}t} - \frac{c}{\sqrt{q_2}}x_{10}\, e^{-\frac{c}{\sqrt{q_2}}t}$$

$$\lambda_1 \approx -c\sqrt{q_2}\; x_{10}\, e^{-\frac{c}{\sqrt{q_2}}t}$$

$$\lambda_2 = \sigma \approx -\left(cx_{10} + \sqrt{q_2}\; x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon}t} + cx_{10}\, e^{-\frac{c}{\sqrt{q_2}}t}$$

Terms with factor $e^{-\sqrt{q_2}/\varepsilon\ t}$ will be important only for small t ($t = O(\varepsilon)$ or less) and appear only in the expressions for the variables (x_2, λ_2) which lose their boundary conditions at $\varepsilon = 0$. It may thus be anticipated that these terms are the zeroth order terms in the boundary layer expansions; and that terms with factor $e^{-c/\sqrt{q_2}\ t}$ are the zeroth order terms of the outer expansion. This will now be shown by obtaining these expansions.

Solution by Matched Asymptotic Expansions

The zeroth order outer system is just the reduced problem, as has been shown earlier.

$$\dot{x}_{1_0} = x_{2_0} \qquad ; \quad x_{1_0}(0) = x_{10}$$

$$\dot{\lambda}_{1_0} = c\lambda_{2_0} \qquad ; \quad \lambda_{1_0}(\infty) = 0$$

$$0 = -cx_{1_0} + \lambda_{2_0}$$

$$0 = q_2\, x_{2_0} - \lambda_{1_0}$$

Solution of this simple system is

$$x_{1_0} = x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$x_{2_0} = -\frac{c}{\sqrt{q_2}}\, x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$\lambda_{1_0} = - c \sqrt{q_2}\, x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$\lambda_{2_0} = \sigma_0 = x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

These functions do not satisfy the initial conditions but do satisfy the terminal conditions and hence a boundary layer will be needed at the initial point only.

The zeroth order boundary layer (inner) system is obtained by using transformation (3.12) and then setting $\varepsilon = 0$:

$$x'_{1_I} = 0 \qquad ; \quad x_{1_I}(0) = x_{10}$$

$$\lambda'_{1_I} = 0$$

$$x'_{2_I} = - c\, x_{1_I} + \lambda_{2_I} \qquad ; \quad x_{2_I}(0) = x_{20}$$

$$\lambda'_{2_I} = q_2\, x_{2_I} - \lambda_{1_I}$$

where $(\ )' = \frac{d(\ )}{d\tau_0}$. Of the four constants of integration for this system, two are determined by the boundary conditions, one is needed to suppress the unstable root of the characteristic equation, and the fourth, say K, is to be determined by the matching conditions. The solution is

$$x_{1_I} = x_{10}$$

$$x_{2_I} = K\, e^{-\sqrt{q_2}\, \tau_0} + (x_{20} - K)$$

$$\lambda_{1_I} = q_2\ (x_{20} - K)$$

$$\lambda_{2_I} = \sigma_I = c\ x_{10} - K\sqrt{q_2}\ \ e^{-\sqrt{q_2}\ \tau_o}$$

The inner and outer solutions are now matched according to (3.24):

$$x_1 :\ x_{10} = x_{10}$$

$$x_2 :\ - \frac{c}{\sqrt{q_2}}\ x_{10} = x_{20} - K$$

$$\lambda_1 :\ - c\sqrt{q_2}\ x_{10} = q_2\ (x_{20} - K)$$

$$\lambda_2 :\ c\ x_{10} = c\ x_{10}$$

These conditions are satisfied unambiguously by the unique value

$$K = x_{20} + \frac{c}{\sqrt{q_2}}\ x_{10}$$

(In the previous section it was shown that the matching of x_2 and λ_2 is redundant.) Thus the zeroth order inner expansion is

$$x_{1_I} = x_{10}$$

$$x_{2_I} = - \frac{c}{\sqrt{q_2}}\ x_{10} + \left(\frac{c}{\sqrt{q_2}}\ x_{10} + x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon}\ t}$$

$$\lambda_{1_I} = - c \sqrt{q_2}\, x_{10}$$

$$\lambda_{2_I} = \sigma_I = c\, x_{10} - \left(c\, x_{10} + \sqrt{q_2}\, x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon} t}$$

Finally, the zeroth order of the uniform expansion obtained according to (3.25) is

$$x_{1_U} = x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$x_{2_U} = \left(\frac{c}{\sqrt{q_2}} x_{10} + x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon} t} - \frac{c x_{10}}{\sqrt{q_2}}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$\lambda_{1_U} = - c \sqrt{q_2}\, x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

$$\lambda_{2_U} = \sigma_U = - \left(c\, x_{10} + \sqrt{q_2}\, x_{20}\right) e^{-\frac{\sqrt{q_2}}{\varepsilon} t} + c\, x_{10}\, e^{-\frac{c}{\sqrt{q_2}} t}$$

which agrees with the first term expansion of the exact solution. A useful approximation to the solution has thus been easily obtained.

This example shows that application of the method of matched asymptotic expansions in effect "splits up" a given problem into relatively easier problems, and it is this characteristic which makes the method attractive for complex problems. The method may be continued to obtain first order and higher approximations.

Numerical Example

To gain insight into the ability of the uniform zeroth approximation of the example to approximate the exact solution, consider the specific case

$$c = 1, \sqrt{q_2} = 1, x_{10} = x_{20}, \varepsilon = \frac{1}{10}$$

so that ε is only one order of magnitude smaller than the other system parameters. For this case, the exact solution is

$$\frac{x_{1_E}}{x_{10}} = -.2746\, e^{-8.873t} + 1.2746\, e^{-1.127t}$$

$$\frac{x_{2_E}}{x_{10}} = 2.436\, e^{-8.873t} - 1.436\, e^{-1.127t}$$

$$\frac{\sigma_E}{x_{10}} = -2.436\, e^{-8.873t} + 1.436\, e^{-1.127t}$$

the outer solution is

$$\frac{x_{1_0}}{x_{10}} = e^{-t}$$

$$\frac{x_{2_0}}{x_{10}} = -e^{-t}$$

$$\frac{\sigma_0}{x_{10}} = e^{-t}$$

the inner solution is

$$\frac{x_{1_I}}{x_{10}} = 1$$

$$\frac{x_{2_I}}{x_{10}} = 2\,e^{-10t} - 1$$

$$\frac{\sigma_I}{x_{10}} = 1 - 2\,e^{-10t}$$

and the uniform solution is

$$\frac{x_{1_U}}{x_{10}} = e^{-t}$$

$$\frac{x_{2_U}}{x_{10}} = -e^{-t} + 2e^{-10t}$$

$$\frac{\sigma_U}{x_{10}} = e^{-t} - 2e^{-10t}$$

The various solutions are plotted on Figures 1, 2, 3. Semi-log paper is used to amplify the behavior at small t (boundary layer). The figures show that agreement between the exact and the approximate solutions is best at very small t and at very large t and worst in the overlap region $\varepsilon \leq t \leq 10\varepsilon$. It appears that the inner solution is better than the uniform solution at small t and that the outer solution is better at large t. It is seen, however, that the uniform solution gives a reasonable approximation to the exact solution; for smaller orders of ε this approximation would, of course, be improved.

The cost is computed from

$$J = \frac{1}{2}\int_0^\infty (q_2\, x_2^{\,2} + \sigma^2)dt$$

$$\frac{J}{x_{10}^{\;2}} = \int_0^\infty \left(\frac{x_2}{x_{10}}\right)^2 dt$$

so that

$$\left(\frac{J}{x_{10}^{\;2}}\right)_E = 0.5496$$

$$\left(\frac{J}{x_{10}^{\;2}}\right)_0 = 0.5$$

$$\left(\frac{J}{x_{10}^{\;2}}\right)_U = 0.3364$$

Thus the approximate solutions are unconservative (underestimate the cost) with the outer solution giving a better value than the uniform one.

<u>A Special Case</u>

To illustrate that for singular perturbation problems a slight modification of the system may lead to dramatic differences in the solution characteristics, the special case of $x_{10} = 0$, $c = 0$ is now considered. The zeroth, first, and second order terms of the uniform asymptotic expansions will be obtained. In this case $\lambda_1 \equiv 0$ so that the system to be solved is

$$\dot{x}_1 = x_2 \qquad ; \quad x_1(0) = 0$$

$$\varepsilon\dot{x}_2 = \sigma \qquad ; \quad x_2(0) = x_{20}$$

$$\varepsilon\dot{\sigma} = q_2 x_2 \qquad ; \quad \sigma(\infty) = 0$$

The exact solution is

$$x_1 = \frac{\varepsilon x_{20}}{\sqrt{q_2}}\left(1 - e^{-\frac{\sqrt{q_2}}{\varepsilon}t}\right)$$

$$x_2 = x_{20} \; e^{-\frac{\sqrt{q_2}}{\varepsilon} t}$$

$$\sigma = -\sqrt{q_2}\, x_{20} \; e^{-\frac{\sqrt{q_2}}{\varepsilon} t}$$

To solve the outer problem asymptotically, to second order, set

$$x_1 \sim \sum_{i=0}^{\infty} \alpha_i(t)\varepsilon^i \; ; \; x_2 \sim \sum_{i=0}^{\infty} \beta_i(t)\varepsilon^i \; ; \; \sigma \sim \sum_{i=0}^{\infty} \sigma_i(t)\varepsilon^i$$

in the system equations and retain only second order terms and lower:

$$\dot{\alpha}_0 + \dot{\alpha}_1\varepsilon + \dot{\alpha}_2\varepsilon^2 = \beta_0 + \beta_1\varepsilon + \beta_2\varepsilon^2$$

$$\dot{\beta}_0\varepsilon + \dot{\beta}_1\varepsilon^2 = \sigma_0 + \sigma_1\varepsilon + \sigma_2\varepsilon^2$$

$$\dot{\sigma}_0\varepsilon + \dot{\sigma}_1\varepsilon^2 = q_2\beta_0 + q_2\beta_1\varepsilon + q_2\beta_2\varepsilon^2$$

To zeroth order:

$$\left.\begin{array}{l} \dot{\alpha}_0 = \beta_0 \\ 0 = \sigma_0 \\ 0 = q_2\beta_0 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_0 = C_0 \\ \beta_0 = 0 \\ \sigma_0 = 0 \end{array}$$

where C_0 is an undetermined constant. The first order terms are

$$\left.\begin{array}{l} \dot{\alpha}_1 = \beta_1 \\ \dot{\beta}_0 = \sigma_1 \\ \dot{\sigma}_0 = q_2\beta_1 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_1 = C_1 \\ \beta_1 = 0 \\ \sigma_1 = 0 \end{array}$$

The second order terms are

$$\left.\begin{aligned} \dot{\alpha}_2 &= \beta_2 \\ \dot{\beta}_1 &= \sigma_2 \\ \dot{\sigma}_1 &= q_2\beta_2 \end{aligned}\right\} \Longrightarrow \begin{aligned} \alpha_2 &= C_2 \\ \beta_2 &= 0 \\ \sigma_2 &= 0 \end{aligned}$$

The outer solution to second order is then

$$x_{1_0} = C_o + C_1\varepsilon + C_2\varepsilon^2$$

$$x_{2_0} = 0 \quad ; \quad \sigma_o = 0$$

The boundary layer equations are

$$x_1' = \varepsilon x_2 \quad ; \quad x_1(0) = 0$$

$$x_2' = \sigma \quad ; \quad x_2(0) = x_{20}$$

$$\sigma' = q_2 x_2$$

where $(\;)' = \frac{d(\;)}{d\tau}$ and $t = \varepsilon\tau$. To solve these to second order set

$$x_1 \sim \sum_{i=0}^{\infty} \delta_i(\tau)\varepsilon^i \quad ; \quad x_2 \sim \sum_{i=0}^{\infty} \eta_i(\tau)\varepsilon^i \; ; \; \sigma \sim \sum_{i=0}^{\infty} \nu_i(\tau)\varepsilon^i$$

in the boundary layer equations and retain terms up to and including second order:

$$\delta_0' + \delta_1'\varepsilon + \delta_2'\varepsilon^2 = \eta_0\varepsilon + \eta_1\varepsilon^2$$
$$\eta_0' + \eta_1'\varepsilon + \eta_2'\varepsilon^2 = \nu_0 + \nu_1\varepsilon + \nu_2\varepsilon^2$$
$$\nu_0' + \nu_1'\varepsilon + \nu_2'\varepsilon^2 = q_2\eta_0 + q_2\eta_1\varepsilon + q_2\eta_2\varepsilon^2$$

To zeroth order:

$$\left.\begin{array}{r} \delta_0' = 0 \\ \eta_0' = \nu_0 \\ \nu_0' = q_2\eta_0 \\ \delta_0(0) = 0 \\ \eta_0(0) = x_{20} \end{array}\right\} \Longrightarrow \begin{array}{l} \delta_0 = 0 \\ \eta_0 = \dfrac{1}{\sqrt{q_2}}\left[(K_0 + \sqrt{q_2}\, x_{20})e^{\sqrt{q_2}\,\tau} - K_0 e^{-\sqrt{q_2}\,\tau}\right] \\ \nu_0 = (K_0 + \sqrt{q_2}\, x_{20})\, e^{\sqrt{q_2}\,\tau} + K_0 e^{-\sqrt{q_2}\,\tau} \end{array}$$

The first order terms are

$$\left.\begin{array}{r} \delta_1' = \eta_0 \\ \eta_1' = \nu_1 \\ \nu_1' = q_2\eta_1 \\ \delta_1(o) = 0 \\ \eta_1(o) = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \delta_1 = \dfrac{1}{q_2}\left[(K_0 + \sqrt{q_2}\, x_{20})\,(e^{\sqrt{q_2}\,\tau} - 1) + K_0\,(e^{-\sqrt{q_2}\,\tau} - 1)\right] \\ \eta_1 = \dfrac{K_1}{\sqrt{q_2}}\left(e^{\sqrt{q_2}\,\tau} - e^{-\sqrt{q_2}\,\tau}\right) \\ \nu_1 = K_1\left(e^{\sqrt{q_2}\,\tau} + e^{-\sqrt{q_2}\,\tau}\right) \end{array}$$

The second order terms are

$$\left.\begin{array}{r} \delta_2' = \eta_1 \\ \eta_2' = \nu_2 \\ \nu_2' = q_2\eta_2 \\ \delta_2(o) = 0 \\ \eta_2(o) = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \delta_2 = \dfrac{K_1}{q_2}\left(e^{\sqrt{q_2}\,\tau} + e^{-\sqrt{q_2}\,\tau}\right) - \dfrac{2K_1}{q_2} \\ \eta_2 = \dfrac{K_2}{\sqrt{q_2}}\left(e^{\sqrt{q_2}\,\tau} - e^{-\sqrt{q_2}\,\tau}\right) \\ \nu_2 = K_2\left(e^{\sqrt{q_2}\,\tau} + e^{-\sqrt{q_2}\,\tau}\right) \end{array}$$

The boundary layer solution to second order is

$$x_{1_I} = \delta_1\varepsilon + \delta_2\varepsilon^2$$

$$x_{2_I} = \eta_o + \eta_1\varepsilon + \eta_2\varepsilon^2$$

$$\sigma_I = \nu_o + \nu_1\varepsilon + \nu_2\varepsilon^2$$

The constants C_o, C_1, C_2, K_o, K_1, K_2 are now determined from matching. Since all orders of the boundary layer expansions must remain finite as $\tau\to\infty$,

$$K_o = -\sqrt{q_2}\ x_{20}\ ;\ K_1 = 0\ ;\ K_2 = 0$$

so that

$$x_{1_I} = \frac{x_{20}}{\sqrt{q_2}}\left(1 - e^{-\sqrt{q_2}\,\tau}\right)\varepsilon$$

$$x_{2_I} = x_{20}\ e^{-\sqrt{q_2}\,\tau}$$

$$\sigma_I = -\sqrt{q_2}\ x_{20}\ e^{-\sqrt{q_2}\,\tau}$$

Matching with the outer solution gives

$$C_o = 0\ ;\ C_1 = \frac{x_{20}}{\sqrt{q_2}}\ ;\ C_2 = 0$$

so that

$$x_{1_0} = \frac{x_{20}}{\sqrt{q_2}}\ \varepsilon$$

$$x_{2_0} = 0$$

$$\sigma_0 = 0$$

The uniformly valid expansions are

$$x_{1_U} = \frac{x_{20}}{\sqrt{q_2}}\left(1 - e^{-\frac{\sqrt{q_2}}{\varepsilon}t}\right)\varepsilon$$

$$x_{2_U} = x_{20}\, e^{-\frac{\sqrt{q_2}}{\varepsilon}t}$$

$$\sigma_U = -\sqrt{q_2}\, x_{20}\, e^{-\frac{\sqrt{q_2}}{\varepsilon}t}$$

Thus the boundary layer expansions are the same as the uniform expansions and the outer expansion has contributed nothing. Also note that the second order terms are all zero so that higher order terms will be zero also, that is

$$\alpha_i = \beta_i = \sigma_i = \delta_i = \eta_i = \nu_i = 0 \;\forall\; i \geq 2$$

This is verified by noting that the uniform expansions above are precisely the exact solutions. It may be concluded that for this special case, the method of matched asymptotic expansions has not resulted in computational simplification.

CONCLUDING REMARKS

Singular perturbation theory and the method of matched asymptotic expansions have been applied to the state regulator problem of optimal control theory. Conditions have been stated under which the method is applicable. The method is illustrated by using it to obtain an approximate solution of a simple, specific, singularly perturbed state regulator problem. It is found that the method is easy to apply and results in a uniform approximation which gives good agreement with the exact solution in the entire range of interest.

REFERENCES

1. Athans, A.; and Falb, P.L.: Optimal Control, McGraw-Hill, 1966.

2. Wasow, W. R.: Asymptotic Expansions for Ordinary Differential Equations, Interscience, 1965.

3. O'Malley, R. E.: Topics in Simgular Perturbations, Advances in Mathematics, vol. 2, 1968.

4. Cole, J. D.: Perturbation Methods in Applied Mathematics, Blaisdell, 1968.

5. Van Dyke, M.: Perturbation Methods in Fluid Mechanics, Academic, 1964.

6. Kelley, H. J.: Singular Perturbations for a Meyer Variational Problem, AIAA Journal, vol. 8, no. 6, June 1970.

7. Kelley, H. J.; and Edelbaum, T. N.: Energy Climbs, Energy Turns, and Asymptotic Expansions, Journal of Aircraft, vol. 7, no. 1, Jan-Feb. 1970

8. Hadlock, C. R.: Singular Perturbations of a Class of Two Point Boundary Value Problems Arising in Optimal Control, Coordinated Science Laboratory Report R-481, July 1970.

9. Kokotovic, P. V.; and Sannuti, P.: Singular Perturbation Method for Reducing the Model Order in Optimal Control Design, IEEE Trans. on Automatic Control, vol. 14, no. 4., August 1968.

10. Sannuti, P.: Singular Perturbation Method in the Theory of Optimal Control, Coordinated Science Laboratory Report R-379, May 1968.

11. Rogers, R. O.; and Sworder, D. D.: Suboptimal Control of Linear Systems Derived from Models of Lower Dimension, AIAA Journal, vol. 9, no. 8, August 1971.

12. Leitmann, G.: An Introduction to Optimal Control, McGraw-Hill, 1966.

13. Harris, W. A., Jr.: Singular Perturbations of Two Point Boundary Value Problems, Journal of Math. and Mech., vol. 11, no. 3, 1962.

14. Hoppensteadt, F. C.: Singular Perturbation on the Infinite Interval, Trans. American Math. Soc., vol. 123, 1966.

15. Buti, G.: The Singular Perturbation Theory of Differential Equations in Control Theory, Periodica Polytechnica, vol. 10, no. 2, 1965.

16. Haddad, A. H.; and Kokotovic, P. V.: Note on Singular Perturbations of Linear State Regulators, IEEE Trans. on Automatic Control, vol. 17, no. 6, June 1971.

17. Carrier, G. F.; Pearson, C. E.: Ordinary Differential Equations, Blaisdell, 1968.

18. Vasileva, A. B.: Asymptotic Behavior of Solutions to Certain Problems Involving Nonlinear Differential Equations Containing a Small Parameter Multiplying the Highest Derivatives, Russian Math. Surveys, vol. 18, no. 3, 1963 (English translation).

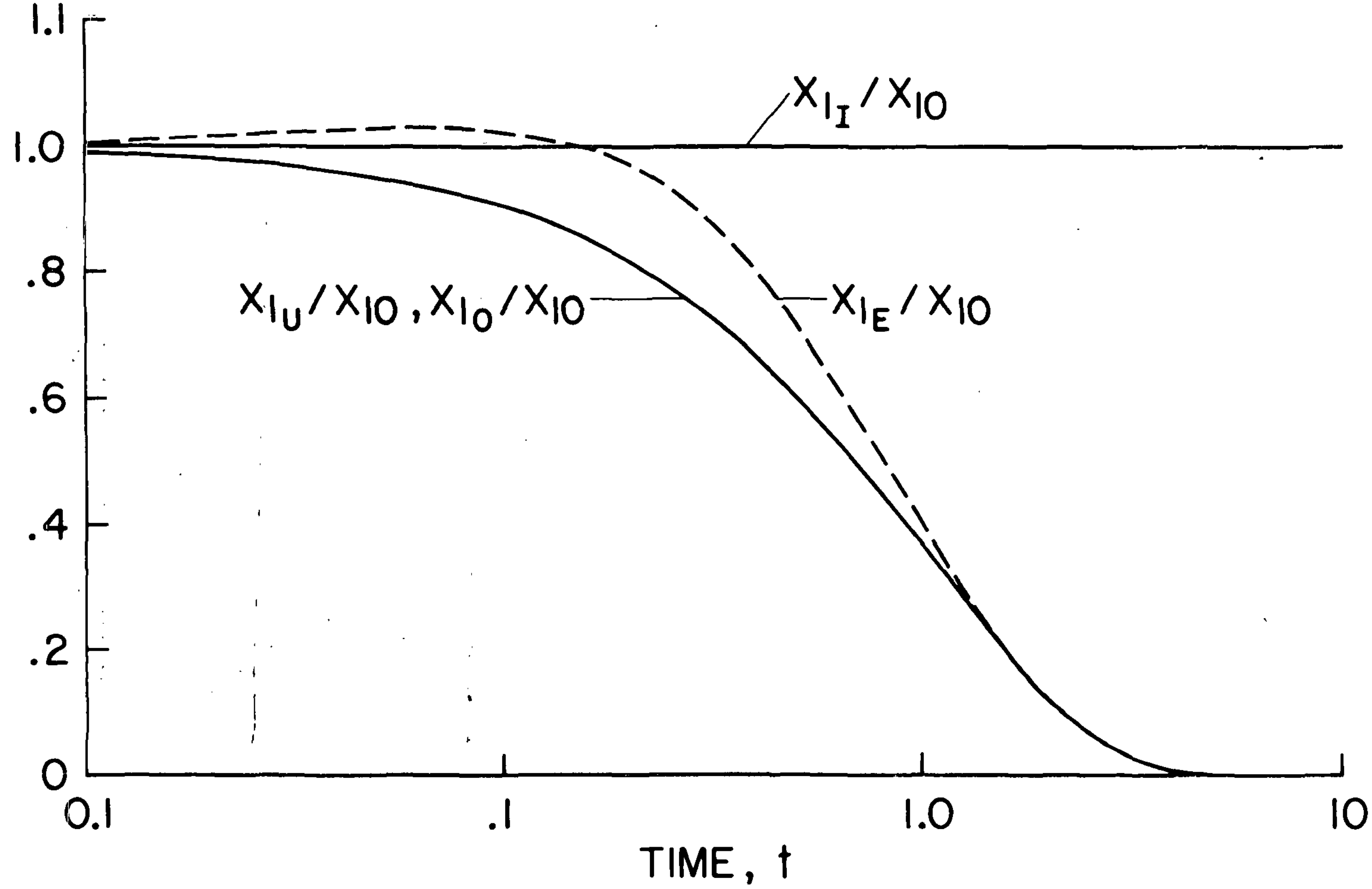

Figure 1. Comparison of solutions for first state variable, x_1.

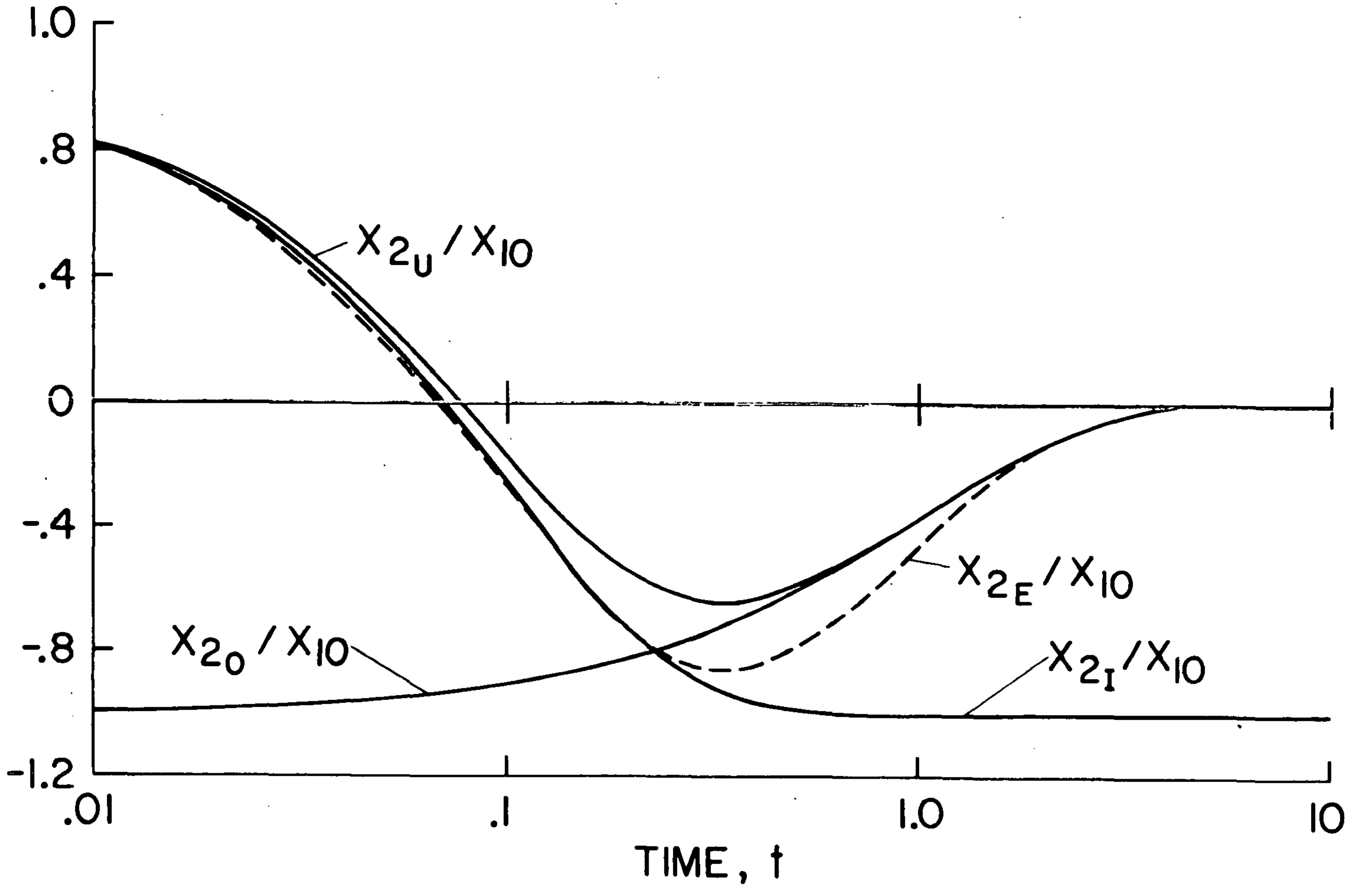

Figure 2. Comparison of solutions for second state variable, x_2.

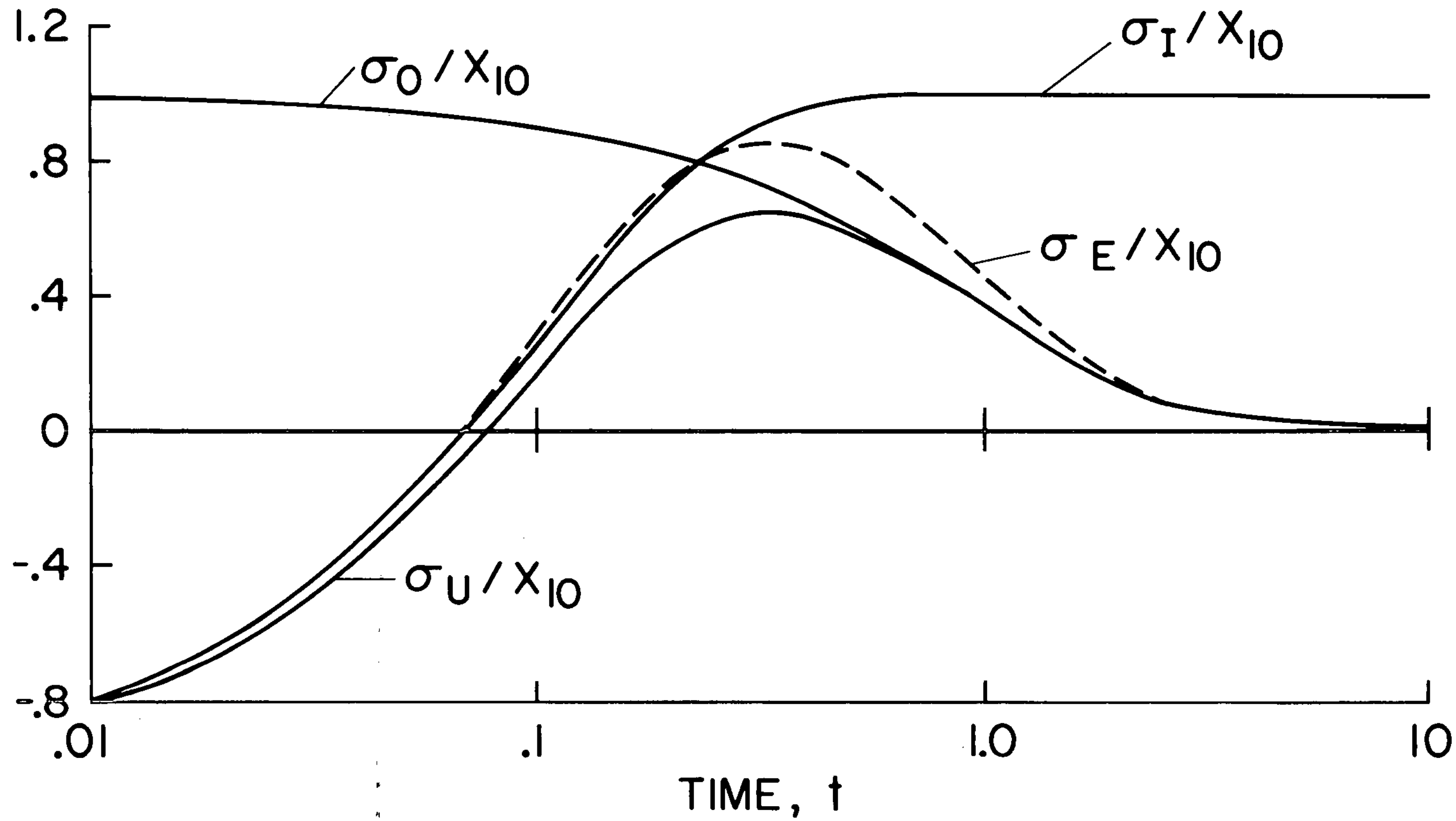

Figure 3. Comparison of solutions for control variable, τ.

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