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ANALYTIC SHORT PERIOD LUNAR AND SOLAR PERTURBATIONS OF ARTIFICIAL SATELLITES

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David Fisher

ABSTRACT

The short period luni-solar theory of Kozai is generalized for arbitrary obliquity of the ecliptic and inclination of the moon's orbit to the ecliptic. Analytic first order lunar perturbations to the elements are derived. The theory is illustrated by an application to the communication satellite Intelsat 3F3.

ANALYTIC SHORT PERIOD LUNAR AND SOLAR PERTURBATIONS OF ARTIFICIAL SATELLITES

1. INTRODUCTION

In order to improve the accuracy of the computation of the orbits of satellites by analytic means, it is important to take into account lunar and solar perturbations. Thus the order of magnitude of the lunar perturbations of short period in the semi-major axis is

$$\delta a/a = 1.1 \times 10^{-7} a^3/a_e^3 \tag{1}$$

where a is the semi-major axis of the satellite and a_e the radius of the Earth. For example, for a close Earth satellite with the semi-major axis equal to about 7000 km, the order of magnitude of the perturbation will be about a meter, while for a communication satellite with semi-major axis of 42,000 kilometers, the order of magnitude of the lunar perturbations is about 1.4 kilometers. This is substantiated by the observations of an arc of Intelsat 3F3 data, Figure 1.

In order to provide insight into our analysis, the luni-solar theories first developed by Kozai 1959, 1966 are briefly sketched.

In 1959, Kozai showed that the luni-solar disturbing function R′ is expressible in a form which we may write symbolically as

$$R' = K(a, e, i) K'(a', e', i') \cos(\ell, g, h, \ell', g', h') \tag{2}$$

In Equation 2, the elements of the satellite are the unprimed quantities. Thus a, e, and i represent the semi-major axis, eccentricity, and inclination of the satellite orbit to the equator, while ℓ, g, and h refer to the mean anomaly, argument of perigee, and right ascension of the node of the satellite. The primed quantities represent the corresponding elements of the sun or the moon.

The elements i' and h', when they refer to the inclination of the moon's orbit to the equator and the right ascension of the moon's node along the equator, show large non-linear variations with time. In 1966, Kozai completely eliminated i' in all terms and h' in most of the significant terms, by means of semi-analytic formulas of the type

$$\sin^2 i' \cos 2h' = .156 + .065 \cos N + .007 \cos 2N \qquad (3)$$

Here N is the longitude of the moon's node along the ecliptic. The quantity N is an angle which varies essentially linearly with time. Using formulas like those of Equation (3) the disturbing function is representable in the form

$$R' = CK(a, e, i) \cos (\ell, g, h, \lambda', \ell', N)$$

where C is a constant and λ' is the mean longitude of the disturbing body.

In this report, the constants appearing in Equation 3 are replaced by more general formulas of the type

$$\sin^2 i' \cos 2h' = (\epsilon, J) + (\epsilon, J) \cos N + (\epsilon, J) \cos 2N \qquad (4)$$

Here ϵ equals the obliquity of the ecliptic and J the inclination of the moon's orbit to the ecliptic. Both ϵ and J are very slowly varying quantities.

Using formulas of this kind, the disturbing function of the moon is derived in a general form

$$R' = K(a, e, i)\, K'(a', e', \epsilon, J) \cos(\ell, g, h, \ell', N, \lambda') \tag{5}$$

The main advantages of using these transformations are that they can be applied to other three-body configurations than the Earth-satellite moon system, and that the accuracy of the transformation can be improved by using more precise values of ϵ and J.

A program written in APL language was prepared to evaluate the coefficient K' of Eq. 5. The author used this language in a remote control unit for developing the program, a listing of which is available at the Goddard Space Flight Center APL public library under the function name MNCLA. To my knowledge this is the first instance of a program in analytic celestial mechanics conducted exclusively from a remote control terminal. We learn to appreciate the remarkable capability of APL in handling vectors and matrices.

Using this program for the coefficients K', with the same initial conditions as Kozai it was found that Kozai's development was correct with the exception of a single error in sign of a term of small magnitude. Thus the correct coefficient of the term with argument $\Omega - N - 2\lambda'$ is +.003, rather than -.003 as given by Kozai.

In this report, instead of applying the theory to close Earth satellites, we shall apply it to communication satellites, which are strongly influenced by the sun and the moon, to the extent indicated by Eq. 1.

The results given in this report lead to the conclusion that the analytic formulas presented are accurate and convenient for obtaining the solar and lunar perturbations on a wide selection of satellites.

2. ANALYSIS

The first step in the process of finding the analytic perturbations in the elements consists of finding and expanding the disturbing function. The next step is usually either to set up the Lagrangian equations of motion and solve these, or alternatively, we may define a determining function and use the method of canonical variables. In this report, the latter course is chosen.

Following the procedure of Kozai, 1966, we first represent the disturbing function R' in the form

$$R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \left(\frac{a'}{r'}\right)^3 \sum_{q,q',\nu} A_{\nu q} A'_{\nu q'} c_{\nu q' q} \cos (q\phi + q'\phi' + \nu\theta) \qquad (6)$$

where q, q', and ν take the values given in Table 1.

The nomenclature and symbols appearing in Eq. (6) are described in the section on notations, appendix 2.

The functions $A_{\nu q}$, $A'_{\nu q'}$, and the constants $c_{\nu q' q}$ are given in Table 1. The index q takes on the two values 0 and 2, the index q' takes the value -2, 0, 2, while the index ν varies over the range -2, -1, 0, 1, 2, as indicated in Table 1.

The expansion of the disturbing function, Eq. (6) now begins. First, we give q' the values -2, 0, and 2 as indicated in Table 1. Then by means of Cayley's tables, 1861, the disturbing function R' of Eq. (6) is expanded in terms of ℓ', the mean anomaly of the disturbing body to obtain as in Fisher, 1971,

TABLE 1

q	q'	ν	$A_{\nu q}$	$c_{\nu q' q}$	$A'_{\nu q'}$	$\frac{\partial A_{\nu q}}{\partial(\cos i)}$
0	0	0	$1 - 3/2 \sin^2 i$	1/4	$1 - 3/2 \sin^2 i'$	$3 \cos i$
0	2	0		3/8	$\sin^2 i'$	
0	0	1	$\sin 2i$	3/16	$\sin 2i'$	$-2 \cos 2i/\sin i$
0	2	-1		-3/8	$\sin i' \cos^2 i'/2$	
0	2	1		3/8	$\sin i' \sin^2 i'/2$	
0	2	-2	$\sin^2 i$	3/8	$\cos^4 i'/2$	$-2 \cos i$
0	0	2		3/16	$\sin^2 i'$	
0	2	2		3/8	$\sin^4 i'/2$	
2	-2	0	$\sin^2 i$	9/32	$\sin^2 i'$	$-2 \cos i$
2	0	0		3/8	$1 - 3/2 \sin^2 i'$	
2	2	0		9/32	$\sin^2 i'$	
2	-2	2	$\cos^4 i/2$	3/4	$\cos^4 i'/2$	$\cos^2 i/2$
2	0	2		3/8	$\sin^2 i'$	
2	2	2		3/4	$\sin^4 i'/2$	
2	2	-2	$\sin^4 i/2$	3/4	$\cos^4 i'/2$	$-\sin^2 i/2$
2	0	-2		3/8	$\sin^2 i'$	
2	-2	-2		3/4	$\sin^4 i'/2$	
2	-2	1	$\sin i \cos^2 i/2$	3/4	$\sin i' \cos^2 i'/2$	$\frac{1 - \cos i - 2\cos^2 i}{2 \sin i}$
2	0	1		-3/8	$\sin 2i'$	
2	2	1		-3/4	$\sin i' \sin^2 i'/2$	
2	2	-1	$\sin i \sin^2 i/2$	-3/4	$\sin i' \cos^2 i'/2$	$-\frac{(1 + \cos i - 2\cos^2 i)}{2 \sin i}$
2	0	-1		3/8	$\sin 2i'$	
2	-2	-1		3/4	$\sin i' \sin^2 i'/2$	

$$R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \sum \left\{ \begin{aligned} & A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{00} \cos \nu\theta \\ & + A_{\nu 0} A'_{\nu 0} c_{\nu 00} B'_{i' 0} \cos (i'\ell' + \nu\theta) \\ & + A_{\nu 0} A'_{\nu 2} c_{\nu 20} B'_{i' 2} \cos (i'\ell' + 2g' + \nu\theta) \\ & + A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{00} \cos (2\phi + \nu\theta) \\ & + A_{\nu 2} A'_{\nu 0} c_{\nu 02} B'_{i' 0} \cos (i'\ell' + 2\phi + \nu\theta) \\ & + A_{\nu 2} A'_{\nu -2} c_{\nu -22} B'_{i' ,-2} \cos (i'\ell' - 2g' + 2\phi + \nu\theta) \\ & + A_{\nu 2} A'_{\nu 2} c_{\nu 22} B'_{i' 2} \cos (i'\ell' + 2g' + 2\phi + \nu\theta) \end{aligned} \right. \tag{7}$$

The range of the indices i' and ν are given in Tables 1 and 2. No confusion should arise between the use of i' as an index or i' as an inclination.

TABLE 2

$B'_{i'0}$	i'	$B'_{i'-2}$	i'	$B'_{i'2}$	i'
$53/16\, e'^3$	-3	$845/48\, e'^3$	-5	$1/48\, e'^3$	-1
$9/4\, e'^2$	-2	$17/2\, e'^2$	-4	0	0
$3/2\, e' + 27/16\, e'^3$	-1	$7/2\, e' - 123/16\, e'^3$	-3	$-1/2\, e' + 1/16\, e'^3$	1
$(1-e'^2)^{-3/2}$	0	$1 - 5/2\, e'^2$	-2	$1 - 5/2\, e'^2$	2
$3/2\, e' + 27/16\, e'^3$	1	$-1/2\, e' + 1/16\, e'^3$	-1	$7/2\, e' - 123/16\, e'^3$	3
$9/4\, e'^2$	2	0	0	$17/2\, e'^2$	4
$53/16\, e'^3$	3	$1/48\, e'^3$	1	$845/48\, e'^3$	5

The coefficients $B'_{i'q'}$ arise from the expansion by means of Cayley's tables, Cayley, 1861, and are listed in Table 2 to order e'^3.

In order to simplify the notation we define

$$
\begin{aligned}
Q_0 &= A'_{\nu 0}\, c_{\nu 00}\, B'_{00} \cos \nu\theta \\
&+ A'_{\nu 0}\, c_{\nu 00}\, B'_{i0} \cos (i'\ell' + \nu\theta) \\
&+ A'_{\nu 2}\, c_{\nu 20}\, B'_{i'2} \cos (i'\ell' + 2g' + \nu\theta)
\end{aligned}
\qquad
\begin{aligned}
P_1 &= A'_{\nu 0}\, c_{\nu 02}\, B'_{00} \cos (2\phi + \nu\theta) \\
&+ A'_{\nu 0}\, c_{\nu 02}\, B'_{i'0} \cos (i'\ell' + 2\phi + \nu\theta) \\
&+ A'_{\nu-2}\, c_{\nu-22}\, B'_{i-2} \cos (i'\ell' - 2g' + 2\phi + \nu\theta) \\
&+ A'_{\nu 2}\, c_{\nu 22}\, B'_{i2} \cos (i\ell' + 2g' + 2\phi + \nu\theta)
\end{aligned}
\tag{8}
$$

Hence Eq. (7) is replaced by

$$
R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \sum \{A_{\nu 0} Q_0 + A_{\nu 2} P_1\} \tag{9}
$$

The indices and range of summation in this equation as well as those following are the same as in Eq. (7).

By elementary trigonometry using the definition of ϕ, Eq. (9) can be expanded in the form

$$R' = n'^2 m' a^2 \left(\frac{r}{a}\right)^2 \sum \{A_{\nu 0} Q_0 + A_{\nu 2} Q_1 \cos 2f - A_{\nu 2} \bar{Q}_1 \sin 2f\} \tag{10}$$

Q_1 is obtained from P_1 by replacing the variable ϕ by g, the argument of perigee. $\bar{Q}_1$, is derived from Q_1 by replacing the cosine functions by sine functions, keeping the same arguments as in Q_1.

The quantities Q_0, Q_1, and $\bar{Q}_1$ are no longer functions of the mean anomaly and are consequently slowly varying quantities which may be held constant when integrating with respect to the mean anomaly.

To find the short period disturbing function, the mean value of R' found by averaging with respect to the mean anomaly is subtracted from R'. Consequently, the short period disturbing function R'_p is given by

$$R'_p = n'^2 m' a^2 \sum \left\{A_{\nu 0} Q_0 \left[\frac{r^2}{a^2} - \left(1 + \frac{3}{2} e^2\right)\right] + A_{\nu 2} Q_1 \left[\frac{r^2}{a^2} \cos 2f - \frac{5}{2} e^2\right] - A_{\nu 2} \bar{Q}_1 \frac{r^2}{a^2} \sin 2f\right\}, \tag{11}$$

since the mean values of r^2/a^2, $r^2/a^2 \cos 2f$, and $r^2/a^2 \sin 2f$ are $1 + 3/2\, e^2$, $5/2\, e^2$, and 0 respectively.

From the method of canonical variables applied to satellite theory, Brouwer and Clemence, 1961, we define the determining function S by means of the partial differential equation

$$\frac{\partial S}{\partial \ell} = \frac{1}{n} R_p' \tag{12}$$

For arbitrary eccentricities, it is convenient to integrate Eq. (12) by transforming to the eccentric anomaly E. We use the relations

$$\begin{aligned} \frac{r}{a} &= 1 - e \cos E, & \frac{r}{a} \cos f &= \cos E - e \\ d\ell &= \frac{r}{a}\, dE, & \frac{r}{a} \sin f &= (1 - e^2)^{1/2} \sin E \end{aligned} \tag{13}$$

Consequently, we find that

$$S = \frac{n'^2\, m'\, a^2}{n} \sum \{A_{\nu 0}\, Q_0\, T_0 + A_{\nu 2}\, Q_1\, T_1 + A_{\nu 2}\, \overline{Q}_1\, T_2\} \tag{14}$$

with

$$\begin{aligned} T_0 &= \left(-2e + \frac{3}{4} e^3\right) \sin E + \frac{3}{4} e^2 \sin 2E - \frac{e^3}{12} \sin 3E \\ T_1 &= -\frac{5}{2} e \left(1 - \frac{e^2}{2}\right) \sin E + \frac{1}{2}\left(1 + \frac{e^2}{2}\right) \sin 2E - \frac{e}{6} \sin 3E \\ T_2 &= (1 - e^2)^{1/2} \left[\frac{5}{2} e \cos E - \frac{1}{2}(1 + e^2) \cos 2E + \frac{e}{6} \cos 3E\right] \end{aligned} \tag{15}$$

R_p' of Eq. (11) can also be expressed in terms of eccentric anomaly since

$$\frac{r^2}{a^2} \cos 2f = \frac{3}{2} e^2 - 2e\cos E + \left(1 - \frac{e^2}{2}\right) \cos 2E$$

$$\frac{r^2}{a^2} \sin 2f = (1 - e^2)^{1/2} (\sin 2E - 2e) \tag{16}$$

The perturbations in the Delaunay elements are then found by the formulas, Brouwer and Clemence, 1961,

$$\delta L = \frac{\partial S}{\partial \ell} \qquad \delta \ell = -\frac{\partial S}{\partial L}$$

$$\delta G = \frac{\partial S}{\partial g} \qquad \delta g = -\frac{\partial S}{\partial G} \tag{17}$$

$$\delta H = \frac{\partial S}{\partial h} \qquad \delta g = -\frac{\partial S}{\partial H}$$

Substituting for S, using Eqs. (14) and (12), we have

$$\delta L = \frac{1}{n} R_p'$$

$$\delta G = -\frac{2 n'^2 m'}{n} \sum \{A_{\nu 2} \overline{Q}_1 T_1 + A_{\nu 2} Q_1 T_2\} \tag{18}$$

$$\delta H = -\frac{n'^2 m'}{n} \sum \{\nu \, [A_{\nu 0} \overline{Q}_0 T_0 + A_{\nu 2} (\overline{Q}_1 T_1 + Q_1 T_2)]\}$$

where $\overline{Q}_0$ indicates the complement of Q_0 found by replacing the cosine functions in Q_0 by sine functions while preserving the arguments.

The perturbations in the angular elements are defined by

$$\delta \ell = - \frac{\partial S}{\partial L} = - \left\{ \frac{\partial S}{\partial a} \frac{\partial a}{\partial L} - \frac{\partial S}{\partial e} \frac{\partial e}{\partial L} \right\}$$

$$\delta g = - \frac{\partial S}{\partial G} = - \left\{ \frac{\partial S}{\partial e} \frac{\partial e}{\partial G} + \frac{\partial S}{\partial (\cos i)} \frac{\partial (\cos i)}{\partial G} \right\} \tag{19}$$

$$\delta h = - \frac{\partial S}{\partial H} = - \frac{\partial S}{\partial (\cos i)} \frac{\partial (\cos i)}{\partial H}$$

From the definition of S we have

$$\frac{\partial S}{\partial a} = \frac{7}{2} \frac{S}{a}$$

$$\frac{\partial S}{\partial e} = \frac{n'^2 m' a^2}{n} \sum \left\{ A_{\nu 0} Q_0 \frac{\partial T_0}{\partial e} + A_{\nu 2} Q_1 \frac{\partial T_1}{\partial e} + A_{\nu 2} \bar{Q}_1 \frac{\partial T_2}{\partial e} \right\} \tag{20}$$

From the relations between the Delaunay and the Kepler elements we have

$$\partial a / \partial L = 2/n a$$

$$\partial e / \partial L = - [n a^2 e (1 - e^2)]^{-1}$$

$$\partial e / \partial G = (n a^2 e)^{-1} (1 - e^2)^{-3/2} \tag{21}$$

$$\partial (\cos i) / \partial G = - \cos i \, (n a^2)^{-1} (1 - e^2)^{-1/2}$$

$$\partial (\cos i) / \partial H = (n a^2)^{-1} (1 - e^2)^{-1/2}$$

From the definitions of T_0, T_1, and T_2, Eq. (15), we give the differential coefficients of these functions in the tabular form

	$\partial T_0/\partial e$	$\partial T_1/\partial e$		$\partial T_2/\partial e$
sin E	$-2 + 9/4\, e^2$	$-5/2\,(1 - 3/2\, e^2)$	$(1 - e^2)^{-1/2} \cos E$	$-5/2\,(1 - 2\, e^2)$
sin 2 E	$3/2\, e$	$e/2$	$(1 - e^2)^{-1/2} \cos 2E$	$e/2\,(1 - 3\, e^2)$
sin 3 E	$-e^2/4$	$-1/6\,(1 - 3/2\, e^2)$	$(1 - e^2)^{-1/2} \cos 3E$	$-1/6\,(1 - 2\, e^2)$
a/r sin E	$-3/4\, e^2$	$-1/2\,(1 + e^2/2)$	$a/r\,(1 - e^2)^{-1/2}$	$5\, e/4$
a/r sin 2 E	$-e + e^3/2$	$-e + e^3/2$	$a/r\,(1 - e^2)^{-1/2} \cos E$	$-1/2\,(1 + e^2)$
a/r sin 3 E	$3/4\, e^2$	$1/2\,(1 + e^2/2)$	$a/r\,(1 - e^2)^{-1/2} \cos 2E$	$e/2$
			$a/r\,(1 - e^2)^{-1/2} \cos 3E$	$1/2\,(1 + e^2)$
a/r sin 4 E	$-e^3/8$	$-e/4\,(1 - e^2/2)$	$a/r\,(1 - e^2)^{-1/2} \cos 4E$	$-e/4$

(22)

Using the functions in Table 1, and Eqs. (17)-(22) the perturbations in the elements may be calculated. It should be recalled that it is necessary to use mean elements of the satellite and the disturbing body as required by first order perturbation theory. For the case where either the sun or the moon is the disturbing body the appropriate elements are given in the Explanatory Supplement, 1961.

The element i' for the sun is the obliquity of the ecliptic, a slowly varying quantity. The element h' for the sun is negligible. Consequently, the solar perturbations can be found by direct application of Eqs. (17) and (18).

However, the elements i' and h' for the moon show large non-linear variations. The analytic treatment for these elements is discussed in the next section.

3. LUNAR PERTURBATIONS

As shown by Kozai, 1966 the trigonometric relations useful in dealing with i' and h' for the moon are given by

$$\cos i' = \cos\epsilon \cos J - \sin\epsilon \sin J \sin N$$

$$\sin i' \sin h' = \sin J \sin N \qquad (23)$$

$$\sin i' \cos h' = \sin\epsilon \cos J + \cos\epsilon \sin J \cos N$$

Here i' and h' are the inclination of the moon's orbit to the equator and the longitude of the moon's node along the equator respectively. The elements ϵ, J and N are the obliquity of the ecliptic, the inclination of the moon's orbit to the ecliptic, and the longitude of the moon's node along the ecliptic. The elements ϵ and J vary very slowly while N is essentially a linear function of time.

As a consequence of Eq. (23), we find

$$A'_{\nu q'} \cos [\beta - (q' + \nu) h']$$

$$= \sum_{p=-2}^{2} (C'_{\nu q' p} + s' S'_{\nu q' p}) (\sin \nu)^{|p|} \cos (\beta + p N) \qquad (24)$$

where β is an angle which does not contain h'. The integer s' is defined by

$$s' = +1 \quad \text{for} \quad (q' + \nu)\, p < 0$$

$$s' = 0 \quad \text{for} \quad (q' + \nu)\, p = 0$$

$$s' = -1 \quad \text{for} \quad (q' + \nu)\, p > 0$$

For convenience we define the functions

$$C_{\nu q' 0} = C'_{\nu q' 0}; \qquad S_{\nu q' 0} = S'_{\nu q' 0} = 0$$

$$\text{for } p \neq 0 \quad C'_{\nu q' p} = \frac{1}{2} C_{\nu q' p} \tag{25}$$

$$S'_{\nu q' p} = \frac{1}{2} S_{\nu q' p}$$

If we furthermore define the quantities d_0, d_1, and d_2 by the formulas

$$d_0 = \frac{1}{2\,(1 + \cos\epsilon \cos J)} \left[1 + \frac{1}{2} \frac{\sin^2\epsilon \sin^2 J}{(1 + \cos\epsilon \cos J)^2} \right]$$

$$d_1 = \frac{\sin\epsilon}{2\,(1 + \cos\epsilon \cos J)^2} \tag{26}$$

$$d_2 = \frac{\sin^2\epsilon}{4\,(1 + \cos\epsilon \cos J)^3}$$

we arrive at Table 3.

TABLE 3*

$C_{\nu q'p}$ and $S_{\nu q'p}$ For Given Pairs ν, q'

	(0,0)		
$C_{\nu q'0}$	$(1 - 3/2 \sin^2 \epsilon)(1 - 3/2 \sin^2 J)$		
$C_{\nu q'1}$	$-3/2 \sin 2\epsilon \cos J$		
$C_{\nu q'2}$	$3/4 \sin^2 \epsilon$		
	(2,0), (0,-2), (0,2), (-2,0)		
$C_{\nu q'0}$	$\sin^2 \epsilon (1 - 3/2 \sin^2 J)$		
$C_{\nu q'1}$	$\sin 2\epsilon \cos J$	$S_{\nu q'1}$	$2 \sin \epsilon \cos J$
$C_{\nu q'2}$	$1/2 (1 + \cos^2 \epsilon)$	$S_{\nu q'2}$	$\cos \epsilon$
	(1,0), (-1,0)		
$C_{\nu q'0}$	$\sin 2\epsilon (1 - 3/2 \sin^2 J)$		
$C_{\nu q'1}$	$2 \cos 2\epsilon \cos J$	$S_{\nu q'1}$	$2 \cos \epsilon \cos J$
$C_{\nu q'2}$	$- 1/2 \sin 2\epsilon$	$S_{\nu q'2}$	$- \sin \epsilon$
	(-1,2), (1,-2)		
$C_{\nu q'0}$	$1/2 \sin \epsilon [\cos J + \cos \epsilon (1 - 3/2 \sin^2 J)]$		
$C_{\nu q'1}$	$1/2 (\cos \epsilon + \cos 2\epsilon \cos J)$	$S_{\nu q'1}$	$1/2 (1 + \cos \epsilon \cos J)$
$C_{\nu q'2}$	$- 1/8 \sin 2\epsilon$	$S_{\nu q'2}$	$- 1/4 \sin \epsilon$
	(1,2), (-1,-2)		
$C_{\nu q'0}$	$\sin^3 \epsilon \cos J (1 - 5/2 \sin^2 J) d_0 + 3/2 \sin^2 \epsilon \cos \epsilon (\sin^2 J) d_1$		
$C_{\nu q'1}$	$\sin^3 \epsilon \cos J d_1 + \sin^2 \epsilon \cos \epsilon (1 + 2 \cos^2 \epsilon) d_0$	$S_{\nu q'1}$	$3 \sin^3 \epsilon d_0$
$C_{\nu q'2}$	$3/2 \sin \epsilon \cos J (1 + \cos^2 \epsilon) d_0$	$S_{\nu q'2}$	$3/2 \sin 2\epsilon \cos J d_0 + 3/2 \sin^2 \epsilon d_1$
C	$+ \sin^2 \epsilon \cos \epsilon (1/2 + \cos^2 J) d_1 + \sin^3 \epsilon \cos J d_2$		
	(2,-2), (-2,2)		
$C_{\nu q'0}$	$1/4 (1 + \cos \epsilon \cos J)^2 + 1/8 \sin^2 \epsilon \sin^2 J$		
$C_{\nu q'1}$	$- 1/2 \sin \epsilon (1 + \cos \epsilon \cos J)$		
$C_{\nu q'2}$	$1/8 \sin^2 \epsilon$		
	(2,2), (-2,-2)		
$C_{\nu q'0}$	$1/4 (1 - \cos \epsilon \cos J)^2 + 1/8 \sin^2 \epsilon (1 - 32 \cos^2 J d_0^2) \sin^2 J$		
$C_{\nu q'1}$	$1/2 \sin \epsilon (1 - \cos \epsilon \cos J)$	$S_{\nu q'1}$	$4 \sin^3 \epsilon \cos J d_0^2$
$C_{\nu q'2}$	$1/8 \sin^2 \epsilon (1 + 32 \cos^2 J d_0^2)$	$S_{\nu q'2}$	$4 \sin^3 \epsilon \cos J d_0 d_1 + 1/2 \sin^2 \epsilon \cos \epsilon (1 + 2 \cos^2 J) d_0^2$

*The subscripts ν and q' are chosen to conform with Table 1. The numbers in parenthesis are pairs of values of v and q'.

If we now apply Eqs. (24) and (25) and use the values of $C_{\nu q'p}$ and $S_{\nu q'p}$ from Table 3 we may write R' of Equation (9) in the form

$$R' = n'^2 m' \left(\frac{r}{a}\right)^2 \sum A_{\nu q} Q_{q'\nu i'p} \cos [q\phi + \nu h + q'\lambda' + (i' - q') \ell' + pN] \qquad (27)$$

where

$$Q_{q'\nu i'p} = c_{\nu q'q} \, B'_{i'q'} \, (C'_{\nu q'p} + s' \, S'_{\nu q'p}) \, (\sin J)^{|p|}.$$

The coefficients $Q_{q'\nu i'p}$, for the initial conditions $\epsilon = 23\overset{\circ}{.}444$ and $J = 5\overset{\circ}{.}1454$, were evaluated by the machine program discussed above, and are given in Appendix 1.

The next step in obtaining the perturbations is to derive the determining function as outlined above. Corresponding to Eq. (14) we find for the moon

$$\begin{aligned} S = \frac{n'^2 m' a^2}{n} \sum \{ & A_{\nu 0} \, Q_{q'\nu iq} \, T_0 \cos (\nu h + q'\lambda' + (i' - q') \ell' + pN) \\ & + A_{\nu 2} \, Q_{q'\nu i'p} \, [T_1 \cos (2\omega + \nu h + q'\lambda' + (i' - q') \ell' + pN) \\ & + T_2 \sin (2\omega + \nu h + q'\lambda' + (i' - q') \ell' + pN)]\} \end{aligned} \tag{28}$$

T_0, T_1, and T_2 and their differential coefficients are given by Eqs. (15) and (22) respectively.

The lunar perturbations of the Delaunay elements are then found by applying the canonical formulas given by Eqs. (18) and (19) to the lunar determining function S, given by Eq. (28).

4. PERTURBATIONS IN THE SEMI-MAJOR AXIS OF INTELSAT 3F3

A convenient way of illustrating the theory presented above is to consider the perturbations in the semi-major axis of Intelsat 3F3. Osculating elements of this satellite were available over a 21-day arc, nearly all of the observations being taken at 2-day intervals.

The definition of the Delaunay element L is given by

$$L = (\mu a)^{1/2} = n a^2 \tag{29}$$

From Eq. (29) we find

$$\delta a = \frac{2na^2}{\mu} \delta L \tag{30}$$

Using Eq. (18), the Vanguard system of units where $\mu = 1$, and the Earth's equatorial radius a_e as a normalizing factor, we have

$$\delta a / a_e = 2 \left(\frac{a}{a_e}\right)^2 R_p' \tag{31}$$

The eccentricity and inclination of Intelsat 3F3 are about .005 and 0°.34, respectively. This permits us to derive a very abbreviated expression for the short period disturbing function which is adquate for the semi-major axis. Thus we find using Eq. (27) and the results of appendix 1 that

$$\delta a/a = 1.102 \times 10^{-7} \left(\frac{a}{a_e}\right)^3 \cos^4 \frac{i}{2} \{.682 \cos (2\ell + 2g + 2h - 2\lambda') + .131 \cos (2\ell + 2g - 2\lambda' - \ell')\} \tag{32}$$

The factor 1.102×10^{-7} arises from $2n'^2m'$ for the moon in Vanguard units.

The perturbations obtained by applying Eq. (32) to the Intelsat 3F3 data are plotted in Figure 1. Agreement between the observed and calculated values of the semi-major axis is adequate, since only the major perturbations are considered.

It should be recalled that Intelsat 3F3 is a synchronous satellite with the period of a day. Since the satellite was observed at equal multiples of a day, the mean anomaly ℓ increased by multiples of 2π for each observation, thus appearing as a constant angle in Eq. (32). Consequently as shown by Eq. (32), the perturbations in the semi-major axis should show a dominant semi-monthly period. This is borne out by the observations, (see Fig. 1).

5. RESULTS AND CONCLUSIONS

A general first order short period theory is given for obtaining perturbations analytically either for the earth, satellite, moon, sun system or a configuration of a similar type.

A computer program was written in APL to apply this theory. The program was used to compute coefficients needed for the disturbing function for the specific case of $\epsilon = 23^\circ\!.444$ and $J = 5^\circ\!.1454$, and the results shown to be in general agreement with the work of Kozai, 1966.

The theory is illustrated by computing the perturbations in the semi-major axis of Intelsat 3F3 with good agreement obtained with observations.

The analysis shows that short period lunar perturbations vary from about one meter for close earth satellites to about 1.4 kilometers for synchronous type satellites. The lunar perturbations as well as those due to the sun can be computed analytically by the formulas of this report.

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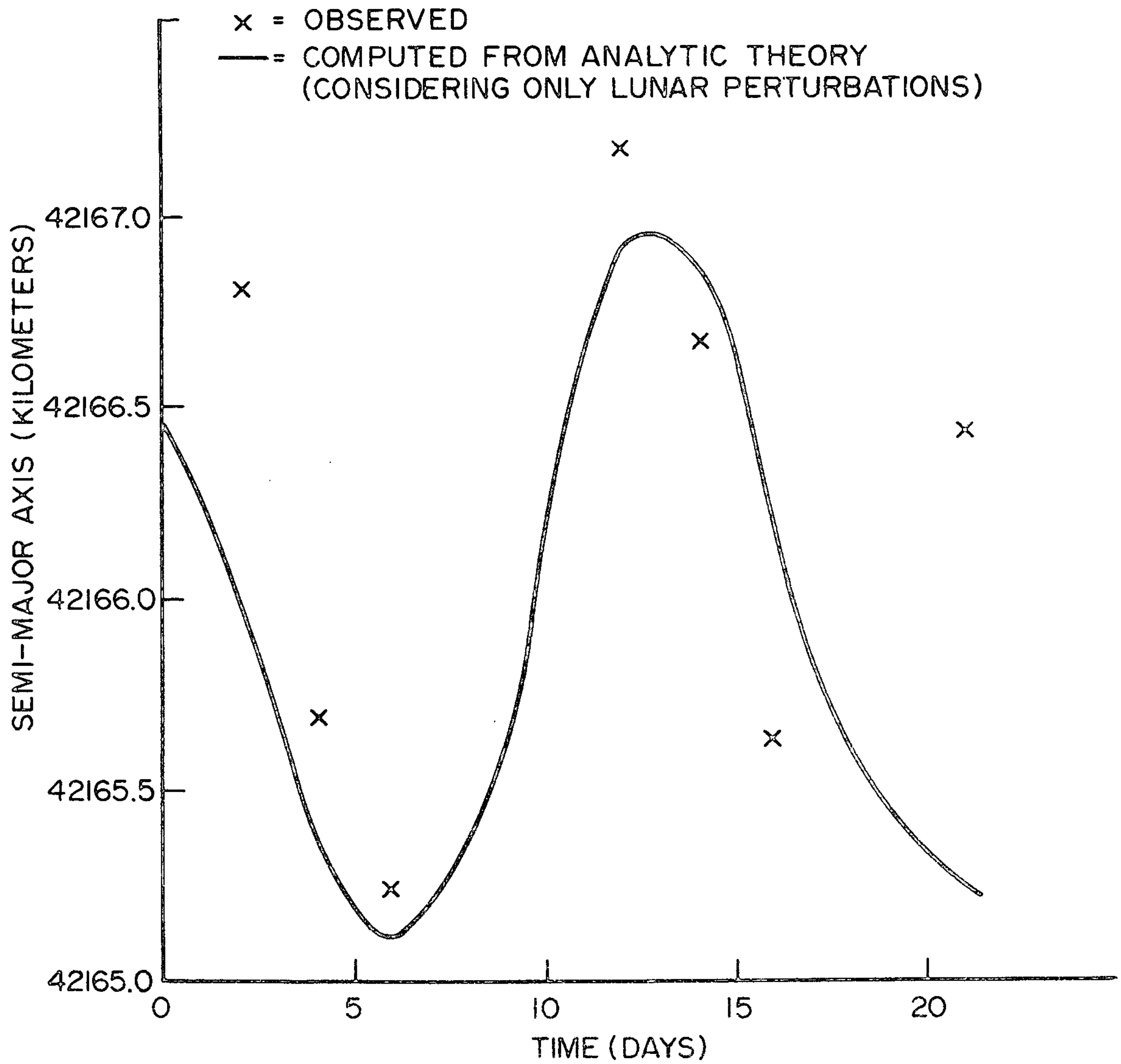

Figure 1. Semi-major axis vs time for satellite intelsat 3F3.

APPENDIX 1

Table 1 – q = 0, q' = 0, ν = 0

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
-2N	6.54221E¯8	8.09422E¯7	9.86229E¯6	1.19897E¯4	9.86229E¯6	8.09422E¯7	6.54221E¯8
-N	¯6.70160E¯6	¯8.29142E¯5	¯1.01026E¯3	¯1.22818E¯2	¯1.01026E¯3	¯8.29142E¯5	¯6.70160E¯6
O	1.03236E¯4	1.27727E¯3	1.55627E¯2	1.89197E¯1	1.55627E¯2	1.27727E¯3	1.03236E¯4
N	¯6.70160E¯6	¯8.29142E¯5	¯1.01026E¯3	¯1.22818E¯2	¯1.01026E¯3	¯8.29142E¯5	¯6.70160E¯6
2N	6.54221E¯8	8.09422E¯7	9.86229E¯6	1.19897E¯4	9.86229E¯6	8.09422E¯7	6.54221E¯8

Table 2 – q = 0, q' = 0, ν = 1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
h – 2N	¯3.15302E¯7	¯3.90101E¯6	¯4.75313E¯5	¯5.77841E¯4	¯4.75313E¯5	¯3.90101E¯6	¯3.15302E¯7
h – N	1.46961E¯5	1.81825E¯4	2.21542E¯3	2.69330E¯2	2.21542E¯3	1.81825E¯4	1.46961E¯5
h	7.41222E¯5	9.17063E¯4	1.11738E¯2	1.35841E¯1	1.11738E¯2	9.17063E¯4	7.41222E¯5
h + N	¯2.14834E¯6	¯2.65799E¯5	¯3.23859E¯4	¯3.93717E¯3	¯3.23859E¯4	¯2.65799E¯5	¯2.14834E¯6
h + 2N	1.35745E¯8	1.67947E¯7	2.04633E¯6	2.48774E¯5	2.04633E¯6	1.67947E¯7	1.35745E¯8

Table 3 – q = 0, q' = 0, ν = 2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2h – 2N	7.59798E¯7	9.40045E¯6	1.14538E¯4	1.39245E¯3	1.14538E¯4	9.40045E¯6	7.59798E¯7
2h – N	7.00310E¯6	8.66445E¯5	1.05571E¯3	1.28343E¯2	1.05571E¯3	8.66445E¯5	7.00310E¯6
2h	1.60716E¯5	1.98842E¯4	2.42277E¯3	2.94537E¯2	2.42277E¯3	1.98842E¯4	1.60716E¯5
2h + N	¯3.01500E¯7	¯3.73025E¯6	¯4.54507E¯5	¯5.52547E¯4	¯4.54507E¯5	¯3.73025E¯6	¯3.01500E¯7
2h + 2N	1.40829E¯9	1.74238E¯8	2.12297E¯7	2.58091E¯6	2.12297E¯7	1.74238E¯8	1.40829E¯9

Table 4 – q = 0, q' = 2, ν = -2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
-2h+2λ'-2N	1.02865E¯10	0.00000E0	¯8.18772E¯7	2.96139E¯5	5.69560E¯6	7.64454E¯7	8.69209E¯8
-2h+2λ'-N	¯2.20688E¯8	0.00000E0	1.75660E¯4	¯6.35341E¯3	¯1.22194E¯3	¯1.64007E¯4	¯1.86481E¯5
-2h+2λ'	1.18387E¯6	0.00000E0	¯9.42325E¯3	3.40826E¯1	6.55508E¯2	8.79811E¯3	1.00037E¯3
-2h+2λ'+N	¯2.20688E¯8	0.00000E0	1.75660E¯4	¯6.35341E¯3	¯1.22194E¯3	¯1.64007E¯4	¯1.86481E¯5
-2h+2λ'+2N	1.02865E¯10	0.00000E0	¯8.18772E¯7	2.96139E¯5	5.69560E¯6	7.64454E¯7	8.69209E¯8

Table 5 — q = 0, q' = 2, ν = -1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
-h+2λ'-2N	9.91515E¯10	0.00000E0	¯7.89213E¯6	2.85448E¯4	5.48999E¯5	7.36857E¯6	8.37830E¯7
-h+2λ'-N	¯1.01791E¯7	0.00000E0	8.10224E¯4	¯2.93047E¯2	¯5.63614E¯3	¯7.56474E¯4	¯8.60135E¯5
-h+2λ'	¯4.89217E¯7	0.00000E0	3.89401E¯3	¯1.40841E¯1	¯2.70878E¯2	¯3.63568E¯3	¯4.13388E¯4
-h+2λ'+N	9.14849E¯9	0.00000E0	¯7.28190E¯5	2.63377E¯3	5.06549E¯4	6.79882E¯5	7.73048E¯6
-h+2λ'+2N	¯4.26870E¯11	0.00000E0	3.39775E¯7	¯1.22892E¯5	¯2.36357E¯6	¯3.17234E¯7	¯3.60705E¯8

Table 6 — q = 0, q' = 2, ν = 0

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2λ'-2N	9.55721E¯9	0.00000E0	¯7.60722E¯5	2.75143E¯3	5.29180E¯4	7.10256E¯5	8.07584E¯6
2λ'-N	8.80894E¯8	0.00000E0	¯7.01162E¯4	2.53601E¯2	4.87748E¯3	6.54647E¯4	7.44355E¯5
2λ'	2.02158E¯7	0.00000E0	¯1.60911E¯3	5.81995E¯2	1.11934E¯2	1.50236E¯3	1.70824E¯4
2λ'+N	¯3.79245E¯9	0.00000E0	3.01867E¯5	¯1.09181E¯3	¯2.09987E¯4	¯2.81841E¯5	¯3.20462E¯6
2λ'+2N	1.77143E¯11	0.00000E0	¯1.41000E¯7	5.09978E¯6	9.80836E¯7	1.31646E¯7	1.49686E¯8

Table 7 — q = 0, q' = 2, ν = 1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
h+2λ'-2N	3.09887E¯9	0.00000E0	¯2.46660E¯5	8.92135E¯4	1.71583E¯4	2.30296E¯5	2.61854E¯6
h+2λ'-N	1.39548E¯8	0.00000E0	¯1.11076E¯4	4.01747E¯3	7.72675E¯4	1.03707E¯4	1.17918E¯5
h+2λ'	2.07570E¯8	0.00000E0	¯1.65219E¯4	5.97574E¯3	1.14931E¯3	1.54258E¯4	1.75396E¯5
h+2λ'+N	¯4.31711E¯10	0.00000E0	3.43627E¯6	¯1.24285E¯4	¯2.39037E¯5	¯3.20831E¯6	¯3.64795E¯7
h+2λ'+2N	1.47920E¯12	0.00000E0	¯1.17739E¯8	4.25847E¯7	8.19027E¯8	1.09928E¯8	1.24992E¯9

Table 8 — q = 0, q' = 2, ν = 2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
-2h+2λ'-2N	6.52002E¯10	0.00000E0	¯5.18972E¯6	1.87705E¯4	3.61012E¯5	4.84544E¯6	5.50942E¯7
2h+2λ'-N	1.98768E¯9	0.00000E0	¯1.58213E¯5	5.72235E¯4	1.10057E¯4	1.47717E¯5	1.67959E¯6
2h+2λ'	2.16391E¯9	0.00000E0	¯1.72240E¯5	6.22970E¯4	1.19815E¯4	1.60814E¯5	1.82851E¯6
2h+2λ'+N	1.48090E¯12	0.00000E0	¯1.17875E¯8	4.26337E¯7	8.19970E¯8	1.10055E¯8	1.25136E¯9
2h+2λ'+2N	¯3.49203E¯13	0.00000E0	2.77954E¯9	¯1.00532E¯7	¯1.93353E¯8	¯2.59515E¯9	¯2.95077E¯10

Table 9 — q = 2, q' = 0, ν = -2

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
2φ-2h-2N	2.81657E¯9	3.48475E¯8	4.24594E¯7	5.16182E¯6	4.24594E¯7	3.48475E¯8	2.81657E¯9
2φ-2h-N	¯6.03000E¯7	¯7.46049E¯6	¯9.09013E¯5	¯1.10509E¯3	¯9.09013E¯5	¯7.46049E¯6	¯6.03000E¯7
2φ-2h	3.21432E¯5	3.97685E¯4	4.84553E¯3	5.89075E¯2	4.84553E¯3	3.97685E¯4	3.21432E¯5
2φ-2h+N	1.40062E¯5	1.73289E¯4	2.11142E¯3	2.56686E¯2	2.11142E¯3	1.73289E¯4	1.40062E¯5
2φ-2h+2N	1.51960E¯6	1.88009E¯5	2.29077E¯4	2.78490E¯3	2.29077E¯4	1.88009E¯5	1.51960E¯6

Table 10 — q = 2, q' = 0, ν = -1

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
2φ-h-2N	2.71489E¯8	3.35895E¯7	4.09266E¯6	4.97548E¯5	4.09266E¯6	3.35895E¯7	2.71489E¯8
2φ-h-1	¯4.29667E¯6	¯5.31597E¯5	¯6.47717E¯4	¯7.87434E¯3	¯6.47717E¯4	¯5.31597E¯5	¯4.29667E¯6
2φ-h	1.48244E¯4	1.83413E¯3	2.23476E¯2	2.71682E¯1	2.23476E¯2	1.83413E¯3	1.48244E¯4
2φ-h+N	2.93922E¯5	3.63649E¯4	4.43083E¯3	5.38659E¯2	4.43083E¯3	3.63649E¯4	2.93922E¯5
2φ-h+2N	¯6.30604E¯7	¯7.80202E¯6	¯9.50626E¯5	¯1.15568E¯3	¯9.50626E¯5	¯7.80202E¯6	¯6.30604E¯7

Table 11 — q = 2, q' = 0, ν = 0

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
2φ-2N	9.81332E¯8	1.21413E¯6	1.47934E¯5	1.79845E¯4	1.47934E¯5	1.21413E¯6	9.81332E¯8
2φ-N	¯1.00524E¯5	¯1.24371E¯4	¯1.51539E¯3	¯1.84226E¯2	¯1.51539E¯3	¯1.24371E¯4	¯1.00524E¯5
2φ	1.54855E¯4	1.91591E¯3	2.33441E¯2	2.83796E¯1	2.33441E¯2	1.91591E¯3	1.54855E¯4
2φ-N	¯1.00524E¯5	¯1.24371E¯4	¯1.51539E¯3	¯1.84226E¯2	¯1.51539E¯3	¯1.24371E¯4	¯1.00524E¯5
2φ+2N	9.81332E¯8	1.21413E¯6	1.47934E¯5	1.79845E¯4	1.47934E¯5	1.21413E¯6	9.81332E¯8

Table 12 — q = 2, q' = 0, ν = 1

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
2φ+h-2N	6.30604E¯7	7.80202E¯6	9.50626E¯5	1.15568E¯3	9.50626E¯5	7.80202E¯6	6.30604E¯7
2φ+h-N	¯2.93922E¯5	¯3.63649E¯4	¯4.43083E¯3	¯5.38659E¯2	¯4.43083E¯3	¯3.63649E¯4	¯2.93922E¯5
2φ+h	¯1.48244E¯4	¯1.83413E¯3	¯2.23476E¯2	¯2.71682E¯1	¯2.23476E¯2	¯1.83413E¯3	¯1.48244E¯4
2φ+h+N	4.29667E¯6	5.31597E¯5	6.47717E¯4	7.87434E¯3	6.47717E¯4	5.31597E¯5	4.29667E¯6
2φ+h+2N	¯2.71489E¯8	¯3.35895E¯7	¯4.09266E¯6	¯4.97548E¯5	¯4.09266E¯6	¯3.35895E¯7	¯2.71489E¯8

Table 13 — q = 2, q' = 0, ν = 2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2φ+2h-2N	1.51960E¯6	1.88009E¯5	2.29077E¯4	2.78490E¯3	2.29077E¯4	1.88009E¯5	1.51960E¯6
2φ+2h-N	1.40062E¯5	1.73289E¯4	2.11142E¯3	2.56686E¯2	2.11142E¯3	1.73289E¯4	1.40062E¯5
2φ+2h	3.21432E¯5	3.97685E¯4	4.84553E¯3	5.89075E¯2	4.84553E¯3	3.97685E¯4	3.21432E¯5
2φ+2h+N	¯6.03000E¯7	¯7.46049E¯6	¯9.09013E¯5	¯1.10509E¯3	¯9.09013E¯5	¯7.46049E¯6	¯6.03000E¯7
2φ+2h+2N	2.81657E¯9	3.48475E¯8	4.24594E¯7	5.16182E¯6	4.24594E¯7	3.48475E¯8	2.81657E¯9

Table 14 — q = 2, q' = 2, ν = -2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2φ-2h-2λ'-2N	¯5.90153E¯10	¯5.19030E¯9	¯3.86706E¯8	¯2.01065E¯7	5.55909E¯9	0.00000E0	¯6.98406E¯13
2φ-2h-2λ'-N	2.50272E¯9	2.20110E¯8	1.63994E¯7	8.52675E¯7	¯2.35750E¯8	0.00000E0	2.96180E¯12
2φ-2h-2λ'	3.65701E¯6	3.21628E¯5	2.39630E¯4	1.24594E¯3	¯3.44480E¯5	0.00000E0	4.32782E¯9
2φ-2h-2λ'+N	3.35918E¯6	2.95434E¯5	2.20115E¯4	1.14447E¯3	¯3.16426E¯5	0.00000E0	3.97536E¯9
2φ-2h-2λ'+2N	1.10188E¯6	9.69087E¯6	7.22023E¯5	3.75411E¯4	¯1.03794E¯5	0.00000E0	1.30400E¯9

Table 15 — q = 2, q' = -2, ν = -1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2φ-h-2λ'-2N	2.49984E¯9	2.19857E¯8	1.63805E¯7	8.51694E¯7	¯2.35478E¯8	0.00000E0	2.95839E¯12
2φ-h-2λ'-N	¯7.29591E¯7	¯6.41662E¯6	¯4.78074E¯5	¯2.48571E¯4	6.87255E¯6	0.00000E0	¯8.63421E¯10
2φ-h-2λ'	3.50793E¯5	3.08516E¯4	2.29862E¯3	1.19515E¯2	¯3.30437E¯4	0.00000E0	4.15140E¯8
2φ-h-2λ'+N	2.35837E¯5	2.07414E¯4	1.54535E¯3	8.03493E¯3	¯2.22152E¯4	0.00000E0	2.79097E¯8
2φ-h-2λ'+2N	5.23708E¯6	4.60592E¯5	3.43167E¯4	1.78427E¯3	¯4.93319E¯5	0.00000E0	6.19773E¯9

Table 16 — q = 2, q' = -2, ν = 0

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2φ-2λ'-2N	1.12264E¯8	9.87346E¯8	7.35627E¯7	3.82484E¯6	¯1.05750E¯7	0.00000E0	1.32857E¯11
2φ-2λ'-N	¯2.40347E¯6	¯2.11381E¯5	¯1.57490E¯4	¯8.18859E¯4	2.26400E¯5	0.00000E0	¯2.84434E¯9
2φ-2λ'	1.28118E¯4	1.12677E¯3	8.39508E¯3	4.36496E¯2	¯1.20684E¯3	0.00000E0	1.51619E¯7
2φ-2λ'+N	5.58266E¯5	4.90985E¯4	3.65811E¯3	1.90201E¯2	¯5.25872E¯4	0.00000E0	6.60670E¯8
2φ-2λ'+2N	6.05688E¯6	5.32692E¯5	3.96885E¯4	2.06357E¯3	¯5.70542E¯5	0.00000E0	7.16791E¯9

Table 17 – q = 2, q' = -2, ν = 1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2ϕ+h-2λ'-2N	7.21411E¯8	6.34468E¯7	4.72714E¯6	2.45784E¯5	¯6.79549E¯7	0.00000E0	8.53740E¯11
2ϕ+h-2λ'-N	¯1.54610E¯5	¯1.35976E¯4	¯1.01310E¯3	¯5.26753E¯3	1.45638E¯4	0.00000E0	¯1.82970E¯8
2ϕ+h-2λ'	8.26777E¯4	7.27136E¯3	5.41756E¯2	2.81682E¯1	¯7.78801E¯3	0.00000E0	9.78434E¯7
2ϕ+h-2λ'+N	1.72027E¯4	1.51295E¯3	1.12723E¯2	5.86095E¯2	¯1.62045E¯3	0.00000E0	2.03582E¯7
2ϕ+h-2λ'+2N	¯1.67566E¯6	¯1.47371E¯5	¯1.09800E¯4	¯5.70896E¯4	1.57843E¯5	0.00000E0	¯1.98303E¯9

Table 18 – q = 2, q' = -2, ν = 2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2ϕ+2h-2λ'-2N	1.73842E¯7	1.52891E¯6	1.13912E¯5	5.92278E¯5	¯1.63754E¯6	0.00000E0	2.05730E¯10
2ϕ+2h-2λ'-N	¯3.72963E¯5	¯3.28014E¯4	¯2.44389E¯3	¯1.27068E¯2	3.51321E¯4	0.00000E0	¯4.41376E¯8
2ϕ-2h-2N	2.00075E¯3	1.75962E¯2	1.31102E¯1	6.81653E¯1	¯1.88465E¯2	0.00000E0	2.36775E¯6
2ϕ+2h-2λ'+N	¯3.72963E¯5	¯3.28014E¯4	¯2.44389E¯3	¯1.27068E¯2	3.51321E¯4	0.00000E0	¯4.41376E¯8
2ϕ+2h-2λ'+2N	1.73842E¯7	1.52891E¯6	1.13912E¯5	5.92278E¯5	¯1.63754E¯6	0.00000E0	2.05730E¯10

Table 19 – q = 2, q' = 2, ν = -2

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2ϕ-2h+2λ'-2N	2.05730E¯10	0.00000E0	¯1.63754E¯6	5.92278E¯5	1.13912E¯5	1.52891E¯6	1.73842E¯7
2ϕ-2h+2λ'-N	¯4.41376E¯8	0.00000E0	3.51321E¯4	¯1.27068E¯2	¯2.44389E¯3	¯3.28014E¯4	¯3.72963E¯5
2ϕ-2h+2λ'	2.36775E¯6	0.00000E0	¯1.88465E¯2	6.81653E¯1	1.31102E¯1	1.75962E¯2	2.00075E¯3
2ϕ-2h+2λ'+ N	¯4.41376E¯8	0.00000E0	3.51321E¯4	¯1.27068E¯2	¯2.44389E¯3	¯3.28014E¯4	¯3.72963E¯5
2ϕ-2h+2λ'+2N	2.05730E¯10	0.00000E0	¯1.63754E¯6	5.92278E¯5	1.13912E¯5	1.52891E¯6	1.73842E¯7

Table 20 – q = 2, q' = 2, ν = -1

	-3ℓ'	-2ℓ'	-ℓ'	0	ℓ'	2ℓ'	3ℓ'
2ϕ-h+2λ'-2N	1.98303E¯9	0.00000E0	¯1.57843E¯5	5.70896E¯4	1.09800E¯4	1.47371E¯5	1.67566E¯6
2ϕ-h+2λ'-N	¯2.03582E¯7	0.00000E0	1.62045E¯3	¯5.86095E¯2	¯1.12723E¯2	¯1.51295E¯3	¯1.72027E¯4
2ϕ-h+2λ'	¯9.78434E¯7	0.00000E0	7.78801E¯3	¯2.81682E¯1	¯5.41756E¯2	¯7.27136E¯3	¯8.26777E¯4
2ϕ-h+2λ'+N	1.82970E¯8	0.00000E0	¯1.45638E¯4	5.26753E¯3	1.01310E¯3	1.35976E¯4	1.54610E¯5
2ϕ-h+2λ'+2N	¯8.53740E¯11	0.00000E0	6.79549E¯7	¯2.45784E¯5	¯4.72714E¯6	¯6.34468E¯7	¯7.21411E¯8

Table 21 – q = 2, q' = 2, $\nu = 0$

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
$2\phi+2\lambda'-2N$	7.16791E¯9	0.00000E0	¯5.70542E¯5	2.06357E¯3	3.96885E¯4	5.32692E¯5	6.05688E¯6
$2\phi+2\lambda'-N$	6.60670E¯8	0.00000E0	¯5.25872E¯4	1.90201E¯2	3.65811E¯3	4.90985E¯4	5.58266E¯5
$2\phi-2\lambda'$	1.51619E¯7	0.00000E0	¯1.20684E¯3	4.36496E¯2	8.39508E¯3	1.12677E¯3	1.28118E¯4
$2\phi+2\lambda'+N$	¯2.84434E¯9	0.00000E0	2.26400E¯5	¯8.18859E¯4	¯1.57490E¯4	¯2.11381E¯5	¯2.40347E¯6
$2\phi+2\lambda'+2N$	1.32857E¯11	0.00000E0	¯1.05750E¯7	3.82484E¯6	7.35627E¯7	9.87346E¯8	1.12264E¯8

Table 22 – q = 2, q' = 2, $\nu = 1$

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
$2\phi+h+2\lambda'-2N$	¯6.19773E¯9	0.00000E0	4.93319E¯5	¯1.78427E¯3	¯3.43167E¯4	¯4.60592E¯5	¯5.23708E¯6
$2\phi+h+2\lambda'-N$	¯2.79097E¯8	0.00000E0	2.22152E¯4	¯8.03493E¯3	¯1.54535E¯3	¯2.07414E¯4	¯2.35837E¯5
$2\phi+h+2\lambda'$	¯4.15140E¯8	0.00000E0	3.30437E¯4	¯1.19515E¯2	¯2.29862E¯3	¯3.08516E¯4	¯3.50793E¯5
$2\phi+h+2\lambda'+N$	8.63421E¯10	0.00000E0	¯6.87255E¯6	2.48571E¯4	4.78074E¯5	6.41662E¯6	7.29591E¯7
$2\phi+h+2\lambda'+2N$	¯2.95839E¯12	0.00000E0	2.35478E¯8	¯8.51694E¯7	¯1.63805E¯7	¯2.19857E¯8	¯2.49984E¯9

Table 23 – q = 2, q' = 2, $\nu = 2$

	$-3\ell'$	$-2\ell'$	$-\ell'$	0	ℓ'	$2\ell'$	$3\ell'$
$2\phi+2h+2\lambda'-2N$	1.30400E¯9	0.00000E0	¯1.03794E¯5	3.75411E¯4	7.22023E¯5	9.69087E¯6	1.10188E¯6
$2\phi+2h+2\lambda'-N$	3.97536E¯9	0.00000E0	¯3.16426E¯5	1.14447E¯3	2.20115E¯4	2.95434E¯5	3.35918E¯6
$2\phi+2h+2\lambda'$	4.32782E¯9	0.00000E0	¯3.44480E¯5	1.24594E¯3	2.39630E¯4	3.21628E¯5	3.65701E¯6
$2\phi+2h+2\lambda'+N$	2.96180E¯12	0.00000E0	¯2.35750E¯8	8.52675E¯7	1.63994E¯7	2.20110E¯8	2.50272E¯9
$2\phi+2h+2\lambda'+2N$	¯6.98406E¯13	0.00000E0	5.55909E¯9	¯2.01065E¯7	¯3.86706E¯8	¯5.19030E¯9	¯5.90153E¯10

APPENDIX 2

NOTATIONS

$A_{\nu q}$ function of i

$A'_{\nu q'}$ function of i'

$A'_{\nu q' p}$ function of ϵ and J

B_{iq} function of e

$B'_{i' q'}$ function of e'

$C_{\nu q'}$ function of ϵ, J

G Delaunay variable = $(\mu a (1 - e^2))^{1/2}$

H Delaunay variable = $(\mu a (1 - e^2))^{1/2} \cos i$

J Inclination of moon's orbit to ecliptic

L Delaunay variable = $(\mu a)^{1/2}$

N Longitude of lunar node along ecliptic

R' disturbing function

$S_{\nu q'}$ function of ϵ and J

a semi-major axis of satellite

a_e mean equatorial radius of earth

$c_{\nu q' q}$ constants of the disturbing function

e eccentricity of the satellite

e' eccentricity of the disturbing body

f true anomaly of satellite

g argument of perigee of satellite

g' argument of perigee of disturbing body

h right ascension of node of the satellite

h' right ascension of the disturbing body

i inclination of satellite orbit

i' inclination of the disturbing body

ℓ mean anomaly of satellite

ℓ' mean anomaly of the disturbing body

m' ratio of mass of the disturbing body to mass of earth plus mass of the disturbing body

n mean motion of satellite

n' mean motion of disturbing body

p summation index

q summation index

q' summation index

r geocentric distance of satellite

r' geocentric distance of the disturbing body

ϵ obliquity of ecliptic

ϕ argument of latitude of satellite $= f + g$

ϕ' argument of latitude of moon $= f' + g'$

λ' mean longitude of the disturbing body $= \ell' + g' + h'$

μ gravitational constant