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THE EXACT SOLUTION OF THE MONOENERGETIC TRANSPORT EQUATION FOR CRITICAL CYLINDERS

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THE EXACT SOLUTION OF THE MONOENERGETIC TRANSPORT EQUATION FOR CRITICAL CYLINDERS

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SUMMARY

An analytic solution for the critical, monoenergetic, bare, infinite cylinder is presented. The solution is obtained by modifying a previous development based on a neutron density transform and Case's singular eigenfunction method. Numerical results for critical radii and the neutron density as a function of position are included and compared with the results of other methods.

I. INTRODUCTION

The exact solutions for the critical dimensions of multiplying media and the associated neutron density distributions can be used as standards in evaluating the more widely used approximate methods of transport analysis such as discrete ordinates or spherical harmonics. Of interest here is the exact solution for the monoenergetic, bare, infinite cylinder. Critical radii for monoenergetic bare cylinders have been determined by several methods. Carlson and Bell[1] interpolated between values calculated with the extrapolated endpoint and variational methods to derive what have long served as reference values. More recently, approximate solutions have been obtained by Hendry,[2] using Fourier expansions, and by Hembd,[3] using a Fourier transformation of the integral equation (I T_n Method). Mitsis,[4] using neutron density transforms, was able to apply Case's method[5] to obtain exact analytical solutions for spheres and cylinders. Gibbs[6] generalized this work to an arbitrary convex geometry and showed a reduction to the Mitsis solutions for the two specialized geometries. Numerical results for critical spheres were presented by Mitsis; however, a recent bibliography[7] of transport theory does not include any reference to numerical results derived from the analytical solution for the critical cylinder. Initial investigation under the present study indicated that the Mitsis formulation for critical cylinders is not convergent.

The solution of the critical cylinders presented here was obtained

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by reformulating the spatial function and the continuum expansion coefficient of the Mitsis solution to achieve convergence. Numerical results for critical radii and the asymptotic and total neutron densities as a function of position are included.

II. ANALYSIS

This analysis includes the development of a neutron density transform for the infinite, rotationally invariant cylinder, the solution of the transformed transport equation by the Case singular eigenfunction expansion technique, and the application of this solution to the critical cylinder problem. The analysis follows that of Mitsis, the primary features of which are included here for completeness and to provide a background for introducing the modifications necessary for obtaining a solution.

A. Development of Neutron Density Transform

In general, the spatial distribution of the neutron density is given by Eq. (1) where c denotes the mean number of neutron secondaries per collision.

$$\rho(r) = \frac{c}{4\pi} \int_{r'} \frac{\rho(\underline{r}')e^{-|\underline{r}-\underline{r}'|}}{|\underline{r} - \underline{r}'|^2} d^3r' \tag{1}$$

Considering the geometry of interest shown in Fig. 1, if we place the position vector $\underline{r}$ along the x-axis and represent $\underline{r}'$ by the cylindrical coordinates (t, α, a), we can obtain the following relationships:

$$x^2 = r^2 + t^2 - 2rt \cos \alpha \tag{2a}$$

$$x^2 + z^2 = |\underline{r} - \underline{r}'|^2 \tag{2b}$$

$$|\underline{r} - \underline{r}'|^2 = r^2 + t^2 - 2rt \cos \alpha + z^2 \tag{2c}$$

Substitution of Eq. (2c) into Eq. (1) yields

$$\rho(r) = \frac{c}{2\pi} \int_0^R t\rho(t)dt \int_0^{2\pi} d\alpha \int_0^\infty \frac{\exp\left[-\sqrt{r^2 + t^2 - 2rt \cos \alpha + z^2}\right]dz}{r^2 + t^2 - 2rt \cos \alpha + z^2} \tag{3}$$

The integral over z is reduced in the following manner:

$$\int_0^\infty \frac{\exp\left[-\sqrt{r^2 + t^2 - 2rt\cos\alpha + z^2}\right]dz}{r^2 + t^2 - 2rt\cos\alpha + z^2} = \int_0^\infty \frac{\exp\left[-\sqrt{x^2 + z^2}\right]dz}{x^2 + z^2} \tag{4a}$$

$$= \frac{1}{x^2}\int_0^\infty \frac{\exp\left[-x\sqrt{1 + (z/x)^2}\right]dz}{1 + (z/x)^2} \tag{4b}$$

$$= \frac{1}{x^2}\int_0^\infty \frac{dz}{\sqrt{1 + (z/x)^2}} \int_x^\infty \exp\left[-u\sqrt{1 + (z/x)^2}\right]du \tag{4c}$$

Changing the order of integration, letting $v = z/x$, and using an identity(8) for $K_o(u)$, the modified Bessel function of the second kind, Eq. (4c) becomes

$$\frac{1}{x}\int_x^\infty du \int_0^\infty \frac{\exp\left[-u\sqrt{1 + v^2}\right]}{\sqrt{1 + v^2}} dv = \frac{1}{x}\int_x^\infty K_o(u)du \tag{4d}$$

Finally, letting $y = u/x$, Eq. (4d) becomes

$$\int_1^\infty K_o(xy)dy = \int_1^\infty K_o\left(y\sqrt{r^2 + t^2 - 2rt\cos\alpha}\right)dy \tag{4e}$$

Substitution of Eq. (4e) into Eq. (3) gives

$$\rho(r) = \frac{c}{2\pi}\int_0^R t\rho(t)dt \int_0^{2\pi} d\alpha \int_1^\infty K_o\left(y\sqrt{r^2 + t^2 - 2rt\cos\alpha}\right)dy \tag{5}$$

The integral over α is performed by first splitting the integral over t into two integrals ranging from 0 to r and from r to R and then applying the addition theorem(9) for K_o.

$$K_o\left(\sqrt{r^2 + t^2 - 2rt\cos\alpha}\right) = \sum_{-\infty}^{\infty} e^{in\alpha} \begin{cases} K_n(r)I_n(t) & r \geq t \\ K_n(t)I_n(r) & r \leq t \end{cases} \tag{6}$$

The exponential in Eq. (6) leads to sine and cosine terms for all orders other than zero. These terms integrate to zero in Eq. (5) and we are left with a 2π contribution for the integral over α. Thus $\rho(r)$ becomes

$$\rho(r) = c\int_1^\infty dy\left[\int_0^r K_o(ry)I_o(ty)t\rho(t)dt + \int_r^R K_o(ty)I_o(ry)t\rho(t)dt\right] \tag{7}$$

Making the substitution $y = 1/\mu$ we obtain

$$\rho(r) = c\int_0^1 \frac{d\mu}{\mu^2}\left[\int_0^r K_o(r/\mu)I_o(t/\mu)t\rho(t)dt + \int_r^R K_o(t/\mu)I_o(r/\mu)t\rho(t)dt\right] \tag{8}$$

Eq. (8) is Eq. (5.5-1) of Mitsis.[4] It is in the form of a kernel $K(r,t)$ which is related to a Green's function $G(r,t;\mu)$

$$\rho(r) = c\int_t t\rho(t)K(r,t)dt \qquad 0 \le t \le R \tag{9}$$

$$K(r,t) = \int_0^1 \frac{1}{\mu^2} G(r,t;\mu)d\mu \tag{9a}$$

$$G(r,t;\mu) = \begin{cases} K_o(r/\mu)I_o(t/\mu) & r \ge t \\ K_o(t/\mu)I_o(r/\mu) & r \le t \end{cases} \tag{9b}$$

which is associated[10] as indicated below with the operator $[d^2/dr^2 + 1/r(d/dr) - 1/\mu^2]$. Next Mitsis introduces a <u>pseudo</u> neutron distribution function $\Phi(r,\mu)$ by

$$\Phi(r,\mu) = c\int_0^R t\rho(t)G(r,t;\mu)dt \tag{10}$$

$$= c\int_0^r t\rho(t)K_o(r/\mu)I_o(t/\mu)dt + c\int_r^R t\rho(t)K_o(t/\mu)I_o(r/\mu)dt \tag{10a}$$

Operation on $\Phi(r,\mu)$ by multiplying $(1/\mu^2)$ and integration over μ returns the neutron density.

$$\rho(r) = \int_0^1 \frac{\Phi(r,\mu)}{\mu^2}\, d\mu \tag{11}$$

Differentiation of Eq. (10a) with respect to r and use of the relation[8]

$$K_{n+1}(z)I_n(z) + I_{n+1}(z)K_n(z) = 1/z$$

yields

$$\frac{\partial\Phi(r,\mu)}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi(r,\mu)}{\partial r} - \frac{1}{\mu^2}\Phi(r,\mu) = -c\int_0^1 \frac{\Phi(r,\mu')d\mu'}{\mu'^2} \tag{12}$$

The boundary conditions for Eq. (12) follow from the definition for $\Phi(r,\mu)$ in Eq. (10a). Setting $r = 0$ causes the first integral to vanish as the second integral remains finite. At the outer boundary we have

$$K_0(R/\mu)\Phi'(R,\mu) + \frac{1}{\mu}K_1(R/\mu)\Phi(R,\mu) = 0 \qquad \mu \geq 0 \tag{13}$$

Eqs. (10), (11) define a transform pair with the function $\Phi(r,\mu)$ obeying the inhomogeneous differential equation (12). The inhomogeneity in Eq. (12) is proportional to the neutron density by Eq. (11).

B. Development of Eigenfunctions by Case's Method

Mitsis goes on to determine a complete set of eigenfunctions for this transform. He assumes a set of separable solutions of the form:

$$\Phi_\nu(r,\mu) = R_\nu(r)\eta_\nu(\mu) \tag{14}$$

Substitution of Eq. (14) into Eq. (12) yields

$$R_\nu''(r) + \frac{1}{r}R_\nu'(r) - \frac{1}{\nu^2}R_\nu(r) = 0 \tag{15}$$

and

$$\left(\frac{1}{\mu^2} - \frac{1}{\nu^2}\right)\eta_\nu(\mu) = c\int_0^1 \frac{1}{\mu'^2}\eta_\nu(\mu')d\mu'; \qquad \mu > 0 \tag{16}$$

with $(1/\nu^2)$ being the separation constant. With the boundary condition of $R_\nu(o)$ finite, the solution of Eq. (15) is restricted to the modified

Bessel function of the first kind.

$$R_\nu(r) = I_0(r/\nu) \tag{17}$$

Following Case's method,(5) $\eta_\nu(\mu)$ is normalized by taking

$$\int_0^1 \frac{1}{\mu^2}\,\eta_\nu(\mu)d\mu = 1 \tag{18}$$

after which Eq. (16) is written as

$$(\nu^2 - \mu^2)\eta_\nu(\mu) = c\nu^2\mu^2; \qquad 0 \le \mu \le 1 \tag{19}$$

For ν in the interval $(-1,1)$, singularities occur at $\nu = \pm\mu$. Accounting for these singularities, the solution of Eq. (19) becomes

$$\eta_\nu(\mu) = \frac{cP\nu^2\mu^2}{\nu^2 - \mu^2} + \zeta(\nu)\delta(\nu - \mu) + \xi(\nu)\delta(\nu + \mu) \tag{20}$$

where P indicates that principal values are to be taken when integrating expressions containing $\eta_\nu(\mu)$ and $\zeta(\nu),\xi(\nu)$ are arbitrary functions. It can be seen in Eq. (19) that $\eta_\nu(\mu) = \eta_\nu(-\mu)$. Applying this symmetry in Eq. (20) requires that $\zeta(\nu) = \xi(\nu)$. Thus we can replace these functions in Eq. (20) by $\nu^2\lambda(\nu)$ to obtain

$$\eta_\nu(\mu) = \frac{cP\nu^2\mu^2}{\nu^2 - \mu^2} + \nu^2\lambda(\nu)\left[\delta(\mu - \nu) + \delta(\mu + \nu)\right]; \qquad \mu > 0 \tag{21}$$

For ν not in $(-1,1)$ the solution of Eq. (19) is

$$\eta_0(\mu) = \frac{c\nu_0^2\mu^2}{\nu_0^2 - \mu^2} \tag{22}$$

where $\pm\nu_0$ are the roots of

$$c\nu\,\tanh^{-1}(1/\nu) = 1 \tag{22a}$$

The discrete and continuum eigenfunctions $\eta_0(\mu)$ and $\eta_\nu(\mu)$ are directly related to those developed by Case for plane geometry, $\phi_{0\pm}(\mu)$ and $\phi_\nu(\mu)$, and share their completeness properties.

$$\eta_0(\mu) = \mu^2[\phi_{0+}(\mu) + \phi_{0-}(\mu)] \tag{23a}$$

$$\eta_\nu(\mu) = \mu^2[\phi_\nu(\mu) + \phi_{-\nu}(\mu)] \tag{23b}$$

where

$$\phi_{0\pm}(\mu) = \frac{c}{2}\frac{\nu_0}{\nu_0 \mp \mu} \tag{23c}$$

$$\phi_\nu(\mu) = \frac{c}{2}\frac{P_\nu}{\nu - \mu} + \lambda(\nu)\delta(\mu - \nu) \tag{23d}$$

and

$$\lambda(\nu) = 1 - c\nu \tanh^{-1}\nu. \tag{23e}$$

Thus an expansion for an even function $\Phi(\mu)$ defined in the range $-1 \leq \mu \leq 1$:

$$\begin{aligned}\Phi(\mu) &= a_0\eta_0(\mu) + \int_{-1}^{1} \overline{A}(\nu)\eta_\nu(\mu)d\nu \\ &= a_0\eta_0(\mu) + \int_0^1 A(\nu)\eta_\nu(\mu)d\nu; \qquad \begin{aligned}\eta_{-\nu}(\mu) &= \eta_\nu(\mu) \\ A(\nu) &= \overline{A}(\nu) + \overline{A}(-\nu)\end{aligned}\end{aligned} \tag{24}$$

can be rewritten as:

$$\Phi(\mu) = \mu^2\left\{a_0[\phi_{0+}(\mu) + \phi_{0-}(\mu)] + \int_0^1 A(\nu)[\phi_\nu(\mu) + \phi_{-\nu}(\mu)]d\nu\right\} \tag{25}$$

Eq. (25) is an expansion of the even function $\psi(\mu) = \Phi(\mu)/\mu^2$ in terms of the complete set $\phi_\nu(\mu)$ with a unique set of coefficients a_0 and $A(\nu)$. We now insert Eqs. (24) and (17) into Eq. (14) to form the general solution of Eq. (12).

$$\Phi(r,\mu) = a_0\eta_0(\mu)I_0(r/\nu_0) + \int_0^1 A(\nu)\eta_\nu(\mu)I_0(r/\nu)d\nu \tag{26}$$

Operation on $\Phi(r,\mu)$ as in Eqs. (11) and (18) returns the neutron density.

$$\rho(r) = a_0I_0(r/\nu_0) + \int_0^1 A(\nu)I_0(r/\nu)d\nu \tag{27}$$

C. Application to the Critical Cylinder

To obtain the solution for the critical cylinder we evaluate $\Phi(r,\mu)$ and its first derivative at $r = R$ and insert them into the boundary condition, Eq. (13), obtaining:

$$a_0\eta_0(\mu)q(\nu_0,\mu) + \int_0^1 A(\nu)\eta_\nu(\mu)q(\nu,\mu)d\nu = 0; \qquad \mu \geq 0 \tag{28}$$

where

$$q(\nu,\mu) = R\left[\frac{1}{\nu} K_0(R/\mu)I_1(R/\nu) + \frac{1}{\mu} K_1(R/\mu)I_0(R/\nu)\right] \tag{29}$$

The general procedure is to separate Eq. (28) into a singular part which can be reduced to a Fredholm type of integral equation to determine the expansion coefficient and a corresponding nonsingular expression for the eignefunction. Toward this end Mitsis introduces the function

$$H(\nu,\mu) = \frac{q(\nu,\mu) - 1}{\nu - \mu} \tag{30}$$

which was found by the authors to give a diverging solution. Here we modify his procedure by reformulating $H(\nu,\mu)$ as

$$H'(\nu,\mu) = \frac{1}{\nu - \mu}\left\{\frac{I_0(R/\mu)}{I_0(R/\nu)} q(\nu,\mu) - 1\right\} \tag{31}$$

$$= \frac{1}{\nu - \mu}\left\{\frac{R}{\nu} I_0(R/\mu)K_0(R/\mu) \frac{I_1(R/\nu)}{I_0(R/\nu)} + \frac{R}{\mu} I_0(R/\mu)K_1(R/\mu) - 1\right\} \tag{31a}$$

Considering the behavior of the modified Bessel functions it can be seen that $H(\nu,\mu)$ is an unbounded function for increasing R/ν whereas $H'(\nu,\mu)$ is bounded for all variables.

$$q(\nu,\mu) = \frac{I_0(R/\nu)}{I_0(R/\mu)} \left\{1 + (\nu - \mu)H'(\nu,\mu)\right\} \tag{32}$$

We also redefine the expansion coefficient.

$$A'(\nu) = A(\nu)I_0(R/\nu) \tag{33}$$

Substitution of Eqs. (32) and (33) into Eq. (28) along with the use of Eqs. (23 a-d) completes the desired separation.

$$\int_0^1 A'(\nu)\phi_\nu(\mu)d\nu = \Phi'(\mu); \qquad \mu \geq 0 \tag{34}$$

$\Phi'(\mu)$ is the nonsingular part defined by

$$\Phi'(\mu) = -a_0[\phi_{0+}(\mu) + \phi_{0-}(\mu)]I_0(R/\mu)q(\nu_0,\mu) - \int_0^1 \frac{c}{2}\,\frac{\nu A'(\nu)\left[2\nu H'(\nu,\mu) + 1\right]}{\nu + \mu}\,d\nu \tag{35}$$

Reduction of (34) to a Fredholm equation proceeds as in the solution for critical slabs(4,11,12) and results in

$$A'(\mu) = \lambda(\mu)g(c,\mu)\Phi'(\mu) - \frac{1}{x^-(\mu)\lambda^+(\mu)}\int_0^1 \frac{\gamma(\nu)\Phi'(\nu)d\nu}{\nu - \mu} \tag{36}$$

and the auxiliary condition

$$\int_0^1 \gamma(\mu)\Phi'(\mu)d\mu = 0 \tag{37}$$

which is the criticality condition for this geometry. Additional functions and their boundary values introduced by this reduction are standard forms(12) with the following definitions.

$$g(c,\mu) = 1/\left\{\lambda^2(\mu) + (c^2\pi^2\mu^2/4)\right\} \tag{38a}$$

$$1/x^-(\mu)\lambda^+(\mu) = g(c,\mu)(\nu_0^2 - \mu^2)(1 - c)x(-\mu) \tag{38b}$$

$$\gamma(\mu) = \frac{c\mu}{2}\,\frac{1}{(\nu_0^2 - \mu^2)(1 - c)x(-\mu)} \tag{38c}$$

$$x(-z) = K(-z)/\{1/x(0) + z\} \tag{39a}$$

$$K(-z) = 1 - \frac{zc}{2}\int_0^1 \frac{[1 - x^2(0)\mu^2]d\mu}{\left(1 - \dfrac{\mu^2}{\nu_0^2}\right)K(-\mu)(\mu + z)} \tag{39b}$$

$$x^2(0) = 1/\nu_0^2(1 - c) \tag{39c}$$

Eqs. (35), (36), and (37) form the solution. Successive approximations are constructed by the following iteration scheme:

$$A_0'(\mu) = 0 \tag{40a}$$

$$\Phi_0'(\mu) = -a_0\left[\phi_0+(\mu) + \phi_0-(\mu)\right] I_0(R/\mu) q(\nu_0,\mu) \tag{40b}$$

$$A_{n+1}'(\mu) = \lambda(\mu) g(c,\mu) \Phi_n'(\mu) - \frac{1}{x^-(\mu)\lambda^+(\mu)} \int_0^1 \frac{\gamma(\nu)\Phi_n'(\nu)d\nu}{\nu - \mu} \tag{40c}$$

$$\Phi_n'(\mu) = \Phi_0'(\mu) - \int_0^1 \frac{\left(\frac{c}{2}\right)\nu A_n'(\nu)\left[2\nu H'(\nu,\mu) + 1\right]d\nu}{\nu + \mu} \tag{40d}$$

$$\int_0^1 \gamma(\mu)\Phi_n'(\mu)d\mu = 0 \tag{40e}$$

For the critical problem a_0 is an arbitrary constant corresponding to the power level and here is normalized to one. At each iteration, a search based on successively finer increments to the radius is conducted until the criticality condition Eq. (40e) is satisfied. The sequence is repeated until the critical radius does not change between successive iterations for $A_n'(\mu)$. The solution is the same as that of Mitsis with exception of the definitions for $A'(\mu)$, $H'(\nu,\mu)$ and the inclusion of the $I_0(R/\mu)$ term in Eq. (40b). The reformulation of $H(\nu,\mu)$ results in a bounded function $H'(\nu,\mu)$ which leads to a convergent solution.

III. NUMERICAL METHODS AND RESULTS

A. Methods

The various functions and parameters present in the solution were calculated by the following methods. The x functions were calculated by the method of Shure and Natelson,[13] Eqs. (39a,b,c). The magnitude of the discrete eigenvalue was determined by Newton's method.[14] Regular Bessel functions were calculated with a recursion relationship.[15] The modified Bessel functions were calculated with a series expansion[16] followed by an asymptotic series for the larger arguments.[16]

The calculation was done on the IBM-7094-II computer. In order to achieve single precision accuracy in the results, the calculation was done in double precision. The discrete eigenvalues and the x functions agreed with the tabulated values of Kowalska.[17] The Bessel functions agreed with the tabulated values in Abramowitz and Stegun[16] to at least nine significant figures.

The calculations were done over a quadrature of twenty-four Gauss-Legendre points. The variation of the critical radius with the number of Gauss points is shown in Table I. This indicates that twenty-four Gauss points are sufficient for seven place accuracy. The singular integral in Eq. (40c) was evaluated by subtracting[12] the singularity and determining the derivative term with Lagrangian interpolation over all points. The critical radius at each iteration for $A_n(\mu)$ was considered converged when the value of the integral in Eq. (40e) fell to less than 10^{-9}. The sensitivity of the criticality condition to the variation of the radius is shown in figure 2.

B. Results

The results presented in Tables II, III, and IV are believed to be precise to seven significant figures for the critical radii and to the number of figures shown for the neutron density values. Table II lists critical radii as a function of c, the mean number of neutron secondaries per collision or the infinite medium multiplication factor. Using the exact values from the present work as reference values, we see that the values of Carlson and Bell[1] are most accurate for the largest system ($c = 1.02$) and least accurate for the smallest system ($c = 2.0$). Carlson and Bell used the extrapolated endpoint method for the larger systems, $R > 1.5$. For the smaller systems they interpolated between values calculated with the extrapolated endpoint and variational methods. The I $T_{\hat{n}}$ method of Hembd[3] appears to be particularly effective. The reduced accuracy of the I $T_{\hat{n}}$ method for large systems is recognized by the author. It is interesting to note that for the smallest system ($c = 2.0$), the approximate solutions[2,3] are more accurate than the original reference value.[1]

The neutron density or total flux was calculated from Eq. (22). Table III lists the neutron density relative to the centerline value. It is seen that the radial drop-off in the neutron density decreases with decrease in the size of the system. Figure 3 shows the neutron density distribution for the case $c = 2.0$ calculated by the method presented here and by various order discrete ordinate calculations. The discrete ordinate calculations were performed with the TDSN program[18] using a moment modified quadrature. The neutron density distribution from the S_{16} calculation agrees very closely with the exact distribution. However, the excess system multiplication calculated by discrete ordinates indicates the increasing importance of the error in the neutron density distribution as the discrete quadrature order is reduced.

The first term of the neutron density, Eq. (22), is the asymptotic density arising from the discrete modes. The asymptotic density corresponds to the diffusion theory solution based on the exact diffusion coefficient. Table IV lists the ratio of the asymptotic to total neutron density as a function of position. The value of this ratio at the outer boundary for the case $c = 1.02$ agrees with the corresponding value for the Milne Problem(12) with $c = 1.0$. As anticipated, the error in the asymptotic density is most severe on the boundary of the smallest system.

In summary, the exact solution for the monoenergetic, bare, infinite cylinder has been presented and compared with approximate solutions of this problem. It was found results(1) previously used as standards were accurate to only three significant figures for small systems (for $c > 1.6$).

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TABLE I. - VARIATION OF CRITICAL RADIUS WITH THE NUMBER OF GAUSS-LEGENDRE QUADRATURE POINTS

Gauss	c = 2	c = 1.02
8	0.668620870	9.043254256
16	.668612711	9.043254733
24	.668612815	9.043254733
40	.668612853	9.043254733

TABLE II. - CRITICAL RADII IN MEAN-FREE PATHS

c	Present work exact	Carlson Bell[1]	Hendry[2] F_3G_8	Hembd[3] I T_4
1.02	9.043255	9.0433	-----	9.04458
1.05	5.411288	5.4118	5.414	5.41152
1.1	3.577391	3.5783	-----	3.57744
1.2	2.287209	2.2884	-----	2.28724
1.4	1.396979	1.3973	-----	1.39699
1.6	1.020839	1.0209	-----	1.02085
1.8	0.807427	0.8067	-----	0.80743
2.0	0.668613	0.6673	0.670	0.66862

TABLE III. - NEUTRON DENSITY AS A FUNCTION OF POSITION

r/R_c	c							
	1.02	1.05	1.1	1.2	1.4	1.6	1.8	2.0
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.25	.9236	.9299	.9361	.9433	.9508	.9550	.9578	.9598
.50	.7118	.7340	.7561	.7819	.8093	.8248	.8352	.8426
.75	.4124	.4522	.4922	.5399	.5918	.6218	.6421	.6570
.85	.2821	.3267	.3718	.4263	.4868	.5223	.5466	.5644
.91	.2033	.2492	.2960	.3535	.4181	.4566	.4830	.5026
.95	.1502	.1958	.2430	.3016	.3686	.4088	.4366	.4572
.98	.1086	.1531	.1999	.2589	.3273	.3688	.3976	.4190
1.0	.0749	.1179	.1641	.2233	.2926	.3351	.3646	.3867

TABLE IV. - RELATIVE ASYMPTOTIC TO TOTAL NEUTRON DENSITY

r/R_c	c							
	1.02	1.05	1.1	1.2	1.4	1.6	1.8	2.0
0.0	1.0000	1.0000	1.0004	1.0024	1.0087	1.0153	1.0214	1.0266
.25	1.0000	1.0001	1.0006	1.0030	1.0102	1.0174	1.0239	1.0294
.50	1.0000	1.0004	1.0018	1.0062	1.0164	1.0258	1.0337	1.0403
.75	1.0006	1.0034	1.0092	1.0205	1.0391	1.0532	1.0641	1.0728
.85	1.0033	1.0110	1.0226	1.0406	1.0652	1.0821	1.0946	1.1043
.91	1.0109	1.0263	1.0443	1.0680	1.0965	1.1146	1.1277	1.1376
.95	1.0294	1.0539	1.0773	1.1041	1.1335	1.1513	1.1639	1.1734
.98	1.0775	1.1084	1.1326	1.1571	1.1824	1.1976	1.2084	1.2165
1.0	1.2318	1.2337	1.2368	1.2424	1.2522	1.2601	1.2664	1.2716

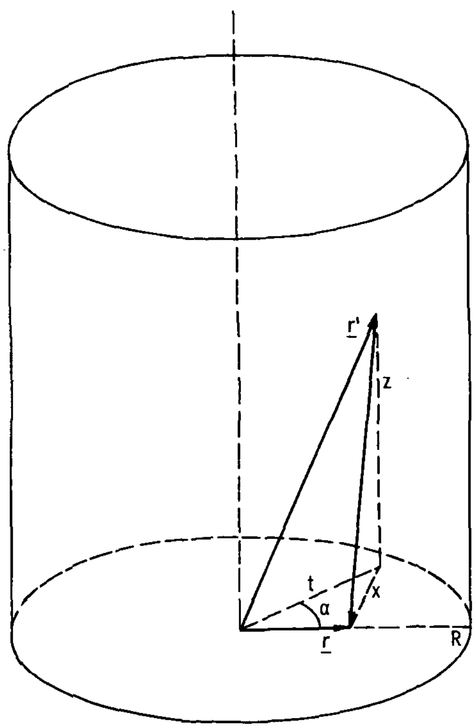

Figure 1. - Cylindrical geometry.

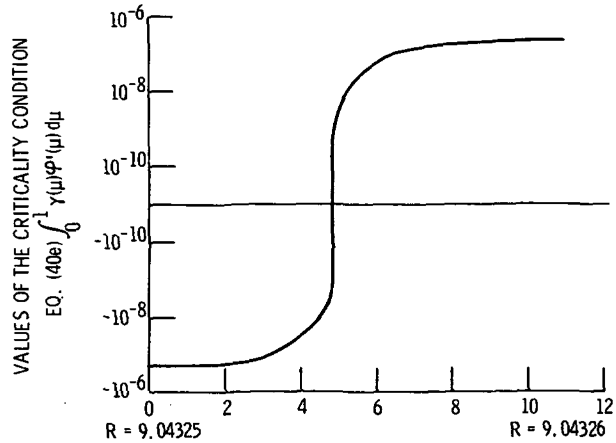

Figure 2. - Sensitivity of the criticality condition to the seventh significant figure in the radius.

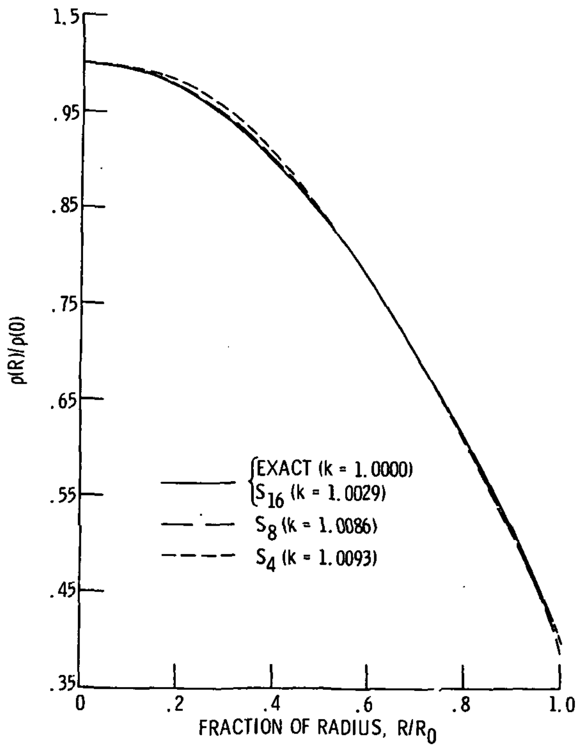

Figure 3. - Neutron density distribution (C = 2.0).