THE PREDICTION OF NONLINEAR LONGITUDINAL COMBUSTION INSTABILITY IN LIQUID PROPELLANT ROCKETS

by

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16. Abstract

An analytical technique is developed to solve nonlinear longitudinal combustion instability problems associated with liquid propellant rocket motors. The analysis produces the transient and limit-cycle behavior of unstable motors and the threshold amplitude required to trigger a linearly stable motor into unstable operation. The limit cycle waveforms are found to exhibit shock wave characteristics for most unstable engine operating conditions. A method of correlating the analytical solutions with experimental data is developed. Calculated results indicate that a second order solution adequately describes the behavior of combustion instability oscillations over a broad range of engine operating conditions, but that higher order effects must be accounted for in order to investigate engine triggering.

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FOREWORD

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ABSTRACT

An analytical technique is developed to solve nonlinear longitudinal combustion instability problems associated with liquid propellant rocket motors. The analysis produces the transient and limit-cycle behavior of unstable motors and the threshold amplitude required to trigger a linearly stable motor into unstable operation. The limit cycle waveforms are found to exhibit shock wave characteristics for most unstable engine operating conditions. A method of correlating the analytical solutions with experimental data is developed. Calculated results indicate that a second order solution adequately describes the behavior of combustion instability oscillations over a broad range of engine operating conditions, but that higher order effects must be accounted for in order to investigate engine triggering.

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SUMMARY

This report describes an analytical technique for the analysis of nonlinear longitudinal combustion instability. The applications of this technique, which is based on the Method of Weighted Residuals, is demonstrated for a liquid propellant rocket combustor with a high impedance injector and a short nozzle. Crocco's time lag theory is used to describe the unsteady combustion response.

The methodology developed in this report can predict the linear stability limits of the engine, triggering limits, and both the transient and final periodic behavior of the combustion instability oscillations. It is shown that the final periodic behavior of the instability is only dependent upon the engine operating conditions (i.e., the Mach number, the combustion parameters n and $\bar{\tau}$, etc.), and is independent of the characteristics of the initial disturbance. Computed results show that in most cases the final waveforms exhibit shock-wave-type behavior, and that the number of shocks present within the combustor is determined by engine operating conditions. It is also found that the waveform of the resulting instability depends upon the proximity of the operating point to engine resonant conditions (i.e., to the minimum point on the linear stability limit). The predicted waveforms are used to develop a technique for correlating the analytical solutions with experimental data.

INTRODUCTION

Experimental observations of rocket motors experiencing high frequency longitudinal combustion instability show that in a majority of cases the oscillations of the gas inside the combustor exhibit shock-wave characteristics. These flow oscillations can be initiated in two ways. In an intrinsically unstable engine the instabilities are spontaneous in nature and they result from any perturbation of the steady state flow field. On the other hand, some engines require the introduction of a finite amplitude disturbance to produce unstable combustion. In either case, the oscillations experience a transient phase prior to the establishment of stable, periodic (i.e., stable limit cycle) waves with oscillation frequencies that are usually close to the frequency of one of the chamber's acoustic modes. These observations suggest that a nonlinear analysis, capable of predicting the limit cycle waveforms and the conditions for which unstable combustion can be initiated by finite amplitude disturbances of the steady state flow, is required.

In this report, the Galerkin method, that is a special application of the Method of Weighted Residuals (see Ref. 1 for discussion of this method), is used to develop an approximate mathematical technique for analyzing the nonlinear behavior of rocket engines susceptible to longitudinal mode combustion instabilities. The desired mathematical techniques are developed by investigating longitudinal combustion instabilities in liquid propellant rocket combustors with a high impedance injector and a short nozzle. The Mach number of the combustor mean flow is assumed to be small. Crocco's pressure sensitive time lag model[2] is used to describe the unsteady combustion process.

The problem is analyzed by solving the conservation equations describing the behavior of large amplitude combustion-driven oscillations in low Mach number mean flows. Because the solution of these equations requires a relatively large amount of computations, a simpler but more restrictive second order analysis is developed concurrently. In this second order analysis, the amplitude of the flow oscillations are restricted to be of the order of the steady state flow Mach number, and terms of order higher than second order are neglected. Hereafter, the former problem formulation will be referred to

as the third order theory, and the latter analysis will be called the second order theory. The applicability of the second order theory will be determined by comparing its results with third order solutions. It will be shown that from a practical point of view, the results predicted by the second order theory are comparable to those found by the solutions of the third order theory.

The results obtained in this investigation are used to develop a technique for correlating the analytical results with experimental data. An empirical method for predicting the nonlinear waveforms is also discussed.

A User's Manual for the required computer programs is included in the appendices of this report.

SYMBOLS

$A_k(t)$, $B_k(t)$, $C_k(t)$, $D_k(t)$	time-dependent coefficients in the series defined in Eqs. (10) through (13)
A_1	semi-empirical peak amplitude of the first harmonic defined in Eq. (28)
B_1, B_2, B_3, B_4	boundary conditions defined in Eqs. (5), (6), (8), and (9)
c	sonic velocity
E_1, E_2, E_3, E_4	flow equations, defined by Eqs. (1), (2), (3), and (7)
$I_1(k,\ell)$, $I_2(k,m,\ell)$, $I_3(k,m,\ell)$, $I_4(k,m,\ell)$, $I_5(k,m,\ell)$	space integrals defined in Eqs. (22) through (26)
k,m,ℓ	summation indices and axial mode numbers
L	combustor length
N	number of terms used in series expansions
n	interaction index
p	dimensionless pressure, $\gamma p^*/\rho_0^* c_0^{*2}$
q	dimensionless acoustic-type velocity, defined in Eq. (14)

T	dimensionless wave period, $T^* c_0^* / L$
t	dimensionless time, $t^* c_0^* / L$
t_0	time correlation parameter
u	dimensionless velocity, u^* / c_0^*
w'	unsteady combustion mass source
z	dimensionless axial coordinate, z^* / L
z_r	dimensionless axial coordinate used in experimental correlation
α	power exponent in experimental correlation
γ	specific heat ratio
$\Delta p'_{max}$	dimensionless peak-to-peak amplitude
δn	vertical displacement at constant $\bar{\tau}$ in the n-$\bar{\tau}$ plane
ϵ	ordering parameter
ν	dimensionless specific volume, $\nu^* \rho_0^*$
ρ	dimensionless density, ρ / ρ_0^*
$\bar{\tau}$	dimensionless pressure sensitive time lag, $\bar{\tau}^* c_0^* / L$
φ	velocity potential
ω	dimensionless angular frequency, $\omega^* L / c_0^*$
ω_1	correlation parameter

Subscripts

e	evaluated at the nozzle entrance
k, m, ℓ	axial mode numbers
LS	evaluated at the linear stability limit
R	evaluated at retarded time, $t = t - \bar{\tau}$
t	time derivative
z	space derivative

4

O   injector face stagnation quantity

Superscripts

′   perturbation quantity

−   steady state quantity

*   dimensional quantity

~   approximate solution

DEVELOPMENT AND SOLUTION OF THE EQUATIONS

Problem Formulation

An analytical technique for investigating the nonlinear stability of combustion-driven axial mode oscillations in liquid propellant rocket combustors is developed. The combustor geometry is shown in Fig. (1). The liquid propellants are injected uniformly through a high impedance injector, converted by a complex combustion process into hot gases, and the gas products are exhausted through a short nozzle. The nondimensional coordinate system is defined with the origin at the injector face and the nozzle entrance plane at $z = z^*/L = 1$. The thermodynamic variables are normalized by the corresponding injector face stagnation quantities, the velocity is nondimensionalized by the injector face steady state stagnation sonic velocity, and time is normalized by a characteristic time defined as the ratio of the combustor length to the injector face stagnation sonic velocity.

In order to develop a problem formulation that is both simple and physically meaningful, the following assumptions are made:

1. The flow is one-dimensional, with the velocity vector parallel to the combustor axis.

2. The mean flow Mach number and its derivative are small.

3. The flow consists of a single constituent perfect gas and liquid droplets of negligible volume.
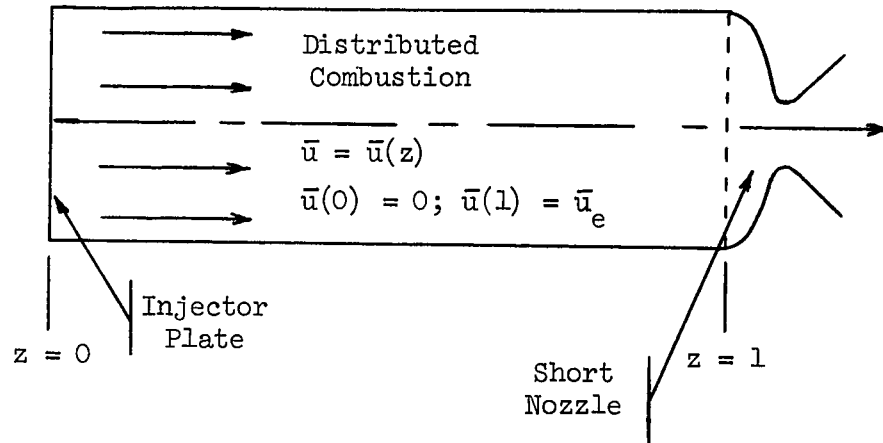
Injector Plate Boundary Condition:

$$u'(0,t) = 0$$

Short Nozzle Boundary Condition

$$\frac{\bar{u} + u'}{\bar{c} + c'} = \text{constant}$$

Figure 1. Combustor Geometry and Boundary Conditions

4. Viscosity, diffusion, and heat conduction are negligible.

5. The liquid droplet specific stagnation enthalpy remains constant as the droplets move through the combustor.

6. The momentum sources arising from gas-droplet interaction are negligible.

Even with these restrictions, the equations describing the behavior of axial combustion instability oscillations are quite complex[3]. A simplification of the analysis results when the relative importance of the various terms in the conservation equations is established by using order of magnitude arguments. In this analysis, the magnitude of each term appearing in the conservation equations is evaluated and all terms whose magnitudes are smaller than a certain threshold value are neglected. To accomplish this task, two ordering parameters are used. One parameter, $\bar{u}_e$, is a measure of the effect of the presence of mean flow upon the oscillations. The second parameter, ϵ, is a measure of the amplitude of the flow oscillations. This investigation is concerned with the behavior of moderate and large amplitude instabilities in combustors having low Mach number mean flows. Consequently, terms of order higher than $O(\bar{u}_e^2)$ and $O(\bar{u}_e \epsilon^2)$ are neglected. Under these restrictions, the equations describing the behavior of the combustion instability oscillations can be written as follows[3]:

1. Continuity:

$$E_1 = v_t' + \bar{u}v_z' + \frac{d\bar{u}}{dz}v' - v'u_z' - u_z' + w_z' + \left[2v'w_z' + \frac{d\bar{u}}{dz}v'^2\right] = 0 \qquad (1)$$

2. Momentum:

$$E_2 = u_t' + \bar{u}u_z' + \frac{d\bar{u}}{dz}u' + u'u_z' + \frac{1}{\gamma}v'p_z' + \frac{1}{\gamma}p_z' = 0 \qquad (2)$$

3. Energy:

$$E_3 = p_t' + \bar{u}p_z' + u'p_z' + \gamma u_z' - \gamma w_z' + \gamma \frac{d\bar{u}}{dz}p' + \gamma p'u_z' + \left[\frac{\gamma(\gamma-1)}{2}\frac{d\bar{u}}{dz}u'^2\right] = 0 \qquad (3)$$

In writing these equations, the specific volume, ν, is used instead of the density, ρ, in order to simplify the numerical solution of the ordinary differential equations that result from the application of the Galerkin method[3].

The term $\frac{\partial w'}{\partial z}$ represents the unsteady mass generation by the burning fuel. In the present analysis this unsteady mass source (or combustion response function, as it is sometimes referred to) is described by Crocco's pressure sensitive time-lag hypothesis[2]. Accordingly, $\frac{\partial w'}{\partial z}$ is then given by the following relation:

$$\frac{\partial w'}{\partial z} = n \frac{d\bar{u}}{dz} \left[p'(z,t) - p'(z,t-\bar{\tau}) \right] \tag{4}$$

where n and $\bar{\tau}$ are the two parameters that Crocco used to describe the unsteady combustion process. The interaction index, n, is a measure of the sensitivity of the combustion process to flow oscillations and the sensitive time lag, $\bar{\tau}$, is representative of the time required for the unsteady combustion process to respond to flow perturbations.

The computed solutions must satisfy the solid wall boundary condition at the injector face; that is

$$B_1 = u'(0,t) = 0 \tag{5}$$

and the quasi-steady short nozzle boundary condition[3,4] at the nozzle entrance

$$B_2 = u'(1,t) - \frac{\gamma-1}{2\gamma} \bar{u}_e p'(1,t) + \left[\bar{u}_e p'^2(1,t) \right] = 0 \tag{6}$$

Additional simplification of the conservation equations is possible when the amplitude of the instability is moderate. Under this condition, it is possible to assume that the ordering parameters $\bar{u}_e$ and ϵ are of the same order of magnitude and all terms of order higher than second (e.g., terms of $O(\bar{u}_e^2 \epsilon)$ or $O(\epsilon^3)$) are negligible; all such terms are bracketed in Eqs. (1) through (3). When these terms are neglected, Eqs. (1) through (3) can be combined[5,6] into the following nonlinear wave equation:

$$E_4 = \varphi_{zz} - \varphi_{tt} - 2\bar{u}\varphi_{zt} - \gamma \frac{d\bar{u}}{dz} \varphi_t - 2\varphi_z\varphi_{zt} - (\gamma-1)\varphi_t\varphi_{zz} - \frac{\partial w'}{\partial z} = 0 \qquad (7)$$

where φ is the velocity potential defined by $\varphi_z = u'(z,t)$. The bracketed term in the short nozzle boundary condition is also neglected, and the boundary conditions are written in terms of the velocity potential φ[6]:

$$B_3 = \varphi_z(0,t) = 0 \qquad (8)$$

$$B_4 = \varphi_z(1,t) + \frac{\gamma-1}{2} \bar{u}_e\varphi_t(1,t) = 0 \qquad (9)$$

It has been shown[3] that solutions of Eqs. (7) through (9) adequately describe the behavior of moderate amplitude axial instabilities, but that solutions of Eqs. (1) through (6) are required to investigate axial mode triggering. Solutions of both formulations of the problem are developed in this report.

Solution Technique

Closed-form mathematical solutions of the equations developed in the preceding section are not known. As a result, it is necessary to resort to the use of either numerical solution techniques or approximate analytical techniques. The former approach is generally quite complex and its application requires excessive computation time; furthermore, the use of numerical solution techniques in general provides little physical insight into the problem. An appealing approximate analytical technique has been developed by Zinn and Powell[6,7] who investigated nonlinear transverse combustion instability problems. In these investigations, the undetermined function form of the Galerkin method, that is a special application of the Method of Weighted Residuals[1,8] (MWR), is used to find the desired solutions.

In order to use the Galerkin method, it is necessary to represent the dependent variables by means of approximate series expansions. The proper choice of the series expansion is critical to the usefulness of the Galerkin method. Various guide lines for the choice of the approximate series expansion are offered in the literature[1,8]. In studies of combustion instabilities it

is convenient to use available experimental data, which indicates that the behavior of high frequency combustion instability oscillations resembles the behavior of the chamber's acoustic modes, as a guide. This information suggests that the dependent variables of the problem be expanded in terms of the natural acoustic modes of the chamber; each having an unknown time-dependent coefficient. Using available acoustic solutions as a guide, the following approximate series representations for the dependent variables are used:

$$\widetilde{v}'(z,t) = \sum_{k=1}^{N} A_k(t) \cos(k\pi z) \tag{10}$$

$$\widetilde{p}'(z,t) = \sum_{k=1}^{N} B_k(t) \cos(k\pi z) \tag{11}$$

$$\widetilde{q}'(z,t) = \sum_{k=1}^{N} C_k(t) \sin(k\pi z) \tag{12}$$

$$\widetilde{\varphi}(z,t) = \sum_{k=1}^{N} D_k(t) \cos(k\pi z) \tag{13}$$

the variable q' represents the "acoustic portion" of the velocity perturbation u' that is given by

$$\widetilde{u}'(z,t) = \left[\frac{\gamma-1}{2\gamma} \bar{u}_e \widetilde{p}'(z,t) - \frac{\gamma^2-1}{8\gamma^2} \bar{u}_e \widetilde{p}'^2(z,t) \right] z + \widetilde{q}'(z,t) \tag{14}$$

The particular choice of an expression for $\widetilde{u}'$, as given in Eq. (14), was dictated by the requirement that the dependent variable satisfy the problem's boundary conditions[3] (i.e., Eqs. (5) and (6)).

The unknown time-dependent mode-amplitudes (e.g., $A_n(t)$, $C_n(t)$, etc.) are determined by the following mathematical procedure. The assumed series

expansions are substituted into the conservation equations and boundary conditions to form differential equation residuals and boundary residuals. If a residual is identically zero, then the corresponding equation or boundary condition is identically satisfied. On the other hand, when the equation or boundary residuals are not identically zero, the residuals are the errors that resulted from using the approximating expansions of the dependent variables. According to the Galerkin method, these errors (i.e., residuals) can be minimized in some average sense by requiring that the residuals satisfy certain orthogonality conditions.[7,6,9] In the solution of the problem defined by Eqs. (1) through (3), the boundary conditions, Eqs. (5) and (6) are identically satisfied by the chosen series expansions, Eqs. (10) through (12), and the required orthogonality conditions are defined by the following relations[3]:

$$\int_0^1 \widetilde{E}_1' \cos(\ell\pi z)\,dz = 0 \qquad \ell = 1,\ldots,N \qquad (15)$$

$$\int_0^1 \widetilde{E}_2' \sin(\ell\pi z)\,dz = 0 \qquad \ell = 1,\ldots,N \qquad (16)$$

$$\int_0^1 \widetilde{E}_3' \cos(\ell\pi z)\,dz = 0 \qquad \ell = 1,\ldots,N \qquad (17)$$

On the other hand, the expansion of the velocity potential used in the second order solution (i.e., Eq. (13)) does not satisfy the quasi-steady short nozzle boundary condition, Eq. (9). In this case, the required orthogonality condition is[3,9]:

$$\int_0^1 \widetilde{E}_3 \cos(\ell\pi z)\,dz - \frac{\gamma-1}{2}\,\bar{u}_e \widetilde{\varphi}_t(1,t)\,\cos(\ell\pi) = 0 \qquad \ell = 1,\ldots,N \qquad (18)$$

The last term in the above equation represents the effect of the nozzle boundary condition residual.

Performing the operations indicated in Eqs. (15) through (17) yields the following system of quasi-linear ordinary differential equations describing the behavior of the unknown mode-amplitude functions:

$$\frac{dA_\ell}{dt} = F_{1_\ell} = (\ell\pi)C_\ell + Q_1 B_\ell - \bar{u}_e A_\ell - n\bar{u}_e (B_\ell - B_{\ell_R})$$

$$+ 2 \sum_{k=1}^{N} \left\{ \bar{u}_e (k\pi) I_1(k,\ell) A_k - Q_1(k\pi) I_1(k,\ell) B_k \right.$$

$$+ \sum_{m=1}^{N} \left[(m\pi) I_2(k,m,\ell) C_k A_m + (m\pi) I_3(k,m,\ell) A_k C_m \right.$$

$$- \bar{u}_e I_3(k,m,\ell) A_k A_m + Q_1(m\pi) I_4(k,m,\ell) B_k A_m$$

$$- Q_1(m\pi) I_4(k,m,\ell) A_k B_m + Q_1 I_3(k,m,\ell) A_k B_m$$

$$- 2n\bar{u}_e I_3(k,m,\ell)(B_k - B_{k_R}) A_m$$

$$\left. \left. + Q_2 I_3(k,m,\ell) B_k B_m - 2Q_2(m\pi) I_4(k,m,\ell) B_k B_m \right] \right\} \tag{19}$$

$$\frac{dB_\ell}{dt} = F_{2_\ell} = -\gamma(\ell\pi)C_\ell - \gamma\bar{u}_e B_\ell - \gamma Q_1 B_\ell + \gamma n\bar{u}_e (B_\ell - B_{\ell_R})$$

$$+ 2 \sum_{k=1}^{N} \left\{ \bar{u}_e (k\pi) I_1(k,\ell) B_k + \gamma Q_1(k\pi) I_1(k,\ell) B_k \right.$$

$$+ \sum_{m=1}^{N} \left[(m\pi) I_2(k,m,\ell) C_k B_m - \gamma(m\pi) I_3(k,m,\ell) B_k C_m \right.$$

$$- \frac{\gamma(\gamma-1)}{2} \bar{u}_e I_2(k,m,\ell) C_k C_m - \gamma Q_1 I_3(k,m,\ell) B_k B_m$$

$$+ (\gamma+1) Q_1(m\pi) I_4(k,m,\ell) B_k B_m - \gamma Q_2 I_3(k,m,\ell) B_k B_m$$

$$+ 2\gamma Q_2 (m\pi) I_4 (k,m,\ell) B_k B_m \Big]\Big\} \tag{20}$$

$$\frac{dC_\ell}{dt} = - 2 \sum_{k=1}^{N} \Big\{ Q_1 I_1 (k,\ell) F_{2_k} + 2Q_2 \sum_{m=1}^{N} I_4 (k,\ell,m) B_k F_{2_m} \Big\} + F_{3_\ell} \tag{21}$$

where

$$F_{3_\ell} = \frac{\ell\pi}{\gamma} B_\ell - \bar{u}_e C_\ell + 2 \sum_{k=1}^{N} \Big\{ - \bar{u}_e (k\pi) I_1 (\ell,k) C_k$$

$$+ \sum_{m=1}^{N} \Big[\frac{1}{\gamma} (m\pi) I_2 (\ell,m,k) A_k B_m - (m\pi) I_2 (k,\ell,m) C_k C_m$$

$$- Q_1 (m\pi) I_4 (k,\ell,m) B_k C_m + Q_1 (k\pi) I_5 (k,\ell,m) B_k C_m$$

$$- Q_1 I_2 (\ell,m,k) B_k C_m \Big]\Big\}$$

In the derivation of Eqs. (19) through (21) a linear steady state velocity distribution, $\bar{u} = \bar{u}_e z$, has been assumed, and the following definitions have been used:

$$I_1 (k,\ell) = \int_0^1 z \sin(k\pi z) \cos(\ell\pi z) \, dz \tag{22}$$

$$I_2 (k,m,\ell) = \int_0^1 \sin(k\pi z) \sin(m\pi z) \cos(\ell\pi z) \, dz \tag{23}$$

$$I_3 (k,m,\ell) = \int_0^1 \cos(k\pi z) \cos(m\pi z) \cos(\ell\pi z) \, dz \tag{24}$$

$$I_4(k,m,\ell) = \int_0^1 z \, \cos(k\pi z) \sin(m\pi z) \cos(\ell\pi z) \, dz \tag{25}$$

$$I_5(k,m,\ell) = \int_0^1 z \, \sin(k\pi z) \sin(m\pi z) \sin(\ell\pi z) \, dz \tag{26}$$

and

$$Q_1 = \frac{\gamma-1}{2\gamma} \bar{u}_e \qquad ; \qquad Q_2 = -\frac{\gamma^2-1}{8\gamma^2} \bar{u}_e$$

The second order solutions are found by performing the operations defined by Eq. (18). The following equations describe the behavior of the mode-amplitude function of the velocity potential:

$$
\frac{d^2 D_\ell}{dt^2} = -(\ell\pi)^2 D_\ell - \gamma \bar{u}_e \frac{dD_\ell}{dt} + \gamma n \bar{u}_e \left[\frac{dD_\ell}{dt} - \frac{dD_\ell}{dt}(t-\bar{\tau}) \right]
$$

$$
+ 2 \sum_{k=1}^N \left\{ \left[2\bar{u}_e(k\pi) I_1(k,\ell) - \frac{\gamma-1}{2} \bar{u}_e (-1)^{k+\ell} \right] \frac{dD_k}{dt} \right.
$$

$$
+ \sum_{m=1}^N \left[(\gamma-1)(m\pi)^2 I_3(k,m,\ell) \frac{dD_k}{dt} D_m \right.
$$

$$
\left. \left. - 2(k\pi)(m\pi) I_2(k,m,\ell) \frac{dD_m}{dt} D_k \right] \right\} \tag{27}
$$

The space integrals defined in Eqs. (22) through (26) are evaluated numerically using a Simpson's rule algorithm[10] (see Appendix A). The nonlinear behavior of axial mode instabilities are found by numerically integrating either Eqs. (19) through (21), or, in the case of moderate amplitude oscillations, Eqs. (27). In order to carry out these computations, engine operating conditions (i.e., γ, $\bar{u}_e$, n and τ), and initial conditions must be specified. The behavior of the mode-amplitude functions is followed through the transient phase to the establishment of periodic oscillations. The perturbation flow field is then calculated using either Eqs. (10) through (12) or Eq. (13). When Eqs. (27) are used to describe the unsteady flow, the

pressure perturbation at any location within the chamber is related to $\widetilde{\varphi}$ by the following second order momentum equation[5,6]:

$$p'(z,t) = \frac{\gamma}{2}\left[\widetilde{\varphi}_z\left(\widetilde{\varphi}_t - 2\right) - \widetilde{\varphi}_z\left(\widetilde{\varphi}_z + 2\bar{u}\right)\right] \qquad (28)$$

A more detailed description of the solution technique outlined in the preceding discussion is included in Appendix B. Typical numerical solutions of these equations are presented and discussed in the following section.

<div align="center">RESULTS AND DISCUSSION</div>

Nonlinear Solutions

Extensive computations have shown that the predicted nonlinear instabilities are dependent upon the engine operating conditions and independent of the nature of the initial disturbances. However, the computation time required to reach limit cycle conditions is reduced when the waveform of the initial disturbance is "close", in some sense, to the waveform of the limit cycle oscillation. For example, the computation time required to reach a discontinuous fundamental mode (1L) limit cycle oscillation is reduced when the assumed initial disturbance has a 1L discontinuous waveform. In this investigation it has been assumed that the engine is operating smoothly until t = 0, at which time a pressure disturbance is impulsively introduced inside the chamber. The velocity perturbation is assumed to be initially zero. Both spacially continuous and spacially discontinuous initial disturbances have been used.

Typical transient and the resulting limit cycle oscillations are shown in Fig. 2. Here, initial continuous fundamental mode perturbations distort themselves into a discontinuous oscillation. When the amplitude of the initial disturbance is larger than the amplitude of the limit cycle oscillation, the transition to shock-wave-type behavior occurs within two cycles. On the other hand, when the initial amplitude is small a longer time period elapses before a shock wave is formed. In either case, the initial disturbances reach the same limit cycle conditions. These data were generated by solutions of the
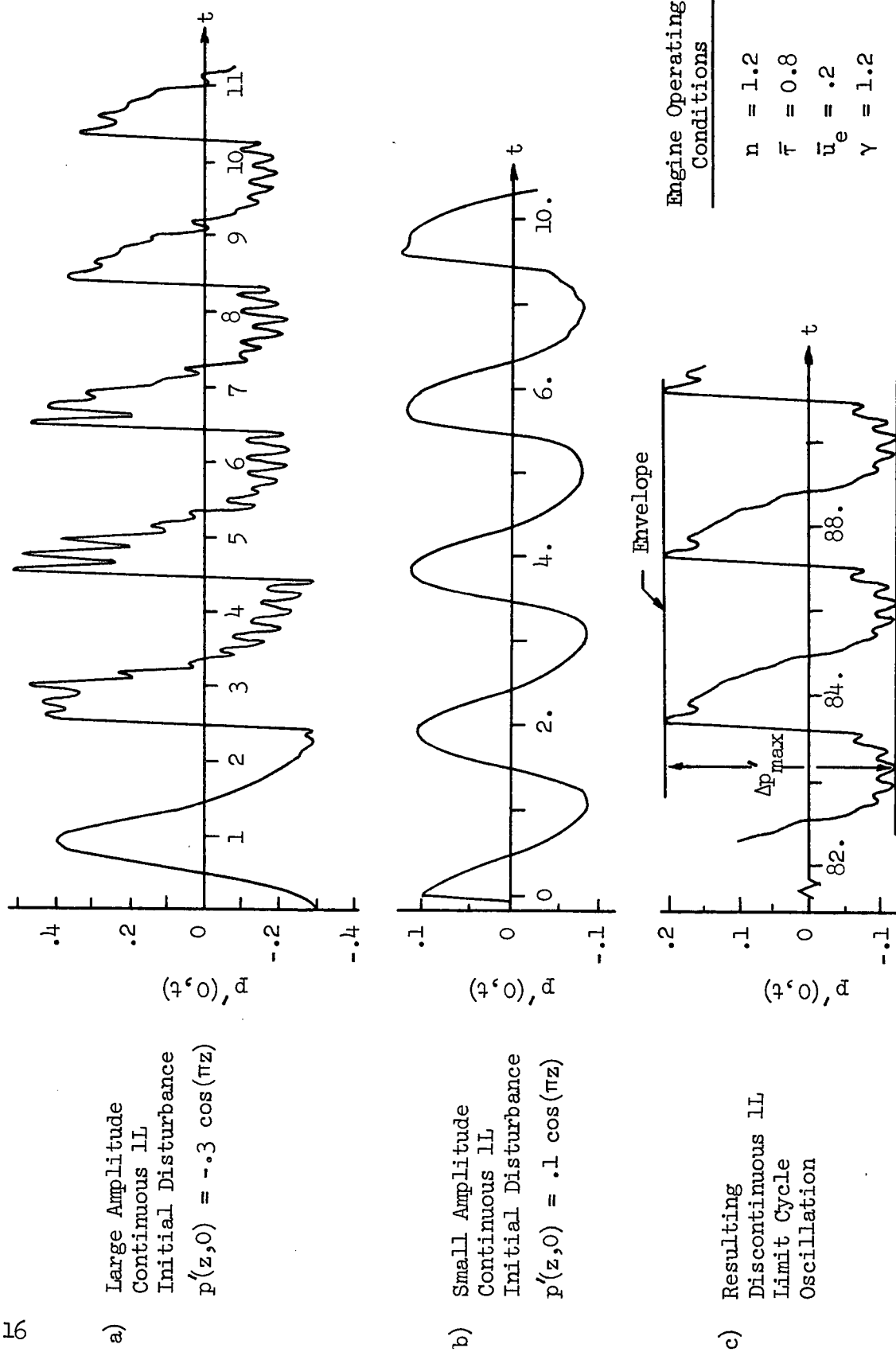
Figure 2. Typical Transient and Limit Cycle Pressure Oscillations for a Given Engine Operating Conditions

second order wave equation (i.e., Eq. (27)); however, the same behavior is exhibited by the solutions of Eqs. (19) through (20). It can also be shown[3] that during the transient period the waves can change from one mode of oscillation to another. Consequently, if for given engine operating conditions the initial disturbance is not in the "proper" mode, then the solutions will adapt themselves to the operating conditions and the final periodic solution will be in the mode of oscillation that is unstable. In other words, no a priori knowledge of the behavior of the resulting instability is required in order to use the analytical technique developed in this report.

The pressure envelope of the combustion instability oscillations is also defined in Fig. 2. The pressure envelope is simply the band of the peak-to-peak pressure amplitudes of the oscillations. The temporal behavior of the pressure envelope will subsequently be used to investigate engine triggering.

It has been found in the course of this investigation that while the second order theory is capable of predicting the behavior of the final instabilities in linearly unstable engines, it is unable to predict the engine's triggering limits. It is shown elsewhere[6] that this difficulty is related to the mathematical structure of the resulting second order equations for the mode amplitudes. In view of these results, it was decided to use the second order theory, that requires considerably less computation time, to investigate the behavior of stable limit cycles in linearly unstable engines, while the third order theory will be used to study the behavior of triggering limits. A justification for this approach is presented in Fig. 3 where predictions of the second order and third order theories, for limit cycle peak-to-peak pressure amplitudes, are compared. It is shown in Fig. 3 that the predictions of both theories are in fair agreement over a wide region of peak-to-peak pressure amplitudes. A possible reason for the observed discrepancies is the different treatment of the nozzle boundary condition in the two theories[3]. It has also been shown[3] that the waveforms predicted by the two theories are in good agreement.

To determine the engine triggering limits, the minimum value of an initial disturbance required to initiate instability in a linearly stable region was determined numerically. For operating conditions where no disturbance can cause instability, the engine is said to be absolutely stable. Due to the
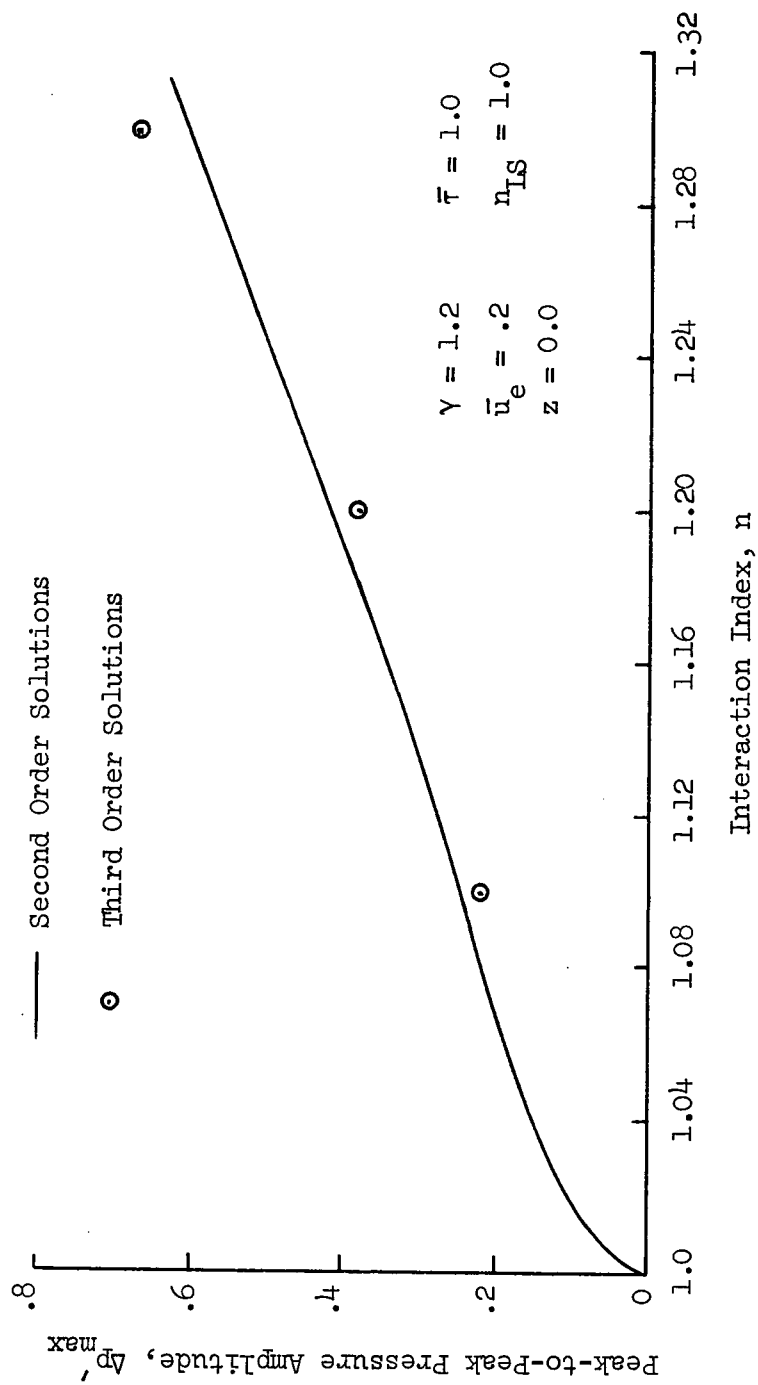
Figure 3. A Comparison of Second and Third Order Solutions

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above-mentioned shortcomings of the second order theory, the third order analysis is used herein to investigate the behavior of the triggering limits; the results of this investigation will determine the manner in which the problem's nonlinearities modify the engine linear stability limits. The behavior of oscillations near a triggering limit are shown in Fig. 4. The upper plot in Fig. 4 was obtained for an initial disturbance larger than the triggering limit; it shows the growth of a pressure envelope. The lower plot in Fig. 4 shows the behavior of an initial disturbance whose magnitude is smaller than the triggering limit; the plot shows the decay of a pressure envelope. The plots presented in Fig. 4 indicate that the threshold disturbance amplitude required to trigger a linearly stable motor, at the operating conditions in question, lies between the amplitudes of these two initial disturbances. The threshold amplitude can be found by requiring a zero growth rate of the threshold disturbance and linearly interpolating the data shown in Fig. 4.

The nonlinear behavior of fundamental mode instabilities can be summarized in an amplitude map of the type shown in Fig. 5. This figure shows linear and nonlinear stability limits, and lines of constant peak-to-peak pressure amplitude. According to Fig. 5, triggering can be obtained in the narrow region between the linear stability limit (solid line) and the nonlinear stability limit (broken line). The small extent of the triggering region is evident at $\bar{\tau} = 1.623$. Here, the vertical displacement of the nonlinear limit from the linear stability limit is only $\delta_n = n - n_{LS} = -.02$. The triggering region for above resonant conditions (i.e., $\bar{\tau} < 1$) is also narrow, and is terminated at $\bar{\tau} = 2/3$ where the second longitudinal mode becomes linearly unstable. It can be shown[3] that the concept of triggering becomes meaningless in a region where one of the modes present in the series expansion is linearly unstable. The significance of the parameter t_o/T, also shown in Fig. 5, will be discussed in the following section.

The objective of the preceding discussions was to provide an indication of the type of data that can be generated by the solution technique developed in this report. Detailed presentations of these and related studies can be found in Ref. 3.

$\Delta p'_{max}(0) = .1545$ 　　　　 $\Delta p'_{max}(35) = .1568$

Peak-to-Peak Pressure Amplitude, $\Delta p'_{max}$

.08

.04

0

-.04

-.08

0　　10　　20　　30　　40

Cycles
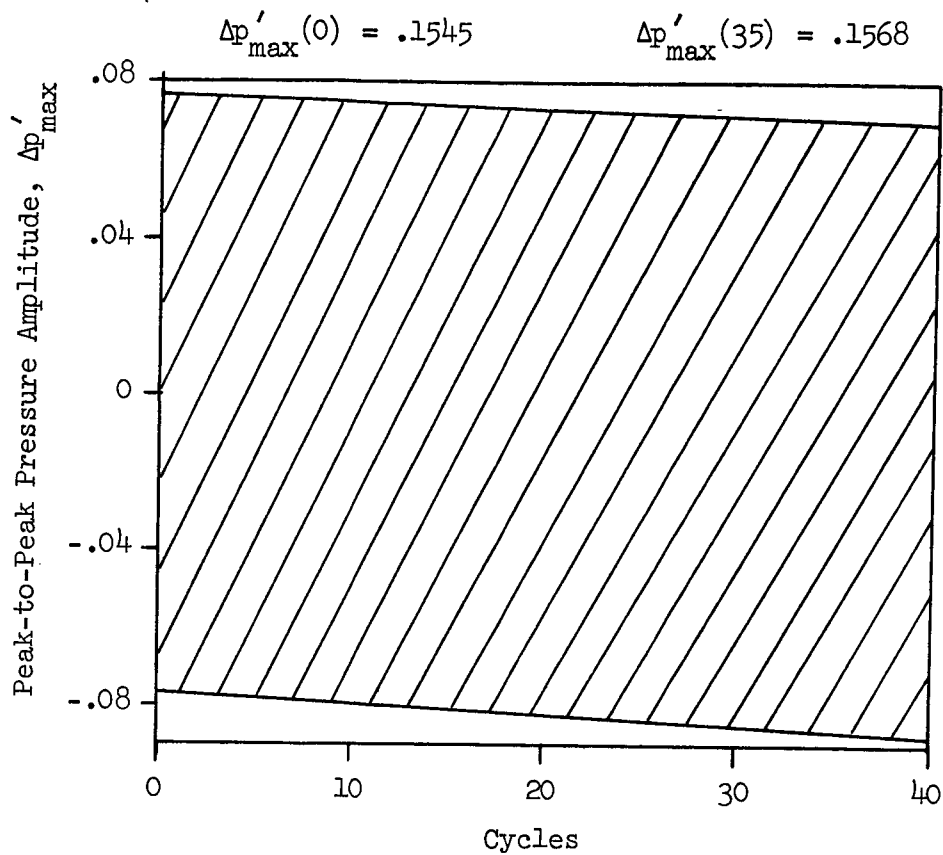
(a)　Initial disturbance above triggering limit

$\Delta p'_{max}(0) = .0883$ 　　　　 $\Delta p'_{max}(35) = .0851$

Peak-to-Peak Pressure Amplitude, $\Delta p'_{max}$

.06

0

-.06

0　　10　　20　　30　　40

Cycles
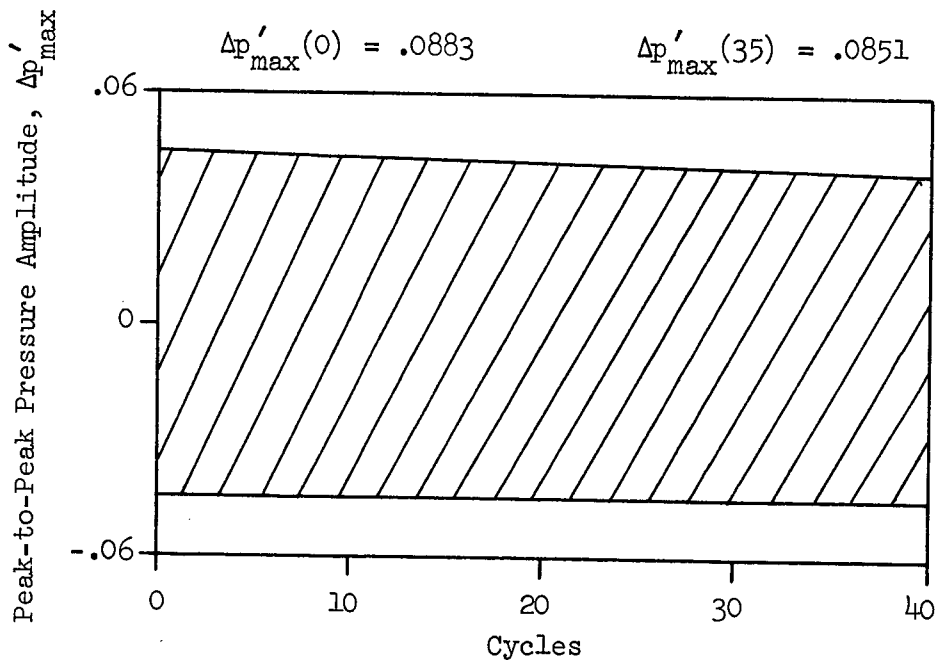
(b)　Initial disturbance below triggering limit

Figure 4.　Perturbation Pressure Envelopes of Fundamental Axial
Mode Instabilities Near a Triggering Limit
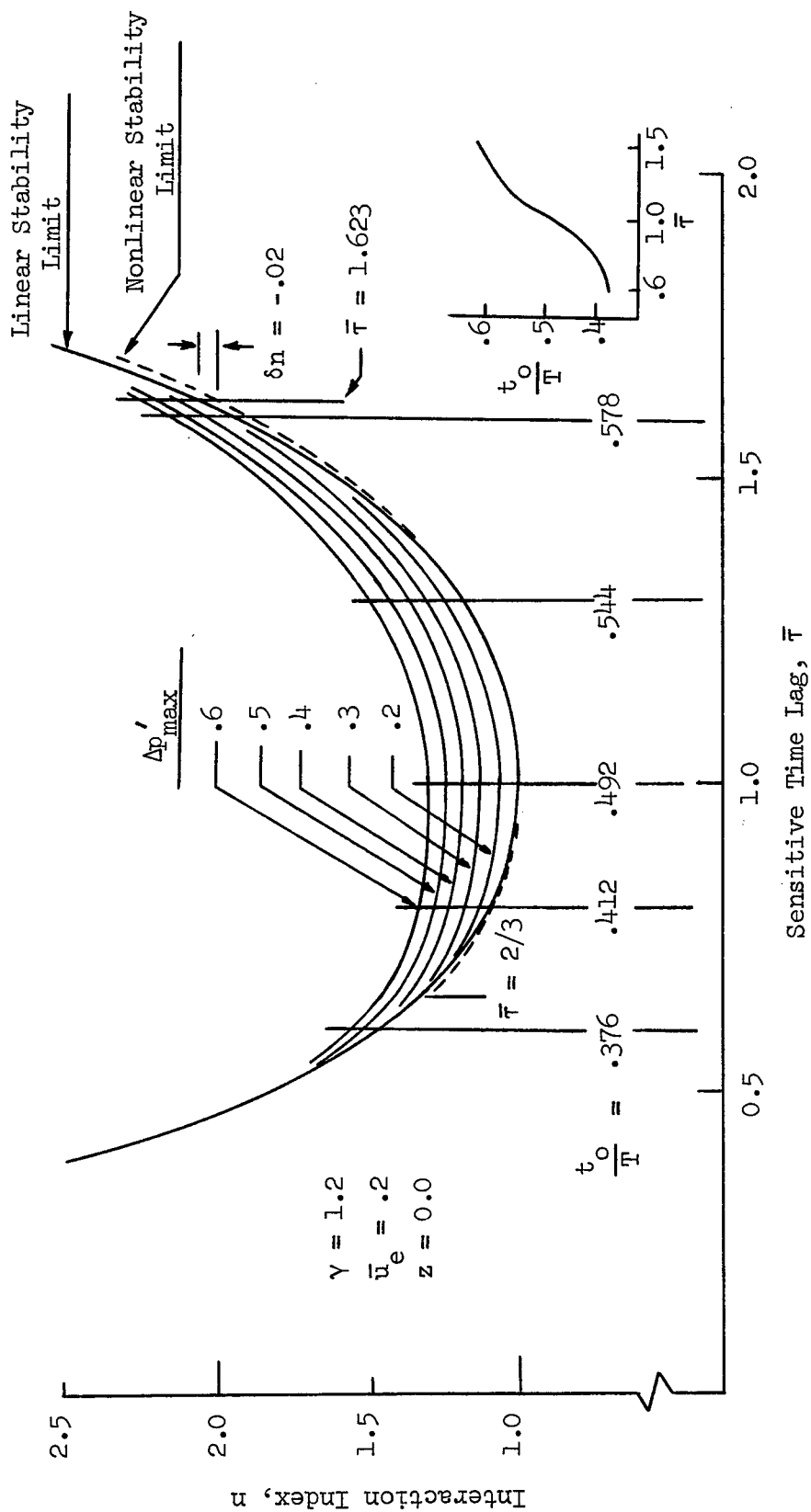
Figure 5. Stability Map

Correlation with Experimental Data

It has been shown in Reference 3 that the shape of the pressure waveforms depends upon engine operating conditions. Specifically, the pressure waveforms are dependent upon the proximity of the operating point to engine resonant conditions (i.e., to n_{min} and $\bar{\tau}_{min}$). The observed behavior of the stable limit cycle pressure oscillations can be used to correlate the analytical results with experimental data. To accomplish this task, two waveform parameters are defined in Fig. 6. In this figure, the solid line shows the numerically computed pressure waveform, and the broken line is the mean pressure waveform used to determine the correlation parameters $\Delta p'_{max}(z_r)$ and $t_o/T(z_r)$; z_r is the normalized axial location for which experimental pressure data is available.

Once z_r is specified, the analytical solution technique can be used to determine both the limit-cycle amplitude map and the dependence of t_o/T on $\bar{\tau}$. Typical results are shown in Fig. 5. The values of $\Delta p'_{max}(z_r)$ and $t_o/T(z_r)$ found from experimental pressure data are then used, in conjunction with the data presented in Fig. 5, to determine the engine operating conditions in terms of n and $\bar{\tau}$.

Semi-Empirical Pressure Waveforms

A semi-empirical method for predicting the pressure waveforms has been developed. The objective of the semi-empirical method is to provide design engineers with a straightforward technique, requiring relatively little computation time, for predicting the nonlinear pressure waveforms. The semi-empirical correlation method is based on the observation[11] that the velocity potential, φ, can be approximated, at least for resonant oscillations, by the following series expansion:

$$\varphi \doteq A_1 \sum_{k=1}^{N} k^{-\alpha} \cos(k\omega_1 t) \cos(k\pi z) \tag{29}$$

where A_1, α, and ω_1 are found from computer-generated data. The nonlinear pressure waveform is then found from Eq. (28).
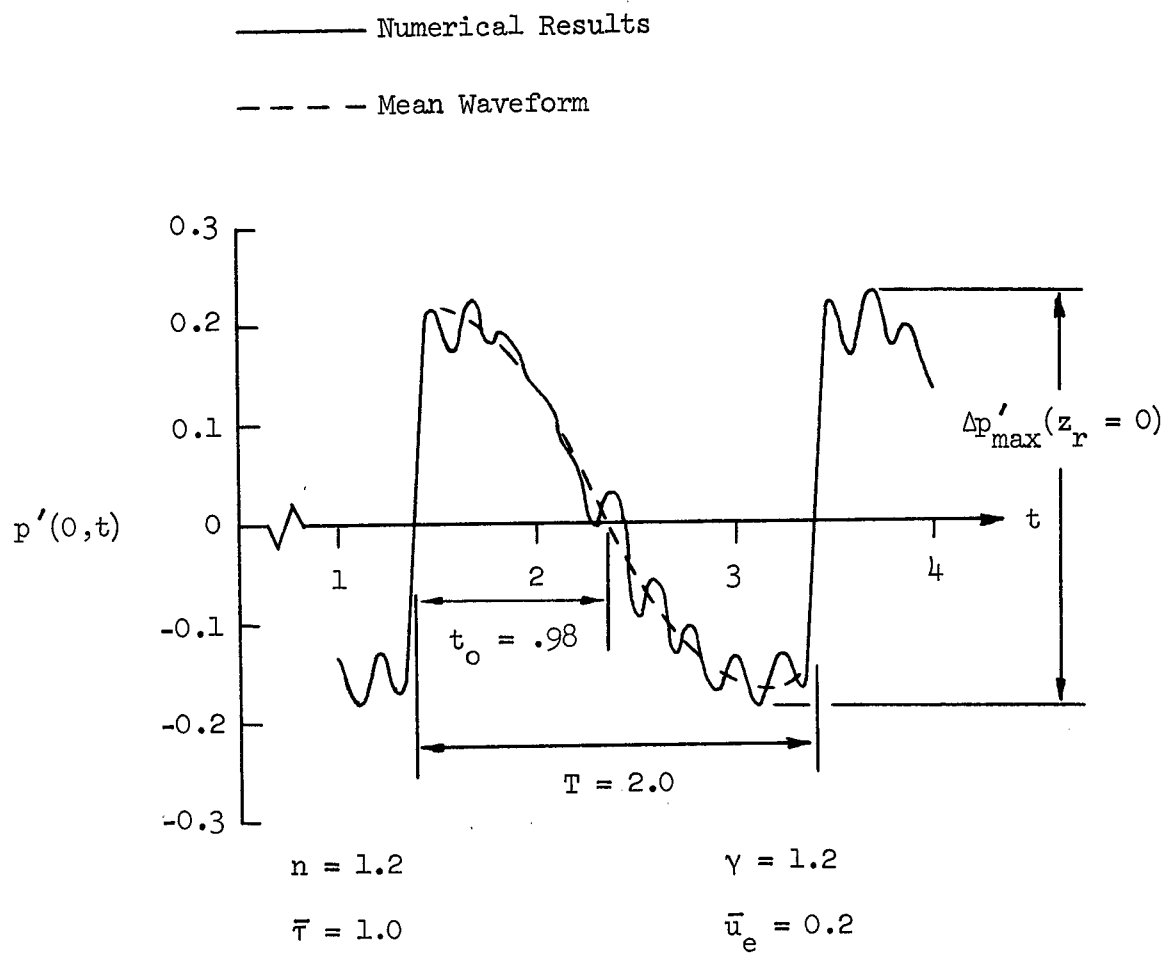
Figure 6. Theoretical Pressure Waveform Used to Determine t_o/T

The parameters A_1, α, and ω_1 are found from the behavior of the mode-amplitude functions computed using Eq. (27) with a five term series expansion (i.e., N = 5 in Eq. (13)). The parameters A_1 and ω_1 are the maximum amplitude and the frequency of the fundamental harmonic, respectively. The exponent, α, accounts for the decrease in maximum amplitudes of the higher harmonics; it is found from an empirical log-log plot of maximum mode amplitude versus the mode number.

Normally, a ten term series (i.e., N = 10, in Eq. (13)) is required to adequately predict the discontinuous waveforms[3]. The required computation time is approximately proportional to the square of the number of terms retained in the series expansion. Consequently, the computation time required for the semi-empirical method is considerably shorter than that required to solve directly for the pressure waveforms using the series solutions containing the unknown time-dependent mode amplitudes.

Semi-empirical pressure waveforms are compared with computer generated solutions in Fig. 7. Ten terms were retained in Eq. (28) in the computation of the semi-empirical waveforms (i.e., N = 10 in Eq. (28)). It is evident from the data shown in Fig. 7 that the semi-empirical method fails to reproduce the waveforms at off resonant oscillations. The probable reasons for this failure are:

1. There is a slight phase shift between the various modes at off-resonant conditions.

2. For off-resonant oscillations, the higher harmonics are both frequency and amplitude modulated.

3. For off-resonant oscillations, the higher harmonics may not obey the amplitude power law found by considering the behavior of the first few mode-amplitude functions.

DISCUSSION AND CONCLUDING REMARKS

An analytical technique has been developed for the analysis of nonlinear longitudinal combustion instabilities in liquid propellant rocket motors. The technique requires relatively little computation time and provides considerable insight into the physics of the problem. The method does not require any
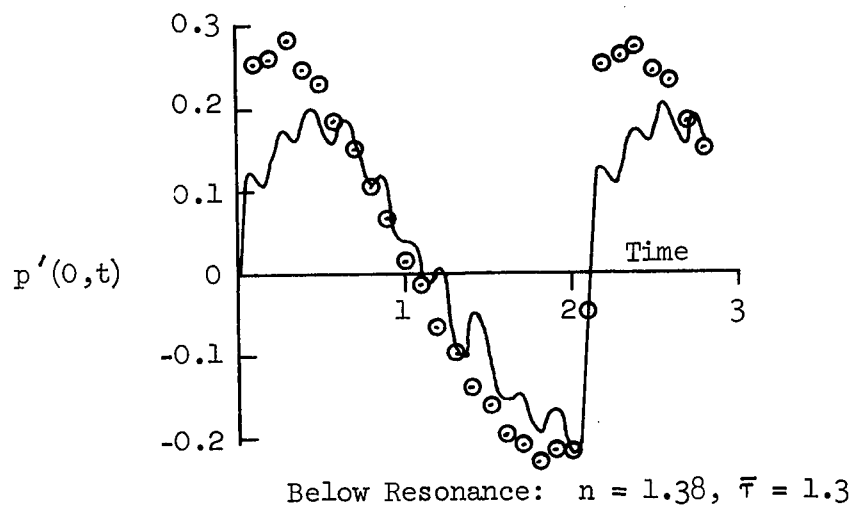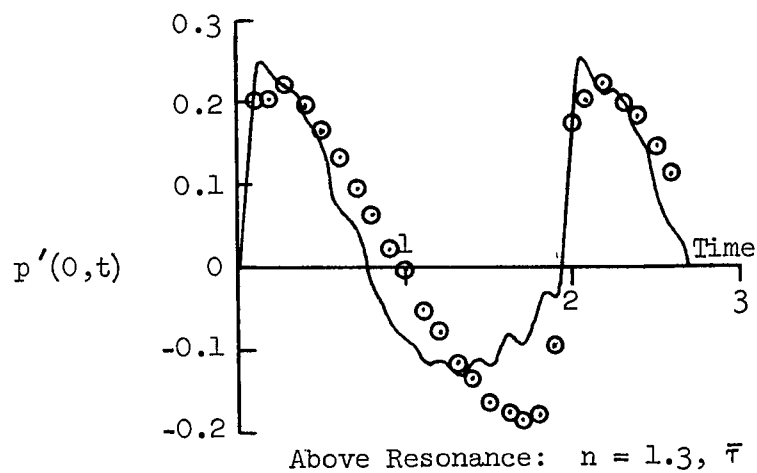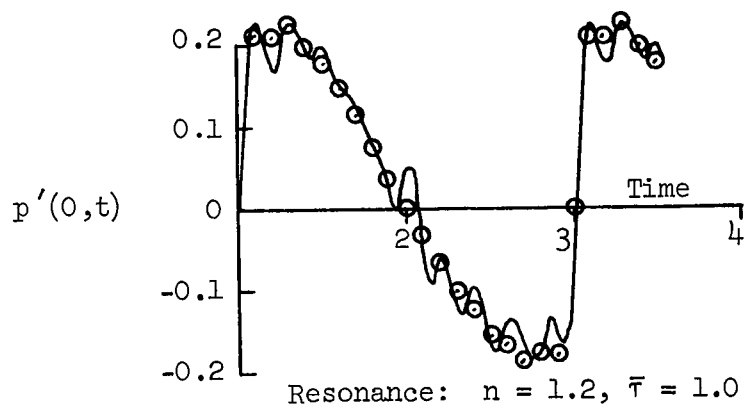
Figure 7. Comparison of Numerical and Semi-Empirical Pressure Waveforms

a priori knowledge of the final form of the instability. The method can predict triggering limits as well as the transient and final periodic behavior of the instability. Results predicted with the aid of this method agree with available experimental data.

Results obtained with a second and a third order analyses show that the second order analyses describes the behavior of longitudinal combustion instability over a broad range of engine operating conditions. The third order theory showed that longitudinal instabilities can be triggered in a very narrow region outside the linear stability limits. The extreme narrowness of the nonlinearly unstable region suggests that from a practical point of view, the longitudinal stability limits of most engines are adequately described by the linear stability limits.

A correlation technique, that can be used to correlate the analytical results with experimental data, and a semi-empirical method for predicting the waveforms of the instability, have been developed.

APPENDIX A

PROGRAM SPAINT: EVALUATES THE SPACE INTEGRALS RESULTING

FROM THE APPLICATION OF THE GALERKIN METHOD

Statement of the Problem

Program SPAINT uses a Simpson's rule integration algorithm to evaluate the space integrals resulting from the application of the Galerkin method. A linear ramp Mach number distribution, $\bar{u}(z) = \bar{u}_e \times z$, is used. The computed integrals are stored in a data file which is used as input data in Program WAVES. The program user must specify the step size to be used in the numerical integration, and the number of terms retained in the series expansion(s) of the dependent variable(s).

The space integrals to be evaluated are given in Eqs. (22) through (26). The following definitions are made for the purpose of computer storage assignment:

Array	Integral	Index (K)	
$T2(1,N,L)$	$= \int_0^1 z \sin(n\pi z) \cos(\ell\pi z)\, dz$	0	(A.1)
$T3(1,N,M,L)$	$= \int_0^1 \sin(n\pi z) \sin(m\pi z) \cos(\ell\pi z)\, dz$	1	(A.2)
$T3(2,N,M,L)$	$= \int_0^1 \cos(n\pi z) \cos(m\pi z) \cos(\ell\pi z)\, dz$	2	(A.3)

Array	Integral	Index (K)	
$T3(3,N,M,L) = \int_0^1 z\cos(n\pi z)\sin(m\pi z)\cos(\ell\pi z)\,dz$		3	(A.4)
$T3(4,N,M,L) = \int_0^1 z\sin(n\pi z)\sin(m\pi z)\sin(\ell\pi z)\,dz$		4	(A.5)

The array indices N, M, and L vary from one to NEQ, where NEQ is the number of terms retained in the series expansion(s) of the dependent variable(s). As coded in this report, NEQ ≤ 10. It is recommended that a value of NEQ = 10 be used regardless of the number of terms in the series. The reason for this choice is discussed in the section of this appendix entitled "Recommendations on Program Usage".

A standard Simpson's rule numerical integration algorithm (see, for example, Conte[10]) is used to evaluate the integrals. In this procedure, the interval [0,1] is divided into 2N subintervals of length h and the integral is evaluated using the following equation:

$$\int_0^1 f(z)\,dz = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 4f_{2N-1} + f_{2N}\right]$$

The error involved in this numerical integration scheme is of the order of h^4. The user specifies h, and h must be such that the interval [0,1] is divided into an even number of subintervals.

Input and Output

The required input data consist of the number of terms in the series expansion(s) of the dependent variable(s), NEQ, and the integration step size, HI. The input data is read into the computer from two data cards:

Card 1: NEQ, integer, is right justified in columns 1-10 (Format I10) and NEQ ≤ 10

Card 2: HI, floating point number, in columns 1-10 (Format F10.0)

The computed integrals are stored in an assigned data file (see the section on the Deck set-up) and are printed in a straightforward output format. The notation used in the printed output is self-explanatory: L, N, and M are array indices (M = 0 for integral (A.1)) and K is the index which defines the integrand (e.g., K = 0 for integral (A.1), etc.).

A typical set of input data and a portion of the printed output are respectively shown in Tables A.1 and A.2.

Deck Set-up

The deck set-up described herein is for the Univac 1108 Exec 8 system used at Georgia Tech. The manner in which data files are assigned might be different at other computer facilities. The important thing to note is that the data file number (I/O unit) assigned to the output data of this program is used as the input data file number in program WAVES. This program uses I/O unit 2 to store the data file. Deck Set-up:

1. Run Card (I.D. Card)

TABLE A.1. Sample Input For Program SPAINT

COLUMN		
1-10	11-20	
10		
.02		

TABLE A.2. Sample Output From Program SPAINT

SPACE INTEGRALS   STEP SIZE = .020   L = 1

OUTPUT FORMAT   INTEGRAL FROM 0 TO 1 OF F(X)

```
K=0 IS F(X) = X*SIN(N*PI*X)*COS(L*PI*X)
K=1 IS F(X) = SIN(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)
K=2 IS F(X) = COS(N*PI*X)*COS(M*PI*X)*COS(L*PI*X)
K=3 IS F(X) = X*COS(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)
K=4 IS F(X) = X*SIN(N*PI*X)*SIN(M*PI*X)*SIN(L*PI*X)
```

K	N=/M	1	2	3	4	5	6	7	8	9	10
0	0	-.7958-01	.2122+00	-.1119+00	.8488-01	.6632-01	.5457-01	-.4643-01	.4044-01	.3583-01	.3219-01
1	1	-.2196-07	.2500+00	-.1317-07	-.9289-10	.7199-09	-.8316-09	-.1478-08	-.4532-08	.6824-09	.3639-08
1	2	-.2500+00	-.3278-07	.2500+00	-.8438-08	-.1608-09	-.1994-09	-.3811-08	.6109-08	.3478-08	.2187-08
1	3	-.1317-07	-.2500+00	-.3504-07	.2500+00	-.1333-07	-.1317-08	.3451-08	-.6779-08	.6782-03	.7750-08
1	4	-.9289-10	.8438-08	-.2500+00	-.2993-07	.2500+00	-.8669-08	.8070-08	.3699-08	.3429-03	.5864-08
1	5	-.7199-09	-.1608-09	.2500+00	.2500+00	-.3095-07	-.2500+00	-.3922-08	-.1686-08	-.1145-09	.2232-08
1	6	-.8316-09	-.1994-09	-.1317-08	-.8669-08	.2500+00	-.3288-07	-.2500+00	-.1153-07	-.2927-09	-.1245-08
1	7	-.1478-08	-.3811-08	.3451-08	.8070-08	-.1686-08	.2500+00	-.3030-07	.2500+00	.1406-07	.1227-07
1	8	-.4532-08	.6109-08	-.6779-08	.3699-08	-.1153-07	.2500+00	.2500+00	-.3452-07	.2500+00	.1918-07
1	9	.6824-09	.3478-08	.6782-08	.3429-08	-.1145-08	-.2927-08	.1406-07	-.3452-07	.3974-07	.2500+00
1	10	-.3639-08	-.2187-08	.7750-08	.5864-08	.9596-06	-.1245-08	.1227-07	-.1918-07	.2500+00	-.3336-07
2	1	.9191-06	-.2500+00	.9509-06	.9505-06	.9330-06	.9656-06	.9692-06	.9561-06	.9652-06	.9636-06
2	2	.2500+00	.9362-06	.2500+00	.2500+00	.2500+00	.9481-06	.9724-06	.9604-06	.9644-06	.9644-06
2	3	.9509-06	.9505-06	.9326-06	.9293-06	.9700-06	.2500+00	.9700-06	.9608-06	.9632-06	.9672-06
2	4	.9668-06	.9596-06	.2500+00	.2500+00	.9688-06	.9545-06	.2500+00	.9545-06	.9604-06	.9620-06
2	5	.9592-06	.9620-06	.9509-06	.9481-06	.9604-06	.9286-06	.9322-06	.9362-06	.9644-05	.9632-06
2	6	.9656-06	.9692-06	.9628-06	.9700-06	.9644-06	.2500+00	.2500+00	.2500+00	.9692-06	.9604-06
2	7	.9672-06	.9604-06	.9724-06	.9688-06	.9632-06	.9545-06	.9517-06	.2500+00	.9517-06	.9523-06
2	8	.9588-06	.9561-06	.9604-06	.9604-06	.9688-06	.9692-06	.9523-06	.9553-06	.9441-06	.9553-06
2	9	.9652-06	.9644-06	.9561-06	.9620-06	.9644-06	.9604-06	.9517-06	.2500+00	.2500+00	.2500+00
2	10	-.9636-06	.9644-06	.9632-06	.9632-06	.9644-06	.9692-06	.9523-06	.2500+00	.9441-06	.9386-06
3	1	.1061+00	-.1989-01	.6366-01	-.2652-01	.1516-01	.9944-02	-.7069-02	.5299-02	-.4127-02	.3307-02
3	2	.9947-01	.4244-01	.6366-02	-.7882-01	-.3647-01	-.2222-01	-.1524-01	.1120-01	.8606-02	-.6836-02
3	3	-.1485+00	-.7295-01	-.2729-01	.1657-01	.8589-01	.4177-01	-.2635-01	-.1855-01	-.1391-01	.1087-01
3	4	-.9234-01	-.1334+00	-.6300-01	-.2022-01	-.2187-01	-.9001-01	.4507-01	-.2906-01	.2081-01	-.1582-01
3	5	.6973-01	-.8290-01	.1263+00	.5770-01	.1609-01	.2518-01	-.9272-01	-.4733-01	-.3097-01	-.2245-01
3	6	.5637-01	.6266-01	.7760-01	.1222+00	.5440-01	.1338-01	.2744-01	.9463-01	.4897-01	-.3239-01
3	7	-.4750-01	-.5108-01	.5853-01	-.7429-01	.1195+00	.5214-01	.1147-01	.2908-01	.9605-01	.5021-01
3	8	-.4113-01	-.4338-01	-.4777-01	.5582-01	-.7203-01	.1176+00	.5050-01	.1005-01	.3032-01	-.9714-01
3	9	-.3631-01	-.3782-01	-.4067-01	-.4551-01	.5391-01	-.7039-01	.1162+00	.4926-01	.8965-02	.3127-01
3	10	-.3253-01	-.3360-01	-.3556-01	-.3876-01	-.4387-01	.5249-01	-.6916-01	.1151+00	.4830-01	.8114-02
4	1	.2122+00	.5968-01	-.4244-01	-.1326-01	-.6064-01	-.3317-02	-.2022-02	-.1328-02	-.9203-03	.6651-03
4	2	-.5968-01	.1698+00	-.4244-01	-.4851-01	-.1658-01	-.8086-02	-.4645-02	-.2943-02	-.1993-02	-.1417-02
4	3	-.4244-01	-.4642-01	.1637+00	-.4310-01	-.5053-01	-.1791-01	-.9007-02	-.5310-02	-.3439-02	.2374-02
4	4	-.1326-01	-.4851-01	-.4310-01	-.1617+00	-.4177-01	-.1658-01	-.1857-01	-.9503-02	-.5691-02	-.3739-02
4	5	-.6064-01	-.4658-01	-.5053-01	-.4177-01	-.1608-01	-.5145-01	-.5195-01	-.5195-01	-.9803-02	.5931-02
4	6	-.3317-02	.8086-02	-.1791-01	-.1857-01	.5440-01	-.4111-01	-.4073-01	-.5224-01	-.1919-01	-.9999-01
4	7	-.2022-02	-.4645-02	-.9007-02	-.5145-01	.5195-01	-.4073-01	-.1600+00	-.4049-01	-.5244-01	.1936-01
4	8	-.1328-02	-.2993-02	-.5053-02	-.1857-01	-.1608-01	-.5224-01	-.4049-01	-.5224-01	-.4033-01	-.5258-01
4	9	-.9203-03	.1993-02	-.3439-02	-.5691-02	-.1895-01	-.1919-01	-.5244-01	-.1598+00	-.1596-01	-.4021-01
4	10	.6651-03	-.1417-02	-.2374-02	-.3739-02	.5931-02	-.9999-02	-.1936-01	-.5258-01	-.4021-01	.1595+00

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2. I/O unit assignment cards.

3. Main Program, MAIN. This program reads the input, calls subroutine SUMM, and outputs the computed integrals.

4. Subroutine SUMM. This program specifies the integrand function, $f(x)$, and calls subroutine SIMPSN.

5. Subroutine SIMPSN. This program performs the Simpson rule integration of $f(x)$. $f(x)$ is defined in the External Real Function Subprogram FOFX.

6. Real Function Subprogram FOFX. This program defines the integrand function $f(x)$ according to the integral index, K.

7. Input Data Cards.

Recommendations on Program Usage

Experience with this program has shown that an integration step size of HI = .02 produces good results. Although NEQ can be varied from 1 to 10, it is recommended that NEQ = 10 be used for the following reason: Using this approach, one data set can be used to compute nonlinear solutions (using program WAVES) for values of NEQ between one and ten. Program WAVES is set-up to use the output generated by program SPAINT in this manner. In summary, it is recommended that values of HI = .02 and NEQ = 10 be used. Approximately 60 seconds of computation time on a U-1108 are required in this case.

```
C
C          THIS PROGRAM EVALUATES THE INTEGRAL OF F(X) FROM 0 TO 1
C          USING SIMPSON RULE
C
C          THE MAIN PROGRAM READS THE INPUT, CALLS SUBROUTINE SUMM,
C          AND OUTPUTS THE COMPUTED INTEGRALS. THE INTEGRALS ARE
C          PRINTED AND STORED IN FILE 3 USING THE FASTRAN SYSTEM.
C          THE F(X) ARE DEFINED WITH THE PRINTED OUTPUT
C          THE SIMPSON RULE INTEGRATION IS PERFORMED IN
C          SUBROUTINE SUMM. THE F(X) ARE DEFINED IN THE EXTERNAL
C          FUNCTION SUBPROGRAM FOFX.
C
C          INPUT DATA
C             CARD 1   IN COL. 1-10 THE STEP SIZE, HI   (ABOUT .01 TO .02)
C             CARD 2   RIGHT JUSTIFIED IN COL. 1-10 THE NUMBER OF
C                      TERMS IN THE SERIES EXPANSION  NEQ< OR = 10
C
C          THE OUTPUT DATA IS DEFINED IN THE PRINTED OUTPUT
C
C          THE COMPUTATION TIME ON THE U-1108 IS ABOUT 60 SEC FOR
C          HI = .02  AND  NEQ = 10.
C
       COMMON/INTER/  T2(1,10,10),T3(4,10,10,10)
  400 FORMAT (8I10)
  402 FORMAT (10X,'OUTPUT FORMAT   INTEGRAL FROM 0 TO 1 OF F(X)',/ )
  403 FORMAT (10X,'K=0 IS F(X) = X*SIN(N*PI*X)*COS(L*PI*X)')
  405 FORMAT (10X,'K=1 IS F(X) = SIN(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)')
  406 FORMAT (10X,'K=2 IS F(X) = COS(N*PI*X)*COS(M*PI*X)*COS(L*PI*X)')
  407 FORMAT (10X,'K=3 IS F(X) = X*COS(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)')
  408 FORMAT (10X,'K=4 IS F(X) = X*SIN(N*PI*X)*SIN(M*PI*X)*SIN(L*PI*X)')
  410 FORMAT (8F10.0)
  430 FORMAT (1H1,   10X,'SPACE INTEGRALS    STEP SIZE = ',F5.3,
     1         4X,'L =',I2,/)
  440 FORMAT (2I5,10E10.4)
  450 FORMAT (/,8X,2HN=,   I5, 9I10, )
  460 FORMAT ('   K     M'/)
  800 FORMAT (5E15.8)
       READ (5,410) HI
       READ (5,400) NEQ
C          INTEGRATION OF SPACE INTEGRALS
       CALL SUMM (NEQ,       HI)
       DO 200 L=1,NEQ
       WRITE (6,430) HI,L
       WRITE (6,402)
       WRITE (6,403)
       WRITE (6,405)
       WRITE (6,406)
       WRITE (6,407)
       WRITE (6,408)
       WRITE (6,450) (I,I=1,NEQ)
       WRITE (6,460)
       M = 0
       K = 1
       J = 0
       WRITE (2,800)       (T2(K,N,L),N=1,NEQ)
```

```
      WRITE (6,440) J,M,(T2(K,N,L),N=1,NEQ)
      DO 220 K=1,4
      DO 230 M=1,NEQ
      WRITE (2,800)      (T3(K,N,M,L),N=1,NEQ)
  230 WRITE (6,440) K,M,(T3(K,N,M,L),N=1,NEQ)
  220 CONTINUE
  200 CONTINUE
      WRITE (2,800) HI
      STOP
      END




      SUBROUTINE SUMM (NEQ,      HI)
      COMMON/INTER/   T2(1,10,10),T3(4,10,10,10)
      NSM=1./HI + 1.01
      DO 100 K=1,5
      IF (K.GT.1) GO TO 1
      MSTP=1
      GO TO 3
    1 MSTP=NEQ
    3 DO 200 L=1,NEQ
      AL = L*3.14159
      DO 210 N=1,NEQ
      AN = N*3.14159
      DO 220 M=1,MSTP
      AM = M*3.14159
      CALL SIMPSN (K,AL,AM,AN,SUM,      NSM,HI)
      IF (K.GT.1) GO TO 4
      T2(K,N,L) = SUM
      GO TO 5
    4 KK = K-1
      T3(KK,N,M,L) = SUM
    5 CONTINUE
  220 CONTINUE
  210 CONTINUE
  200 CONTINUE
  100 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SIMPSN (K,AL,AM,AN,SUM,      NSM,HI)
      EXTERNAL FOFX
      X = 0.0
      SUM = 0.0
      DO 1 I=1,NSM
      C = 1.0
      IF (I.EQ.1) GO TO 2
      IF (I.EQ.NSM) GO TO 2
      C = 4.0
      ID = 2*(I/2) - I
      IF (ID.EQ.0) GO TO 2
      C = 2.0
    2 SUM = SUM + C*FOFX(K,X,AL,AM,AN)
    1 X = X+HI
      SUM = HI*SUM/3.0
      RETURN
      END




      REAL FUNCTION FOFX (K,X,AL,AM,AN)
      GO TO (1,2,3,4,5),K
    1 FOFX = X*SIN(AN*X)*COS(AL*X)
      GO TO 100
    2 FOFX = SIN(AN*X)*SIN(AM*X)*COS(AL*X)
      GO TO 100
    3 FOFX = COS(AN*X)*COS(AM*X)*COS(AL*X)
      GO TO 100
    4 FOFX = COS(AN*X)*SIN(AM*X)*COS(AL*X)*X
      GO TO 100
    5 FOFX = SIN(AN*X)*SIN(AM*X)*SIN(AL*X)*X
  100 CONTINUE
      RETURN
      END
```

APPENDIX B

PROGRAM WAVES: COMPUTES THE COMBUSTION

INSTABILITY OSCILLATION WAVEFORMS

Statement of the Problem

Program WAVES computes the combustion instability oscillation
waveforms for combustors having a linear steady state velocity distri-
bution, $\bar{u}(z) = \bar{u}_e z$, for which $\bar{u}_e$ is small. Before this program can be
used, the space integrals must be evaluated using program SPAINT. The
computed integrals, together with the specification of the engine
operating conditions (i.e., n, $\bar{\tau}$, $\bar{u}_e$, γ, etc.), initial conditions,
and certain program control numbers, make up the required input data
for program WAVES.

Program WAVES performs the following functions:

1. For an initial peak pressure amplitude, initial values of the mode-
    amplitude functions are computed.

2. The time-dependent mode-amplitude functions are found by a Runge-
    Kutta-type numerical integration.

3. Perturbation pressures and velocities are computed.

4. A check for limit cycle conditions is made.

5. Printed and/or plotted output data is generated.

The program provides the user with various options. For instance,
function (3) may be omitted if only the behavior of the mode-amplitude
functions is desired. Similarly, function (4) is omitted when only the

transient behavior of the instabilities is required. The use of these and other user options are discussed in this appendix.

Two nonlinear solutions have been developed in this report: (1) a second order analysis using a nonlinear wave equation, and (2) a large amplitude analysis using a set of three conservation equations. Consequently, two computer programs are required. These programs have been written in a manner which permits a good deal of commonality. In particular, the required input data is the same for all programs. In order to achieve the commonality between the programs, the definitions shown in Table B.1 have been made.

The relations defining the behavior of the functions $A_n(t)$, $B_n(t)$, and $C_n(t)$ are listed in Table B.2.

Program WAVES consist of 11 elements: MAIN, START, POFX, TREND, FLOW, POUT2, POUT, RUNG, EQTN, PRMTRS, and WOUT1. The first seven elements are the same for the three nonlinear solutions. The last four elements are different for each nonlinear solution technique. The functions performed by these elements are discussed in the following paragraphs.

MAIN: Element MAIN serves the twofold functions of (1) reading the data required to compute the nonlinear waveforms, and (2) calling the required subroutines.

START and POFX: These two subroutines provide the initial values of the mode-amplitude functions required for the integration of the ordinary differential equations describing the behavior of the mode-

TABLE B.1. Definition of the Mode-Amplitude Functions

Used in Program WAVES

Array	Parameter	Description
A(N)	$A_n(t)$	Specific volume mode-amplitude function, used only in the large amplitude analysis.
B(N)	$B_n(t)$	Pressure mode-amplitude function. In the analysis using the nonlinear wave equation, $B_n(t)$ represents the time derivative of the velocity potential mode-amplitude function.
C(N)	$C_n(t)$	Acoustic-type velocity mode-amplitude function. In the nonlinear wave equation solutions, $C_n(t)$ represents the velocity potential mode-amplitude function.

TABLE B.2. Equations Governing the Mode-Amplitude Functions

Parameter	Equation Number	
	Wave Equation	Third Order Equations
$A_n(t)$	—	19
$B_n(t)$	27	20
$C_n(t)$	27	21

amplitude functions. It is assumed that the combustor is operating in a steady manner until time $t = 0$, at which time a pressure disturbance is impulsively introduced inside the combustor. The perturbation velocity at $t = 0$ is zero. The user may specify a spacially continuous initial pressure disturbance in any axial mode, or a spacially discontinuous fundamental mode disturbance, with the discontinuity located at $z = .5$ at $t = 0$. The analytical expressions used to find the initial conditions, found by a Fourier analysis of the initial waveform, are given in the following equations:

(1) Spacially Continuous Pulse in the ℓth Axial Mode.

$$C_\ell(t = 0) = 0 \qquad\qquad \ell = 1,\ldots,N \qquad\qquad (B.1)$$

$$B_\ell(t = 0) = \begin{cases} 0 & n \neq \ell \\ p_i & n = \ell \end{cases} \qquad\qquad (B.2)$$

(2) Spacially Discontinuous Pulse.

$$C_\ell(t = 0) = 0 \qquad\qquad \ell = 1,\ldots,N \qquad\qquad (B.3)$$

$$B_\ell(t = 0) = \frac{4p_i}{(\ell\pi)} \sin\left(\frac{\ell\pi}{2}\right) \qquad\qquad (B.4)$$

where in both cases,

$$C_\ell(t) = B_\ell(t) = 0, \quad \text{for} \quad -\bar{\tau} \leq t < 0, \quad \ell = 1,\ldots,N \qquad (B.5)$$

An initial condition on $A_\ell(t)$ is required for the large amplitude

analysis. Consideration of the linear behavior of the system shows

that $\nu = -p/\gamma$; consequently the following relation is used for an

initial condition on $A_\ell(t = 0)$:

$$A_\ell(t = 0) = -B_\ell(t = 0)/\gamma \qquad (B.6)$$

In the solution using the nonlinear wave equation, Eqs. (B.2) and

(B.4) merely approximate the spacial dependence of the initial impulse.

Specifically, these equations are based on a linear representation of

the initial disturbance, and the computed wave amplitude differs by a

factor of γ from the specified p_i.

TREND: This subroutine determines whether or not limit cycle

conditions have been reached. This task is accomplished by evaluating

the summation $S = \sum_{n=1}^{NEQ} B_n(t)$ and examining the behavior of the summation.

Note that S represents the behavior of the injector face pressure.

Subroutine TREND performs the following functions:

1) Determines the maximum (positive) peak amplitude of one cycle of S.

2) Finds two successive average values of S for two cycles, $\bar{S}_1$ and $\bar{S}_2$,

   respectively.

3) Compares the absolute difference, $|\Delta\bar{S}|$, between the two successive

   averages with a user specified percentage, ϵ, of the latter value of the

   average S. If the $|\Delta\bar{S}| < \epsilon\bar{S}_2$ then limit cycle conditions have been

   reached.

4) Makes the appropriate change in the internal program control index

which tells the program that limit cycle conditions have been
reached.

FLOW: Subroutine FLOW computes the summations used to find the
perturbation flow field, outputs the computed pressure and velocity,
and calls subroutine POUT2. The summations computed are:

$$\text{SUMA} = \sum_{n=1}^{\text{NEQ}} A_n(t)\cos(n\pi z)$$

$$\text{SUMB} = \sum_{n=1}^{\text{NEQ}} B_n(t)\cos(n\pi z)$$

$$\text{SUMC} = \sum_{n=1}^{\text{NEQ}} C_n(t)\sin(n\pi z)$$

$$\text{SUMU} = \sum_{n=1}^{\text{NEQ}} (n\pi)C_n(t)\sin(n\pi z)$$

These summations are used in subroutine PRMTRS to calculate the
perturbation flow field.

POUT2 and POUT: Subroutines POUT and POUT2 plot the temporal
behavior of $B(N)$ (the pressure mode-amplitude functions) and the
temporal behavior of the pressure oscillations, respectively. The mode-
amplitude functions to be plotted are specified by the user. The axial
location(s) of the pressure plots are also user specified. The programs
have been developed for use on a CALCOMP plotter.

RUNG: Subroutine RUNG is a modified Runge-Kutta numerical integration algorithm. The modification accounts for the presence of the retarded time variable. In this modification, the retarded variable is treated as a known quantity; that is, it is treated in the same manner as the independent variable. Two algorithms, based on the Runge-Kutta equations developed in reference (12), are used. One algorithm is used to integrate a set of second order O.D.E.'s; the other is used to integrate a set of first order O.D.E.'s. The required expressions are given in the following equations:

(1) First order O.D.E.'s; $y_\ell'^{(j)} = f_\ell^{(j)}\left[y_n, y_n(t-\bar{\tau})\right]$:

$$y_\ell^{(j+1)} = y_\ell^{(j)} + \frac{1}{6}\left\{K_{1_\ell} + K_{4_\ell} + 2\left[K_{2_\ell} + K_{3_\ell}\right]\right\}$$

where

$$K_{1_\ell} = hf_\ell^{(j)}\left[y_n, y_n(t-\bar{\tau})\right]$$

$$K_{2_\ell} = hf_\ell^{(j)}\left[\left(y_n+K_{1_n}/2\right), y_n\left(t-\bar{\tau}+h/2\right)\right]$$

$$K_{3_\ell} = hf_\ell^{(j)}\left[\left(y_n+K_{2_n}/2\right), y_n\left(t-\bar{\tau}+h/2\right)\right]$$

$$K_{4_\ell} = hf_\ell^{(j)}\left[\left(y_n+K_{3_n}\right), y_n\left(t-\bar{\tau}+h\right)\right]$$

and where

$$y_\ell^{(j)} = y_\ell(t)$$

$$y_\ell^{(j+1)} = y_\ell(t+h)$$

and $f_\ell^{(j)}$ is the function evaluated at t.

(2) Second order O.D.E.'s; $y_\ell'' = f_\ell\left[y_n, \ y_n', \ y_n'(t-\bar{\tau})\right]$

$$y_\ell'^{(j+1)} = y_\ell'^{(j)} + \frac{1}{6}\left\{K_{1_\ell} + K_{4_\ell} + 2\left[K_{2_\ell} + K_{3_\ell}\right]\right\}$$

$$y_\ell^{(j+1)} = y_\ell^{(j)} + h\left\{y_\ell'^{(j)} + \frac{1}{6}\left[K_{1_\ell} + K_{2_\ell} + K_{3_\ell}\right]\right\}$$

where

$$K_{1_\ell} = hf_\ell^{(j)}\left[y_n, \ y_n', \ y_n'(t-\bar{\tau})\right]$$

$$K_{2_\ell} = hf_\ell^{(j)}\left[\left(y_n + \frac{h}{2}y_n' + \frac{h}{8}K_{1_n}\right), \ \left(y_n' + \frac{1}{2}K_{1_n}\right), \ y_n'\left(t - \bar{\tau} + \frac{h}{2}\right)\right]$$

$$K_{3_\ell} = hf_\ell^{(j)}\left[\left(y_n + \frac{h}{2}y_n' + \frac{h}{8}K_{1_n}\right), \ \left(y_n' + \frac{1}{2}K_{2_n}\right), \ y_n'\left(t - \bar{\tau} + \frac{h}{2}\right)\right]$$

$$K_{4_\ell} = h f_\ell^{(j)} \left[\left(y_n + h y_n' + \frac{h}{2} K_{3_n} \right), \left(y_n' + K_{3_n} \right), y_n'\left(t - \bar\tau + h \right) \right]$$

and where

$$y_\ell^{(j)} = y_\ell(t) \quad ; \quad y_\ell'^{(j)} = y_\ell'(t)$$

$$y_\ell^{(j+1)} = y_\ell(t+h) \quad ; \quad y_\ell'^{(j+1)} = y_\ell'(t+h)$$

The equations defining the numerical integration of a set of first order O.D.E.'s is used in the solutions of the conservation equations. The second order O.D.E. equations are used to solve the nonlinear wave equation. The functional form of f_ℓ is defined in element EQTN.

In order to use the equations with the retarded variable, the integration step size, h, must be selected such that h divides the time lag, $\bar\tau$, into K equal increments. Thus $\bar\tau = Kh$, and the retarded variables become:

$$y_n(t - \bar\tau) = y_n(t - Kh)$$

$$y_n\left(t - \bar\tau + \frac{h}{2}\right) = y_n\left(t - Kh + \frac{h}{2}\right)$$

$$y_n(t - \bar\tau + h) = y_n(t - Kh + h)$$

It has been found that an integration step size of the order $h \simeq .05$ produces satisfactory results. The program selects the integration step size by forming the ratio $\bar{\tau}/.05$, rounding off the result to the nearest integer, and dividing $\bar{\tau}$ by the resulting integer, that is:

$$\text{integer} = (\bar{\tau}/.05) + .01$$

$$h = \bar{\tau}/\text{integer}.$$

The computation of h is performed in element MAIN.

EQTN: Subroutine EQTN defines the functions, f_ℓ, used in subroutine RUNG to evaluate the K terms. The particular equations defined in EQTN depend upon the problem under consideration (i.e., nonlinear wave equation, etc.). These functions are defined in Table B.2.

PRMTRS: Subroutine PRMTRS uses the summations, SUMA, SUMB, SUMC, and SUMU, computed in subroutine FLOW to calculate the perturbed flow field. The current program is coded to compute the perturbation pressure and velocity, using the following equations:

(1) Nonlinear wave equation solutions:

$$u'(z,t) = -\text{SUMU}$$

$$p'(z,t) = \frac{\gamma}{2} \left[\text{SUMB}(\text{SUMB}-2) + \text{SUMU}(2\bar{u}(z) - \text{SUMU}) \right]$$

(2) Second order conservation equation solutions:

$$p'(z,t) = \text{SUMB}$$

$$u'(z,t) = \text{SUMC} + \frac{\gamma-1}{2\gamma}\, \bar{u}_e Z\, \text{SUMB}$$

(3) Large amplitude solutions:

$$p'(z,t) = \text{SUMB}$$

$$u'(z,t) = \text{SUMC} + \left[1 - \frac{\gamma+1}{4\gamma}\, \text{SUMB}\right] \frac{\gamma-1}{2\gamma}\, \bar{u}_e Z\, \text{SUMB}$$

WOUT1: This program writes the output of the mode-amplitude functions.

Input Data

The required input data consist of the integral values computed by program SPAINT, the engine operating conditions, and program control numbers. The data from program SPAINT is automatically read from data file 2. The remaining data is read from user supplied data cards. These cards are described in this section.

Card 1   (Format 8I10)

Column[1]	Term	Data Type[2]	Information	Restrictions
10	NEQ	I	No. of terms in the series expansion of the dependent variables	≤ 10
20	NX	I	No. of axial locations at which flow field is to be computed	≤ 11
30	LIN	I	LIN = 1 to compute linear solutions LIN $\neq$ 1 nonlinear solutions	
40	IPLOT	I	IPLOT = 1 if any data is to be plotted IPLOT $\neq$ 1 no plots	
50	INPT	I	INPT = 1 to write the space integrals INPT $\neq$ 1 space integrals are not written	

Card 2   (Format 8I10)

Column	Term	Data Type	Information	Restrictions
10	LC1	I	LC1 = 1 to write the mode-amplitude functions LC1 $\neq$ 1 mode-amplitude functions are not written	
20	LC2	I	LC2 = 1 to plot pressure mode-amplitude functions LC2 = 4 no plot of mode-amplitudes	
30	LC4	I	Number of terms to be plotted	≤ 10
40	LC5	I	Incremental index between terms to be plotted	≤ 9

1. For integer data, indicates the column in which data is right justified.
2. I denotes integer data; F denotes floating point (decimal) data.

Card 3 (Format 8I10)

Column	Term	Data Type	Information	Restrictions
10	LP1	I	LP1 = 1 to calculate p' and u' LP1 $\neq$ 1 flow field is not calculated	
20	LP2	I	LP2 = 1 to write p' and u' LP2 $\neq$ 1 p' and u' are not written	
30	LP3	I	LP3 = 1 to plot p' vs t LP3 = 4 no flow field (p') plot	
40	LP4	I	Number of axial locations at which p' vs t is to plotted	≤ 4

Card 4 (Format 8I10)

Column	Term	Data Type	Information	Restrictions
10	NTAU	I	Number of $\bar{\tau}$ to be run	

Card 5 (Format 8F10.0)

Column	Term	Data Type	Information	Restrictions
1-10	UE	F	Exit Mach number	small, $\ll 1$
11-20	GAMMA	F	Specific heat ratio	
21-30	EPS	F	Limit cycle amplitude percent error	EPS = $\mathcal{O}(.01)$

Card 6 (Format 8F10.0)

Column	Term	Data Type	Information	Restrictions
1-10	TBEGIN	F	Normalized time at which output is begun, and at which flow field calculation is started	see discussion

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Column	Term	Data Type	Information	Restrictions
11-20	TEND	F	Normalized time at which computations are terminated	see discussion
21-30	TLMCY	F	Normalized time at which limit cycle check is begun	see discussion
31-40	DELTAT	F	Normalized time increment for output of limit cycle conditions	see discussion
41-50	TSMP	F	Normalized time at which plot of pressure mode-amplitude is begun	
51-60	DELPT	F	Normalized time increment for plot of pressure mode-amplitude, $B(N)$ vs t	see discussion

Discussion of Card 6:

(1) BEGIN must be greater than or equal zero. TEND must be such that the ratio (TEND-TBEGIN)/H is less than 300. This ratio can be estimated using a value of H = .05. Experience has shown that a time increment of TEND-TBEGIN $\doteq$ 12. is sufficient to determine the behavior of the solutions.

(2) If a limit cycle check is not desired, then set TLYMCY > TEND.

(3) DELTAT must be such that DELTAT/H < 300. Usually, a DELTAT $\doteq$ 6 is sufficient to verify that limit cycle conditions have been reached. In this case, approximately three fundamental mode cycles are computed.

(4) If a limit cycle check is made, and if limit cycle conditions are found, TSMP is automatically set equal to the initial time at

which limit cycle conditions are found, if LC2 = 1.

(5) DELPT must be such that DELPT/H < 100. Good results have been obtained using DELPT $\doteq$ 3.9.

(6) If a limit cycle check is made, and limit cycle conditions are not found, the data output begins at TBEGIN and ends at TEND.

Card 7 (Format 8F10.0)

Column	Term	Data Type	Information	Restrictions
1-10	X(1)	F	Axial location at which p' and u' are computed	≤ 1
11-20	X(2)	F	Axial location at which p' and u' are computed	≤ 1
⋮	⋮	⋮	⋮	⋮
⋮	X(NX)	⋮	⋮	⋮

Discussion:

If NX > 8, then two cards are required to complete the input of X(I). In this case, X(9) is in columns 1-10 of card 7B, and so on.

Card 8 (Format 8I10)

This card is included in the data set only when LP3 = 1.

Column	Term	Data Type	Information	Restrictions
10	IPX(1)	I	Index of X(I) at which a p' vs t plot is made	≤ 10
20	IPX(2)	I	Index of X(I) at which a p' vs t plot is made	≤ 10
⋮				
40	IPX(LP4)	I	Index of X(I) at which a p' vs t plot is made	≤ 10

Discussion:

Plots can be made at any four (or fewer) axial locations at which p' is calculated.

Card 9 (8)[1] (F10.0, 2I10)

Column	Term	Data Type	Information	Restrictions
1-10	TAU	F	Sensitive time lag	
20	NNB	I	Number of n to be run at the specified $\bar{\tau}$	≤ 10
30	LCUT	I	Highest mode in which energy feedback is permitted	see discussion

Discussion:

This number is used to eliminate the secondary zones of instability. For fundamental mode investigations, LCUT = 2 is usually appropriate. For $\bar{\tau} > 1$, energy feedback is only permitted in the fundamental mode.

Card 10 (9) (8F10.0)

Column	Term	Data Type	Information	Restrictions
1-10	ANR(1)	F	First value of n	
11-20	ANR(2)	F	Second n	
$\vdots$	$\vdots$		$\vdots$	
$\vdots$	ANR(NNB)	F	Final value of n	

Discussion:

If NNB > 8, then two cards are used to input the ANR(I).

---

1. Number in parenthesis is the card number if card 8 (IPX(I) card) is omitted.

Card 11 (10) (8I10)

Column	Term	Data Type	Information	Restrictions
10	NPI	I	Number of initial disturbances for each n-$\bar{\tau}$ condition	

Card 12 (11) (F10.0, 8I10)

Column	Term	Data Type	Information	Restrictions
1-10	PI	F	Initial disturbance peak amplitude	
20	IPOP	I	If IPOP ≤ 10, then an initial disturbance in the IPOP mode is generated If IPOP = 11, then a spacially discontinuous fundamental mode wave, with the discontinuity at z = .5, is generated	

This completes the description of the input data cards. If NPI > 1, then card 12(11) is repeated NPI times. When NNB > 1, then cards 11(10) and 12(11) must be repeated NNB times. Similarly, when NTAU > 1, card 10(9) through 12(11) must be repeated NTAU times. An example input data set is shown in Table B.3.

Using the input data shown in Table B.3, program WAVES performs the following functions:

1. Nonlinear solutions are found at two axial locations using eight term expansion(s). The exit Mach number is $\bar{u}_e$ = 0.2, and γ = 1.2.

2. The mode-amplitude functions are printed, and the first pressure mode-amplitude function is plotted.

3. The perturbation pressure and velocity are computed at z = 0.0 and

TABLE B.3.   Sample Input Data For Program WAVES

COLUMN					
1-10	11-20	21-30	31-40	41-50	51-60
8	2	2	1	1	
1	1	1	1		
1	1	1	1		
2					
0.2	1.2	0.01			
30.0	42.0	5.0	6.0	30.0	3.9
0.0	0.25				
1					
1.30	2	2			
1.18	1.30				
2					
.025	1				
.05	1				
1					
.15	11				
1.0	1	2			
1.10					
1					
.1	11				

z = 0.25. The results are printed and the temporal dependence of the pressure oscillation is plotted at z = 0.0.

4. A limit cycle check is initiated at t = 5.0. If limit cycle conditions are reached prior to t = 30.0, the required data is output in a time interval of Δt = 6.0 after the establishment of limit cycle conditions. On the other hand, if a limit cycle is not reached by t = 30.0, the data is output in the time interval $30 \leq t \leq 42.0$. The pressure mode-amplitude function is plotted over at time interval of Δt = 3.9.

5. Solutions are to be calculated for two values of $\bar{\tau}$. At the first $\bar{\tau}$ ($\bar{\tau}$ = 1.30), the computations are to be made for two n (n = 1.18 and n = 1.30). The computations at $\bar{\tau}$ = 1.30, n = 1.18 are to be made using two initial disturbances; a .025 and a .05 peak amplitude 1L pressure wave. The computations at $\bar{\tau}$ = 1.30, n = 1.30 are made for a discontinuous 1L pressure wave of peak amplitude equal to .15. At the second $\bar{\tau}$ ($\bar{\tau}$ = 1.0), the computations are made for an n = 1.10 and a discontinuous, .1 peak amplitude pressure wave.

6. In both cases, energy feedback is only permitted in the first two axial modes.

Output Data

The following data output options are available:

(1) INPT = 1 causes the space integrals used in the computations to be written.

(2) LC1 = 1 results in a tabulated output of the mode-amplitude

functions.

(3) LP2 = 1 results in the listing of p′ and u′ as functions of t at each axial location specified by X(I).

(4) LC2 = 1 causes plots of B(N) vs time to be made, with the N's specified by the user.

(5) LP3 = 1 causes plots of p′ vs time to be made at the axial locations X(I) specified by the indices IPX(I).

The output limitations have been discussed in the data input section of this appendix. The output symbols are described in Table B.4. Portions of an example output is shown in Table B.5.

Deck Set-up

The data set described herein is for the Univac 1108 Exec. 8 system as used at Georgia Tech. The important points are:

1. Unit 2 must be assigned to the data file SPAINT.

2. Unit 3 must be assigned to the CALCOMP PLOT subroutines.

It is convenient to group the program elements in the sequence in which they are discussed in the first section of this appendix (i.e., page 37). The program is then adapted to the solution of a particular formulation of the problem (i.e., second order wave equation, etc.) by changing the last four subroutines.

TABLE B.4. Output Symbols

<u>Symbol</u>

A	(1) potential mode-amplitude function, or (2) specific volume mode-amplitude function
AP	time derivative of the potential mode-amplitude
B	pressure mode-amplitude function
C	velocity mode-amplitude function
LINEAR	LINEAR = 1, solutions are linear LINEAR $\neq$ 1, solutions are nonlinear
L	axial mode number
M	axial mode number
N	(1) axial mode number, or (2) interaction index
NEQ	number of terms used in the solutions
P	normalized perturbation pressure
PINITIAL	peak amplitude of the initial disturbance
TAU	sensitive interaction index, $\bar{\tau}$
Z	axial station

TABLE B.5. Sample Output From Program WAVES:
Part of the Space Integrals Used in the Calculations

SPACE INTEGRALS   STEP SIZE = .020   L = 1

OUTPUT FORMAT   INTEGRAL FROM 0 TO 1 OF F(X)

K=0 IS F(X) = X*SIN(N*PI*X)*COS(L*PI*X)
K=1 IS F(X) = SIN(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)
K=2 IS F(X) = COS(N*PI*X)*COS(M*PI*X)*COS(L*PI*X)
K=3 IS F(X) = X*COS(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)
K=4 IS F(X) = X*SIN(N*PI*X)*SIN(M*PI*X)

K	M	N=1	2	3	4	5	6	7	8
0	0	-.7958-01	.2122+00	-.1194+00	.8488-01	-.6632-01	.5457-01	-.4643-01	.4044-01
1	1	-.2196-07	.2500+00	-.1317-07	-.9289-10	-.7199-09	-.8316-09	-.1478-08	-.4532-08
1	2	.2500+00	-.3278-07	.2500+00	-.8438-08	-.1608-08	-.1994-09	-.3811-08	-.6109-08
1	3	-.1317-07	.2500+00	-.3504-07	.2500+00	-.1333-07	-.1317-08	.3451-08	.6779-08
1	4	-.9289-10	.8438-08	.2500+00	.2993-07	.2500+00	-.8669-08	.8070-08	.3699-08
1	5	.7199-09	-.1608-09	-.1333-07	.2500+00	-.3095-07	.2500+00	.3922-08	-.1686-08
1	6	-.8316-09	-.1994-09	-.1317-08	-.8070-08	.2500+00	-.3288-07	.2500+00	-.1153-07
1	7	-.1478-08	.3811-08	.3451-08	.8070-08	.3922-08	.2500+00	-.3030-07	.2500+00
1	8	-.4532-08	.6109-08	.6779-08	.3699-08	-.1686-08	-.1153-07	.2500+00	-.3462-07
2	1	.9191-06	.2500+00	.9509-06	.9668-06	.9592-06	.9656-06	.9672-06	.9580-06
2	2	.2500+00	.9362-06	.2500+00	.9505-06	.9596-06	.9620-06	.9692-06	.9561-06
2	3	.9509-06	.2500+00	.9326-06	.2500+00	.9509-06	.9628-06	.9724-06	.9604-06
2	4	.9668-06	.9505-06	.2500+00	.9298-06	.2500+00	.9481-06	.9700-06	.9608-06
2	5	.9792-06	.9796-06	.9509-06	.2500+00	.9330-06	.2500+00	.9700-06	.9648-06
2	6	.9656-06	.9620-06	.9628-06	.9481-06	.9700-06	.2500+00	.9286-06	.9545-06
2	7	.9672-06	.9692-06	.9724-06	.9700-06	.2500+00	.2500+00	.9322-06	.2500+00
2	8	.9588-06	.9561-06	.9604-06	.9603-06	.9688-06	.9545-06	.2500+00	.9362-06
3	1	-.1061+00	.1989-01	-.6366-01	.2652-01	-.1516-01	-.9944-02	-.706-02	-.5299-02
3	2	-.9947-01	.4244-01	.6630-02	-.7824-01	.3647-01	-.2222-01	-.1524-01	-.1120-01
3	3	-.1485+00	-.7295-01	-.2729-01	-.1657-01	-.8589-01	.4177-01	-.2645-01	.1855-01
3	4	-.9284-01	.1334+00	-.6300-01	-.2022-01	-.2187-01	-.9001-01	.4507-01	-.2906-01
3	5	.6973-01	.8290-01	-.1263+00	-.5770-01	.1609-01	.2518-01	.9272-01	-.4733-01
3	6	-.5637-01	-.6266-01	.7760-01	-.1222+00	-.5040-01	-.1338-01	-.2744-01	.9465-01
3	7	.4750-01	-.5108-01	-.5053-01	-.7429-01	-.1195+00	-.5214-01	-.1147-01	-.2908-01
3	8	-.4113-01	-.4338-01	.4777-01	-.5582-01	-.7203-01	-.1176+00	-.5050-01	-.1005-01
4	1	-.2122+00	-.5968-01	-.4244-01	-.1326-01	-.6064-02	-.3317-02	-.2022-02	-.1328-02
4	2	-.5968-01	.1699+00	.4642-01	-.4851-01	.1658-01	.8086-02	.4645-02	-.2943-02
4	3	-.4244-01	.4642-01	.1637+00	-.4310-01	-.5053-01	.1791-01	-.9007-02	.5310-02
4	4	.1326-01	-.4851-01	.4310-01	-.1617+00	-.4177-01	-.5145-01	-.1857-01	-.9503-02
4	5	-.6064-02	.1658-01	-.5053-01	-.4177-01	-.1608+00	-.4111-01	-.5195-01	-.1895-01
4	6	.3317-02	-.6086-02	.1791-01	-.5145-01	-.4073-01	-.1603+00	-.4073-01	-.5224-01
4	7	-.2022-02	-.4645-02	-.9007-02	-.1857-01	-.5195-01	-.4073-01	-.1600+00	-.4049-01
4	8	.1328-02	-.2943-02	.5310-02	-.9503-02	.1895-01	-.5224-01	-.4049-01	.1598+00

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INITIAL PRESSURE DISTRIBUTION

Z	P
.00000	.09216
.10000	.10684
.20000	.09628
.30000	.09790
.40000	.11386
.50000	.00000
.60000	-.11386
.70000	-.09790
.80000	-.09628
.90000	-.10684
1.00000	-.09216
1.10000	-.10684

TABLE B.5 (cont.). Sample Output From Program WAVES:
Part of the Mode-Amplitude Output

TIME DEPENDENT COEFFICENTS OF THE NONLINEAR WAVE EQUATION   PHI = A(T)*COS(N*PI*Z)

LINEAR= 2   NEQ= 8   N= 1.10000   TAU= 1.0000n
EXIT MACH= .200   GAMMA =1.200   PINITIAL =.1000

| TIME | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 |
TIME	AP1	AP2	AP3	AP4	AP5	AP6	AP7	AP8	AP9	AP10
-.000	.0000	.0000	.0000	.0000	.0000	.0000	.000n	.0000		
-.000	.1273+00	-.4685-06	-.4244-01	.4685-06	.2546-01	-.4685-06	-.1819-01	.4685-06		
.050	.6344-02	-.8014-04	-.2026-02	.7293-04	.1135-02	-.7628-04	-.7025-03	.2252-04		
.050	.1258+00	-.2797-02	-.3715-01	.2498-02	.1759-01	-.2265-02	-.7514-02	.4591-03		
.100	.1252-01	-.2486-02	-.3579-02	.2026-03	.1566-02	-.1221-03	-.6213-03	.7215-06		
.100	.1208+00	-.3786-02	-.2382-01	.2302-02	-.1244-02	-.1102-02	.1013-01	-.1298-02		
.150	.1837-01	-.4508-03	-.4315-02	.2662-03	.1026-02	.4913-04	.8884-04	-.6046-04		
.150	.1124+00	-.4226-02	-.4982-02	-.5158-C4	-.1915-01	.5260-02	.1561-01	-.6534-03		
.200	.2372-01	-.6562-03	-.4052-02	.1742-03	-.1300-03	.2975-03	.6341-03	-.1770-04		
.200	.1011+00	-.3805-02	.1533-01	-.3720-02	-.2457-01	.3592-02	.4439-02	.2285-02		
.250	.2843-01	-.8089-03	-.2844-02	.9987-03	-.1159-02	.3110-03	.4624-03	.108n-03		
.250	.8693-01	-.2079-02	.3196-01	-.6517-02	-.1449-01	-.3353-02	-.1045-01	.2065-02		
.300	.3236-01	-.8415-01	-.9987-03	-.4117-02	-.1432-02	-.7751-06	-.1935-03	-.1084-03		
.300	.6956-01	.9952-03	.4020-01	-.5562-02	.3955-02	-.8178-02	-.1337-01	-.2241-02		
.350	.3534-01	-.6938-03	.1004-02	-.5762-03	.8198-03	-.3670-03	-.60a3-03	-.7294-04		
.350	.4961-01	.4967-02	.3828-01	.5672-03	.1912-01	-.5263-02	-.1806-02	.4180-02		
.400	.3731-01	-.3579-03	.2683-02	.4554-03	.2697-03	-.4309-03	-.3372-03	-.1905-03		
.400	.2088-01	.8170-02	.2771-01	.5168-02	.2221-01	.2927-02	.1161-01	.1090-03		
.450	.3924-01	.7984-04	.36a1-02	-.1124-03	.1167-02	-.1182-02	.3230-03	-.4796-04		
.450	.8549-02	.8882-02	.1163-01	.7883-02	.1199-01	.8625-02	.1241-01	.4977-02		
.500	.3818-01	.4912-03	.3826-02	.2620-03	.1355-02	.2948-03	.6508-03	.1798-03		
.500	-.1049-01	.7307-02	-.5579-02	.6511-02	-.4656-02	.6702-02	-.4855-03	.3078-02		
.550	.3723-01	.7942-03	.3177-02	.4935-03	.7662-03	.4511-03	.2709-03	.1718-03		
.550	-.2703-01	.4702-02	-.1967-01	.2491-02	-.1762-01	-.8130-01	-.133A-01	-.3345-02		
.600	.3550-01	.9996-03	.1943-02	.4992-03	-.2253-03	.2310-03	-.4314-03	-.6971-04		
.600	-.4192-01	.1423-02	-.2882-01	-.2207-02	-.2022-01	-.7306-02	-.1224-01	-.5179-02		

FLOW PRARAMETERS		Z=	.000	
LINEAR= 2	NEQ= 8	N= 1.10000		TAU= 1.0000n
EXIT MACH= .200	GAMMA =1.200	PINITIAL =.1000		

TIME	PRESSURE	VELOCITY
-.000	-.1055+00	.0000
.050	-.11n4+00	.0000
.100	-.11a5+00	.0000
.150	-.11a5+00	.0000
.200	-.1052+0u	.0000
.250	-.9664-01	.0000
.300	-.98n7-01	.0000
.350	-.11n2+00	.0000
.400	-.1213+00	.0000
.450	-.8656-01	.0000
.500	-.2861-02	.0000
.550	.9295-01	.0000
.600	.1479+00	.0000
.650	.14n7+00	.0000
.700	.11n4+00	.0000
.750	.9950-01	.0000
.800	.1125+00	.0000
.850	.13n8+00	.0000
.900	.13n9+00	.0000
.950	.1168+00	.0000
1.000	.10n3+00	.0000
1.050	.1143+00	.0000
1.100	.1176+00	.0000
1.150	.1094+00	.0000
1.200	.9893-01	.0000
1.250	.9826-01	.0000
1.300	.10n7+00	.0000
1.350	.9221-01	.0000
1.400	.7471-01	.0000
1.450	.5890-01	.0000
1.500	.3819-01	.0000
1.550	-.4975-02	.0000
1.600	-.6246-01	.0000
1.650	-.10n2+00	.0000
1.700	-.1010+00	.0000
1.750	-.3786-01	.0000
1.800	-.9498-01	.0000
1.850	-.12n1+00	.0000
1.900	-.13n9+00	.0000
1.950	-.1133+00	.0000
2.000	-.9116-01	.0000
2.050	-.9565-01	.0000

```
C
C     THE SPACE INTEGRALS ARE STORED IN THE ARRAYS T2 AND T3. PROVISION
C     IS MADE FOR ONE N BY N INTEGRAL, AND FOUR N BY N BY N INTEGRALS.
C     MORE INTEGRALS CAN BE TREATED BY CHANGING THE APPROPRIATE DIMENSION
C     STATEMENT.
C
C     THE MODE AMPLITUDES ARE STORED IN THE ARRAYS A, B, AND C. THE RETARDED
C     VARIABLE IS STORED IN ARRAY BS. THE RETARDED VARIABLES REQUIRED AT
C     THE INTEGRATION STEP IN QUESTION ARE STORED IN ARRAYS BR1, BR2, AND BR3.
C     THE TERMS STORED IN THESE ARRAYS DEPENDS ON THE PROBLEM FORMULATION.
C        1. FOR THE NONLINEAR WAVE EQUATION
C           A = BLANK
C           B = TIME DERIVATIVE OF MODE AMPLITUDE
C           C = MODE-AMPLITUDE FUNCTION
C        2. FOR THE SECOND ORDER CONSERVATION EQUATIONS
C           A = BLANK
C           B = PRESSURE MODE-AMPLITUDE
C           C = VELOCITY MODE-AMPLITUDE
C        3. FOR THE LARGE AMPLITUDE ANALYSIS
C           A = SPECIFIC VOLUME MODE-AMPLITUDE
C           B = PRESSURE MODE-AMPLITUDE
C           C = VELOCITY MODE-AMPLITUDE
C
      DIMENSION ANR(10),DATA(2500)
      COMMON/COMP1/ QP1,QP2,QP3,QP4,QP5,QP7
      COMMON/COMS1/ Q1,Q2,GP1,GT1
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,TN(10),IPOP
      COMMON/PLTDA/ TARY(100),BARY(10,100)
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
      COMMON/COM3/ H,HD2,HD6,HD8
      COMMON/COM4/ T2(1,10,10),T3(4,10,10,10)
      COMMON/COM5/ X(11),IPX(4)
      COMMON/COM6/ TSTART,TSTOP,TLYMCY
  400 FORMAT (8I10)
  402 FORMAT (10X,'OUTPUT FORMAT    INTEGRAL FROM 0 TO 1 OF F(X)',/ )
  403 FORMAT (10X,'K=0 IS F(X) = X*SIN(N*PI*X)*COS(L*PI*X)')
  405 FORMAT (10X,'K=1 IS F(X) = SIN(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)')
  406 FORMAT (10X,'K=2 IS F(X) = COS(N*PI*X)*COS(M*PI*X)*COS(L*PI*X)')
  407 FORMAT (10X,'K=3 IS F(X) = X*COS(N*PI*X)*SIN(M*PI*X)*COS(L*PI*X)')
  408 FORMAT (10X,'K=4 IS F(X) = X*SIN(N*PI*X)*SIN(M*PI*X)*SIN(L*PI*X)')
  410 FORMAT (8F10.0)
  430 FORMAT (1H1,   10X,'SPACE INTEGRALS     STEP SIZE = ',F5.3,
     1         4X,'L =',I2,/)
  431 FORMAT (/,8X,2HN=,   I5, 9I10, )
  432 FORMAT (' K     M'/)
  433 FORMAT (2I5,10E10.4)
  450 FORMAT (1H1,///,10X,'DIVERGENT SOLUTION',//)
  460 FORMAT (/,10X,5HTAU= ,F10.5,5X,6HNBAR= ,F10.5,5X,4HUE= ,F10.5,
     1   5X,7HGAMMA= ,F10.5,5X,10HPINITIAL= ,F10.5,
     2//,10X,6HTIME= ,F10.5,5X,6HB(N)= ,E10.4,5X,6HC(N)= ,E10.4)
  800 FORMAT (5E15.8)
  420 FORMAT (F10.0,2I10)
C
C   READ SPACE INTEGRALS FROM FILE 2
C
```

```
      DO 200 L=1,10
      K = 1
      READ (2,800) (T2(K,N,L),N=1,10)
      DO 210 K=1,4
      DO 220 M=1,10
  220 READ (2,800) (T3(K,N,M,L),N=1,10)
  210 CONTINUE
  200 CONTINUE
      READ (2,800) HI
C
C    READ INPUT DATA (EXCEPT COMB. PARAMETERS AND INITIAL DISTURBANCE
C
C    FIRST DATA CARD
C        NEQ = NO. OF TERMS IN EXPANSIONS
C        NX = NO. OF X/L AT WHICH FLOW FIELD CALCULATED
C        LIN = 1 TO CALCULATE LINEAR RESULTS
C        IPLOT = 1 TO PLOT ANY OUTPUT
C        INPT = 1 TO WRITE THE SPACE INTEGRALS READ FROM FILE 2
C    SECOND DATA CARD
C        LC1 = 1 TO WRITE C(N) AND B(N)
C        LC2 = 1 TO PLOT B(N)
C            = 4 NO PLOT OF B(N)
C        LC4 = NUMBER OF TERMS TO BE PLOTTED
C        LC5 = INCRIMENTAL INDEX BETWEEN TERMS TO BE PLOTTED
C    THIRD DATA CARD
C        LP1 = 1 TO CALCULATE U AND P
C        LP2 = 1 TO WRITE U AND P
C        LP3 = 1 TO PLOT P
C            = 4 NO PLOT OF P
C        LP4 = NO. OF X/L AT WHICH P OR U TO BE PLOTTED
C    FORTH DATA CARD
C        NTAU = NO. OF TAU TO BE RUN
C    FIFTH DATA CARD
C        UE = EXIT MACH NUMBER
C        GAMMA = SPECIFIC HEAT RATIO
C        EPS = AMPLITUDE PRECENT ERROR
C    SIXTH DATA CARD
C        TBEGIN = TIME TO START COMPUTATION OF FLOW VARIABLES AND
C                    TO START OUTPUT
C        TEND = STOP TIME
C        TLMCY = START TIME OF LIMIT CYCLE CHECK
C        DELTAT TIME DELTA FOR OUTPUT OF LIMIT CYCLE OSCILLATIONS
C        TSMPI = START TIME FOR PRESSURE MODE-AMPLITUDE PLOT
C        DELPT = TIME DELTA FOR PLOT OF P MODE-AMPLITUDE
C    SEVENTH DATA CARD
C        X(I) = AXIAL LOCATION AT WHICH FLOW FIELD IS TO BE CALCULATED
C    EIGHTH DATA CARD (USED ONLY IF LP3≠4)
C        IPX(I) = INDEX OF X(I) FOR WHICH PRESSURE IS TO BE PLOTTED
C
      READ (5,400) NEQ,NX,LIN,IPLOT,INPT
      READ (5,400) LC1,LC2,    LC4,LC5
      READ (5,400) LP1,LP2,LP3,LP4
      READ (5,400) NTAU
      READ (5,410) UE,GAMMA,EPS
      READ (5,410) TBEGIN,TEND,TLMCY,DELTAT,TSMPI,DELPT
      READ (5,410) (X(I),I=1,NX)
      IF (LP3.EQ.4) GO TO 100
```

```
      READ (5,400) (IPX(I),I=1,LP4)
  100 CONTINUE
C
C   WRITE SPACE INTEGRALS IF INPT = 1
C
      IF (INPT.NE.1) GO TO 110
      DO 700 L=1,NEQ
      WRITE (6,430) HI,L
      WRITE (6,402)
      WRITE (6,403)
      WRITE (6,405)
      WRITE (6,406)
      WRITE (6,407)
      WRITE (6,408)
      WRITE (6,431) (I,I=1,NEQ)
      WRITE (6,432)
      M = 0
      K = 1
      J = 0
      WRITE (6,433) J,M,(T2(K,N,L),N=1,NEQ)
      DO 710 K=1,4
      DO 720 M=1,NEQ
  720 WRITE (6,433) K,M,(T3(K,N,M,L),N=1,NEQ)
  710 CONTINUE
  700 CONTINUE
  110 CONTINUE
C   CALL PLOT SUBROUTINE IF IPLOT = 1
C
      IF (IPLOT.NE.1) GO TO 600
      CALL PLOTS (DATA(1),2500,3)
  600 CONTINUE
C
C   CALCULATION OF SOME TERMS USED IN SOLUTION OF ODES
C
      P2 = 3.14159*3.14159
      QP1=6.28318*UE
      QP2=(GAMMA-1.)*UE/2.
      QP3=(GAMMA-1.)*P2
      QP4=2.*P2
      QP5=GAMMA*UE
      QP7=P2
      Q1 = .5*(GAMMA-1.)*UE/GAMMA
      GP1 = GAMMA + 1.
      GT1 = GAMMA*.5*(GAMMA-1.)*UE
      Q2 = -Q1*.25*GP1/GAMMA
C
      DO 1000 KTAU =1,NTAU
C
C   READ COMBUSTION PARAMETERS
C
      READ (5,420) TAU,NNB,LCUT
      READ (5,410) (ANR(I),I=1,NNB)
C
      LTEMP = TAU/.05 + .01
      H = TAU/LTEMP
      HD2 = H/2.
      HD6 = H/6.
```

```
          HD8 = H/8.
          LTR = (TAU/H) + 1.01
          DO 2000 KK=1,NNB
          READ (5,400) NPI
          ANB= ANR(KK)
          DO 510 LLL =1,NEQ
          TN(LLL) = ANB
          IF (LLL.GT.LCUT) TN(LLL) = 0.0
          IF (TAU.LT.1) GO TO 510
          IF (LLL.NE.1) TN(LLL) = 0.0
      510 CONTINUE
          DO 3000 KKK=1,NPI
          READ (5,420) PI,IPOP
          TSTART = TBEGIN
          TSTOP = TEND
          TSMP = TSMPI
          KMT = 0
          KMTS = DELPT/H + 1.01
          KPLT = 2
          K2 = 2
          LGO = 2
          LOUT = 1
C
          CALL START (LTR,TX,H)
C
          KONTRL = 2
          L = LTR
          TSTOP1 = TSTOP + .10
      340 IF (TX.GT.TSTOP1) GO TO 130
          IF (L.NE.101) GO TO 140
          LTMP = 102 - LTR
          DO 150 L=1,LTR
          DO 160 I=1,NEQ
      160 BS(I,L) = BS(I,LTMP)
      150 LTMP = LTMP + 1
          L = LTR
      140 CONTINUE
C
          TEST = ABS(TX - TLMCY)
          IF (TEST.LT.0.03) K2=1
          IF (K2.NE.1) GO TO 320
          IF (LGO.EQ.1) GO TO 320
          PHIO = 0.0
          DO 900 I=1,NEQ
      900 PHIO = PHIO + B(I)
          CALL TREND (TEST,PHIO,LGO,EPS)
          IF (LGO.EQ.2) GO TO 370
          TSMP = TX
          TSTART = TX
          TSTOP = TX + DELTAT
          TSTOP1 = TSTOP + .10
      370 CONTINUE
      320 CONTINUE
C
          CHECK = ABS(TX-TSTART)
          IF (CHECK.LT.0.04) KONTRL = 1
          IF (KONTRL.NE.1) GO TO 330
```

```
      IF (LC1.NE.1) GO TO 500
      CALL WOUT1 (H,TX)
  500 IF (LC2.EQ.4) GO TO 501
      IF (KMT.GT.KMTS) GO TO 501
      CHK1 = ABS(TX-TSMP)
      IF (CHK1.LE.0.04) KPLT = 1
      IF (KPLT.NE.1) GO TO 501
      KMT = KMT + 1
      TARY(KMT) = TX
      DO 504 KM=1,10
      BARY(KM,KMT) = B(KM)
  504 CONTINUE
      IF (KMT.NE.KMTS) GO TO 501
      CALL POUT  (LC4,LC5,KMT)
      KPLT = 2
  501 IF (LP1.NE.1) GO TO 502
      CALL FLOW (NX,H,TX,LP2,LP3,LP4,LP5,LOUT)
  502 CONTINUE
      IF (LOUT.EQ.2) GO TO 3000
  330 CONTINUE
      L = L+1
      TX = TX + H
      LDO = L-LTR
      LD1 = LDO + 1
      DO 180 I=1,NEQ
      BR1(I) = BS(I,LDO)
      BR3(I) = BS(I,LD1)
  180 BR2(I) = (BR1(I)+BR3(I))/2.
C
      CALL RUNG  (NEQ)
C
      DO 300 I=1,NEQ
      BS(I,L) = B(I)
      CHK1 = B(I)
      CHK2 = C(I)
      IF (CHK1.LT.10.0.AND.CHK2.LT.10.0) GO TO 300
      WRITE (6,450)
      WRITE (6,460) TAU,ANB,UE,GAMMA,PI,TX,CHK1,CHK2
      GO TO 130
  300 CONTINUE
      GO TO 340
  130 CONTINUE
 3000 CONTINUE
 2000 CONTINUE
 1000 CONTINUE
      STOP
      END
```

```
      SUBROUTINE START (LTR,TX,H)
      EXTERNAL POFX
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10),IPOP
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
  400 FORMAT (1H1,//,10X,'INITIAL PRESSURE DISTRIBUTION',//)
  410 FORMAT (12X,'Z', 9X 'P',/)
  420 FORMAT (5X,4F10.5)
      TX = -TAU
      DO 1 N=1,NEQ
      A(N) = 0.0
      C(N) = 0.0
    1 B(N) = 0.0
      DO 100 L =1,LTR
      TX = TX + H
      DO 110 N=1,NEQ
  110 BS(N,L) = 0.0
  100 CONTINUE
      TX = TX - H
      DO 120 I=1,NEQ
      B(I) = POFX (I,PI,IPOP)
      C(I) = 0.0
      A(I) = -B(I)/GAMMA
  120 BS(I,LTR) = B(I)
      WRITE (6,400)
      WRITE (6,410)
      X = 0.0
  150 SUMB = 0.0
      DO 140 I=1,NEQ
      ARG = 3.14159*X*I
      C1= COS(ARG)
      SUMB = SUMB + B(I)*C1
  140 CONTINUE
      P = SUMB
      WRITE (6,420) X,P
      IF (X.GE.1.0) GO TO 200
      X = X + .1
      GO TO 150
  200 CONTINUE
      RETURN
      END




      REAL FUNCTION POFX (I,PI,IPOP)
      IF (IPOP.EQ.11) GO TO 1
C
C         CONTINUOUS WAVE IN IPOP MODE
      POFX = 0.0
      IF (IPOP.EQ.I) POFX = PI
      GO TO 2
    1 CONTINUE
C         DISCONTINUOUS 1L WAVE
      C = 2.*PI
      A    = 1.5708*I
      POFX = C*SIN(A)/A
    2 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE TREND (TEST,PHIO,LGO,EPS)
      DIMENSION PHIMAX(6)
      IF (TEST.GT.0.03) GO TO 1
      K=1
      M=1
    1 IF (M.NE.1) GO TO 10
      IF (PHIO.LE.0) GO TO 4
      PHIBIG = PHIO
      M=2
      GO TO 2
   10 IF (PHIO.LE.PHIBIG) GO TO 3
      PHIBIG=PHIO
      GO TO 2
    3 IF (PHIO.GT.0) GO TO 2
      SIGN = PHIO*PHIM1
      IF (SIGN.GT.0) GO TO 2
      PHIMAX(K)=PHIBIG
      M=1
      K=K+1
    2 PHIM1=PHIO
      IF (K.LE.4) GO TO 4
      AV1=0.0
      AV2=0.0
      DO 5 I=1,2
      AV1= AV1 + PHIMAX(I)
      IP2= I+2
    5 AV2 = AV2 +PHIMAX(IP2)
      K=1
      DELTA = ABS((AV2-AV1)/2.0)
      CHECK=  EPS*AV2/2.0
      IF (DELTA.GT.CHECK) GO TO 4
      LGO=1
      GO TO 6
    4 LGO=2
    6 CONTINUE
      RETURN
      END
```

```fortran
      SUBROUTINE FLOW (NX,H,TX,LP2,LP3,LP4,LP,LOUT)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
      COMMON/COM5/ X(11),IPX(4)
      COMMON/COM6/ TSTART,TSTOP,TLYMCY
      COMMON/COM8/ ABC(303),ORDP(11,303),ORDU(11,303)
  400 FORMAT (1H1, /,10X,'FLOW PRARAMETERS' ,10X,3HZ= ,F6.3,/ )
  410 FORMAT (10X,'LINEAR= ',I2,9X,'NEQ= ',I2, 9X,'N= ',F7.5, 9X,'TAU= '
     1 ,F7.5,/,10X,'EXIT MACH= ',F5.3,3X,'GAMMA =',F5.3, 4X,'PINITIAL ='
     2 ,F5.4,/)
  420 FORMAT (11X, 4HTIME,3X, 8HPRESSURE,3X,8HVELOCITY ,/ )
  430 FORMAT (10X,F7.3,11E10.4)
      TEST = ABS(TX-TSTART)
      IF (TEST.GT.0.03)  GO TO 1
      K=1
    1 CONTINUE
      DO 110 N=1,NX
      A1 = 3.14159*X(N)
      VEL=X(N)*UE
      SUMA = 0.0
      SUMB = 0.0
      SUMC = 0.0
      SUMU = 0.0
      DO 120 I=1,NEQ
      TA= A1*I
      ST = SIN(TA)
      CS = COS(TA)
      SUMA = SUMA + A(I)*CS
      SUMB = SUMB + B(I)*CS
      SUMC = SUMC + C(I)*ST
      SUMU = SUMU + C(I)*I*3.14159*ST
  120 CONTINUE
      CALL PRMTRS (N,K,SUMA,SUMB,SUMC,SUMU,VEL)
  110 CONTINUE
      ABC(K) = TX
      IF (TX.LT.TSTOP) GO TO 300
      LOUT = 2
      KSTOP = K
      IF (LP2.NE.1) GO TO 200
      DO 310 J=1,NX
      KOUNT = 44
      DO 220 L=1,KSTOP
      IF (KOUNT.NE.44) GO TO 210
      WRITE (6,400)  X(J)
      WRITE (6,410) LIN,NEQ,ANB,TAU,UE,GAMMA,PI
      WRITE (6,420)
      KOUNT = 1
  210 CONTINUE
      WRITE (6,430) ABC(L),ORDP(J,L),ORDU(J,L)
      KOUNT = KOUNT + 1
  220 CONTINUE
  310 CONTINUE
  200 IF (LP3.EQ.4) GO TO 300
      CALL POUT2 (LP3,LP4,KSTOP,NX)
  300 CONTINUE
      K = K+1
      RETURN
      END
```

```
      SUBROUTINE POUT2 (LP3,LP4,KSTOP,NX)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM3/ H
      COMMON/COM5/ X(11),IPX(4)
      COMMON/COM8/ ABC(303),ORDP(11,303),ORDU(11,303)
      COMMON/COM9/ ORD(303)
      CALL PLOT (0.0,2.0,-3)
      CALL PLOT (0.0,11.0,3)
      CALL PLOT (1.0,0.5,-3)
      TERMS = NEQ
      NPT = KSTOP
      J1 = NPT + 1
      J2 = NPT + 2
      SIZE = 0.10*NPT
      CALL SCALE (ABC,SIZE,NPT,1)
      DO 1 J=1,LP4
      DO 4 I=1,NX
      ICHK = IPX(J)
      IF (ICHK.NE.I) GO TO 4
      Z = X(I)
      DO 100 M=1,KSTOP
  100 ORD(M) = ORDP(I,M)
      GO TO 110
    4 CONTINUE
  110 CONTINUE
      CALL SCALE (ORD,4.0,NPT,1)
      IF (J.NE.3) GO TO 2
      DELX = SIZE + 4.0
      CALL PLOT (DELX,-6.3,-3)
    2 IF (J.EQ.2.OR.J.EQ.4) GO TO 3
      CALL SYMBOL (2.90,1.80,0.10,32HNORMALIZED PRESSURE TIME HISTORY,
     1            0.0,32)
      CALL SYMBOL (2.30,1.55,0.10, 3HN =,0.0,3)
      CALL SYMBOL (3.30,1.55,0.10, 4HTAU=, 0.0,4)
      CALL SYMBOL (4.50,1.55,0.10, 4HUE =,0.0,4)
      CALL SYMBOL (5.60,1.55,0.10, 6HGAMMA=, 0.0,6)
      CALL SYMBOL (2.30,1.30,0.10, 5HNEQ =,0.0,5)
      CALL SYMBOL (3.30,1.30,0.10, 3HH= ,0.0,3)
      CALL SYMBOL (4.50,1.30,0.10, 4HPI =,0.0,4)
      CALL NUMBER (2.80,1.55,0.10,ANB,0.0,4)
      CALL NUMBER (3.75,1.55,0.10,TAU,0.0,4)
      CALL NUMBER (4.95,1.55,0.10,   UE,0.0,3)
      CALL NUMBER (6.25,1.55,0.10,GAMMA,0.0,3)
      CALL NUMBER (2.80,1.30,0.10,TERMS,0.0,-1)
      CALL NUMBER (3.55,1.30,0.10,H,0.0,3)
      CALL NUMBER (4.95,1.30,0.10,PI,0.0,3)
    3 IF (J.EQ.2.OR.J.EQ.4) DELY = 5.3
      IF (J.EQ.1.OR.J.EQ.3) DELY = 4.0
      DELX = 0.0
      IF (J.EQ.1.OR.J.EQ.3) DELX = 2.0
      CALL PLOT (DELX,DELY,-3)
      CALL SYMBOL (1.80,-1.70,0.14,4HX/L=, 0.0 ,4)
      CALL NUMBER (2.40,-1.70,0.14,   Z,0.0,3)
      CALL FACTOR (0.788)
      CALL AXIS (0.0,0.0, 4HTIME,-4,SIZE,0.0,ABC(J1),ABC(J2))
```

```
      CALL AXIS (0.0,-2.0, 8HPRESSURE,8,4.,90.,ORD(J1),ORD(J2))
      CALL PLOT (0.0,-2.0,-3)
      CALL LINE (ABC,ORD,NPT,1,1,1)
      CALL FACTOR (1.0)
    1 CONTINUE
      RETURN
      END


      SUBROUTINE POUT (NMDE,ISP,NPT)
      DIMENSION COEF(100)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/PLTDA/ TIM(100), BS(10,100)
      COMMON/COM3/ H
      EQN = NEQ
      J1=NPT+1
      J2=NPT+2
      CALL SCALE (TIM,4.0,NPT,1)
      CALL PLOT (0.0,2.0,-3)
      CALL PLOT (0.0,11.0,3)
      CALL PLOT (0.0,0.5,-3)
      KOUNT = 1
      DO 110 I=1,NMDE,ISP
      L = 1
      DO 120 K=1,NPT
      COEF(K)=BS(I,L)
  120 L=L+1
      CALL SCALE (COEF,2.0,NPT,1)
      IF (KOUNT.NE.1) GO TO 1
      CALL SYMBOL (3.0,1.3,0.10,27HTIME DEPENDENT COEFFICIENTS,0.0,27)
      CALL SYMBOL (3.5,1.1,0.10,14HTERM EXPANSION,0.0,14)
      CALL SYMBOL (3.0,0.9,0.10,29HTAU=          NBAR=        H=,0.0,29)
      CALL SYMBOL (3.0,0.7,0.10,20HGAMMA=        MACH=,0.0,20)
      CALL NUMBER (3.0,1.1,0.10, EQN,0.0,-1)
      CALL NUMBER (3.5,0.9,0.10,TAU,0.0,3)
      CALL NUMBER (4.9,0.9,0.10,ANB,0.0,3)
      CALL NUMBER (6.0,0.9,0.10,H,0.0,3)
      CALL NUMBER (5.1,0.7,0.10,UE  ,0.0,3)
      CALL NUMBER (3.7,0.7,0.10,GAMMA,0.0,3)
      CALL PLOT (2.0,0.5,-3)
    1 Y=2.5
      CALL PLOT (0.0,Y,-3)
      CALL AXIS (.0,.0,4HTIME,-4,4.,0.0,TIM(J1),TIM(J2))
      CALL AXIS (0.0,-1.0,5HBN(T),5,2.0,90.0,COEF(J1),COEF(J2))
      CALL SYMBOL (4.5,0.0,0.10,2HN=,0.0,2)
      TERM = I
      CALL NUMBER (4.8,0.0,0.10,TERM, 0.0,-1)
      CALL PLOT (0.0,-1.0,-3)
      CALL LINE (TIM,COEF,NPT,1,0,1)
      CALL PLOT (0.0,1.0,-3)
      IF (KOUNT.NE.3) GO TO 20
      CALL PLOT ( 8.0,-8.0,-3)
      KOUNT =1
      GO TO 110
   20 KOUNT = KOUNT + 1
  110 CONTINUE
      CALL PLOT (8.0,0.0,-3)
      CALL PLOT (0.0,0.0,999)
      RETURN
      END
```

Section of the Program Used to Solve the Nonlinear Wave Equation

```
      SUBROUTINE RUNG (NEQ)
C
C   INTEGRATION OF SECOND ORDER ODE WITH RETARDED VARIABLE.
C   USE WITH SECOND ORDER WAVE EQUATION ANALYSIS.
C
      EXTERNAL EQTN
      DIMENSION R(10,4),BB(10),BPB(10),BB1(10),BPB1(10)
      COMMON/COM2/ BP(10),  B(10),R1(10),R2(10),R3(10),BS(10,110),DM(10)
      COMMON/COM3/ H,H2,HD6,H8
      DO 100 I=1,NEQ
      R(I,1) = H*EQTN(I, R1,B,BP)
      BPB(I) = BP(I)+ R(I,1)/2.
  100 BB(I) = B(I) + H2*BP(I) + H8*R(I,1)
      DO 110 I=1,NEQ
      R(I,2) = H*EQTN(I, R2  ,BB,BPB)
      BPB1(I)= BP(I)+ R(I,2)/2.
  110 BB1(I)= B(I)+ H2*BP(I) +H8*R(I,1)
      DO 120 I=1,NEQ
      R(I,3)= H*EQTN(I,  R2 ,BB1,BPB1)
      BPB(I)= BP(I)+R(I,3)
  120 BB(I) = B(I) + H*BP(I) +H2*R(I,3)
      DO 130 I=1,NEQ
  130 R(I,4)= H*EQTN(I, R3  ,BB,BPB)
      DO 140 I=1,NEQ
      B(I) = H*(BP(I)+(R(I,1)+R(I,2)+R(I,3))/6.) + B(I)
  140 BP(I)= (R(I,1)+2.*(R(I,2)+R(I,3))+R(I,4))/6. +BP(I)
      RETURN
      END


      REAL FUNCTION EQTN(L,YPR,Y,YP)
C
C   SECOND ORDER WAVE EQUATION
C
      DIMENSION Y(10),YP(10),YPR(10)
      COMMON/COMP1/ Q1,Q2,Q3,Q4,Q5,Q7
      COMMON/COM4/ T2(1,10,10),T3(4,10,10,10)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      D1 = -L*L*Q7*Y(L) - Q5*(YP(L) - T(L)*(YP(L) - YPR(L)))
      SUM=.0
      DO 100 N=1,NEQ
      S1 = Q1*N *YP(N)*T2(1,N,L)
      S2 =((-1)**(N+L ))*YP(N)*Q2
      SUM1=.0
      IF (LIN.EQ.1) GO TO 1
      DO 110 M=1,NEQ
      S3 = Q3* M*M *YP(N)*Y(M)*T3(2,N,M,L)
      S4 = Q4* N*M *Y(N)*YP(M)*T3(1,N,M,L)
  110 SUM1= SUM1+S3-S4
    1 CONTINUE
  100 SUM = SUM + SUM1 +S1-S2
      EQTN = D1 +2.*SUM
      RETURN
      END
```

```
      SUBROUTINE PRMTRS (N,K,SUMA,SUMB,SUMC,SUMU,VEL)
C
C   SUBROUTINE FOR CALCULATING FLOW PARAMETERS FOR WAVE EQUATION
C
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM5/ X(11),IPX(4)
      COMMON/COM8/ ABC(303),ORDP(11,303),ORDU(11,303)
      ORDU(N,K) = -SUMU
      IF (LIN.EQ.1) GO TO 1
      ORDP(N,K) = GAMMA*(SUMB*(SUMB-2.)+SUMU*(2.*VEL-SUMU))/2.
      GO TO 2
    1 ORDP(N,K)  = GAMMA*(-SUMB + VEL*SUMU)
    2 CONTINUE
      RETURN
      END




      SUBROUTINE WOUT1 (H,TX)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
      COMMON/COM6/ TSTART,TSTOP,TLYMCY
  420 FORMAT (3X,F7.3,10E10.4)
  430 FORMAT (1H )
  440 FORMAT (1H1,10X,'TIME DEPENDENT COEFFICENTS OF THE ',
     1        'NONLINEAR WAVE EQUATION    PHI = A(T)*COS(N*PI*Z)',/ )
  451 FORMAT (5X,'TIME       AP1       AP2         AP3         AP4       AP5
     1     AP6        AP7        AP8        AP9        AP10')
  452 FORMAT (5X,'TIME       A1        A2          A3          A4        A5
     1     A6         A7         A8         A9        A10')
  410 FORMAT (10X,'LINEAR= ',I2,9X,'NEQ= ',I2, 9X,'N= ',F7.5, 9X,'TAU= '
     1 ,F7.5,/,10X,'EXIT MACH= ',F5.3,3X,'GAMMA =',F5.3, 4X,'PINITIAL ='
     2 ,F5.4,/)
      TEST = ABS(TX-TSTART)
      IF (TEST.GT.0.030) GO TO 10
      K = 16
   10 IF (K.NE.16) GO TO 2
      WRITE (6,440)
      WRITE (6,410) LIN,NEQ,ANB,TAU,UE,GAMMA,PI
      WRITE (6,452)
      WRITE (6,451)
      K=1
    2 WRITE (6,430)
      WRITE (6,420) TX,(C(I),I=1,NEQ)
      WRITE (6,420) TX,(B(I),I=1,NEQ)
      K=K+1
      RETURN
      END
```

Section of the Program Used in the Analysis of Large Amplitude
Oscillations

```fortran
      SUBROUTINE RUNG   (NEQ)
C
C   INTEGRATION OF FIRST ORDER ODE WITH RETARDED VARIABLE
C   USE WITH LARGE AMPLITUDE ANALYSIS
C
      DIMENSION RA1(10),RA2(10),RA3(10),RA4(10),RB1(10),RB2(10),RB3(10),
     1RB4(10),RC1(10),RC2(10),RC3(10),RC4(10),A1(10),A2(10),B1(10),
     2B2(10),C1(10),C2(10)
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
      COMMON/COM3/ H,HD2,HD6
      CALL EQTN (A,B,C,BR1,RA1,RB1,RC1)
      DO 100 I =1,NEQ
      A1(I) = A(I) + HD2*RA1(I)
      B1(I) = B(I) + HD2*RB1(I)
      C1(I) = C(I) + HD2*RC1(I)
  100 CONTINUE
      CALL EQTN (A1,B1,C1,BR2,RA2,RB2,RC2)
      DO 110 I=1,NEQ
      A2(I) = A(I) + HD2*RA2(I)
      B2(I) = B(I) + HD2*RB2(I)
      C2(I) = C(I) + HD2*RC2(I)
  110 CONTINUE
      CALL EQTN (A2,B2,C2,BR2,RA3,RB3,RC3)
      DO 120 I=1,NEQ
      A1(I) = A(I) + H*RA3(I)
      B1(I) = B(I) + H*RB3(I)
      C1(I) = C(I) + H*RC3(I)
  120 CONTINUE
      CALL EQTN (A1,B1,C1,BR3,RA4,RB4,RC4)
      DO 130 I=1,NEQ
      A(I) = A(I) + HD6*(RA1(I)+RA4(I)+2.*(RA2(I)+RA3(I)))
      B(I) = B(I) + HD6*(RB1(I)+RB4(I)+2.*(RB2(I)+RB3(I)))
      C(I) = C(I) + HD6*(RC1(I)+RC4(I)+2.*(RC2(I)+RC3(I)))
  130 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE PRMTRS (N,K,SUMA,SUMB,SUMC,SUMU,VEL)
C
C   SUBROUTINE FOR CALCULATING FLOW PARAMETERS FOR LARGE AMPLITUDE WAVES
C
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COMS1/ Q1,Q2,GP1,GT1
      COMMON/COM5/ X(11),IPX(4)
      COMMON/COM8/ ABC(303),ORDP(11,303),ORDU(11,303)
      ORDP(N,K) = SUMB
      IF (LIN.EQ.1) GO TO 1
      ORDU(N,K) = SUMC + (Q1 + Q2*SUMB)*SUMB*X(N)
      GO TO 2
    1 ORDU(N,K) = SUMC + Q1*SUMB*X(N)
    2 CONTINUE
      RETURN
      END




      SUBROUTINE WOUT1 (H,TX)
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM2/ B(10),C(10),BR1(10),BR2(10),BR3(10),BS(10,110),A(10)
      COMMON/COMS1/ Q1,Q2,GP1,GT1
      COMMON/COM6/ TSTART,TSTOP,TLYMCY
  420 FORMAT (3X,F7.3,10E10.4)
  430 FORMAT (1H )
  440 FORMAT (1H1,10X,'TIME DEPENDENT COEFFICENTS OF THE ',
     1 'LARGE AMPLITUDE SOLUTIONS',/)
  450 FORMAT (5X,'TIME      B1        B2        B3        B4        B5
     1    B6        B7        B8        B9        B10')
  451 FORMAT (5X,'TIME      C1        C2        C3        C4        C5
     1    C6        C7        C8        C9        C10')
  452 FORMAT (5X,'TIME      A1        A2        A3        A4        A5
     1    A6        A7        A8        A9        A10')
  410 FORMAT (10X,'LINEAR= ',I2,9X,'NEQ= ',I2, 9X,'N= ',F7.5, 9X,'TAU= '
     1 ,F7.5,/,10X,'EXIT MACH= ',F5.3,3X,'GAMMA =',F5.3, 4X,'PINITIAL ='
     2 ,F5.4,/)
      TEST = ABS(TX-TSTART)
      IF (TEST.GT.0.030) GO TO 10
      K = 12
   10 IF (K.NE.12) GO TO 2
      WRITE (6,440)
      WRITE (6,410) LIN,NEQ,ANB,TAU,UE,GAMMA,PI
      WRITE (6,452)
      WRITE (6,450)
      WRITE (6,451)
      K=1
    2 WRITE (6,430)
      WRITE (6,420) TX,(A(I),I=1,NEQ)
      WRITE (6,420) TX,(B(I),I=1,NEQ)
      WRITE (6,420) TX,(C(I),I=1,NEQ)
      K=K+1
      RETURN
      END
```

```
      SUBROUTINE EQTN (A,B,C,BR,RA,RB,RC)
C
C     LARGE AMPLITUDE EQUATION
C
      DIMENSION A(10),B(10),C(10),BR(10),RA(10),RB(10),RC(10),F1(10),
     1F2(10),F3(10)
      COMMON/COMS1/ Q1,Q2,GP1,GT1
      COMMON/FLODA/ NEQ,UE,ANB,TAU,GAMMA,PI,LIN,T(10)
      COMMON/COM4/ T2(1,10,10),T3(4,10,10,10)
      DO 100 L=1,NEQ
      PIL = L*3.14159
      S01 =PIL*C(L) + Q1*R(L) - T(L)*UE*(B(L) - BR(L))
      S0 = -UE*A(L) + S01
      R0 = -GAMMA*(S01+ UE*B(L))
      U0 = PIL*B(L)/GAMMA - UE*C(L)
      SUMN1 = 0.0
      SUMN2 = 0.0
      SUMN3 = 0.0
      DO 110 N=1,NEQ
      PIN = N*3.14159
      S1 =  PIN*T2(1,N,L)*A(N)
      S2 = PIN*T2(1,N,L)*R(N)
      R1 =  PIN*T2(1,N,L)*B(N)
      U1 =   PIN*T2(1,L,N)*C(N)
      SUMM1 = 0.0
      SUMM2 = 0.0
      SUMM3 = 0.0
      IF (LIN.EQ.1) GO TO 200
      DO 130 M=1,NEQ
      PIM = M*3.14159
      S3 = PIM*T3(1,N,M,L)*C(N)*A(M)
      S4 = PIM*T3(2,N,M,L)*A(N)*C(M)
      S5 =   T3(2,N,M,L)*A(N)*A(M)
      S6 = PIM*T3(3,N,M,L)*(B(N)*A(M) - B(M)*A(N))
      S8 =   T3(2,N,M,L)*A(N)*B(M)
      S9 =   T3(2,N,M,L)*(B(N)-BR(N))*A(M)
      S10 =   T3(2,N,M,L)*B(N)*B(M)
      S11 = PIM*T3(3,N,M,L)*B(N)*B(M)
      S12 = S10 - 2.*S11
      R2 = PIM*T3(1,N,M,L)*C(N)*B(M)
      R3 =   PIM*T3(2,N,M,L)*B(N)*C(M)
      R4 =   T3(1,N,M,L)*C(N)*C(M)
      U2 = PIM*T3(1,L,M,N)*A(N)*B(M)
      U3 =   T3(1,N,L,M)*C(N)*C(M)*PIM
      U4 =   PIM*T3(3,N,L,M)*B(N)*C(M)
      U5 =   PIN*T3(4,N,L,M)*B(N)*C(M)
      U6 =   T3(1,L,M,N)*B(N)*C(M)
      SUMM1 = SUMM1 + S3 + S4 - UE*S5 + Q1*(S6+S8) -2.*T(N)*UE*S9
     1         + Q2*S12
      SUMM2 = SUMM2 + R2 - GAMMA*R3 - GT1*R4 - Q1*(GAMMA*S10 - GP1*S11)
     1         - Q2*GAMMA*S12
      SUMM3 = SUMM3 + U2/GAMMA - U3 - Q1*(U4 - U5 + U6)
  130 CONTINUE
  200 CONTINUE
      SUMN1 = SUMN1 + SUMM1 + UE*S1 - Q1*S2
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      SUMN2 = SUMN2 + SUMM2 + UE*R1 + GAMMA*Q1*S2
      SUMN3 = SUMN3 + SUMM3 - UE*U1
  110 CONTINUE
      F1(L) = S0 + 2.*SUMN1
      F2(L) = R0 + 2.*SUMN2
      F3(L) = U0 + 2.*SUMN3
  100 CONTINUE
      DO 300 L=1,NEQ
      U8N = 0.0
      DO 310 N=1,NEQ
      U8 = T2(1,L,N)*F2(N)
      U8M = 0.0
      IF (LIN.EQ.1) GO TO 320
      DO 330 M=1,NEQ
      U9 = T3(3,N,L,M)*B(N)*F2(M)
  330 U8M = U8M + U9
  320 CONTINUE
      U8N = U8N - Q1*U8 - Q2*U8M*2.
  310 CONTINUE
      RA(L) = F1(L)
      RB(L) = F2(L)
      RC(L) = F3(L) + 2.*U8N
  300 CONTINUE
      RETURN
      END
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