

X-751-71-384

PREPRINT

NASA TM-X-65888

65888

# A CONTRIBUTION TO THE CONTROLLABILITY OF TIME-LAG SYSTEMS

(NASA-TM-X-65888) A CONTRIBUTION TO THE  
CONTROLLABILITY OF TIME-LAG SYSTEMS A.K.  
Choudhury (NASA) Aug. 1971 20 p CSCL 12A

N72-25561

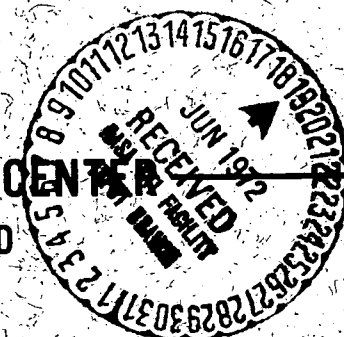
Unclas  
G3/19 30236

A. K. CHOUDHURY

AUGUST 1971

**GSFC**

**GODDARD SPACE FLIGHT CENTER**  
**GREENBELT, MARYLAND**



Reproduced by  
**NATIONAL TECHNICAL  
INFORMATION SERVICE**  
U.S. Department of Commerce  
Springfield VA 22151

X-751-71-384

A CONTRIBUTION TO THE CONTROLLABILITY  
OF TIME-LAG SYSTEMS

A. K. Choudhury

August 1971

Goddard Space Flight Center  
Greenbelt, Maryland

PRECEDING PAGE BLANK NOT FILMED

CONTENTS

	Page
ABSTRACT. . . . .	v
1. INTRODUCTION . . . . .	1
2. DEFINITIONS . . . . .	2
3. REPRESENTATION FOR THE SOLUTION OF EQUATION (1) . . . . .	6
4. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE CONTROLLABILITY . . . . .	10
Theorem 1 . . . . .	10
Proof of Necessity . . . . .	11
Proof of Sufficiency . . . . .	12
REFERENCES. . . . .	16

PRECEDING PAGE BLANK NOT FILMED

A CONTRIBUTION TO THE CONTROLLABILITY  
OF TIME-LAG SYSTEMS

A. K. Choudhury

ABSTRACT

N. a. s. c. of complete controllability of a class linear time-invariant delay-differential system are presented, and the result is expressed in an algebraic form.

# A CONTRIBUTION TO THE CONTROLLABILITY OF TIME-LAG SYSTEMS

## 1. INTRODUCTION

Consider the system

$$\dot{x}(t) = A x(t) + B x(t - h) + C u(t), \quad t > 0 \quad (1)$$

where  $x(t) \in R^n$ ,  $A$  and  $B$  are constant  $n \times n$  matrices.  $C$  is a constant  $n \times m$  matrix, and  $u(t)$  is a continuous  $m$ -dimensional vector, the control function;  $h$  is a positive number, the delay. This paper concerns a property of "complete controllability" which is a particular case of complete controllability to an arbitrary function. The latter concept has been studied by V. M. Popov [6], who gave a necessary and sufficient condition of controllability in this sense in an algebraic form.

In this paper we shall show that the necessary condition of complete controllability to an arbitrary function is also a necessary condition of the particular property of complete controllability provided  $A$  and  $B$  satisfy some additional conditions. If the rank of the matrix  $B$  is 1 or  $n$  and the pair  $(A, B)$  is completely controllable, then the above additional conditions are automatically satisfied and in these cases the n. a. s. c. of controllability to an arbitrary function is also a n. a. s. c. of complete controllability. Earlier works on the above subject were done by F. M. Kirillova and C. V. Čuracova [8], and L. Weiss [5]. L. Weiss also gave a necessary and sufficient condition of controllability involving functional condition.

## 2. DEFINITIONS

The use of operator symbolism is of some use in the solution of delay-differential equations. We introduce the operators  $s$  and  $z$  by the relations

$$s x(t) = \frac{d x(t)}{d t} \quad (2)$$

$$z x(t) = x(t - h). \quad (3)$$

For any real numbers  $k$  and  $\omega$ , we define

$$(k z)^\omega x(t) = k^\omega x(t - \omega h) \quad (4)$$

$$(k s)^\omega x(t) = k^\omega \frac{d^\omega x(t)}{d t^\omega}. \quad (5)$$

We define the sums and products of these operators in the usual way:

$$(s + z) x(t) = s x(t) + z x(t) \quad (6)$$

$$(s z) x(t) = s(z x(t)), \quad (7)$$

etc.

Two operators are called equal, if when applied to an arbitrary function, they produce the same result. With these definitions, the commutative, associative, and distributive laws hold. It can be shown that

$$z^{\omega_1} z^{\omega_2} x(t) = z^{\omega_2} z^{\omega_1} x(t) = z^{\omega_1 + \omega_2} x(t) = x(t - (\omega_1 + \omega_2) h). \quad (8)$$

We introduce the matrix  $r(s)$ , given by

$$r(s) = (sI - A)^{-1} B = \frac{P(s)}{\det(sI - A)}, \quad (9)$$

where  $p(s)$  is a  $n \times n$  matrix each of whose elements are polynomials in  $s$  of degree less than or equal to  $(n - 1)$ .

The set of all real functions  $f: [t_1, t_2] \rightarrow R^n$  having continuous  $k^{\text{th}}$  derivatives is said to belong to  $C^k [t_1, t_2]$ .

Let us introduce the operator

$$L(t; f(t - h)), \quad t_1 \leq t \leq t_2,$$

$$f(\cdot) \in C^0 [t_1 - h, t_2 - h],$$

which is defined as follows

$$L(t; f(t - h)) \triangleq p(s) F(t - h); \quad t_1 \leq t \leq t_2, \quad (10)$$

where  $F(\cdot) \in C^n [t_1 - h, t_2 - h]$  and satisfies the differential equation

$$\det(sI - A) F(t - h) = f(t - h); \quad t_1 \leq t \leq t_2$$

and  $p(s)$  is defined in equation (9).

If  $g(t) \in C^0 [-h, 0]$  and

$$g(t) = \det(sI - A)^{\ell} \phi(t), \quad \ell > n, \quad -h \leq t \leq 0,$$

then

$$L(t; g(t - h)) = p(s) \det(sI - A)^{\ell-1} \phi(t - h), \quad (11)$$

$$0 \leq t \leq h.$$

We define  $L^2(t; f(t-h))$  by the following relation

$$L^2(t; f(t-2h)) \triangleq L(t; f_1(t-h)); \quad t_1 \leq t \leq t_2, \quad (12)$$

where the function  $f_1(\cdot): [t_1-h, t_2-h] \rightarrow R^n$  is given by

$$f_1(t) = L(t; f(t-h)), \quad t_1-h \leq t \leq t_2-h \quad (13)$$

and the operator  $L(t; f(\cdot))$  is defined as above. Combining equations (12) and (13) we have

$$L^2(t; f(t-2h)) = L(t; L(t-h; f(t-2h))), \quad t_1 \leq t \leq t_2. \quad (14)$$

In general we define

$$L^{r+1}(t; f(t-(r+1)h)), \quad t_1 \leq t \leq t_2$$

by the relation

$$L^{r+1}(t; f(t-(r+1)h)) \triangleq L(t; f_r(t-h)), \quad t_1 \leq t \leq t_2, \quad (15)$$

where  $f_r(t)$  is given by

$$f_r(t) = L^r(t; f(t-rh)). \quad (16)$$

By  $L(0; g(t-h))$  we shall denote the value of  $L(t; g(t-h))$  at  $t=0$ , and similarly by  $L^3(2h; g(t-3h))$  we shall denote the value of  $L^3(t; g(t-3h))$  at  $t=2h$  etc. With obvious modifications the domain of  $L(t; g(t-h))$  can be extended to a matrix function.



We introduce the expression  $G(s, z)$  strongly related to the transfer-function of (1) by the following relation

$$G(s, z) = (sI - A - Bz)^{-1} C \quad (18)$$

where  $s, z$  are two auxiliary variables. We observe that the usual transfer-function of (1) is

$$G(s) = G(s, e^{-sh}). \quad (19)$$

Using explicit expression of the matrix  $(sI - A - Bz)^{-1} C$ , we can rewrite equation (18) as

$$G(s, z) = (sI - A - Bz)^{-1} C = \frac{(p_1(s) v(z), p_2(s) v(z), \dots, p_m(s) v(z))}{\det(sI - A - Bz)} \quad (20)$$

where  $v(z)$  is the column vector

$$v(z) = \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ \vdots \\ z^{n-1} \end{pmatrix} \quad (21)$$

and  $p_j(s)$ ,  $j = 1, 2, \dots, m$  are polynomial matrices of the form

$$p_j(s) = \sum_{i=0}^{n-1} p_{ji} s^i,$$

where  $p_{ij}$ ,  $i = 0, 1, \dots, (n - 1)$  are constant  $n \times n$  matrices.

### Complete Controllability

The system (1) is called completely controllable if for every  $\epsilon > 0$ , ( $\epsilon < h$ ), there exists a number  $T \geq h$  and a continuous control function  $u(t)$ ,  $t \in [0, T]$ , such that the corresponding solution  $x(t; g)$  of equation (1) for the initial function  $g \in C^0[-h, 0]$  satisfies the target condition

$$x(t; g) = 0, \quad t \in [T - h + \epsilon, T].$$

Finally, to simplify the calculations, we introduce the notation

$$\Delta(s) = \det(sI - A). \quad (22)$$

### 3. REPRESENTATION FOR THE SOLUTION OF EQUATION (1)

In this section we shall obtain the representation for the solution of equation (1) in terms of operators introduced in the previous section, and the representation is obtained by combining the results obtained in [6] and [2].

Equation (1) can be expressed as

$$(sI - A - Bz)x(t) = Cu(t), \quad t > 0. \quad (23)$$

Let

$$x(t) = \omega_1(t) + \omega_2(t) + \dots + \omega_m(t).$$

Then the above equation can be expressed as

$$\sum_{i=1}^m [(s I - A - B z) \omega_i(t) - C_i u_i(t)] = 0, \quad (24)$$

where  $C_i, i = 1, 2, \dots, m$  are the  $m$ -columns of the  $n \times m$  matrix  $C$  and  $u_1, u_2, \dots, u_m$  are the  $m$ -components of the column vector  $u(t)$ .

Multiplying equation (20) on the left by  $(sI - A - Bz) \times \det(sI - A - Bz)$ , we obtain

$$\begin{aligned} & (s I - A - B z) (p_1(s) v(z), p_2(s) v(z), \dots, p_m(s) v(z)) \\ & = (C_1, C_2, \dots, C_m) \det(s I - A - B z). \end{aligned} \quad (25)$$

The above relation is true for all  $s, z$  for which  $\det(sI - A - Bz) \neq 0$ . However, it can be shown as in [6] that the above relation is true for all  $s, z$ . Equation (25) can also be written as

$$\begin{aligned} (s I - A - B z) p_i(s) v(z) &= C_i \det(s I - A - B z), \\ i &= 1, 2, \dots, m. \end{aligned} \quad (26)$$

Let us consider the following scalar delay-differential equations

$$\begin{aligned} \det(s I - A - B z) \xi_i(t) &= u_i(t), \quad t > 0, \quad \xi_i(t) = 0, \quad t \in [-nh, 0] \\ i &= 1, 2, \dots, m \end{aligned} \quad (27)$$

and the functions

$$\omega_i(t) = p_i(s) v(z) \xi_i(t), \quad t > -h \quad (28)$$

where  $\xi_i(t)$  are the solutions of equation (27) for the vanishing initial conditions.

By virtue of relation (26), we observe that the  $\omega_i(t)$ , satisfy the differential equations

$$\begin{aligned} (s I - A - B z) \omega_i(t) &= C_i u_i, \quad t > 0 \\ i &= 1, 2, \dots, m \end{aligned} \quad (29)$$

and the initial conditions

$$\begin{aligned} \omega_i(t-h) &= z \omega_i(t) = p_i(s) z v(z) \xi_i(t) = 0, \quad t \in [0, h] \\ i &= 1, 2, \dots, m. \end{aligned} \quad (30)$$

Therefore the function

$$x(t) = \sum_{i=1}^m \omega_i(t) = \sum_{i=1}^m p_i(s) v(z) \xi_i(t), \quad t > 0 \quad (31)$$

is the solution of equation (1) for the vanishing initial condition. Any general solution of equation (1) will be given by the above considered particular solution, plus a solution of the homogeneous equation, obtained from equation (1) for  $u = 0$ . Thus any solution of equation (1) can be expressed as

$$x(t) = \sum_{i=1}^m p_i(s) v(z) \xi_i(t) + x(t; g), \quad t > 0 \quad (32)$$

where the  $\xi_i(t)$  are the solutions of equation (27), and  $x(t; g)$  is the solution of equation (1) for  $u = 0$  and the initial function

$$x(t) = g(t) = \Delta^{\ell}(s) \phi(t), \quad t \in [-h, 0]$$

and is given by [2]

$$x(t; g) = p^r(s) \Delta^{\ell-r}(s) \phi(t - rh) + [W_r(t, D) X_r(\tau)]_{\tau=0}, \quad (33)$$

$$t \in [(r-1)h, rh],$$

where

$$D = \frac{d}{d\tau},$$

$$X_r(\tau) = \Delta^{\ell-(r-1)}(D) \phi(\tau) - p(D) \Delta^{\ell-r}(D) \phi(\tau - h), \quad (34)$$

$$\begin{aligned} W_r(t, D) = & L^{r-1}(t; e^{A(t-(r-1)h)}) \Delta^{r-1}(D) \\ & + L^{r-2}(t; e^{A(t-(r-2)h)}) \Delta^{r-1}(D) + \dots \\ & + L(t; e^{A(t-h)}) \Delta^{r-1}(D) \\ & + L^{r-2}(t; e^{A(t-(r-1)h)}) Y_1(D) \Delta^{r-2}(D) \\ & + L^{r-3}(t; e^{A(t-(r-2)h)}) Y_1(D) \Delta^{r-2}(D) \\ & + \dots + L(t; e^{A(t-2h)}) Y_1(D) \Delta^{r-2}(D) \\ & + L^{r-3}(t; e^{A(t-(r-1)h)}) Y_2(D) \Delta^{r-3}(D) \\ & + L^{r-4}(t; e^{A(t-(r-2)h)}) Y_2(D) \Delta^{r-3}(D) + \dots \\ & + L(t; e^{A(t-3h)}) Y_2(D) \Delta^{r-3}(D) + \dots \\ & + L^2(t; e^{A(t-(r-1)h)}) Y_{r-3}(D) \Delta^2(D) \end{aligned}$$

$$\begin{aligned}
& + L(t; e^{A(t-(r-2)h)}) Y_{r-3}(D) \Delta^2(D) \\
& + L(t; e^{A(t-(r-1)h)}) Y_{r-2}(D) \Delta(D) \\
& + e^{A(t-(r-1)h)} Y_{r-1}(D) + e^{A(t-(r-2)h)} Y_{r-2}(D) \Delta(D) \\
& + \dots + e^{A(t-h)} Y_1(D) \Delta^{r-2}(D) + e^{At} \Delta^{r-1}(D),
\end{aligned} \tag{35}$$

and  $Y_{r-1}(D)$  is given by the following recurrence relation

$$\begin{aligned}
Y_{r-1}(D) = & p^{r-1}(D) - L^{r-1}((r-1)h; e^{A(t-(r-1)h)}) \Delta^{r-1}(D) \\
& - L^{r-2}((r-1)h; e^{A(t-(r-1)h)}) Y_1(D) \Delta^{r-2}(D) \\
& - L^{r-3}((r-1)h; e^{A(t-(r-1)h)}) Y_2(D) \Delta^{r-3}(D) \\
& - \dots - L((r-1)h; e^{A(t-(r-1)h)}) Y_{r-2}(D),
\end{aligned} \tag{36}$$

where  $p(D)$  and  $\Delta(D)$  are defined in equations (9) and (22).

#### 4. NECESSARY AND SUFFICIENT CONDITIONS OF COMPLETE CONTROLLABILITY

In this section we shall present sufficient condition of complete controllability and show that the sufficient condition is also a necessary condition provided  $A$  and  $B$  satisfy some additional conditions.

##### Theorem 1

The sufficient condition of controlling any solution of equation (1) to the target function  $W(t) = 0$ ,  $t \in [(n-1)h + \epsilon, nh]$  is that there does not exist any  $n$ -vector

$$d(s) = d_0 + s d_1 + s^2 d_2 + \dots + s^m d_m,$$

where  $d_0 \neq 0$ , and  $d_0, d_1, d_2, \dots, d_m$  are  $n$ -dimensional constant vectors, such that

$$d^T(s) G(s) = 0. \quad (37)$$

The above condition is also necessary provided that

$$d^T(s) p^n(s) \quad \text{and} \quad d^T(s) W_n(t, D)$$

are not simultaneously zero.

#### Proof of Necessity

We prove by contradiction. Suppose that there exists a polynomial vector  $d(s)$  such that

$$d^T(s) G(s) = 0,$$

and

$$d^T(s) p^n(s) \quad \text{and} \quad d^T(s) W_n(t, D)$$

are not simultaneously zero, and any solution of equation (1) is controllable to the target function, i.e.,

$$W(t) = 0, \quad t \in [(n-1)h + \epsilon, nh]. \quad (38)$$

Combining equations (38) and (32), we have

$$0 = \sum_{i=1}^m p_i(s) v(z) \xi_i(t) + p^n(s) \Delta^{\ell-n}(s) \phi(t - nh)$$

$$+ [W_n(t, D) X_n(\tau)]_{\tau=0}, t \in [(n-1)h + \epsilon, nh]. \quad (39)$$

Since  $d^T(s) G(s) = 0$  implies that

$$d^T(s) p_i(s) = 0, \quad i = 1, 2, \dots, m,$$

we have from equation (39)

$$\begin{aligned} 0 &= d^T(s) p^n(s) \Delta^{\ell-n}(s) \phi(t - nh) \\ &+ [d^T(s) W_n(t, D) X_n(\tau)]_{\tau=0}, \\ &t \in [(n-1)h + \epsilon, nh]. \end{aligned} \quad (40)$$

Since  $d^T(s) p^n(s)$  and  $d^T(s) W_n(t, D)$  are not simultaneously equal to zero, we can choose  $\phi \in C^\ell[-h, 0]$ , and hence the initial function  $g(\cdot) \in C^0[-h, 0]$ , such that

$$\begin{aligned} &d^T(s) p^n(s) \Delta^{\ell-n} \phi(t - nh) \\ &+ [d^T(s) W_n(t, D) X_n(\tau)]_{\tau=0} \neq 0, \quad t \in [(n-1)h + \epsilon, nh], \end{aligned} \quad (41)$$

and hence a contradiction.

### Proof of Sufficiency

Since  $d^T(s) G(s) \neq 0$ , we have

$$d^T(s) p_i(s) \vee (e^{-sh}) \neq 0$$

for at least one  $i$ , say  $i = j$ . Therefore as in [6], it follows that  $\det p_j(s) \neq 0$  and the remaining part of the proof of sufficiency follows as in [6].



Cor 1. If rank  $B = n$ , then the n. a. s. c. of complete controllability of the system (1) is that there does not exist any non-zero  $n$ - vector  $d(s) = d_0 + s d_1 + s^2 d_2 + \dots + s^m d_m$ ,  $d_0 \neq 0$  such that

$$d^T(s) G(s) = 0.$$

Proof. Suppose that  $d^T(s) p^n(s) = 0$ , where  $p(s)$  is defined in equation (9). We observe that equation (9) and  $d^T(s) p^n(s) = 0$ , imply that

$$(d_0^T + s d_1^T + s^2 d_2^T + \dots + s^m d_m^T) (A r(s) + B)^n = 0. \quad (42)$$

Dividing both sides of equation (42) by  $s^m$  and taking limit as  $s \rightarrow \infty$ , we obtain noting that  $r(s) \rightarrow 0$  as  $s \rightarrow \infty$ ,

$$d_m^T B^n = 0. \quad (43)$$

Equation (43) contradicts that rank  $B$  is  $n$  and therefore  $d^T(s) p^n(s) \neq 0$ , n. a. s. c. follows.

Cor 2. If the pair  $(A, B)$  is completely controllable, and the rank of the matrix  $B$  is one, then the n. a. s. c. of controllability of the system (1) is that there does not exist any  $n$ - vector  $d(s) = d_0 + s d_1 + s^2 d_2 + \dots + s^m d_m$ ,  $d_0 \neq 0$ , such that

$$d^T(s) G(s) = 0.$$

Proof. Suppose that

$$d^T(s) p^n(s) = 0 \quad (44)$$

$$d^T(s) W_n(t, D) = 0 \quad (45)$$

We observe that equations (44) - (45) imply that

$$d^T(s) x(t; g) = 0, \quad \forall g \in C^0[-h, 0], \quad t \geq (n-1)h. \quad (46)$$

Since  $\text{rank } B = 1$ ,  $B$  can be expressed as

$$B = b c^T, \quad (47)$$

where  $b, c$  are  $n$ -vectors and  $c^T$  denotes the transpose of the column vector  $c$ .

Now

$$d^T(s) p^n(s) = d^T(s) (sI - A)^{-1} b c^T \rho^{n-1}, \quad (48)$$

where  $\rho = c^T (sI - A)^{-1} b$  is a scalar quantity and not equal to zero, since the pair  $(A, bc^T)$  is completely controllable.

Therefore equation (48) implies that

$$d^T(s) (sI - A)^{-1} b c^T = 0. \quad (49)$$

Using equations (9) and (49) and taking limit as  $s \rightarrow \infty$ , we obtain the following equations

$$d_m^T B = 0 \quad (50)$$

$$d_{m-1}^T B + d_m^T A B = 0 \quad (51)$$

$$d_{m-2}^T B + d_{m-1}^T A B + d_m^T A^2 B = 0 \quad (52)$$

$$d_1^T B + d_2^T A B + d_3^T A^2 B + \dots + d_m^T A^{m-1} B = 0 \quad (53)$$

$$(d_0^T + d_1^T A + d_2^T A^2 + \dots + d_m^T A^m) A^j B = 0 \quad (54)$$

$$j = 0, 1, 2, 3, \dots$$

From equations (50) - (54) and (1), we observe that

$$d^T(s) x(t; g) = d^T(A) x(t; g), \quad t > 0 \quad (55)$$

and

$$s^i d^T(s) x(t; g) = d^T(A) A^i x(t; g), \quad t > 0. \quad (56)$$

Using Cayley-Hamilton theorem and equations (46), (55), (56), we observe that

$$\det(sI - A) d^T(A) x(t; g) = 0 \quad (57)$$

$$\forall g \in C^0[-h, 0], \quad \forall t > 0$$

and

$$d^T(A) x(t; g) = 0, \quad t \in [(n-1)h, nh] \quad (58)$$

$$\forall g \in C^0[-h, 0].$$

We observe that  $d^T(A) x(t; g)$  satisfies the differential equation (57) for  $t > 0$  and is identically zero in  $[(n-1)h, nh]$  and therefore

$$d^T(A) x(t; g) = 0, \quad \forall t \geq 0. \quad (59)$$

But we could choose  $g$  such that

$$x(0; g) = d(A).$$

Hence from equation (59) we obtain

$$d^T(A) d(A) = 0 \quad (60)$$

which is a contradiction. Hence

$$d^n(s) p^n(s) \quad \text{and} \quad d^T(s) W(t, D)$$

are not identically zero in this case and consequently follows the n. a. s. c. of complete controllability.

## REFERENCES

1. R. Bellman and K. L. Cooke, Differential Difference Equations, Academic Press, New York, 1963.
2. A. K. Choudhury, Necessary and Sufficient Conditions of Pointwise Completeness of Linear Time-invariant Delay-Differential Systems. X-Document No. X-751-71-213. GSFC, Greenbelt, Maryland.
3. A. K. Choudhury, Algebraic and Transfer-Function Criteria of Fixed-Time Controllability of Delay-Differential Systems (to appear).
4. A. K. Choudhury, N. a. s. c. of Fixed-Time Complete Controllability and the Structure of Linear Delay Systems. pp 184-188 Proc. Fourth Annual Princeton Conf. on Information Sciences and Systems.
5. L. Weiss, "On the Controllability of Delay-Differential Systems," J. Siam Control 5, pp 575-587, 1967.
6. V. M. Popov, "On the Property of Reachability for Some Delay-Differential Equations." University of Maryland Technical Report, R-70-08. Dept. of Electrical Engineering.
7. V. M. Popov, "Pointwise Complete and Pointwise Degenerate Linear Time-invariant Delay Differential Systems," Technical Report R-71-03, University of Maryland.
8. F. M. Kirillova and S. V. Čurakova, "On the Problem of Controllability of Linear Systems with After Effect" (Russian), Diff. Urav., 3 pp 436-445, (1967).