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Mode Coupling and Wave Particle Interactions
for Unstable Ion Acoustic Waves*

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A theory for the spatial development of linearly unstable, coupled waves is presented in which both quasi-linear and mode coupling effects are treated in a self-consistent manner. Steady state excitation of two waves (with frequencies ω_1, ω_2) is assumed at the boundary x = 0, the plasma being homogeneous in the y and z directions. Coupled equations are derived for the x dependence of the amplitudes of the primary waves (ω_1, ω_2) and the secondary waves, $n\omega_1 + m\omega_2$ (n and m being integers), correct through terms of second order in the wave amplitude, $e\phi/T_e$, but without the usual approximation of small growth rates. This general formalism is then applied to the case of coupled ion acoustic waves driven unstable by an ion beam streaming in the direction of the x axis. If the modifications of the ion beam by the waves ("quasi-linear" effects) are ignored, "explosive" instabilities (singularities in all of the amplitudes at finite x) are found, even when all of the waves have positive energy. If these wave-particle interactions are included, the solutions are no longer singular, and all of the amplitudes have finite maxima, at locations in reasonable agreement with experimental results of Taylor and Ikezi.

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I. INTRODUCTION

This calculation was motivated by observations of Taylor and Ikezi[1] on the propagation of ion acoustic waves in presence of ion streaming. In these experiments, a Double Plasma (DP) machine[2] was used to produce a broad beam of ions streaming through a plasma with a velocity, V, somewhat larger than the ion acoustic speed $c_s = (T_e/M)^{1/2}$. Signals applied to the grid separating the driver plasma (the source of the ions) from the target plasma excite ion acoustic waves which propagate into the target plasma. Over a range of frequencies which depends on V, the ion beam makes the ion acoustic waves unstable,[10] and spatial growth is observed.

If two sinusoidal signals, with frequencies ω_1, ω_2, both lying within the unstable range, are simultaneously applied to the grid, it is observed that these waves grow; saturate at distances of 50 to 100 Debye lengths from the grid; and then decay. In addition, a number of waves with frequencies $n\omega_1 + m\omega_2$, where n and m are integers, also grow; saturate, further downstream; and then likewise decay. The observed variation of the primary waves and the largest of the secondary waves are shown in Fig. 1.

At first sight, these results seem eminently reasonable. Since all of the ion acoustic waves have approximately the same velocity, strong mode coupling should be expected, with a consequent coupling of energy from the primary waves (which are driven by the free energy of the ion beam) to the collection of secondary (beat) waves. Thus, mode coupling should explain the observed saturation and decay of the waves.

In fact, however, a straightforward mode coupling calculation leads to a somewhat surprising result. All of the waves, both primary and secondary, become infinite at finite distances from the grid, as shown in Fig. 2.

This apparent violation of energy conservation is, of course, a consequence of neglecting the wave particle interactions, i.e., the perturbation of the beam by the growing waves. If the beam distribution function is taken to be the same at all x, an infinite energy reservoir is available, and there is no a priori limit on the wave amplitudes. While "explosive" instabilities are well known in the theory of mode coupling when some of the waves have a "negative" energy,(3) the present case does not fall into that category.(4) Apparently, linear instability of the individual waves combines with the amplification resulting from the mode coupling to produce the singular behavior at finite distance.

When the "quasi-linear" modification in the beam distribution function (or, more precisely, in the plasma dielectric function) is taken into account, the amplitudes remain finite and show a spatial variation conistent with the experimental results. Although the agreement is not completely quantitative at present, additional experimental observations are planned.

We note the following features of our analysis, some of which have also been considered in earlier papers(7) on mode coupling:

1. We consider waves which are linearly unstable.

2. We study the "boundary value" problem (real ω, complex k, in the vernacular) rather than the more traditional, but experimentally less accessible, initial value problem (real k, complex ω).

3. We eschew the usual restrictions to very small growth rate ($\gamma \ll \omega$ or $\mathrm{Im}\, k \ll \mathrm{Re}\, k$) or a very weak beam ($n_{beam} \ll n_{plasma}$). (Neither is satisfied in the Taylor-Ikezi experiments.)

4. We analyze the problem in configuration space rather than wave-number space.

5. We solve, self-consistently, the coupled mode coupling (wave-wave

interaction) and quasi-linear (wave-particle interaction) problems.

Since the beam in the DP machine is very broad (of order 100 times Debye length) and the primary waves have $\underline{k}$ parallel to the drift velocity, with ω_1, ω_2 below the ion plasma frequency, a one dimensional formulation in terms of electrostatic waves is justified. The kinetic theory analysis required in order to treat adequately the wave particle interaction is given in Section II and coupled equations for the wave amplitudes are derived, assuming no external electric or magnetic fields. These equations involve, in an essential way, both the self-consistent (quasi-linear) d.c. electric field, E_o, generated by the waves and the change, $\Delta\varepsilon$, in the plasma dielectric constant due to the wave-induced modification of the beam distribution function. Equations for E_o and $\Delta\varepsilon$ are derived in Section III. The results are specialized to the case of ion acoustic waves in Section IV, and in Section V numerical solutions for the wave amplitudes are given, first for the case where $\Delta\varepsilon$ is neglected, leading to the explosive instabilities, and then for the complete problem, with $\Delta\varepsilon$ included. As already noted,[9] the latter results are in qualitative agreement with the experimental data.

II. KINETIC FORMULATION

We start with the Vlasov and Poisson equations, assuming no external electric or magnetic fields. For each species we have

$$\partial f/\partial t + v\partial f/\partial x + (q/m)E\ \partial f/\partial v = 0 \quad . \tag{1}$$

(To minimize clutter in the notation, we omit species subscripts whenever this will not cause ambiguity.) Poisson's equation is written

$$\partial E/\partial x - 4\pi \int dv\ nqf = 0 \tag{2}$$

where the symbol $\int$ denotes sum over species as well as integration over velocity, and n is the average density of a species. Since we assume a steady state system, with excitation at frequencies ω_1 and ω_2 , the time dependence of f and E can be written as a generalized Fourier sum over all frequencies of the form

$$\omega_{nm} = n\omega_1 + m\omega_2 \quad . \tag{3}$$

The x dependence remains to be found, but it is convenient to separate out that part which follows from linear theory. Accordingly, we set

$$f(x,v,t) = f_o(x,v) + \textstyle\sum_\omega f_\omega(x,v)\ \exp[i(k_\omega x-\omega t)] \tag{4}$$

$$E(x,t) = E_o(x) + \textstyle\sum_\omega E_\omega(x)\ \exp[i(k_\omega x-\omega t)] \tag{5}$$

where the sum over ω stands for a sum over all values, positive and negative, of the integers n and m in (3), excluding those which would make $\omega = 0$, since the time independent terms are treated separately. We assume that, for given ω , the linear dispersion relation

$$\varepsilon(k_\omega,\omega) = 0 \tag{6}$$

has at most one unstable root,

$$k_\omega = \kappa_\omega - i\beta_\omega \quad ,$$

with $\beta_\omega > 0$. Here

$$\varepsilon(k,\omega) = 1 - \int dv(\omega_p/k)^2\, F'_o(v)\ (v-u)^{-1} \quad , \qquad u = \omega/k \tag{7}$$

is the dielectric function corresponding to x = 0, with $f_o(0,v) \equiv F_o(v)$. At other values of x we write the dielectric function as $\varepsilon + \Delta\varepsilon$ where

$$\Delta\varepsilon(k,\omega;x) = - \int dv(\omega_p/k)^2\ [\partial\Delta f_o(x,v)/\partial v]\ (v-u)^{-1}$$

$$\Delta f_o = f_o(x,v) - f_o(0,v) = f_o(x,v) - F_o(v) \quad . \tag{8}$$

[Choosing k_ω to be the root of $\varepsilon + \Delta\varepsilon$ rather than ε simplifies the formalism somewhat but greatly complicates the subsequent (numerical) solution of the mode coupling equations due to the consequence x-dependence of k_ω and hence of the matrix elements.] The contour of integration in (7) and (8) is, as usual, the real axis, when Im k is negative, the function being analytically continued into the upper half k plane.[5] At x = 0, the boundary values

$$f_o(0,v) = F_o(v) \ , \quad \text{with} \quad \int dv\ F_o(v) = 1$$

and

$$E_{\omega 1}(0) = E_{\omega 1} \ , \qquad E_{\omega 2}(0) = E_{\omega 2} \tag{9}$$

are given, all other amplitudes being 0 there. In particular, we note that the d.c. electric field, $E_o(x)$, is a self-consistent field, generated from nonlinear effects, and not the consequence of any external fields. Although the amplitudes E_ω are most convenient for purposes of analysis, we note that the experimentally measured amplitudes are in fact the quantities

$$\tilde{E}_\omega(x) = 2E_\omega \exp(\beta_\omega x) \quad , \tag{10}$$

so we shall give our final equations and numerical results in terms of these.

To solve the coupled, nonlinear equations which result from substituting the Fourier series (4) and (5) into (1) and (2), we make an expansion in the field amplitudes, treating all of the E_ω and f_ω as proportional to a small parameter, η . (The dimensionless small quantity is actually $|eE_\omega/k_\omega T_e|$.) In addition, our ordering ansatz assumes that the amplitudes f_ω and E_ω are slowly varying, in the sense that $(\partial f_\omega/\partial x)$ and $(\partial E_\omega/\partial x)$ are of order[6] η^2 . We shall see that Δf_o , defined in (8), is at most of order η compared to f_o and that E_o is of order η^2 .

Substituting (4) and (5) into (1) and (2), we obtain equations for the amplitudes f_ω and E_ω by equating to zero the individual Fourier coefficients:

$$f_\omega = (i/k)(v-u)^{-1} \{(qE_\omega/m)\ \partial f_o/\partial v + v\partial f_\omega/\partial x +$$

$$+ (q/m) \textstyle\sum'_{\omega'}\ (\partial f_{\omega'}/\partial v)\ E_{\omega''} \exp(i\Delta kx) + (qE_o/m)\ \partial f_\omega/\partial v\} \quad , \tag{11}$$

$$ikE_\omega + dE_\omega/dx \;=\; 4\pi \int dv\ nq\ f_\omega \tag{12}$$

where

$$k \equiv k_\omega\ , \quad k' \equiv k_{\omega'}\ , \quad k'' \equiv k_{\omega''}\ , \quad \omega'' = \omega-\omega'\ , \quad \Delta k \equiv k' + k'' - k \quad . \tag{13}$$

The prime on the summation symbol means that $\omega' = 0$ and $\omega'' = 0$ are excluded. From the time independent terms of (1) and (2) we obtain

$$v\ \partial(\Delta f_o)/\partial x + (qE_o/m)\ \partial f_o/\partial v = (-q/m) \textstyle\sum'_\omega\ (\partial f_\omega/\partial v)E_{-\omega} \exp(2\beta x) \quad , \tag{14}$$

$$dE_o/dx = 4\pi \int dv\ nq\ \Delta f_o \quad . \tag{15}$$

So far, of course, our equations are exact. To obtain an approximate formulation, correct through terms of order η^2 , we take advantage of the fact that the first term in the bracket on the right hand side of (11) is of order η ; the second and third are of order η^2 ; and the last is of order η^3 (assuming, as will be established later, that E_o is of order η^2). Consequently, in (11) we can drop the last term and substitute in the second and third terms

$$f_\omega = (i/k)(v-u)^{-1} (qE_\omega/m)(\partial f_o/\partial v) \quad , \tag{16}$$

thus obtaining an expression for f_ω correct to order η^2 :

$$f_\omega (x,v) = (i/k)(v-u)^{-1} (q/m) \{E_\omega(\partial f_o/\partial v) + i(v/k)(v-u)^{-1} (\partial F_o/\partial v)(dE_\omega/dx) + $$
$$+ (iq/m) \textstyle\sum'_{\omega'} (E_{\omega'}E_{\omega''}/k') \exp(i\Delta kx)(\partial/\partial v)[F'_o(v)/(v-u')]\}. \tag{17}$$

In the second and third terms of (17) we have replaced $f_o(x,v)$ by $F_o(v)$; since these terms are already of order η^2 , this introduces errors of order η^3 , which we are dropping here. We now integrate this expression for f_ω over v and substitute it on the right hand side of (12), obtaining a typical set of mode coupling equations,

$$D(\omega)\ dE_\omega/dx = ik\Delta\varepsilon\ E_\omega + \textstyle\sum_{\omega'} G(\omega',\omega'')\ E_{\omega'}E_{\omega''} \exp(i\Delta kx) \quad . \tag{18}$$

Here $\Delta\varepsilon$ is given by (8);

$$D(\omega) = -k\ \partial\varepsilon(k,\omega)/\partial k = -2 - u \int dv(\omega_p/k)^2\ F'_o(v)(v-u)^{-2} \quad ; \tag{19}$$

and the mode coupling matrix elements, G (cf. the $\varepsilon^{(2)}$ of Sagdeev and Galeev[7]), are given by

$$G(\omega',\omega'') = \int dv(\omega_p{}^2/kk')(q/m)\ F'_o/(v-u)^2(v-u') \quad . \tag{20}$$

While (18) is similar in form to the usual (temporal) mode coupling equations with x in place of t as independent variable, we note two important differences:

i) The wave-particle effects (modification of f_o by the waves) appear in the linear term involving $\Delta\varepsilon$.

ii) The quantities u and k are complex, i.e., we have made no assumption of small growth rates.

Before we can attempt to solve (18), we must adjoin to it an equation for $\Delta\varepsilon$, which is derived in the next section. For later reference, we note that $G(\omega',\omega'')$ can be replaced by an expression which is symmetric in the ω' and ω'' modes by taking half the sum of $G(\omega',\omega'')$ and $G(\omega'',\omega')$:

$$\tilde{G}(\omega',\omega'') = \frac{1}{2}\,[G(\omega'',\omega') + G(\omega',\omega'')]$$

$$= (2k'k'')^{-1} \int dv(\omega_p{}^2 q/m)\ [F'_o(v)/(v-u)(v-u')(v-u'')] \cdot [1 + v\Delta k/k(v-u)] \tag{21}$$

If we also go over from the E_ω to the physically measured amplitudes $\tilde{E}_\omega$, defined in (10), we have as our basic mode coupling equations

$$D(\omega)\ d\tilde{E}_\omega/dx = [ik\Delta\varepsilon + \beta_\omega D(\omega)]\ \tilde{E}_\omega + \frac{1}{2}\sum{}'_{\omega'}\ \tilde{G}(\omega',\omega'')\ \tilde{E}_{\omega'}\tilde{E}_{\omega''}\ \exp(i\Delta\kappa x) \quad . \tag{22}$$

The coefficients D and $\tilde{G}$, defined by (19) and (21), are independent of x and, for a given set of frequencies, involve simply some integrals related to the plasma dispersion function[8]; explicit evaluations of these are discussed in Appendix I. On the other hand, the coefficient $\Delta\varepsilon$, defined by (8), is a function of x, to whose determination we now turn.

III. DETERMINATION OF E_o AND $\Delta\varepsilon$

If we drop the $\Delta\varepsilon$ term in (22), then the coupled differential equations for the wave amplitudes, $\tilde{E}_\omega$, constitute a closed set, which does not involve the time-independent variables f_o and E_o of (14) and (15). However, as already noted, the resulting amplitudes, $\tilde{E}_\omega$, diverge at finite x so the quasi-linear effects, represented in (22) by $\Delta\varepsilon$, must be included. This requires, in essence, the simultaneous solution of the quasi-linear problem, represented by (14) and (15), together with the mode coupling problem, represented by (22). Moreover, the quasi-linear problem is more difficult than the usual one in two respects:

i) We are dealing with a discrete spectrum of coherent waves and hence cannot take advantage of the formal simplifications associated with averages over the phases of a continuous spectrum of random waves.

ii) We are interested in the eventual application to ion acoustic waves and must therefore take proper account of both ions and electrons.

Our analysis of the effect of the waves on f_o is complementary to the approach of Dupree and Weinstock,[7] who consider the actual particle orbit modifications. We first discuss this quasi-linear sub-problem.

On the right side of (14) we can, to lowest order in η, replace f_ω by its leading term, (16), which gives an equation for Δf_o:

$$v\,\partial(\Delta f_o)/\partial x = -(q/m)(\partial/\partial v)\,\{f_o E_o + (iq/2m)\,{\textstyle\sum_\omega'}\,|\tilde{E}_\omega|^2\, f'_o/k(v-u)\} \tag{23}$$

where, as before,

$$f_o(x,v) = F_o(v) + \Delta f(x,v); \qquad F_o(v) = f_o(0,v) \quad .$$

If we regard the wave intensity,

$$I_\omega(x) \equiv |\tilde{E}_\omega|^2/2$$

as a known function of x , then we can use (15) to express E_o as an integral, over x and v , of Δf_o ; substituting this into (23) leads to a differential-integral equation for f_o , which can be solved by iteration on Δf_o . To lowest order, this is equivalent to simply replacing f_o by F_o on the right-hand side of (23), thus obtaining an explicit expression for Δf_o , and then substituting this into (15), thereby obtaining an ordinary differential equation for $E_o(x)$. Actually, it is more convenient to work with the derivative of this equation,

$$d^2E_o/dx^2 = 4\pi \int dv\; nq\; \partial(\Delta f_o)/\partial x$$

$$= \int dv(\omega_p{}^2/v)\; \{E_o F'_o + (iq/2m) \sum I_\omega(\partial/\partial v)[F'_o/k(v-u)]\} \; . \qquad (24)$$

Note that, notwithstanding the v^{-1} factor in the integrand of (24), this integral is non-singular: since Δf_o must be finite at $v = 0$, the right side of (23) must vanish at $v = 0$, and so the integrand of (24) has no singularity at $v = 0$. Of course, when we subsequently break up the right hand side of (24), individual pieces may be singular, but there will be no difficulty provided we use the same definition of the $v = 0$ singularity in all terms. We shall choose the Cauchy principal value definition.

Our final approximation within this quasi-linear sub-problem is to neglect the left side of (24). This charge neutrality approximation is justified if the scale for E_o is large compared to the Debye length; it is clear from our numerical results that this is in fact the case for the ion acoustic wave application. We then have from (24)

$$E_o(x) = - \{i \int dv(\omega_p{}^2 q/2mv) \sum_\omega I_\omega(\partial/\partial v)[F'_o/k(v-u)]\} \cdot \{ \int dv\; \omega_p{}^2\; F'_o/v\}^{-1} \qquad (25)$$

which shows that the scale of E_o is essentially that of the I_ω and that, like I_ω, E_o is of order η^2. [It then follows from (23) that $\partial(\Delta f_o)/\partial x$ is of order η^2 and hence that Δf_o is of order η; this justifies, a posteriori, our procedure of iterating (23) with respect to Δf_o.]

The sum over ω in (25) is one we shall encounter often. It proves convenient to combine the terms ω and $-\omega$, reducing the sum to one over $\omega > 0$, and to define

$$P(\omega,v) = (i/2)(\partial/\partial v)\ \{[(kv-\omega)^{-1} - (k^*v-\omega)^{-1}]\ F'_o\}$$

$$= -\text{Im}\ (\partial/\partial v)\ [F'_o/k(v-u)] \quad , \tag{26}$$

where we have used the property[5] $k_{-\omega} = -k^*_\omega$. Then

$$E_o(x) = \{ \int dv\ (\omega_p{}^2 q/mv) \sum_{\omega'>0} I_{\omega'}\ P(\omega',v)\} \cdot \{ \int dv\ \omega_p{}^2\ F'_o/v\}^{-1} \quad . \tag{27}$$

Having found E_o, we can return to (23) and find Δf_o (replacing f_o by F_o on the right side). However, for the mode coupling problem, (22), we need only $\Delta\varepsilon$, computed from Δf_o according to (8). From (23) we have then

$$\partial(\Delta\varepsilon)/\partial x = -\int dv(\omega_p/k)^2\ [\partial(\Delta f_o)/\partial x](v-u)^{-2}$$

$$= \int dv[\omega_p/k(v-u)]^2\ (q/mv)\ \{E_oF'_o + (q/m) \sum_{\omega'>0} I_{\omega'}\ P(\omega',v)\} \quad .$$

Since E_o is, itself, a sum over I_ω, as we see from (27), we have, finally,

$$(\partial/\partial x)\ \Delta\varepsilon(\omega,x) = -(1/k^2) \sum_{\omega'>0} H(\omega,\omega')\ I(\omega') \tag{28}$$

where

$$H(\omega,\omega') = [\int dv(\omega_p{}^2 q/m)\ F'_o/v(v-u)^2]\ [\int dv(\omega_p{}^2 q/m)\ P(\omega',v)/v]\ [\int dv\ \omega_p{}^2\ F'_o/v]^{-1}$$

$$- [\int dv(\omega_p q/m)^2\ P(\omega',v)/v(v-u)^2] \tag{29}$$

gives a set of coefficients which are independent of x, like the coefficients D and $\tilde{G}$ in (22).

In fact, we may consider $\Delta\varepsilon(\omega,x)$ as an additional amplitude, to be treated on the same basis as the $\tilde{E}_\omega(x)$. Like the amplitudes $\tilde{E}_\omega$, $\Delta\varepsilon$ is of order η , not η^2 , as one might suppose. In fact, from (28) we see that $\partial(\Delta\varepsilon)/\partial x$ is of order η^2 , which implies that $\Delta\varepsilon$ itself is of order η . Adjoining (29) to (22) gives us the desired formulation in which mode coupling and quasi-linear effects are included in a self-consistent, albeit approximate, fashion. Our basic equations are thus the set (22) plus (28),

$$d\tilde{E}_\omega/dx = [\beta_\omega + ik\Delta\varepsilon/D]\tilde{E}_\omega + \sum_{\omega'>\omega''}{}' [\tilde{G}(\omega',\omega'')/D]\, \tilde{E}_{\omega'}\tilde{E}_{\omega''} \exp(i\Delta\, x)$$

$$d(\Delta\varepsilon)/dx = -\,(1/k^2) \sum_{\omega'>0} H(\omega,\omega')\, I(\omega') \tag{30}$$

with

$$I(\omega) = |\tilde{E}_\omega|^2$$

and the constant coefficients, D, $\tilde{G}$, H defined by (19), (21), and (29). We discuss in the next section the specialization of the formalism to the case of ion acoustic waves.

IV. UNSTABLE ION ACOUSTIC WAVES

We consider an infinite, homogeneous "beam" of ions, with density n_b, streaming with velocity $V > c_s$ through an infinite, homogeneous plasma consisting of stationary "plasma" ions (density n_p) and electrons (density $n = n_b + n_p$). At $x = 0$ (the location of the exciting grid in the plasma) we assume the ion distribution functions to be streaming Maxwellians,

$$F_{oi}(v) = \sum_j (\mu_j/\pi^{1/2} a_j) \exp[-(v-V_j)^2/a_j^2] \tag{31}$$

where j ranges over the two ion "species", beam and plasma, with

$$V_b = V \quad , \qquad V_p = 0$$

$$\mu_b = n_b/n = 1 - \mu_p$$

and

$$a_j = (2T_j/M)^{1/2} \quad .$$

For the electrons, we simply assume a single Maxwellian,

$$F_{oe}(v) = (\pi^{1/2} a)^{-1} \exp[-(v-V_e)^2/a_e^2]$$

$$a_e = (2\, T_e/m_e)^{1/2} \quad . \tag{32}$$

It is to be expected that for ion acoustic waves driven unstable by an ion beam with $V \gtrsim c_s$, the details of the electron distribution function do not matter much when $T_e/T_i \gg 1$, which is the case we want to consider here. Thus, using for the electrons a double humped Maxwellian, with streaming velocities 0 and V, should not give different results. Since we assume a steady state current, the electron current must balance the ion current, i.e., with the single Maxwellian (32) we should choose $V_e = \mu_b V_b$. However, the error due to choosing, instead, $V_e = 0$ is of order $(m/M)^{1/2}$. This is most easily seen for the dielectric function $\varepsilon(k,\omega)$. From (7), (31), and (32) we have

$$\varepsilon = 1 - k^{-2} \{\omega_{pe}^2 Z'[(u-V_e)/a_e]/a_e{}^2 + \sum \mu_j\omega_{pi}^2 Z'[(u-V_j)/a_j]/a_j{}^2\} \tag{33}$$

where Z is the plasma dispersion function[8] and $\sum_j$ denotes summation over the ions (plasma and beam). At this point, it is convenient to choose our units of x and t so that the quantities

$$k_b = (4\pi ne^2/T_e) , \qquad c_s = (T_e/M)^{1/2} , \tag{34}$$

and $\omega_{pi} = k_b c_s$ are all equal to 1. Then

$$a_j{}^2 = 2\tau_j , \qquad a_e{}^2 = 2/\delta$$

where

$$\tau_j = T_j/T_e \ll 1 , \qquad \delta = m_e/M_i ,$$

and we have

$$\varepsilon = 1 - k^{-2} \{(1/2) Z'[(u-V_e)(\delta/2)^{1/2}] + \sum (\mu_j/2\tau_j) Z'(U_j)\}$$

where

$$U_j = (u-V_j)/(2\tau_j)^{1/2} . \tag{35}$$

Since

$$Z'[(u-V_e)(\delta/2)^{1/2}] = - 2 + \theta(\delta^{1/2})$$

it does not matter whether we set V_e equal to $\mu_b V_b$ or 0 . Thus, our final expression is

$$\varepsilon = 1 + k^{-2} [1 - \sum (\mu_j/2\tau_j) Z'(U_j)] \tag{36}$$

and $\varepsilon = 0$ defines k_ω for $\omega > 0$, with[5] $k_{-\omega} = -k^*_\omega$.

For the coefficient $D(\omega)$, defined by (19), the electronic contribution is smaller than the ionic ones by a factor of $\delta^{1/2}$, as shown in Appendix I, so we may drop it altogether. Then

$$D(\omega) = -2 - (u/k^2) \sum \mu_j (2\tau_j)^{-3/2} Z''(U_j) \tag{37}$$

[which also follows from differentiating (36)]. In the remaining coefficients, $\tilde{G}$ and H, the contribution of a given species contains factors of $\omega_p{}^2 q/m$ or $(\omega_p q/m)^2$ from which it might appear that the electron contributions dominate those of the ions, making the results sensitive to the choice of electronic distribution function. We show in Appendix I that this is not the case: the electron and ion terms are of equal order so far as δ is concerned, although the electron terms tend to be smaller by some power of (T_i/T_e). In any case, the changes in the electron terms consequent upon choosing $V_e = 0$ rather than $\mu_b v_b$ are of order $\delta^{1/2}$. Explicit expressions for $\tilde{G}$ and H can most easily be given in terms of a multi-variable generalization of the Z' function,

$$Y_n(s_1, s_2, \ldots s_n) = \pi^{-1/2} \int dt \, [(t-s_1)(t-s_2)\ldots(t-s_n)]^{-1} (d/dt)\, e^{-t^2} \quad . \tag{38}$$

Such expressions are presented in equations (A10) through (A15) in Appendix I. Specifically, we obtain a dimensionless form of the basic equations (30) if we introduce a dimensionless potential, ϕ_ω :

$$\tilde{E}_\omega = -ik\,\Phi_\omega = -i(kT/e)\,\phi_\omega \quad . \tag{39}$$

Then (30) becomes, in our dimensionless units $(\omega_{pi} = k_D = c_s = 1)$

$$d\phi_\omega/dx = (\beta_\omega + ik\Delta\varepsilon/D)\phi_\omega + \frac{1}{2} \sum_{\omega' > \omega''} M(\omega,\omega',\omega'')\, \phi_{\omega'}\, \phi_{\omega''} \exp[i\Delta\kappa x] \quad , \tag{40}$$

$$d\Delta\varepsilon(\omega,x)/dx = - \sum_{\omega' > 0} h(\omega,\omega')\, |\phi_{\omega'}|^2 \quad , \tag{41}$$

where $D(\omega)$ is given by (37); and

$$M(\omega,\omega',\omega'') = i\, L(\omega,\omega',\omega'')/kD(\omega) \quad , \tag{42}$$

L and h being given in terms of Y in Appendix I.

In the following section we give the results of the numerical solution of the coupled equations (40) and (41) for the case of ion acoustic waves, with parameters chosen to approximate the conditions of the Taylor-Ikezi experiments. We shall see that if the quasi-linear effects, represented by (40), are neglected - i.e., if we set $\Delta\varepsilon \equiv 0$ in (40) - then an explosive instability results; including $\Delta\varepsilon$ gives results in qualitative agreement with the experiments.[1]

Regarding the character of the explosive instability, we cannot, as noted earlier, properly speak of the "energy", positive or negative, of the individual waves when the imaginary part of the phase velocity is not infinitesimally small. However, our contention that the explosive instability is not the familiar one associated with negative energy waves[7] can be properly stated in the following way. Having computed the matrix elements $M(\omega,\omega',\omega'')$ using (42), (37), and (A10), we may consider the equations (40) with linear terms β_ω <u>and</u> $\Delta\varepsilon$ set equal to zero. We find the amplitudes to be well-behaved, oscillatory functions, with no singularities, i.e., they behave like coupled waves whose energies all have the same sign. If, now, we introduce the unstable linear terms, $\beta\phi$ with $\beta > 0$, we immediately find an explosive instability, the value of x at the singularity being smaller the larger the value of β. Finally, we include the $i\Delta\varepsilon\phi$ term, which has a negative real part which grows like $\int dx|\phi_\omega|^2$ and hence prevents a singularity at finite x.

V. NUMERICAL RESULTS

We consider the case of ion acoustic waves with excitation frequencies $\omega_1 = 0.4$, $\omega_2 = 0.25$. (As in Section IV, we use units in which ω_{pi}, k_D, and c_s are equal to 1.) The beam strength is

$$\mu_b = n_b/(n_p+n_b) = 0.23 \quad ;$$

the streaming velocity is

$$V_b = 1.67 \quad ;$$

and the temperature ratios are

$$\tau_b = T_b/T_e = 0.1 \ ; \qquad \tau_p = T_p/T_e = 0.066 \quad .$$

(These correspond to the Taylor-Ikezi experiments which had $T_e = 3$ eV , $n = 10^9$ cm^{-3} .) The values of u, k, D, L, M, and h for this choice of parameters are given in Appendix I.

The results are best given in terms of the dimensionless potentials, ϕ_ω , defined in (39). If we drop the $\Delta\varepsilon$ term in (40), then for three waves (ω_1,ω_2, and $\omega_3 = \omega_1-\omega_2$) we obtain the results shown in Fig. 2. (The initial amplitudes of the ϕ_ω , as given in the figure caption, correspond approximately to those of the Taylor-Ikezi experiment.) The explosive instability is apparent and this feature persists if we include more waves, with frequencies ω_{nm} given by (3). Including the $\Delta\varepsilon$ term in (40) and solving the coupled set (40) and (41) for $\Delta\varepsilon$ and the wave amplitudes gives the results shown in Fig. 3 for three waves. The inclusion of more waves does not greatly change the behavior of the first three. Comparison with Fig. 1 indicates a qualitative agreement with the Taylor-Ikezi experiments. Pending further experiments, we cannot say whether the discrepancies are to be attributed to third order effects, neglected here, or to experimental errors.

VI. CONCLUSIONS AND DISCUSSION OF RESULTS

We find that a straightforward extension of mode-coupling theory to waves of well-defined phase which are linearly unstable leads to explosive instabilities (infinite amplitudes at finite distance). This phenomenon appears to be different from the superficially similar one associated with the coupling of positive and negative energy waves, in that here the energy required to feed the explosive instability comes directly from the free energy (in our case, the ion streaming) responsible for linear instability. Of course, in the case of negative energy waves, the ultimate source of energy is similar, so the distinction may be more formal than physical. When the "quasi-linear" modification in free energy due to the wave growth is taken into account, the amplitudes remain bounded and show a qualitative agreement with experiments on ion acoustic waves. Although we deal here only with the boundary-value problem (real ω, complex k) entirely similar results hold for the complementary initial value problem (real k, complex ω).

We conclude that, in general, both mode coupling and quasi-linear effects must be included in even the lowest order non-linear theoretical treatment of linearly unstable waves. We have considered here only the case of coherent waves (meaning well-defined phase), but similar conclusions should obtain in the case of a continuous spectrum of waves with random phases.

We are indebted to James Drake for assistance with the calculations.

FOOTNOTES AND REFERENCES

1. R. J. Taylor and H. Ikezi, "Observation of Strong Ion Acoustic Wave-Wave Interaction", UCLA Plasma Physics Report R-53 (1969); Bull. Am. Phys. Soc. 14, 1066 (1969).

2. R. J. Taylor and K. R. MacKenzie, in Proceedings of the International Conference on Physics of Quiescent Plasmas (Ecole Polytechnique, Paris, 1969), Pt. 3, p. 57; R. J. Taylor, D. R. Baker, and H. Ikezi, Phys. Rev. Letters 24, 205 (1970); R. J. Taylor, Ph.D. Thesis, UCLA (1970).

3. A considerable literature on this subject exists, dating back to 1964. A recent article by H. Wilhelmsson, et al., J. Math. Phys. 11, 1738 (1970) references the earlier work.

4. The usual definition of wave energy as $[\partial(\varepsilon\omega)/\partial\omega]\ E^2/8\pi$ [Landau and Lifshitz, Electrodynamics of Continuous Media (Pergamon Press, Oxford, 1960), p. 253] is valid only when the imaginary part of ε and its roots can be neglected, an approximation we do not make. A more detailed discussion of why the explosive instability encountered here is not simply that associated with the interaction of positive and negative energy waves is given below, following Eq. (41).

5. For $\omega > 0$, a root, u_ω, of the dispersion equation (7) with $\mathrm{Im}\ u_\omega > 0$ corresponds to $\mathrm{Im}\ k_\omega < 0$ (since $k_\omega = \omega/u_\omega$). There the representation (7), with real integration path, is correct. If u_ω is a root with $\mathrm{Im}\ u_\omega > 0$, then u^*_ω will also, formally, be a root, but it cannot be associated with $\omega > 0$ since it leads to $\mathrm{Im}\ k > 0$ and the representation (7) is not correct in the upper half k plane. (As noted, the proper analytic continuation must be used there.) However, u^*_ω is an acceptable root for $\omega < 0$, since it again gives $\mathrm{Im}\ k < 0$. Thus, $u_{-\omega} = u^*_\omega$ which

leads to the usual result [required by the reality of $E(x,t)$ and $f(x,v,t)$] that $k_{-\omega} = -k^*_{\omega}$. In the subsequent analysis, we shall encounter many integrals with one or more factors of $(v-u)^{-1}$. In each of these, the contour in the velocity plane is taken to be the real axis, with u above the real axis for $\omega > 0$, below it for $\omega < 0$. For real u we will choose the Cauchy principal value, as explained in more detail following Eq. (24).

6. Statements regarding the "order" of any functions of v are really to be understood as referring to the various integrals over v which occur in the subsequent analysis.

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APPENDIX I

CALCULATION OF INTEGRALS AND COEFFICIENTS

We summarize here the algebraic considerations involved in computing the coefficients D, $\tilde{G}$, and H which appear in (30), as well as the dielectric function ε , Eq. (7), which determines k_ω , when the distribution functions are those specified in (31) and (32).

The first point to be considered is that some of the integrals involve $\omega_p^{\,2}$, some $\omega_p^{\,2}q/m$, and some $(\omega_p q/m)^2$. This might make it appear that the electron contributions dominate the ion terms in some of the coefficients. While specific calculations show that this is not the case, it is useful to establish this in a general way, as follows.

Consider first the coefficient $D(\omega)$, given by (19). We see that the contribution of a given species, of mass m , temperature T , is proportional to

$$m^{-1} \int dv\ F'_o(v)/(v-u)^2 = m^{-1} \int dv\ F''_o(v)/(v-u) \quad . \tag{A1}$$

We introduce a dimensionless variable, $t = v/a$, where a denotes the thermal velocity for that species, and make use of the fact that $F_o(v)$ can be written as

$$F_o(v) = g(t)/a$$

and hence

$$F''_o(v) = g''(t)/a^3$$

where g is independent of T or m . The expression (A1) has then the form

$$(ma^3)^{-1} \int dt\ g''(t)(t-u/a)^{-1} \tag{A2}$$

where the integral is bounded and generally of order 1. Since $ma^2 = 2T$, the contribution of a given species to D is proportional to $m^{1/2}/T^{3/2}$ and the electron contribution is negligible.

We next examine $\tilde{G}$, given by (21). Here, the contribution of a given species to the term independent of Δk is proportional to

$$m^{-2} \int dv\ F'_o/(v-u)(v-u')(v-u'') \sim (ma^2)^{-2} \int dt\ g'/(t-u/a)(t-u'/a)(t-u''/a) \tag{A3}$$

and hence varies as T^{-2}, independent of m. The same result holds for the term proportional to Δk, since

$$v/(v-u) = t/(t-u/a) \tag{A4}$$

is, again, independent of m. Thus, we expect the electron contributions to $\tilde{G}$ to be smaller than the ion ones by $(T_i/T_e)^2$; in any case, they do not dominate.

Finally, we consider H, defined by (29), which we write as

$$H = (H_1H_2 - H_3H_4)/H_3 \tag{A5}$$

where

$$\begin{aligned} H_1 &= \int dv\ (\omega_p^2 q/m)\ F'_o/v(v-u)^2 \\ H_2 &= \int dv\ (\omega_p^2 q/m)\ P(\omega',v)/v \\ H_3 &= \int dv\ \omega_p^2\ F'_o/v \\ H_4 &= \int dv\ (\omega_p q/m)^2\ P(\omega',v)/v(v-u)^2 \quad . \end{aligned} \tag{A6}$$

Using the same method of analysis as above, and noting that

$$P(\omega',v) \equiv - \text{Im}\ (\partial/\partial v)[F'_o/k'(v-u')] = a^{-4}\ \hat{P}$$

with

$$\hat{P} = \text{Im}\ (\partial/\partial t)[g'/k'(t-u'/a)]$$

we have for the contribution of one species to H_1 through H_4 the following:

$$H_1: \quad m^{-2} \int dv\ F'_o/v(v-u)^2 \sim T^{-2} \int dt\ g'/t(t-u/a)^2$$

$$H_2: \quad m^{-2} \int dv\ P/v \sim T^{-2} \int dt\ \hat{P}/t$$

$$H_3: \quad m^{-1} \int dv\ F'_o/v \sim T^{-1} \int dt\ g'/t$$

$$H_4: \quad m^{-3} \int dv\ P/v(v-u)^2 \sim T^{-3} \int dt\ \hat{P}/t(t-u/a)^2 \quad .$$

As with $\tilde{G}$, the electron terms are, if anything, smaller than the ion terms, by powers of (T_i/T_e) .

In all of these integrals, of course, denominators like $(t-u/a)$ really cause no trouble, since the contour of integration is along the real t axis and $\mathrm{Im}\, u$ is non-zero. In the electron terms where we will make the approximation $u/a_e \to 0$, we let

$$(t-u/a)^{-1} \to P(t-u/a)^{-1} \pm i\pi\ \delta(t-u/a) \tag{A7}$$

where the sign agrees with that of $\mathrm{Im}\, u$. The $1/t$ factors are to be taken in the Cauchy principal value sense [cf. the discussion following (24)]. For denominators of higher order, like $(t-u/a)^{-2}$, integration by parts (together with a separation into partial fractions, if necessary) reduces the problem to (A7). Alternatively, one can use

$$(t-z)^{-2} = (d/dz)(t-z)^{-1} \quad .$$

For the electronic terms in these various integrals, we have $u/a_e \sim c_s/a_e \sim (m_e/M_i)^{1/2}$ so they can, to good approximation, be evaluated at $u = 0$, the corrections being of order $(m_e/M_i)^{1/2}$. The same consideration shows that we can also take the electron streaming velocity, V_e , in (32) as equal to zero, rather than $\mu_b V_b$, since with V_b of order c_s this, too, represents a correction of order $(m_e/M_i)^{1/2}$.

We now give explicit expressions for the coefficients $\tilde{G}$ and H in terms of dimensionless integrals. First, we define a multi-argument generalization of the plasma dispersion[6] (or, more precisely, of its derivative):

$$Y_n(s_1,s_2,\ldots s_n) \equiv -\int dt\ [(t-s_1)(t-s_2)\ldots(t-s_n)]^{-1}\ G'(t)$$

$$G(t) = e^{-t^2}/\pi^{1/2} \quad . \tag{A8}$$

The contour of integration is understood to be the real axis when s is complex, whether Im s is positive or negative. For real s, the Cauchy principal value is to be taken. If all of the arguments coincide, we have

$$Y_n(s,s,\ldots s) = (d/ds)^n\ Z(s) \quad .$$

In particular

$$Y_1(s) = Z'(s) \quad .$$

[Here Z is to be interpreted as the Z_+ function (defined by

$$Z_+(s) = \pi^{-1/2}\int dt\ e^{-t^2}\ (t-s)^{-1}$$

for Im s > 0 and analytically continued into the lower half s plane) for Im s > 0 ; as the Z_- function $\left(Z_-(s) = Z_+(s^*)^*\right)$ for Im s < 0; and as Re Z for s real.] If two of the arguments coincide, we have

$$Y_n(s_1,s_1,s_2,\ldots s_{n-1}) = (\partial/\partial s_1)\ Y_{n-1}(s_1,s_2,\ldots s_n) \quad . \tag{A9}$$

We also introduce a symbol for the Gaussian,

$$g(s) = e^{-s^2}/\pi^{1/2} \quad .$$

Then the distribution functions (31) and (32) are all of the form

$$F_o = \mu g(t)/a \ ; \qquad dF_o/dv = \mu g'/a^2$$

with

$$s = (v-V)/a \quad .$$

For denominators like $(v-u), v$ we have

$$v-u = a(s-U) \; ; \qquad U = (u-V)/a$$

$$v = a(s+W) \; ; \qquad W = V/a \quad .$$

Thus

$$P(\omega',v) = (\mu/a^4) \text{ Im } (\partial/\partial s)[g'/k'(s-U')] \quad .$$

We then have from (21)

$$2k'k'' \tilde{G}(\omega',\omega'') = (\omega_{pi}^2 e/M) \int ds \; (q/e)(M/m)^2 \; (\mu/a^4)$$

$$\times \; [g'(s)/(s-U)(s-U')(s-U'')][(1+\Delta k/k) + (\Delta k/k)(u/a)/(s-U)]$$

where M denotes ion mass and m, μ, a, g carry an implicit species index. We have also

$$ma^2/M = 2T/M = 2c_s^{\,2}(T/T_e) \quad .$$

Going over to the dimensionless units $(\omega_{pi} = 1, \; c_s = 1)$ we then have

$$L(\omega,\omega',\omega'') \equiv - \; 2k'k'' \tilde{G} \; M/e = - \; (1/4) \; \{ \; \textstyle\sum_j \; (\mu_j/\tau_j^{\,2}) \; [(1+\Delta k/k) \; Y_3(U_j,U'_j,U''_j)$$

$$+ \; (\Delta k/k)[u/(2\tau_j)^{1/2}] \; Y_4(U_j,U_j,U'_j,U''_j) \; - \; 8(1+\Delta k/k) \; \} \qquad \text{(A10)}$$

where we have used the fact that $Z'''(0) = 8$.

For the coefficient H we have from (A5) and (A6)

$$H = (e/M)^2 \; h = (e/M)^2 \; (h_1h_2/h_3-h_4) \qquad \text{(A11)}$$

with

$$h_1 = \omega_{pi}^2 \int ds \; (\mu/a^4)(q/e)(M/m)^2 \; g'/(s+W)(s-U)^2$$

where the $(s+W)$ denominator is to be taken as a Cauchy principal value, so that, in dimensionless units.

$$h_1 = (1/4)\ \{\ \sum\ (\mu_j/\tau_j{}^2)\ U_3(U_j,U_j;-W)\ -\ 8\ \}\quad ; \tag{A12}$$

$$h_2 = -\omega_{pi}^2 \int ds\ (\mu/a^4)(q/e)(M/m)^2(s+W)^{-1}(\partial/\partial s)[g'/k'(s-U')]$$

$$= -(1/4)\ \text{Im} \int ds\ \mu(T_e/T)^2\ (q/e)\ g'/k'(s+W)^2\ (s-U')\quad ,$$

or

$$h_2 = -(1/4)\ \text{Im}\ \{\ \sum_j\ (\mu_j/\tau_j{}^2)\ Y_3(U'_j,-W_j,-W_j)\ -\ 8\ \}\quad ; \tag{A13}$$

$$h_3 = \omega_{pi}^2 \int ds\ (\mu/a^2)(M/m)\ g'/(s+W)$$

or

$$h_3 = (1/2)\ \{\ \sum\ (\mu_j/\tau_j)\ R_e\ Z'\ (-W_j)\ -\ 2\}\ ; \tag{A14}$$

and

$$h_4 = -\omega_{pi}^2 \int ds\ (\mu/a^6)(M/m)^3(s+W)^{-1}\ (s-U)^{-2}\ \text{Im}\ (\partial/\partial s)\ g'/k'(s-U)$$

$$= -(\omega_{pi}^2/8) \int ds\ \mu(T_e/T)^3\ [(s+W)^{-2}(s-U)^{-2} + 2(s+W)^{-1}(s-U)^{-3}]\ \text{Im}\ g'/k'(s-U')$$

or

$$h_4 = -(1/16i)\ \sum\ (\mu_j/\tau_j{}^3)\ [Y_5(-W_j,-W_j,U_j,U_j,U'_j)/k'$$

$$+\ 2Y_5(-W_j,U_j,U_j,U_j,U'_j)/k'\ -\ (U'_j \rightarrow U'^*_j\ ,\ \ k' \rightarrow k'^*)]\quad . \tag{A15}$$

With the choice of parameters given in the caption of Fig. 1, the values of u, k, and D obtained from (36) and (37) for the three wave approximation are given in Table I. Note that $\text{Im}\ u$ is typically 20% of $\text{Re}\ u$, which is the reason we have eschewed the usual approximation of infinitesimal $\text{Im}\ u$. The other coefficients needed for the system (40) and (41) are the $h(\omega,\omega')$ defined by (A11) through (A15). In the case of three waves, there are only three mode coupling coefficients: $M(\omega_1) \equiv M(\omega_1,\ \omega_2,\ \omega_3)$; $M(\omega_2) = M(\omega_2,\omega_1,-\omega_3)$; and $M(\omega_3) = M(\omega_3,\omega_1-\omega_2)$. These, together with the associated L values, are also listed in Table I. Since (40) requires $ik\Delta\varepsilon/D$ it is convenient to tabulate the quantity

$$\xi(\omega,\omega') \equiv - ik\, h(\omega,\omega')/D(\omega) \quad , \tag{A16}$$

which is given in Table II for the case of three waves. As is to be expected from our earlier discussion in this appendix, ξ is dominated by the ion terms because of the T_e/T and $(T_e/T)^2$ factors in h_1 through h_4 as well as the $(m_e/M_i)^{1/2}$ weighting of the respective contributions to D .

Table I. Mode coupling coefficients for three waves. The input parameters are given in the caption of Fig. 1.

ω	u	k	$D(\omega)$	$L(\omega)$	$M(\omega)$
0.40	0.945+0.179i	$0.41-7.8\times10^{-2}i$	18.3-11.3i	-2.39-3.83i	-0.57-0.007i
0.25	0.966+0.197i	$0.25-5.1\times10^{-2}i$	52.5-25.6i	0.73+3.23i	1.13-0.30i
0.15	0.974+0.203i	$0.15-3.1\times10^{-2}i$	156.3-68.7i	1.59+7.73i	1.22-0.19i
0.0	0.979+0.206i				

Table II. Quasilinear coefficients, $\xi(\omega,\omega')$. The input parameters are given in the caption of Fig. 1.

	$\omega'_1 = .40$	$\omega'_2 = 0.25$	$\omega'_3 = 0.15$
$\omega_1 = .40$	-.90 + .34i	-.49 + .25i	-.28 + .16i
$\omega_2 = 0.25$	-.48 + .15i	-.27 + .12i	-.16 + .077i
$\omega_3 = 0.15$	-.27 + .077i	-.16 + .064i	-.089 + .042i

FIGURE CAPTIONS

Fig. 1. Wave amplitudes as a function of distance from the grid in the Taylor-Ikezi experiments, with frequencies $\omega_1 = 0.4\ \omega_{pi}$; $\omega_2 = 0.25\ \omega_{pi}$; $\omega_3 = \omega_1 - \omega_2$. The beam ion, plasma ion, and electron temperatures are $T_b = 0.2$ eV; $T_p = 0.3$ eV; $T_e = 3$ eV. Also, $V = 1.67\ c_s$ and the ratio of beam density to plasma density is $n_b/n = 0.23$, with $n \simeq 10^9\ \mathrm{cm}^{-3}$.

Fig. 2. Result of pure mode coupling calculation (sans quasi-linear effects) for the same parameter values given in Fig. 1. The initial amplitudes are $\phi_1 = 0.01$, $\phi_2 = 0.02$. An explosive instability (all $\phi_\omega \to \infty$) occurs at the point marked E.I.

Fig. 3. Solution of the coupled equations (40) and (41) including both mode coupling and quasi-linear effects, when three wave amplitudes are included. The parameter values are the same as in Fig. 2, but no trace of the explosive instability remains, so that a linear, rather than logarithmic, scale can be used for ϕ .

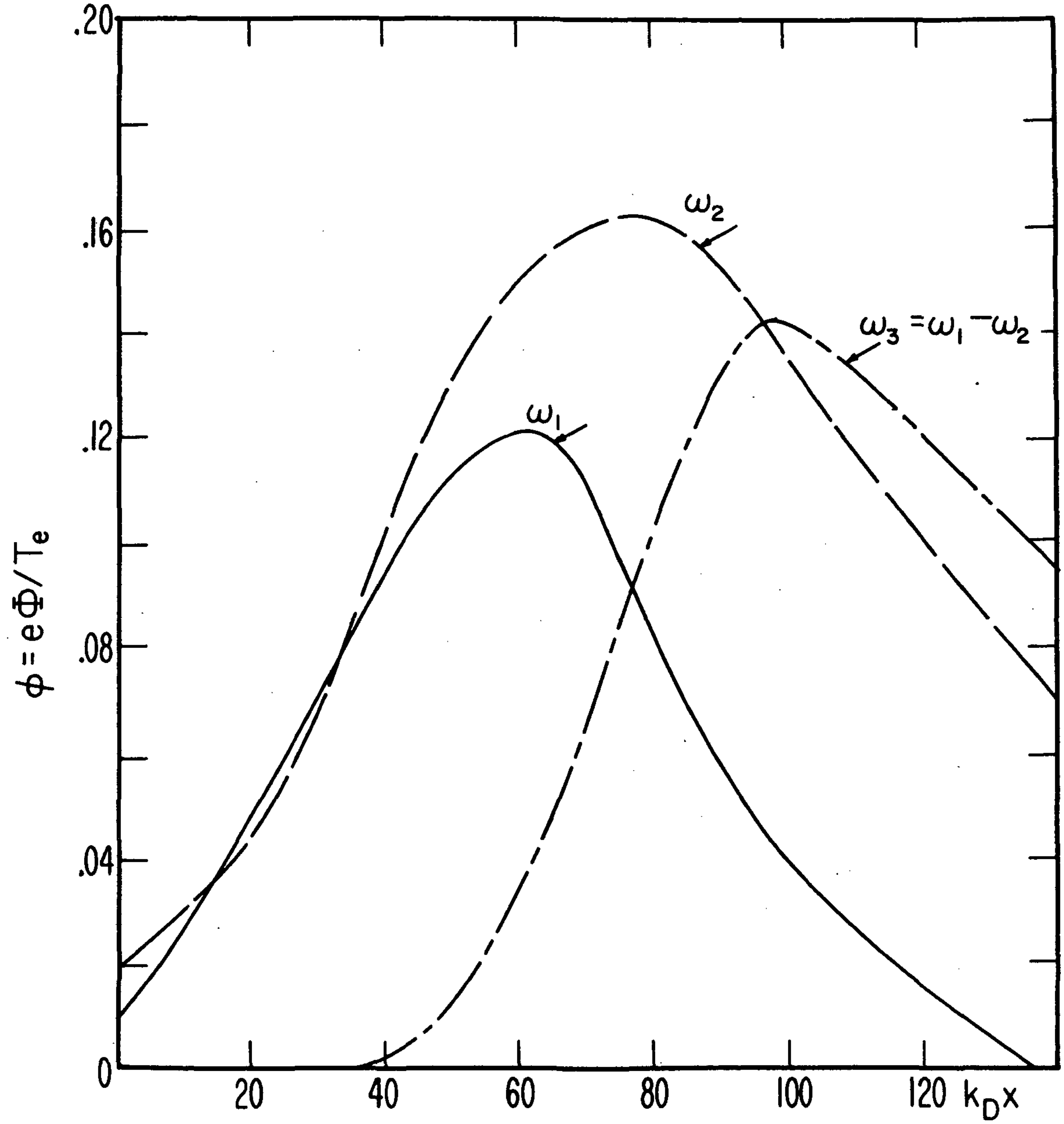
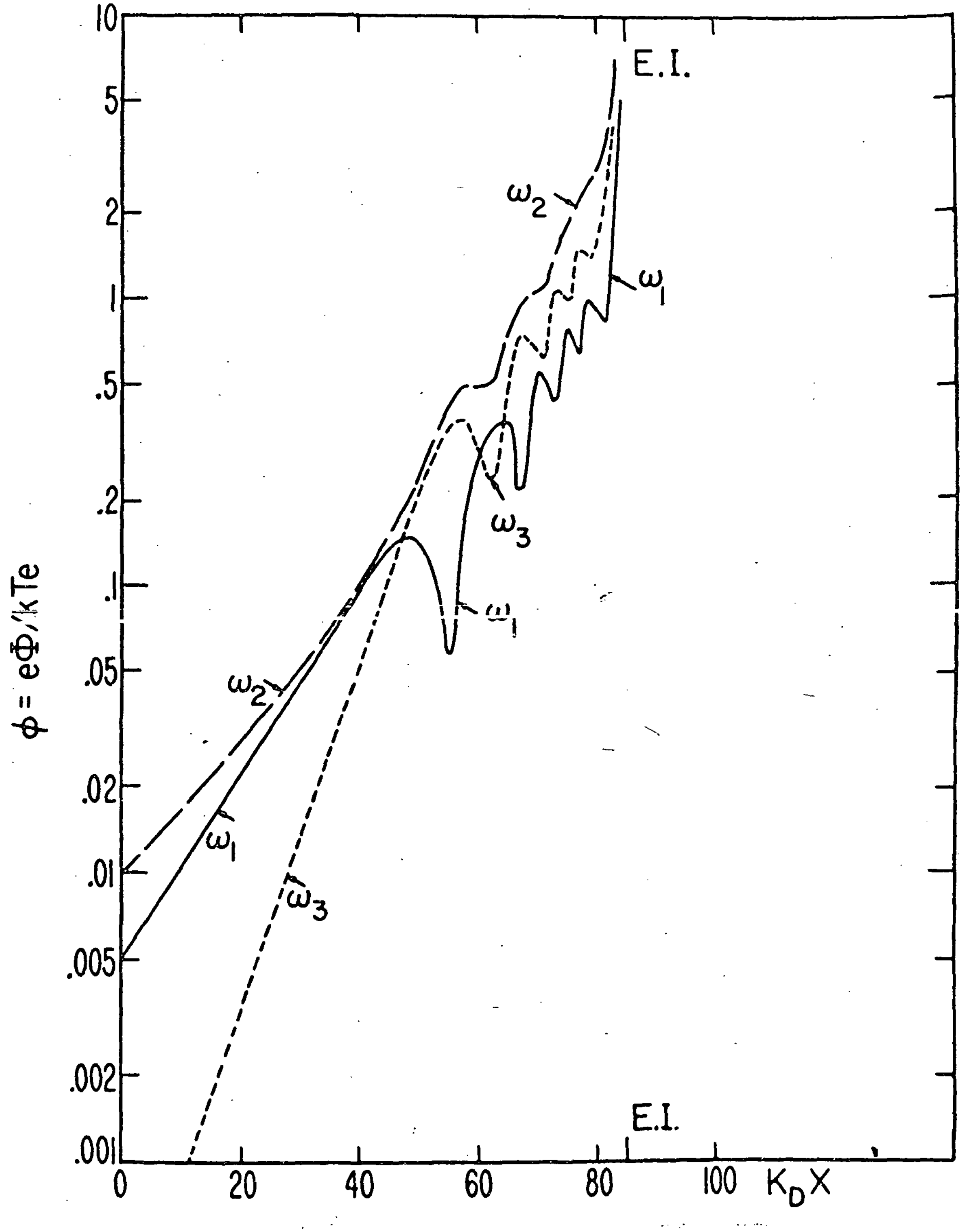


Fig. 1. Wave amplitudes as a function of distance from the grid in the Taylor-Ikezi experiments, with frequencies $\omega_1 = 0.4\ \omega_{pi}$; $\omega_2 = 0.25\ \omega_{pi}$; $\omega_3 = \omega_1 - \omega_2$. The beam ion, plasma ion, and electron temperatures are $T_b = 0.2$ eV; $T_p = 0.3$ eV; $T_e = 3$ eV. Also, $V = 1.67\ c_s$ and the ratio of beam density to plasma density is $n_b/n = 0.23$, with $n \simeq 10^9\ \text{cm}^{-3}$.



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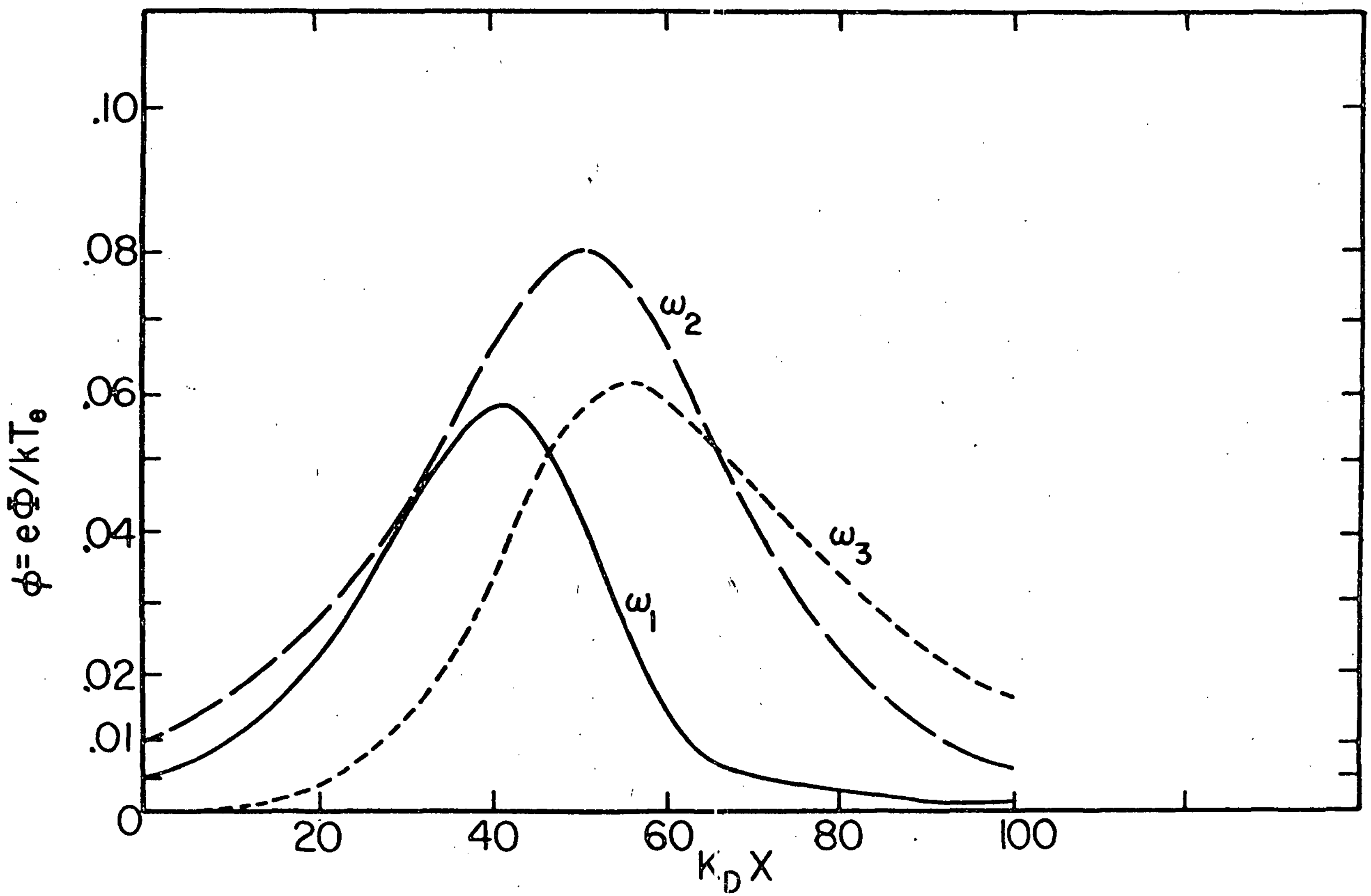

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