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CONJUGATE GRADIENT DETERMINATION OF OPTIMAL PLANE CHANGES FOR A CLASS OF THREE-IMPULSE TRANSFERS BETWEEN NONCOPLANAR CIRCULAR ORBITS

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16. ABSTRACT

A particular type of three-impulse transfer between two circular orbits is analyzed. The possibility of three plane changes is recognized, and the problem is to optimally distribute these plane changes to minimize the sum of the individual impulses. Numerical difficulties and their solution are discussed. Numerical results obtained from a conjugate gradient technique are presented for both the case where the individual plane changes are unconstrained and for the case where they are constrained. Possibly not unexpectedly, multiple minima are found. The techniques presented could be extended to the finite burn case, but primarily the contents are addressed to preliminary mission design and vehicle sizing.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

TECHNICAL MEMORANDUM X-64662

CONJUGATE GRADIENT DETERMINATION OF OPTIMAL PLANE CHANGES FOR A CLASS OF THREE-IMPULSE TRANSFERS BETWEEN NONCOPLANAR CIRCULAR ORBITS

INTRODUCTION

Recently the problem of optimally distributing three plane changes when impulsively transferring between two noncoplanar, circular orbits was reconsidered. The particular version studied involves a velocity impulse at the inner circular orbit that places the vehicle at the perigee of an ellipse whose apogee is greater than the radius of the outer circular orbit. At the apogee of this ellipse, another velocity impulse places the vehicle on an ellipse whose perigee coincides with the outer circular orbit. At the perigee of this ellipse, another velocity impulse occurs to place the vehicle in the outer circular orbit. Each of the three impulses can involve a plane change, and the problem is to minimize the total velocity impulse, i.e., the sum of the velocity impulses, by defining the optimal plane change at each impulse. This problem is interesting from two standpoints. First, when the radii of the two circular orbits and the transfer ellipse apogee are close in magnitude, obtaining numerical solutions with current computer programs which are demonstrably very good programs was very difficult, if not impossible, for reasons discussed later. Secondly, the near-earth on-orbit maneuvering done by the Space Shuttle vehicle or Space Tug could involve just exactly the conditions leading to numerical difficulties; therefore, it is important for planning and other purposes that accurate numerical results be obtainable and available. The first point primarily instigated this report, and led to the development of an extremely good numerical algorithm.

PROBLEM DESCRIPTION

To fix ideas, consider the sketch in Figure 1 of the geometry involved. At point 1, sufficient velocity is imparted to the vehicle to achieve perigee velocity of the first half of the transfer ellipse. At point 2, velocity is added to obtain apogee velocity of the second half of the transfer ellipse. At point 3, the vehicle velocity is circularized. The following formulas define the various velocities:

$$V_c^2 = \frac{\mu}{r_c} \quad ,$$

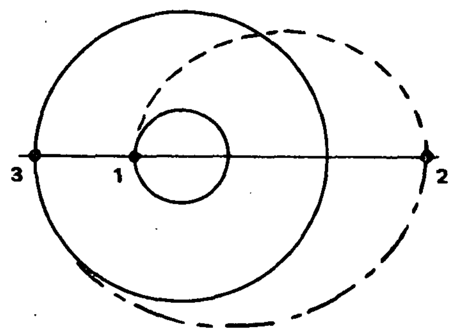

Figure 1. Geometry of the orbital transfer.

$$V_p^{\,2} = \frac{2\mu r_a}{r_p(r_a + r_p)} \quad ,$$

and

$$V_a^{\,2} = \frac{2\mu r_p}{r_a(r_a + r_p)} \quad ,$$

where the subscripts c, a, and p refer to circular, apogee, and perigee, respectively, and μ is the gravitational parameter of the earth.

The velocity impulses in question are

ΔV_1 from V_{c_1} to V_{p_1} ,

ΔV_2 from $V_{a_2^-}$ to $V_{a_2^+}$,

and

ΔV_3 from V_{p_3} to V_{c_3} ,

where $V_{a_2^-}$ and $V_{a_2^+}$ refer to the apogee velocities of the first half and second half of the transfer ellipse. Including the possibility of noncollinear velocity impulses, the ΔV's may be written as

$$\Delta V_1^{\,2} = \frac{\mu}{r_1} + \frac{2\mu r_2}{r_1(r_1 + r_2)} - 2\left(\frac{\mu}{r_1}\right)^{1/2}\left[\frac{2\mu r_2}{r_1(r_1 + r_2)}\right]^{1/2} \cos\vartheta_1 \quad ,$$

$$\Delta V_2^{\,2} = \frac{2\mu r_1}{r_2(r_1 + r_2)} + \frac{2\mu r_3}{r_2(r_2 + r_3)}$$
$$- 2\left[\frac{2\mu r_1}{r_2(r_1 + r_2)}\right]^{1/2}\left[\frac{2\mu r_3}{r_2(r_2 + r_3)}\right]^{1/2} \cos\vartheta_2 \quad ,$$

and

$$\Delta V_3^{\,2} = \frac{2\mu r_2}{r_3(r_2 + r_3)} + \frac{\mu}{r_3} - 2\left(\frac{\mu}{r_3}\right)^{1/2} \left[\frac{2\mu r_2}{r_3(r_2 + r_3)}\right]^{1/2} \cos \vartheta_3 \quad ,$$

where ϑ_1 defines the angle between the plane of the initial, inner circular orbit and the plane of the first half of the transfer ellipse; ϑ_2 defines the angle between the planes of the first half of the transfer ellipse and the second half of the transfer ellipse; and ϑ_3 defines the angle between the plane of the second half of the transfer ellipse and the plane of the outer circular orbit. Factoring out $\frac{\mu}{r_1}$ and performing other manipulations allow these equations to be written:

$$\Delta V_1^{\,2} = \frac{\mu}{r_1}\left\{1 + \frac{2r_2}{(r_1 + r_2)} - 2\left[\frac{2r_2}{(r_1 + r_2)}\right]^{1/2} \cos \vartheta_1\right\} \quad ,$$

$$\Delta V_2^{\,2} = \left(\frac{\mu}{r_1}\right)\left[\frac{2r_1^{\,2}}{r_2(r_1 + r_2)}\right]\left\{1 + \frac{r_3(r_1 + r_2)}{r_1(r_2 + r_3)} - 2\left[\frac{r_3(r_1 + r_2)}{r_1(r_2 + r_3)}\right]^{1/2} \cos \vartheta_2\right\} \quad ,$$

and

$$\Delta V_3^{\,2} = \left(\frac{\mu}{r_1}\right)\left(\frac{r_1}{r_3}\right)\left[1 + \frac{2r_2}{(r_2 + r_3)} - 2\left(\frac{2r_2}{r_2 + r_3}\right)^{1/2} \cos \vartheta_3\right]$$

The total ΔV is

$$\Delta V_{total} = \Delta V_1 + \Delta V_2 + \Delta V_3 \quad ,$$

and if it is referenced to the initial circular velocity, $\sqrt{\frac{\mu}{r_1}}$, ΔV_{total} may be written as

$$\Delta V_T' = \frac{\Delta V_{total}}{\sqrt{\frac{\mu}{r_1}}} = \left[1 + \frac{2r_2}{(r_1 + r_2)} - 2\left(\frac{2r_2}{r_1 + r_2}\right)^{1/2} \cos\vartheta_1\right]^{1/2}$$

$$+ \left[\frac{2r_1^2}{r_2(r_1 + r_2)}\right]^{1/2} \left\{1 + \frac{r_3(r_1 + r_2)}{r_1(r_2 + r_3)}\right.$$

$$\left. - 2\left[\frac{r_3(r_1 + r_2)}{r_1(r_2 + r_3)}\right]^{1/2} \cos\vartheta_2\right\}^{1/2}$$

$$+ \left(\frac{r_1}{r_3}\right)^{1/2} \left[1 + \frac{2r_2}{(r_2 + r_3)}\right.$$

$$\left. - 2\left(\frac{2r_2}{r_2 + r_3}\right)^{1/2} \cos\vartheta_3\right]^{1/2} .$$

Making the following replacements:

$$H_1 = \left(\frac{2r_2}{r_1 + r_2}\right)^{1/2} ,$$

$$H_2 = \left[\frac{2r_1^2}{r_2(r_1 + r_2)}\right]^{1/2} ,$$

$$H_3 = \left[\frac{r_3(r_1 + r_2)}{r_1(r_2 + r_3)}\right]^{1/2} ,$$

$$H_4 = \left(\frac{r_1}{r_3}\right)^{1/2} ,$$

and

$$H_5 = \left(\frac{2r_2}{r_2 + r_3}\right)^{1/2} ,$$

the nondimensional (normalized) total impulse (also referred to as the payoff) is

$$\Delta V_T' = (1 + H_1^2 - 2H_1 \cos \vartheta_1)^{1/2} + H_2(1 + H_3^2 - 2H_3 \cos \vartheta_2)^{1/2}$$
$$+ H_4(1 + H_5^2 - 2H_5 \cos \vartheta_3)^{1/2} \quad . \tag{1}$$

The problem statement reads as follows: Choose ϑ_1, ϑ_2, and ϑ_3 to minimize $\Delta V_T'$ subject to the equality constraint

$$\vartheta_1 + \vartheta_2 + \vartheta_3 = \vartheta_T \quad , \tag{2}$$

where ϑ_T is the angle between the planes of the initial and final circular orbits, and the inequality constraints are

$$\vartheta_1 \geq 0, \ \vartheta_2 \geq 0, \ \vartheta_3 \geq 0 \quad . \tag{3}$$

Reasons based on physical grounds indicate that the inequality constraints can be neglected in the mathematical solution. However, in a later section, it is demonstrated how even more restrictive inequality constraints are treated and solved. The equality constraint is easily accounted for by solving equation (2) for ϑ_3, for example, and substituting into equation (1). The minimization then proceeds with only two free variables, ϑ_1 and ϑ_2. Since the free variables enter into $\Delta V_T'$ in the arguments of trigonometric functions, a numerical procedure is indicated for the minimization process. Invariably the gradient of the payoff will enter into the minimization process either to establish a descent direction or to indicate when a minimum has been achieved, since a necessary condition for a minimum is that the gradient equal zero. The gradient of $\Delta V_T'$ is

$$\frac{\partial \Delta V_T'}{\partial \vartheta_1} = \frac{H_1 \sin \vartheta_1}{(1 + H_1^2 - 2H_1 \cos \vartheta_1)^{1/2}}$$

$$- \frac{H_4 H_5 \sin (\vartheta_T - \vartheta_1 - \vartheta_2)}{\left[1 + H_5^2 - 2H_5 \cos (\vartheta_T - \vartheta_1 - \vartheta_2)\right]^{1/2}} \qquad (4a)$$

$$\frac{\partial \Delta V_T'}{\partial \vartheta_2} = \frac{H_2 H_3 \sin \vartheta_2}{(1 + H_3^2 - 2H_3 \cos \vartheta_2)^{1/2}}$$

$$- \frac{H_4 H_5 \sin (\vartheta_T - \vartheta_1 - \vartheta_2)}{\left[1 + H_5^2 - 2H_5 \cos (\vartheta_T - \vartheta_1 - \vartheta_2)\right]^{1/2}} \quad . \qquad (4b)$$

Comparing equations (1) and (4) shows that the same radicals are involved in each. Previous experience showed that when the apogee of the transfer ellipse was much larger than the radius of the inner circular orbit, the numerical minimization proceeds without incident. However, for the close-in orbit transfers considered herein, one and possibly two difficulties arise: (1) the first evolves because the radicals of equations (1) and (4) involve the differencing of quantities, each of which is about two in magnitude, and this subtraction results in a serious loss of significant digits; and (2) if the numerical minimization procedure is a zero finding algorithm wherein the minimum is defined by finding the roots of equation (4), the Jacobian of equation (4) can be involved. As it results, for the close-in transfers considered here, two of the angles are very small. In terms of the small difference between the cosines of these small angles and one, it can be shown that terms appear in the Jacobian whose order of magnitude is the square of these differences. Typically, for differences of the order 10^{-3}, terms obtained by subtraction appear in the Jacobian of the order of 10^{-6}. This extreme loss of precision dictates that most standard gradient or Newton-Raphson techniques will behave erratically and, when they do converge, will converge to false minima. This type of performance was in fact observed with some very good existing computer programs.

The answers to both difficulties lie in a detailed inquiry into and analysis of the underlying conditions causing the loss of precision. This task is simplified if an analysis of the Jacobian can be eliminated and consideration can be limited to the payoff and its gradient. This is the case for conjugate gradient numerical minimization techniques of which a Sorensen [1] modified Fletcher-Reeves [2] version was chosen for the numerical results reported here. The version used here will be discussed and compared extensively with other techniques in a separate report. Let it suffice to say that conjugate gradient algorithms proceed along successive paths of descent with local minima of the payoff being found along each path. At each iteration, only the payoff and its gradient need be known. Assuming accurate numerical information, successive iterations are guaranteed to decrease the payoff until final convergence is achieved. A basic property is that a payoff quadratic in n variables will be minimized in n iterations.

NUMERICAL ANALYSIS

The numerical analysis begins by observing that H_1, H_3, and H_5 are all greater than one for the problem considered. Taking H_1 as typical, let

$$H_1 = 1 + \Delta_1 \quad . \tag{5}$$

Now,

$$\begin{aligned} H_1^2 - 2H_1 \cos \vartheta_1 &= H_1^2 - 2H_1 \cos \vartheta_1 + \cos^2 \vartheta_1 - \cos^2 \vartheta_1 \\ &= (H_1 - \cos \vartheta_1)^2 - \cos^2 \vartheta_1 \quad . \end{aligned}$$

Thus,

$$\begin{aligned} (1 + H_1^2 - 2H_1 \cos \vartheta_1)^{1/2} &= \left[1 + (H_1 - \cos \vartheta_1)^2 - \cos^2 \vartheta_1\right]^{1/2} \\ &= \left[(H_1 - \cos \vartheta_1)^2 + \sin^2 \vartheta_1\right]^{1/2} \quad . \end{aligned}$$

Substituting equation (5) into the right side of the preceding equation yields

$$(1 + H_1^2 - 2H_1 \cos \vartheta_1)^{1/2} = \left[(1 + \Delta_1 - \cos \vartheta_1)^2 + \sin^2 \vartheta_1\right]^{1/2} .$$

Further, let $\cos \vartheta_1 = 1 - \delta_1$; then,

$$(1 + H_1^2 - 2H_1 \cos \vartheta_1)^{1/2} = \left\{ \left[1 + \Delta_1 - (1 - \delta_1)\right]^2 + \sin^2 \vartheta_1 \right\}^{1/2}$$

$$= \left[(\Delta_1 + \delta_1)^2 + \sin^2 \vartheta_1\right]^{1/2} . \quad (6a)$$

Similarly,

$$(1 + H_3^2 - 2H_3 \cos \vartheta_2)^{1/2} = \left[(\Delta_3 + \delta_3)^2 + \sin^2 \vartheta_2\right]^{1/2} . \quad (6b)$$

and

$$(1 + H_5^2 - 2H_5 \cos \vartheta_3)^{1/2} = \left[(\Delta_5 + \delta_5)^2 + \sin^2 \vartheta_3\right]^{1/2} . \quad (6c)$$

Equations (6) are accurately computable if Δ_i and δ_i are accurately available. To obtain Δ_i, consider H_1 as typical again. From its definition,

$$H_1 = \left(\frac{2r_2}{r_1 + r_2}\right)^{1/2} = \left(\frac{r_2 + r_2 + r_1 - r_1}{r_2 + r_1}\right)^{1/2}$$

$$= \left(1 + \frac{r_2 - r_1}{r_2 + r_1}\right)^{1/2} = (1 + \alpha_1)^{1/2} ,$$

where

$$\alpha_1 = \frac{r_2 - r_1}{r_2 + r_1} .$$

Since $\alpha_1 < 1$, the binomial series converges and represents H_1:

$$H_1 = (1 + \alpha_1)^{1/2} = 1 + \frac{1}{2}\alpha_1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\alpha_1^2}{2!} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{\alpha_1^3}{3!} - \ldots + \ldots \quad .$$

Since $H_1 = 1 + \Delta_1$,

$$\Delta_1 = \frac{1}{2}\alpha_1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\alpha_1^2}{2!} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{\alpha_1^3}{3!} + \ldots (-1)^{n+1}\frac{1}{2^{2(n-1)}}\frac{(2n-3)!}{(n-2)!n!}\alpha_1^n + \ldots \quad n = 2, 3, \ldots \quad . \tag{7a}$$

Similar series hold for H_3 and H_5:

$$\Delta_3 = \frac{1}{2}\alpha_3 - \frac{1}{2^2}\frac{\alpha_3^2}{2!} + \ldots \frac{(-1)^{n+1}}{2^{2(n-1)}}\frac{(2n-3)!}{(n-2)!n!}\alpha_3^n + \ldots \quad n = 2, 3, \ldots \tag{7b}$$

and

$$\Delta_5 = \frac{1}{2}\alpha_5 - \frac{1}{2^2}\frac{\alpha_5^2}{2!} + \ldots \frac{(-1)^{n+1}(2n-3)!}{2^{2(n-1)}(n-2)!n!}\alpha_5^n + \ldots \quad n = 2, 3 \ldots \quad , \tag{7c}$$

where

$$\alpha_3 = \frac{r_2(r_3 - r_1)}{r_1(r_2 + r_3)}$$

and

$$\alpha_5 = \frac{r_2 - r_3}{r_2 + r_3} \quad .$$

All these series are valid for $|\alpha_i| < 1.$

The Maclaurin series for cos ϑ is

$$\cos \vartheta = 1 - \frac{\vartheta^2}{2!} + \frac{\vartheta^4}{4!} + \ldots (-1)^{n+1} \frac{\vartheta^{2(n-1)}}{(2n-2)!}$$

$$+ \ldots \quad n = 1, 2, 3, \ldots \quad ,$$

so that

$$\delta_1 = 1 - \cos \vartheta_1 = \frac{\vartheta_1^2}{2!} - \frac{\vartheta_1^4}{4!} + \ldots (-1)^n \frac{\vartheta_1^{2n}}{(2n)!}$$

$$+ \ldots \quad n = 1, 2, 3, \ldots \quad , \tag{8a}$$

$$\delta_3 = \frac{\vartheta_2^2}{2!} - \frac{\vartheta_2^4}{4!} + \ldots (-1)^n \frac{\vartheta_2^{2n}}{(2n)!} + \ldots \quad n = 1, 2, 3, \ldots \quad , \tag{8b}$$

and

$$\delta_5 = \frac{\vartheta_3^2}{2!} - \frac{\vartheta_3^4}{4!} + \ldots (-1)^n \frac{\vartheta_3^{2n}}{(2n)!} + \ldots \quad n = 1, 2, 3, \ldots \quad . \tag{8c}$$

These series are valid for all finite ϑ.

The series in equations (7) and (8) are alternating series so that the error committed by truncating them at any term is less than the value of the first term in the remainder. Assuming that equation (7) will be used only if $|\alpha_i| \leq 0.1$, an estimate of the number of terms required to limit the error to $\le 10^{-18}$ is obtained as follows. It is easily seen that the magnitude of each coefficient, a_n, in equation (7) is $\le \frac{1}{2^n}$. Thus,

$$|a_n| (0.1)^n \le \frac{1(0.1)^n}{2^n} = 10^{-18}$$

or

$$-n - n \log 2 = -18;$$

however,

$$\log 2 \approx 0.3 \quad .$$

Therefore,

$$1.3\, n = 18$$

or

$$n \approx 14.$$

Therefore, 13 terms of the series in equation (7) are more than sufficient to compute Δ_i with an error less than 10^{-18} when $\alpha_i \le .1$. The magnitude of the coefficients in equation (8) is $\frac{|\vartheta_i|^{2n}}{(2n)!}$. Limiting $|\vartheta_i| \le 2$ ($\approx 114.6°$), an error estimate for 13 terms is

$$\frac{2^{28}}{(28)!} \approx \frac{2.7 \times 10^{8}}{3 \times 10^{29}} = 0.9 \times 10^{-21} \quad .$$

That is, 13 terms are more than sufficient to limit the error to less than 10^{-18}. No effort was made to sharpen these error estimates or economize the series used.

UNCONSTRAINED NUMERICAL EXAMPLES

The foregoing simple analysis is sufficient to eliminate the difficulties discussed earlier. Interestingly, numerical results indicate multiple solutions, the solution obtained being dependent on the starting point since a gradient technique will, or at least should, go to the bottom of whatever valley it begins in. Recalling the implicit inequality constraints of equation (3), the solution points lie on a plane as shown in Figure 2. The payoff along the boundary of this plane can be explicitly computed. As it results the true minima lie on the interior of this plane. These were found by beginning the algorithm at each of the vertices in turn and proceeding to the local minimum. All of the minima lie close to a vertex, and only a maximum of three was ever found, even though in some cases many starting points within the plane were attempted. The global minimum is found by comparison of the local minima. It is not the purpose of this report to present an analysis of close-in orbital transfer; therefore, Tables 1 through 8 are presented only to show the speed and accuracy with which solutions are obtained for representative situations. A general feature of these tables is that the global minimum occurs when the maximum plane change takes place at the maximum radius, i.e., apogee, of the transfer ellipse.

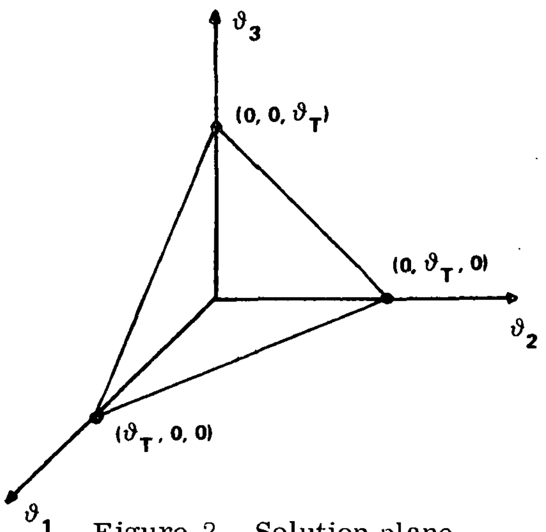

Figure 2. Solution plane.

CONSTRAINED NUMERICAL EXAMPLES

An easily applied transformation technique can be used to account for even more general inequality constraints than those in equation (3). Consider inequalities of the form

$$L_i \leq \vartheta_i \leq U_i \qquad i = 1, 2, 3 \qquad ,$$

where L_i and U_i are lower and upper bounds respectively on ϑ_i. Then, the following transformations automatically satisfy these inequalities:

$$\vartheta_i = L_i + (U_i - L_i) \sin^2 \chi_i \quad ,$$

where χ_i becomes the new minimization variables. These transformations need to be carefully applied. To avoid an unnecessarily complicated explanation, the application used here is to consider explicit limitations on ϑ_1 and ϑ_2 of the form

$$\vartheta_1 = c_1 \sin^2 \chi_1$$

and

$$\vartheta_2 = c_2 \sin^2 \chi_2 \quad ,$$

where $c_1 > 0$ and $c_2 > 0$ and $c_1 + c_2 \leq \vartheta_T$.

These introduce the partials $\frac{\partial \vartheta_1}{\partial \chi_1}$ and $\frac{\partial \vartheta_2}{\partial \chi_2}$ into the gradient equations (4) in a simple way; i.e., $\frac{\partial \Delta V_T'}{\partial \chi_1} = \frac{\partial \Delta V_T'}{\partial \vartheta_1} \frac{\partial \vartheta_1}{\partial \chi_1}$ and $\frac{\partial \Delta V_T'}{\partial \chi_2} = \frac{\partial \Delta V_T'}{\partial \vartheta_2} \frac{\partial \vartheta_2}{\partial \chi_2}$.

The constrained minimization results depend on the values given c_1 and c_2, and it might be expected that either, both, or neither ϑ_1 and ϑ_2 lie on their respective boundaries. Table 9, which can be compared to Table 1, illustrates all four possibilities. It shows that this type of constrained minimization is no more difficult than unconstrained minimization.

CONCLUSIONS

The numerical difficulties of determining optimal plane changes associated with a particular class of three-impulse transfer between noncoplanar, circular orbits have been eliminated using simple series. The conjugate gradient algorithm developed resulted in an ideal tool to obtain very accurate solutions, simply and rapidly, for unconstrained and constrained angles.

TABLE 1. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 150 N. MI. CIRCULAR ORBITS INCLINED AT 28.5 DEGREES (TRANSFER APOGEE IS 200 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	28.5	0.56E-03	0.50013379
1	1.46114	1.46114	25.5777	0.15E-08	0.49216636
2	1.51796	1.40452	25.5775	0.43E-09	0.49216479
3	1.49644	1.33453	25.6690	0.53E-11	0.49216410
4	1.49324	1.33551	25.6712	0.11E-12	0.49216410
5	1.49344	1.33683	25.6697	0.50E-18	0.49216410
6	1.49344	1.33683	25.6697	0.0	0.49216410
0	0.0	28.5	0.0	0.28E-03	0.49333864
1	0.0	27.8431	0.656886	0.28E-03	0.49091133
2	1.22383	26.6175	0.658645	0.20E-08	0.48613730
3	1.22387	26.6016	0.674486	0.20E-12	0.48613706
4	1.22401	26.6016	0.674347	0.73E-13	0.48613706
5	1.22423	26.6013	0.674492	0.72E-19	0.48613706
0	28.5	0.0	0.0	0.29E-03	0.50096085
1	27.5403	0.0	0.959736	0.29E-03	0.49823819
2	1.23454	26.2510	1.01448	0.27E-06	0.48621170
3	1.43016	26.2553	0.814521	0.39E-07	0.48617471
4	1.42698	26.3974	0.675637	0.44E-07	0.48615935
5	1.22422	26.6023	0.673492	0.74E-11	0.48613706
6	1.22423	26.6013	0.674492	0.11E-16	0.48613706
7	1.22423	26.6013	0.674493	0.39E-17	0.48613706
8	1.22423	26.6013	0.674492	0.43E-23	0.48613706

a. Gradient $= \left(\frac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\frac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 2. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 150 N. MI. CIRCULAR ORBITS INCLINED AT 60 DEGREES (TRANSFER APOGEE IS 200 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	60.0	0.45E-03	1.0051436
1	0.546747	0.546747	58.9065	0.17E-05	1.0000558
2	0.707780	0.404563	58.8877	0.61E-07	0.99991549
3	0.688537	0.376323	58.9351	0.63E-08	0.99991119
4	0.678625	0.383077	58.9383	0.34E-09	0.99991071
5	0.676701	0.381311	58.9420	0.10E-12	0.99991069
6	0.676738	0.381284	58.9420	0.14E-18	0.99991069
7	0.676738	0.381284	58.9420	0.50E-20	0.99991069
0	0.0	60.0	0.0	0.22E-03	0.99138951
1	0.0	59.6739	0.326106	0.22E-03	0.98974128
2	0.641237	59.0316	0.327157	0.24E-08	0.98646520
3	0.641244	59.0287	0.330028	0.24E-14	0.98646515
4	0.641246	59.0287	0.330026	0.10E-14	0.98646515
5	0.641251	59.0287	0.330028	0.44E-21	0.96640515
0	60.0	0.0	0.0	0.23E-03	1.0103875
1	59.6427	0.0	0.357284	0.23E-03	1.0086236
2	59.2477	0.394300	0.357992	0.84E-09	1.0067450
3	59.2456	0.394303	0.360086	0.68E-14	1.0067449
4	59.2456	0.394308	0.360082	0.20E-14	1.0067449
5	59.2456	0.394313	0.360086	0.44E-21	1.0067449

a. Gradient $= \left(\frac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\frac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 3. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 150 N. MI. CIRCULAR ORBITS INCLINED AT 28.5 DEGREES (TRANSFER APOGEE IS 151 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	28.5	0.56E-03	0.49593920
1	0.799631	0.799631	26.9007	0.14E-07	0.49066539
2	0.734489	0.865430	26.9001	0.23E-09	0.49065984
3	0.742474	0.875737	26.8818	0.36E-11	0.49065974
4	0.741365	0.876595	26.8820	0.17E-12	0.49065974
5	0.741536	0.876955	26.8815	0.55E-18	0.49065974
6	0.741536	0.876956	26.8815	0.40E-26	0.49065974
0	0.0	28.5	0.0	0.28E-03	0.49155631
1	0.0	28.4847	0.015318	0.28E-03	0.49150473
2	0.703079	27.7816	0.0153418	0.54E-09	0.48894035
3	0.703080	27.7813	0.0155842	0.18E-16	0.48894035
4	0.703080	27.7813	0.0155842	0.17E-16	0.48894035
5	0.703082	27.7813	0.0155842	0.29E-21	0.48894035
0	28.5	0.0	0.0	0.29E-03	0.49675259
1	28.4811	0.0	0.0189021	0.29E-03	0.49669797
2	27.4330	1.04801	0.0189459	0.12E-08	0.49389062
3	27.4324	1.04801	0.0196373	0.50E-15	0.49389061
4	27.4324	1.04801	0.0196367	0.48E-15	0.49389061
5	27.4323	1.04805	0.0196374	0.50E-23	0.49389061

a. Gradient $= \left(\frac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\frac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 4. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 150 N. MI. CIRCULAR ORBITS INCLINED AT 60 DEGREES (TRANSFER APOGEE IS 151 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	60.0	0.45E-03	1.0001036
1	0.351493	0.351493	59.2970	0.66E-08	0.99659350
2	0.346132	0.356896	59.2970	0.47E-11	0.99659319
3	0.346273	0.357045	59.2967	0.14E-14	0.99659319
4	0.346270	0.357047	59.2967	0.31E-17	0.99659319
5	0.346270	0.357047	59.2967	0.73E-25	0.99659319
0	0.0	60.0	0.0	0.22E-03	0.99470577
1	0.0	59.9931	0.00687260	0.22E-03	0.99467142
2	0.341285	59.6518	0.00688441	0.14E-09	0.99293887
3	0.341285	59.6518	0.00689966	0.88E-19	0.99293887
0	60.0	0.0	0.0	0.23E-03	1.0053444
1	59.9928	0.0	0.00716288	0.23E-03	1.0053089
2	59.6249	0.367966	0.00717615	0.83E-09	1.0035026
3	59.6248	0.367966	0.00721726	0.11E-18	1.0035026
4	59.6248	0.367966	0.00721726	0.12E-18	1.0035026
5	59.6248	0.367966	0.00721726	0.30E-19	1.0035026

a. Gradient $= \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 5. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 110 N. MI. CIRCULAR ORBITS INCLINED AT 28.5 DEGREES (TRANSFER APOGEE IS 150 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	28.5	0.57E-03	0.49650885
1	0.638691	0.638691	27.2226	0.15E-06	0.49342843
2	0.975791	0.312526	27.2117	0.55E-07	0.49334451
3	0.947248	0.197258	27.3555	0.23E-07	0.49332974
4	0.807265	0.231993	27.4607	0.43E-08	0.49331817
5	0.789268	0.208230	27.5025	0.46E-10	0.49331723
6	0.793417	0.206754	27.4998	0.24E-12	0.49331722
7	0.793476	0.206919	27.4996	0.11E-13	0.49331722
8	0.793554	0.206908	27.4995	0.34E-20	0.49331722
0	0.0	28.5	0.0	0.28E-03	0.49361548
1	0.0	27.9421	0.557934	0.28E-03	0.49160491
2	0.689414	27.2518	0.558777	0.27E-09	0.48909007
3	0.689421	27.2465	0.564077	0.25E-13	0.48909004
4	0.689460	27.2465	0.564038	0.72E-14	0.48909004
5	0.689507	27.2464	0.564078	0.18E-21	0.48909004
0	28.5	0.0	0.0	0.29E-03	0.496674
1	27.8294	0.0	0.670588	0.29E-03	0.49454219
2	27.6133	0.215837	0.670905	0.42E-11	0.49397078
3	27.6122	0.215837	0.672001	0.13E-14	0.49397078
4	27.6122	0.215846	0.671992	0.15E-15	0.49397078
5	27.6121	0.215851	0.672001	0.13E-21	0.49397078

a. Gradient $= \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 6. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 110 N. MI. CIRCULAR ORBITS INCLINED AT 60 DEGREES (TRANSFER APOGEE IS 150 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	60.0	0.46E-03	1.0041913
1	0.233672	0.233672	59.5327	0.70E-05	1.0023947
2	0.411512	0.0950075	59.4935	0.86E-06	1.0021102
3	0.399695	0.0697055	59.5306	0.28E-06	1.0020961
4	0.367167	0.0848975	59.5479	0.23E-06	1.0020868
5	0.344792	0.0753710	59.5798	0.42E-08	1.0020823
6	0.346985	0.0740348	59.5790	0.72E-11	1.0020822
7	0.347022	0.0740964	59.5789	0.14E-11	1.0020822
8	0.347115	0.0740836	59.5788	0.13E-17	1.0020822
9	0.347115	0.0740836	59.5788	0.81E-26	1.0020822
0	0.0	60.0	0.0	0.22E-03	0.99617013
1	0.0	59.7323	0.267709	0.22E-03	0.99481282
2	0.334814	59.3970	0.268159	0.73E-09	0.99311326
3	0.334815	59.3957	0.269472	0.28E-14	0.99311324
4	0.334818	59.3957	0.269470	0.86E-15	0.99311324
5	0.334820	59.3957	0.269472	0.86E-23	0.99311324
0	60.0	0.0	0.0	0.23E-03	1.0052485
1	59.7194	0.0	0.280566	0.23E-03	1.0038420
2	59.6447	0.0745448	0.280749	0.43E-10	1.0034762
3	59.6444	0.0745449	0.281102	0.21E-15	1.0034762
4	59.6444	0.0745451	0.281102	0.10E-16	1.0034762
5	59.6444	0.0745452	0.281102	0.20E-22	1.0034762

a. Gradient $= \left(\frac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\frac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 7. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 110 N. MI. CIRCULAR ORBITS INCLINED AT 28.5 DEGREES (TRANSFER APOGEE IS 111 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	28.5	0.57E-03	0.49310948
1	0.167574	0.167574	28.1649	0.15E-08	0.49199596
2	0.172128	0.163036	28.1648	0.38E-11	0.49199583
3	0.171911	0.162796	28.1653	0.58E-14	0.49199583
4	0.171901	0.162804	28.1653	0.46E-18	0.49199583
5	0.171902	0.162805	28.1653	0.81E-26	0.49199583
0	0.0	28.5	0.0	0.29E-03	0.49218163
1	0.0	28.4841	0.0158555	0.29E-03	0.49212881
2	0.169402	28.3147	0.0158615	0.91E-11	0.49155075
3	0.169402	28.3148	0.0158285	0.11E-17	0.49155075
4	0.169402	28.3148	0.0158285	0.94E-18	0.49155075
5	0.169402	28.3148	0.0158285	0.11E-21	0.49155075
0	28.5	0.0	0.0	0.29E-03	0.49327239
1	28.4835	0.0	0.0165287	0.29E-03	0.49321892
2	28.3165	0.166959	0.0165348	0.22E-10	0.49268099
3	28.3166	0.166959	0.0164769	0.32E-16	0.49268099
4	28.3166	0.166959	0.0164770	0.27E-16	0.49268099
5	28.3166	0.166958	0.0164769	0.21E-21	0.49268099

a. Gradient $= \left(\frac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\frac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 8. DISTRIBUTION OF PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 110 N. MI. CIRCULAR ORBITS INCLINED AT 60 DEGREES (TRANSFER APOGEE IS 111 N. MI.)

Iteration	ϑ_1	ϑ_2	ϑ_3	Gradient[a]	Payoff
0	0.0	0.0	60.0	0.46E-03	1.0001054
1	0.0735228	0.0735228	59.8530	0.50E-06	0.99936690
2	0.0766450	0.0704655	59.8529	0.26E-10	0.99936641
3	0.0765758	0.0703931	59.8530	0.59E-13	0.99936641
4	0.0765724	0.0703964	59.8530	0.73E-16	0.99936641
5	0.0765723	0.0703963	59.8530	0.81E-21	0.99936641
0	0.0	60.0	0.0	0.23E-03	0.99887366
1	0.0	59.9930	0.00698963	0.23E-03	0.99883858
2	0.0762936	59.9167	0.00699231	0.94E-10	0.99845353
3	0.0762936	59.9167	0.00697960	0.34E-19	0.99845353
0	60.0	0.0	0.0	0.23E-03	1.0011622
1	59.9929	0.0	0.00705916	0.23E-03	1.0011269
2	59.9221	0.0708007	0.00706168	0.15E-09	1.0007720
3	59.9222	0.0708007	0.00704546	0.11E-18	1.0007720
4	59.9222	0.0708007	0.00704546	0.55E-19	1.0007720

a. Gradient $= \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

TABLE 9. DISTRIBUTION OF CONSTRAINED PLANE CHANGES FOR IMPULSIVE TRANSFER BETWEEN 100 AND 150 N. MI. CIRCULAR ORBITS INCLINED AT 28.5 DEGREES (TRANSFER APOGEE IS 200 N. MI.)

Iteration	ϑ_1 [a]	ϑ_2 [a]	ϑ_3 [a]	Gradient[b]	Payoff
Case 1: 5 degree limit on ϑ_1, 5 degree limit on ϑ_2					
0	1.25	1.25	26.0	0.92E-06	0.49218937
1	1.4919609	1.2670539	25.740985	0.69E-08	0.49216474
2	1.4929680	1.3367815	25.670251	0.23E-11	0.49216410
3	1.4934366	1.3368224	25.669741	0.13E-16	0.49216410
4	1.4934363	1.3368255	25.669738	0.12E-17	0.49216410
5	1.4934366	1.3368256	25.669738	0.40E-24	0.49216410
Case 2: 1 degree limit on ϑ_1, 5 degree limit on ϑ_2					
0	0.25	1.25	27.0	0.43E-04	0.49432670
1	0.99999878	1.1942897	26.305712	0.12E-07	0.49229098
2	0.99999969	1.2747884	26.225212	0.48E-12	0.49228996
3	1.0000000	1.2747736	26.225226	0.45E-17	0.49228996
4	1	1.2747719	26.225228	0.12E-21	0.49228996
Case 3: 5 degree limit on ϑ_1, 1 degree limit on ϑ_2					
0	1.25	0.25	27	0.10E-04	0.49291871
1	1.9253040	0.78144550	25.793250	0.13E-05	0.49229192
2	1.4313804	0.83753959	26.231080	0.57E-07	0.49221594
3	1.5509740	0.99627414	25.952752	0.54E-07	0.49218585
4	1.4770807	0.99641748	26.026502	0.27E-09	0.49218394
5	1.4778344	1.0000000	26.022166	0.55E-12	0.49218345
6	1.4776119	1	26.022388	0.11E-18	0.49218345
7	1.4776123	1	26.022388	0.15E-17	0.49218345
8	1.4776120	1	26.022388	0.35E-24	0.49218345
Case 4: 1 degree limit on ϑ_1, 1 degree limit on ϑ_2					
0	0.25	0.25	28.0	0.52E-04	0.49502071
1	0.98877934	0.76936973	26.741851	0.97E-07	0.49236042
2	0.99685410	0.98658947	26.516556	0.57E-08	0.49230787
3	0.99990077	0.99604523	26.504054	0.38E-09	0.49230480
4	0.99991152	0.99849260	26.501596	0.22E-09	0.49230450
5	0.99999996	0.99999993	26.500000	0.71E-13	0.49230427
6	1.0000000	1.0000000	26.500000	0.19E-18	0.49230427
7	1.0000000	1.0000000	26.50000	0.61E-19	0.49230427
8	1	1	26.5	0.21E-27	0.49230427

a. Initially, $\chi_1 = \chi_2 = 30$ degrees

b. Gradient $= \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_1}\right)^2 + \left(\dfrac{\partial \Delta V_T'}{\partial \vartheta_2}\right)^2$

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APPROVAL

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CONJUGATE GRADIENT DETERMINATION OF OPTIMAL PLANE CHANGES FOR A CLASS OF THREE-IMPULSE TRANSFERS BETWEEN NONCOPLANAR CIRCULAR ORBITS

By Roger R. Burrows

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This document has also been reviewed and approved for technical accuracy.

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