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STABILITY OF ROTATING MAGNETIC STARS AND PLANETS AS JUPITER AND THE ORIGIN OF CONVECTIVE MOTION

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ABSTRACT

Based on the method of the energy principle, the effect of the Coriolis force on the stability of rotating hydromagnetic systems is examined and the condition for instability is derived. It is shown that, in the rotating systems, the effect of this force is to inhibit the onset of convective motion and that, once the condition for instability is fulfilled, over-stable states are produced with respect to this motion. Such states seem to be effective on the onset of turbulent convective motion in the systems.

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I. INTRODUCTION

When a rotating magnetic star or planet is in hydrostatic equilibrium, this state is expressed by

$$\rho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\nabla P + \frac{1}{c}\vec{j} \times \vec{B} + \rho\vec{g} \qquad (1)$$

$$\mathrm{curl}\ \vec{B} = \frac{4\pi}{c}\vec{j} \qquad (2)$$

and

$$\mathrm{div}\ \vec{B} = 0, \qquad (3)$$

where ρ, $\vec{\Omega}$, $\vec{r}$, P, c, $\vec{j}$, $\vec{B}$ and $\vec{g}$ are the mass density, the angular velocity, the position vector, the pressure, the speed of light, the electric current, the magnetic field and the gravity force, respectively.

If we take into account the finiteness of the electrical conductivity of the system, though very high, it is shown that the magnetic fields inside the system would necessarily decay gradually as a result of the Joule dissipation. The time for this decay is given by

$$\tau = \frac{4\pi\ \sigma}{c^2} L^2, \qquad \text{(c.g.s.e.s.u)}$$

where σ and L are the electrical conductivity and the characteristic length of the system under consideration, respectively (e.g., Cowling, 1946, 1953). If we assume that the electrical conductivity σ is numerically given as $\sim 10^{12}$ e.s.u., for instance, in the interior of the earth and Jupiter, the above times for these two planets are estimated as $\sim 10^4$ and $\sim 10^6$ years, respectively. These magnitudes are too short to explain the magnetism of the earth and Jupiter by using the idea of "fossil" magnetism as proposed by Cowling (1946).

In order to maintain the magnetism of the earth, for example, we should, therefore, consider the re-generation process of magnetic fields in its interior. As we have shown above, it is evident that the system, in which there is no fluid motion, is not capable of preventing its own magnetic field from decaying completely. Some type of convective motions, which seem to be effective on re-generation or amplification of magnetic fields, must, therefore, exist in order to explain the origin and maintenance of the magnetic fields of stellar objects such as the sun, Jupiter and the earth.

2. STABILITY IN THE NON-ROTATING SYSTEMS

Since the onset of convective motions is closely related to the stability of the systems, it is worthwhile to consider the criterion on their stability. A general method for studying hydromagnetic stability problems has been established with respect to the non-rotating systems (Bernstein et al., 1958). The criterion on the onset of instability in such systems is given by a following inequality:

$$\int \vec{\xi}^* \vec{F}(\vec{\xi})\, dv > 0$$

In fact, any system which fulfilles this inequality is always unstable (Bernstein et al., 1958). As will be later discussed, $\vec{\xi}$ expresses a given displacement vector as applied to the system under consideration and $\vec{\xi}^*$ is its complex conjugate. $\vec{F}(\vec{\xi})$ is given by equation (2-20) in the paper of Bernstein et al. (1958) and gives a generalized force induced by $\vec{\xi}$ (see (5) later).

3. STABILITY IN THE ROTATING SYSTEMS

When we investigate the stability of rotating hydrogmagnetic systems, we need consider the effect of the Coriolis force. Thus the discussion on stability of these systems is different from the forgoing one as made by Bernstein et al. (1958).

For the rotating systems, the equation of motion is given as

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} + 2\rho\, \vec{\Omega} \times \frac{\partial \vec{\xi}}{\partial t} = \vec{F}(\vec{\xi}) \tag{4}$$

and

$$\vec{F}(\vec{\xi}) = \text{grad}\ [\gamma\ P\ \text{div}\ \vec{\xi} + (\vec{\xi}\cdot\text{grad})\ P] + \vec{j} \times \vec{Q} - \vec{B} \times \text{curl}\ \vec{Q} + [\text{div}\ (\rho\vec{\xi})]\ \text{grad}\ \phi, \tag{5}$$

where $\vec{\xi}$, t, ϕ and γ are the displacement vector due to perturbation, the time, the gravitational potential and the ratio of two specific heats. Here $\vec{Q}(\vec{\xi})$ is given by

$$\vec{Q}(\vec{\xi}) = \text{curl } (\vec{\xi} \times \vec{B}) \tag{6}$$

In the above equation, gradϕ consists of two terms as the gravitational and the centrifugal forces, but, in general, the contribution of centrifugal force is negligibly small (e.g., Chandrasekhar, 1961). In case which the system is rotating so fast like pulsars, the force $\vec{F}(\vec{\xi})$ necessarily becomes a function of Ω^2 as suggested by Steinitz (1965). In this paper, we, however, will not consider such cases.

Here we assume that $\vec{\xi}$ is expressed as

$$\vec{\xi} = \vec{\xi}_o (\vec{r}) e^{i\omega t}, \tag{7}$$

where $\vec{\xi}_o (\vec{r})$ is a complex positional vector. By substituting (7) into (4), we obtain

$$- \omega^2 \rho \vec{\xi} + 2i\omega\rho (\vec{\Omega} \times \vec{\xi}) = \vec{F}(\vec{\xi}) \tag{8}$$

By multiplying scalarly $\vec{\xi}^*$ (a complex conjugate of $\vec{\xi}$) with (8), we futher obtain

$$- \omega^2 \rho \vec{\xi}^* \vec{\xi} + 2i\omega\rho \vec{\xi}^* (\vec{\Omega} \times \vec{\xi}) = \vec{\xi}^* \cdot \vec{F}(\vec{\xi}) \tag{9}$$

It is known that $\vec{F}(\vec{\xi})$ is self-adjoint and therefore the right side of (10) is always real (e.g., Bernstein et al., 1958). This equation is here integrated over the volume V, in which the system under consideration is involved, with the boundary condition as

$$\vec{n} \cdot \vec{\xi} = 0,$$

where $\vec{n}$ is the unit vector perpendicular to the surface which encloses the above volume. Thus we obtain

$$- \omega^2 \int \rho \; \vec{\xi}^* \vec{\xi} \; dv + 2\omega \int \rho \vec{\xi}^* \; i(\vec{\Omega} \times \vec{\xi}) \; dv$$

$$= \int \vec{\xi}^* \cdot \vec{F}(\vec{\xi}) \; dv \qquad (10)$$

It is clear that the first term on the left side of the above equation is real and positive.

Since the vectorial operator $i[\vec{\Omega} \times \cdots\cdot]$ is Hermitian (see Appendix 1), the second term on the left side of (10) is also real. Thus, we can solve the above equation algebraically with respect to ω.

The solution of (10) for ω is given by

$$\omega = \frac{1}{\int \rho \; \vec{\xi}^* \vec{\xi} \; dv} \left\{ \int i \; \rho \; \vec{\xi}^* \; (\vec{\Omega} \times \vec{\xi}) \; dv \right.$$

$$\left. \pm \left([\int i \; \rho \; \vec{\xi}^* \; (\vec{\Omega} \times \vec{\xi}) \; dv]^2 - [\int \rho \; \vec{\xi}^* \vec{\xi} \; dv] \; [\int \vec{\xi}^* \; \vec{F} \; (\vec{\xi}) \; dv]^{\frac{1}{2}} \right\} . \qquad (11)$$

In order that ω is real, it is necessary that

$$\int \vec{\xi}^* \cdot \vec{F} \; (\vec{\xi}) \; dv \angle \; 0 \qquad (12)$$

because

$$\int \rho \; \vec{\xi}^* \vec{\xi} \; dv > 0 \text{ and } [\int i \; \rho \; \vec{\xi}^* \; (\vec{\Omega} \times \vec{\xi}) \; dv]^2 > 0$$

Inequality (12) gives a condition that the system under consideration is stable. In so far as this inequality is fulfilled, the system is always stable.

Even if inequality (12) is not fulfilled, the system is also stable when

$$[\int i\ \rho\ \vec{\xi}^*(\vec{\Omega}\times\vec{\xi})\ dv]^2 - [\int \rho\ \vec{\xi}^*\vec{\xi}\ dv]\ [\int \vec{\xi}^*\cdot\vec{F}(\vec{\xi})\ dv] > 0. \tag{13}$$

In consequence, the system is <u>unstable</u> when the above two inequalities, (12) and (13), are not fulfilled: when inequaltiy

$$[\int i\rho\ \vec{\xi}^*(\vec{\Omega}\times\vec{\xi})\ dv]^2 \Big/ [\int \rho\ \vec{\xi}^*\vec{\xi}\ dv] < \int \vec{\xi}^*\cdot\vec{F}(\vec{\xi})\ dv \tag{14}$$

is fulfilled, the system is always unstable. The left side of the above inequality is always positive unless $\vec{\Omega} = 0$.

Although the form of $\vec{\xi}$ is not specified in the above inequality, the result of (14) gives a general criterion on instability of the system. Only when this inequality is fulfilled, the system is unstable. It should be remarked that the lowest value of the onset of instability increases in proportion with Ω^2 as is deduced from the left side of (14). This means that the effect of the Coriolis force is to inhibit the onset of convection.

4. <u>DISCUSSION ON THE ONSET OF CONVECTIVE MOTION</u>

Since the left side of (14) is always positive, it is clear that the effect of the Coriolis force is to stabilize the system by inhibiting the onset of convective motion.

In other words, the rotating system would, in general, be stabilized due to the action of the Coriolis force, because the lowest value for the onset of instability is raised by the amount of $[\int i\rho\vec{\xi}^{*}\ (\vec{\Omega}\times\vec{\xi})\ dv]^2 / [\int \rho\vec{\xi}^{*}\vec{\xi}\ dv]$ (> 0) in comparison with that for the non-rotating systems. The stabilizing effect by the Coriolis force has also been found in the case for the rotating fluid heated below (Chandrasekhar, 1953).

Once inequality (14) is fulfilled, the system moves to an over-stable state as is deduced from (11). Then convective motion is set up in the system, but is oscillatory in nature.

In these above discussions, we had better remarked some restriction on the applicability of the formulae considered in the last section, because the present discussion cannot be applied to the case, in which the effect of the Coriolis force is so strong that the same perturbation $\vec{\xi}$ cannot be used for both the rotating and non-rotating systems. Namely, our present result is only useful for the rotating systems where the term of the Coriolis force can be treated as a small perturbation with respect to the non-rotating systems.

As shown above, the onset of over-stability is usually inhibited by the effect of the Coriolis force. However, once the state of over-stability is reached, convective motion is generated and, later on, seems to become turbulent due to the further growth of over-stability in the system. Since the build-up of convective motion is highly inhibited in comparison with the case for the non-rotating systems, it can be said that the rotating stars, in which there exists convective motion, are relatively rare.

At present, we have known that, although stars are more or less rotating, all of them do not necessarily have their own magnetic fields. In the very early history of their evolution, the most stars might have captured the magnetic fields in the interstellar space and therefore they must have had their own magnetic fields during their young ages. As discussed earlier in this paper, these fields, however, might have decayed throughout their evolution by the Joule dissipation unless there existed convective motion in their interior. Since the effect of the Coriolis force is to inhibit the onset of convective motion, the rotation of stars must have worked as an obstracle on the maintenance of their own magnetic fields while the systems are stable. This fact seems to be one of the reasons why, at present, very limited number of stars and planets only have their own magnetic fields.

For the first time, the effect of the Coriolis force is to inhibit the onset of convective motion. Once this motion is set up, it seems, however, that the role of this force enforces to build up turbulent motion as a result of over-stability. Important is that turbulent motion in the rotating systems is non-reflectionsymmetric on account of the action of this force (e.g., Braginskii, 1965; Steenbeck and Krause, 1969). Thus the onset of turbulent motion in the rotating systems seems to be very important from the view point on the dynamo action for the maintenance of the magnetic fields of stellar objects (e.g., Parker, 1969, 1970).

5. CONCLUDING REMARKS

In this paper, we have shown that the effect of the Coriolis force is to inhibit the onset of convection in the rotating hydromagnetic systems. Since, as is evident from (14), the lowest state on the onset of convective motion is proportional to Ω^2, the effect of the Coriolis force becomes larger as the magnitude of Ω increases. In this case, the state which is not stable is always over-stable. Turbulent motion which seems to develop from the oscillatory growth of convective motion would be effective on the dynamo action on magnetic fields in the systems.

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APPENDIX 1

The second term on the left-hand side of (10) is rewritten as

$$\int \vec{\xi}^* \, i(\vec{\Omega} \times \vec{\xi}) \, dv = \Omega \int \vec{\xi}^* \, i(\vec{e} \times \vec{\xi}) \, dv, \qquad \text{(A-1)}$$

where $\vec{e}$ is the unit vector in the direction of $\vec{\Omega}$.

Let us examine whether the above term fulfIles the following equality

$$\int \bar{\varphi} \, A \, \psi \, dv = \int \overline{A \, \psi} \, \varphi \, dv, \qquad \text{(A-2)}$$

where φ and ψ are functions of complex variables and A is the operator (see Schiff, 1968). In our case,

$$\bar{\varphi} = \vec{\xi}^*, \quad \psi = \vec{\xi}$$

and

$$A = i[\vec{e} \times \cdots].$$

By using these definitions, we calculate the right-hand side of (A-2):

$$- \Omega \int \overline{i(\vec{e} \times \vec{\xi})} \cdot \vec{\xi} \, dv$$

$$= \Omega \int i \, \bar{\vec{\xi}} \, (\vec{e} \times \vec{\xi}) \, dv = \Omega \int i(\vec{e} \times \bar{\vec{\xi}}) \, \vec{\xi} \, dv$$

$$= \Omega \int i \, \vec{\xi}^* \, (\vec{e} \times \vec{\xi}) \, dv \qquad \text{(A-3)}$$

This calculation thus shows that expression (A-1) is Hermitian: i.e., the operator A is Hermitian. This means that $\Omega \int (\vec{e} \times \bar{\vec{\xi}}) \, \vec{\xi} \, dv$ is <u>purely imaginary</u>.

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