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"INVESTIGATION OF MASS TRANSFER BETWEEN TWO PARALLEL WALLS AT DIFFERENT TEMPERATURES BY A MOMENT METHOD"

T. N. Sloat,
R. H. Edwards
and
R. L. Collins

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ABSTRACT

One dimensional flow between two fixed parallel walls composed of the same substance but at different temperatures and spaced a distance ℓ apart is considered. The hot plate is the evaporating surface (source) and the cold plate is the condensing surface (sink). The vapor between the two plates is assumed to be a monatomic gas consisting of Maxwell molecules. Lees' moment method is used to obtain a set of six non-linear equations whose solution, subject to the boundary conditions of this problem, is possibly valid from free molecular to continuum conditions.

Both the non-linear equations and a linearized approximation to them are solved.

The non-linear problem required the solution of six simultaneous and ordinary non-linear differential equations with three boundary conditions given at each wall. An iterative numerical procedure was used to match these boundary conditions. For the continuum limit (Reynolds number large), the vapor leaving the hot plate was found to accelerate rapidly to an equilibrium velocity. In the vicinity of the cold wall, the vapor first decelerated, then experienced a slight terminal acceleration. In the rarefied limit (Reynolds number very small), the vapor velocity was found to be essentially constant across the flow field.

The linearized problem in closed form under the assumption of a small mean velocity is solved. Large and small Knudsen numbers are examined. In both the rarefied and continuum limits, the mean velocity was found to be constant across the flow field. For given emission temperatures and density ratios at the two surfaces, the mean speed between the plates varied with Reynolds number because of the effects of molecular collisions.

The evaporation coefficient is defined here as the ratio of the actual mass flux to the difference between the Knudsen effluxes from the two surfaces. Its value is nearly one for the range of Knudsen numbers considered in both the linear and non-linear problems.

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NOMENCLATURE

B_1 = mass constant

B_2 = momentum constant

B_3 = energy constant

c_i = vector particle velocity

C_i = relative particle velocity

$\bar{C}$ = mean molecular speed $\left(\sqrt{\frac{8RT}{\pi}}\right)$

d = distance between two plates

$E_1, E_2 = 1 + \mathrm{erf}\left[u_1/\sqrt{2RT_1}\right],\ 1 + \mathrm{erf}\left[-u_2/\sqrt{2RT_2}\right]$

f, f_1 = velocity distribution functions for "probe" and colliding particles, respectively

f_1, f_2 = components of two stream Maxwellian

F_i = external force acting on a single particle

h = specific enthalpy

H = total enthalpy

k = Boltzmann constant

Kn = Knudsen number (λ/d)

m = molecular mass

M = Mach number (u/a)

n = number of molecules per unit volume

n_1, n_2 = number density functions in two stream Maxwellian

$N_{1,2}$ = small perturbation to $n_{1,2}$

$N_- = N_1 - N_2$

$N_+ = N_1 + N_2$

p = pressure ($p = \rho RT$)

$\vec{q}$ = heat flux vector

Q = arbitrary function of particle velocity

R = gas constant

Re = Reynolds number $\left(\frac{\rho ud}{\mu}\right)$

S = speed ratio ($u/\sqrt{2RT}$)

t = time

$t_{1,2}$ = small perturbations to $T_{1,2}$

$t_- = t_1 - t_2$

$t_+ = t_1 + t_2$

T = temperature

T_1, T_2 = temperature functions in two stream Maxwellian

u = mean velocity in x direction

u_1, u_2 = velocity functions in two stream Maxwellian

$U = \sqrt{RT_I}$ or $\sqrt{RT_{II}}$

U_1, U_2 = small perturbations to u_1, u_2

x = coordinate in x direction

$X_1, X_2 = \exp\left[-u_{1,2}^2/2RT_{1,2}\right]$

α = the evaporation coefficient

$\beta_{1,2} = 2RT_{1,2}$

γ = ratio of specific heats (c_p/c_v)

ϵ = small parameter

λ = mean free path

μ = viscosity

ν = kinematic viscosity

$\nu_{1,2} = u_{1,2}/\sqrt{2\pi}$

ρ = gas density

σ_{ij} = stress tensor

$\tau_{ij} = \sigma_{ij} + p\,\delta_{ij}$

Subscripts

I indicates hot wall

II indicates cold wall

Chapter 1

INTRODUCTION

When the state of the vapor at a gas-liquid interface is at the saturated temperature and pressure corresponding to surface temperature, an equilibrium situation exists. There is an exact balance between the two molecular processes of evaporation and condensation. If the vapor and liquid surfaces are not in equilibrium, then, a net flux of molecules either condenses on or evaporates from the surface.

In recent years many publications have appeared dealing with evaporation from and condensation on a solid or liquid surface. Evaporation either into a near vacuum or into a gas where there are small deviations from equilibrium at the liquid-vapor interface have been the two areas most thoroughly studied. Little effort has been directed towards investigating the evaporation and condensation process over a wide pressure ratio range.

Prior to the space program, interest in low pressure evaporation and condensation grew from the study of thin film deposition, molecular distillation, and other aspects of vacuum technology. In the last decade the field has expanded to include the study of numerous materials which may evaporate or sublime in a space environment. Such processes produced both forces and heat fluxes at the surface-vapor interface which could be of concern to the spacecraft designer. For example, these forces could overcome the small gravitational torques required for operation of a gravity gradient satellite.

Initial work in the field of low pressure evaporation was conducted by Langmuir [1] in 1913. He was interested in sublimation of incandescent light filaments and developed a semi-empirical expression for the rate of evaporation. Since that time many more experimental studies have been performed to evaluate the evaporation (sublimation) coefficients of a variety of materials. A compilation of experimental evaporation coefficients obtained through 1961 is given by Paul [2]. He indicated that the majority of materials which have been studied evaporate into a vacuum at or near the maximum rate given by the Knudsen-Langmuir expression for effusive flow

$$\dot{m} = 1/4 \, \rho \, \overline{C} \tag{1}$$

where $\overline{C} = \sqrt{\frac{8RT}{\pi}}$ and ρ is the vapor density. In a vacuum environment, the experimental evaporation coefficient is obtained by dividing the measured or calculated mass flux by that given in equation (1). Materials which evaporate at significantly lower rates than the maximum were generally characterized as those which exist in the vapor in forms different from the condensate. These conclusions were in agreement with an earlier study made by Knacke and

Stranski [3]. Paul also noted that unclean surfaces and experimental errors tend to reduce the evaporation coefficient so that in some cases the experimentally determined value may be lower than the true value.

Experimental studies undertaken to determine the evaporation coefficient for 2-ethyl hexyl phthalate in a near vacuum environment led Hickman and Trevoy [4] to notice the effect of a small back pressure on the evaporation coefficient. They found that at a limiting vapor pressure of 1 μ the evaporation coefficient was near unity while a two order of magnitude increase in the vapor pressure reduced the coefficient to 0.75. In a later study with water, Hickman [5] again observed the influence of back pressure on the rate of evaporation. He obtained an evaporation coefficient of 0.25, which was considerably higher than most values given previously.

A number of theoretical investigations have been conducted in which evaporation from a surface was studied with a wide range of back pressures. In 1936 Crout [6] considered the one dimensional problem of evaporation of a monatomic vapor from a surface. He was primarily interested in evaluating the gas properties at the vapor-surface interface. To do this, Crout developed a modified Maxwellian distribution incorporating four constants. The values of the constants were determined by balancing the gas-liquid mass, momentum, and energy fluxes and specifying the rate of evaporation.

A similar analysis was performed by Schrage [7] in 1953. However, he assumed a different form for the distribution function near the surface. The outflow ($u > 0$) was described by a Maxwellian distribution function f_o corresponding to the emitting surface temperature. The back flow ($u < 0$) was assumed to be represented by $f_o(1+BU)$ where B is related to the mass flux and U is the random molecular velocity perpendicular to the surface. As with Crout, the objective of Schrage's analysis was to evaluate gas properties at the interface, but the mass flux was an undetermined parameter. Schrage studied both monatomic and polyatomic vapors.

More recently, Collins and Edwards [8] studied evaporation from a spherical surface into a vacuum or into a pure vapor under strong nonequilibrium conditions. The object of this investigation was to determine the effect of molecular back scatter in the encompassing vapor cloud on the rate of evaporation for strong nonequilibrium conditions. The continuum assumption was applied and both monatomic and diatomic vapors were considered. A Grad representation for the distribution function was used to permit the connection of the surface boundary conditions and the gas dynamics in a consistent manner. It was found that for evaporation into a vacuum with infinite Reynolds number, the evaporation coefficient was independent of surface temperature and equal to 0.8116 for a monatomic gas and 0.7778 for a diatomic gas. Evaporation into a homogeneous vapor was also studied. The evaporation coefficient was found to be greater than one in certain cases. In both problems the rate of evaporation was determined not specified as it was in Crout's [6] and Schrage's [7] case.

Within the last two years three papers have used Lees' moment technique [9] to approximate evaporation and condensation phenomenon.

Patton and Springer [10] studied two quasi-steady problems:

i) evaporation from a plane surface into a vapor

ii) flow between two parallel plates at different temperatures.

In their analysis the vapor is treated as an ideal gas composed of monatomic Maxwell molecules, with a Lees' representation of the distribution function. They employed four moments (mass, x-momentum, energy, and x-heat flux where x is the direction of motion) of the Boltzmann equation and solved a linearized form of the resulting equations in order to obtain an analytical representation for the mass flux in terms of Knudsen number for the two problems considered.

In a similar manner, Sampson and Springer [11] investigated the evaporation of a spherical drop into a pure vapor. Again four moments were taken and the resulting equations linearized to obtain the mass flux. They also considered droplet evaporation into a gas-vapor mixture.

Both condensation of a vapor on a flat surface and evaporation from and condensation on a spherical drop were studied by Shankar [12]. Besides solving both problems using the quasi-steady assumption, he also obtained a solution for the nonsteady flat plate condensation problem. Shankar used the same four moments that were employed by Springer and his associates to determine the mass flux for both problems. Furthermore, he solved the liquid-vapor interface (flat surface) problem with a six moment method and found essentially the same expression for the mass flux as given by the four moment procedure. He did not consider the two plate problem.

The advantage of Lees' [9] multiple moment kinetic theory technique is that it affords a method of solution which is in some circumstances valid over the range of flow conditions from free molecular to continuum. The corresponding equations are, however, so complex that even with a one dimensional evaporation problem it is necessary to linearize the moment equations in order to obtain an analytical solution. There is, as always, a question regarding the range of validity of the linearization. For example, the linearized four moment method used by Springer and his associates and Shankar to solve the flate plate and spherical problems does not allow for a near equilibrium condition to exist with mass flux. Further, the small parameter used to linearize the equations is the mean velocity.

Thus, a problem arises in the case of sumultaneous small velocity and high Reynolds number limits.

In the present investigation the problem of flow between two parallel plates at different temperatures is solved numerically for some cases using Lees' moment method. The influence of the induced vapor cloud on the evaporation rate and the vapor motion between the two plates can be studied over a range of flow conditions. Six moment equations are used along with the proper boundary conditions.

Both the non-linear and linearized problem are solved. The non-linear problem is a two point boundary value problem which is solved numerically. The solution to the two plate problem is given analytically for the case where the equations may be linearized as small deviations from equilibrium.

Several simplifying assumptions are made. They are: the two plates and the vapor are composed of the same substance; the vapor is a monatomic gas consisting of Maxwell molecules and obeys the perfect gas law; the accommodation coefficient is unity, i.e., every molecule that strikes the surface will be absorbed by it; and a Maxwellian distribution corresponding to surface temperature with zero mean velocity describes the molecules emitted by a surface. The possible effect of surface structure on the evaporation and condensation rates is neglected.

Chapter 2

KINETIC THEORY FORMULATION

2.1 The Boltzman Equation

In fluid flow fields where large gradients occur, translation non-equilibrium effects are observed by the presence of viscous stress and heat flux. For such flows the Maxwell distribution function does not adequately represent the translational molecular velocity and it is necessary to obtain other expressions describing the physical process. Consideration of the conservation of mass, momentum and energy in the absence of external forces results in five differential equations describing a larger number of dependent variables. For Newtonian liquids and perfect gases at normal densities it is possible to empirically justify additional constitutive equations and an equation of state which allow the number of dependent variables in the conservation equations to be reduced to five. For gas flows at lower density the constitutive equations are not valid and it is necessary to approach the problem from a different point of view.

A number of methods have been developed to describe gas flows over a range of gas dynamic regimes. The most complete of these is to follow all of the particles in their collisions throughout the flow field. Such attempts have met with little success because of the great complexity, although Monte Carlo techniques which follow representative molecules are being used successfully in some problems. In lieu of following the dynamical trajectories of separate particles it is mathematically feasible to represent an approximate variation of the particle distribution function throughout the flow. Such an approximate formulation is applied in this investigation and is described briefly below.

The variation of the molecular distribution function is governed by the Boltzmann equation. This equation can be derived either by introducing appropriate time averages into Liouville's equation for an N particle system [9] or by writing an equation for the rate of change of the number of particles in a given velocity range (Vincenti and Kruger [13]). It has the form:

$$\frac{\partial f}{\partial t} + c_j \frac{\partial f}{\partial x_j} + \frac{\partial}{\partial c_j}(F_j f) = \int_{-\infty}^{\infty} \int_{dA_c} [f(c_i')f(c_j') - f(c_i)f(c_j)] V dA_c \, d\vec{c}_j, \qquad (2)$$

where f is the distribution function, V is the relative velocity between colliding molecules, and dA_c is the generalized differential collision cross section. Integration is over the velocity space of the colliding molecules. From left to right the terms of this equation may be interpreted as the rate of change of the number of molecules of class c_i which results from convection, external forces, and collisions with other molecules. There are two implicit limitations of the Boltzmann equation [13]. First the range of intermolecular forces of the gas must be small compared to molecular separation which must in turn be small compared to the mean free path. Such a limitation corresponds to the assumption of a thermally perfect gas. Second the distribution function must not change appreciably over a distance of the order of the range of interparticle forces or time interval of the order of the duration of a representative collision.

For the majority of gas dynamic problems of interest these limitations cause no problem. The Boltzmann equation is difficult to solve because of the numbers of molecules involved and because of the complexity introduced by the non-linear collision integral. As a result it is usually necessary to introduce approximate methods. A number of techniques have been developed which exploit the possibility of linearizing the collision integral term. Unfortunately, none of these methods give results applicable over the range of flow regimes from free molecular to continuum and it is necessary to introduce another form of analysis.

2.2 Maxwell's Equation of Transfer

The difficulty inherent in an attempt to solve the Boltzmann equation directly is not the only motivation to find another kinetic theory formulation. Maxwell recognized that it is not the distribution function itself that is of interest but certain lower moments which correspond to physical variables of interest. As a result he developed an integral equation of transfer for any quantity Q which is a function of particle velocity. In general, such an expression may be derived either by considering the sources of change of Q in the physical space or by multiplying the Boltzmann equation (2) by Q and integrating over velocity space. The resulting equation is known as Maxwell's equation of transfer and takes the form:

$$\frac{\partial \overline{Q}}{\partial t} + \frac{\partial}{\partial x_j} (\overline{c_j Q}) - \overline{F_j \frac{\partial Q}{\partial c_j}} = \Delta[Q]$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{dA_c} (Q' - Q) f f_1 V \, d\vec{c} \, d\vec{c}_1 \, dA_c \qquad (3)$$

where $(Q' - Q)$ is the change in $Q(c_i)$ resulting from a molecular collision and V is the relative velocity between colliding molecules. Integration is performed over the velocity space of both the probe particle of interest (unsubscripted) and the colliding particle of a different class (subscripted 1). As before, dA_c represents the general expression for the differential collision cross-section.

As with the Boltzmann equation for the distribution function the terms above may be interpreted from left to right as the rate of change of Q in a fixed volume due to particle convection, external forces, and collisions. In this expression the distribution function does not appear explicitly but rather as a weighting function, the form of which is discussed subsequently. In this manner the Boltzmann equation for the distribution function is not satisfied locally but rather in some average sense. Such an approach is analogous to the integral techniques used in solving boundary layer equations.

The equation of transfer given above cannot, in general, be reduced any further without a knowledge of the distribution function and the details of the collision process. However, if $Q(c_i)$ represents the mass, momentum, or energy per molecule, these equations simplify. For such functional forms of Q conservation of mass, momentum, or energy during elastic impact require that the right hand side of equation (3) be zero. For these collisional invariants a system of equations is obtained which is independent of the collision process. These equations constitute a coupled set of five differential equations for the conservation of mass, momentum, and energy of a monatomic gas occupying a fixed volume in physical space. Depending on the form taken for the distribution function it is generally necessary to consider additional moments to obtain the proper number of differential equations for the undetermined functions in the distribution function. It is in evaluating these "higher" moments that the details of the collision process must be specified.

2.3 Maxwell's Inverse Fifth Molecular Repulsion Model

The collision integral on the right hand side of equation (3) may, in principle, be solved for any arbitrary distribution function and law of force between colliding molecules. Practically speaking it is desirable to choose a law of repulsion which affords the greatest mathematical simplicity yet retains the non-linear character of the collision integral and the short range interaction behavior implicit in the derivation of the Boltzmann equation. These considerations prompted Maxwell to suggest an inverse fifth power repulsion law which takes the form

$$F = m_1 m_2 K r^{-5} \tag{4}$$

where K is a constant, r is the distance between centers of molecules, and m_1 and m_2 are the molecular mass.

The inverse fifth power repulsion law does not provide a particularly accurate description of intermolecular forces, however, it does allow one to simplify the collision integral. For this repulsion law the relative speed of the colliding molecules vanishes under the integral and the collision integral may be written as

$$\Delta[Q] = \sqrt{(m_1 + m_2)K} \int\int f f_1 J \, d\vec{c} \, d\vec{c}_1 \tag{5a}$$

where

$$J = \int_0^{\infty} \int_0^{2\pi} (Q' - Q) \, d\varepsilon \; a \, da \tag{5b}$$

and ϵ and a are parameters describing the collision process. Thus for Maxwellian molecules the collision integral may be interpreted as the value of J averaged over the velocity space of the two participating classes of molecules. Furthermore, J is proportional to the value of Q and is independent of the velocity distribution function.

2.4 Lees' Bimodal Velocity Distribution Function

All that remains to complete the kinetic theory formulation is an expression for the velocity distribution function. Although the form of the distribution function to be used in solving the equation of transfer is not unique, basic requirements to be satisifed are:

i) It must have the "two-sided" character essential to rarefied gas flows.

ii) It must be capable of providing a smooth transition from rarefied flows to the Navier-Stokes regime.

iii) It should lead to the simplest possible set of differential equations and boundary conditions consistent with requirements i) and ii).

Guided by the limiting solution for free molecular flow, Lees [9] suggested that the distribution function take the form of a "two-sided" Maxwellian. At a given point the contributions of the two "sides" are determined by line of sight. This is illustrated schematically in Figure 1 for a spherical body placed in an unbounded

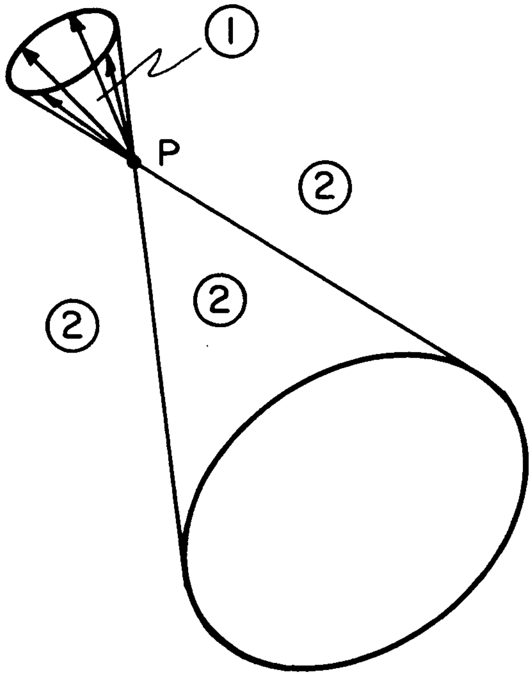

Figure 1 Cone of Influence for Two Stream Maxwellian

free-molecular gas with diffuse reflection at the body surface. At a point P, particles with velocity vectors lying within conical Region I are described by a Maxwellian corresponding to the velocity and temperature of the surface. The distribution function for the remaining particles emanating from Region II is the free-stream Maxwellian.

In the general case the two regions are determined by the line of sight principle and the distribution function takes the form

$\vec{c}$ in Region I

$$f = f_1 = n_1(\vec{r}, t)\left[\frac{1}{2\pi R T_1(\vec{r}, t)}\right]^{3/2} \exp\left\{-\frac{[\vec{c} - \vec{u}_1(\vec{r}, t)]^2}{2 R T_1(\vec{r}, t)}\right\}$$

$$(6)$$

$\vec{c}$ in Region II

$$f = f_2 = n_2(\vec{r}, t)\left[\frac{1}{2\pi R T_2(\vec{r}, t)}\right]^{3/2} \exp\left\{-\frac{[\vec{c} - \vec{u}_2(\vec{r}, t)]^2}{2 R T_2(\vec{r}, t)}\right\}.$$

In these two expressions $n_{1,2}(\vec{r}, t)$, $\vec{u}_{1,2}(\vec{r}, t)$, and $T_{1,2}(\vec{r}, t)$ are ten undetermined functions of space and time. It is then necessary to determine these functions by solution of ten simultaneous moment equations. When these functions are specified, all macroscopic quantities of interest can be evaluated.

Chapter 3

MOMENT EQUATIONS AND BOUNDARY CONDITIONS FOR THE FLAT PLATE EVAPORATION - CONDENSATION PROBLEM

3.1 Problem Definition and the Distribution Function

The present investigation considers the two plate problem. The surfaces are maintained at unequal temperatures and external forces are ignored. Emphasis is placed on understanding the physical behavior of a single component vapor between the two bounding surfaces. It is assumed that the two surfaces are maintained at constant temperature. The hot wall is the evaporating surface (source) and the cold wall is the condensing surface (sink). It is further assumed that the vapor between them is monatomic, obeys the perfect gas law, and consists of Maxwell molecules. The process is quasi-steady.

The problem is illustrated schematically in Figure 2. The hot plate, at temperature T_I, is located at $x = 0$ and the cold plate, at temperature T_{II}, is placed at $x = \ell$. For this problem gradients of physical variables and parameters in a direction parallel to the plates are zero. As a result, the expression for the two-stream distribution function may be written

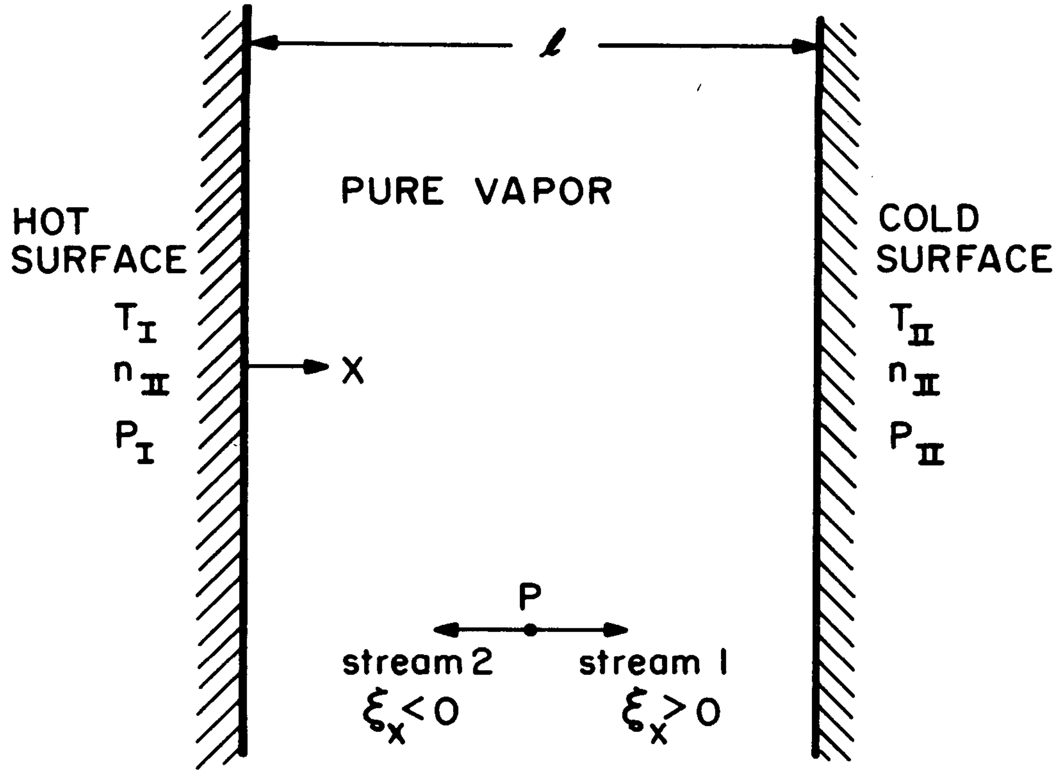

Figure 2 The Two Plate Problem

$c_x > 0$:

$$f = f_1(x) = n_1(x)\left[\frac{1}{2\pi RT_1(x)}\right]^{3/2} \exp\left\{-\frac{(c_x - u_1(x))^2 + c_y^2 + c_z^2}{2RT_1(x)}\right\}$$

(7)

$c_x < 0$:

$$f = f_2(x) = n_2(x)\left[\frac{1}{2\pi RT_2(x)}\right]^{3/2} \exp\left\{-\frac{(c_x - u_2(x))^2 + c_y^2 + c_z^2}{2RT_2(x)}\right\}.$$

In these two expressions, $n_{1,2}(x)$, $u_{1,2}(x)$, and $T_{1,2}(x)$ are six undetermined functions of the independent variable x. With this form of the distribution function the six parameters may be evaluated by simultaneous solution of six independent moment equations.

3.2 Equations of Transfer

In obtaining these equations, it is desirable to consider the lowest moments of the Boltzmann equation for two reasons: first, they allow the greatest degree of mathematical simplicity; and second, the "lower" equations generally involve moments of the distribution function which may be interpreted as physical variables of interest. With this in mind, the six lowest independent moment equations will be developed; this will involve choosing successive forms for $Q(c_i)$ that represent the lowest powers and combinations of molecular velocities.

As discussed previously, choosing $Q(c_i)$ to be the molecular mass, momentum, and energy results in moment equations which are conservation equations for the respective quantities. For these values of $Q(c_i)$ the collisional term is zero and in the absence of external forces the generalized moment equation (3) becomes successively

$$\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x_j}(\rho\,\overline{c_j}) = 0 \tag{8}$$

$$\frac{\partial}{\partial t}(\rho\,\overline{c_i}) + \frac{\partial}{\partial x_j}(\rho\,\overline{c_j c_i}) = 0 \tag{9}$$

$$\frac{\partial}{\partial t}(\rho\,\overline{c^2}) + \frac{\partial}{\partial x_j}(\rho\,\overline{c_j c^2}) = 0 \tag{10}$$

where $c^2 = \sum_{i=1}^{3} c_i^2$. For the steady-state one dimensional problem of interest, time and the y- and z-derivatives are zero. Thus, the first three independent moment equations are:

$$Q_1 = m: \qquad \frac{d}{dx}(\rho\,\overline{c_x}) = 0 \tag{11}$$

$$Q_2 = mc_x: \qquad \frac{d}{dx}(\rho\,\overline{c_x^2}) = 0 \tag{12}$$

$$Q_3 = mc^2/2: \qquad \frac{d}{dx}(\rho\,\overline{c_x c^2}) = 0\,. \tag{13}$$

Expressing these equations in terms of physical variables is facilitated by equating the molecular velocity to the sum of the mean and thermal velocities

$$c_x = \overline{c_x} + C_x = u + C_x \; . \qquad (14)$$

In terms of the thermal velocity the following moments of the distribution function can be identified:

$$\sigma_{ij} \equiv -\rho \, \overline{C_i C_j} \equiv \tau_{ij} - p \, \delta_{ij} \qquad (15)$$

$$\frac{3}{2} kT \equiv \frac{1}{2} \rho \, \overline{C^2} \qquad (16)$$

$$\dot{q}_i \equiv \frac{1}{2} \rho \, \overline{C_i C^2} \; . \qquad (17)$$

In addition, the following equations are appropriate for the distribution function and gas model assumed

$$n = \int f \, d\bar{c} \qquad (18)$$

$$p = nkT = \rho RT \qquad (19)$$

$$h = e + \frac{p}{\rho} = \frac{5}{2} RT \; . \qquad (20)$$

By using equations (13) to (20), the first three moment equations may be rewritten as

$$\frac{d}{dx} [\rho u] = 0 \qquad (21)$$

$$\frac{d}{dx} [\rho u^2 - \sigma_{xx}] = 0 \qquad (22)$$

$$\frac{d}{dx} \left[\rho u \left(\frac{3}{2} RT + \frac{u^2}{2} \right) + \dot{q}_x - \sigma_{xx} u \right] = 0 \; . \qquad (23)$$

Integration of these equations yields

$$\rho u = mB_1 \tag{24}$$

$$\rho u^2 - \sigma_{xx} = mB_2 \tag{25}$$

$$\rho u\left(\frac{3}{2}RT + \frac{u^2}{2}\right) + \dot{q}_x - \sigma_{xx}u = mB_3 \tag{26}$$

where B_1, B_2, and B_3 are integration constants.

The next higher moment equations are obtained by setting

$$Q_4(c_i) = mc_jc_k .$$

The resulting equation of transfer is interpreted as an expression for the flux of momentum, mc_k, in the x_j direction. For this value of Q the collision term on the right hand side of equation (3) is non-zero and is evaluated by Lees [9] for Maxwell molecules. In this problem, equation (3) simplifies to

$$\frac{d}{dx}[\, m\,\overline{c_xc_jc_k}\,] = \frac{p}{\mu}\tau_{jk} \tag{27}$$

where j and k may independently assume values from 1 to 3. Three non-trivial equations are obtained for j = k, although due to symmetry in the y and z directions the two independent equations resulting are

$$Q_{4a} = mc_x^2: \qquad \frac{d}{dx}[\,\overline{mc_x^3}\,] = \frac{p}{\mu}\tau_{xx} \tag{28}$$

$$Q_{4b} = mc_y^2: \qquad \frac{d}{dx}\left[m\overline{c_x c_y^2}\right] = \frac{p}{\mu}\tau_{yy}\,. \tag{29}$$

Adding twice equation (29) to equation (28) gives

$$\frac{d}{dx}\left[m\overline{c_x(c_x^2+c_y^2+c_z^2)}\right] = \frac{p}{\mu}\left(\sum_i \sigma_{ii} + 3p\right) = 0$$

which is identical to the energy equation (13). Thus of equations (13), (28), and (29), only two are independent; the choice of the two to be used in the solution for the arbitrary functions will be made on the basis of simplicity.

The fifth equation of transfer is obtained by setting $Q_4(c_i) = m\,c_j(c^2/2)$ and may be interpreted as the flux of energy in the j-direction. Lees [9] has evaluated the corresponding collision term for Maxwellian molecules and in this case the resulting moment equation becomes

$$Q_5 = m\,c_x(c^2/2): \qquad \frac{d}{dx}\left[m\overline{c_x^2\,\frac{c^2}{2}}\right] = \frac{p}{\mu}\left[-\frac{2}{3}\dot{q}_x + \tau_{xx}u\right]. \tag{30}$$

The sixth independent equation of transfer is obtained by choosing $Q_6(c_i) = mc_j^3$. The meaning of this equation is less physical, as is typical of higher moments. It represents the flux of mc_j^2 in the j-direction. The collision integral is evaluated in Appendix A and the moment equation reduces to

$$Q_6 = mc_x^3: \qquad \frac{d}{dx}\left[m\overline{c_x^4} \right] = \frac{p}{\mu}\left[\dot{q}_x + 3u\tau_{xx} - \frac{3}{2}m\overline{C_x^3} \right]. \tag{31}$$

In summary, the six independent differential equations of transfer to be used are

$$\frac{d}{dx}\left[m\overline{c_x} \right] \equiv \frac{d}{dx}\left[\rho u \right] = 0 \tag{11}$$

$$\frac{d}{dx}\left[m\overline{c_x^2} \right] \equiv \frac{d}{dx}\left[\rho u^2 - \sigma_{xx} \right] = 0 \tag{12}$$

$$\left.\begin{array}{l} \dfrac{d}{dx}\left[m\overline{c_x c^2} \right] \equiv \dfrac{d}{dx}\left[\rho u\left(\dfrac{3}{2}\dfrac{p}{\rho} + \dfrac{u^2}{2} \right) + \dot{q}_x - \sigma_{xx}u \right] = 0 \\[3ex] \dfrac{d}{dx}\left[m\overline{c_x^3} \right] - \dfrac{p}{\mu}\tau_{xx} = 0 \\[3ex] \dfrac{d}{dx}\left[m\overline{c_x c_y^2} \right] - \dfrac{p}{\mu}\tau_{xx} = 0 \end{array}\right\}\text{two of three} \qquad \begin{array}{r} (13) \\[3ex] (28) \\[3ex] (29) \end{array}$$

$$\frac{d}{dx}\left[m\overline{c_x^2\frac{c^2}{2}} \right] + \frac{p}{\mu}\left[\frac{2}{3}\dot{q}_x - \tau_{xx}u \right] = 0 \tag{30}$$

$$\frac{d}{dx}\left[m\overline{c_x^4} \right] - \frac{p}{\mu}\left[\dot{q}_x + 3u\tau_{xx} - \frac{3}{2}m\overline{C_x^3} \right] = 0\,. \tag{31}$$

3.3 Expression of Physical Variables and Moment Equations in Terms of Parameters of the Distribution Function

In order to solve the above equations for the parameters n_1, n_2,

u_1, u_2, T_1, T_2 as functions of x, it is necessary to express the equations in terms of these variables. In principle this involves nothing more than evaluating the prescribed moments of equation (7) and substituting them into the equations of transfer. In performing these operations and in reducing the moment equations to their simplest form, a significant amount of algebraic manipulation is required which adds nothing to the understanding of the problem. Therefore, the contributing physical variables and higher moments of the distribution function are evaluated and the final forms of the equations of transfer are presented without details of the intermediate algebraic steps.

The appropriate moments of the bimodal distribution function (7) were obtained by using integrals summarized in Appendix B. Because of the recurrence of certain functional forms in all moments, it is convenient to define

$$E_1 \equiv 1 + \mathrm{erf}\left[\frac{u_1(x)}{\sqrt{2RT_1(x)}}\right] \qquad (32a)$$

$$E_2 \equiv 1 + \mathrm{erf}\left[\frac{-u_2(x)}{\sqrt{2RT_2(x)}}\right] \qquad (32b)$$

$$X_1 \equiv \exp\left[-u_1^2(x)/2RT_1(x)\right] \qquad (32c)$$

$$X_2 \equiv \exp\left[-u_2^2(x)/2RT_2(x)\right] \qquad (32d)$$

and to replace the variables $T_1(x)$ and $T_2(x)$ with

$$\beta_1 = 2RT_1(x) \tag{32e}$$

$$\beta_2 = 2RT_2(x) \; . \tag{32f}$$

With these definitions, the physical variables and higher moments may be expressed as

$$n = \int f \, d\vec{c} = \frac{1}{2} n_1 E_1 + \frac{1}{2} n_2 E_2$$

$$nu = \int f c_x d\vec{c} = \frac{1}{2} n_1 \left(u_1 E_1 + \sqrt{\frac{\beta_1}{\pi}} \, X_1 \right) + \frac{1}{2} n_2 \left(u_2 E_2 - \sqrt{\frac{\beta_2}{\pi}} \, X_2 \right) = B_1$$

$$\sigma_{xx} = -m \int f C_x^2 d\vec{c} = -m \int f (c_x - u)^2 d\vec{c}$$

$$= -\frac{1}{2} mn_1 \left\{ \left[\frac{\beta_1}{2} + u_1^2 \right] E_1 + u_1 \sqrt{\frac{\beta_1}{\pi}} \, X_1 \right\}$$

$$-\frac{1}{2} mn_2 \left\{ \left[\frac{\beta_2}{2} + u_2^2 \right] E_2 - u_2 \sqrt{\frac{\beta_2}{\pi}} \, X_2 \right\} + mnu^2$$

$$\sigma_{yy} = \sigma_{zz} = -m \int f C_y^2 \, d\vec{c}$$

$$= -\frac{1}{4} mn_1 \beta_1 E_1 - \frac{1}{4} mn_2 \beta_2 E_2$$

$$p = -\sum_i (\sigma_{ii}/3) = \frac{1}{6} mn_1 \left[\left(u_1^2 + \frac{3}{2} \beta_1 \right) E_1 + u_1 \sqrt{\frac{\beta_1}{\pi}} X_1 \right]$$

$$+ \frac{1}{6} mn_2 \left[\left(u_2^2 + \frac{3}{2} \beta_2 \right) E_2 - u_2 \sqrt{\frac{\beta_2}{\pi}} X_2 \right] - \frac{1}{3} mnu^2$$

$$\dot{q}_x = \frac{m}{2} \int C_x C^2 f \, d\vec{c}$$

$$= \frac{1}{4} mn_1 \left\{ - \left[\frac{5}{2} (u-u_1)\beta_1 + (u-u_1)^3 \right] E_1 + \left[2\beta_1 + u_1^2 + 3(u^2 - uu_1) \right] \sqrt{\frac{\beta_1}{\pi}} X_1 \right\}$$

$$- \frac{1}{4} mn_2 \left\{ \left[\frac{5}{2} (u-u_2)\beta_2 + (u-u_2)^3 \right] E_2 + \left[2\beta_2 + u_2^2 + 3(u^2 - uu_2) \right] \sqrt{\frac{\beta_2}{\pi}} X_2 \right\}$$

$$\int f c_x^3 d\vec{c} = \frac{1}{2} n_1 \left\{ \left(u_1^3 + \frac{3}{2} u_1 \beta_1 \right) E_1 + \left(\beta_1 + u_1^2 \right) \sqrt{\frac{\beta_1}{\pi}} X_1 \right\}$$

$$+ \frac{1}{2} n_2 \left\{ \left(u_2^3 + \frac{3}{2} u_2 \beta_2 \right) E_2 - \left(\beta_2 + u_2^2 \right) \sqrt{\frac{\beta_2}{\pi}} X_2 \right\}$$

$$\int f c_x c_y^2 d\vec{c} = \frac{1}{4} n_1 \beta_1 \left[u_1 E_1 + \sqrt{\frac{\beta_1}{\pi}} X_1 \right] + \frac{1}{4} n_2 \beta_2 \left[u_2 E_2 - \sqrt{\frac{\beta_2}{\pi}} X_2 \right]$$

$$\int f c_x^2 \frac{c^2}{2} d\vec{c} = \frac{1}{4} n_1 \left\{ \left[\frac{5}{4} \beta_1^2 + 4u_1^2 \beta_1 + u_1^4 \right] E_1 + \left[u_1^3 + \frac{7}{2} u_1 \beta_1 \right] \sqrt{\frac{\beta_1}{\pi}} X_1 \right\}$$

$$+ \frac{1}{4} n_2 \left\{ \left[\frac{5}{4} \beta_2^2 + 4u_2^2 \beta_2 + u_2^4 \right] E_2 - \left[u_2^3 + \frac{7}{2} u_2 \beta_2 \right] \sqrt{\frac{\beta_2}{\pi}} X_2 \right\}$$

$$\int f c_x^4 d\vec{c} = \frac{1}{2} n_1 \left\{ \left[\frac{3}{4} \beta_1^2 + 3\beta_1 u_1^2 + u_1^4 \right] E_1 + \left[u_1^3 + \frac{5}{2} u_1 \beta_1 \right] \sqrt{\frac{\beta_1}{\pi}} X_1 \right\}$$

$$+ \frac{1}{2} n_2 \left\{ \left[\frac{3}{4} \beta_2^2 + 3\beta_2 u_2^2 + u_2^4 \right] E_2 - \left[u_2^3 + \frac{5}{2} u_2 \beta_2 \right] \sqrt{\frac{\beta_2}{\pi}} X_2 \right\}$$

$$\int f\, C_x^3\, d\vec{c} = \int f(c_x - u)d\vec{c}$$

$$= \frac{1}{2} n_1 \left\{ - \left[\frac{3}{2} \beta_1 + (u-u_1)^2 \right](u-u_1)E_1 + (\beta_1 + u_1^2 + 3u^2 - 3uu_1)\sqrt{\frac{\beta_1}{\pi}}\, X_1 \right\}$$

$$+ \frac{1}{2} n_2 \left\{ - \left[\frac{3}{2} \beta_2 + (u-u_2)^2 \right](u-u_2)E_2 - (\beta_2 + u_2^2 + 3u^2 - 3uu_2)\sqrt{\frac{\beta_2}{\pi}}\, X_2 \right\} .$$

Therefore, the three conservation equations become

$$\frac{d}{dx}\left[n_1\left(u_1E_1 + \sqrt{\frac{\beta_1}{\pi}}\, X_1\right) + n_2\left(u_2E_2 - \sqrt{\frac{\beta_2}{\pi}}\, X_2\right)\right] = \frac{d}{dx}\left[2B_1\right] = 0 \quad (33)$$

$$\frac{d}{dx}\left[n_1\left\{\left(\frac{\beta_1}{2} + u_1^2\right)E_1 + u_1\sqrt{\frac{\beta_1}{\pi}}\, X_1\right\} + n_2\left\{\left(\frac{\beta_2}{2} + u_2^2\right)E_2 - u_2\sqrt{\frac{\beta_2}{\pi}}\, X_2\right\}\right.$$

$$= \frac{d}{dx}\left[2B_2\right] = 0 \quad (34)$$

$$\frac{d}{dx}\left[n_1\left\{\left(\frac{5}{2}u_1\beta_1 + u_1^3\right)E_1 + (2\beta_1 + u_1^2)\sqrt{\frac{\beta_1}{\pi}}\, X_1\right\}\right.$$

$$\left. + n_2\left\{\left(\frac{5}{2}u_2\beta_2 + u_2^3\right)E_2 - \left(2\beta_2 + u_2^2\right)\sqrt{\frac{\beta_2}{\pi}}\, X_2\right\} = \frac{d}{dx}\left[4B_3\right] = 0 . \quad (35)$$

Similarly, the remaining moment equations may be simplified by combining terms and by utilizing the integration constants B_1, B_2, and B_3 to yield

$$\frac{d}{dx}\left[n_1(u_1^3 + \frac{3}{2}u_1\beta_1)E_1 + n_1(\beta_1 + u_1^2)\sqrt{\frac{\beta_1}{\pi}}\, X_1\right.$$

$$+ n_2(u_2^3 + \frac{3}{2} u_2\beta_2)E_2 - n_2(\beta_2 + u_2^2)\sqrt{\frac{\beta_2}{\pi}}X_2 \Big] \tag{36}$$

$$- \frac{2}{3} \frac{p}{\mu}\Big[2\ nu^2 - 2B_2 + \frac{1}{2}(n_1\beta_1E_1 + n_2\beta_2E_2)\Big] = 0$$

$$\frac{d}{dx}\Big[n_1\beta_1\Big(u_1E_1 + \sqrt{\frac{\beta_1}{\pi}}X_1\Big) + n_2\beta_2\Big(u_2E_2 - \sqrt{\frac{\beta_2}{\pi}}X_2\Big)\Big]$$

$$+ \frac{2}{3} \frac{p}{\mu}\Big[2\ nu^2 - 2B_2 + \frac{1}{2}(n_1E_1\beta_1 + n_2E_2\beta_2)\Big] = 0 \tag{37}$$

$$\frac{d}{dx}\Big[n_1\Big(\frac{5}{4}\beta_1^2 + 4u_1^2\beta_1 + u_1^4\Big)E_1 + n_1\Big(u_1^3 + \frac{7}{2}u_1\beta_1\Big)\sqrt{\frac{\beta_1}{\pi}}X_1$$

$$+ n_2\Big(\frac{5}{4}\beta_2^2 + 4u_2^2\beta_2 + u_2^4\Big)E_2 - n_2\Big(u_2^3 + \frac{7}{2}u_2\beta_2\Big)\sqrt{\frac{\beta_2}{\pi}}X_2\Big] \tag{38}$$

$$+ \frac{4}{3} \frac{p}{\mu}\Big[2B_3 - u(B_2 + n_1\beta_1E_1 + n_2\beta_2E_2)\Big] = 0$$

$$\frac{d}{dx}\Big[n_1\Big(\frac{3}{4}\beta_1^2 + 3\beta_1u_1^2 + u_1^4\Big)E_1 + n_1\Big(u_1^3 + \frac{5}{2}u_1\beta_1\Big)\sqrt{\frac{\beta_1}{\pi}}X_1$$

$$+ n_2\Big(\frac{3}{4}\beta_2^2 + 3\beta_2u_2^2 + u_2^4\Big)E_2 - n_2\Big(u_2^3 + \frac{5}{2}u_2\beta_2\Big)\sqrt{\frac{\beta_2}{\pi}}X_2\Big] \tag{39}$$

$$+ 2\frac{p}{\mu}\Big\{-u\Big[B_2 + \frac{1}{4}(n_1\beta_1E_1 + n_2\beta_2E_2)\Big] + 2B_3 - \frac{3}{4}n_1\beta_1\Big(u_1E_1 + \sqrt{\frac{\beta_1}{\pi}}X_1\Big)$$

$$- \frac{3}{4}n_2\beta_2\Big(u_2E_2 - \sqrt{\frac{\beta_2}{\pi}}X_2\Big)\Big\} = 0.$$

3.4 Non-Dimensional Equations and Boundary Conditions

Non-dimensionalization of the physical variables and moment equations is accomplished by introducing the following definitions

$$x = d\,\bar{x}$$

$$n_{1,2} = n_I\,\bar{n}_{1,2}$$

$$u_{1,2} = U\,\bar{u}_{1,2}$$

$$\beta_{1,2} = U^2\bar{\beta}_{1,2}$$

where the characteristic velocity $U = \sqrt{RT_I}$ and variables subscripted I are evaluated at the surface of the hot plate. Note the cold plate could just as easily be used as a reference. The viscosity law for a gas composed of Maxwell molecules (Appendix A, equation (A14)) is

$$\mu = \frac{T}{T_I}\,\mu_I = \bar{T}\,\mu_I\,.$$

With this result, the coefficient $\frac{p}{\mu} \cdot \frac{d}{U}$ becomes

$$\frac{p}{\mu}\,\frac{d}{U} = \frac{\bar{n}\,Re}{\sqrt{\gamma}\,M} = \bar{n}\,Re$$

where the Reynolds number is defined by

$$Re \equiv \frac{\rho_I\,U d}{\mu_I} \tag{40}$$

and $\sqrt{\gamma}\,M = 1$ because $M = \dfrac{U}{\sqrt{\gamma R T_I}}$.

For spherical molecules the relation between viscosity and the mean free path λ is

$$\mu = \frac{1}{2}\rho \bar{C} \lambda .$$

Hence, the Reynolds number in this problem can be related to Knudsen number (Kn) by

$$Re = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{d}{\lambda} = \frac{\sqrt{\pi/2}}{Kn} \tag{41}$$

where $Kn = \lambda/d$.

In terms of the non-dimensional quantities, the integrated conservation equations and physical variables of interest become

continuity

$$\frac{\bar{n}_1}{2}\left[\bar{u}_1 E_1 + \sqrt{\frac{\bar{\beta}_1}{\pi}}\, X_1\right\} + \frac{\bar{n}_2}{2}\left\{\bar{u}_2 E_2 - \sqrt{\frac{\bar{\beta}_2}{\pi}}\, X_2\right] = \bar{B}_1 = \bar{n}\bar{u} \tag{42}$$

x-momentum

$$\frac{\bar{n}_1}{2}\left[\left(\frac{\bar{\beta}_1}{2} + \bar{u}_1^2\right)E_1 + \bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\, X_1\right] + \frac{\bar{n}_2}{2}\left[\left(\frac{\bar{\beta}_2}{2} + \bar{u}_2^2\right)E_2 - \bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\, X_2\right]$$

$$= \bar{B}_2 \tag{43}$$

energy

$$\frac{\bar{n}_1}{4}\left[\left(\frac{5}{2}\bar{u}_1\bar{\beta}_1 + \bar{u}_1^3\right)E_1 + \left(2\bar{\beta}_1 + \bar{u}_1^2\right)\sqrt{\frac{\bar{\beta}_1}{\pi}}\, X_1\right]$$

$$+\ \frac{\bar{n}_2}{4}\left[\left(\frac{5}{2}\bar{u}_2\bar{\beta}_2+\bar{u}_2^{3}\right)E_2-\left(2\bar{\beta}_2+\bar{u}_2^{2}\right)\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]=\bar{B}_3 \qquad (44)$$

$$\bar{n}=\frac{1}{2}\bar{n}_1E_1+\frac{1}{2}\bar{n}_2E_2=\bar{p}/\bar{T} \qquad (45)$$

pressure

$$\bar{p}=\frac{1}{6}\bar{n}_1\left[\left(\bar{u}_1^{2}+\frac{3}{2}\bar{\beta}_1\right)E_1+\bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]$$

$$+\ \frac{1}{6}\bar{n}_2\left[\left(\bar{u}_2^{2}+\frac{3}{2}\bar{\beta}_2\right)E_2-\bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]-\frac{1}{3}\bar{n}\,\bar{u}^{2} \qquad (46)$$

normal shear stress

$$\bar{\tau}_{xx}=-\frac{1}{3}\bar{n}_1\left[\bar{u}_1^{2}E_1+\bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]-\frac{1}{3}\bar{n}_2\left[\bar{u}_2^{2}E_2-\bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]+\frac{2}{3}\bar{n}\,\bar{u}^{2} \qquad (47)$$

heat flux

$$\bar{q}_x\doteq\frac{1}{4}\bar{n}_1\left\{-\left[\frac{5}{2}(\bar{u}-\bar{u}_1)\bar{\beta}_1+(\bar{u}-\bar{u}_1)^{3}\right]E_1+\left[2\bar{\beta}_1+\bar{u}_1^{2}+3(\bar{u}^{2}-\bar{u}\,\bar{u}_1)\right]\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right\}$$

$$-\ \frac{1}{4}\bar{n}_2\left\{\left[\frac{5}{2}(\bar{u}-\bar{u}_2)\bar{\beta}_2+(\bar{u}-\bar{u}_2)^{3}\right]E_2+\left[2\bar{\beta}_2+\bar{u}_2^{2}+3(\bar{u}^{2}-\bar{u}\,\bar{u}_2)\right]\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right\}. \qquad (48)$$

The evaporation coefficient is given by

$$\alpha \equiv \frac{\rho u}{\frac{1}{4}\rho_I\sqrt{\frac{8RT_I}{\pi}} - \frac{1}{4}\rho_{II}\sqrt{\frac{8RT_{II}}{\pi}}} = \frac{2\sqrt{\pi}\ \overline{B}_1}{\left(\overline{n}_I\sqrt{\overline{\beta}_I} - \overline{n}_{II}\sqrt{\overline{\beta}_{II}}\right)} . \qquad (49)$$

The remaining four moment equations are

x-momentum flux

$$\frac{d}{d\overline{x}}\left\{ \overline{n}_1\left[\overline{u}_1^3 + \frac{3}{2}\overline{u}_1\overline{\beta}_1\right]E_1 + \overline{n}_1(\overline{\beta}_1 + \overline{u}_1^2)\sqrt{\frac{\overline{\beta}_1}{\pi}}X_1 + \overline{n}_2\left[\overline{u}_2^3 + \frac{3}{2}\overline{u}_2\overline{\beta}_2\right]E_2 \right.$$

$$\left. - \overline{n}_2(\overline{\beta}_2 + \overline{u}_2^2)\sqrt{\frac{\overline{\beta}_2}{\pi}}X_2\right\} - \frac{2}{3}\,\mathrm{Re}\,\overline{n}\left\{2\overline{n}\,\overline{u}^2 - 2\overline{B}_2 + \frac{1}{2}(\overline{n}_1\overline{\beta}_1E_1 + \overline{n}_2\overline{\beta}_2E_2)\right\} \qquad (50)$$

$$= 0$$

y-momentum flux

$$\frac{d}{d\overline{x}}\left[\overline{n}_1\overline{\beta}_1\left(\overline{u}_1E_1 + \sqrt{\frac{\overline{\beta}_1}{\pi}}X_1\right) + \overline{n}_2\overline{\beta}_2\left(\overline{u}_2E_2 - \sqrt{\frac{\overline{\beta}_2}{\pi}}X_2\right)\right]$$

$$+ \frac{2}{3}\,\mathrm{Re}\,\overline{n}\left[2\overline{n}\,\overline{u}^2 - 2\overline{B}_2 + \frac{1}{2}(\overline{n}_1\overline{\beta}_1E_1 + \overline{n}_2\overline{\beta}_2E_2)\right] = 0 \qquad (51)$$

x-energy flux

$$\frac{d}{d\overline{x}}\left[\overline{n}_1\left(\frac{5}{4}\overline{\beta}_1^2 + 4\overline{u}_1^2\overline{\beta}_1 + \overline{u}_1^4\right)E_1 + \overline{n}_1\left(\overline{u}_1^3 + \frac{7}{2}\overline{u}_1\overline{\beta}_1\right)\sqrt{\frac{\overline{\beta}_1}{\pi}}X_1\right.$$

$$\left. + \overline{n}_2\left(\frac{5}{4}\overline{\beta}_2^2 + 4\overline{u}_2^2\overline{\beta}_2 + \overline{u}_2^4\right)E_2 - \overline{n}_2\left(\overline{u}_2^3 + \frac{7}{2}\overline{u}_2\overline{\beta}_2\right)\sqrt{\frac{\overline{\beta}_2}{\pi}}X_2\right]$$

$$+ \frac{4}{3}\,\mathrm{Re}\,\overline{n}\left[2\overline{B}_3 - \overline{u}(\overline{B}_2 + \overline{n}_1\overline{\beta}_1E_1 + \overline{n}_2\overline{\beta}_2E_2)\right] = 0 \qquad (52)$$

x-flux of c_x^2

$$\begin{aligned}
&\frac{d}{d\bar{x}}\Big[\bar{n}_1\Big(\frac{3}{4}\bar{\beta}_1^2+3\bar{\beta}_1\bar{u}_1^2+\bar{u}_1^4\Big)E_1+\bar{n}_1\Big(\bar{u}_1^3+\frac{5}{2}\bar{u}_1\bar{\beta}_1\Big)\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\\
&+\bar{n}_2\Big(\frac{3}{4}\bar{\beta}_2^2+3\bar{\beta}_2\bar{u}_2^2+\bar{u}_2^4\Big)E_2-\bar{n}_2\Big(\bar{u}_2^3+\frac{5}{2}\bar{u}_2\bar{\beta}_2\Big)\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\Big]\\
&+2\,\mathrm{Re}\,\bar{n}\Big\{-\bar{u}\Big[\bar{B}_2+\frac{1}{4}(\bar{n}_1\bar{\beta}_1E_1+\bar{n}_2\bar{\beta}_2E_2)\Big]+2\bar{B}_3\\
&-\frac{3}{4}\bar{n}_1\bar{\beta}_1\Big(\bar{u}_1E_1+\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\Big)-\frac{3}{4}\bar{n}_2\bar{\beta}_2\Big(\bar{u}_2E_2-\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\Big)\Big\}=0. \qquad (53)
\end{aligned}$$

The boundary conditions for this problem are established directly from the assumptions related to the nature of the flow at the two walls. In the absence of any conclusive evidence to the contrary it is assumed that all incident molecules are absorbed and thermally accommodated at the wall surface. Furthermore, molecules are emitted from the surfaces with zero mean velocity and with a local Maxwellian distribution corresponding to the plate temperature and the number density of the saturated vapor at that temperature.

Mathematically these assumptions are equivalent to the following boundary conditions

at $\bar{x} = 0$

$$n_1 = n_I \qquad \bar{n}_1 = 1$$

$$u_1 = 0 \qquad \bar{u}_1 = 0 \qquad (54a)$$

$$T_1 = T_I \qquad \bar{\beta}_1 = 2$$

at $\bar{x} = 1$

$$\begin{aligned} n_2 &= n_{II} & \bar{n}_2 &= n_{II}/n_I \\ u_2 &= 0 & \bar{u}_2 &= 0 \\ T_2 &= T_{II} & \bar{\beta}_2 &= 2T_{II}/T_I \end{aligned} \tag{54b}$$

and $T_I > T_{II}$.

Chapter 4

THE LINEARIZED TWO PLATE PROBLEM

In order to find an analytical solution to the non-linear equations (42) to (44) and (50) to (53), subject to the boundary conditions (54a) and (54b), a small deviation from equilibrium is considered. This approach, and the resulting equations are identical to those given by Shankar [12] but different boundary conditions will be applied. The first order perturbation solution is found by using

$$\begin{aligned} \bar{n}_1 &= 1 + \bar{N}_1 \epsilon + \ldots \\ \bar{n}_2 &= 1 + \bar{N}_2 \epsilon + \ldots \\ \bar{T}_1 &= 1 + \bar{t}_1 \epsilon + \ldots \\ \bar{T}_2 &= 1 + \bar{t}_2 \epsilon + \ldots \\ \bar{u}_1 &= \epsilon \bar{U}_1 + \ldots \\ \bar{u}_2 &= \epsilon \bar{U}_2 + \ldots \end{aligned} \tag{55}$$

where ϵ is a small parameter and $\bar{N}_{1,2}$, $\bar{t}_{1,2}$, and $\bar{U}_{1,2}$ are of $0(1)$. Definitions (32a) through (32f) become

$$\begin{aligned} \bar{\beta}_1 &= 2(1 + \bar{t}_1 \epsilon + \ldots) \\ \bar{\beta}_2 &= 2(1 + \bar{t}_2 \epsilon + \ldots) \\ E_1 &= 1 + \sqrt{\frac{2}{\pi}}\, \bar{U}_1 \epsilon + \ldots \\ E_2 &= 1 - \sqrt{\frac{2}{\pi}}\, \bar{U}_2 \epsilon + \ldots \end{aligned} \tag{56}$$

$$X_1 = 1 + 0(\epsilon^2) + \dots$$

$$X_2 = 1 + 0(\epsilon^2) + \dots$$

where $\overline{\beta} = 2\overline{T}$. The boundary conditions are

at $\overline{x} = 0$

$$N_1(0) = \frac{(n_{II} - n_I)}{n_{II}} = -\frac{\Delta N}{n_{II}} = -\Delta\overline{N} \tag{57a}$$

$$t_1(0) = -\frac{(T_{II} - T_I)}{T_{II}} = -\frac{\Delta T}{T_{II}} = -\Delta\overline{T}$$

$$U_1(0) = 0$$

at $\overline{x} = 1$

$$N_2(1) = 0$$

$$t_2(1) = 0 \tag{57b}$$

$$U_2(1) = 0$$

where $\Delta\overline{N}$ and $\Delta\overline{T}$ are of $0(\epsilon)$. The cold wall conditions are used as the references values in solving the linearized problem. This is consistent with Springer and Patton [10].

By placing equations (55) and (56) into the moment equations (42) to (44) and (50) through (53), we obtain

$$2(\overline{N}_1 - \overline{N}_2) + \overline{T}_1 - \overline{T}_2 + 2\sqrt{\frac{\pi}{2}}\,(\overline{U}_1 + \overline{U}_2) = \overline{B}_1'' \tag{58}$$

$$\overline{N}_1 + \overline{N}_2 + \overline{T}_1 + \overline{T}_2 + 2\sqrt{\frac{2}{\pi}}\,(\overline{U}_1 - \overline{U}_2) = 2\overline{B}_2' \tag{59}$$

$$\bar{N}_1 - \bar{N}_2 + 3/2(\bar{T}_1 - \bar{T}_2) + 5/4\sqrt{\frac{\pi}{2}}\ (\bar{U}_1 + \bar{U}_2) = \bar{B}'_3 \tag{60}$$

$$\frac{d}{d\bar{x}}\left(\bar{N}_1 - \bar{N}_2 + 3/2(\bar{T}_1 - \bar{T}_2) + 3/2\sqrt{\frac{\pi}{2}}\ (\bar{U}_1 + \bar{U}_2)\right) = -\sqrt{\frac{\pi}{2}}\ \frac{(\bar{U}_1 - \bar{U}_2)}{3Kn} \tag{61}$$

$$\frac{d}{d\bar{x}}\left(\bar{N}_1 - \bar{N}_2 + 3/2(\bar{T}_1 - \bar{T}_2) + \sqrt{\frac{\pi}{2}}\ (\bar{U}_1 + \bar{U}_2)\right) = 1/3\sqrt{\frac{\pi}{2}}\ \frac{(U_1 - \bar{U}_2)}{Kn} \tag{62}$$

$$\frac{d}{d\bar{x}}\left(5(\bar{N}_1 + \bar{N}_2) + 10(\bar{T}_1 + \bar{T}_2) + 12\sqrt{\frac{2}{\pi}}\ (\bar{U}_1 - \bar{U}_2)\right) = \frac{1}{3Kn}\left[2(N_1 - N_2) - 7(T_1 - T_2)\right] \tag{63}$$

$$\frac{d}{d\bar{x}}\left[\left(3(\bar{N}_1 + \bar{N}_2) + 6(\bar{T}_1 + \bar{T}_2) + 8\sqrt{\frac{2}{\pi}}\ (\bar{U}_1 - \bar{U}_2)\right)\right] = \frac{1}{2Kn}\left(2(N_1 - N_2) - (T_1 - T_2)\right) \tag{64}$$

where

$$\bar{B}''_1 = 4\sqrt{\frac{\pi}{2}}\ \bar{B}_1$$

$$\bar{B}_2 = 1 + \bar{B}'_2$$

$$\bar{B}'_3 = \sqrt{\frac{\pi}{2}}\ \bar{B}_3$$

$$Re = \sqrt{\frac{\pi}{2}}\ \frac{1}{Kn}\ . \tag{41}$$

and Kn = λ/d. Define

$$\begin{aligned} \bar{N}_+ &= \bar{N}_1 + \bar{N}_2 & \bar{N}_- &= \bar{N}_1 - \bar{N}_2 \\ \bar{t}_+ &= \bar{t}_1 + \bar{t}_2 & \bar{t}_- &= \bar{t}_1 - \bar{t}_2 \\ \bar{\nu}_+ &= \bar{\nu}_1 + \bar{\nu}_2 & \nu_- &= \bar{\nu}_1 - \bar{\nu}_2 \end{aligned} \qquad (65)$$

where $\bar{U}_i = \sqrt{2\pi}\,\bar{\nu}_i$. As previously indicated, only two of the three equations (60) to (62) are independent. The six equations that Shankar [12] used to study condensation at a liquid-vapor interface are utilized in this work. With the aid of the above definitions, we have

$$2N_- + t_- + 2\pi\nu_+ = \bar{B}_1'' \qquad (66)$$

$$N_+ + t_+ + 4\nu_- = \bar{B}_2' \qquad (67)$$

$$7\bar{t}_- - 2\bar{N}_- = 2\bar{B}_3'' \qquad (68)$$

$$\frac{d}{dx}(2\bar{N}_- + 3\bar{t}_- + 2\pi\bar{\nu}_+) = 2/3\,\frac{\pi}{Kn}\,\bar{\nu}_- \qquad (69)$$

$$\frac{d}{dx}(5\bar{N}_+ + 10\bar{t}_+ + 24\bar{\nu}_-) = -2/3\,\frac{\bar{B}_3''}{Kn} \qquad (70)$$

$$\frac{d}{dx}(3\bar{N}_+ + 6\bar{t}_+ + 16\bar{\nu}_-) = \frac{1}{Kn}(3\bar{t}_- - \bar{B}_3'') \qquad (71)$$

where equation (68) is found by combining equations (58) and (60). It should be noted that equation (68) is exactly $q^{(1)}$ = constant where q (heat flux) = $q^{(1)}\epsilon + \cdots$.

The solution to the above equations is

$$\bar{t}_- = De^{-\Lambda\bar{x}} + Ee^{+\Lambda\bar{x}} + \frac{\bar{B}_3''}{5} \tag{72a}$$

$$\bar{t}_+ = \alpha_1 - \frac{2\bar{B}_3''}{15Kn}\bar{x} + 3D\sqrt{\frac{2}{5\pi}}\, e^{-\Lambda\bar{x}} - 3E\sqrt{\frac{2}{5\pi}}\, e^{+\Lambda\bar{x}} \tag{72b}$$

$$\bar{\nu}_- = 3/2\; E\sqrt{\frac{5}{2\pi}}\, e^{+\Lambda\bar{x}} - 3/2\; D\sqrt{\frac{5}{2\pi}}\, e^{-\Lambda\bar{x}} \tag{72c}$$

$$\bar{\nu}_+ = \frac{\bar{B}_1''}{2\pi} + \frac{\bar{B}_3''}{5\pi} - \frac{4D}{\pi}\, e^{-\Lambda\bar{x}} - \frac{4E}{\pi}\, e^{+\Lambda\bar{x}} \tag{72d}$$

$$\bar{N}_- = 7/2\; D\, e^{-\Lambda\bar{x}} + 7/2\; E\, e^{+\Lambda\bar{x}} - 3/10\; \bar{B}_3'' \tag{72e}$$

$$\bar{N}_+ = \bar{B}_2' - \alpha_1 + \frac{2\bar{B}_3''\bar{x}}{15Kn} + 3D\left(2\sqrt{\frac{5}{2\pi}} - \sqrt{\frac{2}{5\pi}}\right)e^{-\Lambda\bar{x}} + 3E\left(\sqrt{\frac{2}{5\pi}} - 2\sqrt{\frac{5}{2\pi}}\right)e^{+\Lambda\bar{x}} \tag{72f}$$

where $\Lambda = \frac{1}{2Kn}\sqrt{\frac{5\pi}{2}}$ and D, E, and α_1 are constants to be determined. By using the definitions given in (65), we find that equations (72a) to (72f) can be written as

$$2\bar{N}_1 = \bar{B}_2' - \alpha_1 - 3/10\ \bar{B}_3'' + \frac{2\bar{B}_3''\bar{x}}{15Kn} + \left(7/2 + \frac{24}{\sqrt{10\pi}}\right) D\, e^{-\Lambda\bar{x}} + \left(7/2 - \frac{24}{\sqrt{10\pi}}\right) E\, e^{+\Lambda\bar{x}} \tag{73a}$$

$$2\bar{N}_2 = \bar{B}_2' - \alpha_1 + 3/10\ \bar{B}_3'' + \frac{2\bar{B}_3''\bar{x}}{15\ Kn} + \left(\frac{24}{\sqrt{10\pi}} - 7/2\right) D\, e^{-\Lambda\bar{x}} - \left(7/2 + \frac{24}{\sqrt{10\pi}}\right) E\ e^{+\Lambda\bar{x}} \tag{73b}$$

$$2\bar{t}_1 = \alpha_1 - \frac{2\bar{B}_3''\bar{x}}{15Kn} + \left(1 + \frac{6}{\sqrt{\pi 10}}\right) D\, e^{-\Lambda\bar{x}} + \left(1 - \frac{6}{\sqrt{10\pi}}\right) E e^{\Lambda\bar{x}} + \frac{\bar{B}_3''}{5} \tag{73c}$$

$$2\bar{t}_2 = \alpha_1 - \frac{2\bar{B}_3''\bar{x}}{15\ Kn} - \frac{\bar{B}_3''}{5} + \left(\frac{6}{\sqrt{10\pi}} - 1\right) D\, e^{-\Lambda\bar{x}} - \left(1 + \frac{6}{\sqrt{10\pi}}\right) E e^{\Lambda\bar{x}} \tag{73d}$$

$$2\bar{\nu}_1 = \frac{\bar{B}_1''}{2\pi} + \frac{\bar{B}_3''}{5\pi} - \left(\frac{4}{\pi} + 3/2\sqrt{\frac{5}{2\pi}}\right) D\, e^{-\Lambda\bar{x}} - \left(\frac{4}{\pi} - \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right) E e^{\Lambda\bar{x}} \tag{73e}$$

$$2\bar{\nu}_2 = \frac{\bar{B}_1''}{2\pi} + \frac{\bar{B}_3''}{5\pi} - \left(\frac{4}{\pi} - \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right) D\, e^{-\Lambda\bar{x}} - \left(\frac{4}{\pi} + \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right) E\, e^{\Lambda\bar{x}}. \tag{73f}$$

Application of the boundary equations (57a) and (57b) to equations (73a) to (73f) yields

$$-2\Delta\bar{N} = \bar{B}_2' - \alpha_1 - 3/10\ \bar{B}_3'' + \left(\frac{7}{2} + \frac{24}{\sqrt{10\pi}}\right) D + \left(\frac{7}{2} - \frac{24}{\sqrt{10\pi}}\right) E \tag{74a}$$

$$0 = \bar{B}'_2 - \alpha_1 + 3/10\ \bar{B}''_3 + \left(\frac{24}{\sqrt{10\pi}} - \frac{7}{2}\right) De^{-\Lambda} - \left(\frac{7}{2} + \frac{24}{\sqrt{10\pi}}\right) Ee^{+\Lambda} + \frac{2\bar{B}''_3}{15\ Kn} \tag{74b}$$

$$-2\Delta\bar{T} = \alpha_1 + \frac{\bar{B}''_3}{5} + \left(1 + \frac{6}{\sqrt{10\pi}}\right)D + \left(1 - \frac{6}{\sqrt{10\pi}}\right)E \tag{74c}$$

$$0 = \alpha_1 - \frac{2\bar{B}''_3}{15\ kn} - \frac{\bar{B}''_3}{5} + \left(\frac{6}{\sqrt{10\pi}} - 1\right)De^{-\Lambda} - \left(1 + \frac{6}{\sqrt{10\pi}}\right)Ee^{\Lambda} \tag{74d}$$

$$0 = \frac{\bar{B}''_1}{2\pi} + \frac{\bar{B}''_3}{5\pi} - \left(\frac{4}{\pi} + \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right)D - \left(\frac{4}{\pi} - \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right)E \tag{74e}$$

$$0 = \frac{\bar{B}''_1}{2\pi} + \frac{\bar{B}''_3}{5\pi} - \left(\frac{4}{\pi} - \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right)De^{-\Lambda} - \left(\frac{4}{\pi} + \frac{3}{2}\sqrt{\frac{5}{2\pi}}\right)Ee^{\Lambda} \tag{74f}$$

These six equations can be solved for the unknown constants $\bar{B}''_1$, $\bar{B}'_2$, $\bar{B}''_3$, D, E, and α.

Let $D = D'e^{\Lambda/2}$ and $E = E'e^{-\Lambda/2}$. Then:

$$\bar{B}''_3 = \frac{\Delta\bar{N}\left(-\frac{24}{\sqrt{10\pi}}\sinh\frac{\Lambda}{2} - 4\cosh\frac{\Lambda}{2}\right) + \Delta\bar{T}\left(\frac{96}{\sqrt{10\pi}}\sinh\frac{\Lambda}{2} + 14\cosh\frac{\Lambda}{2}\right)}{\frac{-132}{5\sqrt{10\pi}}\sinh\frac{\Lambda}{2} - 4\cosh\frac{\Lambda}{2} - \frac{8}{\sqrt{10\pi}}\ \frac{\sinh\Lambda/2}{Kn} - \frac{6}{5}\ \frac{\cosh\Lambda/2}{Kn}} \tag{75a}$$

$$D' = E' = \frac{\left(\frac{2}{5} + \frac{2}{15}\ \frac{1}{Kn}\right)\Delta\bar{N} + \left(\frac{3}{5} + \frac{2}{15}\ \frac{1}{Kn}\right)\Delta\bar{T}}{\frac{-132}{5\sqrt{10\pi}}\sinh\frac{\Lambda}{2} - 4\cosh\frac{\Lambda}{2} - \frac{8}{\sqrt{10\pi}}\ \frac{\sinh\Lambda/2}{Kn} - \frac{6}{5}\ \frac{\cosh\Lambda/2}{Kn}} \tag{75b}$$

$$\alpha_1 = \left(\frac{1}{5} + \frac{2}{15}\ \frac{1}{Kn}\right)\bar{B}_3'' + 2D' \cosh\frac{\Lambda}{2} + \frac{12\,D'}{\sqrt{10\pi}} \sinh\frac{\Lambda}{2} \tag{75c}$$

$$\bar{B}_2' = -\frac{\bar{B}_3''}{10} + 9D' \cosh\frac{\Lambda}{2} + \frac{60D'}{\sqrt{10\pi}} \sinh\frac{\Lambda}{2} \tag{75d}$$

$$\bar{B}_1'' = -\frac{2}{5}\bar{B}_3'' + 16D' \cosh\frac{\Lambda}{2} + 6\pi\sqrt{\frac{5}{2\pi}}\, D' \sinh\frac{\Lambda}{2} \tag{75e}$$

Now, it is possible to examine the two limiting extremes Kn → 0 and Kn → ∞. For Kn → 0, we have

$$\bar{B}_1'' = -1.66\ \Delta\bar{P}\ , \quad \Delta\bar{P} = (\Delta\bar{N} + \Delta\bar{T})$$

$$\bar{B}_1 = -\frac{\sqrt{2\pi}}{2\pi}\ (.83)\ \Delta\bar{P} \tag{76a}$$

$$\bar{B}_2' = -\Delta\bar{P} \tag{76b}$$

$$\bar{B}_3'' \rightarrow 0 \tag{76c}$$

and for Kn → ∞, we find that

$$\bar{B}_1'' = -2\Delta\bar{N} - \Delta\bar{T}$$

$$\bar{B}_1 = -\frac{1}{4}\sqrt{\frac{2}{\pi}}\ (-2\Delta\bar{N} - \Delta\bar{T}) \tag{77a}$$

$$\bar{B}_2' = -\Delta\bar{P} \tag{77b}$$

$$\bar{B}_3'' = \Delta\bar{N} - \frac{7}{2}\Delta\bar{T}\ . \tag{77c}$$

When Kn → ∞ (free molecular limit), the mass flux $\bar{B}_1$ to $0(\epsilon)$ agrees exactly with the Knudsen-Langmuir expression. In the continuum limit Kn →0 the mass flux to $0(\epsilon)$ becomes independent of the temperature difference and depends only on the pressure difference. This result agrees in form with Fuchs [14] and Shankar [12].

Finally, the behavior of the density and velocity in the flow field between the two plates is established. The number density is defined by

$$\bar{n} = 1/2(n_1 E_1 + n_2 E_2) \, . \tag{45}$$

In terms of perturbed quantities, (45) becomes

$$\bar{\rho} = 1 + (1/2 \, N_+ + \nu_-)\epsilon + \cdots \tag{78}$$

where $\rho = mn$. From equations (72c) and 72f), one finds

$$\bar{\rho} = 1 + \left[\frac{\bar{B}'_2}{2} - \frac{\alpha_1}{2} + \frac{\bar{B}''_3 \bar{x}}{15 \, \mathrm{Kn}} + \left(\frac{12}{\sqrt{10\pi}} - \frac{3}{2}\sqrt{\frac{5}{2\pi}} \right) D' e^{-\Lambda \bar{x}/2} + \left(-\frac{12}{\sqrt{10\pi}} + \frac{3}{2}\sqrt{\frac{5}{2\pi}} \right) D' e^{\Lambda \bar{x}/2} \right] \epsilon + \cdots \tag{79}$$

where $D = D' e^{\Lambda/2}$, $E = E' e^{-\Lambda/2}$,

$$D' = E' = \frac{\Delta N \left(\frac{2}{5} + \frac{2}{15} \frac{1}{\mathrm{Kn}} \right) + \Delta T \left(\frac{3}{5} + \frac{2}{15} \frac{1}{\mathrm{Kn}} \right)}{\frac{-132}{5\sqrt{10\pi}} \sinh \frac{\Lambda}{2} - 4 \cosh \frac{\Lambda}{2} - \frac{8}{\sqrt{10\pi}} \frac{\sinh \Lambda/2}{\mathrm{Kn}} - \frac{6}{5} \frac{\cosh \Lambda/2}{\mathrm{Kn}}} \tag{75b}$$

and $\bar{B}_2'$ and α_1 are given by (75c) and (75d) respectively. Equation (79) at the cold wall can be reduced to

$$\bar{\rho} = 1 + \left(- \frac{3}{20} \bar{B}_3'' + \frac{7}{2} D' \cosh \frac{\Lambda}{2} + 3\sqrt{\frac{5}{2\pi}}\, D' \sinh \frac{\Lambda}{2} \right) \epsilon + \cdots . \tag{80}$$

At $\bar{x} = 1$ (cold wall), we find

Kn → 0

$$\bar{\rho} \to 1 - \frac{\frac{2}{15}\left(\frac{7}{2} + \frac{15}{\sqrt{10\pi}} \right)}{\left(\frac{8}{\sqrt{10\pi}} + \frac{6}{5} \right)} \Delta \bar{P} + \cdots \tag{81a}$$

Kn → ∞

$$\bar{\rho} = 1 - \frac{1}{2} \Delta \bar{N} + \cdots . \tag{81b}$$

Similarly, for $\bar{x} = 0$ (hot wall), there results

$$\bar{\rho} = 1 + \left[- \left(\frac{3}{20} + \frac{1}{15\,\mathrm{Kn}} \right) \bar{B}_3'' + \left(\frac{7}{2} \cosh \frac{\Lambda}{2} + \frac{48}{\sqrt{10\pi}} \sinh \frac{\Lambda}{2} - 3\sqrt{\frac{5}{2\pi}} \sinh \frac{\Lambda}{2} \right) D' \right] \epsilon + \cdots$$

and the limits are

$Kn \rightarrow 0$

$$\bar{\rho} = 1 - \frac{\Delta\bar{N}\left(11 + \frac{90}{\sqrt{10\pi}}\right) + \Delta\bar{T}\left(7 + \frac{30}{\sqrt{10\pi}}\right)}{15\left(\frac{8}{\sqrt{10\pi}} + \frac{6}{5}\right)} + \ldots$$

or

$$\bar{\rho} = 1 - \frac{\Delta\bar{N}(27.07) + \Delta\bar{T}(12.36)}{39.42} + \cdots . \tag{82a}$$

$Kn \rightarrow \infty$

$$\bar{\rho} = 1 - \frac{1}{2}\Delta\bar{N} + \cdots . \tag{82b}$$

The mass velocity is

$$\bar{u} = \frac{\bar{n}_1\bar{u}_1 E_1 + \bar{n}_2\bar{u}_2 E_2 + \bar{n}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\, e^{-\bar{u}_1^2/\bar{\beta}_1} - \bar{n}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\, e^{-\bar{u}_2^2/\bar{\beta}_2}}{\bar{n}_1 E_1 + \bar{n}_2 E_2} \tag{83}$$

and in terms of the perturbed quantities $\bar{u}$ becomes

$$\bar{u} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\bar{N}_- + \frac{1}{2}\bar{t}_- + \pi\,\bar{\nu}_+\right)\epsilon + \cdots \tag{84}$$

where $\bar{N}_-$, $\bar{t}_-$, and $\bar{\nu}_+$ are given by (72e), (72a), and (72d) respectively. At any point in the flow field between the two plates and at $\bar{x} = 0$ or $\bar{x} = 1$, (84) reduces to

$$\bar{u} = \frac{1}{2}\sqrt{\frac{2}{\pi}} \cdot \frac{\bar{B}_1''}{2}\epsilon + \cdots = \bar{B}_1\epsilon + \ldots . \qquad (85)$$

The limits for $\bar{B}_1$ as $Kn \to 0$ and $Kn \to \infty$ are given by (76a) and (77a).

The density correction to $0(\epsilon)$ at the hot wall, equation (82a), depends on the relative size of $\Delta\bar{N}$ versus $\Delta\bar{T}$. Both $\Delta\bar{N}$ and $\Delta\bar{T}$ are negative by definition. If the density correction to $0(\epsilon)$ is greater than zero, then the flow is density dominated $\left(\frac{\Delta\bar{T}}{\Delta\bar{N}} < 2.19\right)$; but if the correction to $0(\epsilon)$ is less than zero, the flow is temperature dominated $\left(\frac{\Delta\bar{T}}{\Delta\bar{N}} > 2.19\right)$.

From equation (85), it is apparent that the small parameter ϵ is related to the mean velocity $\bar{u}$ which results from a small deviation from equilibrium. The fact that $\bar{u}$ is constant to $0(\epsilon)$ is not surprising since

$$\bar{\rho} = 1 + \epsilon\bar{\rho}^{(1)} + \epsilon^2\bar{\rho}^{(2)} + \ldots$$

$$\bar{u} = \epsilon\bar{u}^{(1)} + \epsilon^2\bar{u}^{(2)} + \ldots$$

and $\bar{\rho}\bar{u} = \text{const.}$ Therefore, we have

$$\epsilon\bar{u}^{(1)} + \epsilon^2(\bar{\rho}^{(1)}\bar{u}^{(1)} + \bar{u}^{(2)}) + \cdots = \text{const.}$$

which implies

$$\bar{u}^{(1)} = c_1$$

$$\bar{u}^{(2)} = c_2 - \bar{\rho}^{(1)}\bar{u}^{(1)}$$

but $\overline{\rho}^{(1)} \neq$ constant (as previously demonstrated, $Kn \to 0$) so $\overline{u}^{(2)} \neq$ constant.

The characteristic Re, Equation (40), is based on a velocity $U = \sqrt{RT_{II}}$ which is of 0(a) where a is the speed of sound. Re can be written as

$$Re = \frac{\rho_{II} \sqrt{RT_{II}}\, d}{\mu_{II}} = \frac{\rho_{II} u\, d}{\mu_{II}} \cdot \frac{1}{\dfrac{u}{\sqrt{RT_{II}}}} = \frac{Re_{\rho u}}{\sqrt{\gamma} \cdot M_u} \tag{86}$$

where γ = ratio of specific heats, M_u = Mach number, $Re_{\rho u}$ is Reynolds number based on the mass flux ρu at the cold wall and the mean velocity u is small. For extremely small kinematic viscosities ν, large d, or a combination of both, $Re_{\rho u}$ could be large when u is small.

When u is not small, the full non-linear equations must be solved.

Chapter 5

NUMERICAL SOLUTION TECHNIQUE

5.1 Solution of Separated Boundary Value Problem as Initial Value Problem

The six moment equations to be solved for the unknown parameters $n_{1,2}$ $T_{1,2}$ $u_{1,2}$ form a system of first order ordinary non-linear differential equations. Because all dependent variables are present in each of the equations they are completely coupled and must be solved simultaneously. Although three of the equations are integrable, no method was devised to simplify the solution by using the mixed algebraic and differential equations. Attempts to reduce the number of dependent variables by the introduction of groupings were unsuccessful and no feasible analytic integration technique was developed.

Numerical solution of such a system of equations can be accomplished by a number of methods if the values of all the variables are prescribed at one of the boundaries, i.e., if it is an initial value problem. With the present problem three of the boundary conditions are given at each surface and as such represent a separated or two point boundary value problem. Solution of this type of problem generally involves either quasi-linearization as used in the last chapter or reduction to an initial value problem. Because the free molecular limit allows an obvious choice of initial values which systematically vary at higher Reynolds number, the latter approach is used in order to solve the problem for strong non-equilibrium conditions.

The numerical solution begins at the first plate ($\bar{x}=0$). Here, values of $\bar{n}_1$, $\bar{u}_1$ and $\bar{\beta}_1$ are prescribed and the values of the other three variables $\bar{n}_2$, $\bar{u}_2$ and $\bar{\beta}_2$ are assumed. Next, the derivatives of the six functions are evaluated at the wall. With these derivatives known, the six equations can then be simultaneously integrated to $\bar{x}+s$ by using a fourth order Runge-Kutta scheme. Here s is the normalized step size. From this point the process of evaluating the derivatives and of integration is continued to $\bar{x}=1$. At $\bar{x}=1$ the integrated values of $\bar{n}_2$, $\bar{u}_2$ and $\bar{\beta}_2$ are compared to the actual boundary conditions prescribed for the problem. If the integrated values are not sufficiently close to the boundary conditions new values of $\bar{n}_2$, $\bar{u}_2$, and $\bar{\beta}_2$ at $\bar{x}=0$ are calculated and the equations are integrated again. The iteration for the correct values of $\bar{n}_2$, $\bar{u}_2$, and $\bar{\beta}_2$ at $\bar{x}=0$ continues in this manner until the boundary conditions specified for $\bar{n}_2$, $\bar{u}_2$ and $\bar{\beta}_2$ at $\bar{x}=1$ are satisfied. Since the six parameters are evaluated at each step of the integration by using the definitions in the previous chapter, the values of the physical variables of interest are determined throughout the flow field. The specific details of the integration method are briefly outlined in the next two sections.

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The complete computer program is given in Appendix C.

5.2 Evaluation of Local Derivatives

To simplify the mathematics of the problem it is convenient to define the function Y_j which represents the six dependent variables so that

$$\begin{aligned} Y_1 &\equiv \bar{n}_1(x) \\ Y_2 &\equiv \bar{u}_1(x) \\ Y_3 &\equiv \bar{\beta}_1(x) \\ Y_4 &\equiv \bar{n}_2(x) \\ Y_5 &\equiv \bar{u}_2(x) \\ Y_6 &\equiv \bar{\beta}_2(x)\ . \end{aligned} \qquad (87)$$

Furthermore, it is desirable to leave the six independent moment equations in the differential form

$$F_{ij}(Y_j)W_j + G_i(Y_j;\ \mathrm{Re}) = 0\ , \qquad i = j = 1, 6 \text{ (Appendix C)} \qquad (88)$$

where the derivative W_j is

$$\frac{dY_j}{dx} \equiv W_j \qquad j = 1, 6$$

and F_{ij} and G_i are algebraic functions only of the six variables Y_j and the Reynolds number.

The subroutine DRVTV (Y, W, Re) is used to evaluate the local derivative W_j for input values of Y_i and Re (Appendix C). To do this DRVTV calls another subroutine DSIMQ which solves the six equations (88) simultaneously.

5.3 Fourth Order Runge-Kutta Integration

A number of numerical integration techniques were considered; however, because of the simplicity in the present application a fourth order Runge-Kutta integration scheme is employed. For a system of two first order ordinary differential equations

$$y' = g_1(x, y, z) \quad , \quad z' = g_2(x, y, z)$$

the fourth order Runge-Kutta integration for step size s is given by:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) + 0(s^5) \, ,$$

$$z_{n+1} = z_n + \frac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4) + 0(s^5) \, ,$$

where,

$$k_1 = s\, g_1(x_n, y_n, z_n),$$

$$\ell_1 = s\, g_2(x_n, y_n, z_n),$$

$$k_2 = s\, g_1(x_n + \frac{1}{2}s, \; y_n + \frac{1}{2}k_1, \; z_n + \frac{1}{2}\ell_1)$$

$$\ell_2 = s\, g_2(x_n + \frac{1}{2}s, \; y_n + \frac{1}{2}k_1, \; z_n + \frac{1}{2}\ell_1) \, ,$$

$$k_3 = s\,g_1(x_n + \frac{1}{2}s,\ y_n + \frac{1}{2}k_2,\ z_n + \frac{1}{2}\ell_2),$$

$$\ell_3 = s\,g_2(x_n + \frac{1}{2}s,\ y_n + \frac{1}{2}k_2,\ z_n + \frac{1}{2}\ell_2),$$

$$k_4 = s\,g_1(x_n + s,\ y_n + k_3,\ z_n + \ell_3),$$

$$\ell_4 = s\,g_2(x_n + s,\ y_n + k_3,\ z_n + \ell_3)\,.$$

This system of equations is now generalized to allow integration of six simultaneous equations.

Equation (88) can be rearranged to

$$Y'_j = -(F^{-1})_{ij}G_i \equiv g_i(\overline{x}, Y_j) \qquad i = j = 1, 6 \qquad (89)$$

where the value of g_i is obtained simply by calling DRVTV (Y,g,Re). Y_i at $\overline{x}+s$ is found from

$$Y_i(\overline{x}+s) = Y_i(\overline{x}) + \frac{1}{6}(k_{i1} + 2(k_{i1} + k_{i3}) + k_{i4}) + 0(s^5) \qquad (90)$$

where

$$k_{i1} = sg_i(\overline{x}, Y_j) \qquad (91a)$$

$$k_{i2} = sg_i(\overline{x} + \frac{1}{2}s,\ Y_j + \frac{1}{2}k_{j1}) \qquad (91b)$$

$$k_{i3} = sg_i(\overline{x} + \frac{1}{2}s,\ Y_j + \frac{1}{2}k_{j2}) \qquad (91c)$$

$$k_{i4} = sg_i(\overline{x} + s,\ Y_j + k_{j3}) \qquad i = j = 1, 6. \qquad (91d)$$

To obtain explicit values of the k's in the program, use is made of the flexibility of the subroutine DRVTV. At point $\bar{x}$ values Y_j and $W_i(Y_j)$ are known, hence

$$k_{i1} = sW_i(Y_j) .$$

By defining

$$z_i = Y_i + \frac{1}{2} k_{i1} ,$$

the subroutine DRVTV (z, W, Re) is called to obtain

$$W_i(Y_j + \frac{1}{2} k_{j1}) = g_i(Y_j + \frac{1}{2} k_{j1})$$

hence

$$k_{i2} = sW_i(Y_j + \frac{1}{2} k_{j1}) .$$

Similarly, the remaining terms k_{i3} and k_{i4} are obtained by successively setting z_i equal to $Y_i + \frac{1}{2} k_{i2}$ and $Y_i + k_{i3}$, and calling for $W_i(Y_j + \frac{1}{2} k_{j2})$ and $W_i(Y_j + k_{j3})$. Once these are known $Y_i(\bar{x}+s)$ is obtained from equation (90). The step size is chosen to be 1/Re or 0.01, which ever is smaller.

Chapter 6

RESULTS OF NUMERICAL SOLUTION TO THE NON-LINEAR TWO PLATE PROBLEM

In this chapter the two plate problem is solved in certain cases for strong non-equilibrium conditions by using the numerical integration procedure discussed in the last section. Three cases are considered:

Case I, $T_{II}(d) = 1/2\ T_I(0),\ n_{II}(d) = 1/2\ n_I(0)$ (92a)

Case II, $T_{II}(d) = 1/10\ T_I(0),\ n_{II}(d) = 1/2\ n_I(0)$ (92b)

Case III, $T_{II}(d) = 1/2\ T_I(0),\ n_{II}(d) = 1/10\ n_I(0)$ (92c)

For all cases, $u_1(0) = u_2(d) = 0$. Case I is dealt with in depth whereas the other two cases are discussed only to illustrate certain differences that occur at low Reynolds number. For large Re, all cases considered behave in the same manner. The reference values are those at the hot wall.

The direction of integration is opposite to that of the vapor flow. This is analogous to the situation that exists in numerically solving the one dimensional viscous shock equations, Von Mises [15]. In the shock problem the direction of flow is supersonic to subsonic. Because a saddle point exists on the subsonic side of the shock and a nodal point on the supersonic side, the direction of integration is always taken to be subsonic to supersonic. An attempt was made to determine if such singularities existed at the hot and cold walls but due to the complex nature of the equations no definite results were obtained. Despite this, the numerical procedure employed to solve cases I, II and III simply will not march forward from the hot wall without eventually blowing up at some point in the flow field for Re $\geq$ 2. Therefore, the cold plate is positioned at $\bar{x} = 0$.

The six moment equations used in the numerical analysis are equations (42), (43), (50), (51), (52), and (53). These equations along with the expressions for density (45), pressure (46), τ_{xx} (47), heat flux (48), and the evaporation coefficient (49) give a complete picture of the flow field subject to the boundary conditions prescribed (54).

As pointed out before, each case is solved as an initial value problem. A one dimensional array, ALF, contains six elements corresponding to $\bar{n}_1(0)$, $\bar{u}_1(0)$, $\bar{\beta}_1(0)$, $\bar{n}_2(1)$, $\bar{u}_2(1)$, and $\bar{\beta}_2(1)$. These are the boundary conditions specified by (54) and either (92a), (92b), or (92c) depending on the case considered. Note that the subscript (1) and (2) are reversed when the cold wall becomes $x = 0$. YO(4), YO(5), and YO(6) are the initial values guessed at $\bar{x} = 0$ for $\bar{n}_2(0)$, $\bar{u}_2(0)$, and $\bar{\beta}_2(0)$. With ALF(1), ALF(2), ALF(3), YO(4), YO(5), and YO(6) given and a Re specified, the numerical integration proceeds forward from $\bar{x} = 0$ to $\bar{x} = 1$ in steps of 0.01 by using a fourth order Runge-Kutta scheme (Appendix C). For case I, ALF(1) = 1/2, ALF(2) = 0, ALF(3) = 1 at the cold wall ($\bar{x} = 0$) and ALF(4) = 1, ALF(5) = 0, ALF(6) = 2 at the hot wall ($\bar{x} = 1$). Note that Y_1 to Y_6 (equation (87)) is defined in computer language as Y(J), J = 1 to 6. If the values for Y(4), Y(5), and Y(6) at $\bar{x} = 1$ are not equal to ALF(4), ALF(5), and ALF(6) within a specified error, 0.001, then an iterative scheme is devised to change the initially guessed values YO(4), YO(5), and YO(6) until $|\mathrm{ALF(J)} - \mathrm{Y(J)}| \leq$ 0.001, J = 4, 5, 6. The details of the iterative scheme are given in Appendix C.

The integration step size was changed to see if it had any effect on the result. Two schemes were tried: D = 0.001 across the entire flow field and D = 0.001 near the two walls and D = 0.01 in the rest of the flow field. The numerical results were essentially unchanged from those given by using D = 0.01.

Figures 3, 4, and 5 indicate how $\bar{n}_2(0)$, $\bar{u}_2(0)$, and $\bar{\beta}_2(0)$ change as Re increases from 0.01 to 100 for case I. Similarly, Figures 6, 7, and 8 show how $\bar{n}_1(1)$, $\bar{u}_1(1)$, and $\bar{\beta}_1(1)$ behave over this Re range. It is worthwhile to note that $\bar{n}_2(0)$, $\bar{u}_2(0)$, and $\beta_2(0)$ approach limiting values 0.6702, -.4105, and 1.553 respectively for Re $\geq$ 10. $\bar{n}_1(1)$ (Figure 6) is a bell shaped curve which has a peak value of 0.835 at Re = 1.8 and approaches 0.646 as Re → 100. $\bar{u}_1(1)$ (Figure 7) has a

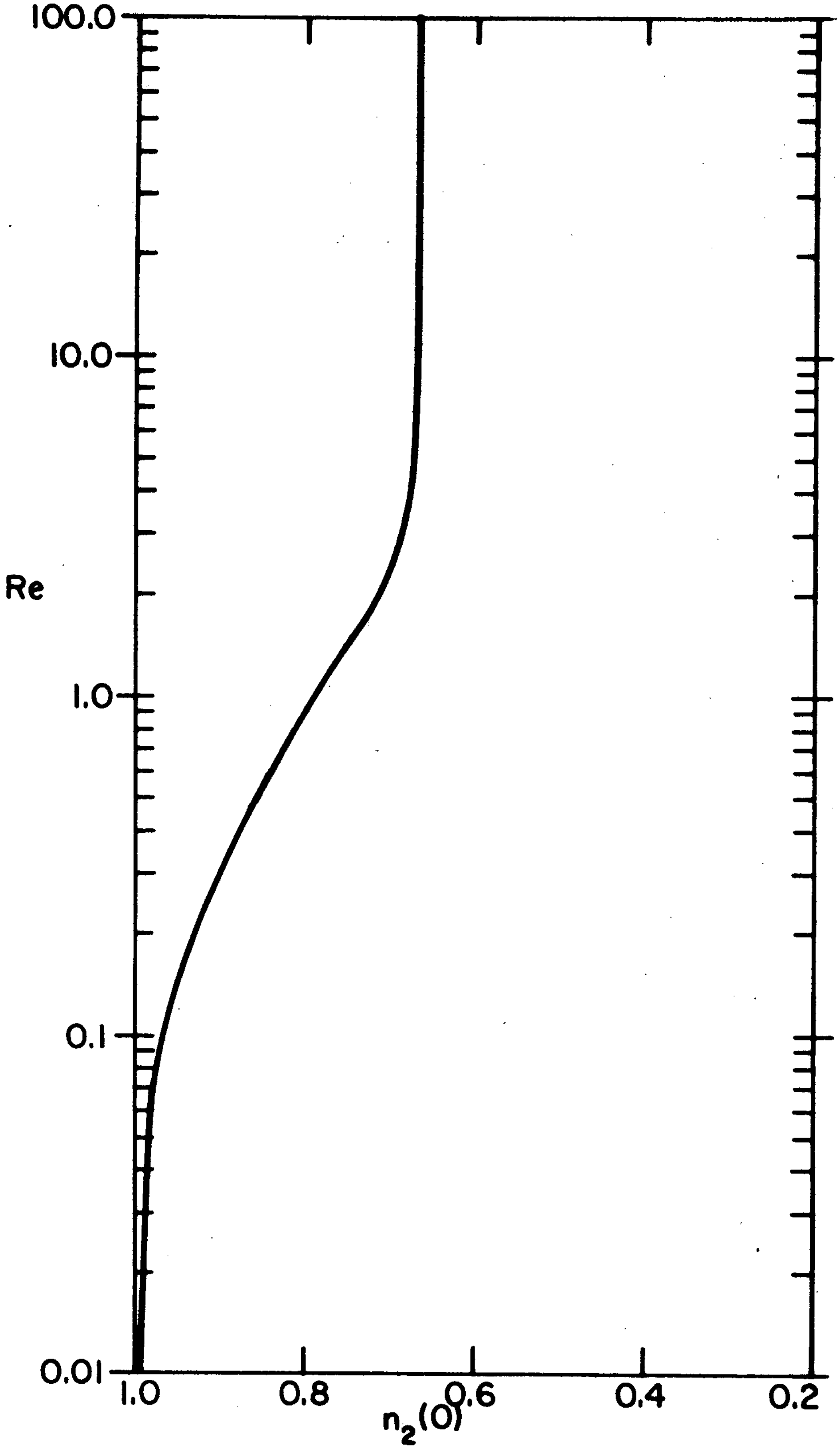


Figure 3 $\bar{n}_2(o)$ vs Re

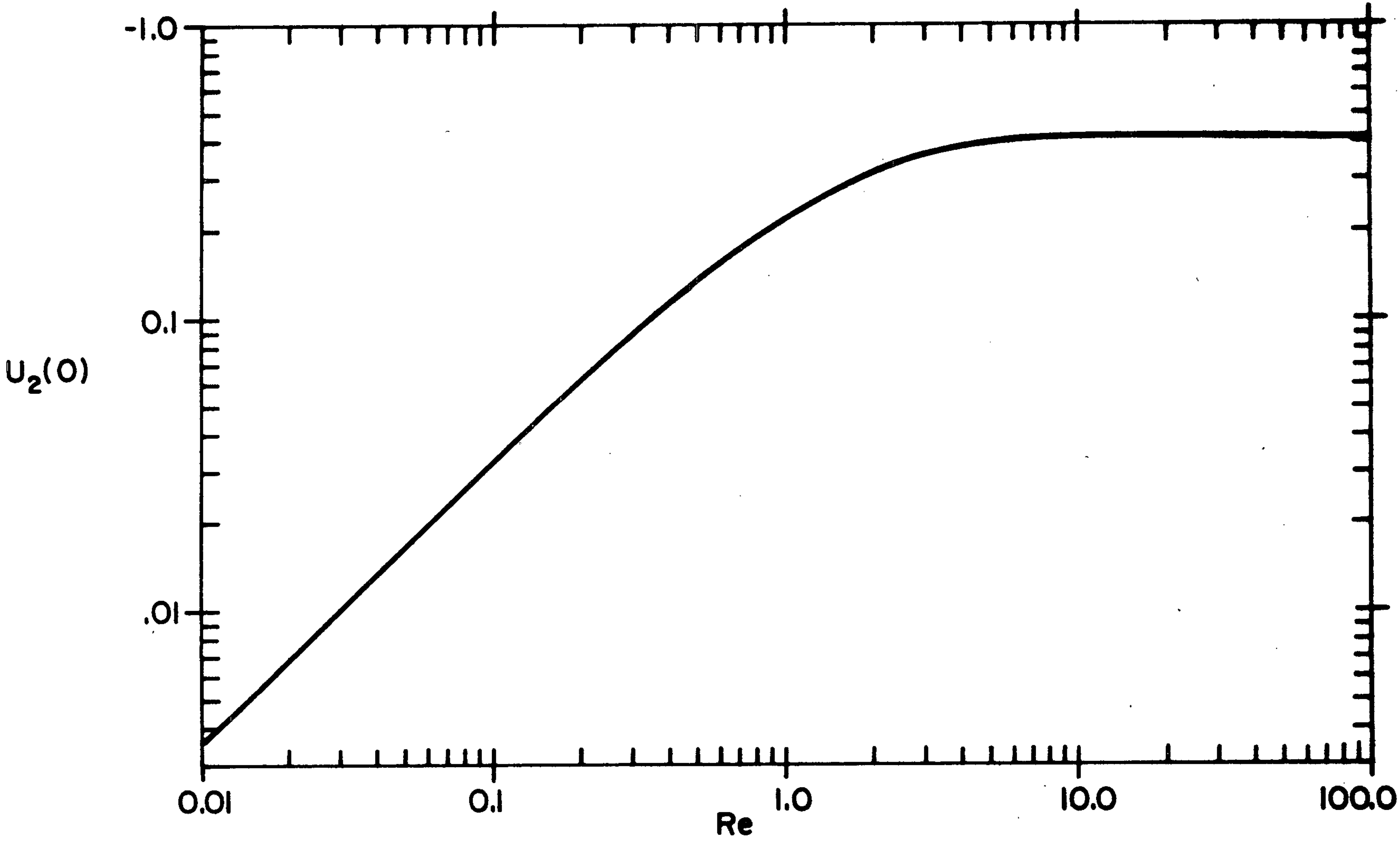


Figure 4 $\bar{u}_2(o)$ vs Re

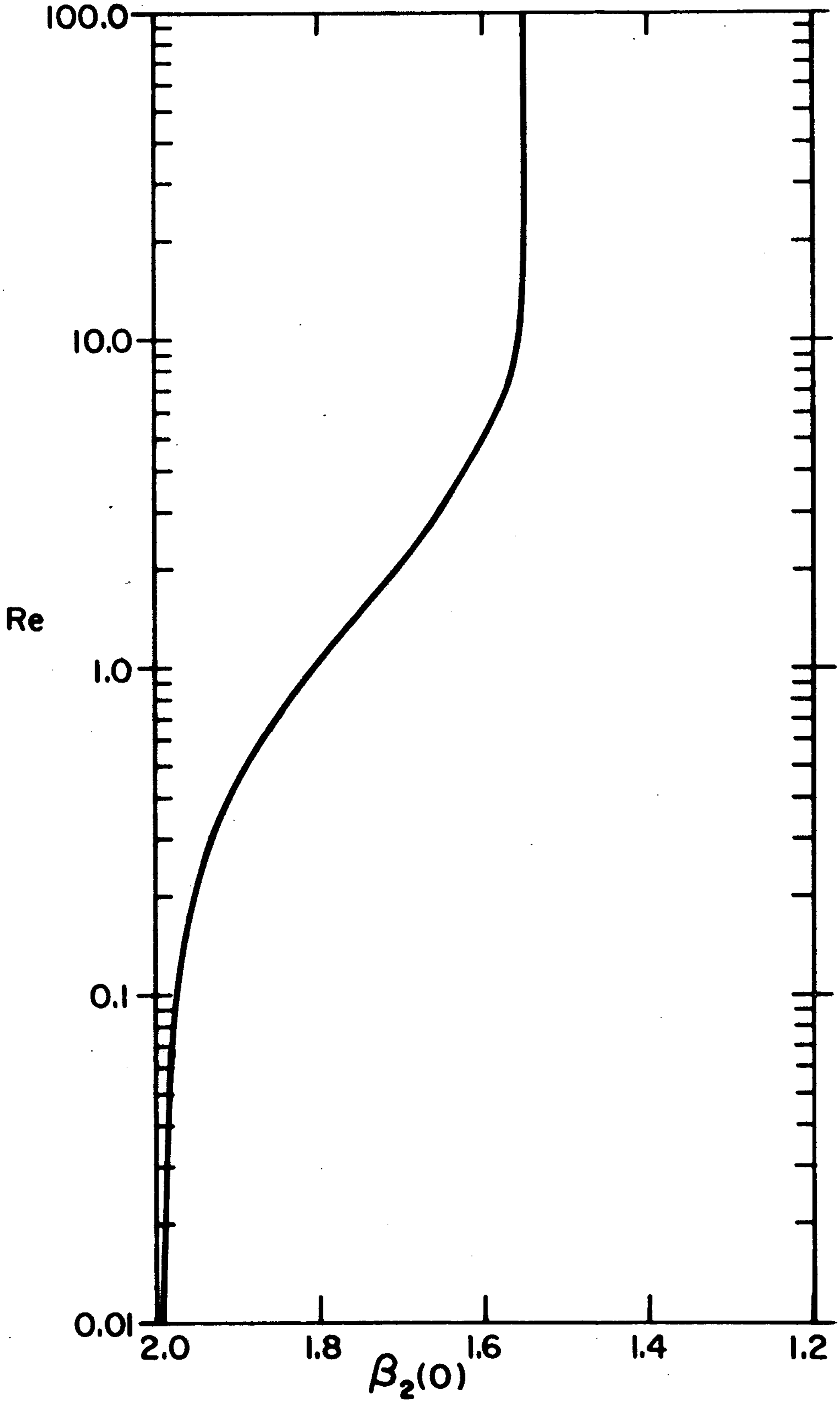


Figure 5 $\bar{\beta}_2(o)$ vs Re

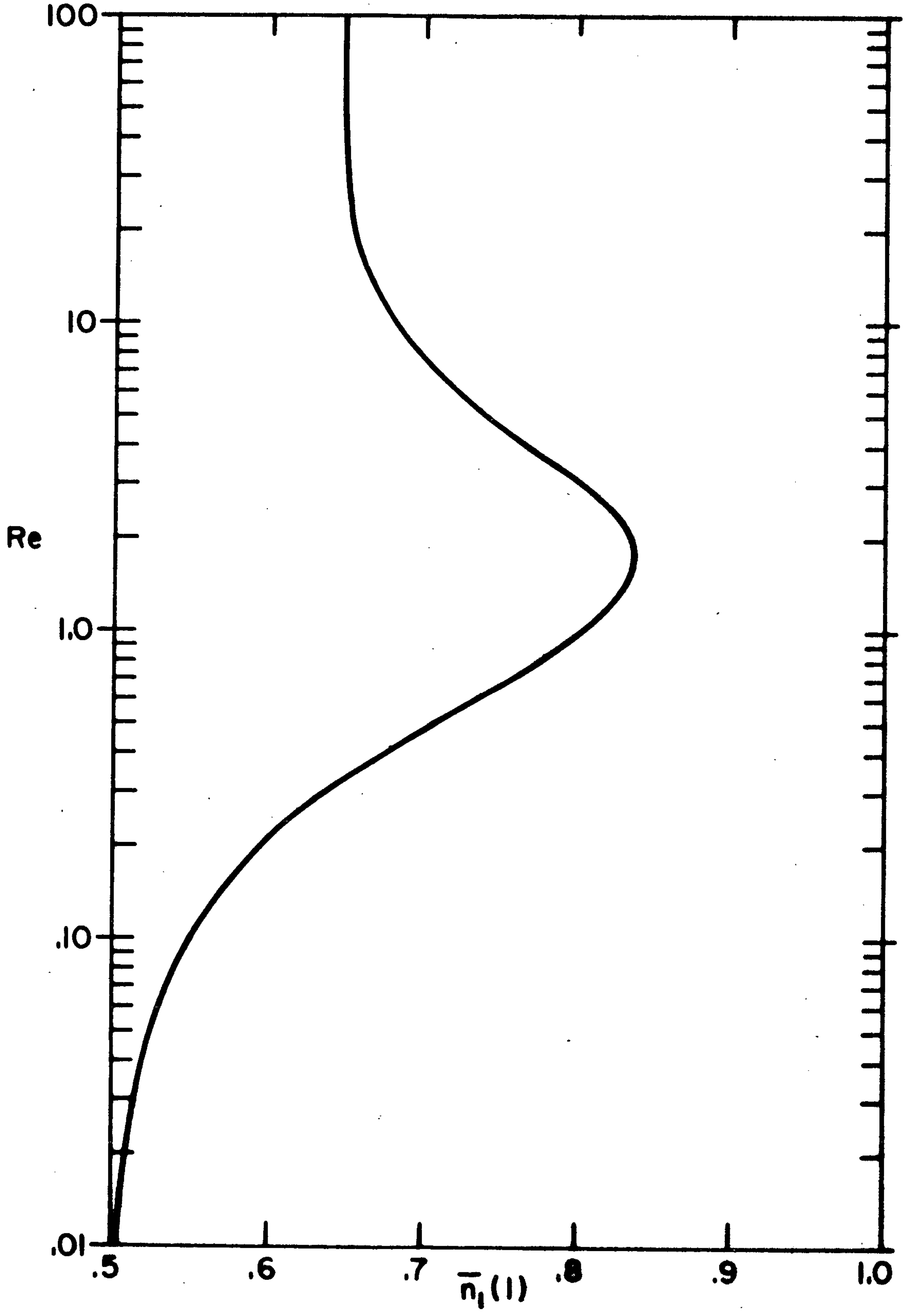


Figure 6 $\bar{n}_1(1)$ vs Re

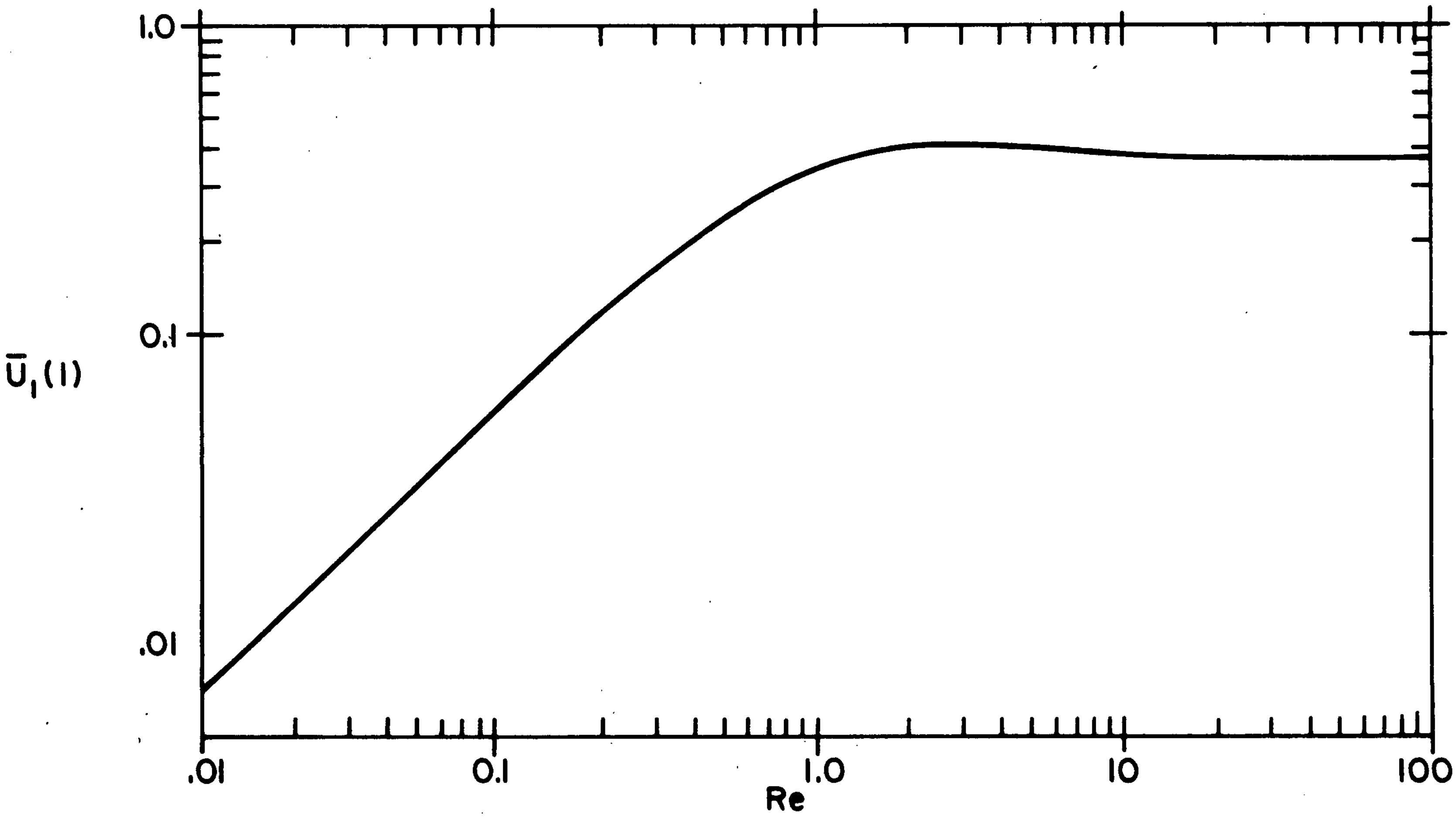


Figure 7 $\bar{u}_1(1)$ vs Re

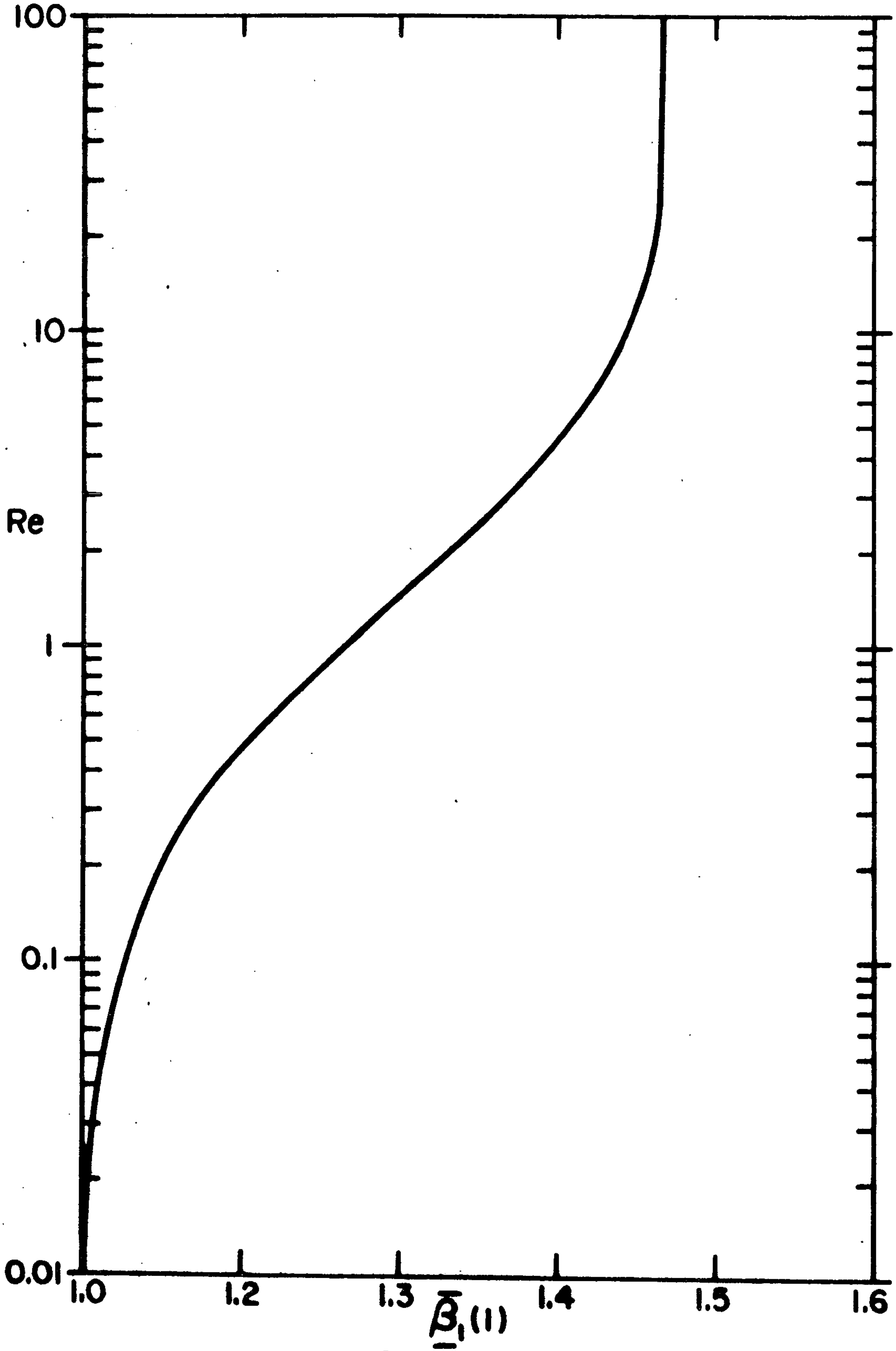


Figure 8 $\bar{\beta}_1(1)$ vs Re

maximum value of 0.405 at Re = 3.0 and decreases to 0.37 at Re = 100. Finally, $\bar{\beta}_1(1)$ (Figure 8) approaches 1.73 as Re → 100.

The behavior of the mean vapor velocity $\bar{u}(\bar{x})$ for case I is illustrated in Figure 9. At Re = 0.01 $\bar{u}$ is constant. As Re increases from Re = 0.01 to Re = 1.0, the curve for $\bar{u}(x)$ has an essentially constant positive slope from $\bar{x} = 0$ to $\bar{x} = 1$. Physically this means the flow accelerated from the hot wall to the cold one. As the magnitude of Re becomes greater than one a point of inflection begins to appear in the $\bar{u}(\bar{x})$ curve. Finally, for Re ≥ 6 the curve $\bar{u}(\bar{x})$ becomes concave upward except at the cold wall where it is concave downward. Physically, this corresponds to a flow which accelerates at the hot wall reaches a maximum value and then the decelerates toward the cold wall. At $\bar{x} = 0.05$ the flow starts to accelerate again. $\bar{u}$ is negative because the direction of integration (cold to hot) is opposite to that of the vapor motion. Velocity curves are plotted up to Re = 12.2. Figure 10 contains three of the $\bar{u}$ curves shown in Figure 9 replotted on a scale comparable to that used in later figures. Figure 11 and Figure 12 indicate how $\bar{\rho}(\bar{x})$ and $\bar{\beta}(\bar{x})$ behave for Re = 12.0.

The curves for $\bar{u}_1(\bar{x})$ and $\bar{u}_2(\bar{x})$ are illustrated for Re = 11.7 in Figure 13. Similarly $\bar{n}_1(\bar{x})$, $\bar{n}_2(\bar{x})$ and $\bar{\beta}_1(\bar{x})$, $\bar{\beta}_2(\bar{x})$ are plotted in Figures 14 and 15 respectively for Re = 11.7. These figures indicate the existence of two regions of rapid change: one at the

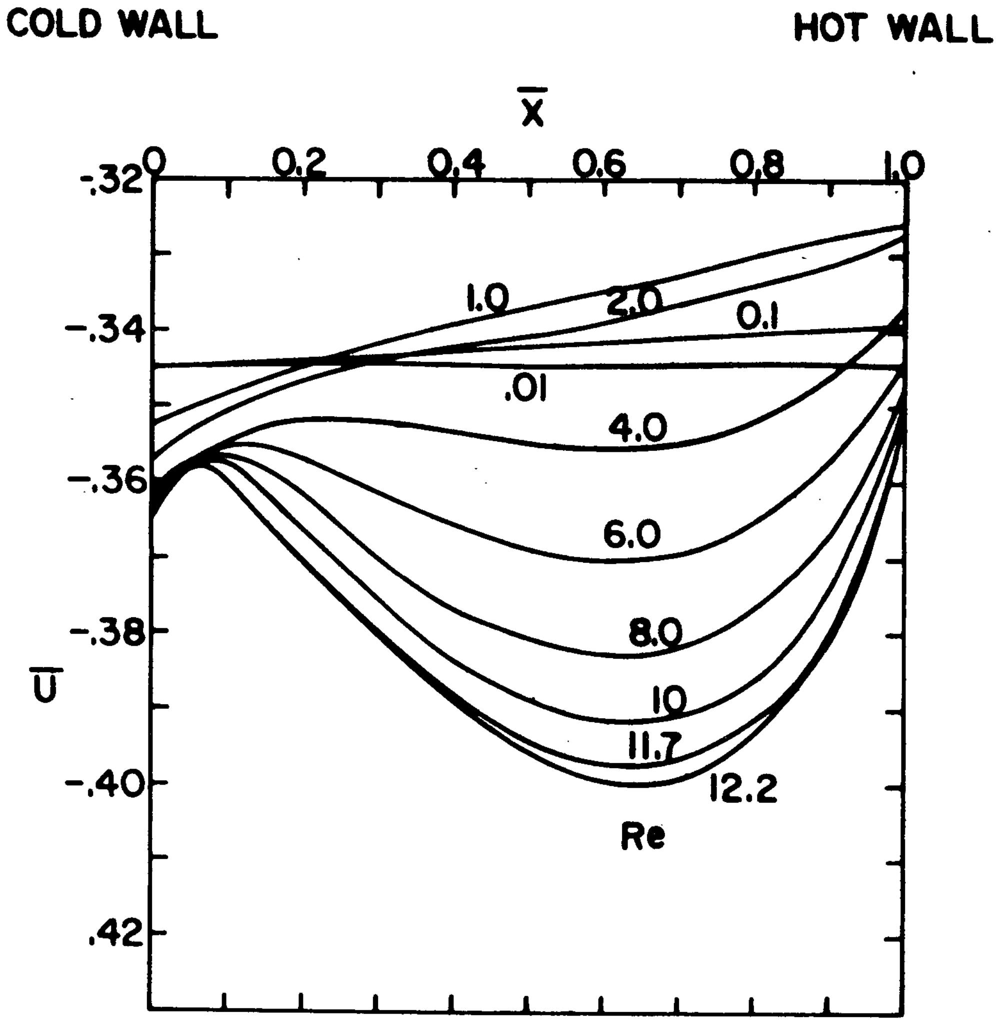


Figure 9 $\bar{u}$ vs $\bar{x}$

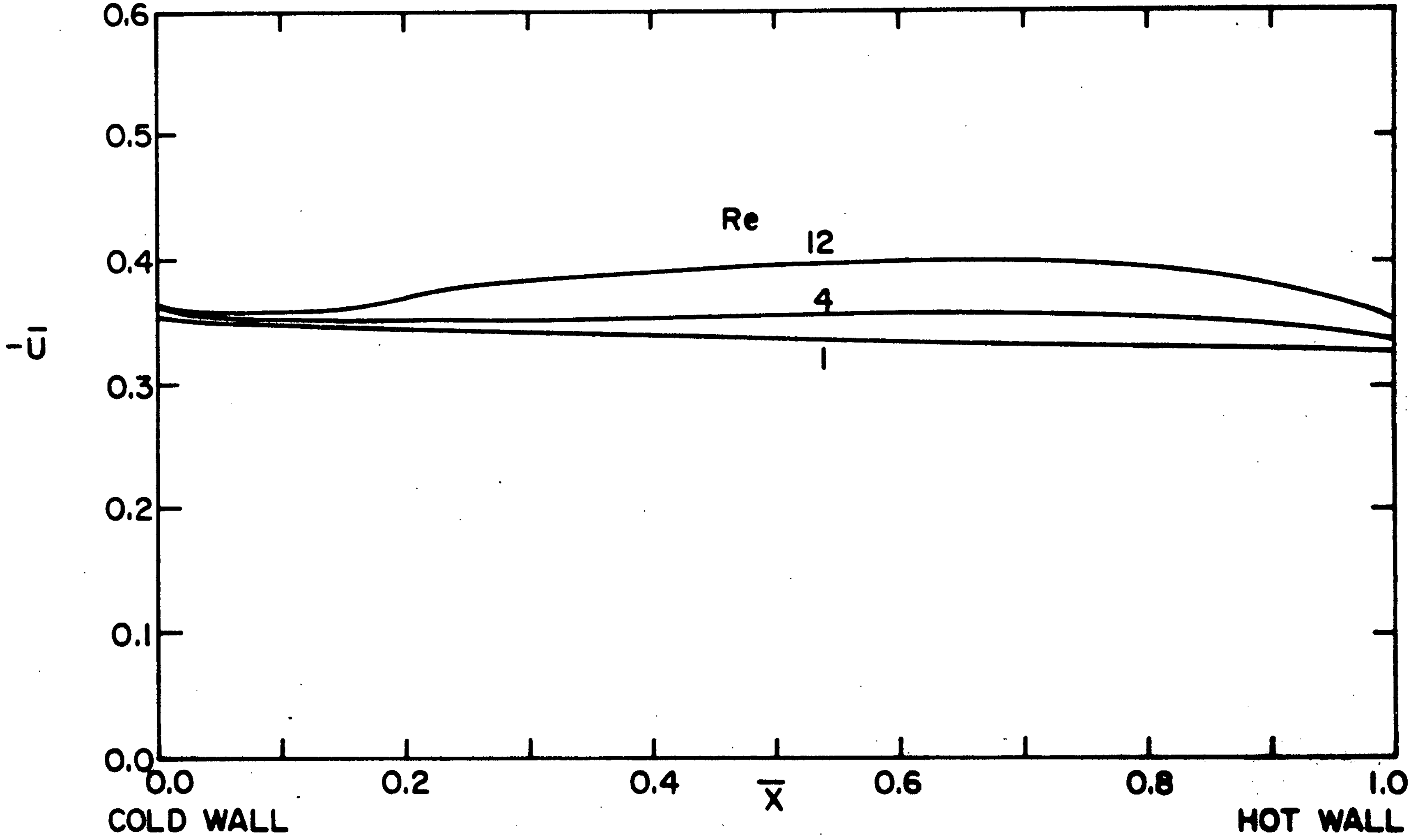


Figure 10 $-\bar{u}$ vs $\bar{x}$

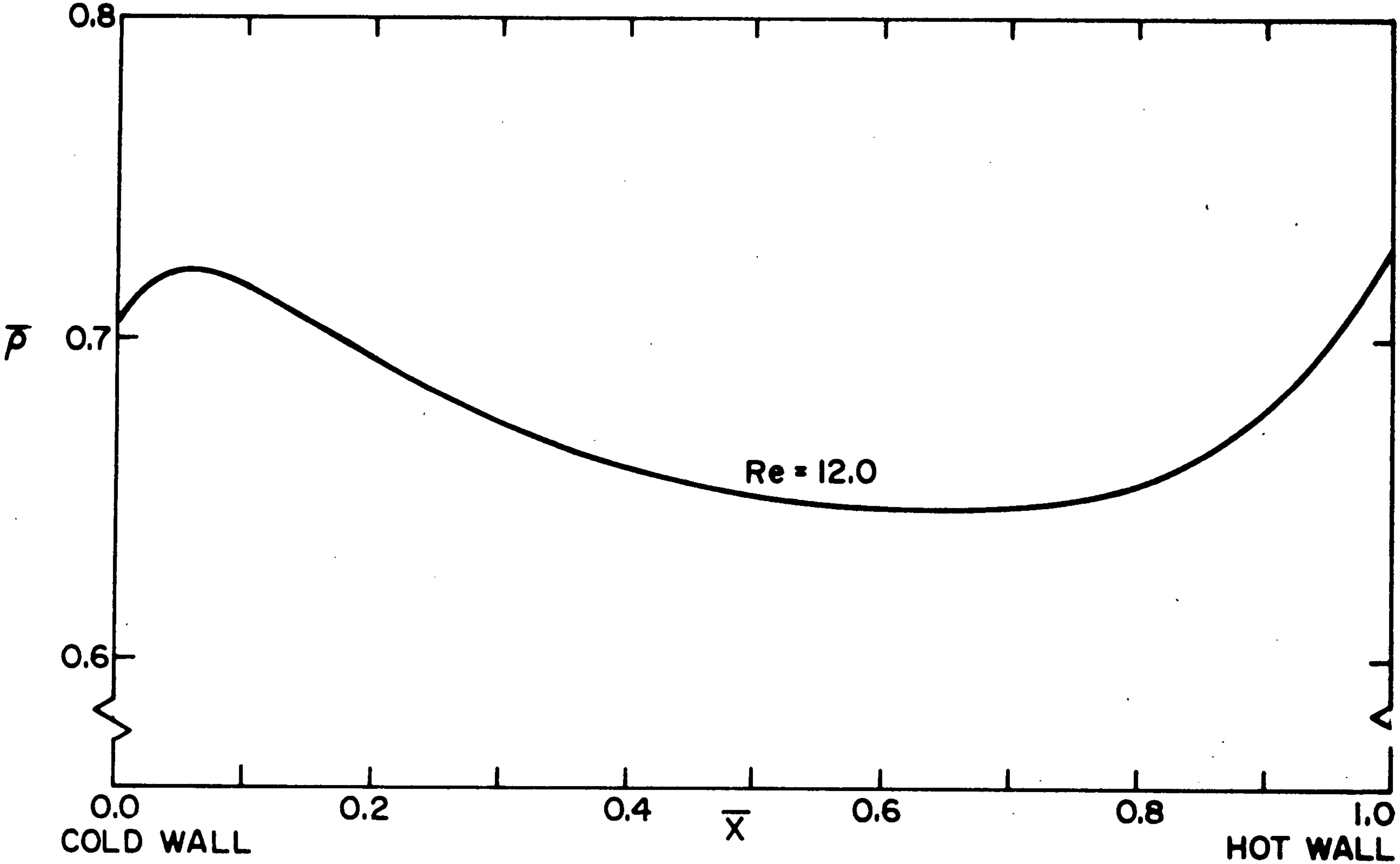


Figure 11 $\bar{\rho}$ vs $\bar{x}$

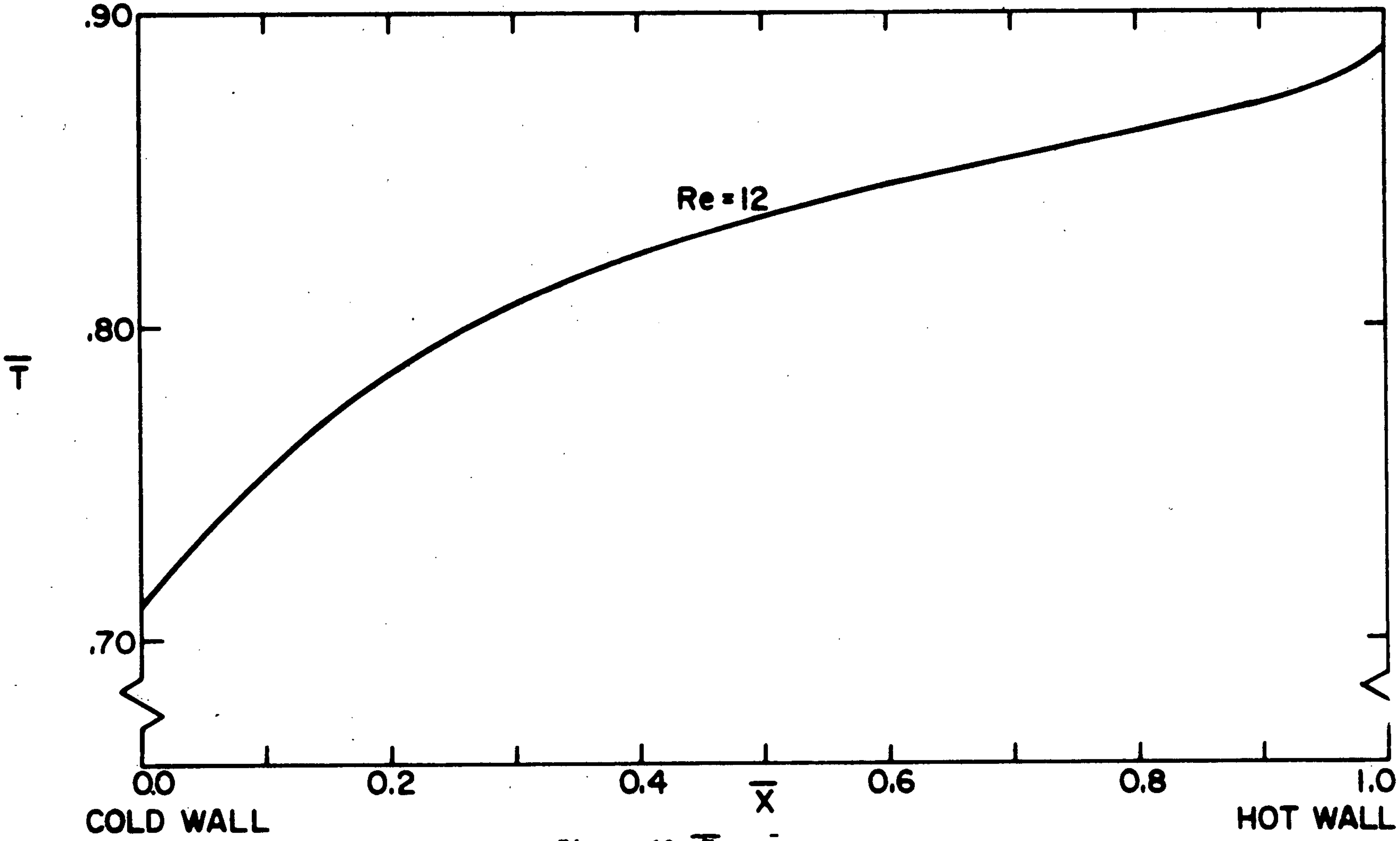


Figure 12 $\overline{T}$ vs $\bar{x}$

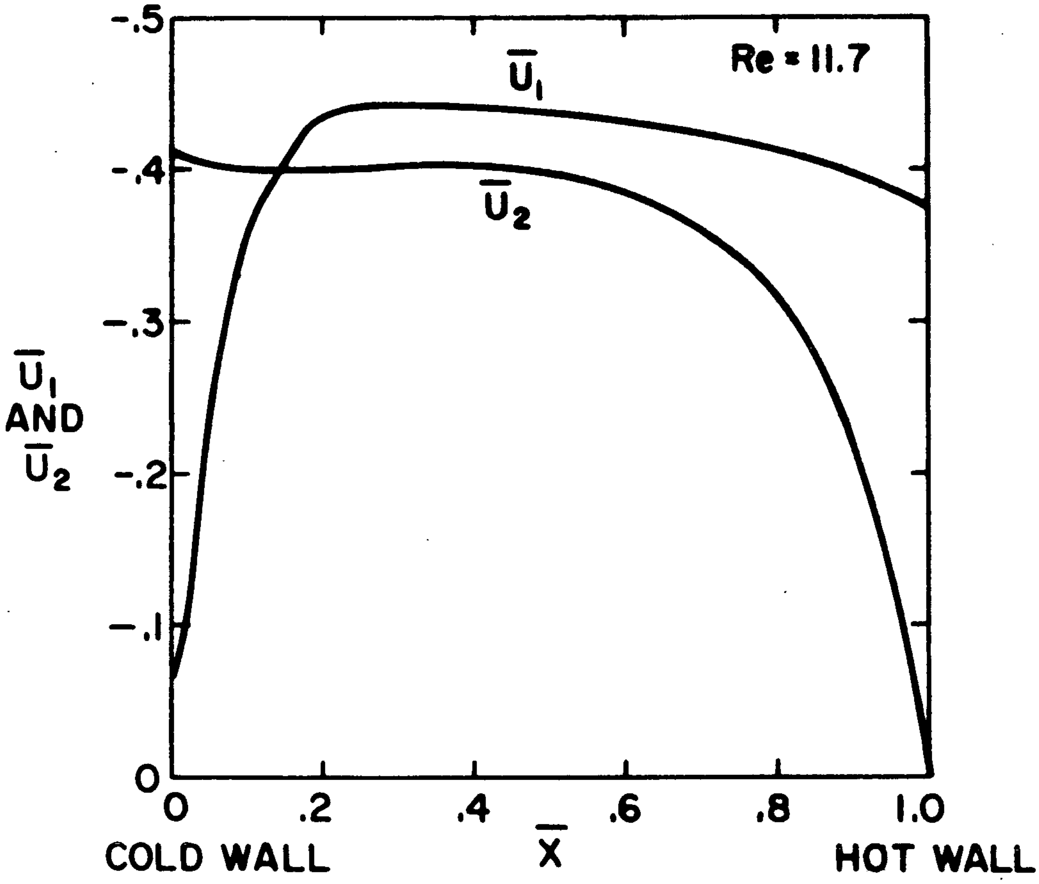


Figure 13 $\bar{u}_1$ and $\bar{u}_2$ vs $\bar{x}$

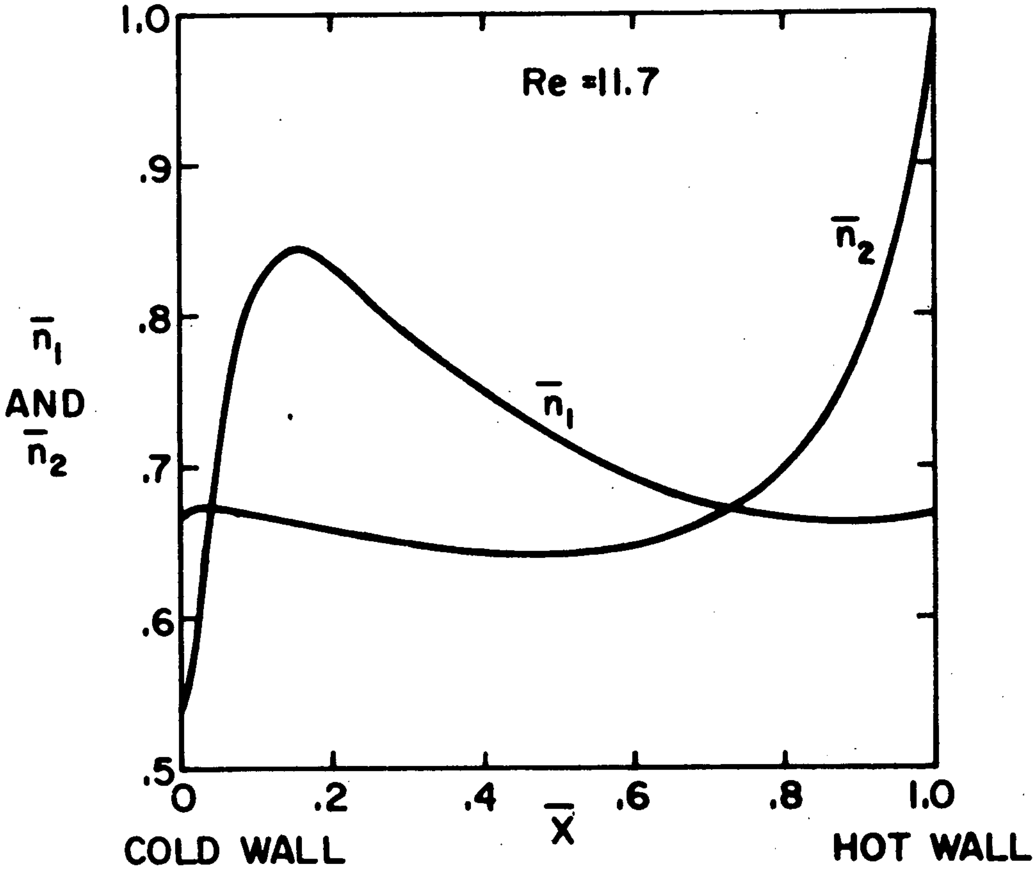


Figure 14 $\bar{n}_1$ and $\bar{n}_2$ vs $\bar{x}$

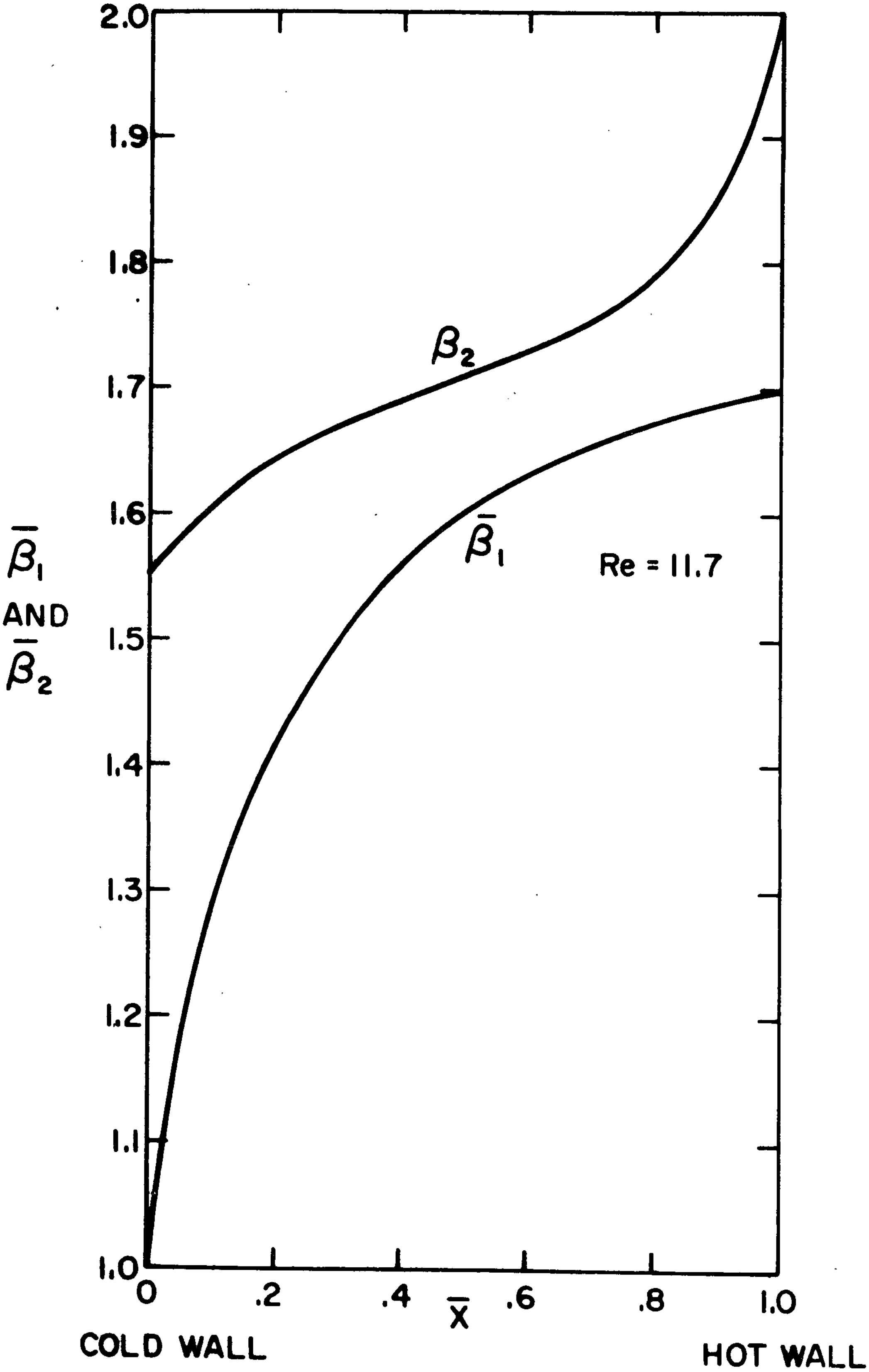


Figure 15 $\bar{\beta}_1$ and $\bar{\beta}_2$ vs $\bar{x}$

cold wall and the other at the hot wall.

Past Re = 12.0, it becomes increasingly difficult to use the method of integration discussed previously. However, at Re = 12.0, $\bar{n}_2(0)$, $\bar{u}_2(0)$, and $\bar{\beta}_2(0)$ have almost reached their limiting values, i.e., their values remain unchanged as Re → ∞. This fact allows use of the integration technique of shooting-splitting. The forward marching scheme used is a fourth order Runge-Kutta with an integration step size D = 0.0001 up to $\bar{x} = 0.005$ and D = 0.001 past 0.005. The integration process is begun at $\bar{x} = 0$ for Re = 100.0 (Kn = 0.0125 by (41)) and the values $\bar{n}_1(0)$, $\bar{u}_1(0)$, $\bar{\beta}_1(0)$, $\bar{n}_2(0)$, $\bar{u}_2(0)$ and $\bar{\beta}_2(0)$ for Re = 12.0 are used to start the integration. As the integration moves forward $\bar{u}_2$ will eventually go to zero at some $\bar{x} < 1$ and the integration process is stopped at that point. Next, the process is started over at $\bar{x} = 0$ but for a value of $\bar{u}_2(0)$ that is 0.01 greater than $\bar{u}_2(0)_{Re = 12.0}$. The new curve for $\bar{u}_2(\bar{x})$ eventually becomes greater than one at some $\bar{x}$. $u_2 = 1$ is used as a convenient cutoff point. If one takes the average between these two values for $\bar{u}_2(0)$ and repeats the process either an up curve or a down curve will result. The averaging process is repeated until $\bar{u}_2(0)$ for the up curve and down curve match to five decimal points. Then at a point $\bar{x} = .15$ (arbitrarily selected, Figure 16), the values of Y_i for the two curves are averaged and a new curve is found. This curve either goes up or down, but more importantly, it extends further out into the flow field than any of the curves starting at $\bar{x} = 0$. If this new curve is up for the sake of argument, then its values of Y_i and those of the down curve which starts at $\bar{x} = 0$ are averaged at $\bar{x} = 0.15$ and a new up or down curve is calculated. In this manner the flow behavior for large Re can be found (Figures 16, 17, 18, 19, 20, 21, 22, 23, and 24).

Each point at which $\bar{u}_2(\bar{x}) = 0$ can be thought of as a position of the hot wall and the corresponding value of Re is found by multiplying $\bar{x} \cdot 100$ since Re is linear in distance. Furthermore, the shooting-splitting technique can be carried out past $\bar{x} > 1$ and the resulting Re would be greater than 100. From Figures 18, 16, and 22 the various values of $\bar{n}_1$, $\bar{u}_1$, and $\bar{\beta}_1$ corresponding to $\bar{u}_2 = 0$ are plotted on Figures 6, 7, and 8. Note that $\bar{x} = 1$ corresponds to Re = 100 in the scale used in Figure 16 to Figure 24. However, each Re corresponding to $\bar{u}_2(\bar{x}) = 0$ can also have $\bar{x} = 1$ just by redefining the scale since it is linear.

Figures 16 to 24 indicate that an equilibrium situation is attained at $\bar{x} = 0.2$. Past this point $\bar{u} = \bar{u}_1 = \bar{u}_2$, $\bar{n}_1 = \bar{n}_2 = \bar{n}$, and $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta} = 2\bar{T}$ as the limits in the above figures illustrate. As Re becomes large the regions of rapid change become thinner (Figures 9 and 18). In the large Re limit the flow field has the following behavior: the velocity $\bar{u}$ accelerates from its values at the hot plate ($\bar{x} = 1$) to an equilibrium speed and then decelerates from equilibrium

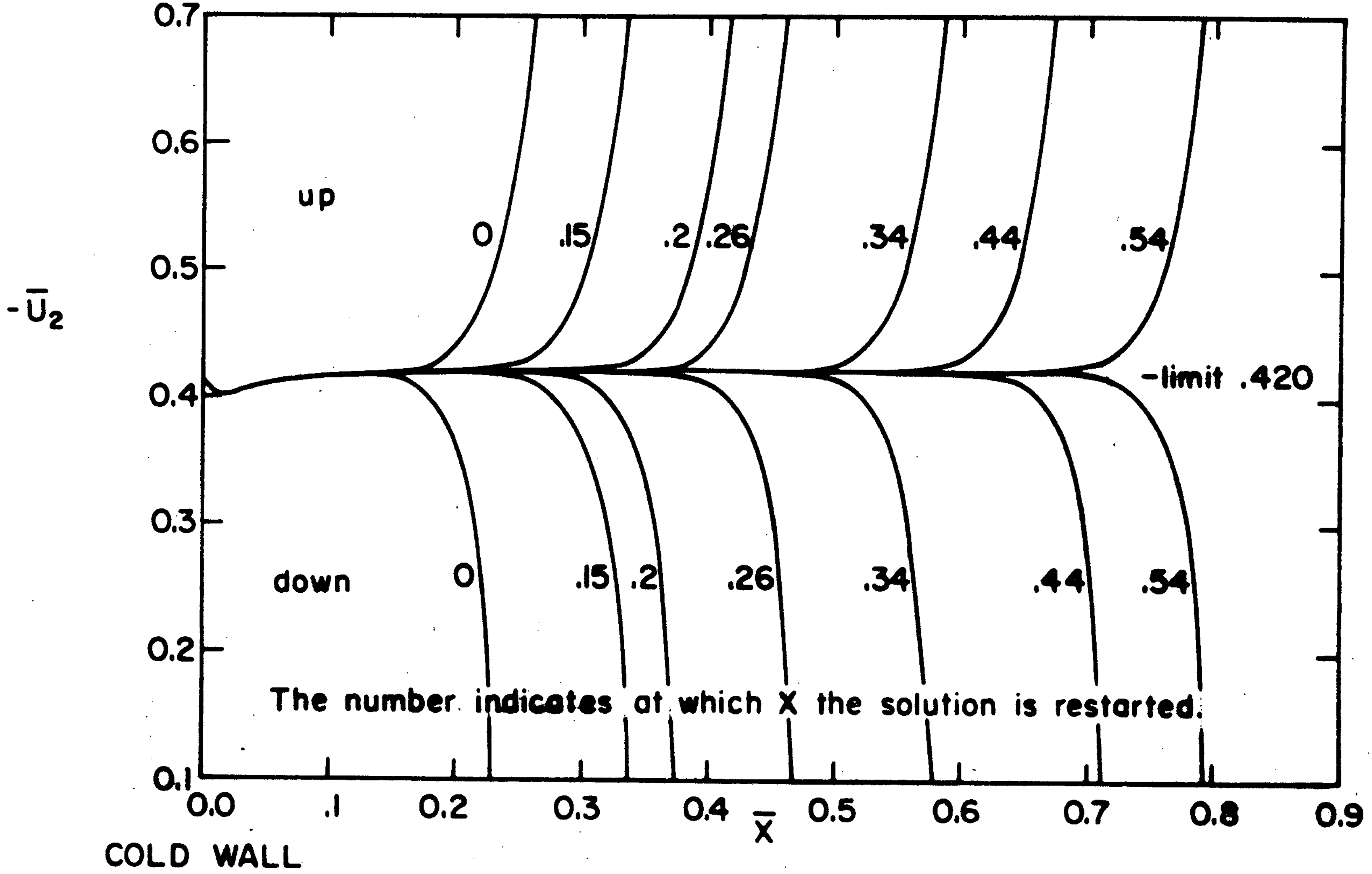


Figure 16 $-\bar{u}_2$ vs $\bar{x}$

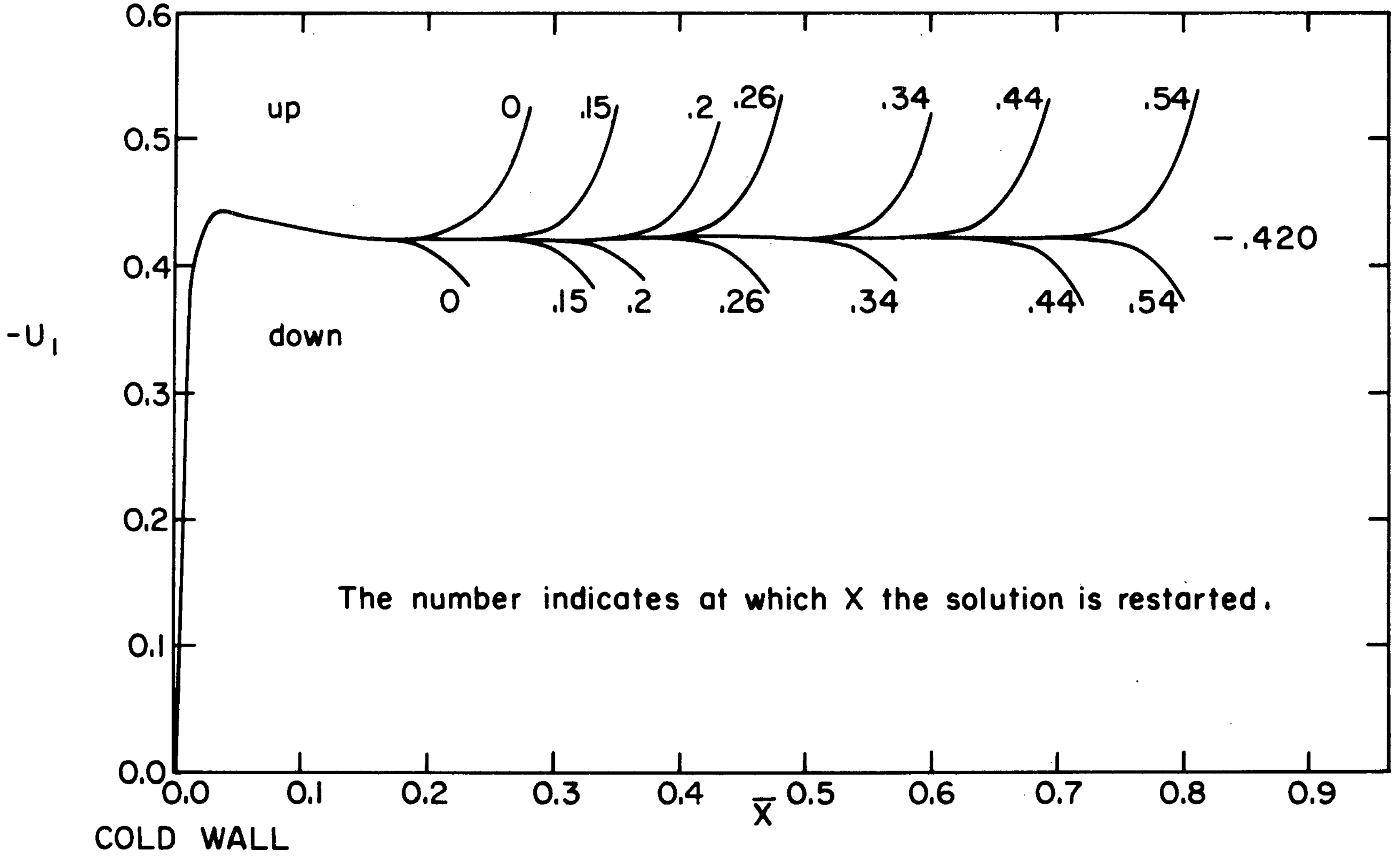
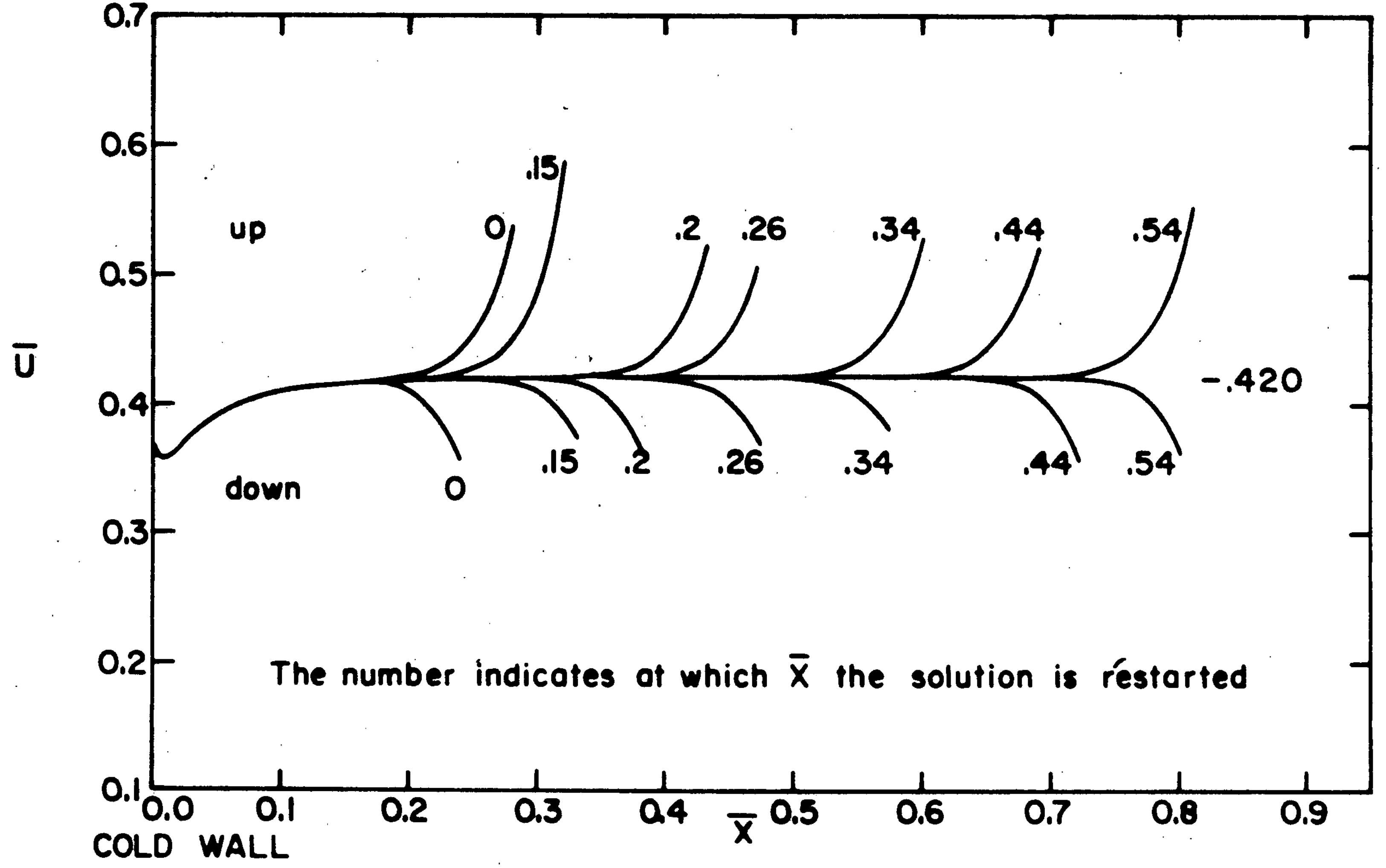


Figure 17 $-\bar{u}_1$ vs $\bar{x}$



The number indicates at which $\overline{X}$ the solution is restarted

Figure 18 $-\bar{u}$ vs $\bar{x}$

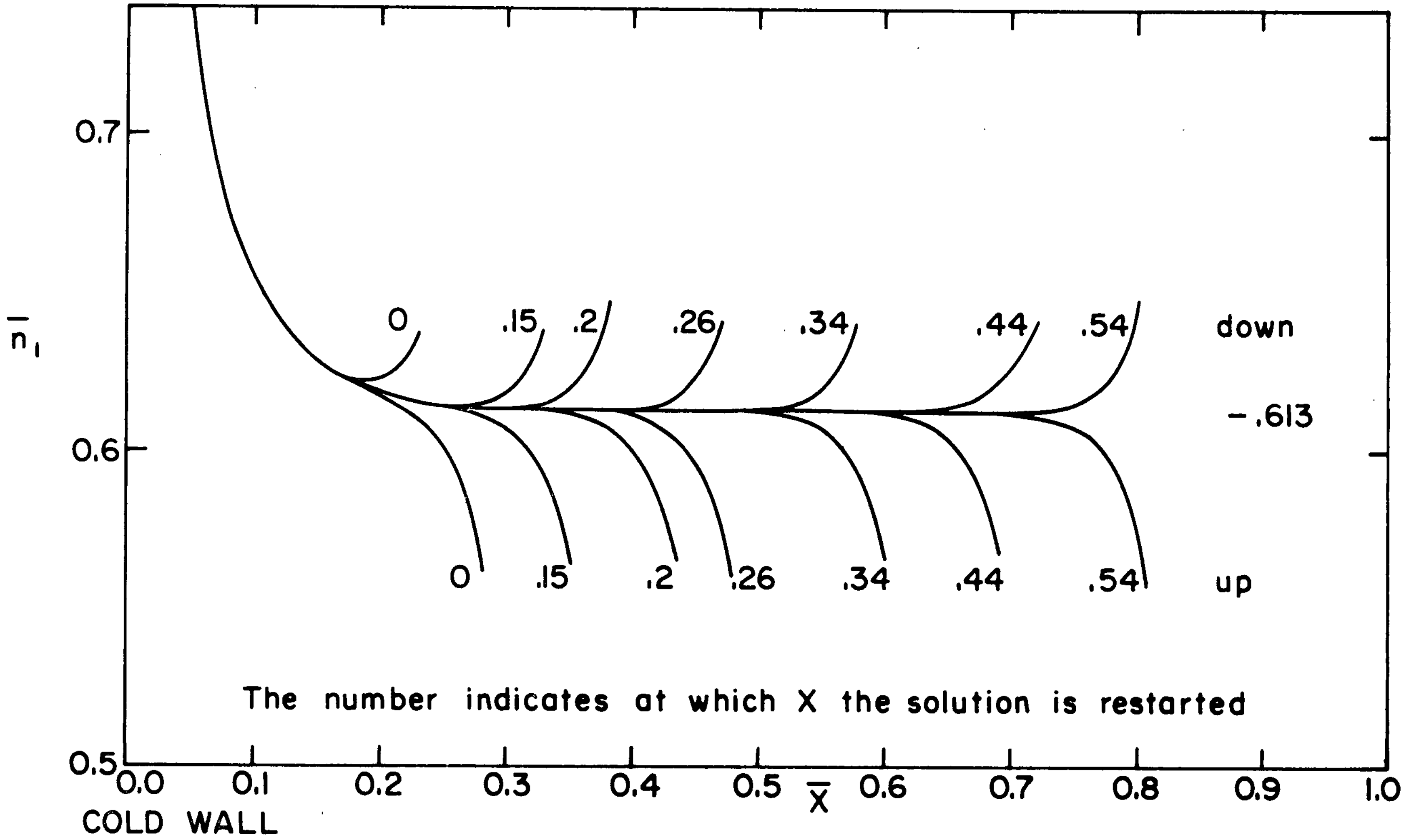


Figure 19 $\bar{n}_1$ vs $\bar{x}$

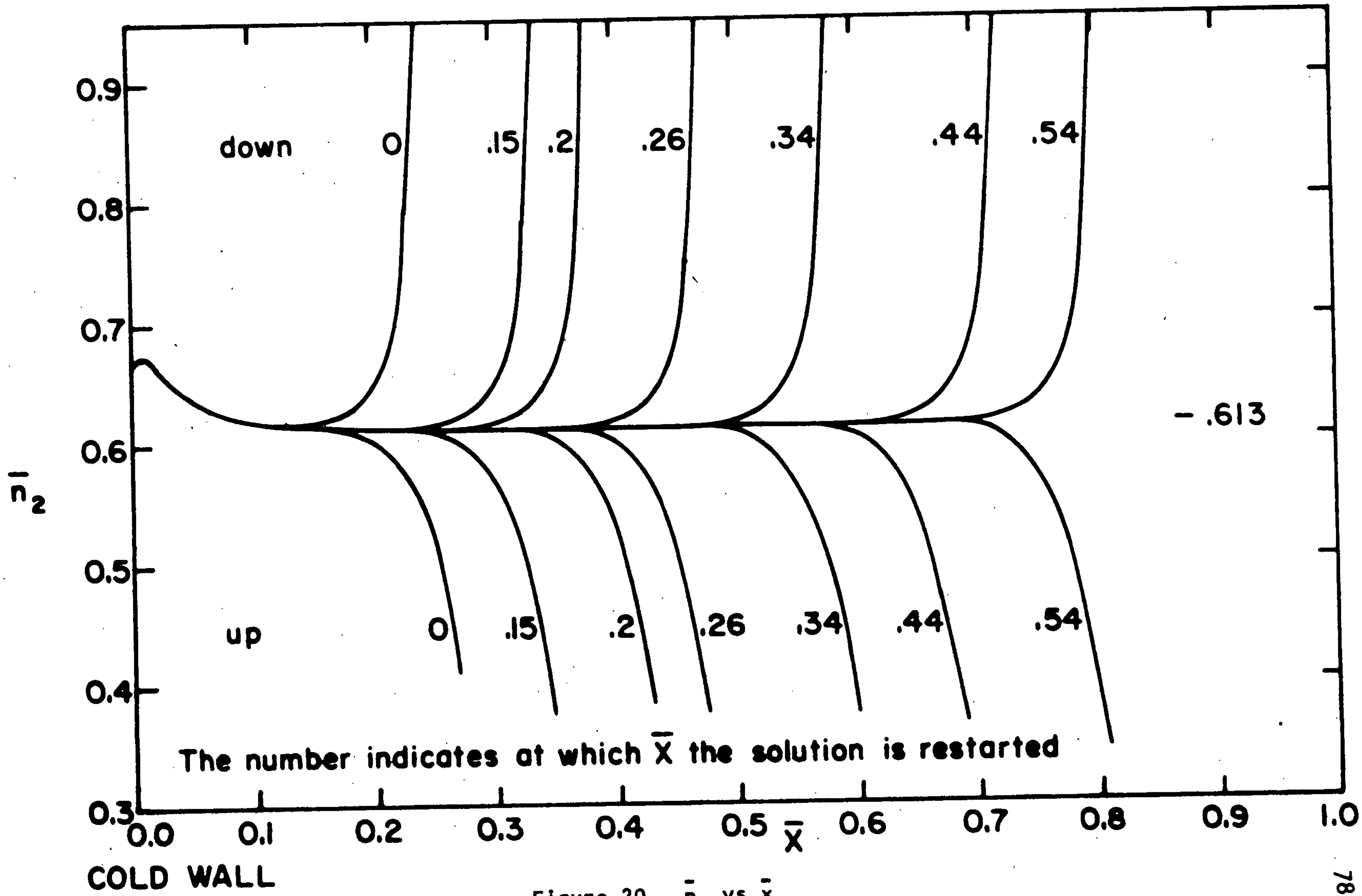


Figure 20 $\bar{n}_2$ vs $\bar{x}$

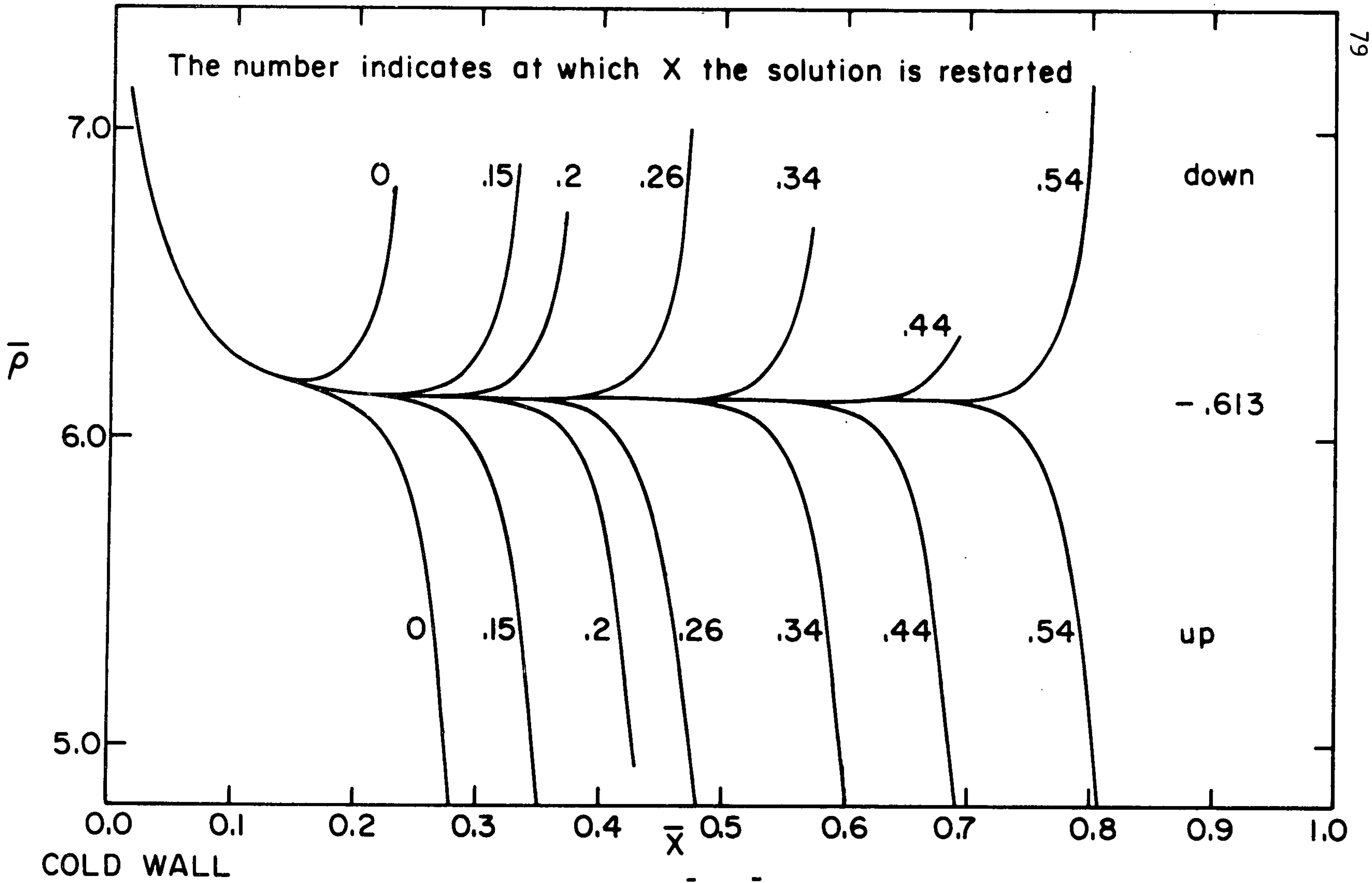


Figure 21 $\bar{\rho}$ vs $\bar{x}$

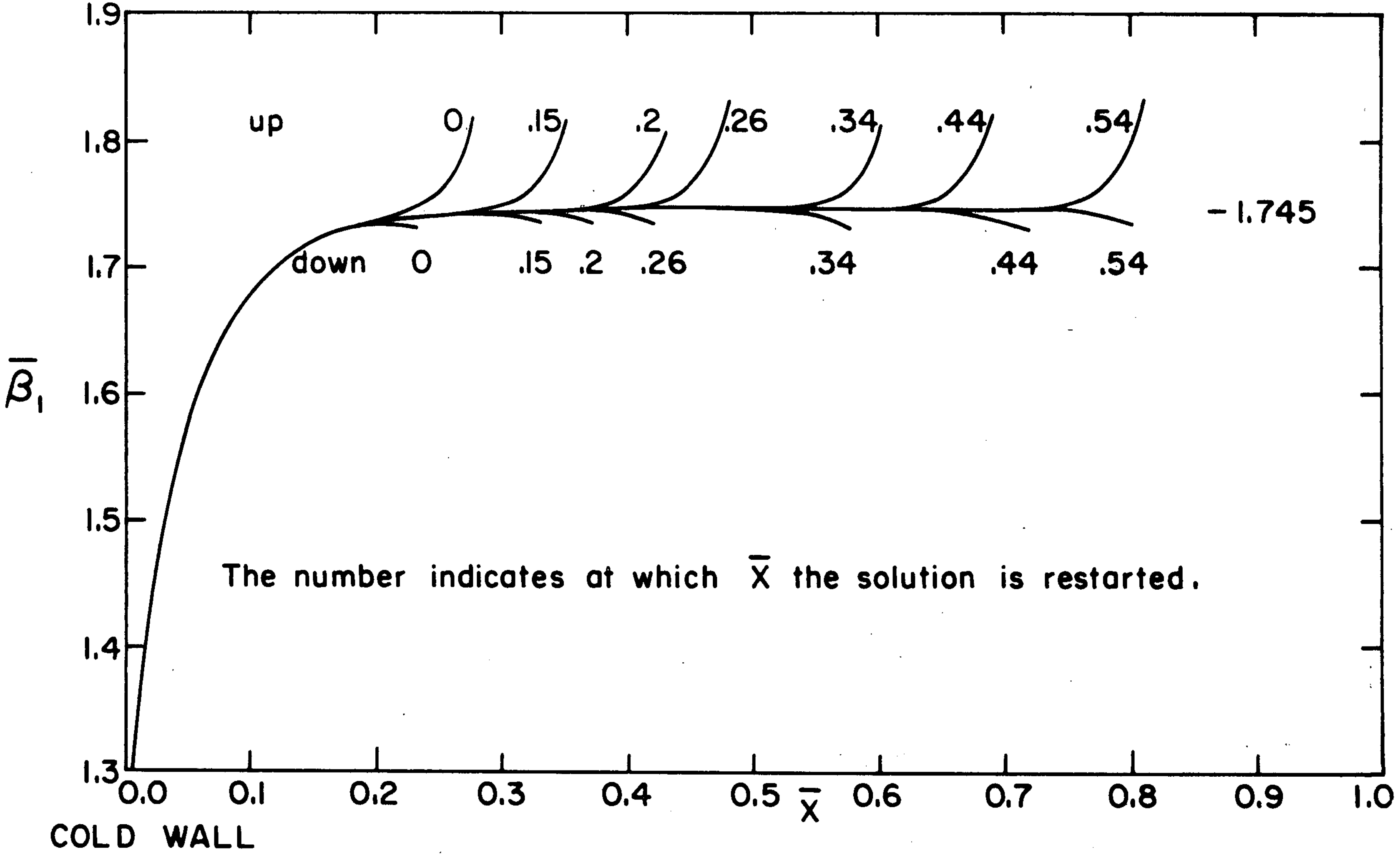


Figure 22 $\bar{\beta}_1$ vs $\bar{x}$

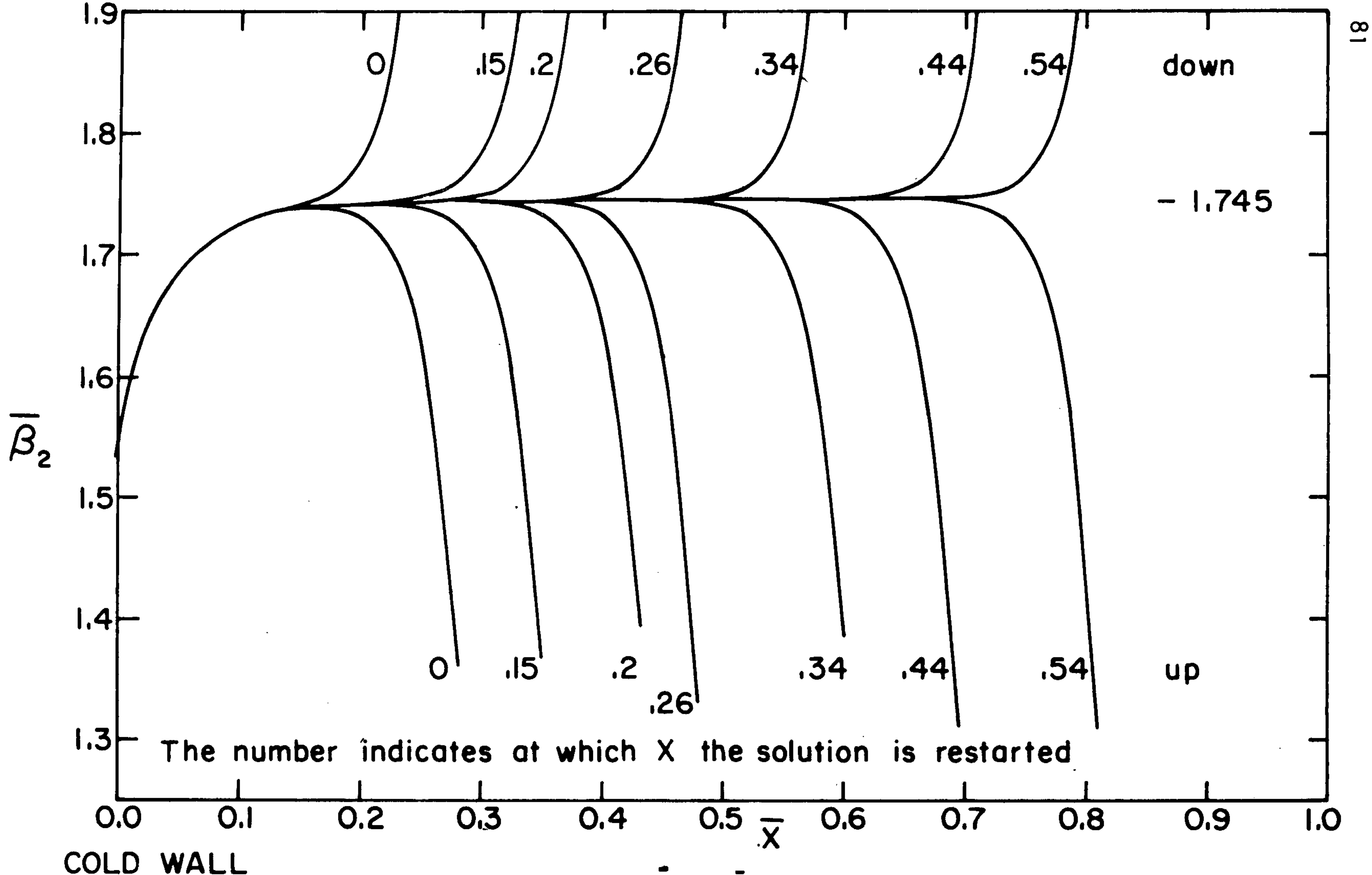


Figure 23 $\bar{\beta}_2$ vs $\bar{x}$

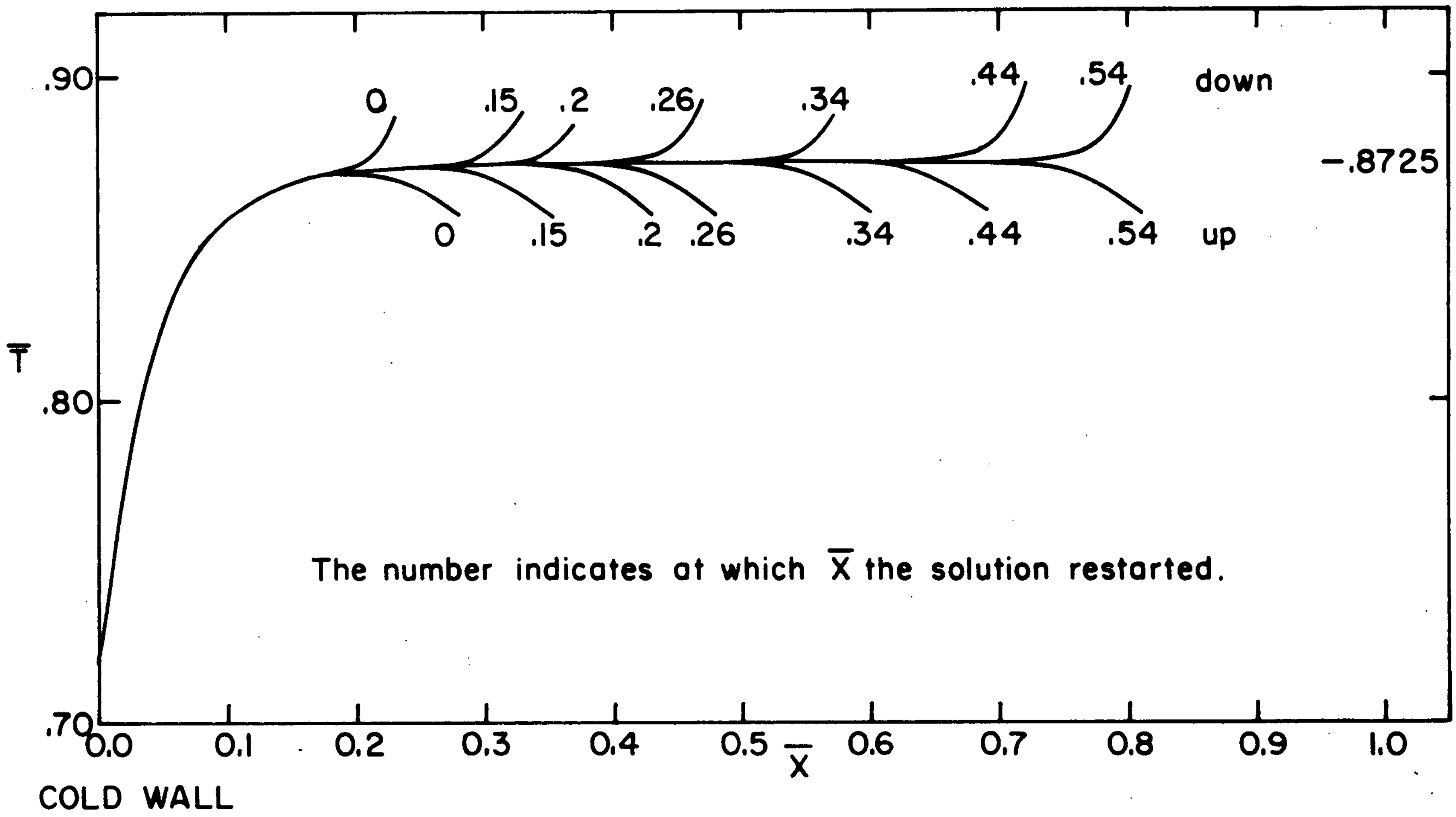


Figure 24 $\bar{T}$ vs $\bar{x}$

as the vapor approaches the cold wall. Very near this plate $\bar{u}$ accelerates again; both the density and temperature at $\bar{x} = 1$ decrease to their respective values in the equilibrium state whereas near the condensing surface the temperature decreases and the density increases to a peak value and then decreases slightly. For $Re \rightarrow \infty$, the total enthalpy (Figure 25) decreases slightly at the hot wall and then approaches equilibrium. Near the condensing surface, it decreases strongly. This flow pattern for large Re is like the result found by Collins and Edwards [16] except they found H to be constant at the evaporating surface. In the low Re limit ($Re = 0.01$) $\bar{u}$, $\bar{\rho}$, $\bar{\beta}$, etc., are found to be essentially constant.

In Figure 26 the velocity $\bar{u}$ is plotted for case III (92c). For $Re = 0.01$ $\bar{u}$ is constant. In the range of Re between 0.1 and 1.0 the vapor velocity $\bar{u}$ decelerates over most of the flow field and then accelerates very near the cold wall. This behavior is completely different from that of case I in the same Re range. For $Re > 1.0$ the $\bar{u}$ curves follow the same pattern as those in Figure 12 (case I). Furthermore, $\bar{n}_2(0)$, $\bar{u}_2(0)$, and $\bar{\beta}_2(0)$ approach limiting values around $Re = 14$ and the shooting-splitting technique can be used.

Case II (92b) has boundary conditions exactly opposite to those of case III. The mean velocity $\bar{u}$ for case II (Figure 27) exhibits the same behavior as $\bar{u}$ in case I (Figure 9) except that there is no acceleration at the hot wall.

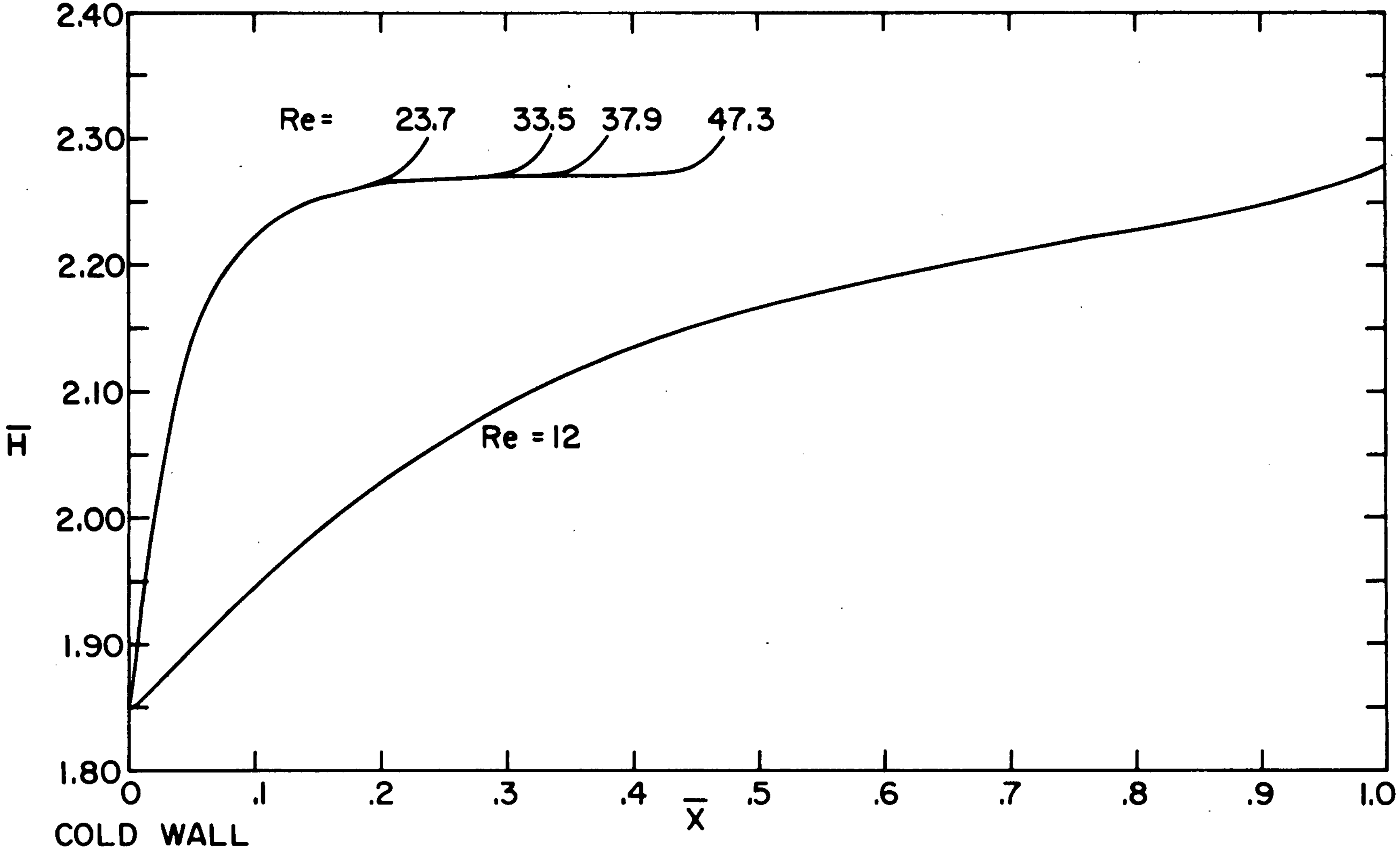


Figure 25 $\bar{H}$ vs $\bar{x}$

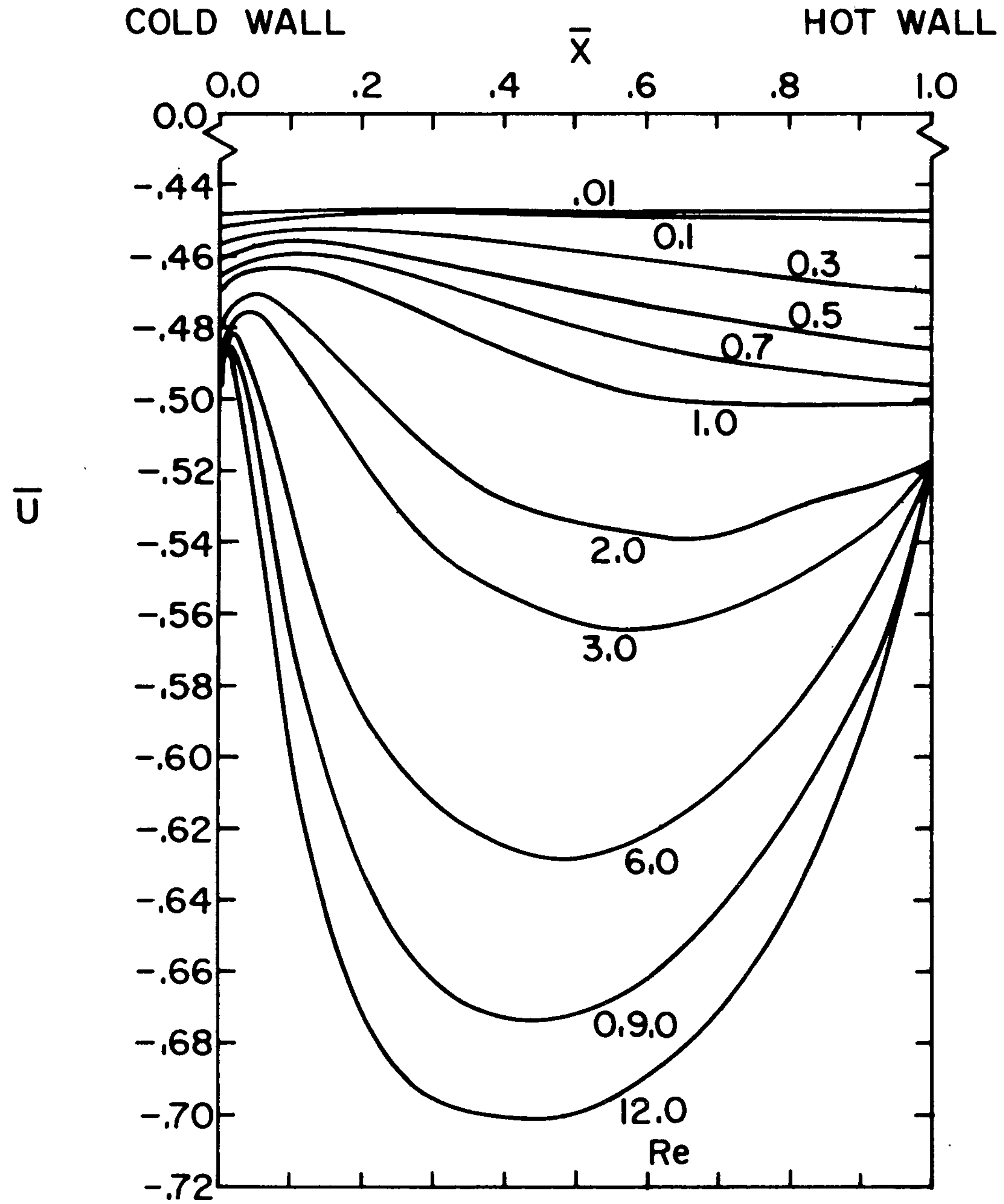


Figure 26 $\bar{u}$ vs $\bar{x}$ Case III

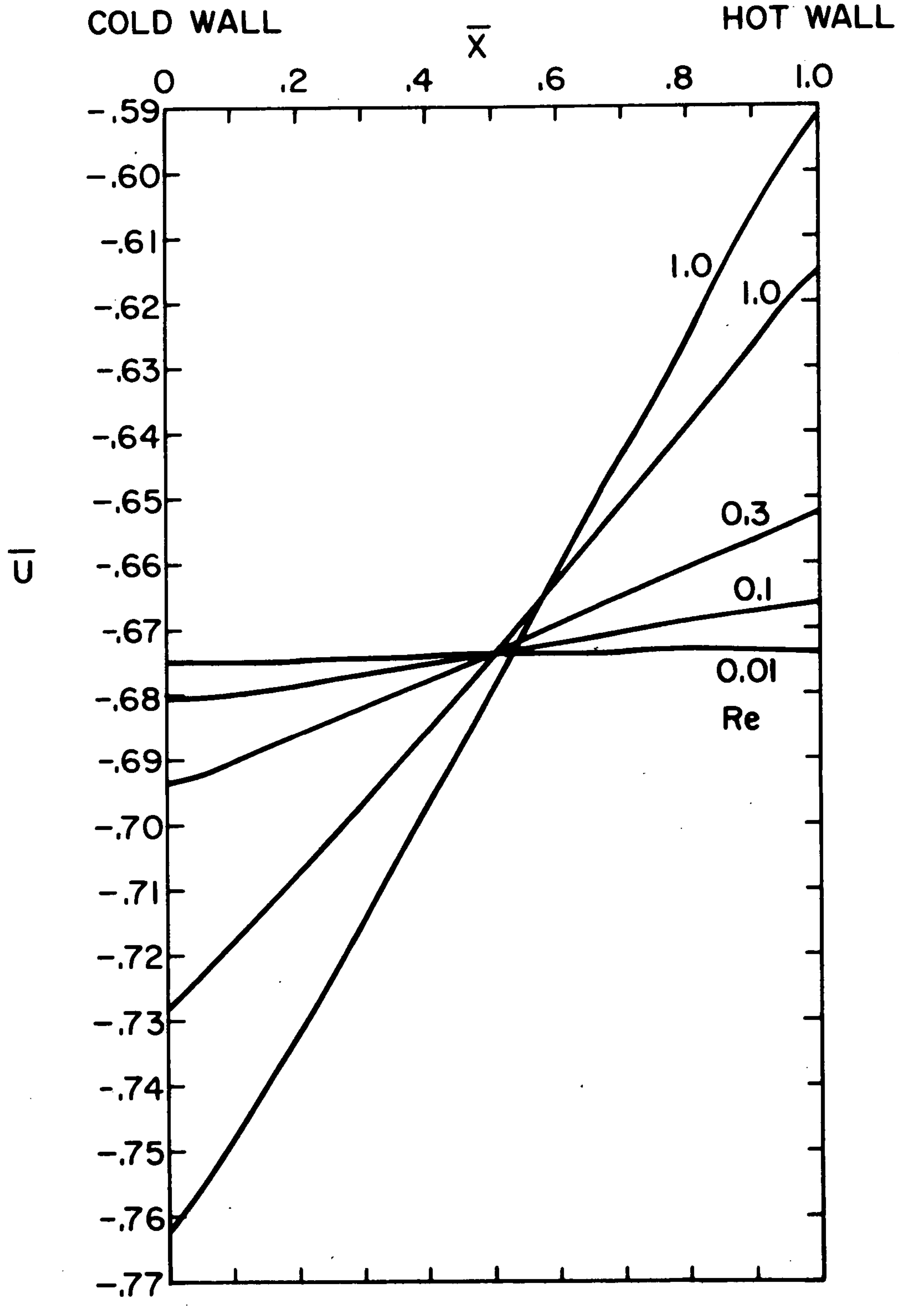


Figure 27 $\bar{u}$ vs $\bar{x}$ Case II

Finally, from equation (86), there results

$$Re_{\rho u} = \sqrt{\gamma}\ \frac{u}{\sqrt{\gamma R T_I}} \cdot \frac{\rho_I \sqrt{R T_I}\, d}{\mu_I} = \sqrt{\gamma}\ M_u\ Re$$

where u = mean velocity. Since $M_u \sim 0(1)$ and $\gamma = 5/3$, $Re_{\rho u} \sim 0(Re)$.

Chapter 7

CONCLUSIONS

The two plate problem is solved for a monatomic vapor composed of Maxwell molecules. Lees' moment method is used to obtain a set of six non-linear moment equations whose solution, subject to the boundary conditions of this problem, is continuous over the range of flow conditions from free molecular to continuum.

To obtain an analytical solution to this problem, small deviations from equilibrium are considered. A first order perturbation analysis is used and the mean velocity $\overline{u}$ is the small parameter in the problem. For Kn $\rightarrow \infty$, the flow properties and the vapor velocity are constant between the two plates and the value of the evaporation coefficient is one. When Kn $\rightarrow 0$, the flow is either density or temperature dominated depending on whether $\Delta\overline{T}/\Delta\overline{N} < 2.19$ or $\Delta\overline{T}/\Delta\overline{N} > 2.19$ respectively. The evaporation coefficient in this limit is 0.83. The $Re_{\rho u}$ based on the mass flux at the cold wall can be small or large depending on the magnitude of d/ν.

The two plate problem for strong non-equilibrium conditions is solved as an initial value problem. The direction of integration is opposite to that of the vapor motion. Three cases are considered. For Re $\rightarrow 0$ the vapor properties and velocity are constant across the flow field. As Re → ∞, two regions of rapid change appear: one at the hot wall and one at the cold wall. The flow accelerates at the hot wall to an equilibrium velocity, decelerates from equilibrium to a local minimum velocity as it approaches the cold wall, and then accelerates again. The vapor temperature and density decrease at the hot wall to their respective equilibrium values. However, at the cold wall the vapor density increases rapidly, reaches a peak and then drops slightly whereas the vapor temperature decreases. Finally, the total enthalpy decreases slightly at the hot surface to an equilibrium value and then drops sharply at the cold wall.

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APPENDIX A

DERIVATION OF EQUATIONS OF TRANSFER FOR $Q(c_i) = mc_j^3$

As previously discussed, Maxwell's inverse fifth power law of molecular repulsion is used in evaluating the collision integral. For such molecules the collision integral becomes

$$\Delta Q = \sqrt{(m_1 + m_2)K} \int\int ff_1 \, J \, d\vec{c} \, d\vec{c}_1 \,, \tag{A.1}$$

where

$$J = \int_0^{\infty} \int_0^{2\pi} (Q'-Q) \, d\epsilon \, a \, da \tag{A.2}$$

and a and ϵ are geometric parameters describing the collision process. The description and interpretation of the parameters appearing in equation (A.2) and the subsequent development have been treated by a number of authors. For details the reader is referred to the work of Lees [9] whose nomenclature and methodology is adopted in this work.

For the moment of interest the difference in Q resulting from a collision is

$$\frac{Q'-Q}{m} = (c_j')^3 - (c_j)^3 . \tag{A.3}$$

To express Q'-Q in terms of c_j, use is made of the expression for c_j'

originally developed by Maxwell [9],

$$c'_j = c_j + (c_1 - c)_j \cos^2(\theta'/2) + \sqrt{V^2 - (c_1 - c)_j^2}\; \sin\theta' \cos(\epsilon + \omega_{jk}) \tag{A.4}$$

where θ', ϵ, and ω_{jk} are angles in various planes which describe the binary collision. By substituting this expression into (A.3), Q'-Q becomes

$$\frac{Q'-Q}{m} = 3c_j^2 a + 3c_j^2 a' \cos\epsilon' + 3c_j a^2 + 6c_j a a' \cos\epsilon' + 3c_j a'^2 \cos^2\epsilon'$$
$$+ a^3 + 3a^2 a' \cos\epsilon' + 3aa'^2 \cos^2\epsilon' + a'^3 \cos^3\epsilon' \tag{A.5}$$

where

$$a = (c_1 - c)_j \cos^2(\theta'/2)$$

$$a' = \frac{1}{2}\sqrt{V^2 - (c_1 - c)_j^2}\; \sin\theta'$$

$$\epsilon' = \epsilon + \omega_{jk} .$$

After placing this expression for Q'-Q into equation (A.2), integration over ϵ is performed first. It is noted that terms proportional to odd powers of $\cos m\epsilon$ or $\sin m\epsilon$ $(m \neq 0)$ integrate to zero over the 2π limit hence terms involving $\cos\epsilon'$ and $\cos^2\epsilon'$ integrate to zero. Using the fact that integration of $\cos^2(\epsilon + \omega_{jk})$ is π and rearranging, the integral of $\frac{Q'-Q}{m}$ over ϵ becomes:

$$\int_0^{2\pi} (c_j'^3 - c_j^3)\,d\epsilon = 2\pi \Big\{ (c_1 - c)_j \Big[c_{j1}^2 + c_j c_{j1} + c_j^2 \Big] \cos^2(\theta'/2)$$

$$- \frac{1}{4} \Big[c_j \Big(\frac{5}{2} (c_1 - c)_j^2 - \frac{3}{2} V^2 \Big) + (c_1 + c)_j (c_1 - c)_j^2 \Big] \sin \theta'$$

$$- \frac{1}{4} (c_1 - c)_j \Big[\frac{5}{2} (c_1 - c)_j^2 - \frac{3}{2} V^2 \Big] \sin^2\theta' \cos^2(\theta'/2) \Big\}. \qquad \text{(A.6a)}$$

Since this expression must be symmetrical with respect to the probe and colliding gas molecules, an equivalent statement of the bracketed term in equation (A.6a) is

$$\Big\{ \Big\} = (c - c_1)_j [c_{j1}^2 + c_j c_{j1} + c_j^2] \cos^2(\theta'/2)$$

$$- \frac{1}{4} \Big[c_{j1} \Big(\frac{5}{2} (c_1 - c)_j^2 - \frac{3}{2} V^2 \Big] + (c + c_1)_j (c_1 - c)_j^2 \Big] \sin^2\theta'$$

$$- \frac{1}{4} (c - c_1)_j \Big[\frac{5}{2} (c_1 - c)_j^2 - \frac{3}{2} V^2 \Big] \sin^2\theta' \cos^2(\theta'/2) \ . \qquad \text{(A.6b)}$$

To obtain symmetry in the expression for J the bracketed terms in equation (A.6a) and (A.6b) are summed and divided by two to yield

$$J = \frac{m}{4} (c_1 + c)_j \Big[\frac{3}{2} V^2 - \frac{9}{2} (c_1 - c)_j^2 \Big] \int_0^{\infty} \pi \sin^2\theta' \, a\, da \ . \qquad \text{(A.7)}$$

Maxwell has evaluated the integral in equation (A.7) and the result is

$$A_2 = \pi \int_0^\infty \sin^2\theta' \, a \, da = 1.3682 . \qquad (A.8)$$

Thus, the equation for J is

$$J = \frac{1}{4} m A_2 (c_1 + c)_j \left[\frac{3}{2} V^2 - \frac{9}{2} (c_1 - c)_j^2 \right] . \qquad (A.9)$$

To allow physical interpretation at a later point in the development it is convenient to express V^2 in terms of thermal velocities since the mean velocity of probe and colliding particles are identical

$$V^2 = \sum_{i=1}^{3} (C_1 - C)_i^2 = \sum_{i=1}^{3} (C_1^2 + C^2 - 2C_1C)_i . \qquad (A.10)$$

In making this substitution the collision term for a single component gas may be written as

$$\Delta Q = \frac{1}{4} m A_2 \sqrt{2mK} \int\int f f_1 (c_1 + c)_j \left[\frac{3}{2} V^2 - \frac{9}{2} (C_1 - C)_j^2 \right] d\bar{c} \, d\bar{c}_1 . \qquad (A.11)$$

Evaluation of the integral using $c_j = C_j + u_j$ yields

$$\Delta Q = \frac{1}{4} m n^2 A_2 \sqrt{2mK} \left\{ \frac{3}{2} \left[\overline{C_{1j} C_1^2} + \overline{C_j C^2} + 2u_j (\overline{C_1^2} + \overline{C^2}) \right] \right.$$

$$\left. - \frac{9}{2} \left[\overline{C_{1j}^3} + \overline{C_j^3} + 2u_j (\overline{C_{1j}^2} + \overline{C_j^2}) \right] \right\} . \qquad (A.12)$$

Again, calling upon the symmetry between probe and colliding particles, there results

$$\Delta Q = \frac{1}{2} m n^2 A_2 \sqrt{2mK} \left[\frac{3}{2} \overline{C_j C^2} + 3u_j \overline{C^2} - \frac{9}{2} \overline{C_j^3} - 9u_j \overline{C_j^2} \right] . \tag{A.13}$$

To simplify this expression use is made of the equation derived by Maxwell for viscosity based on a local full range Maxwellian velocity distribution

$$\mu = \frac{kT}{\frac{3}{2} A_2 \sqrt{2mK}} . \tag{A.14}$$

With this result the collision term becomes

$$\Delta Q = \rho \frac{p}{\mu} \left[\overline{C_j C^2}/2 + u_j \overline{C^2} - \frac{3}{2} \overline{C_j^3} - 3u_j \overline{C_j^2} \right] . \tag{A.15}$$

In terms of the shear stress and heat flux it may be rewritten

$$\Delta Q = \frac{p}{\mu} \left[\dot{q}_j + 3u_j \tau_{jj} - \frac{3}{2} \rho \overline{C_j^3} \right] . \tag{A.16}$$

The corresponding moment equation is

$$\frac{\partial}{\partial t} \left[m \int f c_j^3 d\vec{c} \right] + \frac{\partial}{\partial x_i} \left[m \int f c_i c_j^3 d\vec{c} \right] = \frac{p}{\mu} \left[\dot{q}_j + 3u_j \tau_{jj} - \frac{3}{2} \rho \overline{C_j^3} \right] . \tag{A. 17}$$

APPENDIX B

INTEGRALS USED TO EVALUATE MOMENT EQUATIONS

$$\int_{-\infty}^{c} e^{-\xi^2/\beta}\, d\xi = \frac{1}{2}\sqrt{\pi\beta}\left[1 + \operatorname{erf}\left(\frac{c}{\sqrt{\beta}}\right)\right] \tag{B.1}$$

$$\int_{-\infty}^{c} \xi\, e^{-\xi^2/\beta}\, d\xi = -\frac{\beta}{2}\, e^{-c^2/\beta} \tag{B.2}$$

$$\int_{-\infty}^{c} \xi^2 e^{-\xi^2/\beta}\, d\xi = \frac{1}{4}\,[\pi\beta^3]^{\frac{1}{2}}\left[1 + \operatorname{erf}\left(\frac{c}{\sqrt{\beta}}\right)\right] - \frac{1}{2}\,\beta c e^{-c^2/\beta} \tag{B.3}$$

$$\int_{-\infty}^{c} \xi^3 e^{-\xi^2/\beta}\, d\xi = -\frac{\beta}{2}\,(\beta + c^2)\, e^{-c^2/\beta} \tag{B.4}$$

$$\int_{-\infty}^{c} \xi^4 e^{-\xi^2/\beta}\, d\xi = \frac{3}{8}\,[\pi\beta^5]^{\frac{1}{2}}\left[1 + \operatorname{erf}\left(\frac{c}{\sqrt{\beta}}\right)\right] - \frac{\beta}{2}\left[c^3 + \frac{3}{2}\beta c\right]e^{-c^2/\beta} \tag{B.5}$$

APPENDIX C

COMPUTER PROGRAM

The six moment equations that are solved in the subroutine DRVTV are:

continuity, i = 1

$$(\bar{u}_1 E_1 + \sqrt{\bar{\beta}_1/\pi}\, X_1) \frac{d\bar{n}_1}{d\bar{x}} + (\bar{u}_2 E_2 - \sqrt{\bar{\beta}_2/\pi}\, X_2) \frac{d\bar{n}_2}{d\bar{x}}$$

$$+ \bar{n}_1 E_1 \frac{d\bar{u}_1}{d\bar{x}} + \bar{n}_2 E_2 \frac{d\bar{u}_2}{d\bar{x}} + 1/2 \frac{\bar{n}_1}{\sqrt{\pi \bar{\beta}_1}} X_1 \frac{d\bar{\beta}_1}{d\bar{x}} - 1/2 \frac{\bar{n}_2}{\sqrt{\pi \bar{B}_2}} X_2 \frac{d\bar{\beta}_2}{d\bar{x}} = 0 \tag{C.1}$$

x-momentum equation i = 2

$$\left([\bar{\beta}_1/2 + \bar{u}_1^2] E_1 + \bar{u}_1 X_1 \sqrt{\bar{\beta}_1/\pi} \right) \frac{d\bar{n}_1}{d\bar{x}} + \left(2\bar{n}_1 \sqrt{\frac{\bar{\beta}_1}{\pi}}\, X_1 + 2\bar{u}_1 \bar{n}_1 E_1 \right) \frac{d\bar{u}_1}{d\bar{x}}$$

$$+ \frac{\bar{n}_1 E_1}{2} \frac{d\bar{\beta}_1}{d\bar{x}} + \left([\bar{\beta}_2/2 + \bar{u}_2^2] E_2 - \bar{u}_2 X_2 \sqrt{\bar{\beta}_2/\pi} \right) \frac{d\bar{n}_2}{d\bar{x}}$$

$$+ \left(2\bar{u}_2 \bar{n}_2 E_2 - 2\bar{n}_2 X_2 \sqrt{\frac{\bar{\beta}_2}{\pi}} \right) \frac{d\bar{u}_2}{d\bar{x}} + \frac{\bar{n}_2 E_2}{2} \frac{d\bar{\beta}_2}{d\bar{x}} = 0 \tag{C.2}$$

x-momentum flux, i = 3

$$\left[\left(\bar{u}_1^3 + 1.5\,\bar{u}_1\bar{\beta}_1\right)E_1 + \sqrt{\frac{\bar{\beta}_1}{\pi}}\,(\bar{\beta}_1 + \bar{u}_1^2)X_1\right]\frac{d\bar{n}_1}{d\bar{x}}$$

$$+\left[3\bar{n}_1\bar{u}_1^2E_1 + 1.5\,\bar{n}_1\bar{\beta}_1E_1 + 3\bar{n}_1\bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]\frac{d\bar{u}_1}{d\bar{x}}$$

$$+\left[\frac{3}{2}\bar{n}_1\bar{u}_1E_1 + \frac{3}{2}\bar{n}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]\frac{d\bar{\beta}_1}{d\bar{x}} + \left[\left(\bar{u}_2^3 + 1.5\bar{u}_2\bar{\beta}_2\right)E_2 - \sqrt{\frac{\bar{\beta}_2}{\pi}}\,(\bar{\beta}_2 + \bar{u}_2^2)X_2\right]\frac{d\bar{n}_2}{d\bar{x}}$$

$$+\left[3\bar{n}_2\bar{u}_2^2E_2 + \frac{3}{2}\bar{n}_2\bar{\beta}_2E_2 - 3\bar{n}_2\bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]\frac{d\bar{u}_2}{d\bar{x}} + \frac{3}{2}\bar{n}_2\left[\bar{u}_2E_2 - \sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]\frac{d\bar{\beta}_2}{d\bar{x}}$$

$$-\frac{2Re}{3}\,\bar{n}\left\{2\bar{n}\,\bar{u}^2 - 2\bar{B}_2 + \frac{1}{2}\,(\bar{n}_1\bar{\beta}_1E_1 + \bar{n}_2\bar{\beta}_2E_2)\right\} = 0 \qquad \text{(C.3)}$$

y-momentum flux, i = 4

$$\Big[\bar{\beta}_1(\bar{u}_1E_1 + \sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1)\,\frac{d\bar{n}_1}{d\bar{x}} + \bar{n}_1\bar{\beta}_1E_1\,\frac{d\bar{u}_1}{d\bar{x}} + \left[\bar{n}_1\bar{u}_1E_1 + \frac{3}{2}\bar{n}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]\frac{d\bar{\beta}_1}{d\bar{x}}$$

$$+\left[\bar{\beta}_2\left(\bar{u}_2E_2 - \sqrt{\frac{\bar{\beta}_2}{\pi}}X_2\right)\frac{d\bar{n}_2}{d\bar{x}} + \bar{n}_2\bar{\beta}_2E_2\,\frac{d\bar{u}_2}{d\bar{x}} + \left[\bar{n}_2\bar{u}_2E_2 - \frac{3}{2}\bar{n}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]\frac{d\bar{\beta}_2}{d\bar{x}}\right.$$

$$+\frac{2Re}{3}\,\bar{n}\left[2\,\bar{n}\,\bar{u}^2 - 2\bar{B}_2 + \frac{1}{2}\,(\bar{n}_1\bar{\beta}_1E_1 + \bar{n}_2\bar{\beta}_2E_2)\right] = 0 \qquad \text{(C.4)}$$

x-energy flux, i = 5

$$\left[\left(\frac{5}{4}\bar{\beta}_1^2+4\bar{u}_1^2\bar{\beta}_1+\bar{u}_1^4\right)E_1+\sqrt{\frac{\bar{\beta}_1}{\pi}}\left(\bar{u}_1^3+\frac{7}{2}\,\bar{u}_1\bar{\beta}_1\right)X_1\right]\frac{d\bar{n}_1}{d\bar{x}}+$$

$$\left[\bar{n}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,(6\bar{\beta}_1+4\bar{u}_1^2)X_1+4\bar{n}_1(2\bar{u}_1\bar{\beta}_1+\bar{u}_1^3)E_1\right]\frac{d\bar{u}_1}{d\bar{x}}+$$

$$\left[\bar{n}_1\left(\frac{5}{2}\bar{\beta}_1+4\bar{u}_1^2\right)E_1+4\bar{n}_1\bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]\frac{d\bar{\beta}_1}{d\bar{x}}+$$

$$\left[\left(\frac{5}{4}\bar{\beta}_2^2+4\bar{u}_2^2\bar{\beta}_2+\bar{u}_2^4\right)E_2-\sqrt{\frac{\bar{\beta}_2}{\pi}}\left(\bar{u}_2^3+\frac{7}{2}\,\bar{u}_2\bar{\beta}_2\right)X_2\right]\frac{d\bar{n}_2}{d\bar{x}}+$$

$$\left[-\bar{n}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,(6\bar{\beta}_2+4\bar{u}_2^2)X_2+4\bar{n}_2(2\bar{u}_2\bar{\beta}_2+\bar{u}_2^3)E_2\right]\frac{d\bar{u}_2}{d\bar{x}}+$$

$$\left[\bar{n}_2\left(\frac{5}{2}\bar{\beta}_2+4\bar{u}_2^2\right)E_2-4\bar{n}_2\bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\right]\frac{d\bar{\beta}_2}{d\bar{x}}+$$

$$\frac{4}{3}\,\mathrm{Re}\;\bar{n}\left[\,2\bar{B}_3-\bar{u}(\bar{B}_2+\bar{n}_1\bar{\beta}_1E_1+\bar{n}_2\bar{\beta}_2E_2)\right]=0 \tag{C.5}$$

flux of C_x^2 in x direction, i = 6

$$\left[\left(\frac{3}{4}\bar{\beta}_1^2+3\bar{u}_1^2\bar{\beta}_1+\bar{u}_1^4\right)E_1+\sqrt{\frac{\bar{\beta}_1}{\pi}}\,(\bar{u}_1^3+\frac{5}{2}\,\bar{u}_1\bar{\beta}_1)X_1\right]\frac{d\bar{n}_1}{d\bar{x}}+$$

$$\left[\bar{n}_1\bar{u}_1(4\bar{u}_1^2+6\bar{\beta}_1)E_1+4\bar{n}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,(\bar{u}_1^2+\bar{\beta}_1)X_1\right]\frac{d\bar{u}_1}{d\bar{x}}+$$

$$\left[3\bar{n}_1\left(\frac{\bar{\beta}_1}{2}+\bar{u}_1^2\right)E_1+3\bar{n}_1\bar{u}_1\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\right]\frac{d\bar{\beta}_1}{d\bar{x}}+$$

$$\begin{aligned}
&\Big[\Big(\frac{3}{4}\bar{\beta}_2^2+3\bar{u}_2^2\bar{\beta}_2+\bar{u}_2^4\Big)E_2-\sqrt{\frac{\bar{\beta}_2}{\pi}}\,(\bar{u}_2^3+\frac{5}{2}\,\bar{u}_2\bar{\beta}_2)X_2\Big]\frac{d\bar{n}_2}{d\bar{x}}+\\
&\Big[\bar{n}_2\bar{u}_2(4\bar{u}_2^2+6\bar{\beta}_2)E_2-4\bar{n}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,(\bar{u}_2^2+\bar{\beta}_2)X_2\Big]\frac{d\bar{u}_2}{d\bar{x}}+\\
&\Big[3\bar{n}_2\Big(\frac{\bar{\beta}_2}{2}+\bar{u}_2^2\Big)E_2-3\bar{n}_2\bar{u}_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\Big]\frac{d\bar{\beta}_2}{d\bar{x}}+2\mathrm{Re}\,\bar{n}\Big[+2\bar{B}_3\\
&-\bar{u}\Big[\bar{B}_2+\frac{1}{4}(\bar{n}_1\bar{\beta}_1E_1+\bar{n}_2\bar{\beta}_2E_2)\Big]-\frac{3}{4}\bar{n}_1\bar{\beta}_1\Big(\bar{u}_1E_1+\sqrt{\frac{\bar{\beta}_1}{\pi}}\,X_1\Big)\\
&-\frac{3}{4}\bar{n}_2\bar{\beta}_2\Big(\bar{u}_2E_2-\sqrt{\frac{\bar{\beta}_2}{\pi}}\,X_2\Big)\Big]=0 \qquad\qquad (\mathrm{C.6})
\end{aligned}$$

where

$$E_1 = 1. + \mathrm{erf}\left(\bar{u}_1/\sqrt{\bar{\beta}_1}\right)$$

$$E_2 = 1 - \mathrm{erf}\left(\bar{u}_2/\sqrt{\bar{\beta}_2}\right)$$

$$X_1 = e^{-\bar{u}_1^2/\bar{\beta}_1}$$

$$X_2 = e^{-\bar{u}_2^2/\bar{\beta}_2}$$

$$\bar{n} = \frac{1}{2}(\bar{n}_1E_1+\bar{n}_2E_2)$$

$$\bar{u} = \frac{\bar{n}_2\bar{u}_2E_2+\bar{n}_1\bar{u}_1E_1+\left(\bar{n}_1X_1\sqrt{\frac{\bar{\beta}_1}{\pi}}-\bar{n}_2X_2\sqrt{\frac{\bar{\beta}_2}{\pi}}\right)}{\bar{n}_1E_1+\bar{n}_2E_2}\,.$$

The complete computer program is listed on the following pages.

```
C  MAIN PROGRAM--INTEGRATION FROM COLD TO HOT WALL
       IMPLICIT REAL *8 (A-H,O-Z)
       REAL *8 K,L,M
       DIMENSION ALF(6),YO(6),W(6),B(3),K(6,4),Z(6),
       Y(6),DYO(3),DY1(3),YOSTO(6),Y1STO(6),C(3,3),
       L(6),M(6),DYA(3),F(6,6),G(6)
       INTEGER CTR1,CTR2
       CALL $TIME$ (45)
     3 READ (5,10,END = 9999)(ALF(I), I=1,6),(YO(J),
       J = 4,6),RE
    10 FORMAT (6E12.8)
       A = 1.0/DSQRT (3.141592D0)
    15 CTR2 = 0
    20 X = 0.0
       CTR1 = 0
C      X IS THE INDEPENDENT VARIABLE RANGING FROM 0
       TO 1
C  Y(I) I=1 TO 6 REPRESENTS THE VARIABLES N,U, AND BETA
       Y(1) = ALF(1)
       Y(2) = ALF(2)
       Y(3) = ALF(3)
       Y(4) = YO(4)
       Y(5) = YO(5)
```

```
      Y(6) =  YO(6)
C  THE EVAPORATION COEFFICIENT = EVCOEF; THE TOTAL
   ENTHALPY =  TOTH
      CALL PHYSPR (Y,ALF,RHO,U,P,TEMP,TAUXX,
      QX,TOTH,B,EVCOEF,&800)
      WRITE (6,30) RE, (Y(I), I=1,6),(B(I),I=1,3)
   30  FORMAT (1H1,38X,'FLAT PLATE EVAPORATION-
      CONDENSATION PROBLEM'
     C///1H, 'INITIAL CONDITIONS RE = ', E12.4, '
      Y(1) = ', E12.4,
     C'Y(2) = ', E12.4, ' Y(3) = ', E12.4, ' Y(4) = ',
      E12.4  /1H,
     C21X, 'Y(5) = ', E12.4,  '  Y(6) = ',  E12.4,  '
      B(1) = ', E12.4,  '
     CB(2) = ', E12.4,  '  B(3) = ',  E12.4  ///1H,
      'CTR1     X
     C     D        Y(1)',        '        Y(2)
      Y(3)                      Y(4)',      '
     C       Y(5)        Y(6)'/1H,       17X,   'RHO      U
     C  P         ',   'TEMP        TAUXX      QX
      TOTH
     C     EVCOEF' )
   60 CALL DRVTV(Y,W,RE,F,G,&800)
```

```
C  CTR1 MULTIPLIED BY 100 GIVES THE POSITION X.
C  PRINT OUT OCCURS ONLY AT THE SPECIFIED POSITIONS
   GIVEN BY CTR1
      IF   (CTR1  .EQ.   0)   GO TO 215
      IF   (CTR1  .EQ.   1)   GO TO 215
      IF   (CTR1  .EQ.   2)   GO TO 215
      IF   (CTR1  .EQ.   5)   GO TO 215
      IF   (CTR1  .EQ.  10)   GO TO 215
      IF   (CTR1  .EQ.  15)   GO TO 215
      IF   (CTR1  .EQ.  20)   GO TO 215
      IF   (CTR1  .EQ.  25)   GO TO 215
      IF   (CTR1  .EQ.  30)   GO TO 215
      IF   (CTR1  .EQ.  40)   GO TO 215
      IF   (CTR1  .EQ.  50)   GO TO 215
      IF   (CTR1  .EQ.  60)   GO TO 215
      IF   (CTR1  .EQ.  70)   GO TO 215
      IF   (CTR1  .EQ.  75)   GO TO 215
      IF   (CTR1  .EQ.  80)   GO TO 215
      IF   (CTR1  .EQ.  85)   GO TO 215
      IF   (CTR1  .EQ.  90)   GO TO 215
      IF   (CTR1  .EQ.  95)   GO TO 215
      IF   (CTR1  .EQ.  98)   GO TO 215
      IF   (CTR1  .EQ.  99)   GO TO 215
```

```
      IF    (CTR1   .EQ.    100)    GO TO 215
      IF (X  .LT.  1.0)  GO TO 217
  215 CALL PHYSPR (Y,ALF,RHO,U,P,TEMP,TAUXX,
      QX, TOTH,B,EVCOEF,&800)
      WRITE (6,216) CTR1,X,D, (Y(I),I=1,6),RHO,U,P,
      TEMP, TAUXX,QX,TOTH,EV
     CCOEF
  216 FORMAT (1H,  I4,  1P8E13.4/1H,  11X,  1P8E13.4)
      IF (X - 1.0)  217,  5070,  5060
C INTEGRATE Y(J) TO X+D USING 4TH ORDER RUNGE-KUTTA.
  217 D=0.01
C CORRECT LAST STEP FOR COMPUTER ROUND OFF ERRORS.
      IF  ( (X+D)  .LT.  1.0)  GO TO 218
      D=1.0 - X
      X=1.0
      GO TO 220
  218 X=X+D
      CTR1  =  CTR1 + 1
  220 CONTINUE
C     NEXT EVALUATE THE COEFFICIENTS USED IN
      EXPRESSION FOR Y(J)
      DO  230  I=1,6
      K(I, 1) = D * W(I)
```

```
230    Z(I) = Y(I) + 0.50 * K(I,1)
       CALL DRVTV(Z,W,RE,F,G,&800)
       DO 231 I=1,6
       K(I,2) = D * W(I)
231    Z(I) = Y(I) + 0.50 * K(I,2)
       CALL DRVTV(Z,W,RE,F,G,&800)
       DO 232 I=1,6
       K(I,3) = D * W(I)
232    Z(I) = Y(I) + K(I,3)
       CALL DRVTV(Z,W,RE,F,G,&800)
C      NOW INTEGRATE Y(J) TO X+D USING THESE
       COEFFICIENTS
       DO 233 I=1,6
233    Y(I) = Y(I) + (K(I,1) + D * W(I) ) / 6.0D0 +
      (K(I,2) + K(I,3))/ 3.0D0
       IF (Y(3) .GE.0) GO TO 240
       WRITE (6,1)
     1 FORMAT (1HO, 10X, 'SQUARE ROOT OF NEGATIVE
       NUMBER Y(3)')
       WRITE (6.4) ((F(I,J), J=1,6),W(I),G(I),I=1,6)
     4 FORMAT (1X1P 6E 15.7)
       GO TO 800
   240 IF (Y(6) .GE.0) GO TO 245
```

```
        WRITE (6,2)
   2    FORMAT (1HO,10X, 'SQUARE ROOT OF NEGATIVE
        NUMBER Y(6)')
        WRITE (6,9) ((F(I,J), J=1,6),W(I),G(I),I=1,6)
   9    FORMAT (1X1P6E 15.7)
        GO TO 800
 245    CONTINUE
        IF (X-1.0) 60,60,5060
 5060   WRITE (6,5062) X
 5062   FORMAT (1H1, 10X, 25H X GREATER THAN
        1.0, X = ,E16.7)
 5070   WRITE (6,5080) (B(I), I=1,3) , (ALF(J), J=4,6)
 5080   FORMAT (1H, 'B(I), I=1,3 = ' 1P3E13.4, 4X,
        'ALF(J), J=4,6 = ',  1P3E -
        C13.4)

C ITERATION TECHNIQUE: CHANGE YO(4) FIRST,
C LEAVING YO(5) AND YO(6) UNCHANGED; USE THE
C OLD  VALUE OF YO(4), CHANGE YO(5), AND LEAVE
C YO(6) UNTOUCHED; NEXT USE THE OLD VALUES OF
C YO(4) AND YO(5) AND CHANGE YO(6). FINALLY
C CHANGE YO(4), YO(5), AND YO(6) SIMULTANEOUSLY.
C REPEAT PROCESS UNTIL ABS(ALF(I)-Y(I)) < ERROR.
```

```
     DO 6000  J=4,6
     IF ((DABS(ALF(J) - Y(J))) .GT.  0.001) GO TO
     6010
6000 CONTINUE
     GO TO 3
6010 IF (CTR2  .GT.  0)   GO TO 6020
     CTR2  =  CTR2 + 1
     DO 6011  J=4,6
     YOSTO(J) = YO(J)
6011 Y1STO(J)  =  Y(J)
     DYO(1) = (ALF(4) - Y(4)) / 10.0
     DYO(2) = (ALF(5) - Y(5)) / 10.0
     DYO(3) = (ALF(6) - Y(6)) / 10.0
     DYA(1) = ALF(4) - Y(4)
     DYA(2) = ALF(5) - Y(5)
     DYA(3) = ALF(6) - Y(6)
6012 YO(4)   = YOSTO(4) + DYO(1)
     GO TO 20
6020 IF (CTR2  .GT.  1)  GO TO 6030
     CTR2 = CTR2 + 1
     DY1(1) = Y(4) - Y1STO(4)
     DY1(2) = Y(5) - Y1STO(5)
     DY1(3) = Y(6) - Y1STO(6)
     DO 6021  J=1,3
```

```
 6021  C(J,1) = (DY1(J) / DYO(1) )
       YO(4) = YOSTO(4)
 6013  YO(5) = YOSTO(5) + DYO(2)
       GO TO 20
 6030  IF (CTR2  .GT.  2)  GO TO 6040
       CTR2 = CTR2 + 1
       DY1(1) = Y(4) - Y1STO(4)
       DY1(2) = Y(5) - Y1STO(5)
       DY1(3) = Y(6) - Y1STO(6)
       DO 6031  J=1,3
 6031  C(J,2) = (DY1(J) / DYO(2)  )
       YO(5) = YOSTO(5)
 6014  YO(6) = YOSTO(6) + DYO(3)
       GO TO 20
 6040  DY1(1) = Y(4) - Y1STO(4)
       DY1(2) = Y(5) - Y1STO(5)
       DY1(3) = Y(6) - Y1STO(6)
       DO 6041  J=1,3
 6041  C(J,3) = (DY1(J) / DYO (3)  )
C   SUBROUTINE DMINV IS AN IBM MATRIC INVERSION
C   PROGRAM - DOUBLE PRECISION.
       CALL DMINV (C,3,Q,L,M)
       IF  (Q  .EQ.  0.0)   GO TO 6070
```

3-3

```
       YO(4) = YOSTO(4) + C(1,1) * DYA(1) + C(1,2) *
       DYA(2) + C(1,3) * DYA(3)
       YO(5) = YOSTO(5) + C(2,1) * DYA(1) + C(2,2) *
       DYA(2) + C(2,3) * DYA(3)
       YO(6) = YOSTO(6) + C(3,1) * DYA(1) + C(3,2) *
       DYA(2) + C(3,3) * DYA(3)
       GO TO 15
  6070 WRITE  (6,6071)
  6071 FORMAT (1H1, 10X, 29HDETERMINANT OF C
       EQUALS ZERO)
       GO TO 9999
  9999 CALL EXIT
C  THE FOLLOWING INCREMENTAL CHANGES ARE MADE
C  ONLY IF Y(3) OR Y(6) BECOME NEGATIVE.  THIS IS
C  DONE IN ORDER TO RESTART THE PROGRAM AGAIN.
   800 IF (CTR2 .EQ. 0)  GO TO 9999
       IF (CTR2-2) 801,  802,  803
   801 DYO(1) = -DYO(1)/2.
       ITEK = ITEK + 1
       IF (ITEK .EQ. 5)  GO TO 9999
       GO TO 6012
   802 DYO(2) = -DYO(2)/2.
       ITEK = ITEK + 1
```

```
        IF (ITEK. EQ. 5) GO TO 9999
        GO TO 6013
   803  DYO(3) = -DYO(3)/2.
        ITEK = ITEK + 1
        IF (ITEK.  EQ.  5) GO TO 9999
        GO TO 6014
        END
C SUBROUTINE PHYSPR
        SUBROUTINE PHYSPR (Y,ALF,RHO,U,P,TEMP,
        TAUXX,QX,TOTH,B,EVCOEF,*)
C       PURPOSE - TO EVALUATE PHYSICAL PROPER-
        TIES IN THE FLOW FIELD
        IMPLICIT REAL *8 (A-H,O-Z)
        DIMENSION Y(6),ALF(6),B(3)
        A = 1.0 /DSQRT (3.141592D0)
        IF (Y(3).GE.0) GO TO 1000
        WRITE (6,1001)
  1001  FORMAT (1H0, 10X, 'SQUARE ROOT OF A
        NEGATIVE NUMBER Y(3)   SUB P')
        RETURN 1
  1000  IF (Y(6) .GE. 0) GO TO 1002
        WRITE (6,1003)
  1003  FORMAT (1H0, 10X, 'SQUARE ROOT OF A
```

```
     NEGATIVE NUMBER Y(6)  SUB P')
     RETURN 1
1002 R1 = DSQRT (Y(3))
     R2 = DSQRT (Y(6))
     E1 = 1.0 + DERF (Y(2) / R1)
     E2 = 1.0 + DERF (-Y(5) /R2)
     X1 = DEXP (-(Y(2) ** 2)/Y(3))
     X2 = DEXP (-(Y(5) ** 2) /Y(6))
     ARX1 = A * R1 * X1
     ARX2 = A * R2 * X2
     Y2S = Y(2)**2
     Y5S = Y(5)**2
     RHO = 0.50 * (Y(1) * E1 + Y(4) * E2)
     IF (RHO.LT.0.0) RETURN 1
     B(1) = .50 * (Y(1)*(Y(2)*E1+ARX1) +Y(4)*
     (Y(5) * E2-ARX2))
     U = B(1) / RHO
     US = U**2
     UMY2S = (U - Y(2))**2
     UMY5S = (U - Y(5))**2
     UX2MY2 = 2.0 * U - Y(2)
     UX2MY5 = 2.0 * U - Y(5)
     P = (Y(1)/6.0D0) * ((UMY2S + 1.50*Y(3))*
```

```
 E1 - ARX1 * UX2MY2  )
C+ (Y(4)/6.0D0) * ((UMY5S + 1.50*Y(6))*E2 +
 ARX2 * UX2MY5 )
 TEMP = P/RHO
 TAUXX = -(1.0D0/3.0D0)*(Y(1)*(UMY2S*E1 -
 UX2MY2 * ARX1 ) + Y(4) *
C(UMY5S*E2 + UX2MY5 * ARX2 ) )
 QX = .250 * (Y(1)*(-2.50*Y(3) + UMY2S ) *
 (U-Y(2))*E1 + (2.0*Y(3) +
CY2S + 3.0* (US - U*Y(2)) )*ARX1) - Y(4)*
 ((2.50 * Y(6) + UMY5S ) *
C(U-Y(5))*E2 + (2.0*Y(6) + Y5S + 3.0 *(US -
 U*Y(5)) )* ARX2 )  )
 TOTH = 2.50 * TEMP + .50 * US
 SIGMXX = TAUXX - P
 B(2) = RHO*US - SIGMXX
 B(3) = RHO*U * (1.50 * TEMP + .50* US) -
 U * SIGMXX + QX
 EVCOEF = 2.0 * B(1) / (A* (ALF(1) * (DSQRT
 (ALF(3))) - ALF(4) * (D
CSQRT(ALF(6))) ) )
 RETURN
 END
```

```
C  SUBROUTINE DRVTV
       SUBROUTINE DRVTV(Y,W,RE,F,G,*)
       IMPLICIT REAL *8 (A-H,O-Z)
C      PURPOSE
C         OBTAIN VALUES OF DERIVATIVES WJ AT
          GIVEN X KNOWING YJ AND RE
       DIMENSION Y(6),W(6),B(3),G(6),F(6,6)
       A = 1.0/DSQRT(3.141592D0)
       IF (Y(3) .GE. 0)  GO TO 1000
       WRITE (6,1001)
 1001  FORMAT (1H0,10X, 'SQUARE ROOT OF A
       NEGATIVE NUMBER Y(3)   SUB D')
       RETURN 1
 1000  IF (Y (6) .GE. 0)  GO TO 1002
       WRITE (6,1003)
 1003  FORMAT (1H0, 10X, 'SQUARE ROOT OF A
       NEGATIVE NUMBER Y(6)  SUB D')
       RETURN 1
 1002  R1 = DSQRT (Y(3))
       R2 = DSQRT (Y(6))
       E1 = 1.0 + DERF (Y(2)/R1)
       E2 = 1.0 + DERF (-Y(5)/R2)
       X1 = DEXP (- (Y(2) **2) / Y(3))
       X2 = DEXP (-(Y(5) **2) / Y(6) )
```

```
   ARX1 = A * R1 * X1
   ARX2 = A * R2 * X2
   YAXDR1 = Y(1) * A * X1 / R1
   YAXDR2 = Y(4) * A * X2 / R2
   Y1E = Y(1) * E1
   Y4E = Y(4) * E2
   Y2E = Y(2) * E1
   Y5E = Y(5) * E2
   Y12 = Y(1) * Y(2)
   Y45 = Y(4) * Y(5)
   Y2S = Y(2)**2
   Y5S = Y(5)**2
   RHO = .50 * (Y1E + Y4E)
   B(1) = 0.5 * (Y(1) * (Y2E + ARX1) + Y(4) *
   (Y5E - ARX2) )
   U = B(1)/RHO
   USQ = U**2
   B(2) = .50 * (Y(1) * ( ( .50 * Y(3) + Y2S) *
   E1 + Y(2) * ARX1) +
C  Y(4) * ( ( .50 * Y(6) + Y5S ) * E2 - Y(5) * ARX2))
   B(3) = .250 * (Y(1) * ( ( 2.50 * Y(3) + Y2S ) *
   Y2E + ( 2.0 * Y(3)+ Y2S ) *
C  ARX1 ) + Y(4) * ( ( 2.50 * Y(6) + Y5S) * Y5E -
```

```
C ( 2.0 * Y(6) + Y5S) * ARX2 ) )
  F(1,1) = Y2E + ARX1
  F(1,2) = Y1E
  F(1,3) = 0.5 * YAXDR1
  F(1,4) = Y5E - ARX2
  F(1,5) = Y4E
  F(1,6) = -0.5 * YAXDR2
  F(2,1) = (.50 * Y(3) + Y2S) * E1 + Y(2) * ARX1
  F(2,2) = 2.0 * Y(1) * (Y2E + ARX1)
  F(2,3) = .50 * Y1E
  F(2,4) = (.50 * Y(6) + Y5S) * E2 - Y(5) * ARX2
  F(2,5) = 2.0 * Y(4) * (Y5E - ARX2)
  F(2,6) = .50 * Y4E
  F(3,1) = (Y2S + 1.5 * Y(3))*Y2E + ARX1*
  (Y(3) + Y2S)
  F(3,2) = 3.0 * Y(1) * ((Y2S + 0.50 * Y(3)) *
  E1 + Y(2) * ARX1)
  F(3,3) = 1.5 * Y(1) * (Y2E + ARX1)
  F(3,4) = (Y5S + 1.5 * Y(6)) * Y5E - ARX2 *
  (Y(6) + Y5S)
  F(3,5) = 3.0 * Y(4) * ((Y5S + 0.50 * Y(6))*
  E2 - Y(5) *ARX2)
  F(3,6) = 1.5 * Y(4) * (Y5E - ARX2)
```

```
 F(4,1) = Y(3) * (Y2E + ARX1)
 F(4,2) = Y(3) * Y1E
 F(4,3) = Y(1) * (Y2E + 1.5 * ARX1)
 F(4,4) = Y(6) * (Y5E - ARX2)
 F(4,5) = Y(6) * Y4E
 F(4,6) = Y(4) * (Y5E - 1.5 * ARX2)
 F(5,1) = ( (1.25 * Y(3) + 4.0 * Y2S ) *
 Y(3) + Y2S**2 ) * E1
C + Y(2) * ARX1 * (Y2S + 3.50 * Y(3) )
 F(5,2) = Y(1) * ARX1 * ( 6.0 * Y(3) + 4.0 * Y2S)
 + 4.0 * Y1E * Y(2) * (2.0 * Y(3) + Y2S )
 F(5,3) = Y1E * (2.5 * Y(3) + 4.0 * Y2S) +
 4.0 * Y12 * ARX1
 F(5,4) = (Y(6) * (1.25 * Y(6) + 4.0 * Y5S) +
 Y5S**2 ) * E2 - ARX2
C * Y(5) * (Y5S + 3.50 * Y(6) )
 F(5,5) = -Y(4) * ARX2 * (6.0 * Y(6) + 4.0 *
 Y5S) + 4.0 * Y4E
C * Y(5) * (2.0 * Y(6) + Y5S )
 F(5,6) = Y4E * (2.5 * Y(6) + 4.0 * Y5S) - 4.0 *
 Y45 * ARX2
 F(6,1) = (Y(3) * (0.75 * Y(3) + 3.0 * Y2S) +
 Y2S**2 ) * E1 +
```

```
C Y(2) * ARX1 * ( Y2S + 2.5 * Y(3) )
  F(6,2) = Y12 * (4.0*Y2S + 6.0*Y(3) ) * E1 +
  4.0 * Y(1) * ARX1 * (Y2S + Y(3) )
  F(6,3) = 3.0 * Y1E * (0.5*Y(3) + Y2S) +
  3.0 * Y12 * ARX1
  F(6,4) = ( (.750 * Y(6) + 3.0 * Y5S) * Y(6) +
  Y5S**2 ) * E2 - Y(5)
C  * ARX2 * ( Y5S + 2.5 * Y(6) )
  F(6,5) = Y45 * (4.0*Y5S + 6.0 * Y(6) ) * E2 -
  4.0 * Y(4) * ARX2 * (Y5S + Y(6) )
  F(6,6) = 3.0 * Y4E * (0.5 * Y(6) + Y5S) -
  3.0 * Y45 * ARX2
  G(1) = 0.0
  G(2) = 0.0
  G(3) = -(2.0D0/3.0D0) * RHO * RE * (2.0 *
  RHO * USQ - 2.0 *
CB(2) + .50 * (Y1E * Y(3) + Y4E + Y(6) ) )
  G(4) = -G(3)
  G(5) = (4.0D0/3.0D0) * RHO * RE * (2.0 * B(3)
  - U * (B(2) + Y1E
C* Y(3) + Y4E * Y(6) ) )
  G(6) = 2.0 * RHO * RE * (-U * (B(2) + .250 *
  (Y1E * Y(3) + Y4E
```

```
     C * Y(6) ) ) + 2.0 * B(3) - .750 * Y(1) * Y(3) *
      (Y2E + ARX1 ) -
     C.750 * Y(4) * Y(6) * (Y5E - ARX2 ) )
C  SUBROUTINE DSIMQ IS AN IBM PROGRAM WHICH
C .SOLVES A SYSTEM OF SIMULTANEOUS LINEAR
C  EQUATIONS - DOUBLE PRECISION.
   49  CALL DSIMQ(F,G,6,KS)
       DO 50 I=1,6
   50  W(I) = -G(I)
       IF (KS.EQ. 0)  GO TO 51
       WRITE (6,52)
   52  FORMAT(1HO, 10X, 'KS IS ONE SINGULAR
       SOLUTION')
       RETURN 1
   51  CONTINUE
       RETURN
       END
C  SHOOTING-SPLITTING MAIN PROGRAM WHICH
   STARTS AT X=0.
       IMPLICIT REAL *8 (A-H,O-Z)
       REAL *8 K,L,M
       DIMENSION ALF(6),YO(6),W(6),B(3),F(6,6),
       G(6),Y(6),Z(6),K(6,4)
```

```
      INTEGER CTR1
      READ (5,10) (ALF(I),I=1,6),YO(4),YO(6),RE
   10 FORMAT (6E12.8/3E12.8)
    3 READ (5,31, END = 9999) YO(5)
   31 FORMAT (E12.8)
      A = 1.0/DSQRT(3.141592D0)
   20 X=0.0
      CTR1 = 0
      Y(1) = ALF(1)
      Y(2) = ALF(2)
      Y(3) = ALF(3)
      Y(4) = YO(4)
      Y(5) = YO(5)
      Y(6) = YO(6)
      CALL PHYSPR (Y,ALF,RHO,U,P,TEMP,
      TAUXX,QX,TOTH,B,EVCOEF,&800)
      WRITE (6,3) RE, (Y(I),I=1,6),(B(I),I=1,3)
   30 FORMAT (1H1,38X, 'FLAT PLATE EVAPORATION-
      CONDENSATION PROBLEM '
    C///1H, 'INITIAL CONDITIONS   RE = ',
      E12.6, ' Y(1) = ', E12.6,
    C' Y(2) = ', E12.6, ' Y(3) = ', E12.6, '
      Y(4) = ', E12.6 /H ,
```

$u_2(o)$ is the critical parameter in shooting-splitting technique. It is constant near x=0 and drops off sharply at x=1.

```
     C21X, 'Y(5) = ', E12.6,  '  Y(6) = ', E12.6 , '
      B(1) = ', E12.6,
     CB(2) = ', E12.6 , '   B(3) = ', E12.6 ///1H,
      'CTR1    X
     C   D        Y(1)',    '     Y(2)      Y(3)     Y(4)', '
     C       Y(5)         Y(6)'/1H,     17X, 'RHO     U
     C  P     ',   'TEMP     TAUXX      QX      TOTH','
     C      EVCOEF ' )
  60  CALL DRVTV(Y,W,RE,F,G,&800)
      IF (CTR1  .EQ.  10000)   GO TO 800
 215  CALL PHYSPR(Y,ALF,RHO,U,P,TEMP,
      TAUXX,QX,TOTH,B,EVCOEF,&800)
      WRITE (6,216) CTR1,X,D, (Y(I),I=1,6),
      RHO,U,P,TEMP,TAUXX,QX,TOTH,EVCOEF
 216  FORMAT (1H, I4, 1P8E13.4/1H,11X, 1P8E13.4)
      INTEGRATE Y(J) TO X+D USING 4TH ORDER
      R-K AND VARIABLE STEP SIZE
 217  IF (CTR1 .LT.  50)   GO TO 81
      D = 0.001
      GO TO 82
  81  D = 0.0001
  82  X = X+D
      IF (CTR1  .LT.  50)   GO TO 91
```

```
      CTR1 = CTR1+10
      GO TO 220
   91 CTR1 = CTR1 + 1

C     NEXT EVALUATE THE COEFFICIENTS USED IN
      EXPRESSION FOR Y(J)
      DO 230 I=1,6
      K(I,1) = D * W(I)
  230 Z(I) = Y(I) + 0.50 * K(I,1)
      CALL DRVTV(Z,W,RE,F,G,&800)
      DO 231 I=1,6
      K(I,2) = D * W(I)
  231 Z(I) = Y(I) + 0.50 * K(I,2)
      CALL DRVTV(Z,W,RE,F,G,&800)
      DO 232 I=1,6
      K(I,3) = D * W(I)
  232 Z(I) = Y(I) + K(I,3)
      CALL DRVTV(Z,W,RE,F,G,&800)
C     NOW INTEGRATE Y(J) TO X+D USING THESE
      COEFFICIENTS
      DO 233 I=1,6
  233 Y(I) = Y(I) + (K(I,1) + D*W(I) )/6.0D0 + (K(I,2)
      + K(I,3))/3.0D0
```

```
      IF (Y(5) .GE.  0.0)  GO TO 61
      IF (Y(5) .LE.  -1.0)   GO TO 62
      IF (Y(3) .GE.  0)  GO TO 240
      WRITE (6,1)
    1 FORMAT (1H0, 10X, 'SQUARE ROOT OF
      NEGATIVE NUMBER Y(3)')
      GO TO 800
  240 IF (Y(6) .GE.  0)   GO TO 245
      WRITE (6,2)
    2 FORMAT (1H0, 10X, 'SQUARE ROOT OF
      NEGATIVE NUMBER Y(6) ')
      GO TO 800
  245 CONTINUE
      IF (X-1.0) 60,60,800
 9999 CALL EXIT
  800 WRITE (6,69) (B(I),I=1,3)
   69 FORMAT (1H0, 'B(I), I=1,3='  1P3E15.6)
      GO TO 3
   61 WRITE (6,63)
   63 FORMAT (1H0, 10X, 'Y(5) IS GREATER THAN
      OR EQUAL TO ZERO ')
      WRITE (6,65)  (B(I),I=1,3)
   65 FORMAT (1H0, 'B(I), I=1,3 = '  1P3E16.6)
```

```
      GO TO 3
   62 WRITE (6,64)
   64 FORMAT (1H0, 10X, 'Y(5) IS LESS THAN OR
      EQUAL TO MINUS ONE')
      WRITE (6,67)  (B(I), I=1,3)
   67 FORMAT(1H0, 'B(I),I=1,3 = '  1P3E16.6)
      GO TO 3
      END
C  SHOOTING-SPLITTING MAIN PROGRAM  WHICH STARTS
   AT X = NUMBER.
      IMPLICIT REAL *8 (A-H,O-Z)
      REAL *8  K,L,M
      DIMENSION ALF(6),YO(6),W(6),B(3),F(6,6),G(6),Y(6),Z(6),K(6,4)
       INTEGER CTRI
C  ALF(I) = VALUES OF UP CURVE AT X; YO(I) = VALUES OF DOWN CURVE AT X.
    3 READ (5,10,END=9999) (ALF(I),I=1,6),(YO(I),I=1,6),
      CTR1,X,RE
   10 FORMAT (6E12.8/6E12.8/I5,2E12.8)
      A=1.0/DSQRT (3.141592D0)
      DO 99 I=1,6
   99 Y(I) = (ALF(I) + YO(I))/2.0D0
      CALL PHYSPR(Y,ALF,RHO,U,P,TEMP,
      TAUXX,QX,TOTH,B,EVCOEF,&800)
```

```
      WRITE (6,30) RE, (Y(I),I=1,6),(B(I),I=1,3)
   30 FORMAT (1H1, 38X,'FLAT PLATE EVAPORATION
      -CONDENSATION PROBLEM '
     C///1H, 'INITIAL CONDITIONS  RE = ',
      E12.6, '  Y(1) = ', E12.6 ,
     C'  Y(2) = ', E12.6,  '  Y(3) = ', E12.6,  '
      Y(4) = ', E12.6 /1H ,
     C21X, 'Y(5) = ', E12.6,  '  Y(6) = ', E12.6, '
      B(1) = ', E12.6 ,  '
     CB(2) = ', E12.6 ,  '  B(3) = ', E12.6 ///1H,
      'CTR1     X
     C  D    Y(1)',  '    Y(2)    Y(3)     Y(4) ',  '
     C      Y(5)      Y(6)'/1H,  17X,  'RHO    U
     C P        ',  'TEMP    TAUXX    QX   TOTH','
     C       EVCOEF' )
   60 CALL DRVTV(Y,W,RE,F,G,&800)
      IF (CTR1  .EQ.  20000)  GO TO 800
  215 CALL PHYSPR(Y,ALF,RHO,U,P,TEMP,
      TAUXX,QX,TOTH,B,EVCOEF,&800)
      WRITE (6,216)  CTR1,X,D, (Y(I), I=1,6),RHO,
      U,P,TEMP,TAUXX,QX,TOTH,EVCOEF
  216 FORMAT (1H, I5, 1P8E13.4/1H,11X,1P8E13.4)
      INTEGRATE Y(J) TO X+D USING 4TH ORDER
```

```
          R-K AND VARIABLE STEP SIZE
          D = 0.001
          X = X+D
          CTRI = CTRI + 10
C         NEXT EVALUATE THE COEFFICIENTS USED IN
          EXPRESSION FOR Y(J)
          DO 230  I=1,6
          K(I,1) = D * W(I)
 230      Z(I) = Y(I) + 0.50 * K(I,1)
          CALL DRVTV(Z,W,RE,F,G,&800)
          DO 231  I=1,6
          K(I,2) = D * W(I)
 231      Z(I) = Y(I) + 0.50 * K(I,2)
          CALL DRVTV(Z,W,RE,F,G,&800)
          DO 232 I=1,6
          K(I,3) = D * W(I)
```

```
 232  Z(I) = Y(I) + K(I,3)
      CALL DRVTV(Z,W,RE,F,G,&800)
C     NOW INTEGRATE Y(J) TO X+D USING THESE
      COEFFICIENTS
      DO 233  I=1,6
 233  Y(I) = Y(I) + (K(I,1) + D*W(I))/6.0D0 +
      (K(I,2) + K(I,3))/3.0D0
      IF (Y(5)  .GE.  0.0)    GO TO 61
      IF (Y(5)  .LE.  -1.0)    GO TO 62
      IF (Y(3)  .GE.  0)    GO TO 240
      WRITE (6,1)
    1 FORMAT (1H0, 10X, 'SQUARE ROOT OF
      NEGATIVE NUMBER Y(3)')
      GO TO 800
 240  IF (Y(6)  .GE.  0)    GO TO 245
      WRITE (6,2)
    2 FORMAT (1H0, 10X, 'SQUARE ROOT OF
      NEGATIVE NUMBER Y(6) ')
      GO TO 800
 245  CONTINUE
      IF  (X-2.0)  60,60,800
9999  CALL EXIT
 800  WRITE (6,69)  (B(I), I=1,3)
```

```
69 FORMAT (1H0, 'B(I), I=1,3 = ' 1P3E15.6)
   GO TO 3
61 WRITE (6,63)
63 FORMAT (1H0, 10X, 'Y(5) IS GREATER THAN
   OR EQUAL TO ZERO')
   WRITE (6,65) (B(I), I=1,3)
65 FORMAT (1H0, 'B(I), I=1,3 = ' 1P3E16.6)
   GO TO 3
62 WRITE (6,64)
64 FORMAT (1H0, 10X, 'Y(5) IS LESS THAN OR
   EQUAL TO MINUS ONE')
   WRITE (6,67) (B(I),I=1,3)
67 FORMAT (1H0, 'B(I), I=1,3 = ' 1P3E16.6)
   GO TO 3
   END
```

APPENDIX D

A NAVIER-STOKES TYPE FORMULATION OF THE LEES' MOMENT EQUATIONS

The six moment equations to be used are

continuity

$$\frac{n_1}{2}\left(u_1E_1+\sqrt{\frac{\beta_1}{\pi}}\,X_1\right)+\frac{n_2}{2}\left(u_2E_2-\sqrt{\frac{\beta_2}{\pi}}\,X_2\right)=B_1 \tag{33}$$

x-momentum

$$\begin{aligned}\frac{1}{4}\left(n_1\beta_1E_1+n_2\beta_2E_2\right)+\frac{1}{2}\Bigg(n_1u_1^2E_1+n_1u_1X_1\sqrt{\frac{\beta_1}{\pi}}\\ +n_2u_2^2E_2-n_2u_2X_2\sqrt{\frac{\beta_2}{\pi}}\Bigg)=B_2\end{aligned} \tag{34}$$

energy

$$\begin{aligned}\frac{1}{4}\left(n_1u_1^3E_1+n_2u_2^3E_2\right)+\frac{5}{8}\left(n_1u_1\beta_1E_1+n_2u_2\beta_2E_2\right)\\ +\frac{1}{4}\left(n_1u_1^2X_1\sqrt{\frac{\beta_1}{\pi}}-n_2u_2^2X_2\sqrt{\frac{\beta_2}{\pi}}\right)+\frac{1}{4}\Bigg(2\beta_1n_1X_1\sqrt{\frac{\beta_1}{\pi}}\\ -2\beta_2n_2X_2\sqrt{\frac{\beta_2}{\pi}}\Bigg)=B_3\end{aligned} \tag{35}$$

x-momentum flux

$$\frac{d}{dx}\Big[n_1(u_1^3 + \frac{3}{2}\beta_1 u_1)E_1 + n_1\sqrt{\frac{\beta_1}{\pi}}(\beta_1 + u_1^2)X_1$$

$$+ n_2(u_2^3 + \frac{3}{2}\beta_2 u_2)E_2 - n_2\sqrt{\frac{\beta_2}{\pi}}(\beta_2 + u_2^2)X_2\Big]$$

$$+ \frac{2}{3}\frac{p}{\mu}\Big[-2n\,u^2 + 2B_2 - \frac{1}{2}(n_1\beta_1E_1 + n_2\beta_2E_2)\Big] = 0 \qquad (36)$$

x-energy flux

$$\frac{d}{dx}\Big[n_1\Big(\frac{5}{4}\beta_1^2 + 4u_1^2\beta_1 + u_1^4\Big)E_1 + n_1\sqrt{\frac{\beta_1}{\pi}}(u_1^3 + \frac{7}{2}u_1\beta_1)X_1$$

$$+ n_2\Big(\frac{5}{4}\beta_2^2 + 4u_2^2\beta_2 + u_2^4\Big)E_2 - n_2\sqrt{\frac{\beta_2}{\pi}}(u_2^3 + \frac{7}{2}u_2\beta_2)X_2\Big]$$

$$+ \frac{4}{3}\frac{p}{\mu}\Big[2B_3 - u\,B_2 - u(\beta_1n_1E_1 + \beta_2n_2E_2)\Big] = 0 \qquad (38)$$

x-flux of C_x^2

$$\frac{d}{dx}\Big[n_1\Big(\frac{3}{4}\beta_1^2 + 3\beta_1u_1^2 + u_1^4\Big)E_1 + n_1\sqrt{\frac{\beta_1}{\pi}}(u_1^3 + \frac{5}{2}\beta_1u_1)X_1$$

$$+ n_2\Big(\frac{3}{4}\beta_2^2 + 3\beta_2u_2^2 + u_2^4\Big)E_2 - n_2\sqrt{\frac{\beta_2}{\pi}}(u_2^3 + \frac{5}{2}\beta_2u_2)X_2\Big]$$

$$+\ 2\frac{p}{\mu}\Big[\ -u\left(B_2+\frac{1}{4}\left(n_1\beta_1E_1+n_2\beta_2E_2\right)\right)+2B_3$$

$$-\ \frac{3}{4}\left(n_1\beta_1u_1E_1+n_2\beta_2u_2E_2\right)$$

$$-\ \frac{3}{4}\left(\beta_1n_1X_1\sqrt{\frac{\beta_1}{\pi}}\ -\ \beta_2n_2X_2\sqrt{\frac{\beta_2}{\pi}}\right)\Big]=0 \qquad (39)$$

and

$$n\ =\ \frac{1}{2}\left(n_1E_1+n_2E_2\right) \qquad (D.1)$$

$$T\ =\ \frac{1}{Rn}\Big[\ \frac{n_1}{6}\left\{\left(\left(u-u_1\right)^2+\frac{3}{2}\beta_1\right)E_1-\sqrt{\frac{\beta_1}{\pi}}\left(2u-u_1\right)X_1\right\}$$

$$+\ \frac{n_2}{6}\left\{\left(\left(u-u_2\right)^2+\frac{3}{2}\beta_2\right)E_2+\sqrt{\frac{\beta_2}{\pi}}\left(2u-u_2\right)X_2\right\}\Big] \qquad (D.2)$$

where R is the gas constant, $X_1 = e^{-u_1^2/\beta_1}$, $X_2 = e^{-u_2^2/\beta_2}$, $E_1 = 1 + \mathrm{erf}\ u_1/\sqrt{\beta_1}$, and $E_2 = 1 + \mathrm{erf}\ (-u_2/\sqrt{\beta_2})$. These equations can be rewritten in terms of speed ratio $S = \dfrac{u}{\sqrt{2RT}}$. The result is

continuity

$$\frac{n_1u_1}{2}\left(E_1+\frac{1}{\sqrt{\pi}}\ \frac{X_1}{S_1}\right)+\frac{n_2u_2}{2}\left(E_2-\frac{1}{\sqrt{\pi}}\ \frac{X_2}{S_2}\right)=B_1 \qquad (D.3)$$

x-momentum

$$\frac{n_1u_1^2}{4}\left(\frac{E_1}{S_1^2}+2E_1+\frac{2X_1}{S_1\sqrt{\pi}}\right)+\frac{n_2u_2^2}{4}\left(\frac{E_2}{S_2^2}+2E_2-\frac{2X_2}{S_2\sqrt{\pi}}\right)=B_2 \qquad (D.4)$$

energy

$$\begin{aligned}&\frac{n_1 u_1^3}{4}\left(E_1 + \frac{5}{2}\,\frac{E_1}{S_1^2} + \frac{X_1}{S_1\sqrt{\pi}} + \frac{2X_1}{S_1^3\sqrt{\pi}}\right)\\ &+ \frac{n_2 u_2^3}{4}\left(E_2 + \frac{5}{2}\,\frac{E_2}{S_2^2} - \frac{X_2}{S_2\sqrt{\pi}} - \frac{2X_2}{S_2^3\sqrt{\pi}}\right) = B_3 \qquad \text{(D. 5)}\end{aligned}$$

x-momentum flux

$$\begin{aligned}&\frac{d}{dx}\Bigg[n_1 u_1^3\left(\frac{3}{2}\,\frac{E_1}{S_1^2} + E_1 + \frac{X_1}{S_1\sqrt{\pi}} + \frac{X_1}{\sqrt{\pi}\,S_1^3}\right)\\ &+ n_2 u_2^3\left(\frac{3}{2}\,\frac{E_2}{S_2^2} + E_2 - \frac{X_2}{S_2\sqrt{\pi}} - \frac{X_2}{S_2^3\sqrt{\pi}}\right)\Bigg]\\ &+ \frac{2}{3}\,\frac{p}{\mu}\left[-2nu^2 + 2B_2 - \frac{1}{2}\left(\frac{n_1 u_1^2 E_1}{S_1^2} + \frac{n_2 u_2^2 E_2}{S_2^2}\right)\right] = 0 \qquad \text{(D. 6)}\end{aligned}$$

x-energy flux

$$\begin{aligned}&\frac{d}{dx}\Bigg[n_1 u_1^4\left(\frac{5}{4}\,\frac{1}{S_1^4} + \frac{4}{S_1^2} + 1\right)E_1 + n_1 u_1^4\left(1 + \frac{7}{2}\,\frac{1}{S_1^2}\right)\frac{X_1}{S_1\sqrt{\pi}}\\ &+ n_2 u_2^4\left(\frac{5}{4}\,\frac{1}{S_2^4} + \frac{4}{S_2^2} + 1\right)E_2 - n_2 u_2^4\left(1 + \frac{7}{2}\,\frac{1}{S_2^2}\right)\frac{X_2}{S_2\sqrt{\pi}}\Bigg]\\ &+ \frac{4}{3}\,\frac{p}{\mu}\left[2\cdot B_3 - uB_2 - u\left(\frac{u_1^2 n_1 E_1}{S_1^2} + \frac{u_2^2 n_2 E_2}{S_2^2}\right)\right] = 0 \qquad \text{(D. 7)}\end{aligned}$$

x-flux of C_x^2

$$\frac{d}{dx}\Big[n_1 u_1^4 \Big(\frac{3}{4}\frac{1}{S_1^4} + \frac{3}{S_1^2} + 1 \Big) E_1 + n_1 u_1^4 \Big(1 + \frac{5}{2}\frac{1}{S_1^2} \Big) \frac{X_1}{S_1\sqrt{\pi}}$$

$$+ n_2 u_2^4 \Big(\frac{3}{4}\frac{1}{S_2^4} + \frac{3}{S_2^2} + 1 \Big) E_2 - n_2 u_2^4 \Big(1 + \frac{5}{2}\frac{1}{S_2^2} \Big) \frac{X_2}{S_2\sqrt{\pi}} \Big]$$

$$+ \frac{2p}{\rho}\Big[-u \Big(B_2 + \frac{1}{4}\Big(\frac{n_1 E_1 u_1^2}{S_1^2} + \frac{n_2 u_2^2 E_2}{S_2^2} \Big)\Big)$$

$$+ 2B_3 - \frac{3}{4}\Big(\frac{n_1 u_1^3 E_1}{S_1^2} + \frac{n_2 u_2^3 E_2}{S_2^2} \Big)$$

$$- \frac{3}{4}\Big(\frac{n_1 X_1 u_1^3}{S_1^3\sqrt{\pi}} - \frac{n_2 X_2 u_2^3}{S_2^3\sqrt{\pi}} \Big)\Big] = 0 \qquad \text{(D.8)}$$

and

$$T = \frac{1}{nR}\Big[\frac{n_1 u_1^2 E_1}{4S_1^2} - \frac{n_1 u_1^2}{6S_1\sqrt{\pi}} \Big(2\frac{u}{u_1} - 1 \Big) X_1$$

$$+ \frac{n_2 u_2^2 E_2}{4S_2^2} + \frac{n_2 u_2^2}{6S_2\sqrt{\pi}} \Big(2\frac{u}{u_2} - 1 \Big) X_2 \Big]. \qquad \text{(D.9)}$$

To develop a Navier-Stokes type representation from equations (D.3) to (D.9), a procedure analogous to that used by Chapman and

Enskog is followed here. Let

$$n_1 = n(1 + n_1^*)$$

$$n_2 = n(1 + n_2^*)$$

$$S_1 = S(1 + S_1^*)$$

$$S_2 = S(1 + S_2^*)$$

$$u_1 = u(1 + U_1^*)$$

$$u_2 = u(1 + U_2^*)$$

where the differences

$$n_1 - n_2 = n(n_1^* - n_2^*)$$

$$u_1 - u_2 = u(U_1^* - U_2^*)$$

$$S_1 - S_2 = S(S_1^* - S_2^*)$$

express a small deviation from equilibrium. By placing these expansions into equations (D. 1) and (D. 3) to (D. 9), there results

continuity

$$nu = B_1 \tag{D. 10}$$

x-momentum

$$nu^2\left(1 + \frac{1}{2S^2} + \frac{1}{3}\,\frac{e^{-S^2}}{S\sqrt{\pi}}\left[(2S^2 + 1)(S_1^* - S_2^*) - (n_1^* - n_2^*)\right]\right) = B_2 \tag{D. 11}$$

energy

$$\frac{nu^3}{4}\left(2 + \frac{5}{S^2} - (n_1^* - n_2^*)\left(\frac{1}{3} + \frac{1}{2S^2}\right)\frac{e^{-S^2}}{S\sqrt{\pi}} + \frac{7}{2}(U_1^* - U_2^*)\frac{e^{-S^2}}{S^3\sqrt{\pi}}\right.$$

$$\left. + (S_1^* - S_2^*)\frac{e^{-S^2}}{S\sqrt{\pi}}\left(\frac{2}{3}S^2 + \frac{4}{3} - \frac{7}{2S^2}\right)\right) = B_3 \qquad \text{(D. 12)}$$

x-momentum flux

$$\frac{d}{dx}\left(2 + \frac{3}{S^2}\right)nu^3 = -\frac{2}{3}\frac{p}{\mu}\, nu^2\, \frac{e^{-S^2}}{S\sqrt{\pi}}\left[(2S^2+1)(S_1^* - S_2^*) - n_1^* - n_1^*)\right] \qquad \text{(D. 13)}$$

x-energy flux

$$\frac{d}{dx}\left(2 + \frac{8}{S^2} + \frac{5}{2S^4}\right)nu^4 = -\frac{4}{3}\frac{p}{\mu}\, nu^3\left[\left(S^2 + 1 - \frac{7}{4S^2}\right)(S_1^* - S_2^*)\right.$$

$$\left. - \frac{1}{2}\left(1 + \frac{1}{2S^2}\right)(n_1^* - n_2^*) + \frac{7}{4S^2}(U_1^* - U_2^*)\right]\frac{e^{-S^2}}{S\sqrt{\pi}} \qquad \text{(D. 14)}$$

x-flux of C_x^2

$$\frac{d}{dx}\left(2 + \frac{6}{S^2} + \frac{3}{2S^4}\right)nu^4 = -2\,\frac{p}{\mu}\, nu^3 \cdot \frac{e^{-S^2}}{S\sqrt{\pi}} \cdot$$

$$\left[\left(S^2 + 1 - \frac{1}{4S^2}\right)(S_1^* - S_2^*) - \frac{1}{2}\left(1 + \frac{1}{2S^2}\right)(n_1^* - n_2^*) + \frac{1}{4}\frac{(U_1^* - U_2^*)}{S^2}\right] \qquad \text{(D. 15)}$$

where the combinations

density

$$n_1^* + n_2^* + (n_1^* - n_2^*)\,\mathrm{erf}S + \frac{2S}{\sqrt{\pi}} e^{-S^2} (S_1^* - S_2^*) = 0$$

continuity

$$(U_1^* + U_2^*) + (U_1^* - U_2^*)\,\mathrm{erf}S + (n_1^* - n_2^*) \frac{e^{-S^2}}{S\sqrt{\pi}}$$

$$+ (U_1^* - U_2^*) \frac{e^{-S^2}}{S\sqrt{\pi}} - \frac{2Se^{-S^2}}{\sqrt{\pi}} (S_1^* - S_2^*) - \frac{e^{-S^2}}{S\sqrt{\pi}} (S_1^* - S_2^*) = 0$$

temperature

$$-(S_1^* + S_2^*) - (S_1^* - S_2^*)\,\mathrm{erf}S + \frac{2Se^{-S^2}}{\sqrt{\pi}} (S_1^* - S_2^*) + \frac{e^{-S^2}}{S\sqrt{\pi}} (S_1^* - S_2^*)$$

$$-(n_1^* - n_2^*) \frac{e^{-S^2}}{S\sqrt{\pi}} - (U_1^* - U_2^*) \frac{e^{-S^2}}{S\sqrt{\pi}} - \frac{Se^{-S^2}}{3\sqrt{\pi}} (n_1^* - n_2^*)$$

$$+ \frac{S}{3} \frac{e^{-S^2}}{\sqrt{\pi}} (S_1^* - S_2^*) + \frac{2S^3 e^{-S^2}}{3\sqrt{\pi}} (S_1^* - S_2^*) = 0$$

were used to simplify the above equations. Hence, in combining equations (D.11) and (D.13), the x-momentum equation becomes

$$n u^2 \left(1 + \frac{1}{2S^2} - \frac{1}{2} \frac{\mu}{uP} \frac{d}{dx} \left(2 + \frac{3}{S^2} \right) u^2 \right) = B_2 . \qquad \text{(D.16)}$$

A similar expression can be obtained for the energy equation, but it is less obvious. Define

$$X = \frac{d}{dx}\left(2 + \frac{3}{S^2}\right)u^2$$

$$Y = \frac{d}{dx}\left(2 + \frac{8}{S^2} + \frac{5}{2S^4}\right)u^3$$

$$Z = \frac{d}{dx}\left(2 + \frac{6}{S^2} + \frac{3}{2S^4}\right)u^3 .$$

The differential equations (D. 13) to (D. 15) simplify to

$$X = -\frac{2}{3}Q(S)\,[\,(2S^2 + 1)\Delta S - \Delta N]$$

$$Y = -\frac{4}{3}Q(S)\,u\left[\left(S^2 + 1 - \frac{7}{4S^2}\right)\Delta S - \frac{1}{2}\left(1 + \frac{1}{2S^2}\right)\Delta N + \frac{7}{4}\,\frac{\Delta u}{S^2}\right]$$

$$Z = -2Q(S)\,u\left[\left(S^2 + 1 - \frac{1}{4S^2}\right)\Delta S - \frac{1}{2}\left(1 + \frac{1}{2S^2}\right)\Delta N + \frac{\Delta u}{4S^2}\right]$$

where $Q(S) = \frac{p}{\mu} \cdot u \cdot \frac{e^{-S^2}}{S\sqrt{\pi}}$, $\Delta S = S_1^* - S_2^*$, $\Delta N = n_1^* - n_2^*$, and $\Delta u = U_1^* - U_2^*$. These three equations are solved for ΔS, ΔN, and Δu

in terms of X, Y, and Z. The solution is

$$\Delta u = -\frac{S^2 Y}{Q\,u} + \frac{8}{3}\,\frac{S^2 Z}{Q\,u} - 3\left(S^2 + \frac{1}{2}\right)\frac{X}{Q}$$

$$\Delta S = -\frac{S^2 Y}{2Qu} + \frac{7}{3}\,\frac{ZS^2}{Qu} - 3\left(S^2 + \frac{1}{2}\right)\frac{X}{Q}$$

$$\Delta N = -\frac{S^2(2S^2+1)Y}{2\,Q\,u} + \frac{7}{3}\,\frac{(2S^2+1)S^2 Z}{Qu} - \frac{6XS^2(S^2+1)}{Q} .$$

By placing these results into the energy equation (D.12), we have

$$\frac{nu^3}{4}\left[2 + \frac{5}{S^2} + \frac{\mu}{p\,u}\left[-\frac{3}{2}\,\frac{1}{u}\,\frac{d}{dx}\left(2 + \frac{8}{S^2} + \frac{5}{2S^4}\right)u^3 + \frac{d}{dx}\left(2 + \frac{3}{S^2}\right)u^2\right]\right) = B_3. \qquad \text{(D. 17)}$$

Equations (D. 16) and (D. 17) do not reduce to the one dimensional viscous equations even though they are similar in form. $d/dx\ (2 + 3/S^2)u^2$ is related to a stress term and $d/dx\ (2 + 8/S^2 + 5/2S^4)u^3$ can be thought of as a conduction type term. If the d/dx [ ] terms are zero, then (D. 10), (D. 16) and (D. 17) reduce to the inviscid equations

$$nu = B_1$$

$$n u^2\left(1 + \frac{1}{2S^2}\right) = B_2$$

$$\frac{nu^3}{4}\left(2 + \frac{5}{S^2}\right) = B_3$$

or

$$\rho \cdot u^2 + P = B_2'$$

$$\rho u \cdot \left(\frac{u^2}{2} + \frac{5}{2} RT\right) = B_3'$$

$$\rho u = B_1' \ .$$

The stress term in equations (D.16) and (D.17) is not invariant under a Galilean transformation as is the stress term in the one-dimensional viscous equations. This not surprising since Lees' two sided Maxwellian is referenced to a particular coordinate system.