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**ANALYSIS OF OIL-LUBRICATED,  
FLUID-FILM, THRUST BEARINGS  
WITH ALLOWANCE FOR  
TEMPERATURE DEPENDENT VISCOSITY**

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**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MAY 1972**



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| 16. Abstract<br><p>A preliminary design study was performed to seek a fluid-film thrust bearing design intended to be part of a high-speed, hybrid (rolling element/fluid film) bearing configuration. The base line used is a design previously tested. To improve the accuracy of theoretical predictions of load capacity, flow rate, and friction power loss, an analytical procedure was developed to include curvature effects inherent in thrust bearings and to allow for the temperature rise in the fluid due to viscous heating. Also, a "narrow-groove" approximation in the treatment of the temperature field was formulated to apply the procedure to the Whipple thrust bearing. A comparative trade-off study was carried out assuming isothermal films; its results showed the shrouded-step design to be superior to the Whipple design for the intended application. An extensive parametric study was performed, employing isoviscous calculations, to determine the optimized design, which was subsequently recalculated allowing for temperature effects.</p> |  |                                                             |                                                               |                                                                   |  |
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1. Bearings  
2. Thin films  
3. Pol materials



## FOREWORD

Much of the conceptual foundation of this work was laid in 1969, during which Dr. V. N. Constantinescu was a Visiting Scientist at MTI and related topics were among the popular subjects of discussion. Dr. A. J. Smalley provided the results on the spiral-grooved bearing. Mr. Leo Winn (MTI Program Manager) provided guidance on the design objectives.



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## SUMMARY

This report contains the results of the analytical design study accomplished under NASA Contract No. NAS3-14399. The main objective of this study was to determine the optimum fluid-film thrust bearing design intended to be part of the high-speed, hybrid boost bearing used for jet engine thrust bearing applications.

To improve the accuracy of the analytical predictions, definite improvements were made to the "state of the art" of thrust bearing analyses; accounting for curvature effects, temperature rise in the fluid due to viscous heating, and sudden transition between laminar and turbulent flows, this theory (incompressible, thin-film flow with temperature effects) was extended to the spiral-grooved (Whipple) geometry.

Using these analyses, a comparative trade-off study was made of the shrouded-step and Whipple thrust bearings. It turned out that the shrouded-step bearing design proved more favorable in the present application. During this phase of the study it became apparent that temperature rise in a fluid-film due to frictional dissipation is predominant in bearings with a low operating temperature, small film thickness, and high sliding speed. The principal effects are to increase flow, reduce friction power loss, and reduce load capacity.

An extensive parametric study of the shrouded-step configuration, using iso-viscous calculations, was made. The performance characteristics of this bearing are presented. Subsequent recalculations, allowing for temperature effects, were made on this design and are included herein.

A comparison was also made between the present analyses and actual test data. This comparison indicates excellent agreement on load-carrying capacity at low viscosity (high temperature) over a wide speed range. Using high viscosities (low temperature operation) the agreement is good for speeds up to 10,000 rpm. At high speeds and high viscosity conditions, test data indicates considerably lower load-carrying capacity than that predicted by the analysis.

## INTRODUCTION

Using a fluid-film bearing and a rolling-element bearing in parallel in a hybrid configuration provides the benefit of long-life at high-speed operation (taken for granted with fluid-film bearings) together with low-friction starting under load (characteristic to rolling elements). Rolling-element thrust bearings are in common use in many propulsion and auxilliary devices. Advance concepts continue to press for higher specific impulse and/or power density, such that in the context of bearing requirements, the combination of DN and thrust load is altogether beyond available experience and reasonable technological projection to assure reliable, long-life operations.

With this hybrid concept, a fluid-film shares the total thrust load more with speed increase. In this manner, the rolling-elements are required to operate either under high load at low DN, or at high DN in an essentially lightly loaded condition. Thus the danger of fatigue-failure is substantially reduced. Clearly, the benefits of the hybrid concept can be realized at the expense of higher oil flow and power loss of the fluid-film component. Consequently, one desires to seek the optimum design for each application.

In the preceding phase of the present program, [1]\* [2], an experimental hybrid bearing was designed, built, and tested. The validity of the hybrid approach was verified. However, the requirements of flow and power were appreciable, and the measured data showed that the fluid-film load at low oil temperatures was considerably lower than that predicted by analysis. Since the same analytical procedure was used to "optimize" the design, it was felt desirable to first refine the analytical procedure and then to revise the design with the objective of reducing flow and power to a more reasonable level.

It was recognized that the shortcomings of the original analytical approach consist of 1) quasi-one-dimensional approximations in solving the Reynolds equation; 2) inability to account for the sudden transition between laminar and turbulent flows; and 3) the assumption of a constant lubricant viscosity [3]. The question of sudden laminar-to-turbulent transition was recently considered [4].

By itself, its effects were not large enough to account for the observed discrepancies. For the desired computation, an existing analytical procedure for calculating tilting-shoe thrust bearings with pad distortions [5] is a logical starting point, to which additional refinements are to be added;

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\* Bracketed numbers refer to identically numbered references at end of report.

in this manner considerable established algorithms can be used to advantage. The main improvement sought is the simultaneous consideration of conduction and convection heat transfer effects in the lubricant film.

The configuration of the fluid-film bearing thus far considered is the shrouded-step. An alternate configuration is the Whipple, or spiral-grooved thrust bearing. The latter has been popularly accepted in a large variety of gas lubricated systems. For the present work, these two configurations, augmented by external pressurization, will be separately optimized. Then the optimized designs will be compared and the temperature effects on the better design will be determined.

## DESCRIPTION OF ANALYTICAL APPROACH

To form the basis of the desired design calculations, the problem of viscous heating in the lubricant film and the attendant heat transfer processes was analyzed. The corresponding mathematical derivations are given in Appendix A. The temperature problem of the fluid film of a spiral-grooved bearing is complicated by the groove-ridge gap fluctuations which affect both heat dissipation and heat conduction, and by the circulation pattern related to the inclined grooving geometry which governs heat convection. A "narrow groove" theory, which is consistent with Whipple's solution of the pressure problem, was derived to provide the framework of a practicable computation procedure; this is described in Appendix B. Highlights of these analyses and the manner in which they have been incorporated in the computational procedure will be outlined in the following paragraphs.

Analysis of the thrust bearing fluid film, with allowance for viscous heating and the subsequent viscosity change, has been performed previously for the tilting shoe configuration assuming laminar flow [5]. The iterative algorithm employed by the previous analysis to reconcile pressure-flow and temperature is an essential "prior art". Added features required by the present work include:

1. Capability to treat pockets and pressurized feeding;
2. Allowance for turbulence and transitional flow phenomena;
3. Consideration of both convective and conductive heat transfer modes;
4. Adaptation of these features for the spiral-grooved thrust bearings.

The first two items involve coding of established analytical procedure; the latter two are new analyses to be carried out.

In the thin-film heat transfer analysis, the selected strategy involves reducing the three-dimensional energy equation to a two-dimensional problem by integrating across the film thickness and approximating by their corresponding mean values the convective fluxes and viscosity (which also determines eddy diffusivity in the turbulent case). A crucial aspect in this scheme concerns the apparent temperature for the values of viscosity appearing in the dissipation and the conduction terms. The same question also arises in the Reynolds equation.

Thus the key issue is first to construct Reynolds number sensitive functions which govern:

1. Velocity profiles,
2. Nusselt number (heat transfer coefficient), and
3. Dissipation density;

and then to provide criteria to determine the temperature value at which the mean viscosity appearing with various functions is to be calculated. A "Model Couette Flow" is analyzed "exactly" with the dissipation reduced by a numerical factor for simulating convective cooling. While there are three such terms in the system of governing equations (namely, the coefficient of friction, the Nusselt number, and the dissipation function) mathematical similarity causes the effective temperatures for mean viscosity of the coefficient of friction and the dissipation function to be equal. The two independent effective temperatures\* are parametrically dependent on the wall temperature, the Reynolds number, the Prandtl number, and a "heating parameter". Given a particular oil and an operating wall temperature, there are still two parameters. It turned out, that if the effective temperature is expressed in terms of an interpolation coefficient between the wall and mean film temperatures, the numerical values of these interpolation coefficients are primarily single parameter functions dependent on the Reynolds number, in which the viscosity is based on the friction-temperature. Empirical formulae for these interpolation coefficients were established for MIL-L-7808 oil at wall temperatures of 200 F (93.4 C) and 300 F (149 C). With a simplified allowance for the contribution of pressure gradients to dissipation at the walls, the friction-temperature and conduction-temperature calculated from the "Model Couette Flow", are used directly. Universal functions for heat transfer and dissipation are also assumed to be the same as those of the "Model Couette Flow".

In the treatment of transition from laminar to turbulent flows, a transitional Reynolds number and a fully-turbulent Reynolds number are postulated. Prevailing experimental evidence [4] indicated that the latter is twice the value of the former.

Generalization of the linearized turbulent lubrication theory [6] (LTLT) to include transitional effects is accomplished by defining the distinction between the actual and apparent sliding velocities, the latter being used to calculate the apparent Reynolds number in LTLT. For an actual Reynolds number less than the transitional Reynolds number, the apparent sliding velocity is set to zero, while for an actual Reynolds number in excess of the fully-turbulent Reynolds number, the apparent sliding velocity is equated to the actual sliding velocity. In the transitional range, an elliptical interpolation scheme is used to render an abrupt departure from the laminar condition and a smooth blending into the fully turbulent condition. The

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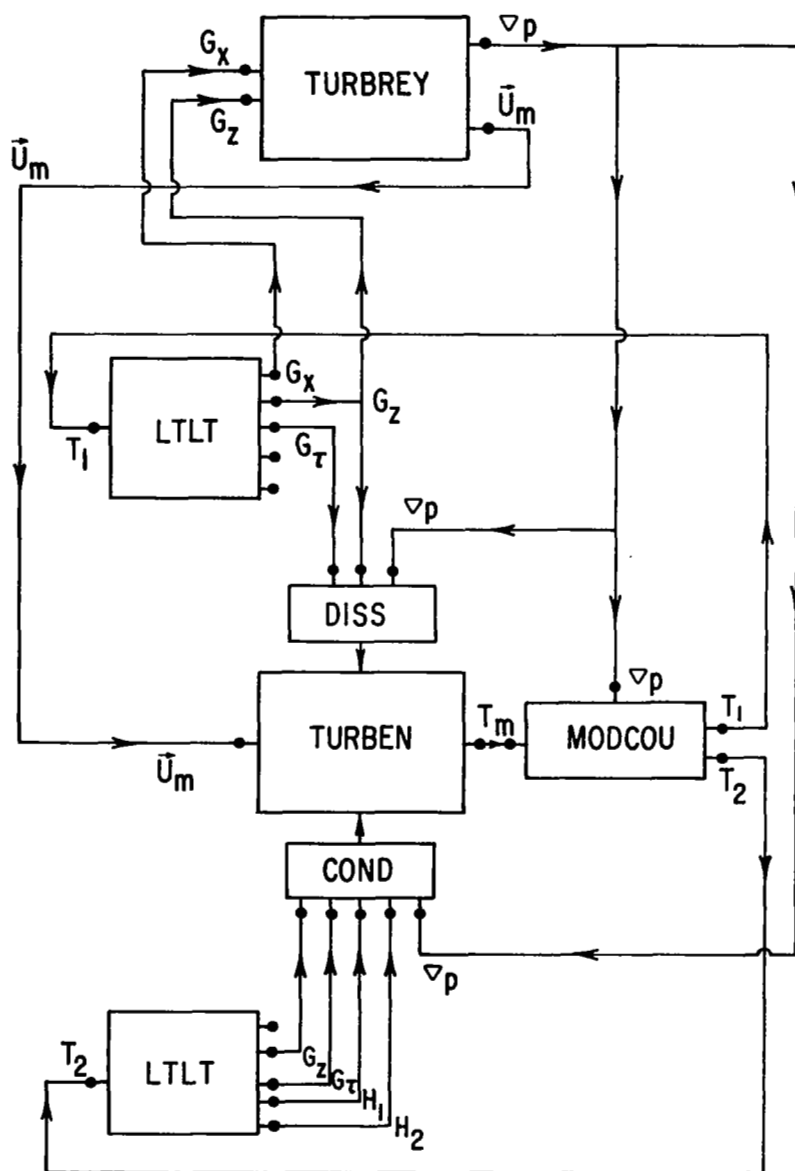
\* For convenience of reference, these two effective temperatures will be given the abbreviated names of friction-temperature and conduction-temperature, respectively.

apparent sliding velocity is used to calculate the Reynolds number, which determines the interpolation coefficients for the friction-temperature and conduction-temperature as well as those governing the universal functions for heat transfer and dissipation. Thus, in the laminar range, the apparent Reynolds numbers for friction and dissipation on one side, and that for heat transfer on the other side, reduce to zero regardless of the fact that they are based on different viscosities; and the temperature interpolation coefficients assume the same asymptotic value of unity. With fluid film heating, the question arises as to what temperature should be used to calculate the transitional and fully turbulent Reynolds numbers. There are not enough experimental data to deal with the transition problem with temperature sensitive viscous fluids (aerodynamicists have dealt with temperature influence on density rather than viscosity) therefore, the tentative plan is to use the friction-temperature for the calculation of both the transitional and the fully-turbulent Reynolds numbers.

The LTLT is now extended to treat fluid-film heating including transitional effects. Two additional universal functions are introduced in addition to the three already needed in the isoviscous problem. The latter being related to the computation of wall friction, mean flow along sliding, and mean flow in the direction normal to sliding. The Reynolds numbers to be used for these universal functions are based on the apparent sliding velocity and mean viscosities, which are respectively determined by the analysis of friction and conduction problems of the "Model Couette Flow". The block diagram on the following page illustrates the scheme involved.

To apply the above method to calculate pressure and temperature fields of the spiral-grooved thrust bearing simultaneously, it is desirable to calculate the gross features accurately, while ignoring local details as much as possible. This point of view was inherent in the "classical" theory of Whipple, a prominent feature of which is the depiction of cross-groove pressure profiles by linear variations. For the present problem, because of viscosity variations, the cross-groove pressure profile is no longer linear, instead it must satisfy a narrow-groove ordinary differential equation. Other elements in the Whipple theory (namely, congruency of pressure field, invariance of normal flux, and continuity of radial flux) are retained. The energy equation requires a different treatment because of the convective effect. In fact, the integration must proceed along a flux line with a suitable inlet condition. Simplification attributable to the narrow groove geometry enters through the condition that the normal flux is invariant. If the overall radial flux of a groove-ridge pair is not zero, the convective path would undergo a radial shift across each groove-ridge pair. This distance, together with the temperature increment along the flux path over a groove-ridge pair, determines the coarse scale temperature gradient. Thus it is not necessary to integrate the energy equation along the same flux path beyond one groove-ridge pair. If the overall radial flow vanishes, then the flux-path returns to the same radius for each groove-ridge pair and the congruency condition of temperature overrides; i.e., the initial condition must be abandoned. In other words, in the absence

| Name    | Function                                                                                    |
|---------|---------------------------------------------------------------------------------------------|
| LTLT    | Generation of Reynolds Number Dependent Universal Functions, Including Transitional Effects |
| MODCOU  | Calculation Friction and Conduction Temperatures                                            |
| TURBREY | Solution of the Turbulent Reynolds Equation                                                 |
| TURBEN  | Solution of the Turbulent Energy Equation                                                   |
| DISS    | Computation of Dissipation                                                                  |
| COND    | Computation of Heat Transfer Coefficient                                                    |



of gross convective flux, the temperature field does not communicate with the outside. Dependence of pressure profile on the temperature field and the need to integrate the energy equation along the convective path constitute the coupling between the Reynolds and energy equations. Except for details in handling the energy equation and the coarse scale extrapolation of both pressure and temperature fields, the scheme illustrated by the previous block diagram remains applicable.

## DISCUSSION OF RESULTS

Before attempting to seek the optimum design, an assessment of the significance of temperature sensitivity is made. This knowledge is practically very useful, since simultaneous calculation of Reynolds and energy is very time consuming and - hopefully - will not be needed all the time during preliminary tradeoff studies. A shrouded-step design - similar to that analyzed previously [1] - is the basis for this evaluation. Principal dimensions of this bearing are shown at the top of Table I. The combination of a small film thickness (2 mils) and low wall temperature is chosen to represent realistic conditions at which temperature sensitivity should be quite severe. A small film thickness restricts the flow, therefore rendering convective cooling ineffective. A low wall temperature keeps viscosity and also its temperature sensitivity high. The lubricant is assumed to be MIL-L-7808 oil in this and all subsequent calculations, in which temperature sensitivity is considered, except where otherwise specified. The viscosity-temperature relation is shown in Appendix A. Calculated load, flow and friction power are compared in Table I according to the method employed in previous work [1, 2, 3] and by the new method with and without temperature sensitivity. In the isoviscous calculation, viscosity is that at the wall temperature (100 F (37.8 C) in this case).

Viscosity reduction - due to heating at 100 F (37.8 C) oil inlet temperature - causes flow to increase substantially, as much as 43% at 20,000 rpm. There is also a slight reduction in load and a somewhat larger reduction in power. The qualitative effects of speed are fairly truthfully depicted by the isoviscous calculations. The results for a wall temperature of 300 F (149 C) at 20,000 rpm indicate similar behavior for load and power; however, viscosity reduction is not apparent here and the flow does not show the large disparity exhibited in the 100 F (37.8 C) case.

Further assessment of these methods of calculation can be made by comparing their results with the experimental data from [2]. The shrouded-step bearing design used in [2] was smaller in size (OD = 5 in., ID = 3.65 in.) and a MIL-L-23699 lubricating oil was employed. Complete description of the bearing and oil are given in [2]. Load-speed relations were calculated with the measured film thickness at each speed for two oil inlet temperatures (assuming a constant viscosity for each case) and are compared with experimental data graphically in Fig. 1. The good agreement of the present theory with the experimental data for the high temperature run, particularly in view of the considerable disparity of the corresponding curve calculated by the previous method, is certainly gratifying. The comparison for the low temperature run is less convincing. Although the new calculation is always in better agreement with the experimental data, the degree of divergence at speeds above 10,000 rpm is very large indeed. It is somewhat surprising to note that the disparity between the results obtained by the method of Ref. 3 and those calculated with the new method assuming an isoviscous lubricant - is much larger than that accountable by temperature effects. One must surmise that the practically important advance of the new method is predominantly with respect to the nonthermal features; namely, variation of the sliding speed with radius, unsymmetrical distribution of radial leakage flows, and

transitional turbulence.

Additional comparisons are given in Table II. The calculated flows are generally larger than those measured. However, the amount of discrepancy is mostly within the effects accountable by uncertainties in the film thickness measurement (at least for the high temperature run). The comparison on power loss is more erratic. For the high-temperature run, the measured power loss is higher at higher speeds even though it is in reasonable agreement with the calculations by either method at low speeds. A likely explanation is the additional rotor torque caused by the churning of oil excess outside the bearing film. For the low temperature run, the measured flow is uniformly lower. The effect of film thickness uncertainty is checked for the high-temperature run at 20,000 rpm using a smaller gap judged to be representative of maximum uncertainty in the displacement gaging system. The corresponding change in load is more than needed to account for the spread between measurement and the present calculation. It also shows the correct trend for the flow but does not improve the comparison in power loss. Lubricant heating effects are checked by allowing for thermal effects in the lubricating film for the low-temperature case at 20,000 rpm. Although it reflects the correct trends in load and power loss, the actual amounts are relatively small, while the flow rate becomes almost twice the measured amount. One possible explanation to account for the disparities in load and flow for the low temperature run is the possibility of thermal crowning of the bearing surfaces. Insufficient bearing feed flow could be another.

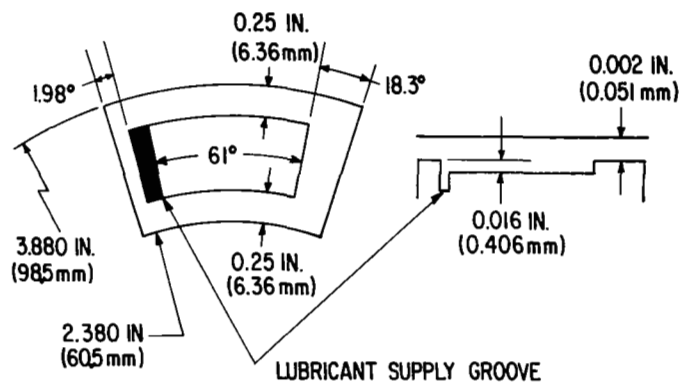
These observations led to the decision to do most of the "searches" and preliminary comparisons with isoviscous calculations. The final calculations, however, will always be made with temperature effects to make sure that the desired load capacity can be realized with power and flow within the assigned bounds.

A comparison was made between the shrouded-step and spiral-grooved configurations. Constraints fixed for both configurations in this comparison are as follows:

|                 |                                                       |
|-----------------|-------------------------------------------------------|
| O.D.            | 6.5" (165 mm)                                         |
| I.D.            | 5.0" (127 mm)                                         |
| Gap             | 1.5 mils (0.038 mm)                                   |
| RPM             | 10,000                                                |
| Supply Pressure | 193 psig ( $1.33 \times 10^6$ newton/m <sup>2</sup> ) |
| Viscosity;      | $0.5 \times 10^{-6}$ lb-sec/in <sup>2</sup> (3.45 cp) |
| MIL-L-7808,     |                                                       |
| 200 F (93.4C)   |                                                       |

Other dimensions of the shrouded-step bearing are shown in Fig. 2a, they were chosen on the basis of previous maximum load designs.

TABLE I. COMPARISON OF CALCULATED LOAD, FLOW AND FRICTION POWER



LUBRICANT MIL-L-7808 OIL

| CALCULATION METHOD |        | ISOVISCIOUS          |                                              |             | TEMPERATURE-DEPENDENT |                                              |             | METHOD OF REF. 3     |                                              |             |
|--------------------|--------|----------------------|----------------------------------------------|-------------|-----------------------|----------------------------------------------|-------------|----------------------|----------------------------------------------|-------------|
| Wall Temperature   | RPM    | Load<br>lb. (newton) | Flow<br>gpm ( $10^{-5}$ m <sup>3</sup> /sec) | Power<br>hp | Load<br>lb. (newton)  | Flow<br>gpm ( $10^{-5}$ m <sup>3</sup> /sec) | Power<br>hp | Load<br>lb. (newton) | Flow<br>gpm ( $10^{-5}$ m <sup>3</sup> /sec) | Power<br>hp |
| 100 F<br>(37.7 C)  | 5,000  | 3,180 (14,150)       | 2.52 (15.9)                                  | 8.37        | 3,057 (13,600)        | 2.97 (18.7)                                  | 7.32        | 2,954 (13,130)       | 1.54 ( 9.66)                                 | 8.96        |
|                    | 10,000 | 6,543 (29,100)       | 4.84 (30.5)                                  | 39.3        | 5,989 (26,600)        | 6.63 (41.8)                                  | 31.0        | 5,745 (25,500)       | 2.76 (17.3 )                                 | 40.3        |
|                    | 15,000 | 10,801 (48,000)      | 7.77 (49.0)                                  | 97.6        | 9,586 (42,600)        | 11.3 (71.2)                                  | 80.1        | 13,094 (58,100)      | 5.94 (37.3 )                                 | 110.0       |
|                    | 20,000 | 15,700 (69,600)      | 11.10 (70.0)                                 | 188.8       | 14,208 (63,300)       | 15.9 (100.0)                                 | 179.6       | 24,992 (111,000)     | 11.1 (69.0 )                                 | 227.0       |
| 300 F<br>(149 C)   | 20,000 | 8,123 (36,100)       | 21.0 (132.0)                                 | 105.1       | 7,862 (35,000)        | 20.7 (130.0)                                 | 101.1       | 12,495 (55,500)      | 21.0 (132.0 )                                | 123.2       |

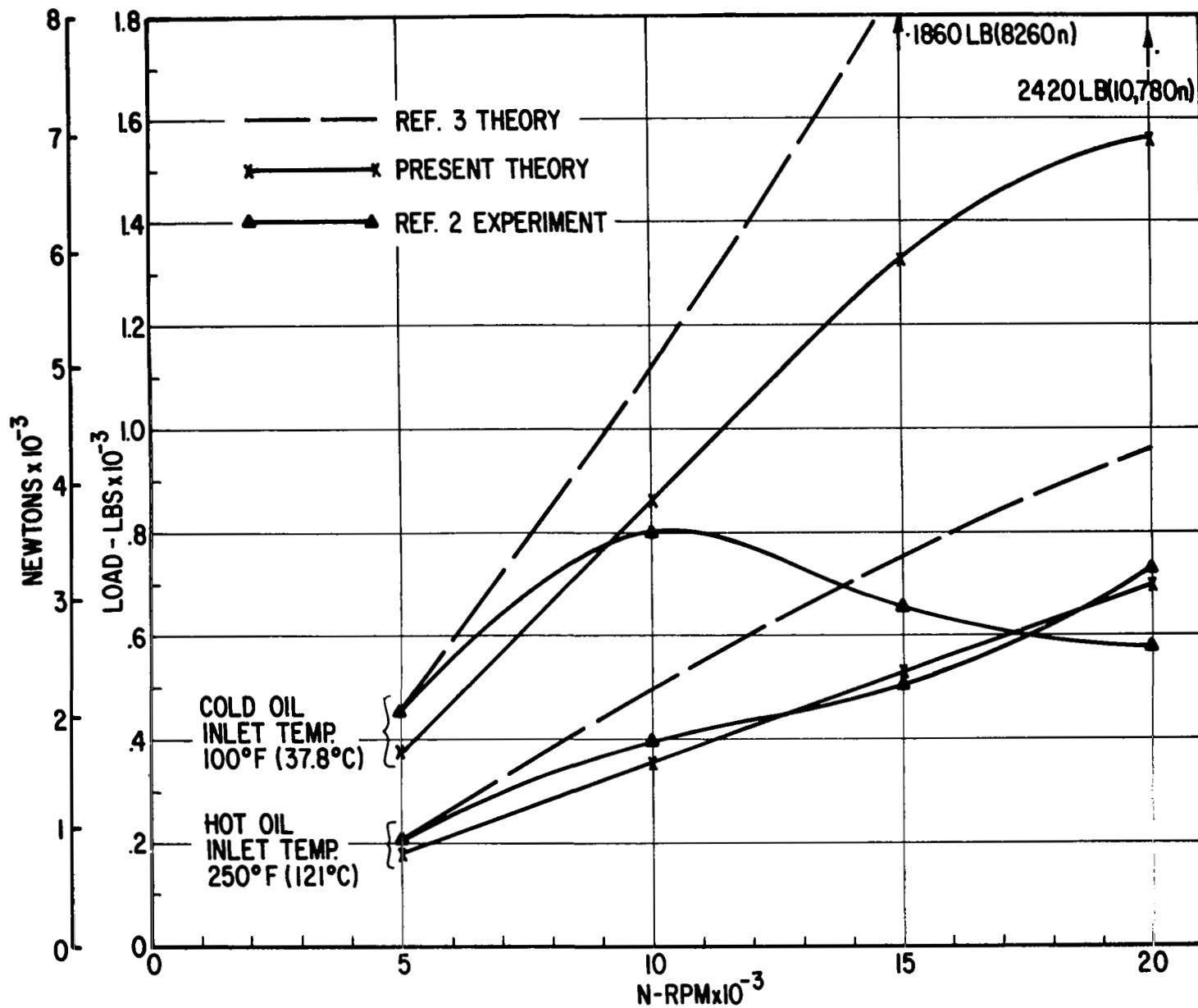
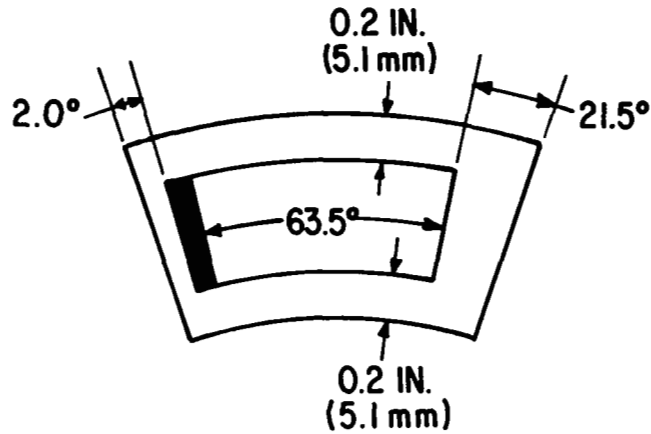


Fig. 1 Comparison of Load Capacity Between Theories and Experiment

TABLE II. COMPARISON OF THEORIES WITH EXPERIMENT

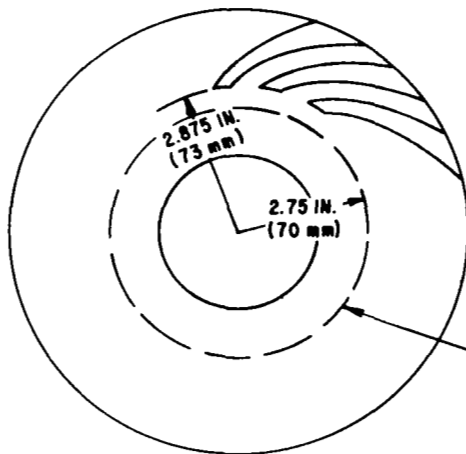
| N(rpm) | h<br>mils | P <sub>f</sub><br>psi                                    | T <sub>f</sub><br>F | Load, lb.    |        |         | Flow, gpm                                   |        |         | Power Loss, hp |        |         |
|--------|-----------|----------------------------------------------------------|---------------------|--------------|--------|---------|---------------------------------------------|--------|---------|----------------|--------|---------|
|        |           |                                                          |                     | Exp.         |        |         | Exp.                                        |        |         | Exp.           |        |         |
|        |           |                                                          |                     | Ref. 3       | Ref. 2 | Present | Ref. 3                                      | Ref. 2 | Present | Ref. 3         | Ref. 2 | Present |
| 5,000  | 3.3       | 30                                                       | 98                  | 456          | 460    | 378     | 2.0                                         | 1.9    | 2.5     | 0.53           | 0.56   | 1.5     |
| 10,000 | 2.1       | 55                                                       | 113                 | 1116         | 800    | 864     | 2.0                                         | 2.1    | 2.4     | 3.8            | 2.8    | 6.2     |
| 15,000 | 2.3       | 56                                                       | 139                 | 1860         | 650    | 1331    | 3.4                                         | 2.4    | 3.9     | 14.            | 8.9    | 17.2    |
| 20,000 | 2.2       | 30                                                       | 115                 | 2420         | 580    | 1567    | 4.5                                         | 3.3    | 4.7     | 36.            | 21.    | 29.     |
| 20,000 | 2.2       | 30                                                       | 115*                | -            | -      | 1416    | -                                           | -      | 6.2     | -              | -      | 22.5    |
| 5,000  | 2.7       | 24                                                       | 228                 | 208          | 200    | 180     | 3.3                                         | 4.2    | 4.7     | 0.45           | 0.59   | 0.32    |
| 10,000 | 2.1       | 28                                                       | 240                 | 500          | 400    | 359     | 3.6                                         | 4.0    | 4.5     | 2.7            | 2.0    | 2.0     |
| 15,000 | 2.3       | 35                                                       | 253                 | 750          | 500    | 534     | 6.7                                         | 5.5    | 7.5     | 6.9            | 10.6   | 6.4     |
| 20,000 | 2.4       | 39                                                       | 277                 | 960          | 730    | 695     | 10.0                                        | 6.1    | 11.5    | 13.3           | 23.    | 12.2    |
| 20,000 | 2.0       | 39                                                       | 277                 | -            | -      | 810     | -                                           | -      | 8.2     | -              | -      | 13.3    |
|        |           |                                                          |                     |              |        |         |                                             |        |         |                |        |         |
| N(rpm) | h<br>mm   | P <sub>f</sub><br>10 <sup>6</sup> newt<br>m <sup>2</sup> | T <sub>f</sub><br>C | Load, Newton |        |         | Flow (10 <sup>-5</sup> m <sup>3</sup> /sec) |        |         |                |        |         |
|        |           |                                                          |                     | Exp.         |        |         | Exp.                                        |        |         |                |        |         |
|        |           |                                                          |                     | Ref. 3       | Ref. 2 | Present | Ref. 3                                      | Ref. 2 | Present |                |        |         |
| 5,000  | 0.084     | 0.207                                                    | 36.7                | 2030         | 2050   | 1680    | 12.6                                        | 12.0   | 15.7    |                |        |         |
| 10,000 | 0.053     | 0.379                                                    | 45.                 | 4970         | 3560   | 3840    | 12.6                                        | 13.2   | 15.1    |                |        |         |
| 15,000 | 0.058     | 0.386                                                    | 59.5                | 8280         | 2890   | 5930    | 21.4                                        | 15.1   | 24.5    |                |        |         |
| 20,000 | 0.056     | 0.207                                                    | 46.1                | 10770        | 2580   | 6970    | 28.3                                        | 20.8   | 29.6    |                |        |         |
| 20,000 | 0.056     | 0.207                                                    | 46.1*               | -            | -      | 6300    | -                                           | -      | 39.0    |                |        |         |
| 5,000  | 0.068     | 0.166                                                    | 109.                | 925          | 890    | 800     | 20.8                                        | 26.5   | 29.6    |                |        |         |
| 10,000 | 0.053     | 0.193                                                    | 115.5               | 2220         | 1780   | 1600    | 22.7                                        | 25.2   | 28.3    |                |        |         |
| 15,000 | 0.058     | 0.242                                                    | 123.                | 3340         | 2220   | 2370    | 42.2                                        | 34.6   | 47.2    |                |        |         |
| 20,000 | 0.061     | 0.269                                                    | 136.                | 4270         | 3250   | 3090    | 63.0                                        | 38.4   | 72.5    |                |        |         |
| 20,000 | 0.051     | 0.269                                                    | 136.                | -            | -      | 3600    | -                                           | -      | 51.6    |                |        |         |

\* Temperature - Dependent

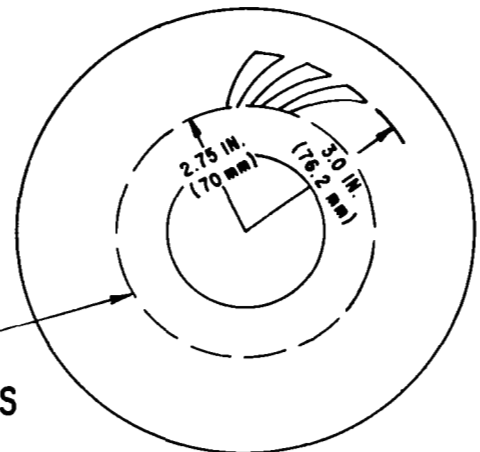


LUBRICANT SUPPLY GROOVE

(a) SHROUDED STEP (4 SECTORS)



(b) INWARD PUMPING SPIRAL-GROOVED THRUST BEARING



(c) OUTWARD PUMPING SPIRAL-GROOVED THRUST BEARING

GROOVE DEPTH = 6 MILS (.015 mm)  
 SPIRAL ANGLE = 18° (FROM CIRCUMFERENCE)  
 GROOVE/RIDGE WIDTH RATIO = 1.0

Fig. 2 Dimensions of Shrouded-Step and Spiral-Grooved Bearings for Comparative Studies

Two versions of spiral-grooved designs were considered. In Fig. 2b, the conventional inward-pumping arrangement is shown with the augmenting pressurized supply at a diameter of 5.5 in. (140 mm). In this arrangement, the outward radial flow is opposed by the pumping action of the spiral grooves. Fig. 2c shows outward-pumping spiral grooves in direct communication with the pressurized supply but separated from the outer diameter by a "land". The dimensions shown are the results of some exploratory trials to obtain the highest load. The comparison of load, flow, and power are as follows:

|                                                 | <u>Shrouded-Step</u><br>(Fig. 2a) | <u>Spiral-Grooved</u><br>(Fig. 2b) | (Fig. 2c)   |
|-------------------------------------------------|-----------------------------------|------------------------------------|-------------|
| Load, lbs (newtons)                             | 2,391 (10630)                     | 1691 (7520)                        | 2105 (9350) |
| Flow, gpm ( $10^{-5} \text{ m}^3 \text{ sec}$ ) | 7.88 (49.7)                       | 2.48 (15.6)                        | 2.16 (13.6) |
| Power, hp                                       | 6.70                              | 9.43                               | 9.36        |

It is seen that the shrouded-step design has a larger load capacity, a larger flow rate, and also a smaller friction power. The larger flow rate of the shrouded-step is mainly due to its additional leakage into the radial slots. One may expect that the spiral-grooved bearing should be more sensitive to temperature effects because of its lesser convective cooling. The smaller friction power of the shrouded-step is due to the smaller overall land area. The larger load capacity of the shrouded-step was somewhat of a surprise, since the spiral-grooved design has been more popular in recent years precisely because of its load capacity. The explanation for this apparent contradiction consists of three points:

- (a) The general favor for spiral-grooved thrust bearings over other types is mainly in gas bearing applications; i.e., gas bearing gyroscopes and Brayton-cycle machinery, where the large compressibility number would tend to limit the unit load to the order of magnitude of the ambient pressure. In a spiral-grooved, gas-lubricated, thrust bearing, this limitation is removed by using a large number of narrow grooves [8]. Since, in the present problem, the lubricant is incompressible, the relative advantage of the spiral-grooved, gas-lubricated, thrust bearing for large compressibility numbers does not apply.
- (b) The spiral-grooved thrust bearing may be designed to be either inward-pumping, with the grooves in communication with the ambient at the outer diameter, or outward-pumping, with the grooves in communication with the ambient at the inner diameter. If the inner/outer diameter ratio is small, say less than 0.6, the load capacity of the inward-pumping design is significantly larger than that of the

outward-pumping design [9, 10]. This feature is usually employed to advantage. In the present situation, the mechanical arrangement of the rolling-element bearing necessitates a relatively larger inner diameter. In fact, the inner/outer diameter ratio of 0.77 used in the calculations is not sufficiently large from the practical point of view.

- (c) The spiral-grooved thrust bearing is primarily a viscous, self-acting bearing. In the present case, there is considerable centrifugal head available to augment (or even replace) the viscous, self-acting effects. Therefore, any advantage related to a viscous, self-acting fluid film would become relatively obscured.

In view of these arguments, it was decided to abandon further considerations of the spiral-grooved fluid-film bearings.

In the next round of optimization, or trade-off study, isoviscous calculations (using oil viscosity at 200 F (93.4 C) were performed to look for the optimum parameters for the shrouded-step thrust bearing. Because of practical limitations to the test equipment the following conditions of operation were imposed for a speed of 20,000 rpm:

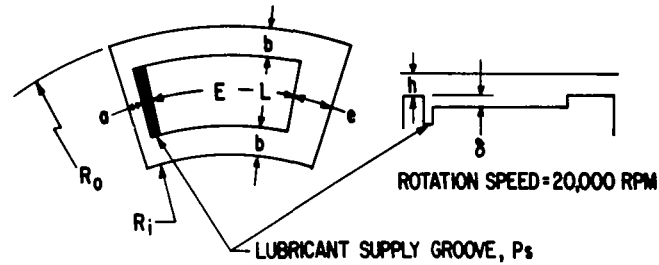
|                |                                                         |
|----------------|---------------------------------------------------------|
| Load Capacity  | > 2,500 lbs (11,120 newtons)                            |
| Friction Power | < 60 hp                                                 |
| Flow           | > 10 gpm ( $63 \times 10^{-5} \text{ m}^3/\text{sec}$ ) |

The results of these calculations are summarized in Table III. In Series A, nine cases consist of the combinations of three pocket depths and three bearing gaps. The chosen supply pressure of 50 psig (345,000 newton/m<sup>2</sup>) is consistent with the standard aircraft lube system. According to the results on flow, which is roughly proportional to the third power of the gap, one readily concludes that the gap has to be in the range of 1.0 - 2.0 mils (0.025 - 0.051 mm). Within this range of bearing gap, the load capacity peaks with the pocket depth somewhere between 12 and 16 mils (0.30 and 0.41 mm). The friction power is practically independent of the pocket depth. Both the load capacity and the friction power are significantly larger than the target values, indicating that the bearing is somewhat oversized. In Series B, the outer radius was adjusted according to the scaling law for the friction power,

$$\text{HP} \sim (R_o^4 - R_i^4),$$

such that a value of 60 hp was obtained for the 1 mil (0.025 mm) gap; setting  $R_o = 3.5 \text{ in}$  (89 mm). With the smaller total annular width, the widths of the pocket and the lands were reduced accordingly. An additional consideration introduced was a higher supply pressure. The higher supply pressure may be drawn from the centrifugal head of the runner. The pocket depth

TABLE III. ISOVISCOUS OPTIMIZATION CALCULATIONS OF SHROUDED-STEP BEARING



| $P_s$<br>psig ( $10^3$ newton/m <sup>2</sup> ) | Series | Case | $R_i$      | $R_o$       | $b$         | $L$         | $a$  | $e$   | $E$   | $\delta$  | $h$          | $W$            | HP    | $Q$                                  |
|------------------------------------------------|--------|------|------------|-------------|-------------|-------------|------|-------|-------|-----------|--------------|----------------|-------|--------------------------------------|
|                                                |        |      | in (mm)    |             |             |             | deg  |       |       | mils (mm) |              | lb. (newton)   | hp    | gpm ( $10^{-5}$ m <sup>3</sup> /sec) |
| 50 (345)                                       | A      | 1    | 3.0 (76.2) | 3.75 (95.2) | 0.238 (6.0) | 0.274 (7.0) | 1.83 | 22.41 | 60.30 | 12 (0.30) | 1 (0.025)    | 7,587 (33,700) | 107.9 | 4.91 (31)                            |
|                                                |        | 2    |            |             |             |             |      |       |       |           | 2 (0.051)    | 3,524 (15,700) | 81.6  | 13.89 (87.5)                         |
|                                                |        | 3    |            |             |             |             |      |       |       |           | 4 (0.102)    | 1,166 (5,190)  | 63.0  | 26.25 (165)                          |
|                                                |        | 4    |            |             |             |             |      |       |       | 16 (0.41) | 1 (0.025)    | 6,448 (28,600) | 107.8 | 4.20 (26.5)                          |
|                                                |        | 5    |            |             |             |             |      |       |       |           | 2 (0.051)    | 3,664 (16,300) | 82.3  | 14.56 (92)                           |
|                                                |        | 6    |            |             |             |             |      |       |       |           | 4 (0.102)    | 1,359 (6,040)  | 62.9  | 30.84 (194)                          |
|                                                |        | 7    |            |             |             |             |      |       |       | 20 (0.51) | 1 (0.025)    | 5,566 (24,800) | 107.6 | 3.64 (23)                            |
|                                                |        | 8    |            |             |             |             |      |       |       |           | 2 (0.051)    | 3,616 (16,100) | 82.9  | 14.45 (91)                           |
|                                                |        | 9    |            |             |             |             |      |       |       |           | 4 (0.102)    | 1,506 (6,700)  | 63.0  | 34.36 (216)                          |
| 50 (345)                                       | B      | 1    |            | 3.50 (89.0) | 0.16 (4.1)  | 0.18 (4.6)  |      |       |       | 16 (0.41) | 1 (0.025)    | 3,193 (14,200) | 60.6  | 4.72 (30)                            |
|                                                |        | 2    |            |             |             |             |      |       |       |           | 2 (0.051)    | 1,466 (6,520)  | 46.1  | 13.27 (84)                           |
|                                                |        | 3    |            |             |             |             |      |       |       |           | 4 (0.102)    | 490 (2,180)    | 35.3  | 25.52 (61)                           |
| 193 (1,330)                                    |        | 4    |            |             |             |             |      |       |       |           | 1 (0.025)    | 3,940 (17,500) | 60.3  | 5.92 (37)                            |
|                                                |        | 5    |            |             |             |             |      |       |       |           | 2 (0.051)    | 1,989 (8,850)  | 45.4  | 18.49 (116)                          |
|                                                |        | 6    |            |             |             |             |      |       |       |           | 4 (0.102)    | 838 (3,730)    | 34.2  | 45.71 (288)                          |
| 120 (827)                                      | C      | 1    |            |             |             |             |      |       |       |           | 1.4 (0.036)  | 2,660 (11,800) | 53.9  | 9.65 (61)                            |
|                                                |        | 2    |            |             |             |             |      |       |       |           | 1.6 (0.041)  | 2,289 (10,200) | 50.7  | 11.8 (74)                            |
|                                                |        | 3    |            |             |             |             |      |       |       |           | 1.8 (0.046)  | 1,977 (8,800)  | 48.0  | 13.9 (88)                            |
|                                                |        | 4    |            | 3.55 (90.2) | 0.175 (4.4) | 0.20 (5.1)  |      |       |       |           | 1.4 (0.036)  | 3,216 (14,300) | 61.4  | 9.66 (61)                            |
|                                                |        | 5    |            |             |             |             |      |       |       |           | 1.6 (0.041)  | 2,795 (12,450) | 57.8  | 11.9 (74)                            |
|                                                |        | 6    |            |             |             |             |      |       |       |           | 1.8 (0.046)  | 2,434 (10,820) | 54.8  | 14.1 (88)                            |
|                                                |        | 7    |            | 3.60 (91.5) | 0.19 (4.8)  | 0.22 (5.6)  |      |       |       |           | 1.4 (0.036)  | 3,812 (16,950) | 69.3  | 9.63 (61)                            |
|                                                |        | 8    |            |             |             |             |      |       |       |           | 1.6 (0.041)  | 3,341 (14,900) | 65.3  | 12.0 (76)                            |
|                                                |        | 9    |            |             |             |             |      |       |       |           | 1.8 (0.046)  | 2,932 (13,050) | 61.9  | 14.1 (90)                            |
|                                                | D      | 1    |            | 3.55 (90.2) | 0.175 (4.4) | 0.20 (5.1)  |      |       |       | 16 (0.41) | 1.45 (0.037) | 3,105 (13,800) | 60.4  | 10.23 (65)                           |
|                                                |        | 2    |            |             |             |             |      |       |       | 14 (0.36) | 1.45 (0.037) | 3,109 (13,800) | 60.2  | 10.21 (65)                           |

was fixed at 16 mils (0.41 mm). Scanning the results of Series B, it is apparent that friction power is no longer a test limitation. The objectives of flow and load capacity are both satisfied for the 1 mil gap with either supply pressure. However, there are obvious advantages to use the largest possible gap. At 50 psig ( $0.345 \times 10^6$  newton/m<sup>2</sup>) supply pressure, the gap is load-capacity limited; whereas, at 193 psig ( $1.33 \times 10^6$  newton/m<sup>2</sup>) supply pressure, the gap is flow limited. Clearly, at an intermediate supply pressure, the maximum gap would be limited simultaneously by load-capacity and flow, and this would be the optimum supply pressure which allows all design objectives to be met with the largest possible gap. Interpolating between the two supply pressures of Series B, this condition was found to be 120 psig ( $0.83 \times 10^6$  newton/m<sup>2</sup>), and the gap would be about 1.5 mils (0.038 mm). At this point, it is noted that the friction power limit has not been reached, therefore a larger bearing may be tolerated so that the largest possible load capacity may be realized within the flow and friction power limitations. Therefore, in Series C, with the supply pressure fixed at 120 psig ( $0.83 \times 10^6$  newton/mm<sup>2</sup>), the possibility of increasing the outer diameter was explored. The widths of the pocket and the lands were again increased in proportion. It is seen, in Series C, that the flow is dependent only on the bearing gap, thus one readily finds by interpolation that the flow allowance of 10 gpm ( $63 \times 10^{-5}$  m<sup>3</sup>/sec) would be met with 1.45 mils (0.037 mm) gap, and the friction power allowance of 60 hp would be satisfied with 3.55 in. (93.7 mm) outer radius. This is essentially the optimized design. Series D contains its verification calculation and also a check on the effects of a slightly shallower pocket. There is virtually no difference in all aspects for the range of pocket depth between 14 to 16 mils (0.35 to 0.41 mm). Finally, the temperature effects are considered for the optimized design and the results are listed as follows:

| <u>Method of Calc.</u>                     | <u>Isoviscous</u> | <u>Temperature Dependent</u> |
|--------------------------------------------|-------------------|------------------------------|
| Load, lbs (newtons)                        | 3,105 (13,800)    | 3,030 (13,450)               |
| Flow, gpm ( $10^{-5}$ m <sup>3</sup> /sec) | 10.23 (64.5)      | 10.87 (68.5)                 |
| Power, hp                                  | 60.4              | 57.3                         |

This comparison indicates that the performance of the optimized shrouded-step bearing is essentially not sensitive to temperature effects at the selected conditions of operation: 1.45 mils (0.037 mm) gap, 200 F (93.4 C) oil inlet temperature, and 20,000 rpm. However, in view of the results shown in Tables I, II and Fig. 1, significant temperature effects can be anticipated at higher speeds and particularly at lower oil inlet temperatures.

As the desired supply pressure for the optimized design is likely to be derived from the centrifugal head of the runner, an appropriate flow restrictor must be designed to correspond to the particular combination of speed, film thickness, and flow. Assuming an orifice type restrictor with a pressure drop equal to 2.46 times the velocity head, the required restrictor in each sector should have a diameter of 0.0575 in. (1.46 mm).

With the fixed restrictor geometry, the performance characteristics of the "optimized" fluid-film bearing would vary with both speed and operating gap. Such off-design performance characteristics are shown in Fig. 3 and Table IV. These calculations were made by the isoviscous method for the MIL-L-7808 oil at 200 F (93.4 C). Thermal film effects were evaluated at 20,000 rpm and were found to be negligible as indicated in Table IV. Fig. 3 shows that the load capacity is very sensitive to the accuracy in setting the bearing gap.

In the shrouded-step design, load is shared between hydrodynamic and hydrostatic effects. The supply pressure required by the optimized design described above is higher than is available from the standard aircraft lube system. However, the centrifugal head of the runner is amply sufficient for this purpose. Since the hydrodynamic effect is roughly proportional to the first power of the speed whereas the centrifugal head is proportional to the second power, higher speed operations favor a purely hydrostatic design. In that event, re-examination of the configuration design in addition to parametric optimization would be desirable. In particular, elimination of the radial slots, which divide the bearing into four sectors, is probably necessary in order to minimize the flow rate.

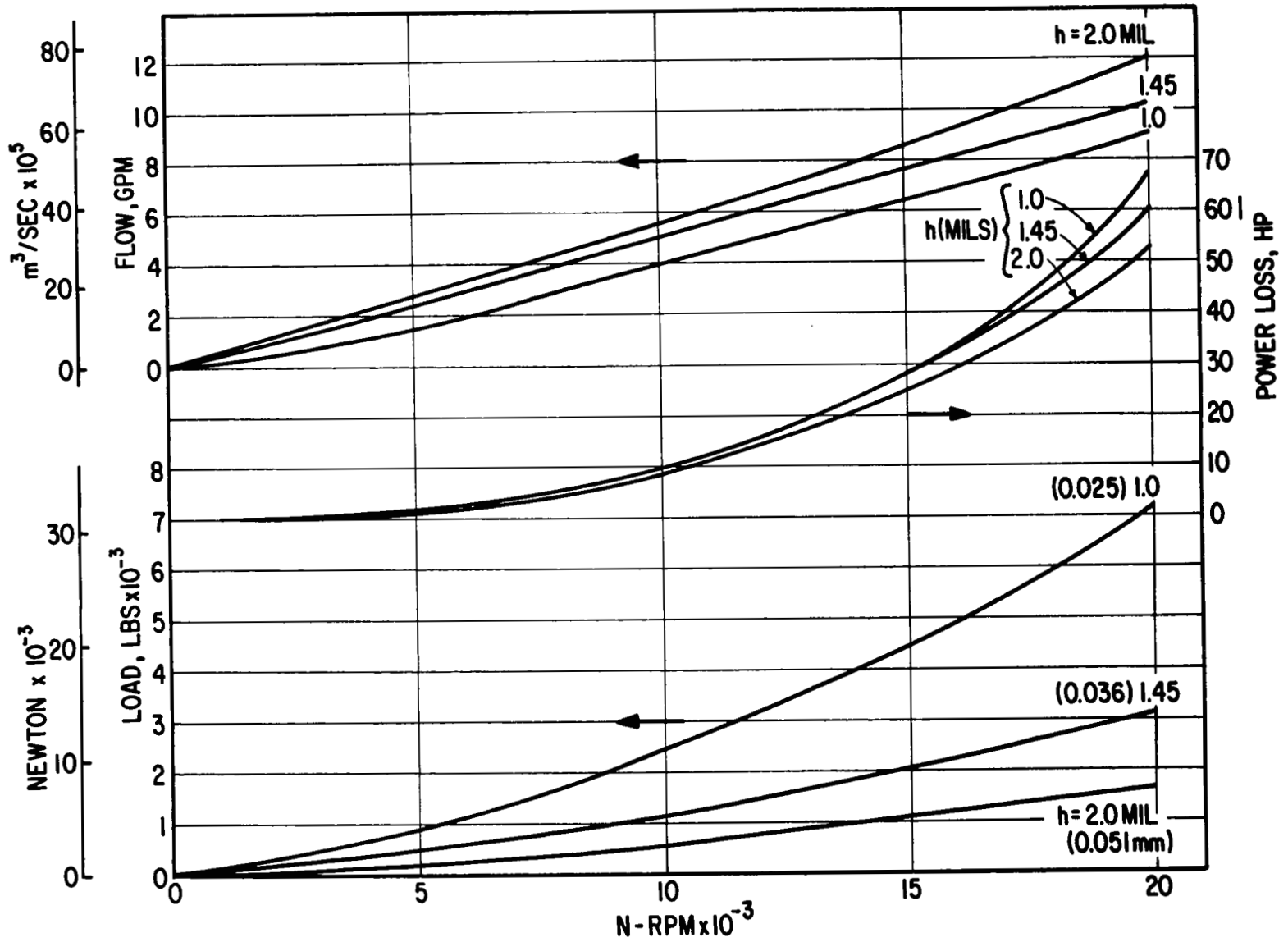


Fig. 3 Performance Data for Optimized "Pocketed-Shrouded-Step Bearing"

TABLE IV. PERFORMANCE DATA FOR OPTIMIZED "POCKETED-SHROUDED-STEP BEARING"

| h,mils<br>N,rpm | 1.0          |         |         |                     | 1.45         |         |         |      | 2.0          |         |         |                     |
|-----------------|--------------|---------|---------|---------------------|--------------|---------|---------|------|--------------|---------|---------|---------------------|
|                 | W(lb)/P(psi) | Q (gpm) | HP (hp) | $\dot{\gamma}$ (hp) | W(lb)/P(psi) | Q (gpm) | HP (hp) | (hp) | W(lb)/P(psi) | Q (gpm) | HP (hp) | $\dot{\gamma}$ (hp) |
| 20,000          | 7153/615     | 9.14    | 67.9    | 9.32                | 3105/120     | 10.2    | 60.4    | 10.4 | 1624/1.0     | 12.0    | 52.8    | 12.2                |
| 20,000*         | 7026/615     | 10.1    | 65.1    | 10.3                | 3120/120     | 10.9    | 57.3    | 11.1 | 1587/1.0     | 12.4    | 50.6    | 12.6                |
| 15,000          | 4452/390     | 6.42    | 28.3    | 3.68                | 2075/100     | 7.60    | 28.3    | 4.36 | 1025/1.0     | 8.51    | 25.0    | 4.88                |
| 10,000          | 2455/220     | 3.95    | 9.20    | 1.01                | 1117/ 74     | 5.12    | 8.35    | 1.30 | 569/10       | 5.51    | 8.49    | 1.40                |
| 5,000           | 888/ 77      | 1.43    | 2.08    | 0.09                | 477/ 35      | 2.31    | 1.51    | 0.15 | 221/10       | 2.72    | 1.15    | 0.17                |

| h,mm<br>N,rpm | 25.4                                                                 |                                  | 36.8                                                                 |                                  | 51                                                                   |                                  |
|---------------|----------------------------------------------------------------------|----------------------------------|----------------------------------------------------------------------|----------------------------------|----------------------------------------------------------------------|----------------------------------|
|               | $\frac{W \text{ (newton)}}{P \text{ (newton)}}$<br>$\frac{m^2}{m^2}$ | $Q$<br>$10^{-5} \frac{m^3}{sec}$ | $\frac{W \text{ (newton)}}{P \text{ (newton)}}$<br>$\frac{m^2}{m^2}$ | $Q$<br>$10^{-5} \frac{m^3}{sec}$ | $\frac{W \text{ (newton)}}{P \text{ (newton)}}$<br>$\frac{m^2}{m^2}$ | $Q$<br>$10^{-5} \frac{m^3}{sec}$ |
| 20,000        | 31,900                                                               | 57.6                             | 13,800                                                               | 64.2                             | 7,230                                                                | 75.6                             |
| 20,000*       | 31,300                                                               | 63.7                             | 13,900                                                               | 68.6                             | 7,060                                                                | 78.0                             |
| 15,000        | 19,800                                                               | 40.5                             | 9,230                                                                | 48.0                             | 4,560                                                                | 53.7                             |
| 10,000        | 10,900                                                               | 24.9                             | 4,970                                                                | 32.3                             | 2,530                                                                | 34.8                             |
| 5,000         | 3,950                                                                | 9.0                              | 2,120                                                                | 14.6                             | 985                                                                  | 17.2                             |

\* Temperature - Dependent

## CONCLUSIONS AND RECOMMENDATIONS

The results of the analyses performed indicate that:

- A. Temperature rise within the fluid-film due to frictional dissipation can significantly affect the bearing performance under the combined conditions of

- Low bearing temperature
- High sliding speed
- Small film thickness

The principal effects in decreasing order of magnitude are

- Increased flow
- Reduced friction power loss
- Reduced load capacity

However, the isoviscous calculations generally give the correct qualitative trends in aspects not directly related to temperature.

- B. The spiral-grooved thrust bearing is less favorable in the present application because:

1. The lubricant is incompressible
2. The annular width of the bearing is small
3. The self-acting portion of the load capacity is not predominating
4. Lesser convective cooling makes it more temperature sensitive

- C. An optimized shrouded-step bearing design is found which satisfies test prescribed limitations. It operates at 1.45 mils (0.037 mm) gap with load capacities at 20,000 and 10,000 rpm respectively of 3,000 lbs. (13,330 newtons) and 1100 lbs. (4,900 newtons). The supply pressure of the optimized shrouded-step at 20,000 rpm should be 120 psig ( $0.83 \times 10^6$  newton/mm<sup>2</sup>).

- D. A purely hydrostatic design which utilizes the available centrifugal head to its fullest extent should be considered. With the full centrifugal head, the bearing area can be substantially smaller and considerable saving in power loss may be realized. This advantage would be particularly significant for higher speed operations.

## LIST OF REFERENCES

1. Winn, L. W., Badgley, R. H., "Development of Long Life Jet Engine Thrust Bearings", NASA CR-72744, June 1970.
2. Winn, L. W., "Development and Test of Long-Life, Hybrid Boost Thrust Bearings for Small, High-Speed Jet Engine Applications", Prepared under Contract N00019-68-C-0269, for Naval Air Systems Command, MTI-69TR17, June 1969.
3. Wachmann, C., Wilcock, D. F., and Yampolsky, "Design of a Process Fluid Turbulent Film Shrouded Step Thrust Bearing for Circulators for High Temperature Helium Cooled Reactors", Bearing and Seal Design In Nuclear Power Meeting, ASME, 1967, pp. 354-370.
4. Constantinescu, V. N., Hsing, F. C., and Pan, C. H. T., "A Procedure For the Analysis of Bearings Operating in the Transition Range Between Laminar and Fully Developed Turbulent Flow", Appendix A to NASA CR-72744 "Development of Long Life Jet Engine Thrust Bearings", June 1970.
5. Malanoski, S. B., Arwas, E. B., and Castelli, V., "A Computer Program for Analysis of Naval, Main Propulsion Thrust Bearings", Vol. 2 of Final Report under Contract Nobs 94457, Prepared for Naval Ships Systems Command, Washington, D. C., 20360, MTI 67TR5, January 9, 1966.
6. Ng, C. W. and Pan, C. H. T., "A Linearized Turbulent Lubrication Theory", Journal of Basic Engineering, Trans. ASME, Series D, Vol. 87, No. 3, September 1965, pp. 675-688.
7. Bootsma, J., "The Effect of Viscosity Variation with Temperature on Performance of Spiral Groove Bearings", Trans. ASLE, Vol. 12, No. 4, October 1969.
8. Pan, C. H. T., "A Universally Valid, Compressible Theory for Gas Lubricated Narrow, Spiral-Groove Thrust Bearings", Presented at 1969 Southampton Symposium, England.
9. Malanoski, S. B. and Pan, C. H. T., "The Static and Dynamic Characteristics of the Spiral-Grooved Thrust Bearing", Journal of Basic Engineering, Trans. ASME, Series D, Vol. 87, No. 3, September 1965, pp. 547-558.
10. Muijderland, E. A., Spiral Groove Bearings, Philips Technical Library (1966).

## APPENDIX A

### INCOMPRESSIBLE THIN FILM FLOW WITH TEMPERATURE EFFECTS

#### 1.0 Statement of Problem

Consider a fluid, which has a temperature dependent viscosity,  $\mu(T)$ , flowing between two nearly parallel surfaces, the temperatures of which are kept constant at  $T_w$ . Let the separation between the surfaces be  $h$ , and let one of the surfaces move tangentially to itself with the velocity  $\vec{V}$ . The mean surface can be described in terms of orthogonal (curvilinear) coordinates  $(x, z)$  respectively parallel and perpendicular to the sliding velocity  $\vec{V}$ . Let the normal distance from the mean surface be  $y$ , then

$$|y| \leq \frac{1}{2} h(x, z)$$

contains the entire thin film flow field of interest.

The momentum equations, having neglected the inertial effects, are

$$\begin{aligned} \frac{\partial \tau_{xy}}{\partial y} &= \frac{\partial}{\partial y} \left[ \mu \left( 1 + \frac{\epsilon}{\nu} \right) \frac{\partial u}{\partial y} \right] = \frac{\partial p}{\partial x} \\ \frac{\partial \tau_{zy}}{\partial y} &= \frac{\partial}{\partial y} \left[ \mu \left( 1 + \frac{\epsilon}{\nu} \right) \frac{\partial w}{\partial y} \right] = \frac{\partial p}{\partial z} \end{aligned} \quad (1)$$

The corresponding boundary conditions are

$$\begin{aligned} u(y = -\frac{h}{2}) &= w(y = \pm \frac{h}{2}) = 0 \\ u(y = \frac{h}{2}) &= V \end{aligned} \quad (2)$$

$V$  being a function of  $z$ .  $\epsilon$  is the eddy diffusivity to account for the turbulent Reynolds stress. The continuity requirement assuming incompressibility

can be written as

$$\text{div} \int_{-h/2}^{h/2} \vec{u} dy + \frac{\partial h}{\partial t} = 0 \quad (3)$$

The boundary conditions depend on the particular physical situation, typically some relation between flow and pressure would be stipulated. Clearly, with a temperature dependent viscosity, one is also required to deal with the energy equation. The appropriate time-independent energy equation, taking into account the turbulent heat transport and assuming heat diffusivity to be numerically equal to momentum diffusivity, can be written for an incompressible thin film to be:

$$\rho C_p \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) T = \frac{\partial}{\partial y} \left[ \kappa \left( 1 + \frac{\epsilon}{\nu} \text{Pr} \right) \frac{\partial T}{\partial y} \right] + \frac{\tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} \quad (4)$$

where,  $\tau^2 = \tau_{xy}^2 + \tau_{zy}^2$ . This equation was previously derived by Lund [1]\*. Turbulent diffusivity may be obtained by empirical correlation of experimental data. There is no unique form of optimum representation without reflecting personal taste. One expression which has been used with a fair amount of success in turbulent film lubrication studies was originally due to Reichardt [2]:

$$\frac{\epsilon}{\nu} = k \left[ \frac{\left( \frac{h}{2} - |y| \right)}{\nu} \sqrt{\frac{\tau}{\rho}} - \delta_\ell^+ \right] \tanh \frac{\left( \frac{h}{2} - |y| \right)}{\delta_\ell^+ \nu} \sqrt{\frac{\tau}{\rho}} \quad (5)$$

where the empirical constants  $k$  and  $\delta_\ell^+$  are usually taken to be 0.4 and 10.7 respectively.

In principle, a complete mathematical statement has been presented. It would be quite feasible to carry out a numerical computation without further simplifications, however, in the interest of achieving maximal computational economy which would be very important in anticipation of extensive design calculations, additional simplifying assumptions will be made.

\*Bracketed numbers refer to identically numbered references at end of this Appendix.

## 2.0 Review of Isoviscous Turbulent Film Analysis

To form the strategy for the development of a practical computation scheme, it is useful to review the isoViscous turbulent film lubrication theory consistent with the general schemes according to equations (1,2,3,5).

Formally, one can write

$$\int_{-h/2}^{h/2} \bar{U} dy = - \frac{h^3}{\mu} \bar{G} \cdot \nabla p + \frac{\bar{V}h}{2} \quad (6)$$

where  $\bar{G}$  is a dyadic coefficient which accounts for the influence of turbulence on velocity profiles.  $\bar{G}$  turns out to be diagonal, the components of which are functions of  $\frac{h^3}{\mu\nu} \frac{\partial p}{\partial z}$ ,  $\frac{h^3}{\mu\nu} \frac{\partial p}{\partial z}$ , and  $\frac{Vh}{\nu}$  [3]\*. In [4], by assuming

$\left| \frac{h \nabla p}{\tau_{xy}(0)} \right| \ll 1$ ,  $\bar{G}$  was linearized and its components were found to depend only on  $\frac{Vh}{\nu}$ . The linearized results, so far as the components of  $\bar{G}$  are concerned, were found to be accurate for a much wider range than the underlying a priori assumption, and therefore can be used up to

$$\left| \frac{h \nabla p}{\tau_{xy}(0)} \right|_{\max} \approx 1.$$

## 3.0 Treatment of Transitional Flow

The components of  $\bar{G}$  reduce to the laminar value,  $\frac{1}{12}$ , only in the limiting condition  $\frac{Vh}{\nu} \rightarrow 0$ . This is a basic shortcoming of the analysis which does not explicitly recognize the critical transition phenomenon. A practical remedy of the situation is to perform a reasonable interpolation between laminar and turbulent conditions [5]. One such reasonable interpolation rule is to regard  $\frac{Vh}{\nu}$  as an effective Reynolds number, while the actual Reynolds number is written as  $\frac{Uh}{\nu}$  to avoid notational confusion, these are related to each other by

$$\frac{V}{U} = f \left( \frac{Uh}{\nu}, R_{cr}, R_f \right), \quad (7)$$

---

\* In [3] the eddy diffusivity used included a "core effect" which was neglected in [4].

where  $R_{cr}$  and  $R_f$  are respectively critical and fully developed Reynolds numbers. According to available experimental data,  $R_f \approx 2R_{cr}$  is a reasonable general rule for a variety of wall shear flows.

$$f = \begin{cases} 0 & \text{for } \left| \frac{Uh}{\nu} \right| < R_{cr} \\ 1 & \text{for } \left| \frac{Uh}{\nu} \right| > R_f \end{cases} \quad (8)$$

For intermediate values of  $\left| \frac{Uh}{\nu} \right|$ , there is no overwhelming argument to favor one kind of interpolation rule over another. The elliptical rule will be used primarily due to personal taste:

$$f = \sqrt{1 - \frac{\left( R_f - \frac{Uh}{\nu} \right)^2}{\left( R_f - R_{cr} \right)^2}} \quad \text{for } R_{cr} < \left| \frac{Uh}{\nu} \right| < R_f \quad (9)$$

Interpolation cannot be performed on the coefficient of friction,  $C_f$ , directly because

$$\lim_{\frac{Vh}{\nu} \rightarrow 0} C_f = \frac{8}{\left( \frac{Vh}{\nu} \right)} \rightarrow \infty$$

Instead, the appropriate shear stress parameter for interpolation is

$$\frac{\tau h}{\mu U} = \frac{C_f}{8} \frac{Vh}{\nu} \quad (10)$$

Note that the correct shear stress would be calculated from  $\frac{\tau h}{\mu U}$  but not from  $C_f$ . In fact, one should take note that  $C_f$  as defined in terms of  $Vh/\nu$  above should be multiplied by  $f$  to obtain the conventional coefficients of friction.

#### 4.0 Mean Film Heat Transfer Analysis

Consider next the possibilities of simplifying the energy equation. To

avoid the need to treat Eq. (4) as a partial differential equation, one can formally integrate it across the film and obtain

$$\rho C_p \operatorname{div} \int_{-h/2}^{h/2} \vec{U} T dy = \left[ \kappa \left( 1 + \frac{\epsilon}{\nu} \operatorname{Pr} \right) \frac{\partial T}{\partial y} \right]_{-h/2}^{h/2} + \int_{-h/2}^{h/2} \frac{\tau^2 dy}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} \quad (11)$$

To make further progress, "drastic" assumptions will now be made to facilitate the computation of Eq. (11). For the left-hand side, the approximation to be imposed amounts to replacing the average of the product by the product of the averages:

$$\int_{-h/2}^{h/2} \vec{U} T dy \approx \vec{U}_m T_m h \quad (12)$$

Since, by definition

$$\vec{U}_m = \frac{1}{h} \int_{-h/2}^{h/2} \vec{U} dy$$

$$T_m = \frac{1}{h} \int_{-h/2}^{h/2} T dy$$

Eq. (12) has omitted

$$\int_{-h/2}^{h/2} \left( \vec{U} - \vec{U}_m \right) \left( T - T_m \right) dy$$

from the right-hand side. The conduction and dissipation terms also must be manipulated into more tractable forms. The first step toward this end involves the introduction of the conduction cooling factor,  $\phi$ , which is required to satisfy the following equation:

$$\frac{d}{dy} \left[ \kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right) \frac{\partial T}{\partial y} \right] + \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} = 0 \quad (14)$$

Eq. (14) can be formally integrated twice:

$$\kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right) \frac{dT}{dy} = - \left[ q_o'' + \int_0^y \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} dy' \right] \quad (15)$$

where  $q_o''$  is the conduction heat flux at film center, and  $y'$  is the dummy variable. It is understood that the integrand is regarded to be dependent on the dummy variable. The second integration yields

$$T = T_o - \int_0^y \frac{dy''}{\kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right)} \left[ q_o'' + \int_0^{y''} \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} dy' \right] \quad (16)$$

Again  $y''$  is a dummy variable. The constants of integration,  $q_o''$  and  $T_o$ , are determined by the boundary conditions

$$T(y = \pm h/2) = T_{\pm w} \quad (17)$$

#### 4.1 The Couette Problem

Before attempting to carry to the end a practical treatment of the conduction and dissipation terms, one can derive some insight by considering the special case of a Couette problem. The system of equations of relevance here are

$$\tau = \tau_c \quad (18)$$

$$T_{+w} = T_{-w} = T_w .$$

These, together with  $\mu(T)$  and Eqs. (5), (15) and (16), completely define the

conduction-dissipation problem ( $\sigma^*$  is regarded to be known). This special case, by virtue of symmetry, requires

$$q''_0 = 0 \quad (19)$$

and one only needs to consider half of the film, say  $0 < y < h/2$ . The crucial assumption to be introduced is the  $y$ -independence of  $\sigma$ . The dimensionless variables of the present problem are

$$\begin{aligned} \bar{y} &= y/h \\ h_o^+ &= \frac{h}{\nu_o} \sqrt{\frac{\tau_c}{\rho}} \\ Q &= \frac{\mu_o q''}{\sigma \tau_c^2 h} \\ \theta &= \frac{\mu_o \kappa (T_o - T)}{\sigma \tau_c^2 h^2} \end{aligned} \quad (20)$$

$\nu_o$  and  $\mu_o$  are respectively the kinematic and dynamic viscosities at  $T_o$ .

The governing equations can now be rewritten as

$$\begin{aligned} \frac{dQ}{d\bar{y}} &= \frac{\mu_o}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} \\ \frac{d\theta}{d\bar{y}} &= \frac{Q}{1 + \frac{\epsilon}{\nu} Pr} \\ \frac{\epsilon}{\nu} &= 0.4 \left\{ \left( \frac{1}{2} - \bar{y} \right) h_o^+ \left( \frac{\mu_o}{\mu} \right) - 10.7 \tanh \left[ \frac{1}{10.7} \left( \frac{1}{2} - \bar{y} \right) h_o^+ \left( \frac{\mu_o}{\mu} \right) \right] \right\} \\ Q(\bar{y} = 0) &= 0 \end{aligned}$$

---

\* Strictly speaking, for the Couette problem,  $\sigma$  should be exactly unity. It is given a general numerical value here in anticipation of generalizing the procedure to include convective heat transfer.

$$\theta(\bar{y} = 0) = 0 \quad (21)$$

$$T = T_o \left( 1 - \frac{\sigma \tau_c^2 h^2}{\mu_o \kappa T_o} \theta \right)$$

$$\mu = \mu(T)$$

This system of equations can be numerically integrated by the "marching" method for every set of input parameters  $(T_o, h_o^+, \frac{\sigma \tau_c^2 h^2}{\mu_o \kappa T_o} \theta)$  to obtain  $Q$   $(0 < y < \frac{1}{2})$ . In particular, one is interested in

$$Q_w = \frac{\mu_o q_w''}{\sigma \tau_c^2 h} \quad (22)$$

$$\theta_w - 2 \int_0^{1/2} \theta d\bar{y} = \frac{\mu_o \kappa (T_m - T_w)}{\sigma \tau_c^2 h^2} \quad (23)$$

$$\frac{1}{Q_w} \left[ \theta_w - 2 \int_0^{1/2} \theta d\bar{y} \right] = \frac{\kappa (T_m - T_w)}{h q_w''} \quad (24)$$

where subscript "w" corresponds to the point  $\bar{y} = \frac{1}{2}$ .

Intuitively one may expect that reasonable accuracy may be realized by assuming a viscosity corresponding to a temperature somewhere between  $T_w$  and  $T_o$ . For instance, the laminar film was fairly accurately calculated by evaluating viscosity at  $T_m$  [6]. However, with the possibility of turbulence, particularly in view of the rather large Prandtl number of typical lubricating oils, it is deemed worthwhile to examine the possibility of mean viscosity approximation in some detail. Suppose one solves the above problem once more with the isoviscous assumption; i.e.  $\mu = \mu_1$  is  $y$ -independent. The solutions are

$$Q^{(i)} = \frac{\mu_o}{\mu_i} \int_0^{\bar{y}} \frac{d\bar{y}'}{(1 + \frac{\epsilon}{\nu})_i} \quad (25)$$

$$\theta^{(i)} = \frac{\mu_o}{\mu_i} \int_0^{\bar{y}} \frac{d\bar{y}''}{(1 + \frac{\epsilon}{\nu} Pr)_i} \int_0^{y''} \frac{dy'}{(1 + \frac{\epsilon}{\nu})_i} \quad (26)$$

Let  $\mu_1$  be chosen such that

$$Q_w^{(1)} = Q_w \quad (27)$$

and  $\mu_2$  be chosen such that

$$\frac{1}{Q_w^{(2)}} \left[ \theta_w^{(2)} - 2 \int_0^{\frac{1}{2}} \theta^{(2)} d\bar{y} \right] = \frac{1}{Q_w} \left[ \theta_w - 2 \int_0^{\frac{1}{2}} \theta d\bar{y} \right] \quad (28)$$

Then  $\mu_1$  and  $\mu_2$  are the effective mean viscosities for calculating the dissipation and conduction terms properly for the Couette problem.

To recount the Couette Problem, given the physical properties of the lubricant, the system is specified by the parameters  $(T_o, h_o + \frac{\sigma \tau_c 2h^2}{\mu_o \kappa T_o})$ . The important solutions are

$$\theta_w = \frac{\mu_o \kappa (T_o - T_w)}{\sigma \tau_c 2h^2} \quad (29)$$

$$Q_w = \frac{\mu_o q_w''}{\sigma \tau_c 2h} \quad (30)$$

$$\frac{1}{Q_w} \left[ \theta_w - 2 \int_0^{\frac{1}{2}} \theta d\bar{y} \right] = \frac{\kappa (T_m - T_w)}{h q_w''} \quad (31)$$

For computational convenience, the input parameters were all related to  $T_0$ . ( $T_i$ ;  $i = 1, 2$ ) can be found such that isoviscous calculations based on  $\mu_i = \mu(T_i)$  would yield the correct values of dissipation and conduction heats; i.e.

$$\int_{-h/2}^{h/2} \frac{\tau_c^2}{\mu(1 + \frac{\epsilon}{\nu})} dy = \frac{\tau_c^2 h}{\mu_1} \int_{-1/2}^{1/2} \frac{d\bar{y}}{(1 + \frac{\epsilon}{\nu})_1} \quad (32)$$

$$\left[ \kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right) \frac{\partial T}{\partial y} \right]_{-h/2}^{h/2}$$

$$= -2 q_w''$$

$$= - \frac{\int_0^{1/2} \frac{d\bar{y}}{(1 + \frac{\epsilon}{\nu})_2}}{\int_0^{1/2} d\bar{y} \int_{\bar{y}}^{1/2} \frac{dy'}{(1 + \frac{\epsilon}{\nu} Pr)_2} \frac{d\bar{y}''}{(1 + \frac{\epsilon}{\nu})_2}} \frac{\kappa}{h} (T_m - T_w) \quad (33)$$

Thus, we may regard  $\frac{T_i - T_w}{T_m - T_w}$  to be determined by  $(T_0, h_o^+, \frac{\sigma \tau_c^2 h^2}{\mu \kappa T_0})$ .

#### 4.2 Typical Numerical Results for the Couette Problem

The mean film heat transfer analysis for the Couette problem as governed by Eqs. (21 through 28) can be summarized in terms of

$$\frac{T_i - T_w}{T_m - T_w} ; \quad i = 1, 2$$

for a specific lubricating oil as functions of

$$\left( T_o, h_o^+, \mathcal{F}_o = \frac{\sigma \tau_c^2 h^2}{\mu_o \kappa T_o} \right) .$$

Numerical results were obtained for (MIL-L-7808 oil) in the following ranges:

$$T_o = 200 \text{ F}, 300 \text{ F}; (93.4\text{C}, 149\text{C})$$

$$0 < h_o^+ < 500$$

$$0 < \mathcal{F}_o = \frac{\sigma \tau_c^2 h^2}{\mu_o \kappa T_o} < 1000$$

Since the last two parameters are related through  $(\tau_c, \mu_o, \text{ and } h)$ , it is useful to recognize the identity

$$\mathcal{F}_o \equiv \frac{\sigma \tau_c^2 h^2}{\mu_o \kappa T_o} \equiv \frac{\sigma \mu \nu^2}{h^2 \kappa T_o} (h_o^+)^4 .$$

Furthermore, in actual problems, one also has practical limitations on the film thickness and sliding speed. The interrelations of these parameters are illustrated in Figure A-1, based on the assumption of a constant temperature (200°F). The temperature-viscosity relation of the oil considered is shown in Figure A-2. A composite plot of the numerical results found is shown in Figure A-3 where  $(\mathcal{F}_o)_{\max}$  is defined along the borders of the conditions  $h < 0.5 \text{ mil}$ ,  $DN < 120,000 \text{ in-rpm}$  as shown in Figure A-1 for 200°F. The remarkable pattern shown in Figure A-3 is the relative insensitivity to  $T_o$  and  $\mathcal{F}_o$ , despite the fact that  $\mu_o$  varies by more than two to one. In particular, lack of strong  $\mathcal{F}_o$  dependence means, so long as one is only interested in the temperature ratios

$$\frac{T_i - T_w}{T_m - T_w} ; \quad i = 1, 2 ,$$

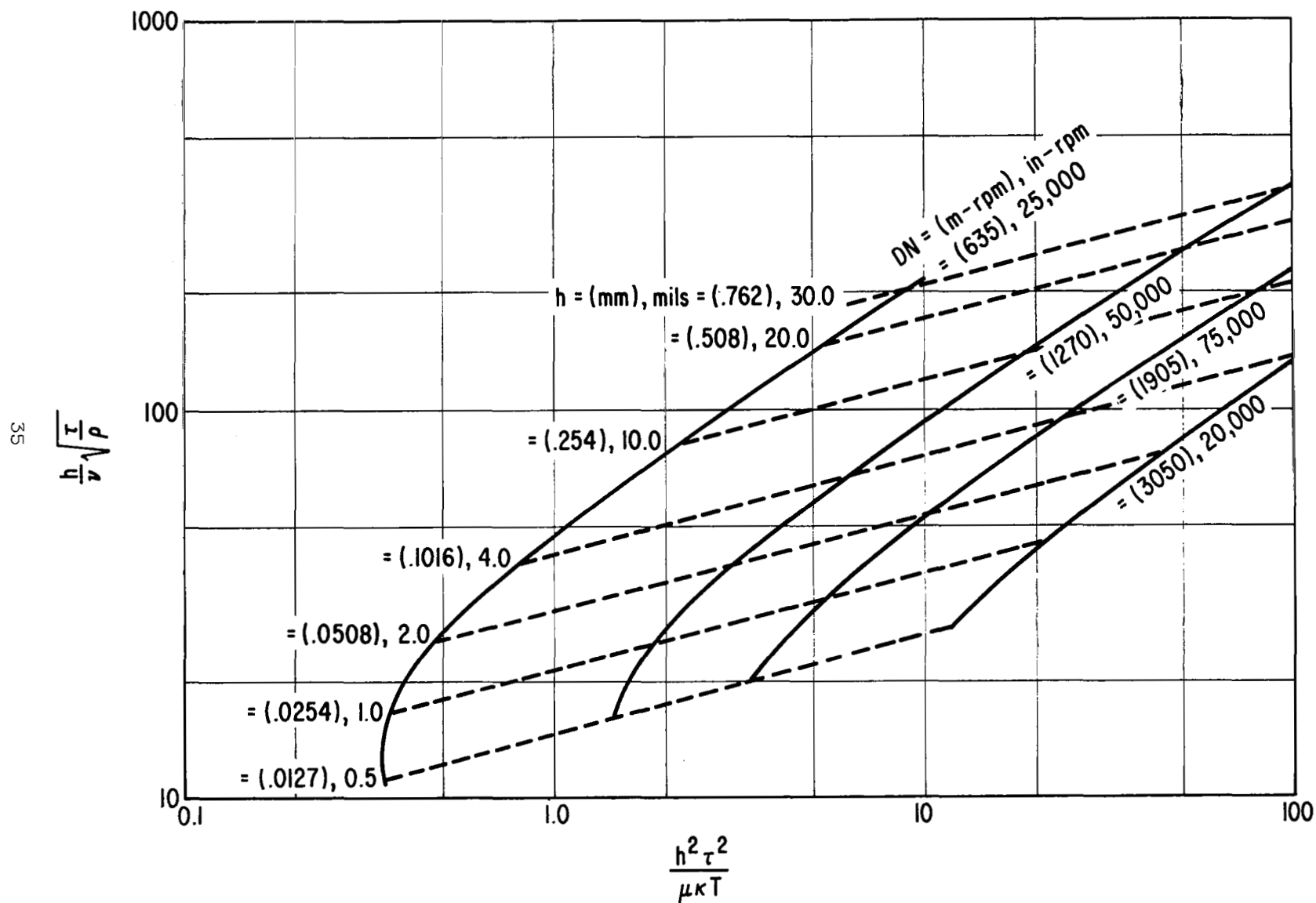


Fig. A-1 Map, Illustrating the Interrelations of the Isoviscous Parameters,  $h^+$  and  $\tilde{\lambda}$  for MIL-L-7808 Oil at 200°F (93.4°C)

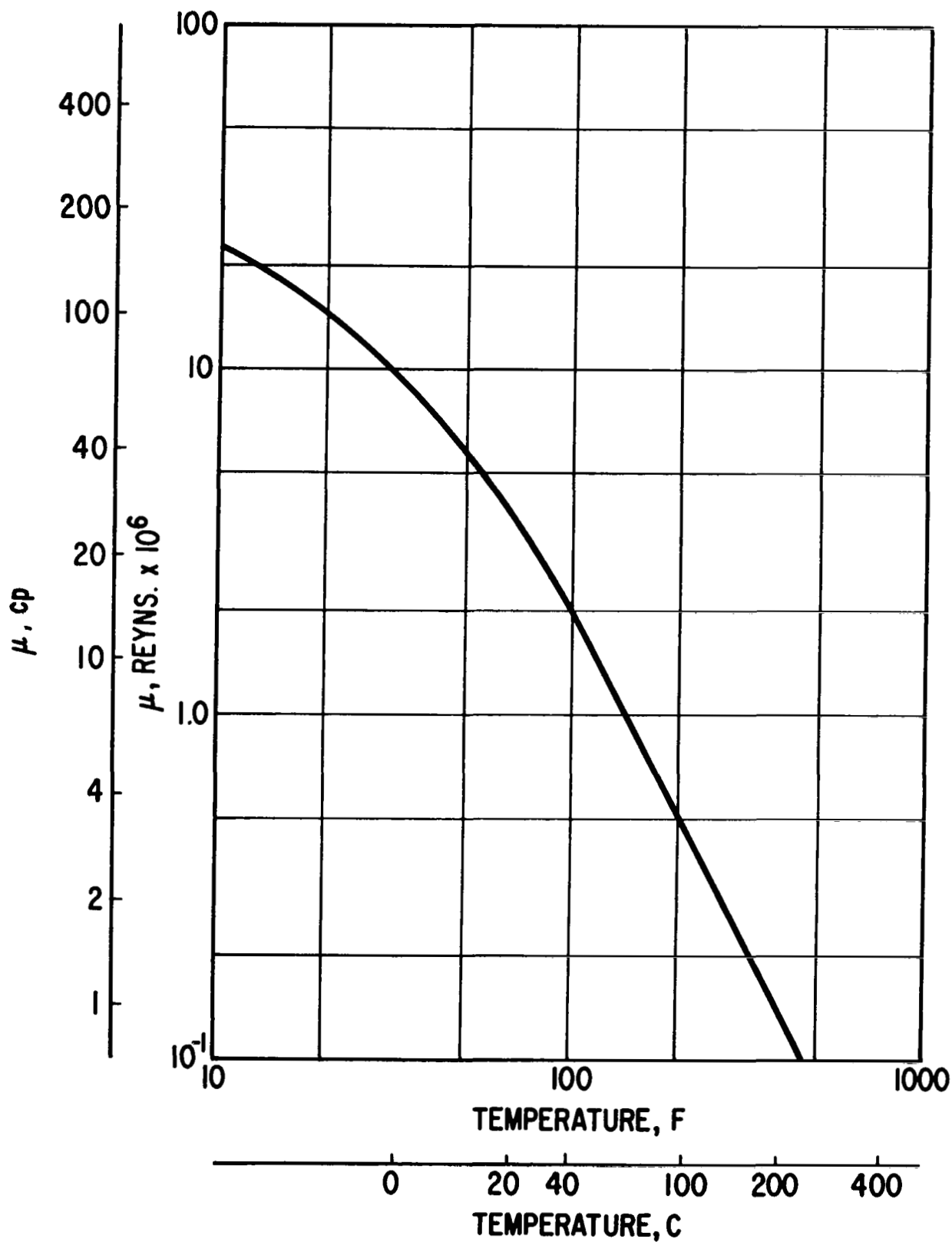


Fig. A-2 Viscosity Versus Temperature — MIL-L-7808 Oil (Diester Type)

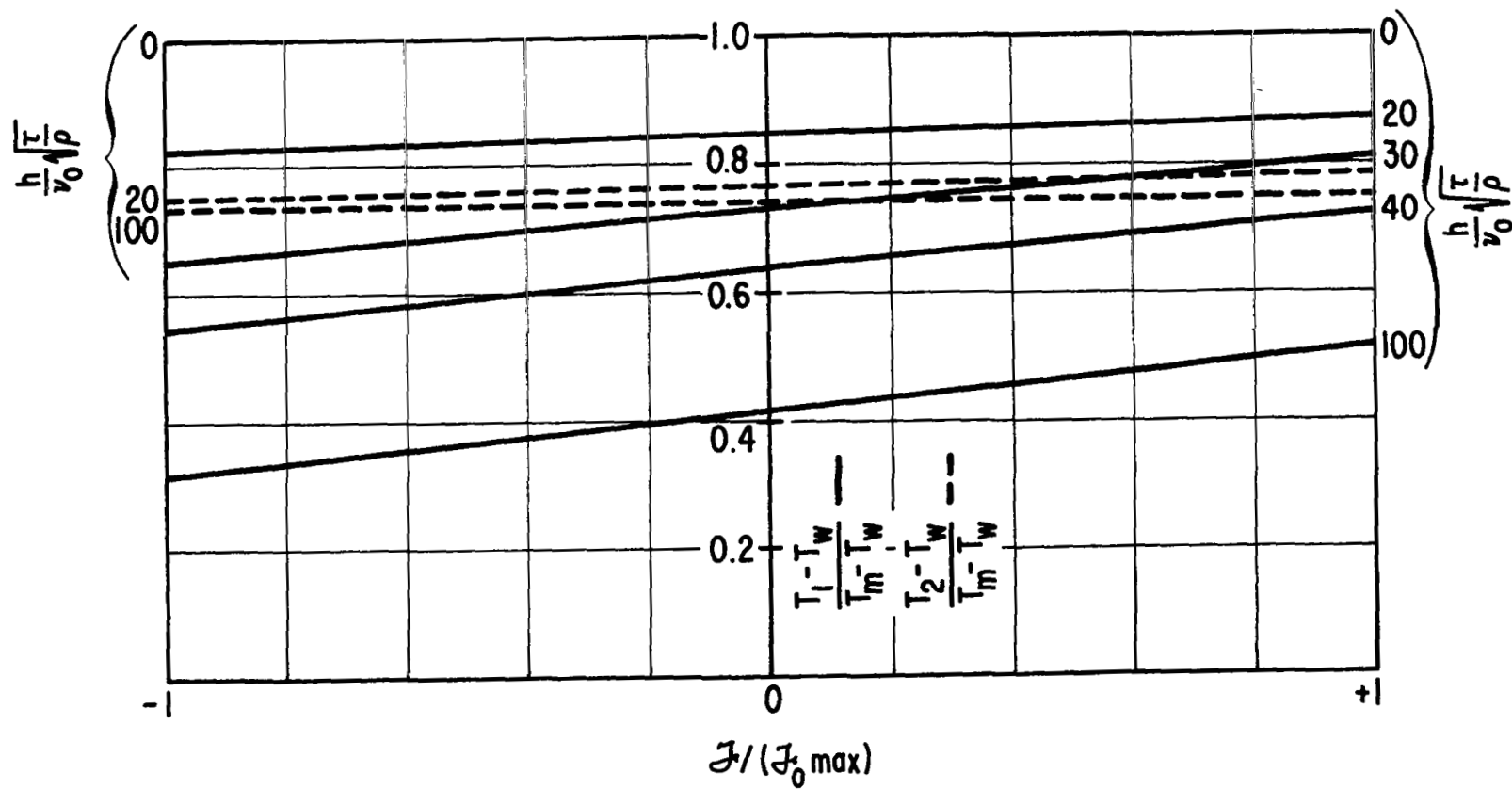


Fig. A-3  $\mathcal{F}$  Independency Curves at  $T_c = 200^\circ\text{F}$  ( $93.4^\circ\text{C}$ );  $\frac{T_i - T_w}{T_m - T_w}$ ;  
 $i = 1, 2$  Versus  $\mathcal{F}/(\mathcal{F}_0 \max)$  for Various  $h_0^+$ .

the limiting condition,  $\mathcal{F}_0 \rightarrow 0$ , can be used satisfactorily. In other words, the "linearized temperature viscosity" relation should give adequate results.

Consequently, one may conveniently use the single parameter relations

$$\frac{T_i - T_w}{T_m - T_w} = \Theta_i \left( \frac{h}{v_1} \sqrt{\frac{f\tau}{\rho}} \right) \quad i = 1, 2$$

as illustrated graphically in Figure A-4, where the recommended empirical formulae are also indicated.

### 4.3 Pressure Gradient Effects

One may reasonably argue that many features of the Couette problem should remain valid even if the shear stress in the film is significantly altered by pressure gradients; i.e., even if  $\left| h \nabla p \right| \approx \left| \tau_c \right|$ . It is desirable that computational procedures can be developed to utilize as much as possible the relatively simple analysis of the Couette problem. The value of this approach depends on if reasonable universal functions can be established and if the determination of  $\frac{T_i - T_w}{T_m - T_w}$  is both convenient and reliable. In this work, the analytical derivations will stress plausibility, rigor will be forsaken in order to achieve computational ease.

Having accepted this utilitarian philosophy, one can proceed to develop the required computation procedure. These questions can be answered by a careful scrutiny of Equations (15, 16, and 17) under the following conditions:

$$\begin{aligned} \tau_{xy} &= \tau_c + y \frac{\partial p}{\partial x} \\ \tau_{zy} &= y \frac{\partial p}{\partial z} \\ \tau^2 &= (\tau_{xy})^2 + (\tau_{zy})^2 \\ T_{+w} &= T_{-w} = T_w \end{aligned} \tag{34}$$

$$\theta_1 = B + h^+ * [C1 + h^+ * (C2 + h^+ C3)]$$

$$h^+ < \sqrt{700}$$

$$\sqrt{700} \leq h^+ \leq 220$$

$$h^+ > 220$$

$$B = 1.0$$

$$C1 = C2 = C3 = 0$$

$$B = 1.05$$

$$C1 = -1.34E-2$$

$$C2 = 8.13E-5$$

$$C3 = -1.68E-7$$

$$B = 0.286$$

$$C1 = -1.59E-4$$

$$C2 = C3 = 0$$

$$\theta_2 = B + h^+ * [C1 + h^+ * (C2 + h^+ C3)]$$

$$h^+ < \sqrt{700}$$

$$\sqrt{700} \leq h^+ \leq 40$$

$$h^+ > 40$$

$$B = 1.0$$

$$C1 = C2 = C3 = 0$$

$$B = 1.01$$

$$C1 = -6.75E-3$$

$$C2 = -1.17E-4$$

$$C3 = 3.14E-6$$

$$B = 0.755$$

$$C1 = -5.0E-5$$

$$C2 = C3 = 0$$

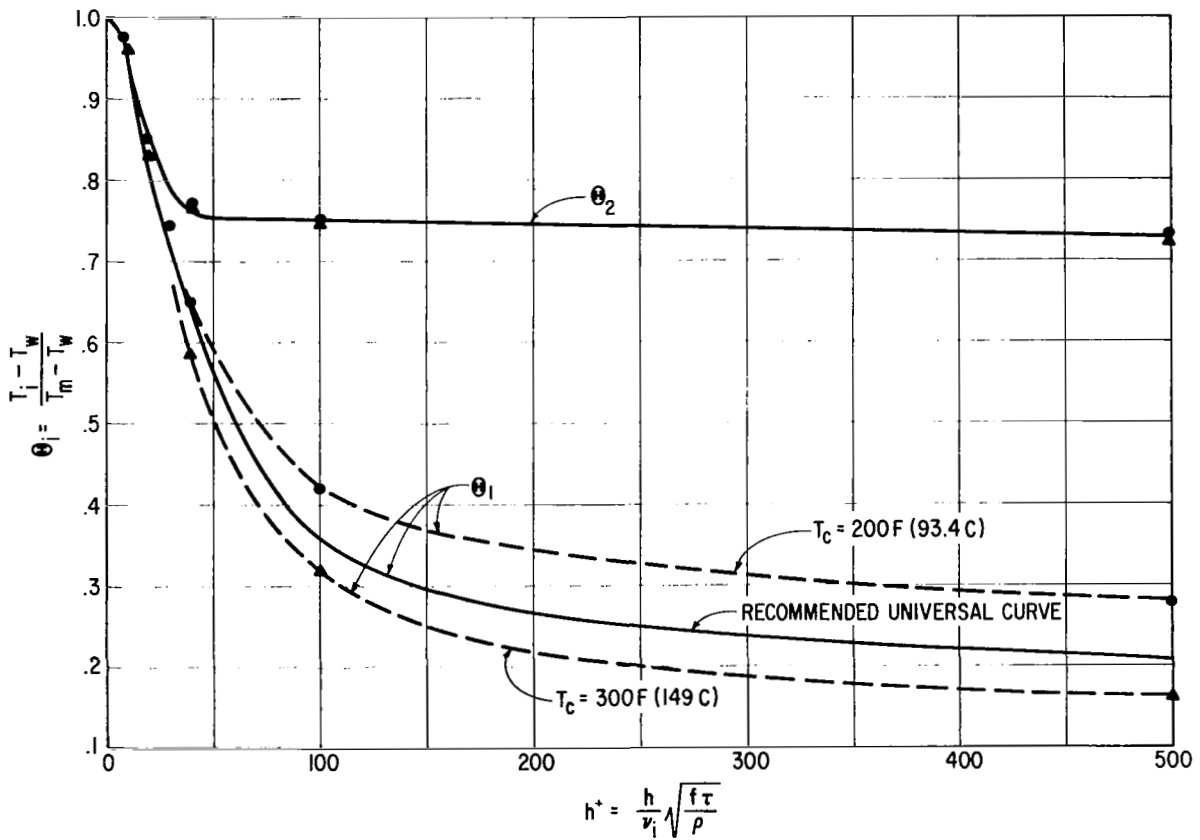


Fig. A-4 · Temperature Ratios Versus  $\frac{h}{v_1} \sqrt{\frac{f\tau}{\rho}}$ , and Recommended Empirical Formulae

The major issue here concerns the significance of  $(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z})$  on the total conduction flux

$$- \left[ \kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right) \frac{\partial T}{\partial y} \right]_{-h/2}^{h/2} = \int_{-h/2}^{h/2} \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} dy \quad (35)$$

and the mean film temperature (after some manipulation)

$$T_m - T_w = \frac{1}{h} \int_{-h/2}^{h/2} \frac{y dy}{\kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right)} \left[ q_o'' + \int_0^y \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} dy' \right] \quad (36)$$

where,

$$q_o'' = - \frac{\int_{-h/2}^{h/2} \frac{dy}{\left( 1 + \frac{\epsilon}{\nu} Pr \right)} \int_0^y \frac{\sigma \tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} dy'}{\int_{-h/2}^{h/2} \frac{dy}{\left( 1 + \frac{\epsilon}{\nu} Pr \right)}} \quad (37)$$

Consider that

$$\begin{aligned} \frac{\tau^2}{\mu \left( 1 + \frac{\epsilon}{\nu} \right)} &= F(y) \\ &= F_e(y) + F_o(y) \end{aligned} \quad (38)$$

$F_e$  and  $F_o$  are respectively even and odd parts of  $F(y)$ . Then

$$- \left[ \kappa \left( 1 + \frac{\epsilon}{\nu} Pr \right) \frac{\partial T}{\partial y} \right]_{-h/2}^{h/2} = \sigma \int_{-h/2}^{h/2} F_e dy \quad (39)$$

And

$$F_e = \frac{1}{2} \left\{ \tau_c^2 + y^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] \right\} \times$$

$$\left\{ \frac{1}{\mu(1 + \frac{\epsilon}{v})} \Big|_y + \frac{1}{\mu(1 + \frac{\epsilon}{v})} \Big|_{-y} \right\} +$$

$$\tau_c y \frac{\partial p}{\partial x} \left\{ \frac{1}{\mu(1 + \frac{\epsilon}{v})} \Big|_y - \frac{1}{\mu(1 + \frac{\epsilon}{v})} \Big|_{-y} \right\}$$

For the laminar case,  $\frac{\epsilon}{v} = 0$ , as the temperature tends to be higher on the side where  $\tau_c$  and  $y \frac{\partial p}{\partial x}$  are additive, the viscosity there would be lower, therefore the cross-product term is expected to be additive to the quadratic terms. Furthermore, skewness in  $\mu$  (due to temperature change) for typical oils tends to increase the coefficient for the quadratic terms. Thus a reasonably equivalent Couette approximation would be

$$F_e \approx \left\{ \tau_c^2 + y^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] \right\} \frac{1}{\mu_c (1 + \frac{\epsilon}{v})_c} \quad (40)$$

Here, omission of the cross-product term is somewhat compensated by the use of a smaller denominator,  $\mu_c (1 + \frac{\epsilon}{v})_c$ , which corresponds to the equivalent Couette problem with

$$\tau_c' \approx \sqrt{\tau_c^2 + \left( \frac{h}{2} \right)^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right]} \quad (41)$$

replacing  $\tau_c$  in the input parameter which accounts for viscosity-temperature

dependence; namely\*

$$\frac{\sigma \tau_c'^2 h^2}{\mu_o \kappa T_o} \quad (42)$$

Together with  $T_o$ , and  $h_o^+$ ,  $\frac{T_{1'} - T_w}{T_m - T_w}$  can now be determined from the previously outlined analysis for the Couette problem, and the dissipation term in Equation (11) can be calculated as

$$\int_{-h/2}^{h/2} \frac{\tau^2}{\mu(1 + \frac{\epsilon}{\nu})} dy \approx \frac{h}{\mu_{1'}} \left\{ \tau_o'^2 G_1(T_{1'}) + h^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] G_2(T_{1'}) \right\} \quad (43)$$

where,

$$G_1(T_{1'}) = 2 \int_0^{1/2} \frac{d\bar{y}}{(1 + \frac{\epsilon}{\nu})_{1'}} \quad (44)$$

$$G_2(T_{1'}) = 2 \int_0^{1/2} \frac{y^{-2} d\bar{y}}{(1 + \frac{\epsilon}{\nu})_{1'}}$$

are universal functions of  $(T_o, h_o^+, \frac{\sigma \tau_c'^2 h^2}{\mu_o \kappa T_o})$ . Using similar reasoning,  $\frac{T_2 - T_w}{T_m - T_w}$  can be found according to  $(T_w, h_w^+, \frac{\sigma \tau_c'^2 h^2}{\mu_w \kappa T_w})$ , then the conduction

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\* Use of  $\tau_c'$  instead of  $\tau_c$  is to account for additional heating due to

$(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z})$ . The turbulence parameter,  $h_o^+$ , is still based on  $\tau_c$ . Thus, one maintains separation between heating and turbulence input parameters. The subtle merit of this fine point will become evident when the momentum problem is considered.

term in Equation (11) can be calculated as

$$\begin{aligned}
 & - \left[ \kappa \left( 1 + \frac{\varepsilon}{\nu} \text{Pr} \right) \frac{\partial T}{\partial y} \right]_{-h/2}^{-h/2} \\
 & = \frac{\tau_c^2 G_1(T_2') + h^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] G_2(T_2')}{\tau_c^2 H_1(T_2') + h^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] H_2(T_2')} \frac{\kappa}{h} (T_m - T_w)
 \end{aligned} \tag{45}$$

where,

$$\begin{aligned}
 G_1(T_2') &= 2 \int_0^{1/2} \frac{d\bar{y}}{(1 + \frac{\varepsilon}{\nu})_{2'}}, \\
 G_2(T_2') &= 2 \int_0^{1/2} \frac{\bar{y}^2 d\bar{y}}{(1 + \frac{\varepsilon}{\nu})_{2'}}, \\
 H_1(T_2') &= 2 \int_0^{1/2} d\bar{y} \int_{\bar{y}}^{1/2} \frac{d\bar{y}'}{(1 + \frac{\varepsilon}{\nu} \text{Pr})_{2'}} \int_0^{\bar{y}'} \frac{d\bar{y}''}{(1 + \frac{\varepsilon}{\nu})_{2'}}. \\
 H_2(T_2') &= 2 \int_0^{1/2} d\bar{y} \int_{\bar{y}}^{1/2} \frac{d\bar{y}'}{(1 + \frac{\varepsilon}{\nu} \text{Pr})_{2'}} \int_0^{\bar{y}'} \frac{\bar{y}''^2 d\bar{y}''}{(1 + \frac{\varepsilon}{\nu})_{2'}}.
 \end{aligned} \tag{46}$$

$(G_1, G_2, H_1, H_2)$  are universal functions of  $\frac{h}{\nu_i} \sqrt{\frac{\tau_c}{\rho}}$ .

In so far as the important effect of  $(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial z})$  is accountable by the use of a slightly larger value of

$$\mathcal{R}_0 = \frac{\sigma \tau_c h^2}{\mu_o \kappa T_o}$$

to find  $T_i$ ; and it has already been established in the preceding section (Figures A-3 and A-4), that  $T_i$  ( $i = 1, 2$ ) are insensitive to  $\mathcal{F}_0$  and that they may be approximated by the one-parameter  $\left(\frac{h}{v_i} \sqrt{\frac{f\tau'}{\rho}}\right)$  relations which are the limiting results for  $\mathcal{F}_0 \rightarrow 0$ ; it becomes evident that an adequate allowance for pressure gradients in the mean-film heat transfer analysis, is to replace  $T_i - T_w$  by

$$T_i' - T_w = \left(\frac{\tau_c'}{\tau_c}\right)^2 \Theta_i = \left\{1 + \left(\frac{h}{2\tau_c}\right)^2 \left[\left(\frac{\partial p}{\partial x}\right)^2\right]\right\} \Theta_i. \quad (47)$$

For fluid films with pressure gradients  $T_i'$  instead of  $T_i$  should be used in the mean-film heat transfer analysis. This heuristic rule is obviously correct for the limiting condition  $\frac{h\nabla p}{\tau_c} \rightarrow 0$ ; it is expected to over-estimate  $|T_i - T_w|$  in the other extreme  $\frac{h\nabla p}{\tau_c} \rightarrow \infty$ .

## 5.0 Coupling with the Momentum Problem

Obviously, it is also advantageous to develop a mean viscosity momentum analysis. The corresponding Couette problem is

$$\frac{du}{dy} = \frac{\tau_c}{\mu(1 + \frac{\varepsilon}{v})} \quad (48)$$

Nondimensionalizing, define

$$\bar{u} = \frac{\mu_0(u-V/2)}{h\tau_c} \quad (49)$$

Then,

$$\frac{d\bar{u}}{d\bar{y}} = \frac{\mu_0}{\mu(1 + \frac{\varepsilon}{v})} \quad (50)$$

with the boundary condition

$$\bar{u}(\bar{y}=0) = 0 \quad (51)$$

should be solved simultaneously with the temperature problem, Equation (21). Noting the identical mathematical properties of  $\bar{u}$  and  $Q$ , one concludes that

$$\frac{\mu_o(u-v/2)}{h\tau_c} \equiv \frac{\mu_o q''}{\sigma\tau_c^2 h}$$

More specifically,

$$\frac{\mu_o v}{h\tau_c} = \frac{2\mu_o q_w''}{\sigma\tau_c^2 h} = 2 \int_0^{1/2} \frac{\mu_o d\bar{y}}{\mu(1 + \frac{\epsilon}{v})} = 2Q_w \quad (52)$$

The equivalent isoviscous momentum problem is therefore the same as the equivalent isoviscous dissipation problem. Consequently,  $T_1'$  is also the mean temperature for momentum analysis. All relevant quantities ( $G_x$ ,  $G_z$ ,  $\frac{\tau_c h}{\mu_1 V}$ ) are to be determined as functions of  $\frac{Vh}{v_1}$ . In fact, isoviscous results of  $G_1 \equiv G_\tau = \frac{\mu V}{\tau_c h}$  and  $G_2 \equiv G_z$  have already been computed [4]. To consolidate symbols, ( $G_\tau$ ,  $G_z$ ) will be used from now on.

## 6.0 Summary

In summary, the governing equations consist of

$$\frac{\partial h}{\partial t} + \text{div} (\vec{U}_m h) = 0 \quad (53)$$

$$\vec{U}_m = \vec{i} \left( - \frac{G_x}{\mu} \right)_{T_1'} h^2 \frac{\partial p}{\partial x} + \frac{1}{2} U \quad + \vec{k} \left( - \frac{G_z}{\mu} \right)_{T_1'} h^2 \frac{\partial p}{\partial z} \quad (54)$$

$$\begin{aligned} & \rho C_p \left[ \frac{\partial}{\partial t} (T_m h) + \text{div} (\vec{U}_m T_m h) \right] \\ &= - \left[ \frac{G_\tau |_{T_2'} + (\frac{h}{\tau_c})^2 |\nabla p|^2 G_z |_{T_2'}}{H_1 |_{T_2'} + (\frac{h}{\tau_c})^2 |\nabla p|^2 H_2 |_{T_2'}} \right] \left( \frac{\kappa}{h} \right) (T_m - T_w) \\ &+ \tau_c^2 h \left[ \frac{G_\tau}{\mu} \right]_{T_1'} + \left( \frac{h}{\tau_c} \right)^2 |\nabla p|^2 \frac{G_z}{\mu} \Big|_{T_1'} \quad (55) \end{aligned}$$

There are five universal functions in above equations, representing velocity and temperature shape factors influencing various terms. Three of these are dependent on the "Couette" Reynolds number,  $Uh/\nu$ , only; the other two also depend on the Prandtl number,  $\mu C_p/\kappa$ . Subscripts " $T_1$ " indicate the effective mean temperature at which the viscosity coefficient is evaluated. For the laminar film, the universal functions are known explicitly

$$\left. \begin{aligned} G_T &= 1 \\ G_x &= G_z = H_1 = \frac{1}{12} \\ H_2 &= \frac{1}{240} \end{aligned} \right\} \quad (56)$$

Note that, for the laminar film, Prandtl number dependence of ( $H_1$ ,  $H_2$ ) disappears. Transition to turbulence may be determined according to the empirical approach described in Section 3. Specifically, assuming that transition to turbulence and fully developed turbulence respectively take place at  $R_{cr} = 700$  and  $R_f = 1400$ , the universal functions ( $G_T$ ,  $G_x$ ,  $G_z$ ) are graphically shown in the ranges  $500 < Uh/\nu < 1,500$  and  $1000 < Uh/\nu$  respectively in Figures A-5 (a and b). Shown in Figure A-6 is the function

$$\frac{h}{\nu} \sqrt{\frac{f\tau_c}{\rho}} = \sqrt{\frac{f}{G_T} \frac{Uh}{\nu}}$$

which is essentially constructed from the relations of  $f$  and  $G_T$  vs.  $\frac{Uh}{\nu}$ ; this function has a special role when the  $T_2'$  related universal functions are sought. This point will be elucidated further later. Two other universal functions, ( $H_1$ ,  $H_2$ ), are Prandtl number dependent in addition to being Reynolds number dependent. For the oil of immediate interest, the pertinent numerical results are shown in Figures A-7 (a and b). To use these universal relations, the effective film temperatures are to be determined according to Equation (48), which is rewritten below

$$T_i' - T_w = \left\{ 1 + \left( \frac{h}{2\tau_c} \right)^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] \right\} \Theta_i \quad (57)$$

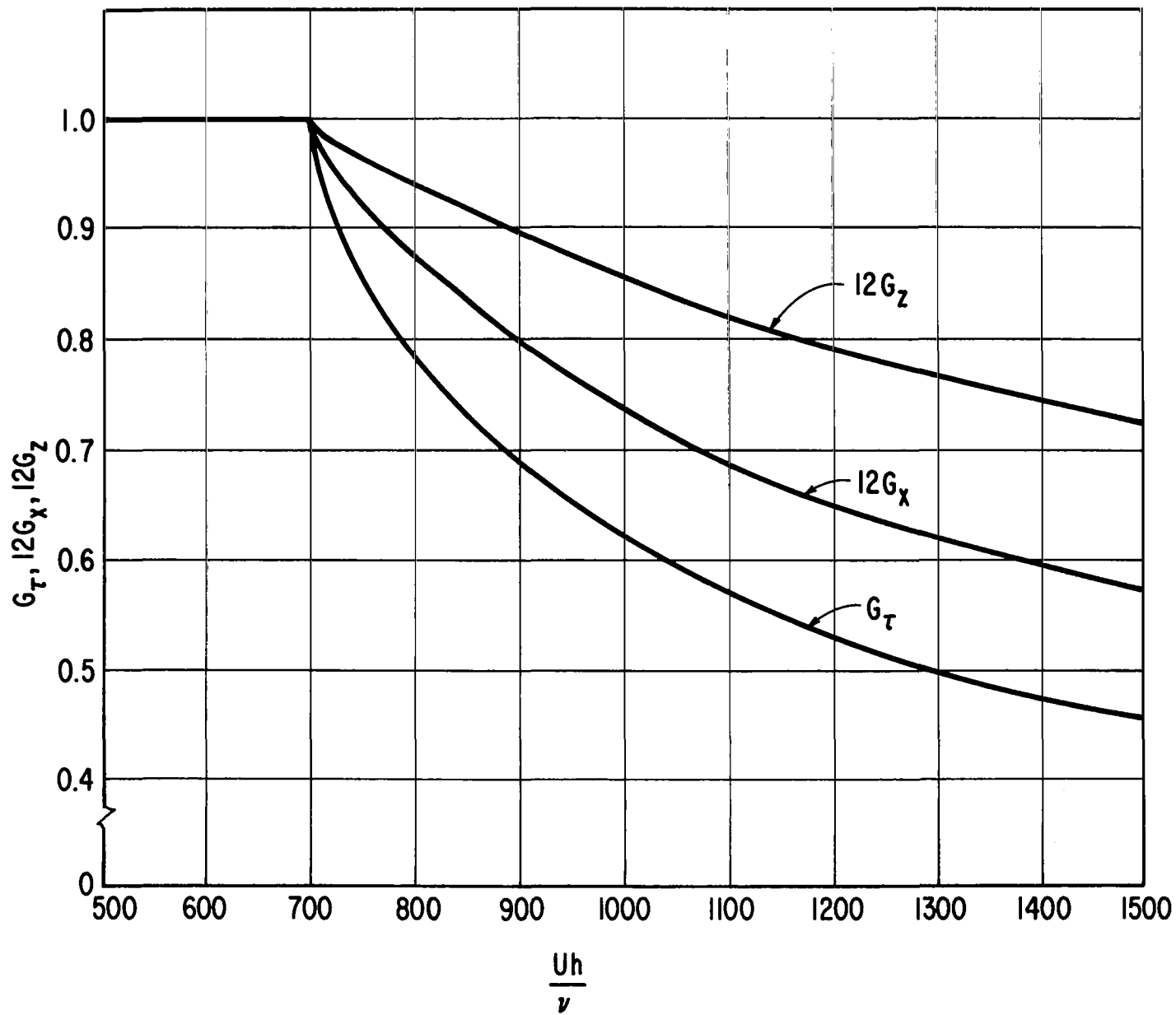


Fig. A-5a Universal Functions for Transition Region ( $700 < \frac{U_h}{\nu} < 1400$ ) Versus  $\frac{U_h}{\nu}$

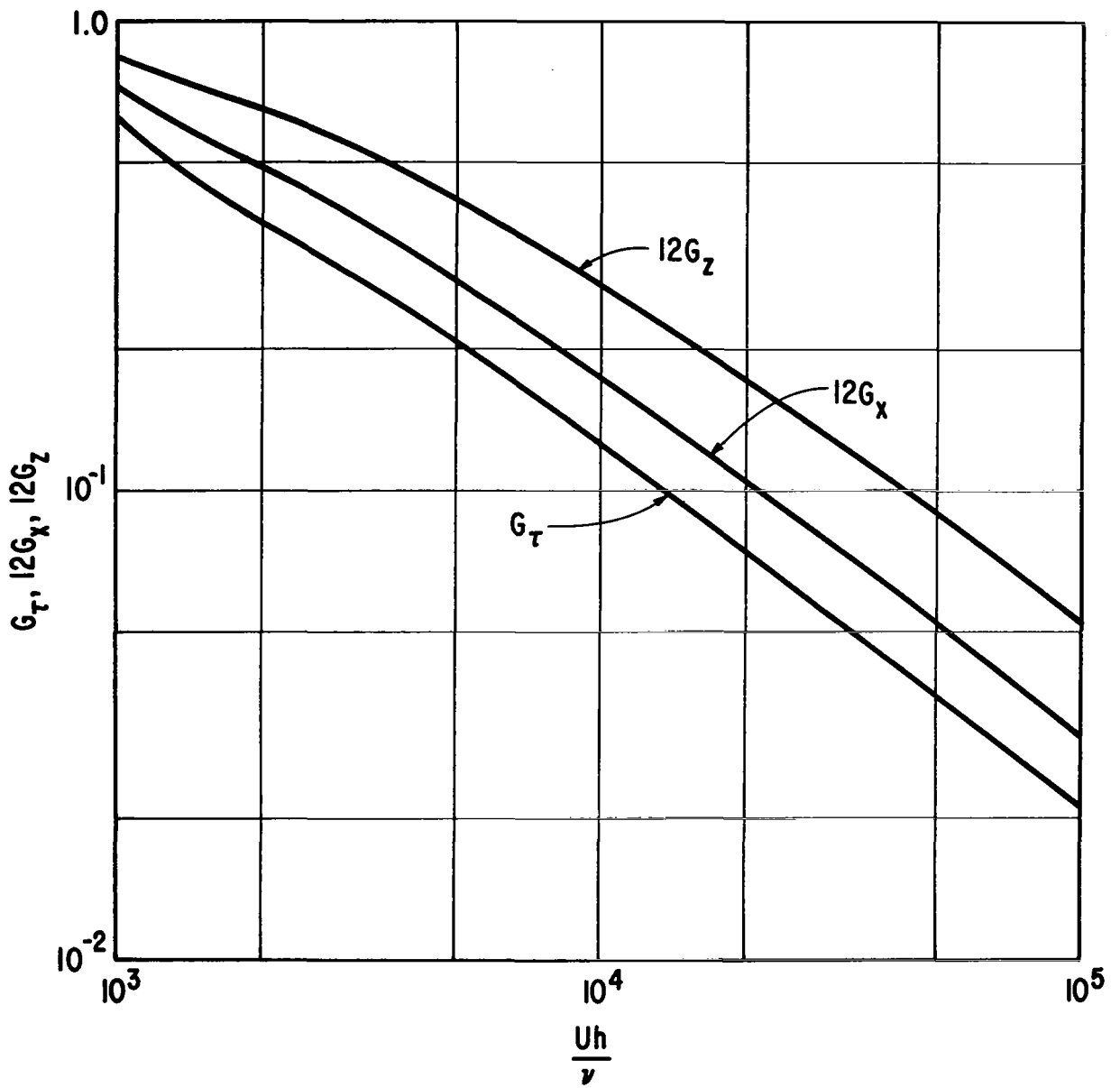


Fig. A-5b Universal Functions for Fully Turbulent Region ( $\frac{U_h}{\nu} > 1400$ ) Versus  $\frac{U_h}{\nu}$

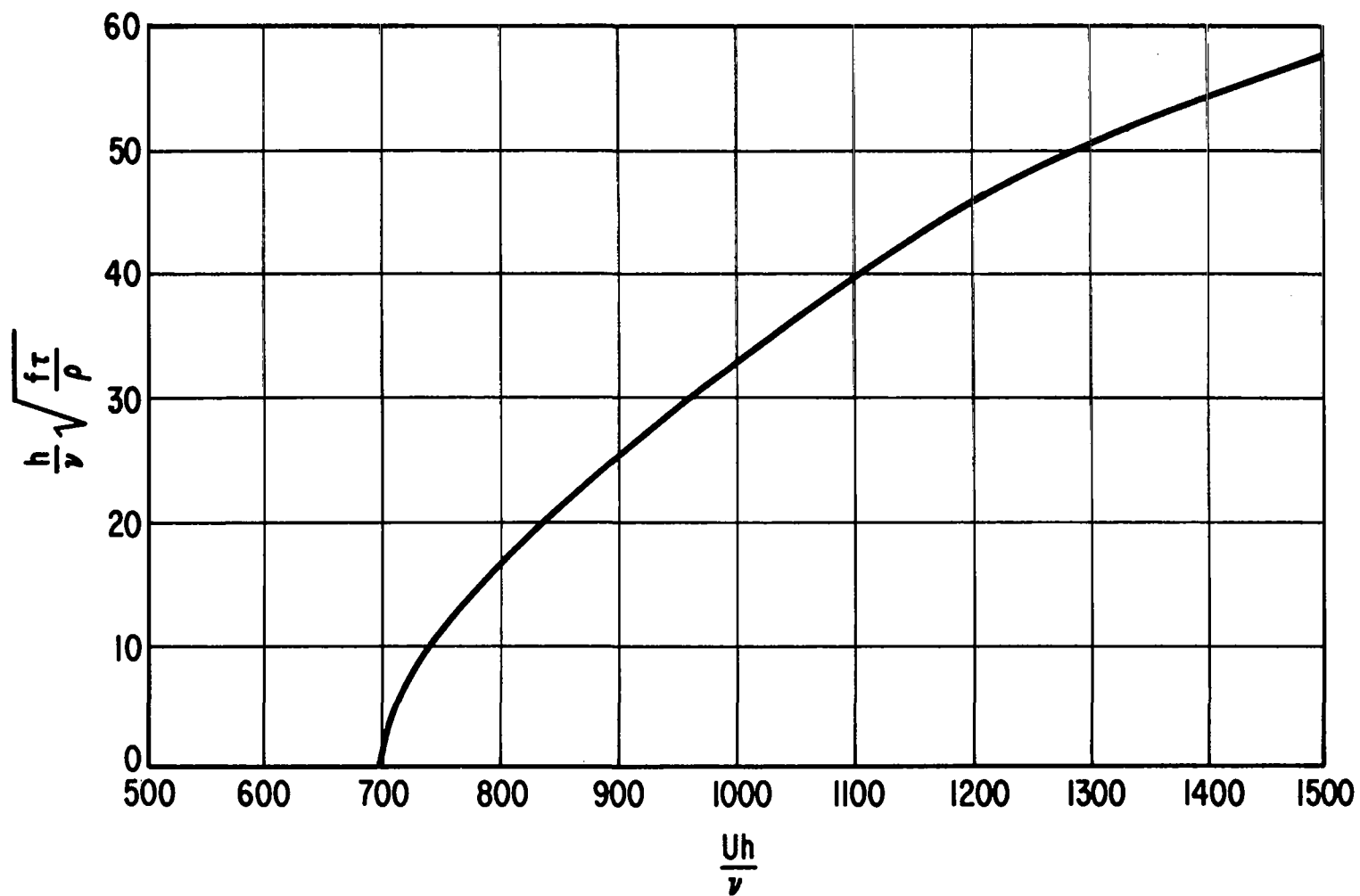


Fig. A-6a  $\frac{h}{v} \sqrt{\frac{f\tau}{\rho}}$  Versus  $\frac{Uh}{v}$  for Transition Region ( $700 < \frac{Uh}{v} < 1400$ )

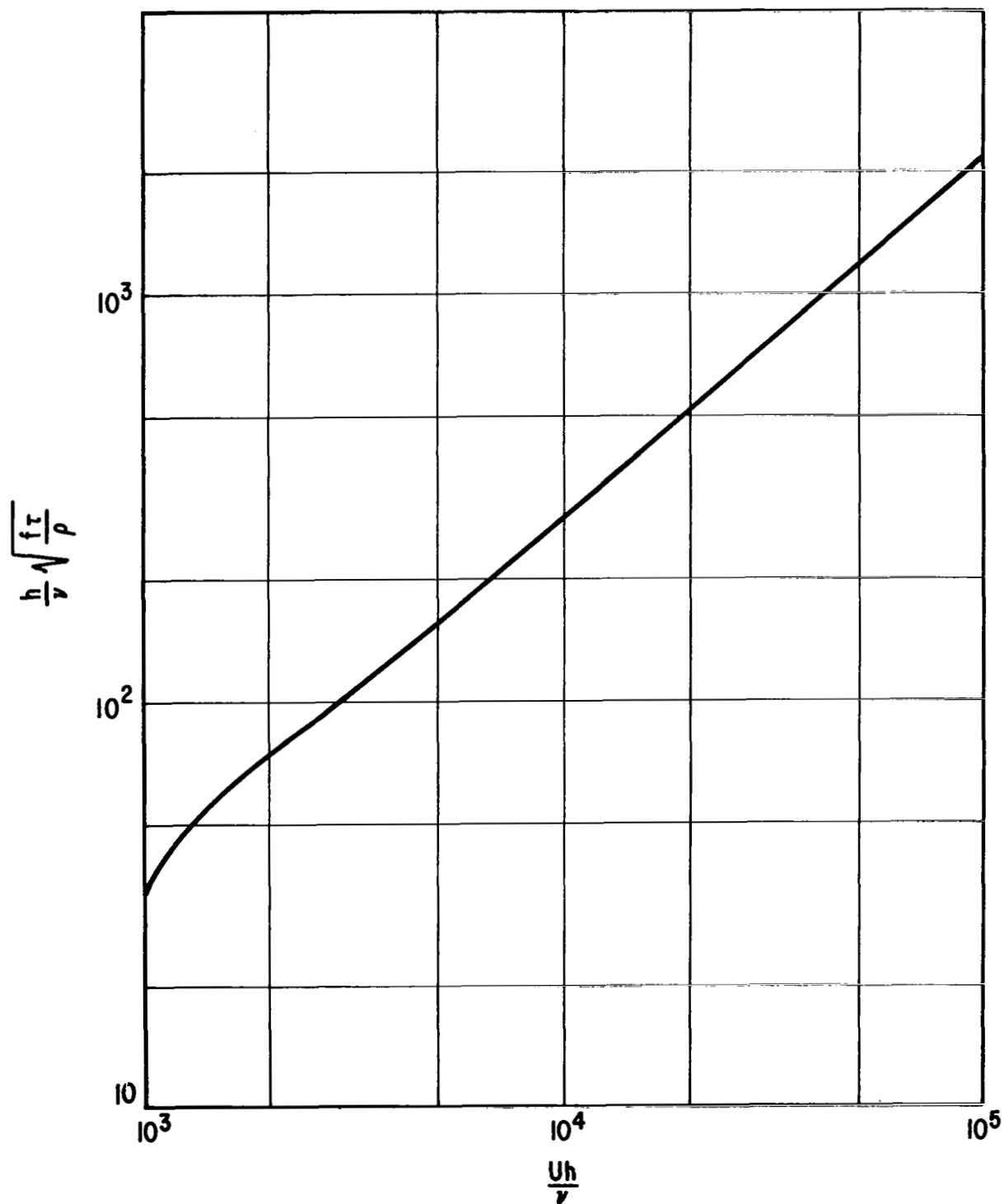


Fig. A-6b  $\frac{h}{v} \sqrt{\frac{f\tau}{\rho}}$  Versus  $\frac{Uh}{v}$  for Fully Turbulent Region ( $\frac{Uh}{v} > 1400$ )

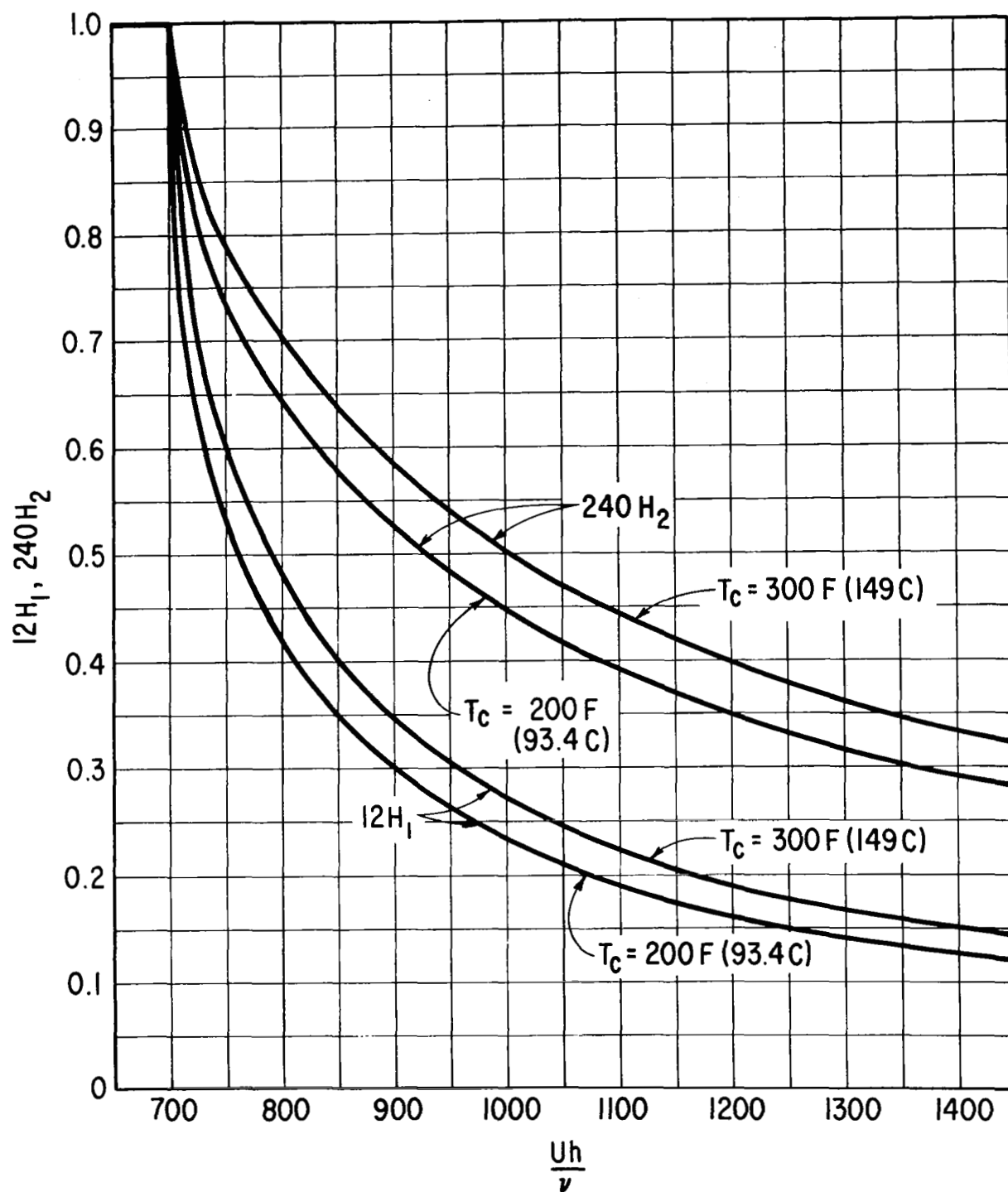


Fig. A-7a Universal Functions for Transition Region ( $700 < \frac{U_h}{\nu} < 1400$ )  
Versus  $\frac{U_h}{\nu}$  for MIL-L-7808 Oil at  $200^\circ\text{F}$  ( $93.4^\circ\text{C}$ ) and  $300^\circ\text{F}$  ( $149^\circ\text{C}$ )

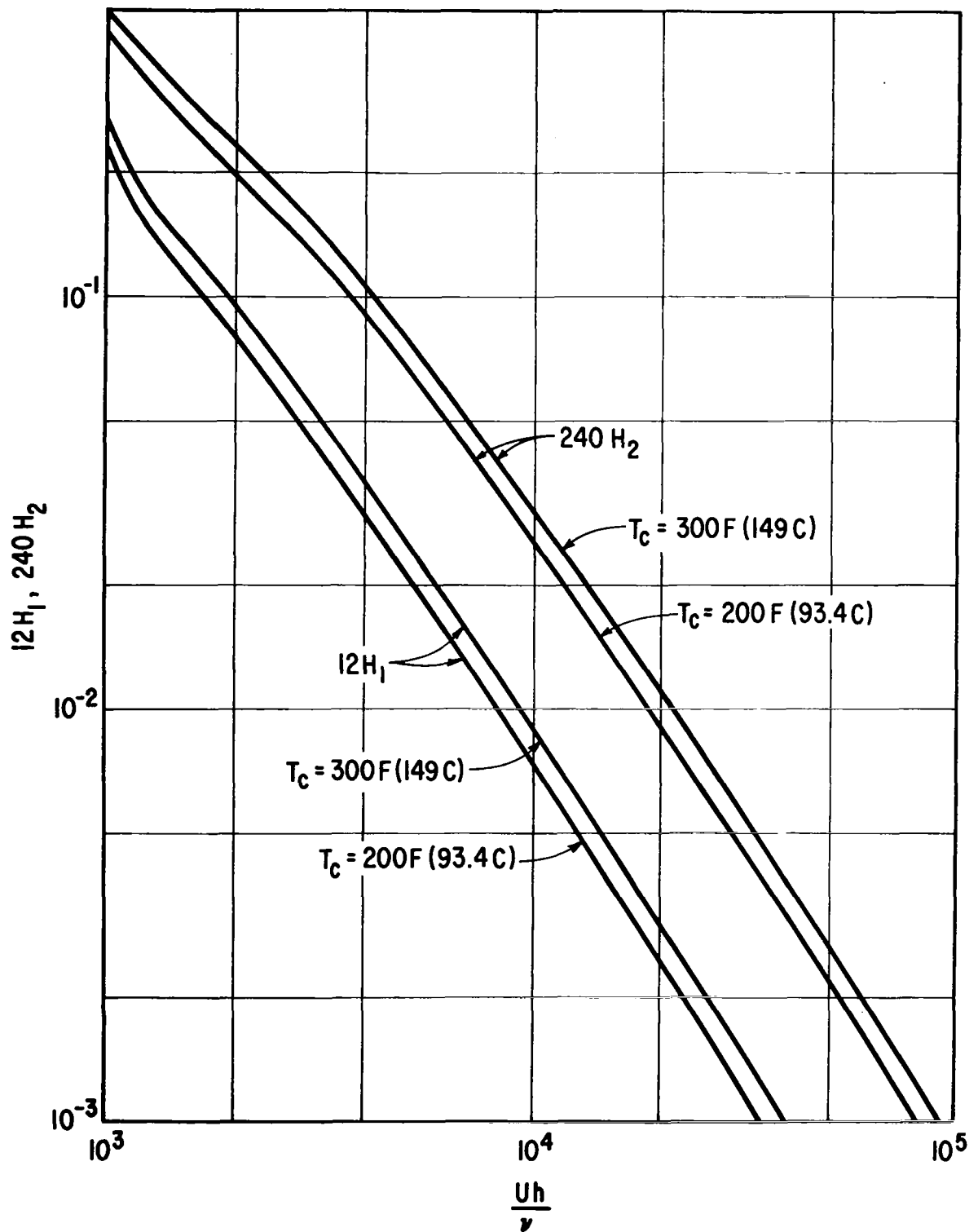


Fig. A-7b Universal Functions for Fully Turbulent Region ( $\frac{U_h}{\nu} > 1400$ )  
Versus  $\frac{U_h}{\nu}$  for MIL-L-7808 Oil at  $200^\circ\text{F} (93.4^\circ\text{C})$  and  $300^\circ\text{F} (149^\circ\text{C})$

$\Theta_i$  are functions of  $\frac{h}{v_i}$ ,  $\frac{f\tau_c}{\rho}$  as shown graphically in Figure A-4.

Equations (53) and (54) are to be solved with appropriate boundary conditions related to pressure and or flow. Equation (55) needs an initial condition for all points in the bearing film as well as inlet values at boundary entry points (the initial condition is obviously not needed for the steady-state problem). For the steady-state problem, an iterative procedure may be applied to converge on the desired solution. In the initial trial, the isothermal condition may be imposed:

$T_m = T_1' = T_2' = T_w$ . Using the corresponding viscosity,  $Uh/v$  is fixed, and all universal functions ( $G_\tau, G_x, G_z; H_1, H_2$ ) can be accordingly determined; then the system of equations are solved to determine  $(p, T_m)$  at all field points. Using the previous value of  $T_1'$  as an input, Equation (57) and Figure A-6 combined will yield improved estimates on  $T_1'$  and  $T_2'$ . Subsequently,  $(G_\tau, G_x, G_z)_{T_1'}$  can be recalculated

to prepare for the next round of iteration. The proper procedure for finding  $(G_\tau, G_z; H_1, H_2)_{T_2'}$  is somewhat subtle. First,  $(f, \tau_c)$  are determined from  $v_1'$ ; then, with  $v_2'$ ,  $\frac{h}{v_2'} \sqrt{\frac{f\tau_c}{\rho}}$  is calculated,

which in turn will yield a value of  $Uh/v$  according to Figure A-6. The latter is then used to determine  $(G_\tau, G_z; H_1, H_2)$  with Figures A-5 and A-7. Note that  $Uh/v$  is related to  $T_2'$  only indirectly through  $\frac{h}{v_2'} \sqrt{\frac{f\tau_c}{\rho}}$ . The improved universal functions provide a more accurate

basis for the calculation of  $(p, T_m)$ , and the successive iterations can be continued until sufficient convergence of the desired results is achieved.

## REFERENCES

1. Lund, J. W. and Arwas, E. B., "A Simultaneous Solution of the Lubrication and the Energy Equations for Turbulent Journal Bearing Films", MTI Report No. 64TR31 (R.1), 1964.
2. Hinze, J. O., Turbulence, McGraw-Hill Book Company, Inc., New York, New York, 1959.
3. Elrod, H. G. and Ng, C. W., "A Theory of Turbulent Fluid Films and Its Application to Bearings", ASME Trans., Series F, Journal of Lubrication Technology, Vol. 89, pp. 346-362, 1967.
4. Ng, C. W. and Pan, C. H. T., "A Linearized Turbulent Lubrication Theory", ASME Trans., Series D, Journal of Basic Engineering, Vol. 87, 1965, pp. 675-688.
5. Constantinescu, V. N., Hsing, F. C., and Pan, C. H. T., "Development of Long Life Jet Engine Thrust Bearings", Appendix A, MTI Report No. 70TR44, 1970.
6. Bootsma, J., "The Effect of Viscosity Variations with Temperature on the Performance of Spiral Groove Bearings". Trans. ASLE, Vol 12, No. 4, October 1969, pp.287-296.

## APPENDIX B

### TURBULENT FILM "WHIPPLE" THEORY WITH TEMPERATURE DEPENDENT VISCOSITY

#### 1.0 Review of Thin Film Theory

The analysis of an incompressible turbulent film with temperature dependent viscosity was given in [1]\*. The system of governing equations is summarized below:

$$\text{div} (\vec{u}_m h) + \frac{\partial h}{\partial t} = 0 \quad (1)$$

$$u_m = - \frac{h^2}{\mu_1} G_{x1} \frac{\partial p}{\partial x} + \frac{U}{2} \quad (2)$$

$$w_m = - \frac{h^2}{\mu_1} G_{z1} \frac{\partial p}{\partial z} \quad (3)$$

$G_x$  and  $G_z$  are known functions of  $Uh/\nu$  [2]; when  $\nu$  is evaluated at temperature  $T_i$ , they are designated as  $G_{xi}$  and  $G_{zi}$ .  $\mu_i$  is  $\mu(T_i)$ . Sensitivity to temperature is exhibited by having to identify  $T_i$ . Laminar-turbulent transition is governed by the numerical value of  $Uh/\nu$ , [1,3]:

| $\frac{Uh/\nu}{}$ | $\frac{f = V/U}{}$                                          |
|-------------------|-------------------------------------------------------------|
| $< R_{cr}$        | 0                                                           |
| $> R_{cr}, < R_f$ | $\{1 - [(R_f - Uh/\nu_1)/(R_f - R_{cr})]^2\}^{\frac{1}{2}}$ |
| $> R_f$           | 1                                                           |

The mean film temperature is governed by [1]:

$$\rho C_p h \left( \frac{\partial}{\partial t} + \vec{U}_m \cdot \nabla \right) T_m = -\Sigma q_w'' + \varphi \quad (4)$$

The wall heat flux,  $q_w''$ , and the dissipation heat,  $\varphi$ , have the ratio

$$\sigma = \frac{\Sigma q_w''}{\varphi} \quad (5)$$

---

\*Bracketed numbers refer to identically numbered references at the end of this Appendix.

Formulae for calculating  $q_w''$  and  $\varphi$  are

$$\Sigma q_w'' = \left( \frac{G_{\tau 2} + F_{p1} G_{z2}}{H_{\tau 2} + F_{p1} H_{p2}} \right) \frac{\kappa}{h} (T_m - T_w) \quad (6)$$

$$\varphi = \left( 1 + \frac{F_{p1} G_{z1}}{G_{\tau 1}} \right) \frac{\mu_1 U^2}{G_{\tau 1} h} \quad (7)$$

where

$$G_{\tau i} = \frac{\mu_i U}{h \tau_c} = 2 \left[ \int_0^{1/2} \frac{d\bar{y}}{(1 + \frac{\varepsilon}{v})} \right]_{T_i} = G_{\tau} \left( \frac{Vh}{v_i} \right) \quad (8)$$

$$H_{\tau i} = 2 \left[ \int_0^{1/2} \frac{\bar{y} d\bar{y}}{(1 + \frac{\varepsilon}{v} Pr)} \int_0^{\bar{y}} \frac{d\bar{y}'}{(1 + \frac{\varepsilon}{v})} \right]_{T_i} = H_{\tau} \left( \frac{Vh}{v_i}, \frac{\mu_i C_p}{\kappa} \right) \quad (9)$$

$$H_{pi} = 2 \left[ \int_0^{1/2} \frac{\bar{y} d\bar{y}}{(1 + \frac{\varepsilon}{v} Pr)} \int_0^{\bar{y}} \frac{\bar{y}'^2 d\bar{y}'}{(1 + \frac{\varepsilon}{v})} \right]_{T_i} = H_p \left( \frac{Vh}{v_i}, \frac{\mu_i C_p}{\kappa} \right) \quad (10)$$

$$F_{pi} = \left( \frac{h^2 G_{\tau i}}{\mu_i U} \right)^2 \left[ \left( \frac{\partial p}{\partial x} \right)^2 + \left( \frac{\partial p}{\partial z} \right)^2 \right] \quad (11)$$

$T_w$  and  $T_m$  are respectively wall and mean-film temperatures.  $T_i$  are established according to an "exact" analysis of the Couette problem to render the appropriate values of the dissipation heat and the Nusselt number, and are known numerically in the form

$$\left( \frac{T_i - T_w}{T_m - T_w} \right)$$

which depends on three parameters  $(T_w, \frac{Vh}{v_w}, \frac{\sigma h^2 \tau_h^2}{\mu_w \kappa T_w})$  for the particular lub-

ricant.  $\tau_h^2$  is

$$\tau_h^2 = \tau_c^2 + \left(\frac{h}{2}\right)^2 \left[ \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2 \right] \quad (12)$$

Numerical results, however, show that it is quite satisfactory to represent

$$\frac{T_i - T_w}{T_m - T_w} = \left(\frac{\tau_h}{\tau_c}\right)^2 \Theta_i \left(\frac{v_h}{v_1}\right) \quad (13)$$

which are more convenient to use. Graphical results of various universal functions,  $(G_T, G_X, G_Z; H_1, H_2)$ , are referred to [1].

## 2.0 Natural Coordinates

The geometrical pattern of the spiral grooves on a thrust bearing is given by the equation

$$\theta_g = \theta_o + \omega_1 t + \text{ctn}\beta \ln (r/r_o) + (m-1) \Delta\theta \quad (14)$$

which relates the instantaneous angular coordinate of the aft-side of the mth ridge with the radial coordinate. It is natural to investigate the pressure and temperature fields with the angular coordinate measured from  $\theta_g$ , i.e.

$$\xi = \theta - \theta_g \quad (15)$$

The coordinate system  $(r, \xi)$  is skewed, as illustrated in Fig. B-1. Fig. B-2 depicts a typical configuration and defines the choosen geometrical conventions. The transformation from the orthogunal system  $(r, \theta)$  to the skewed system  $(r, \xi)$  is:

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial t} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\text{ctn}\beta}{r} & 0 \\ 0 & 1 & 0 \\ 0 & -\omega_1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial t} \end{bmatrix} \quad (r, \xi) \quad (16)$$

Note that a time independent system in  $(r, \xi)$  would not be time independent in  $(r, \theta)$  when the grooved surface is rotating,  $\omega_1 \neq 0$ . Also, the circumferential velocity in the  $(r, \xi)$  system is  $U - \omega_1 r$ .

In the  $(r, \xi)$  system, the divergence operation is most naturally expressed as

$$\text{div } (\vec{A}) = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{\csc \beta}{r} \frac{\partial}{\partial \xi} (A_{\perp}) \quad (17)$$

$(A_r, A_{\perp})$  are components of the Vector  $\vec{A}$ ,  $A_r$  being the radial component, and  $A_{\perp}$  being that perpendicular to the direction of  $\xi = \text{constant}$ . The mean velocity vector  $\vec{U}_m$  accordingly would have the following components:

$$W_m = -\frac{h^2}{\mu_1} G_{z1} \left( \frac{\partial}{\partial r} - \frac{\cot \beta}{r} \frac{\partial}{\partial \xi} \right) p \quad (18)$$

$$U_m = -\frac{h^2}{\mu_1} G_{x1} \frac{1}{r} \frac{\partial p}{\partial \xi} + \frac{(\omega_2 - \omega_1)r}{2} \quad (19)$$

$$\begin{aligned} U_{\perp m} &= U_m \sin \beta - W_m \cos \beta \\ &= -\frac{h^2}{\mu_1} \left[ (G_{x1} \sin \beta + G_{z1} \cos \beta \cot \beta) \frac{1}{r} \frac{\partial}{\partial \xi} - G_{z1} \cos \beta \frac{\partial}{\partial r} \right] p + \frac{(\omega_2 - \omega_1)r}{2} \sin \beta \end{aligned} \quad (20)$$

The first term in the right hand side of Eq. (1) can now be written out as

$$\text{div } (\vec{U} h) = \frac{1}{r} \frac{\partial}{\partial r} (r W_m h) + \frac{\csc \beta}{r} \frac{\partial}{\partial \xi} (U_{\perp m} h) \quad (21)$$

### 3.0 Narrow Groove Approximations

The temperature dependent Whipple thrust bearing problem has been considered for the laminar case in the absence of groove-wise thermal gradient [6]. The objectives of the present analysis include additional considerations of turbulence and groove-wise thermal gradients. The analysis will utilize the narrow-groove approximation

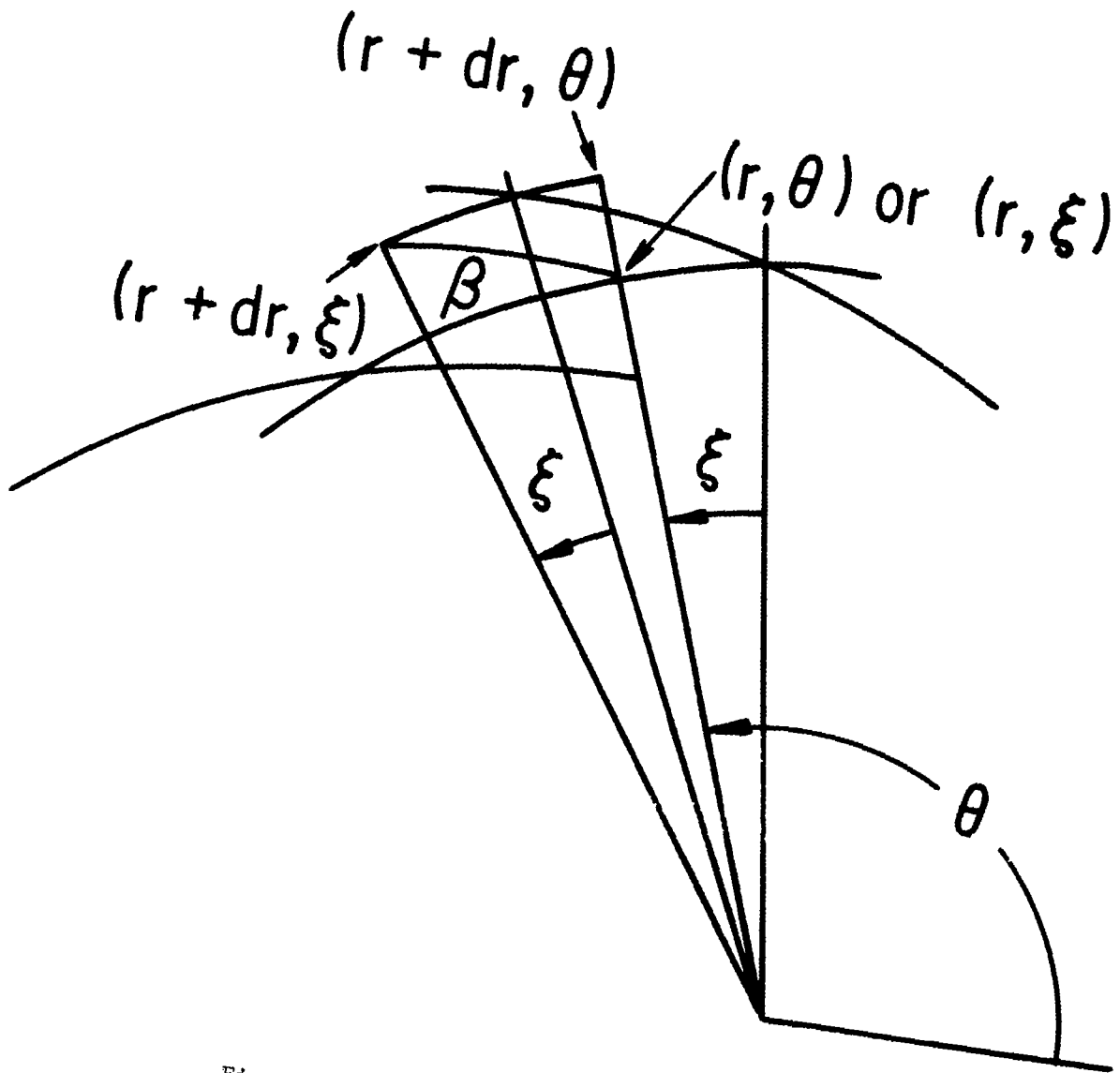
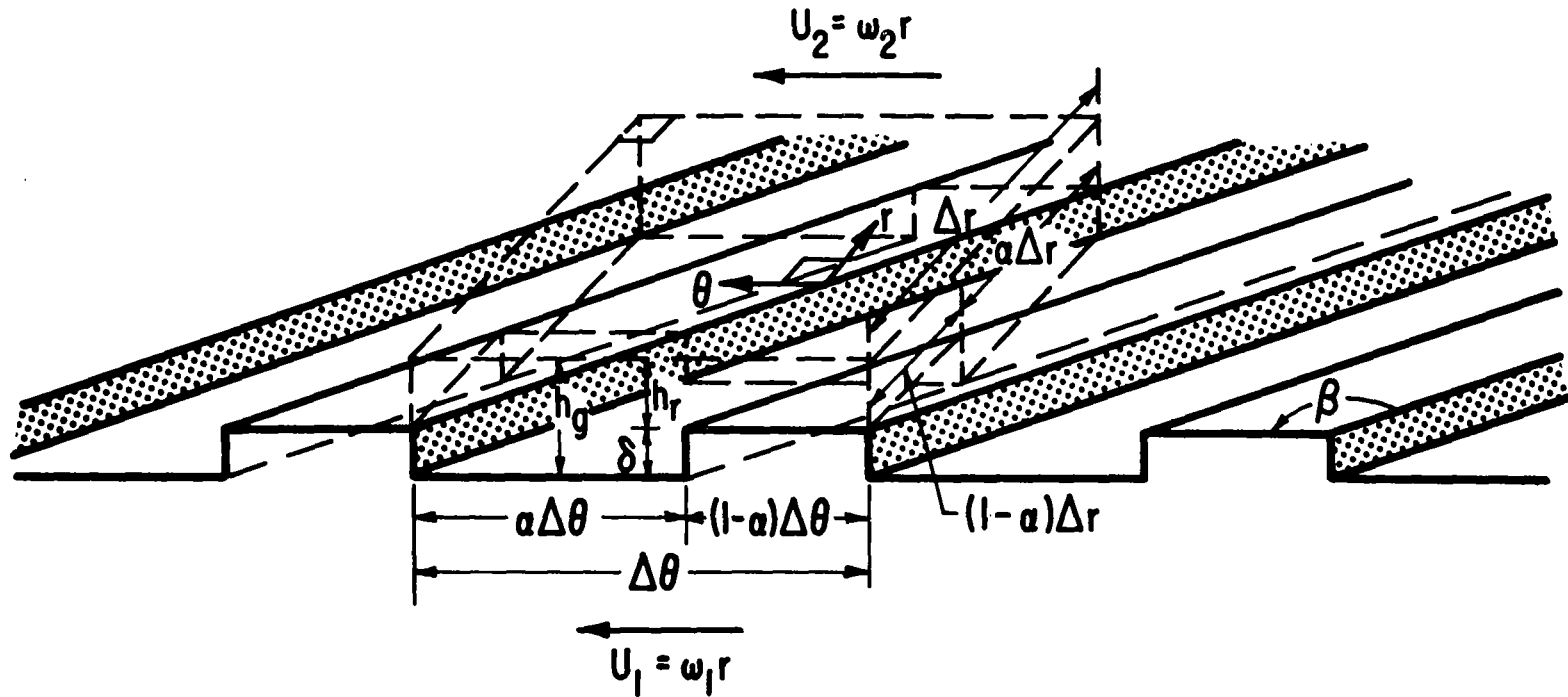


Fig. B-1 Skew Coordinate Representation



| DIMENSIONLESS GROOVE PARAMETERS |                    |
|---------------------------------|--------------------|
| WIDTH FRACTION                  | $a$                |
| SPIRAL ANGLE                    | $\beta$            |
| GAP RATIO                       | $\Gamma = h_g/h_r$ |

Fig. B-2 Spiral Groove Configuration and Macroscopic Control Volume

for the energy equation in a manner similar to Whipple's treatment of the pressure equation.

The basic premise of the narrow-groove approximation is to recognize different natural scales in the  $(r, \xi)$  system [4] for the pressure induced flux components. Thus, Eqs. (18) and (20) can be expressed as

$$\psi_{\perp} = r U_{\perp m} h = \left[ Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} \right] \sin \beta \quad (22)$$

$$\psi_r = r W_m h = Q_r$$

where

$$Q = -\frac{h^3}{\mu_1} \left[ (G_{x1} + G_{z1} \cot^2 \beta) \frac{\partial}{\partial \xi} - G_{z1} \cot \beta r \frac{\partial}{\partial r} \right] p \quad (23)$$

$$Q_r = -\frac{h^3}{\mu_1} G_{z1} \left( r \frac{\partial}{\partial r} - \cot \beta \frac{\partial}{\partial \xi} \right) p \quad (24)$$

Rewriting Eq. (21) with above expressions, one obtains

$$\text{div} (\vec{U}_m h) = \frac{1}{r^2} \left\{ r \frac{\partial Q_r}{\partial r} + \frac{\partial}{\partial \xi} \left[ Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} \right] \right\} \quad (25)$$

The fine scale of the microstructure is predicated by the term  $\partial h / \partial \xi$  in the above equation. Therefore the properly scaled dimensionless coordinates are

$$\zeta = \ln \frac{r}{r_0} \quad (26)$$

$$\eta = \frac{\xi}{\Delta \theta}$$

and Eq. (25) can be more meaningfully rewritten as

$$r^2 \Delta \theta \text{div} (\vec{U}_m h) = \frac{\partial}{\partial \eta} \left[ Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} \right] + \Delta \theta \frac{\partial Q_r}{\partial \zeta} \quad (27)$$

Thus, the governing differential equation becomes

$$\frac{\partial}{\partial \eta} \left[ Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} \right] + \Delta \theta \left( \frac{\partial Q_r}{\partial \zeta} + r^2 \frac{\partial h}{\partial t} \right) = 0 \quad (28)$$

Mathematical formalism of the narrow-groove approximation involves letting  $\Delta \theta \rightarrow 0$  while assuming boundedness of  $\left( \frac{\partial Q_r}{\partial \zeta} + r^2 \frac{\partial h}{\partial t} \right)$  in Eq. (28). Consequently

$$\frac{d}{d\eta} \left[ Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} \right] = 0 \quad , \quad (29)$$

or,

$$Q_{\perp} + \frac{(\omega_2 - \omega_1) r^2 h}{2} = \csc \beta \Psi_{\perp} \quad , \quad (30)$$

where  $\Psi_{\perp}$  depends only on the macrostructure coordinates  $(\zeta, \xi)$  which are regarded as invariants in the macrostructure problem of concern here.

The macrostructure problem can be treated by reverting back to the coarse-scale coordinates  $(\zeta, \xi)$  and averaging over one full period of the fine scale,  $\Delta \theta$ , the dependent variables. Before undertaking this step, it is useful to observe that the macrostructure problem is more naturally described in the earth-bound orthogonal coordinate system  $(r, \theta)$ . The transformation from  $(\zeta, \xi)$  back to  $(r, \theta)$  is the inversion of Eq. (16), namely;

$$\begin{bmatrix} \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial t} \end{bmatrix}_{(\zeta, \xi)} = \begin{bmatrix} 1 & \cot \beta & 0 \\ 0 & 1 & 0 \\ 0 & \omega_1 & 1 \end{bmatrix} \begin{bmatrix} r \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial t} \end{bmatrix}_{(r, \theta)} \quad (31)$$

Rewriting Eq. (28) in coarse-scale, earth-bound, orthogonal coordinates,

$$\frac{\partial}{\partial \theta} \left[ Q_{\perp} + \cot \beta Q_r + \frac{(\omega_1 + \omega_2) r^2 h}{2} \right] + r \frac{\partial Q_r}{\partial r} + r^2 \frac{\partial h}{\partial t} = 0 \quad (32)$$

Averaging over one groove-ridge pair,

$$\frac{\partial}{\partial \theta} \left[ \bar{Q}_{\theta} + \frac{(\omega_1 + \omega_2)}{2} r^2 \bar{h} \right] + r \frac{\partial \bar{Q}_r}{\partial r} + r^2 \frac{\partial \bar{h}}{\partial t} = 0 \quad (33)$$

where,

$$(\bar{\phantom{x}}) \equiv \int_0^1 (\phantom{x}) d\eta \quad (34)$$

$$Q_{\theta} = Q_{\perp} + \cot \beta Q_r = -\frac{h^3}{\mu_1} G_{x1} \frac{\partial p}{\partial \xi} \quad \bullet$$

The macrostructure and microstructures are further related to one another by the following relations:

$$\frac{\partial \bar{p}}{\partial \theta} = \frac{1}{\Delta \theta} \int_0^1 \frac{\partial p}{\partial \eta} d\eta \quad (35)$$

$$r \frac{\partial \bar{p}}{\partial r} = \frac{\partial p}{\partial \xi} - \cot \beta \frac{\partial \bar{p}}{\partial \theta} \quad (36)$$

Since  $\bar{p}$  depends only on coarse-scale coordinates, Eq. (36) indicates that  $\frac{\partial p}{\partial \xi}$  should be treated as an invariant in the micro-structure problem.

Returning to the microstructure problem, Eq. (23) can be integrated with respect to  $\eta$  with the aid of Eq. (22), keeping in mind that  $\psi_{\perp}$  and  $\frac{\partial p}{\partial \xi}$  should be treated as invariants,

$$p = \bar{p} + \Delta \theta \int_0^1 \left\{ \mu_1 \left[ \frac{(\omega_2 - \omega_1) r^2}{2h^2} - \csc \beta \frac{\psi_{\perp}}{h^3} \right] + \cot \beta \frac{\partial p}{\partial \xi} G_{z1} \right\} \frac{d\eta}{(G_{x1} + G_{z1} \cot^2 \beta)} \quad \bullet \quad (37)$$

Use of  $\bar{p}$  as the lower limit integration constant identifies the macropressure field with the pressure at the aft-side of each ridge. Links between micro- and macrostructures according to Eq. (34) are

$$\bar{Q}_\theta = -\frac{1}{\Delta\theta} \int_0^1 \frac{h}{\mu_1} G_{x1} \frac{\partial p}{\partial \eta} d\eta \quad (38)$$

$$\bar{Q}_r = -\frac{\partial p}{\partial \zeta} \int_0^1 \frac{h^3}{\mu_1} G_{z1} d\eta + \frac{\text{ctn}\beta}{\Delta\theta} \int_0^1 \frac{h^3}{\mu_1} G_{z1} \frac{\partial p}{\partial \eta} d\eta \quad (39)$$

Also, from Eq. (30) one obtains

$$\psi_\perp = \sin\beta \bar{Q}_\theta - \cos\beta \bar{Q}_r + \frac{(\omega_2 - \omega_1)}{2} r^2 h \sin\beta \quad (40)$$

Summarizing, one may outline the conceptual computation procedure of the narrow-groove approximation to consist of three steps:

- Given macrostructure variables  $(\bar{Q}_\theta, \bar{Q}_r)$ , one seeks the micro-structure solution. Eq. (40) can be used to find  $\psi_\perp$  immediately. Next,  $\frac{\partial p}{\partial \zeta}$  can be found by eliminating  $\frac{\partial p}{\partial \zeta}$  and  $Q_\perp$  between Eqs. (22), (23), (24), then integrating across, over one groove-ridge width, and using symbols consistent with Eqs. (26) and (34):

$$\frac{\partial p}{\partial \zeta} = \frac{\sin^2\beta \left\{ \int_0^1 \frac{\text{ctn}\beta G_{z1}}{G_{x1} + \text{ctn}^2\beta G_{z1}} \left[ \left( \frac{\omega_2 - \omega_1}{2} \right) r^2 h - \text{csc}\beta \psi \right] d\eta - \bar{Q}_r \right\}}{\int_0^1 \frac{h^3}{\mu_1} \frac{G_{x1} G_{z1}}{G_{x1} + \text{ctn}^2\beta G_{z1}} d\eta} \quad (41)$$

Subsequently, Eq. (37) can be used to calculate  $p$ .

- Eq. (33) can be integrated to yield  $(\bar{Q}_\theta, \bar{Q}_r)$  at new coarse-scale locations.
- Eqs. (35) and (36) now permit calculation of  $\bar{p}$  at new coarse-scale locations.

### 3.1 Pressure Problem

The isoviscous turbulent Whipple problem has been considered [7]. The pressure field can be regarded to display a fine structure  $(p_g, p_r)$ , representing fluctuation corresponding to the presence of grooves, and a coarse structure,  $\bar{p}$ , reflecting the macroscopic group-action of groove-ridge pairs. With turbulence, the fine structure would depend on the local Reynolds numbers  $(\frac{U_{hg}}{\nu}, \frac{U_{hr}}{\nu})$ , where  $U = |\omega_2 - \omega_1| r$ . Temperature dependence of viscosity further requires the Reynolds number to be spatially varying even in the fine scale. The appropriate relations will be developed with the latter feature.

### 3.2 Temperature Problem

To allow for the influence of temperature on viscosity, the temperature problem must be treated again with both micro- and macrostructure view points. The principal objective is to determine the distribution of  $\mu$ , as a function of the fine-scale coordinate,  $\eta$ , at each coarse-scale location  $(r, \theta)$ .

#### 3.2.1 The General Narrow-Groove Approximation

The essence of the narrow-groove approximation in the temperature problem is to neglect the transient enthalpy change in comparison with the convective enthalpy transfer in the microstructure; i.e.

$$\left| h \frac{\partial T_m}{\partial t} \right| < \left| \frac{\bar{\Psi}}{r} \cdot \nabla T_m \right| \quad (42)$$

To emphasize the quasi-static assumption, in place of Eq. (4), the governing equation for the fine structure temperature problem can be rewritten as

$$\left[ \rho C_p \frac{\bar{\Psi}}{r} \cdot \nabla + \Delta \theta \left( \frac{G_{r2} + F_{p1} G_{z2}}{H_{r2} + F_{p1} H_{p2}} \right) \frac{\kappa}{h} \right] (T_m - T_w) = \Delta \theta \left( 1 + \frac{F_{p1} G_{z1}}{G_{r1}} \right) \frac{\mu_1 U^2}{G_{r1} h} \quad (43)$$

$$\text{where } \vec{\nabla} \equiv \Delta\theta \nabla . \quad (44)$$

Now, since the convective derivative can be spatially integrated along the flux line as a total differential, and, in particular, one has

$$\left| \frac{\vec{\Psi}}{ds} \right| = \frac{\Psi_{\perp}}{dn_s} = \frac{\Psi_r}{dr} \quad (45)$$

where  $ds$  is a distance element along the flux line and  $(dn_s, dr)$  are its projections, respectively, on a line normal to the groove sides, and on a radial line, then the convective derivative can be expressed as

$$\begin{aligned} \vec{\Psi} \cdot \vec{\nabla} (T_m - T_w) &= |\vec{\Psi}| \frac{d}{ds} (T_m - T_w) \Delta\theta \\ &= \Psi_{\perp} \frac{d}{dn_s} (T_m - T_w) \Delta\theta \\ &= \frac{\csc\beta \Psi_{\perp}}{r} \frac{d}{d\eta} (T_m - T_w) . \end{aligned} \quad (46)$$

$\frac{d}{d\eta} (T_m - T_w)$  is understood to designate

$$\lim_{ds \rightarrow 0} \frac{(T_m - T_w) \Big|_{s+ds} - (T_m - T_w) \Big|_s}{\eta \Big|_{s+ds} - \eta \Big|_s} . \quad (47)$$

Geometrical interpretations of these concepts are shown in Fig. B-3. The advantage of expressing the convective derivative along  $\Psi_{\perp}$  is in its invariance in the microstructure problem. Thus Eq. (43) can be rewritten as

$$\left( \frac{d}{dn} + N \right) (T_m - T_w) = \Phi \quad (48)$$

where,

$$N = \frac{r^2 \Delta\theta}{\rho C_p \csc\beta \Psi_{\perp}} \left( \frac{G_{\tau 2} + F_{p1} G_{z2}}{H_{\tau 2} + F_{p1} H_{p2}} \right) \frac{\kappa}{h} \quad (49)$$

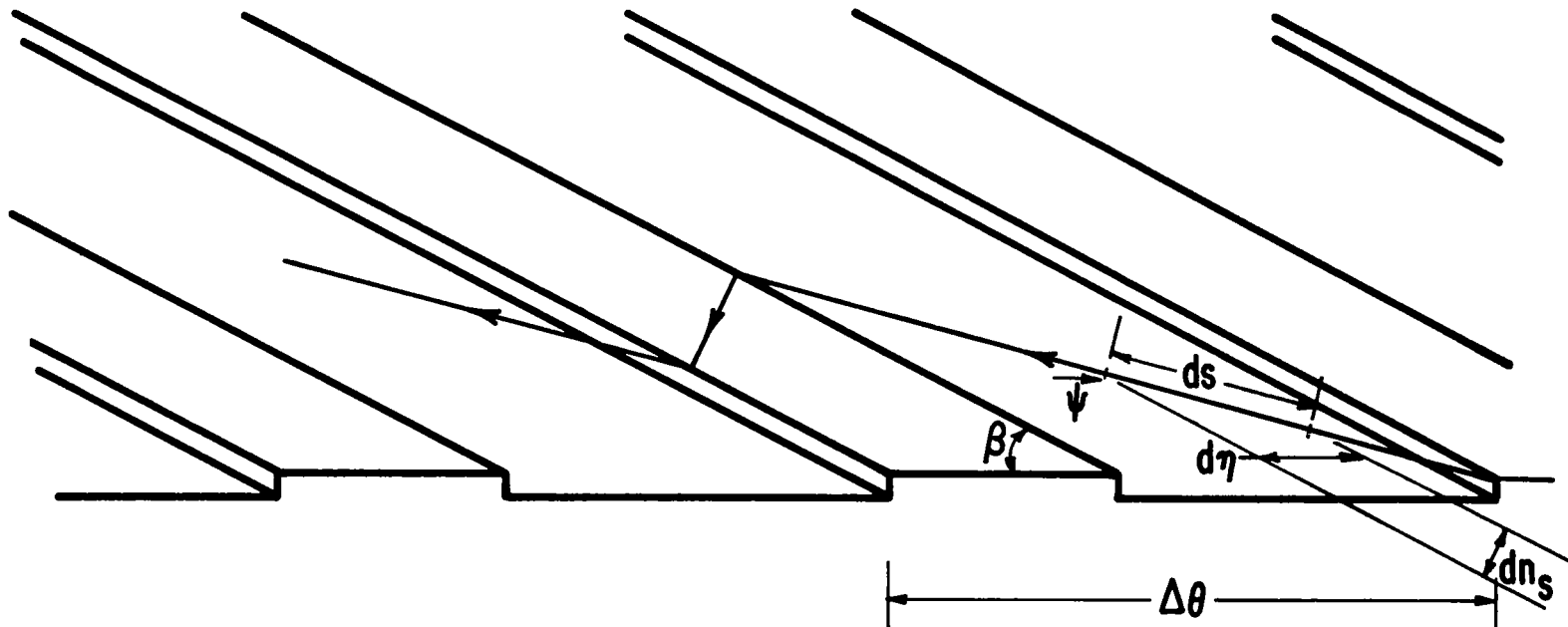


Fig. B-3 Streamline and Convective Path

$$\Phi = \frac{r^2 \Delta \theta}{\rho C_p \csc \beta \Psi_{\perp}} \left( 1 + \frac{F_{p1} G_{z1}}{G_{r1}} \right) \frac{\mu_1 U^2}{G_{r1} h} \quad (50)$$

The link between micro- and macrostructures of the temperature is provided by the relation between the increment of  $T_m$  and the flux trajectory spanning the width of one groove-ridge pair. The increment of  $T_m$  is

$$\Delta T_{mo} = \int_0^1 \frac{d}{d\eta} (T_m - T_w) d\eta \quad (51)$$

The corresponding radial distance traversed by the flux line segment is, on account of Eq. (45)

$$\begin{aligned} \Delta r &= \frac{r \Delta \theta}{\csc \beta \Psi_{\perp}} \int_0^1 \bar{\Psi}_r d\eta \\ &= \frac{r \Delta \theta \bar{\Psi}_r}{\csc \beta \Psi_{\perp}} \quad (52) \end{aligned}$$

The macrostructure temperature convection derivative can be defined as

$$\begin{aligned} &\int_0^1 \rho C_p \frac{\bar{\Psi}}{r} \cdot \nabla T_m d\eta \\ &= \frac{\rho C_p \csc \beta \Psi_{\perp}}{r^2 \Delta \theta} \int_0^1 \frac{dT_m}{d\eta} d\eta \\ &= \frac{\rho C_p \csc \beta \Psi_{\perp}}{r^2 \Delta \theta} \Delta T_m \quad . \end{aligned}$$

Eliminating  $\Psi_{\perp}$  by means of Eq. (52)

$$\int_0^1 \rho C_p \frac{\bar{\Psi}}{r} \cdot \nabla T_m d\eta = \rho C_p \frac{\bar{\Psi}_r}{r} \frac{\Delta T_m}{\Delta r}$$

and since  $\Delta r$  may be regarded as infinitesimal in the macrostructure problem,

one can define

$$\frac{\partial T_{mo}}{\partial r_s} \equiv \frac{\Delta T_{mo}}{\Delta r} \quad (53)$$

Consequently,

$$\int_0^1 \rho C_p \frac{\bar{\psi}}{r} \cdot \nabla T_m d\eta = \rho C_p \frac{\bar{\psi}}{r} \frac{\partial T_{mo}}{\partial r_s} \quad (54)$$

And the locus of the (macrostructure) flux trajectory is, rewriting Eq. (52),

$$\frac{dr_s}{d\xi} = \frac{r \bar{\psi}_r}{\csc \beta \psi_{\perp}} \quad (55)$$

Average each term in Eq. (4) over the width of one groove-ridge pair, one obtains

$$\int_0^1 h \frac{\partial T_m}{\partial t} d\eta + \frac{\bar{\psi}}{r} \frac{\partial T_{mo}}{\partial r_s} + \frac{\csc \beta \psi_{\perp}}{r^2 \Delta \theta} \int_0^1 [N(T_m - T_w) - \Phi] d\eta = 0 \quad (56)$$

In Eqs. (53), (54), and (56), a subtle distinction between the macrostructure point of view and that of the microstructure appears in the use of the partial derivative symbol for the convection term. Thus,  $\partial T_{mo} / \partial r$  should not be evaluated directly from Eq. (51).

There may be some advantage to compute the macrostructure with the divergence formulation. That is, the left hand side of Eq. (4) for an incompressible fluid can be replaced by

$$\rho C_p \left[ \frac{\partial}{\partial t} \{h(T_m - T_w)\} + \text{div} \{\bar{U}_m h (T_m - T_w)\} \right]$$

Making use of Eq. (17), then reverting to the earth-bound coordinate system, instead of Eq. (56), one can write

$$\frac{\partial}{\partial t} \int_0^1 h(T_m - T_w) d\eta + \frac{1}{r} \frac{\partial}{\partial r} \int_0^1 Q_r (T_m - T_w) d\eta + \frac{\partial}{\partial \theta} \int_0^1 \left[ \frac{Q_\theta}{r^2} + \right. \quad (57)$$

$$\left. \frac{(\omega_1 + \omega_2)}{2} h \right] (T_m - T_w) d\eta + \frac{\csc \beta \psi_\perp}{r^2 \Delta \theta} \int_0^1 [N(T_m - T_w) - \Phi] d\eta = 0 .$$

In any case, regardless whether Eq. (56) or (57) should be used, the micro-structure solution is needed to compute Eqs. (51), (52), (53) as well as various integrals in Eq. (56) or (57).

The microstructure solution according to Eq. (48) is known in closed form [8]:

$$T_m - T_w = \exp \left\{ - \int N d\eta \right\} \left[ (\bar{T}_m - T_w) + \int \exp \left\{ \int N d\eta \right\} \Phi d\eta \right] . \quad (58)$$

It is understood that all lower limits of integration are at  $\eta=0$ .  $\bar{T}_m$  is the value of  $T_m$  at  $\eta=0$ . It is important to recall that Eq. (58) describes the fluid film temperature along a flux line which starts from  $(r, \xi)$  and arrives at a congruent groove line at  $(r + \Delta r, \xi + \Delta \theta)$  with  $(\Delta r, \Delta \theta)$  related to each other by Eq. (52). In general, Eq. (58) is an initial-value type solution, rendering

$$(T_m - \bar{T}_m)_\psi = -(\bar{T}_m - T_w) \left[ 1 - \exp \left\{ - \int_0^1 N d\eta \right\} \right] + \exp \left\{ - \int_0^1 N d\eta \right\} \int_0^1 (\Phi \exp \left\{ \int_0^\eta N d\eta \right\}) d\eta . \quad (59)$$

Dependence of this expression on  $\bar{T}_m$  shows up not only explicitly in  $(\bar{T}_m - T_w)$ , but also implicitly in  $N$  and  $\Phi$ . Calculating  $\bar{T}_m$  according to Eq. (59) across the width of one groove-ridge pair, one obtains

$$\Delta T_{mo} = -(\bar{T}_m - T_w) \left[ 1 - \exp \left\{ - \int_0^1 N d\eta \right\} \right] + \exp \left\{ - \int_0^1 N d\eta \right\} \int_0^1 (\Phi \exp \left\{ \int_0^\eta N d\eta \right\}) d\eta . \quad (60)$$

According to the narrow-groove approximation, Eqs. (52) and (60) yield

$$\left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} = \frac{\csc \beta \bar{\Psi}_{\perp}}{\bar{Q}_r} \frac{\Delta T_{mo}}{r \Delta \theta} \quad (61)$$

Eq. (48) can be formally differentiated with respect to  $r$  (constant  $\xi$  or  $\eta$ ) then integrated with respect to  $\eta$  to obtain

$$\left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} = \exp \left\{ - \int_0^{\eta} N d\eta' \right\} \left[ \left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} + \int_0^{\eta} \left\{ \left. \frac{\partial \Phi}{\partial r} \right|_{\xi} - (T_m - T_w) \left. \frac{\partial N}{\partial r} \right|_{\xi} \right\} \exp \left\{ \int_0^{\eta'} N d\eta'' \right\} d\eta' \right] \quad (62)$$

Consequently, one can compute

$$(T_m - \bar{T}_m)_{\text{const. } r} = (T_m - \bar{T}_m)_{\bar{\Psi}} - \frac{r \Delta \theta}{\csc \beta \bar{\Psi}_{\perp}} \left. \frac{\partial T_m}{\partial r} \right|_{\xi} \int_0^{\eta} \bar{Q}_r d\eta' \quad (63)$$

An anomaly of Eq. (63) exists at the condition of  $\bar{\Psi}_r = 0$ , when the average radial convective flux vanishes. Physically, this means the initial temperature  $\bar{T}_m$  can no longer be separately specified. Instead  $T_m$  must be totally periodic, so that one must impose

$$\begin{aligned} \Delta \bar{T}_m &= f_s \left( r, \xi; \eta=1; \bar{T}_m(r, \xi) \right) \Big|_{\bar{\Psi}_r=0} \\ &= -(\bar{T}_m - T_w) \left[ 1 - \exp \left\{ - \int_0^1 N d\eta \right\} \right] + \exp \left\{ - \int_0^1 N d\eta \right\} \int_0^1 \exp \left\{ \int N d\eta \right\} \Phi d\eta \\ &= 0, \end{aligned}$$

or,

$$(\bar{T}_m - T_w)_{\bar{\Psi}_r=0} = \frac{\exp \left\{ - \int_0^1 N d\eta \right\} \int_0^1 \exp \left\{ \int_0^{\eta} N d\eta' \right\} \Phi d\eta}{1 - \exp \left\{ - \int_0^1 N d\eta \right\}} \quad (64)$$

### 3.2.2 The Thrust Bearing Problem

The thrust bearing problem imposes rotational symmetry on the macrostructure and periodicity in the microstructure. The latter, because of the convective character of the governing equation, however, is implicit and is displayed primarily in the  $\theta$ -independence of  $\bar{Q}_r$ ,  $\bar{Q}_\theta$ ,  $\bar{h}$ ,  $\bar{\psi}_L$ , and  $\bar{\Phi}$ . The single exception in the case of  $\bar{\psi}_r=0$  has already been discussed. Here microperiodicity must be imposed even when there is  $\theta$ -dependence in the macrostructure.

The microstructure problem is still governed by Eqs. (59), (60), (61), (62), (63), and (64). The macrostructure problem is now reduced to

$$\frac{\partial}{\partial t} \int_0^1 h(T_m - T_w) d\eta + \frac{1}{r} \frac{\partial}{\partial r} \int_0^1 Q_r(T_m - T_w) d\eta + \frac{\csc\beta \bar{\psi}_L}{r^2 \Delta\theta} \int_0^1 [N(T_m - T_w) - \bar{\Phi}] d\eta = 0. \quad (65)$$

From Eqs. (48), (51), (52), and (53),

$$\begin{aligned} & \frac{\csc\beta \bar{\psi}_L}{r^2 \Delta\theta} \int_0^1 [N(T_m - T_w) - \bar{\Phi}] d\eta \\ &= - \frac{\csc\beta \bar{\psi}_L}{r^2 \Delta\theta} \int_0^1 \frac{d}{d\eta} (T_m - T_w) \Big|_{\text{flux line}} d\eta \\ &= - \frac{\csc\beta \bar{\psi}_L}{r^2 \Delta\theta} \Delta \bar{T}_m \\ &= - \frac{\bar{\psi}_r}{r} \frac{d\bar{T}_m}{dr_s} \\ &= - D(r, t). \end{aligned} \quad (66)$$

This relation simplifies Eq. (65) somewhat. The general problem requires specification of the initial condition

$$\bar{T}_m = (r, t = 0)$$

and the influx boundary condition

$$\bar{T}_m(r_i, t > 0)$$

(if  $\bar{\Psi}_r \neq 0$ ). If  $\bar{\Psi}_r = 0$ , because  $\Delta\bar{T}_m = 0$ ,  $D = 0$ .

The procedure to solve the steady-state thrust bearing problem can be further examined in some details. Substituting Eq. (66) into Eq. (65) and dropping the time-dependent term,

$$\frac{1}{r} \frac{d}{dr} \int_0^1 Q_r (T_m - T_w) d\eta = D(r) \quad (67)$$

Starting with the influx boundary condition  $\bar{T}_m(r_i)$ , Eq. (59) can be calculated. Subsequently, utilizing Eqs. (60), (61), (62), and (63), the quadrature

$$C(r) = \int_0^1 Q_r (T_m - T_w) d\eta \quad (68)$$

can be performed at  $r_i$ .  $D(r_i)$  can also be calculated from Eqs. (59) and (66). Eq. (67) can now be integrated to obtain  $\bar{T}_m(r_i + \Delta r)$  which in turn determines  $\bar{T}_m(r_i + \Delta r)$  through Eqs. (59), (60), (61), (62), (63), and (68). The procedure can be continued until  $T_o(r)$  as well as  $T_m(r_i, 0 \leq \eta \leq 1)$  are determined for all  $r$  of interest. If  $\bar{\Psi}_r = 0$ , then  $D(r) = 0$ , and  $\Delta\bar{T}_m = 0$ . A special procedure must be devised to calculate Eq. (63).

### 3.3 Summary of the Pressure-Temperature Problem for the Steady-State Operation of Thrust Bearings

The foregoing derivations are comprised of the elements required to include heat-transfer and turbulence (including transition effects) considerations into the computation of spiral-grooved bearings in a manner consistent with the narrow groove approximation. The steady-state operation of thrust bearings is of immediate interest. To provide the basis for coding the required computer program,

the relevant equations are reiterated in the following.

Conceptually, two major steps are involved. The first one concerns the simultaneous treatment of pressure and temperature problems. An iterative process will be followed, beginning with an initial pressure-flow approximation, e.g. the isothermal solution, the temperature problem would be solved first. Then the influence of temperature on viscosity would be introduced or updated to seek an improved solution of the pressure problem, and the iterative process would be continued until sufficient accuracy is realized. Alternately, one may employ the Newton-method of iteration to seek improved pressure and temperature fields simultaneously. The sequential iteration scheme is selected because of proven experience of its viability. The second major step involves the connection between the microstructure and macrostructure problems. This step would be carried out for the temperature and pressure problems separately for each step of iteration.

### 3.3.1 Coordinate Systems

The macrostructures are depicted in terms of the usual cylindrical system  $(r, \theta)$ . In the steady-state problem, one can affix the coordinates to the grooved surface. Accordingly, the relative rotation  $\omega = \omega_2 - \omega_1$  is of interest only. The radius may be normalized by the outer radius of the thrust bearing:

$$\bar{r} = \ln (r/r_0) \quad . \quad (69)$$

The  $(\bar{r}, \theta)$  system is orthogonal and coarse scaled; i.e. it has only the capability of referring to the spatial averages of various quantities over a multiplicity of grooves. Furthermore, for the thrust bearing with a parallel film, rotational symmetry prevails in the macrostructure; i.e.

$$\frac{\partial F}{\partial \theta} = 0 \quad (70)$$

for all  $F$  representing a groove-wise averaged quantity.

The microstructures are depicted in terms of skewed or spiraled coordinate

system  $(r, \xi)$ , where  $\xi$  is the angular coordinate measured from the aft-side of the nearest ridge. Generally, one is only interested in the range

$$|\xi| < \Delta\theta \quad (71)$$

which is the angular span of one groove-ridge pair. The radius can again be normalized by the outer radius:

$$\zeta = \ln (r/r_o) \quad (72)$$

Symbolic distinction between Eqs. (69) and (72) is intended to distinguish the skewed system,  $(\zeta, \xi)$ , from the orthogonal system,  $(\bar{r}, \theta)$ . It is also natural to "stretch"  $\xi$  to study the microstructure with a fine scale:

$$\eta = \xi/\Delta\theta \quad (73)$$

which is the principal independent variable (of the microstructure problem).

### 3.3.2 Dependent Variables

The pressure field  $p$  can be regarded to consist of the macrostructure

$$\bar{p}(\bar{r}) \quad (74)$$

which may be regarded as the pressure at the aft-side of a ridge, which is approximately located at  $(\bar{r})$ , the microstructure

$$p'(\eta; \bar{r}) = p - \bar{p} \quad (75)$$

which is the detailed pressure distribution within the width of a groove-ridge pair approximately located at  $(\bar{r})$ . Note that  $(\bar{r})$  appears in Eq. (75) as a parameter. The actual spatial coordinate is  $(\eta)$ .

The temperature field  $T_m$ , which is the mean fluid film temperature, can be

similarly expressed in terms of the macrostructure

$$\bar{T}_m(\bar{r}) \quad (76)$$

which is the value of  $T_m$  at the aft-side of a ridge approximately located at  $(\bar{r})$ , and the microstructure

$$T'_m(\eta; \bar{r}) = T_m - \bar{T}_m \quad (77)$$

### 3.3.3 Pressure Problem

The rotational symmetry of the thrust bearing permits considerable simplification in the equations previously derived. That is, dropping  $\partial F / \partial \theta$ , Eq. (36) becomes

$$\frac{d\bar{p}}{d\bar{r}} = \frac{\partial p}{\partial \zeta} = p' \quad (78)$$

In the fine scale, rotational symmetry is reflected in the periodic condition in the microstructure. Thus, from Eq. (37),

$$\csc \beta \psi_{\perp} = \frac{\frac{1}{2} \omega r^2 \int_0^1 \frac{\mu_1}{h^2 G_{\eta}} d\eta + \cot \beta P' \int_0^1 \frac{G_{z1}}{G_{\eta}} d\eta}{\int_0^1 \frac{\mu_1}{h^3 G_{\eta}} d\eta} \quad (79)^*$$

where, for brevity, it has been written

$$G_{\eta} = G_{x1} + G_{z1} \cot^2 \beta \quad (80)$$

Rearranging Eq. (41) and using Eq. (80), one finds

$$\bar{Q}_r = \cot \beta \left[ \frac{1}{2} \omega r^2 \int_0^1 h \frac{G_{z1}}{G_{\eta}} d\eta - \csc \beta \psi_{\perp} \int_0^1 \frac{G_{z1}}{G_{\eta}} d\eta \right] - P' \int_0^1 \frac{h^3}{\mu_1} \frac{G_{x1} G_{z1}}{G_{\eta}} d\eta \quad (81)$$

---

\*This relation may be refined by allowing for dynamic head losses across the groove sides.

And, finally, the governing d.e. for the macrostructure is obtained by dropping  $\theta$ - and  $t$ - derivatives in Eq. (33):

$$\frac{d\bar{Q}_r}{dr} = 0 \quad (82)$$

In the sequential iteration method,  $(\mu_1, G_{x1}, G_{z1})$  are determined by the microstructure temperature solution in the previous step, and may be regarded as functions of  $\eta$  here.

#### 3.3.4 Temperature Problem

In the temperature problem, viscous heat dissipation and heat transfer are both dependent on the microstructure pressure gradient in the form

$$\begin{aligned} \left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2 &= \left(\frac{1}{r} \frac{\partial p}{\partial \theta}\right)^2 + \left(\frac{\partial p}{\partial r}\right)^2 \\ &= \left(\frac{1}{r\Delta\theta} \frac{\partial p'}{\partial \eta}\right)^2 + \left(\frac{p'}{r} - \frac{\text{ctn}\beta}{r\Delta\theta} \frac{\partial p'}{\partial \eta}\right)^2 \\ &= \frac{1}{r^2} \left[ \left(\frac{\text{csc}\beta}{\Delta\theta} \frac{\partial p'}{\partial \eta}\right)^2 - \frac{2 \text{ctn}\beta}{\Delta\theta} \frac{\partial p'}{\partial \eta} p' + (p')^2 \right]. \end{aligned} \quad (83)$$

Thus it is necessary to have the explicit expression for  $\partial p'/\partial \eta$ , which can be obtained by rearranging Eqs. (22) and (23):

$$\frac{1}{\Delta\theta} \frac{\partial p'}{\partial \eta} = \frac{1}{G_\eta} \left( \frac{1}{2} \omega r^2 \frac{\mu_1}{h^2} - \text{csc}\beta \frac{\mu_1}{h^3} + \text{ctn}\beta G_{z1} p' \right). \quad (84)$$

These relations together with the procedures outlined in [1] will permit computations of  $(\mu_i, G_{\tau i}, H_{\tau i}, H_{p i}, F_{p i})$  for  $(i = 1, 2)$  according to Eqs. (8 through 11). Again, at this point, results from the previous step of iteration would be used. Terms related to heat transfer and viscous dissipation as previously given by Eqs. (49) and (50) are listed below:

$$N = \frac{r^2 \Delta \theta}{\rho C_p \csc \beta \Psi_{\perp}} \left( \frac{G_{\tau 2} + F_{p1} G_{z2}}{H_{\tau 1} + F_{p1} G_{p2}} \right) \frac{\mu}{h} \quad (85)$$

$$\Phi = \frac{r^2 \Delta \theta}{\rho C_p \csc \beta \Psi_{\perp}} \left( 1 + \frac{F_{p1} G_{z2}}{G_{\tau 1}} \right) \frac{\mu_1 U^2}{G_{\tau 1} h} \quad (86)$$

The microstructure of the fluid film temperature as given along a flux line is, previously given as Eq. (59),

$$(T'_m)_{\Psi} = -(\bar{T}_m - T_w) \left[ 1 - \exp\left\{-\int_0^{\eta} N d\eta\right\} \right] + \exp\left\{-\int_0^{\eta} N d\eta\right\} \int_0^{\eta} (\Phi \exp\left\{\int_0^{\eta} N d\eta\right\}) d\eta \quad (87)^*$$

The temperature profile at constant  $r$  is

$$(T'_m)_{\text{const. } r} = (T'_m)_{\Psi} - \frac{r \Delta \theta}{\csc \beta \Psi_{\perp}} \left. \frac{\partial T_m}{\partial r} \right|_{\xi} \int_0^{\eta} Q_r d\eta' \quad (88)$$

$Q_r$  can be found by recombining Eqs. (22), (23), and (24) as

$$Q_r = \left[ \csc \beta \left( \frac{1}{2} \omega r^2 h - \csc \beta \Psi_{\perp} \right) - p' \frac{h^3}{\mu_1} G_{x1} \right] \frac{G_{z1}}{G_{\eta}} \quad (89)$$

The increment of  $\bar{T}_m$  across a groove-ridge pair is

$$\begin{aligned} \Delta T_{mo} &= (T'_m)_{\Psi, \eta=1} \\ &= -(\bar{T}_m - T_w) \left[ 1 - \exp\left\{-\int_0^1 N d\eta\right\} \right] + \exp\left\{-\int_0^1 N d\eta\right\} \int_0^1 \Phi \exp\left\{\int_0^{\eta} N d\eta\right\} d\eta \quad (90) \end{aligned}$$

Beyond this point, the applicable formulae are different depending on whether or not  $\bar{Q}_r = 0$  (and  $\Delta T_{mo} = 0$ ).

---

\*This equation may be refined to allow for dynamic head loss across groove-sides.

### 3.3.4.1 $\bar{Q}_r \neq 0$

The temperature gradient along constant  $\xi$  is (renumerating Eqs. (62) and (61)):

$$\left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} = \exp\left\{-\int_0^{\eta} N d\eta'\right\} \left[ \left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} + \int_0^{\eta} \left\{ \left. \frac{\partial \bar{q}}{\partial r} \right|_{\xi} - (T_m - T_w) \left. \frac{\partial N}{\partial r} \right|_{\xi} \right\} \exp\left\{\int_0^{\eta'} N d\eta''\right\} d\eta' \right] \cdot \quad (91)$$

where,

$$\left. \frac{\partial \bar{T}_m}{\partial r} \right|_{\xi} = \frac{\csc \beta \psi_{\perp}}{\bar{Q}_r} \frac{\Delta T_{mo}}{r \Delta \theta} \quad (92)$$

Eqs. (87) through (92) together define the microstructure problem. Eq. (92) is in fact also the governing equation for the macrostructure problem, which is symbolically rewritten as

$$r \frac{d\bar{T}_m}{dr} = \frac{\csc \beta \psi_{\perp}}{\bar{Q}_r} \frac{\Delta T_{mo}}{\Delta \theta} \quad (93)$$

An initial value would have to be specified for  $\bar{T}_m$ . Schematic graphical interpretations of these relations are shown in Fig. B-4a.

### 3.3.4.2 $\bar{Q}_r \equiv 0$

For this condition, there is no net convection, and the microstructure must be periodic along the flux line. That is,  $\Delta T_{mo} = 0$ , or from Eq. (90)

$$\bar{T}_m = T_w + \frac{\exp\left\{-\int_0^1 N d\eta\right\}}{1 - \exp\left\{-\int_0^1 N d\eta\right\}} \int_0^1 \bar{q} \exp\left\{\int_0^{\eta} N d\eta'\right\} d\eta \quad (94)$$

Combining with Eq. (87), one obtains

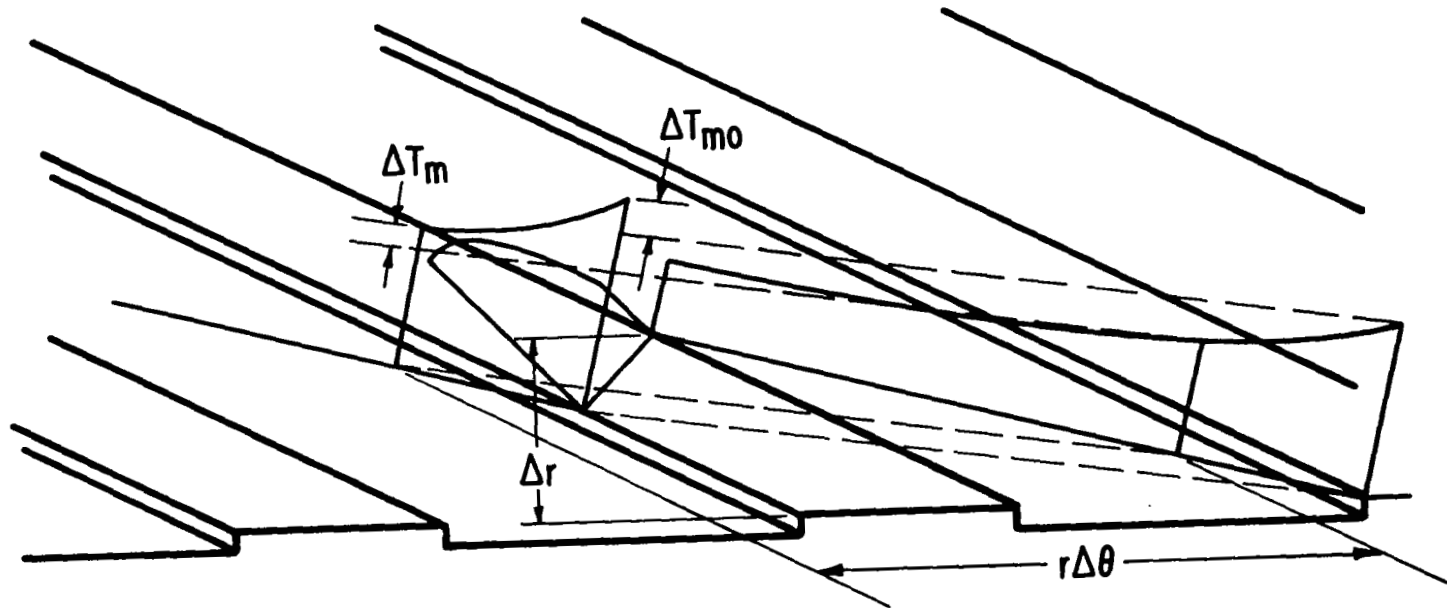


Fig. B-4a Temperature Profile in Bearing with Net Radial Flow

$$(T_m - T_w)_\psi = \exp \left\{ - \int_0^1 N d\eta \right\} \left[ \frac{\exp \left\{ - \int_0^1 N d\eta \right\} \int_0^1 \Phi \exp \left\{ \int_0^\eta N d\eta' \right\} d\eta}{1 - \exp \left\{ - \int_0^1 N d\eta \right\}} + \int_0^1 \Phi \exp \left\{ \int_0^\eta N d\eta' \right\} d\eta \right] \quad (95)$$

which can be formally differentiated with respect to  $r$  to obtain

$$\left. \frac{\partial T_m}{\partial r} \right|_\xi = \left. \frac{\partial}{\partial r} \right|_\xi (T_m - T_w)_\psi \quad (96)$$

Then this expression and Eq. (87) can be substituted into Eq. (88) to yield the microstructure. Schematic graphical interpretations of these relations are shown in Fig. B-4b.

### 3.3.4.3 Pure "Conduction" Problem

In the limiting case,  $N \rightarrow \infty$ , convective heat transfer can be neglected altogether, and Eq. (48) is reduced to

$$T_m = T_w + \frac{\Phi}{N} \quad (97)$$

and, accordingly,

$$\sigma = 1 \quad \bullet \quad (98)$$

A convenient first estimate of  $N$  can be based on the laminar, isoviscous, Couette-like flow, which is

$$N \approx \frac{12 r^2 \Delta \theta \kappa}{\rho C_p \csc \beta \psi_\perp h} \approx \frac{24 \Delta \theta \kappa}{\rho C_p \omega h^2} \quad \bullet \quad (99)$$

For the strongly turbulent condition, application of Reynolds analogy would

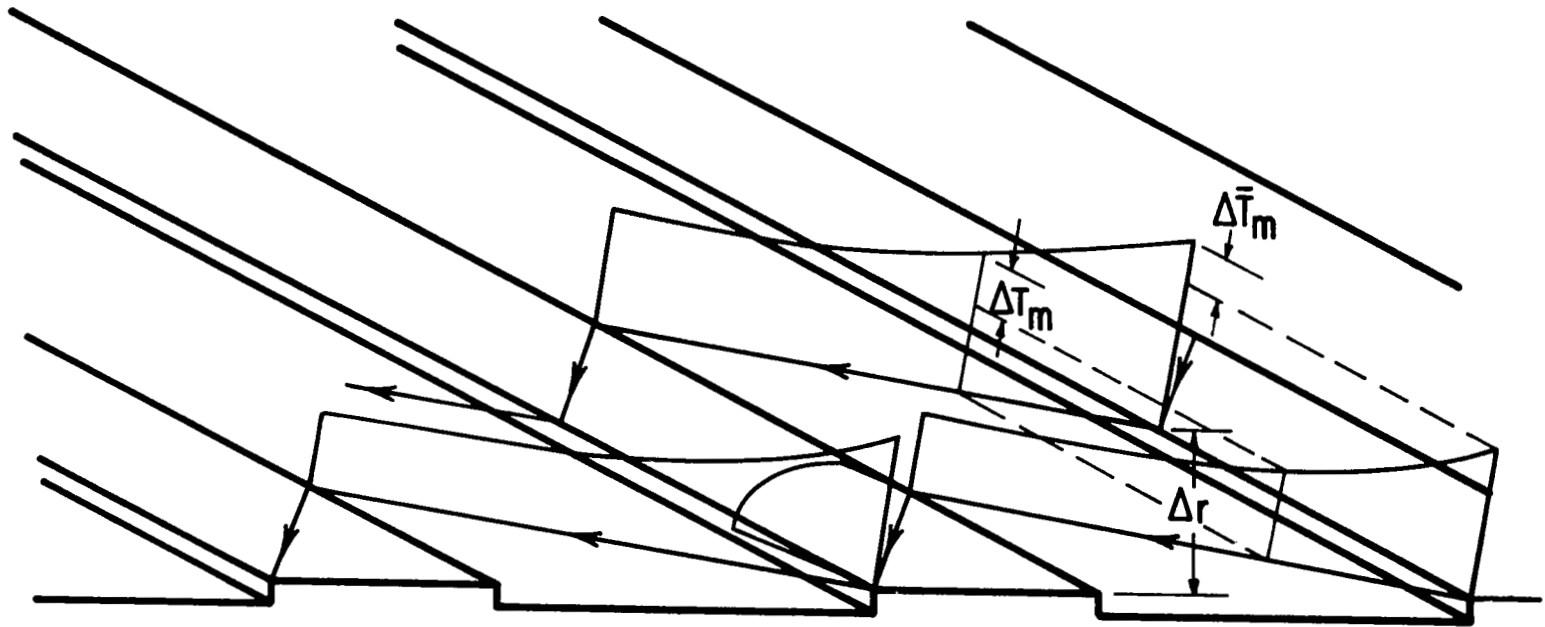


Fig. B-4b Temperature Profile in Bearing with Zero Net Radial Flow

yield

$$N \approx 0.2\pi \Delta\theta \left(\frac{r}{h}\right) \sqrt{\frac{C_f}{2}} \quad (100)$$

where  $C_f$  is the turbulent Couette friction factor.

## REFERENCES

1. Appendix - A this report.
2. Ng, C. W. and Pan, C. H. T., "A Linearized Turbulent Lubrication Theory," Trans ASME, Journal of Basic Engineering, Vol. 87, Series D, No. 3, September 1965, pp. 675-688.
3. Constantinescu, V. N., Hsing, F. C., and Pan, C. H. T., "Development of Long Life Jet Engine Thrust Bearings," Appendix A, MTI Report 70TR44, 1970.
4. Pan, C. H. T., "Compressible Narrow Groove Theory - An Approximate Analysis of Gas Lubricated Spiral-Grooved Bearings at Large Compressibility Numbers," MTI Report 68TR70, 1968.
5. Pan, C. H. T., "A Universally Valid Compressible Theory for Gas-Lubricated, Narrow, Spiral-Grooved Thrust Bearings," MTI Report 69TR19, 1969.
6. Bootsma, J., "The Effect of Viscosity Variations with Temperature on the Performance of Spiral Groove Bearings," Trans. ASLE, Vol. 12, Issue 4, October 1969, pp. 287-296.
7. Vohr, J. H. and Chow, C. Y., AFAPL-TR-65-45, Part VIII, Rotor-Bearing Dynamics Design Technology, Spiral-Grooved Floating Ring Journal Bearing, Air Force Aero Propulsion Laboratory, Wright-Patterson Air Force Base, Ohio, April, 1969.
8. Kamke, E. "Differential Equations," Chelsea Publishing Co., Third Edition, 1959.

## NOMENCLATURE

|                                             |                                                                                                                       |
|---------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
| $G_f$                                       | = coefficient of friction described after (9)                                                                         |
| $G_p$                                       | = specific heat                                                                                                       |
| $\mathcal{F}, \mathcal{F}_e, \mathcal{F}_o$ | = See section 4.3, Eq.(38)                                                                                            |
| $f = \frac{V}{U}$                           | = See (7)                                                                                                             |
| $G$                                         | = dyadic coefficient described after (6)                                                                              |
| $G_x$                                       | = turbulent coefficient for flow in x direction                                                                       |
| $G_z = G_2$                                 | = turbulent coefficient for flow in z direction,                                                                      |
| $G_r = G_1$                                 | = defined in (44)                                                                                                     |
| $H_1, H_2$                                  | = defined by (46)                                                                                                     |
| $h$                                         | = normal thickness of fluid film                                                                                      |
| $h_o^+$                                     | = $\frac{h}{v_o} \sqrt{\frac{\tau_c}{\rho}}$ ; See (20)                                                               |
| $Pr$                                        | = Prandtl number, $\mu C_p / \kappa$ , dimensionless                                                                  |
| $p$                                         | = fluid pressure, psi                                                                                                 |
| $Q$                                         | = $\frac{\mu_o q''}{\sigma \tau_c^2 h}$ ; See (20)                                                                    |
| $Q_w$                                       | = $\frac{\mu_o q_w''}{\sigma \tau_c^2 h}$ ; See (22)                                                                  |
| $Q^{(i)}$                                   | = defined by (25)                                                                                                     |
| $Q_w^{(i)}$                                 | = See (27), (28)                                                                                                      |
| $q''$                                       | = conduction heat flux                                                                                                |
| $q_o''$                                     | = conduction heat flux at film center                                                                                 |
| $R_f$                                       | = critical value of Reynold number corresponding to onset of complete turbulence, or upper value of transition region |

|                |                                                                                                               |
|----------------|---------------------------------------------------------------------------------------------------------------|
| $R_{cr.}$      | = critical value of Reynolds number corresponding to onset of turbulence, or lower value of transition region |
| $T$            | = temperature                                                                                                 |
| $T_o$          | = temperature at film center                                                                                  |
| $T_w$          | = surface (wall) temperature                                                                                  |
| $T_{\pm w}$    | = temperature of wall at $y = \pm h/2$ , See (17)                                                             |
| $U$            | = surface velocity used to define actual Reynolds number                                                      |
| $\vec{U}$      | = total vector velocity of fluid                                                                              |
| $\bar{u}$      | $= \frac{\mu_o (u - \frac{V}{2})}{h\tau_c}$                                                                   |
| $u, v, w$      | = fluid velocity corresponding to directions $x, y, z$                                                        |
| $V$            | = $f(z)$ , velocity of surface; also used in definition of effective Reynolds number - See Sec. 3.            |
| $\vec{V}$      | = tangential velocity of surface                                                                              |
| $x, z$         | = orthogonal (curvilinear) coordinates                                                                        |
| $y$            | = normal distance from the mean surface                                                                       |
| $\bar{y}$      | = $y/h$ ; See (20)                                                                                            |
| $y', y''$      | = dummy variables used in (15) and (16)                                                                       |
| $\epsilon$     | = eddy diffusivity                                                                                            |
| $\theta$       | $= \frac{\mu_o \kappa (T_o - T)}{\sigma \tau_c h^2} ; \text{ See (20)}$                                       |
| $\theta_w$     | = defined by (23)                                                                                             |
| $\theta^{(i)}$ | = defined by (26)                                                                                             |

|                                   |                                                               |
|-----------------------------------|---------------------------------------------------------------|
| $\theta_w^{(i)}$                  | = See (28)                                                    |
| $k$                               | = thermal conductivity                                        |
| $\mu$                             | = absolute viscosity of fluid                                 |
| $\mu_i$                           | = See (25)                                                    |
| $\nu$                             | = kinematic viscosity                                         |
| $\sigma$                          | = conduction cooling factor defined by (13)                   |
| $\rho$                            | = fluid density                                               |
| $\tau$                            | = shear stress                                                |
| $\tau_c$                          | = Couette shear stress                                        |
| $\tau_{xy}, \tau_{zy}$            | = shear stress components                                     |
| $(1 + \frac{\epsilon}{\nu})_i$    | = See (25)                                                    |
| $(1 + \frac{\epsilon}{\nu} Pr)_i$ | = See (26)                                                    |
| $\nabla$                          | = grad operator used in (6)                                   |
| $\Theta$                          | = function notation; see sec. 4.2 and Fig. 4                  |
| $( )_i$                           | = refers to $( )$ at $i=1,2$                                  |
| $( )_m$                           | = average or mean value of $( )$                              |
| $( )_o$                           | = film center value of $( )$                                  |
| $( )_w$                           | = refers to $( )$ at wall                                     |
| $( )'$                            | = related to pressure gradient effects of $( )$ ; see sec 4.3 |