THE USE OF THE THREE DIMENSIONAL COHERENCY MATRIX IN ANALYZING THE POLARIZATION PROPERTIES OF PLANE WAVES

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ABSTRACT

A technique for analyzing the polarization properties of plane waves is developed which offers a number of advantages over methods currently used in the analysis of both ground and satellite observations of waves. This technique reduces the computations required to find the wave normal vector, is less sensitive to common noise sources, and is amenable to analog implementation. This technique here is applied specifically to the analysis of a proton whistler, but may also be used in most studies of ULF, ELF, and VLF magnetic wave phenomena.

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INTRODUCTION

The study of the polarization properties of plane waves is an important tool in space plasma physics. It is of value in identifying the type of wave, determining the origin of the wave, and interpreting the propagation characteristics of the medium through which the wave travels. Polarization analysis is currently employed in studying a wide range of waves including chorus, hiss, lightning whistlers and hydromagnetic waves in the magnetosphere, as well as various waves detected in the magnetosheath, bow shock, and interplanetary medium.

Fowler et al. (1967), and Rankin and Kurtz (1970) deal with the two dimensional analysis of quasimonochromatic polarized waves. Following an analysis presented in Born and Wolf (1970), these authors demonstrate that the polarization properties for a quasimonochromatic wave represented by:

$$H_x(t) = a_x(t)\exp\{i[\omega t+\phi_x(t)]\}$$

$$H_y(t) = a_y(t)\exp\{i[\omega t+\phi_y(t)]\}$$

may be expressed in terms of the "polarization parameters"

$$\tan 2\theta = \frac{2\bar{a}_x\bar{a}_y}{\bar{a}_x^2 - \bar{a}_y^2}\cos(\bar{\phi}_y-\bar{\phi}_x)$$

$$\sin 2\beta = \frac{2\bar{a}_x\bar{a}_y}{\bar{a}_x^2 + \bar{a}_y^2}\sin(\bar{\phi}_y-\bar{\phi}_x)$$

where the bars represent time averages, θ is the angle between the major axis of the polarization ellipse and the X-axis, and $\tan\beta$ is the ratio of the minor to the major axis of the ellipse. The sign of β indicates the sense of polarization: $\beta>0$ for counterclockwise and $\beta<0$ for clockwise rotation of the perturbation vector about the Z-axis.

Fowler et al. (1967) show how these polarization parameters may be obtained in terms of the elements of the two dimensional coherency matrix:

$$J = \begin{vmatrix} \langle H_x(t)H_x^*(t)\rangle & \langle H_x(t)H_y^*(t)\rangle \\ \langle H_y(t)H_x^*(t)\rangle & \langle H_y(t)H_y^*(t)\rangle \end{vmatrix}$$

where the angular brackets represent time averages, and the asterisk indicates the corresponding complex conjugate.

This technique is directly applicable to the analysis of plane waves if the X and Y axes can be chosen to lie in the plane of the wave. In general, however, when a plane wave is detected by measuring its three dimensional vector time series,the plane of the wave will not coincide with any of the three orthogonal measurement planes. We then have to deal with the three dimensional coherency matrix:

$$J = \begin{vmatrix} \langle H_x(t)H_x^*(t)\rangle & \langle H_x(t)H_y^*(t)\rangle & \langle H_x(t)H_z^*(t)\rangle \\ \langle H_y(t)H_x^*(t)\rangle & \langle H_y(t)H_y^*(t)\rangle & \langle H_y(t)H_z^*(t)\rangle \\ \langle H_z(t)H_x^*(t)\rangle & \langle H_z(t)H_y^*(t)\rangle & \langle H_z(t)H_z^*(t)\rangle \end{vmatrix}$$

In order to utilize the results of the two dimensional analysis,it is necessary to determine the plane of polarization of the perturbation vector or, equivalently, the wave normal vector.

TECHNIQUE AND THEORY

Let us first examine the coherency matrix defined in terms of the three functions $H_x(t)$, $H_y(t)$, and $H_z(t)$ which are analytic representations of the measured signals and, as such, are complex. It is this fact which complicates the generation of the coherency matrix in the time domain, since the generation of the analytic representation of a real function involves a convolution integral (Bracewell, 1965). In the frequency domain, however, the analytic representation of the real signal is easily obtained by multiplying the Fourier transform of the real signal by the Heaviside step function:

$$S(f) = \begin{matrix} 0 & f<0 \\ 1 & f\geq 0 \end{matrix}$$

We may obtain the frequency domain representation of the three time series, after they have been digitized, by use of the Fast Fourier Transform algorithm of Cooley and Tukey (1965). Then, by use of the generalized power theorem:

$$\int_{-\infty}^{\infty} g(t)h^*(t)dt = \int_{-\infty}^{\infty} G(f)H^*(f)\,df$$

we may extend the definition of the coherency matrix to the frequency domain. The resulting matrix is:

$$J = \begin{vmatrix} \langle H_x(f)H_x^*(f)\rangle & \langle H_x(f)H_y^*(f)\rangle & \langle H_x(f)H_z^*(f)\rangle \\ \langle H_y(f)H_x^*(f)\rangle & \langle H_y(f)H_y^*(f)\rangle & \langle H_y(f)H_z^*(f)\rangle \\ \langle H_z(f)H_x^*(f)\rangle & \langle H_z(f)H_y^*(f)\rangle & \langle H_z(f)H_z^*(f)\rangle \end{vmatrix}$$

where the angular brackets now represent the average over all positive frequencies.

The coherency between two signals is defined in terms of the elements of this matrix as:

$$C_{ij} = \frac{J_{ij}}{\sqrt{J_{ii}J_{jj}}}$$

The coherency has a value of one for signals which are highly interrelated (i.e. completely polarized signals), and a value of zero for signals which are not interrelated (i.e. noise).

In order to understand what types of signals enter the various matrix elements, we will look at the calculation of the X-Y element $\langle H_x(f)H_y^*(f)\rangle$. For any given frequency the transform coefficients for the X and Y axes may be represented as:

$$H_x(f) = A_f\cos\theta_f + iA_f\sin\theta_f$$

$$H_y(f) = B_f\cos\phi_f + iB_f\sin\phi_f$$

where θ_f and ϕ_f are the phases of the signals referenced to the same arbitrary zero point. The product $H_x(f)H_y^*(f)$ for this frequency reduces to:

$$H_x(f)H_y^*(f) = A_fB_f\cos(\theta_f-\phi_f) + iA_fB_f\sin(\theta_f-\phi_f) \quad (1)$$

From this expression it is obvious how two signals with varying phase relations to one another will enter the off diagonal elements of the coherency matrix. The diagonal elements of the matrix will all be real positive quantities since there is no phase difference between a function and its conjugate.

With the above information we may examine how some typical signals will enter the coherency matrix. First, we will consider incoherent signals. For incoherent signals the off diagonal terms average to zero (since the phase relations are random), and, thus, these signals will enter only in the diagonal terms of the matrix. The coherency matrix for a non-isotropic incoherent signal is given by:

$$J = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix}$$

It is interesting to note that if we rotate into any other coordinate system (by use of a similarity transformation) the signals in that coordinate system are no longer incoherent, that is, the off diagonal terms are no longer zero except in a singular case to be discussed below. To illustrate this we will rotate the incoherent signal found by letting b = c = 0 in the above matrix into a system defined by $\theta=\phi=\psi=\pi/4$:

$$\begin{pmatrix} .147 & -.853 & .5 \\ .853 & -.147 & -.5 \\ .5 & .5 & .707 \end{pmatrix} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} .147 & -.853 & .5 \\ .853 & -.147 & .5 \\ .5 & -.5 & .707 \end{pmatrix} =$$

$$a^2 \begin{pmatrix} .025 & .129 & .074 \\ .129 & .725 & .427 \\ .074 & .427 & .250 \end{pmatrix}$$

The coherencies in this coordinate system are:

$$C_{xy} = 1 \qquad C_{xz} = 1 \qquad C_{yz} = 1$$

Thus, in the rotated system the incoherent signal appears as a completely coherent signal on all axes. The incoherent isotropic noise source is of interest in a number of studies. This source may be represented by multiplying the identity matrix by a constant, i.e.:

$$J = a^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

This is a special case in which the above analysis of the effects of coordinate transformations fail since the identity matrix is an invariant under a similarity transformation. This signal will, therefore, remain incoherent in any coordinate system.

If the off diagonal elements of the coherency matrix are non-zero, then, by definition, the signals are at least partially coherent (although a coordinate system may exist in which the signals are incoherent). The phase relations between the coherent parts of the signals determine how they enter the off diagonal terms as was shown in equation (1). From this equation we see that the only signals which enter the imaginary part of the coherency matrix are signals which are in quadrature to one another, that is, signals which have phase relations of $\pm\pi/2$. All other signals (incoherent, in-phase, and out-of-phase coherent) enter the real part of the coherency matrix. This property provides a useful method of analyzing plane waves in the presence of noise sources.

We will now consider a monochromatic plane wave which has a counterclockwise sense of polarization about the wave normal vector. In the principal axes system for this wave (the system in which Z is along the wave normal and $\theta = 0$), the time series for this wave may be represented by:

$$H_x(t) = a \exp\{i[\omega t+\phi_x]\}$$

$$H_y(t) = b \exp\{i[\omega t+\phi_y]\}$$

$$H_z(t) = 0$$

where $\phi_y - \phi_x = \pi/2$ for a counterclockwise sense of polarization in the principal axes system.

The coherency matrix for this wave is shown to be (Fowler et al., 1967):

$$J = \begin{vmatrix} a^2 & iab & 0 \\ -iab & b^2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 0 \end{vmatrix} + iab \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

where we have extended the two dimensional analysis to three dimensions.

The coherency matrix in any other coordinate system (i.e. the system in which the actual measurements occur) may be obtained by the similarity transformation:

$$J' = RJR^{-1}$$

where R is a general coordinate transformation matrix (Goldstein,1965). The advantage of starting in the principal axes coordinate system is that it illustrates the two properties of the coherency matrix which may be used to determine the wave normal direction.

The property which is currently used by a number of investigators and is a reasonably straight forward extension of the two dimensional analysis, is that, in the principal axes system, the real symmetric part of the coherency matrix is diagonal. We note that since the coherency matrix is a Hermitian matrix, the real part of the matrix is symmetric and the imaginary part is antisymmetric. The technique used, then, is to diagonalize the real symmetric part of the coherency matrix, use the eigenvector associated with the minimum eigenvalue as the wave normal direction, and then rotate the full coherency matrix so that the new Z-axis corresponds to the wave normal direction. At this point the two dimensional analysis may be applied to the X-Y submatrix to obtain the polarization parameters for the plane wave. This technique of analyzing plane waves is discussed in a number of current papers, Means (1971), McPherron et al. (1972). Although this technique of analysis is a reasonably straight forward extension of the two dimensional technique, it has two serious drawbacks. It uses the real symmetric part of the coherency matrix, the part that is most susceptable to interference from unwanted signals, to determine the wave normal direction and, although the techniques for determining the eigenvectors and eigenvalues are readily available, the mathematical steps involved are extensive.

We will now consider the second property, the relatively simple antisymmetric imaginary part of the coherency matrix in the principal axes system and how it may be used to determine the wave normal direction. We may rotate the matrix

$$J_I = ab \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

to another coordinate system by the use of the similarity transformation:

$$J_I' = ab\ RJ_IR^{-1}$$

where we will use:

$$R = \begin{vmatrix} A & D & G \\ B & E & H \\ C & F & I \end{vmatrix}$$

This rotation gives:

$$J_I' = ab \begin{vmatrix} 0 & AE-BD & FA-CD \\ BD-AE & 0 & FB-CE \\ CD-FA & CE-FB & 0 \end{vmatrix}$$

evaluating this matrix in terms of the coefficients for the general rotation matrix given in terms of the Eulerian angles (ϕ,θ,ψ), Goldstein (1965) gives:

$$J_I' = ab \begin{vmatrix} 0 & \cos\theta & -\sin\theta\cos\psi \\ -\cos\theta & 0 & \sin\theta\sin\psi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & 0 \end{vmatrix}$$

The fact that J_I' is independent of the rotation ϕ about the Z-axis is easily demonstrated by applying this rotation separately. Thus, any rotation about the wave normal vector $\hat{k}$ (the Z-axis in this coordinate system) doesn't change the imaginary part of the coherency matrix.

Referring again to Goldstein (1965), we can interpret the components of J_I' in terms of the components of the wave normal vector $\hat{k}$:

$$J_I' = ab \begin{vmatrix} 0 & k_z & -k_y \\ -k_z & 0 & k_x \\ k_y & -k_x & 0 \end{vmatrix} \equiv \begin{vmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{vmatrix}$$

Thus, we may obtain the components of the wave normal vector directly from the imaginary part of the coherency matrix by using:

$$k_x^2 + k_y^2 + k_z^2 = 1$$

$$J_{xy}^2 + J_{xz}^2 + J_{yz}^2 = a^2b^2$$

and

$$k_x = \frac{J_{yz}}{ab} \qquad k_y = \frac{-J_{xz}}{ab} \qquad k_z = \frac{J_{xy}}{ab}$$

It is important to note that, as a result of the original assumptions, this is the wave normal direction for a counterclockwise, elliptically polarized, plane wave. Since the sense of polarization of the wave changes if we let $\hat{k}' = -\hat{k}$, this poses no real restriction on the analysis. If a wave with a counterclockwise rotation has a wave normal vector $\hat{k}$, a wave with the same wave normal vector but clockwise rotation will, in the above analysis, result in a wave normal vector $-\hat{k}$.

In plasma physics the sense of polarization of plane waves is always referenced to the magnetic field direction $\hat{B}$. A wave is considered right-handed if it rotates in a counterclockwise direction when looking down $\hat{B}$, left-handed if it rotates in a clockwise sense. The determination of the sense of polarization of the wave in plasma physics terms is easily interpreted from the dot product of $\hat{k}$, the wave normal direction defined above, and $\hat{B}$, the magnetic field direction. For right-handed waves $\hat{k}\cdot\hat{B}$ will be greater than zero, whereas for left-handed waves $\hat{k}\cdot\hat{B}$ will be less than zero.

Once the wave normal vector is determined, it is necessary to rotate the coherency matrix into a coordinate system such that one axis of the new system contains this vector. If we assume that this axis is the Z-axis in the new coordinate system, then the polarization parameters for the wave may be determined by applying the two dimensional analysis to the X-Y submatrix of the coherency matrix. As was shown earlier the particular choice of X-Y axis is arbitrary, that is, a rotation about the Z-axis doesn't affect the imaginary part of the coherency matrix. Since the angle that the major axis of the ellipse makes with the $\hat{k}\cdot\hat{B}$ plane is important in a number of the theories, it is advantageous to choose the X-axis in the $\hat{k}\cdot\hat{B}$ plane and the Y-axis perpendicular to this plane. The resulting right-handed orthogonal coordinate system is one in which the Z-axis lies along the wave normal vector $\hat{k}$ and the X-axis lies in the $\hat{k}\cdot\hat{B}$ plane.

In this new system the magnitudes of the three Z components of the coherency matrix (J_{xz}, J_{yz}, J_{zz}) are related to the noise sources which enter the real part of the coherency matrix. It is possible to solve for the noise sources directly if some assumptions are made about the type of noise and coordinate system in which it enters the signal (i.e. incoherent instrument noise). The usefulness of this procedure is questionable unless there is strong reason to believe that the major sources of the noise are of the assumed type. Of more general usefulness is the signal to noise ratio defined by:

$$SNR = \frac{J_{xx} + J_{yy}}{J_{zz}}$$

This quantity is a useful indicator of the reliability of the polarization parameters.

There is one type of plane wave for which this analysis is invalid, the linearly polarized plane wave. The coherency matrix for a monochromatic linearly polarized plane wave in the principal axes coordinate system is:

$$J_L = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

where the power in the linearly polarized signal (a^2) is given by the trace of J_L. Since the linearly polarized plane wave results in a pure real coherency matrix the above analysis cannot be applied. One may, however, apply a general rotation matrix (R) to this coherency matrix, and, in a manner similar to that used in analyzing the imaginary part of the coherency matrix, determine the direction of the linear axis. Doing this:

$$J_L = \begin{vmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{vmatrix}$$

From J_L we may obtain the components of the major perturbation vector $\hat{L}$ and the linear power as follows:

$$T_r J_L = a^2(L_x^2 + L_y^2 + L_z^2) = a^2$$

$$L_x = \sqrt{J_{xx}/a^2}$$

$$L_y = J_{xy}/a^2 L_x$$

$$L_z = J_{xz}/a^2 L_x$$

Thus, we can analyze the plane waves without resorting to the diagonalization of the coherency matrix, even for linear polarization.

The use of the imaginary part of the coherency matrix in determining the wave normal vector offers a number of advantages over techniques which use the real part. The first advantage is a result of the fact that the imaginary part of the coherency matrix is susceptible to only one source of interference, signals which are in quadrature to one another. This form of interference is the least likely form one is likely to encounter from the instrumentation used to detect these waves. Other advantages result from the fact that it is not necessary to do an eigen analysis in order to determine the wave normal vector. This results in a large saving in computation when digital techniques of analysis are used as well as allows the analog implementation of the technique.

ANALOG IMPLEMENTATION

There have been some attempts at analyzing plane polarized waves using analog techniques (Smith, 1970). The necessity of performing an eigenvector analysis in order to determine the wave normal vector and sense of polarization has, until now, imposed severe restrictions on the implementation of analog data processing systems. The analysis of the imaginary part of the coherency matrix just presented provides a technique which does not involve the eigenvector analysis. In this analysis,it was shown that the imaginary part of the coherency matrix was directly related to the components of the wave normal vector in the following way:

$$J_I = iab \begin{vmatrix} 0 & k_z & -k_y \\ -k_z & 0 & k_x \\ k_y & -k_x & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & I_{xy} & I_{xz} \\ I_{yx} & 0 & I_{yz} \\ I_{zx} & I_{zy} & 0 \end{vmatrix}$$

From the independent components of this matrix we may form a polarization vector $\vec{P}$ as follows:

$$\vec{P} = \hat{i}I_{yz} - \hat{j}I_{xz} + \hat{k}I_{xy} = ab(\hat{i}k_x + \hat{j}k_y + \hat{k}k_z)$$

This vector has a magnitude directly proportional to the area of the polarization ellipse formed by the plane wave and a direction which corresponds to the wave normal for a counterclockwise elliptically polarized plane wave.

In the current studies in magnetospheric physics, the wave normal direction and the sense of polarization are primary pieces of information used in comparing wave theory with experiment. These characteristics may be determined from the polarization vector by normalizing $\vec{P}$ to get $\hat{k}$ and then determining the sense from $\hat{k}\cdot\hat{B}$, where $\hat{B}$ is the dc magnetic field. The positive dot product corresponds to right-hand polarized waves and the negative product corresponds to left-hand waves. In addition to these quantities the magnitude of $\vec{P}$ (being directly proportional to the area of the ellipse) is a useful indicator.

In order to form the polarization vector it is necessary to measure the three quadrature components of the coherency matrix I_{xy}, I_{xz}, I_{yz}. A functional block diagram of the necessary processing for the analog determination of the components of the polarization vector $\vec{P}$ is given in Figure 1. The basic technique for measuring the quadrature function is that found in Bendat and Piersol (1966). We note that these determinations can be carried out on board spacecraft with simple state-of-the-art circuitry. The particular averaging constant, bandwidth, and modulation frequency used will depend upon the frequency band to be analyzed and the recording technique used.

DIGITAL IMPLEMENTATION

The digital implementation of the analysis of plane polarized waves involves four major steps:

A. The generation of the three dimensional coherency matrix.

B. The determination of the wave normal vector.

C. The rotation of the coherency matrix to a system where the Z-axis is along the wave normal vector.

D. The determination of the polarization parameters using the X-Y submatrix.

Descriptions of systems designed to determine the wave normal vector using the real part of the coherency matrix are given by Means (1971), and McPherron et al. (1972). It is relatively simple to modify these systems to determine the wave normal vector from the quadrature components of the coherency matrix, and then test the technique. The results of two of these tests, one on simulated data and one on actual data, are presented here.

The simulated data was generated on the computer using samples of a plane polarized, right-handed, signal with a wave normal vector of (0.5, 0.5, 0.707). For this test random gaussian numbers, with varying average amplitudes, were added to these samples. The resulting data, which has a variable signal to noise ratio, was run through the modified system. The results of this run are summarized in Table I. We note that as the signal to noise ratio decreases the coherency and polarized power decrease, the angle between the measured wave normal vector and the actual vector increase and sin 2β becomes less reliable.

The sample of data presented in Figure 2 is that of a proton whistler detected by the OGO-6 search coil magnetometer on September 19, 1969 at 1133 UT. An analysis of this whistler, using the real part of the coherency matrix for the wave normal determination, is presented in the paper by Chan et al. (1972).

This whistler provides a good test case because of the number of different mixtures of signals present. There is a strong right-handed signal from 0.06 sec to 0.15 sec. During this time the coherence, signal to noise ratio (SNR) and polarized power are all high. Correspondingly, the angle between $\hat{k}$ and $\hat{B}$ and the value of sin 2β show little variability. From 0.15 seconds to 0.29 seconds there is a mixture of right and left-hand waves. During this time the SNR, coherence, and polarized power all decrease with time. This is reflected in the higher variability observed in the $\hat{k}\cdot\hat{B}$ angle. The change from a dominantly right-handed signal to dominantly left-handed signal occurs between 0.29 and 0.49 seconds. We note that the SNR and the coherence dip in this region, the polarized power generally decreases, sin 2β changes from right-handed to left-handed, and the angle between $\hat{B}$ and $\hat{k}$ is quite variable. From 0.40 to 0.49 the amplitude of the right-handed signal decreasesrapidly while the amplitude of the left-handed signal remains roughly constant. This results in increasing values of the coherence, SNR, and polarized power. In addition sin 2β and the $\hat{k}\cdot\hat{B}$ angle show less variability. Finally,the left-handed signal slowly decays, and disappears at approximately 1.6 seconds. During this interval as the SNR and polarized power decrease, the coherence, sin 2β and $\hat{k}\cdot\hat{B}$ remain relatively constant until the SNR falls below 5, then the coherence decreases, and becomes more variable and the values for sin 2β and the angle between $\hat{k}$ and $\hat{B}$ become more variable.

In comparing these results with those obtained by using the real part of the coherency matrix to determine the wave normal vector, we find that meaningful results are obtained for a slightly longer interval of time, corresponding to a slightly lower SNR. This technique fails where there are equal contributions of right and left-handed signals in the analysis band, as occurs from 0.29 to 0.40 seconds. This is to be expected since the resultant signal corresponds to a linearly polarized plane wave.

The analog implementation would give results similar to those obtained here for the wave normal vector. In the digital implementation, however, we are not restricted to the broadband analysis of the proton whistler presented here for test purposes. We may determine the polarization parameters for narrower bands of frequencies by filtering the data before generating the coherency matrix. We note that, in the frequency domain, this corresponds to averaging over bands of frequency and not over all frequencies. With this technique we may separate the proton and electron branches of the whistler, except where they both lie in the same analysis band.

SUMMARY

The analysis of the polarization properties of plane polarized waves involves a study of the three dimensional coherency matrix. In presenting the coherency matrix we have shown how various signals will enter the elements of the matrix. In particular, we have shown that incoherent signals enter the diagonal elements of the matrix, that coherent signals with phase relations of $n\pi$ radians (where n is any integer) enter the diagonal and the real off diagonal terms and that coherent signals with phase relations of $m\frac{\pi}{2}$ radians (where m is an odd integer) enter the diagonal and imaginary off diagonal elements. We also emphasized that signals which are incoherent in one coordinate system may be highly coherent when viewed in any other coordinate system.

There are two possible techniques of analyzing the coherency matrix for plane polarized signals in an arbitrary coordinate system. The method currently

used by a number of investigators involves the diagonalization of the real symmetric part of the coherency matrix. This method has two important shortcomings, the number of computations necessary to diagonalize the matrix and the susceptibility of the real part of the coherency matrix to noise sources.

The method described in this paper utilized the relatively simple form of the imaginary part of the coherency matrix in the principal axes coordinates. It is shown that, in a general coordinate system, the imaginary part of the coherency matrix may be expressed as:

$$J_I = iab \begin{vmatrix} 0 & k_z & -k_y \\ -k_z & 0 & k_x \\ k_y & -k_x & 0 \end{vmatrix}$$

where the $\hat{k}$'s are the components of the wave normal vector in this coordinate system. Thus,we may obtain the wave normal vector directly from the imaginary part of the coherency matrix. In addition to the saving in computation obtained by removing the need to diagonalize a matrix, the only sources of signals which enter the imaginary part of the coherency matrix are signals which are in quadrature. This greatly reduces the possible sources of interference since most sources of noise on instruments will be either incoherent or in phase coherent sources.

The fact that the components of the wave normal vector are given directly by the terms in the imaginary part of the coherency matrix may be used to develop an analog technique to determine the wave normal vector. This may be especially useful on satellite experiments where the data transmission capabilities preclude the transmission of the complete spectrum of information available to the various experiments.

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TABLE I

TEST RESULTS ON COMPUTER GENERATED SIGNALS

SNR	COHERENCE	sin2β	$\hat{k}\cdot\hat{z}$	% POLARIZATION
∞	1.00	.637	1.000	100
400	.98	.636	1.000	95
100	.95	.616	1.000	95
50	.91	.666	.999	91
40	.93	.586	.999	94
30	.83	.634	.998	89
20	.80	.698	.997	85
10	.81	.674	.978	82
5	.71	.561	.935	76

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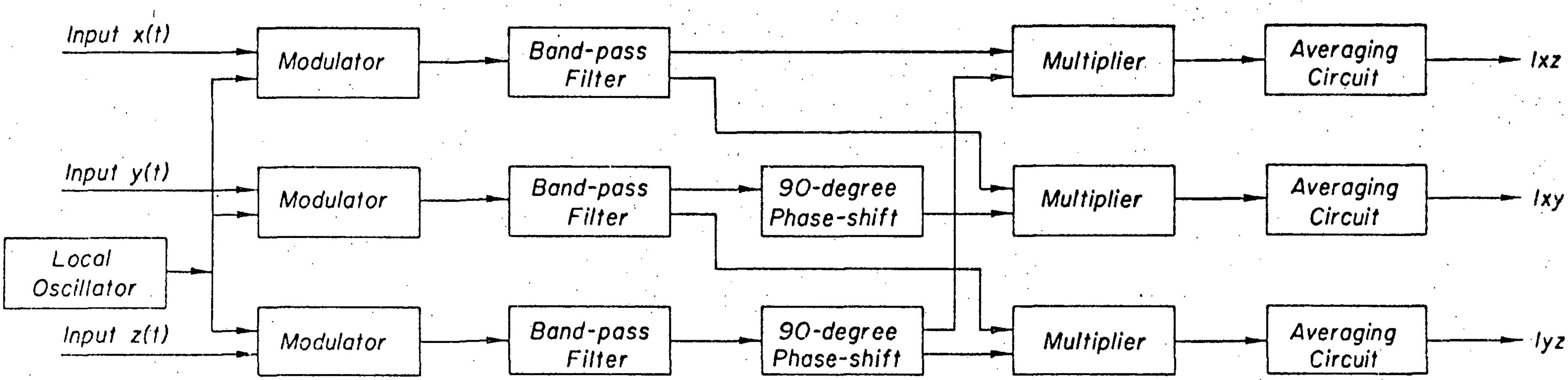
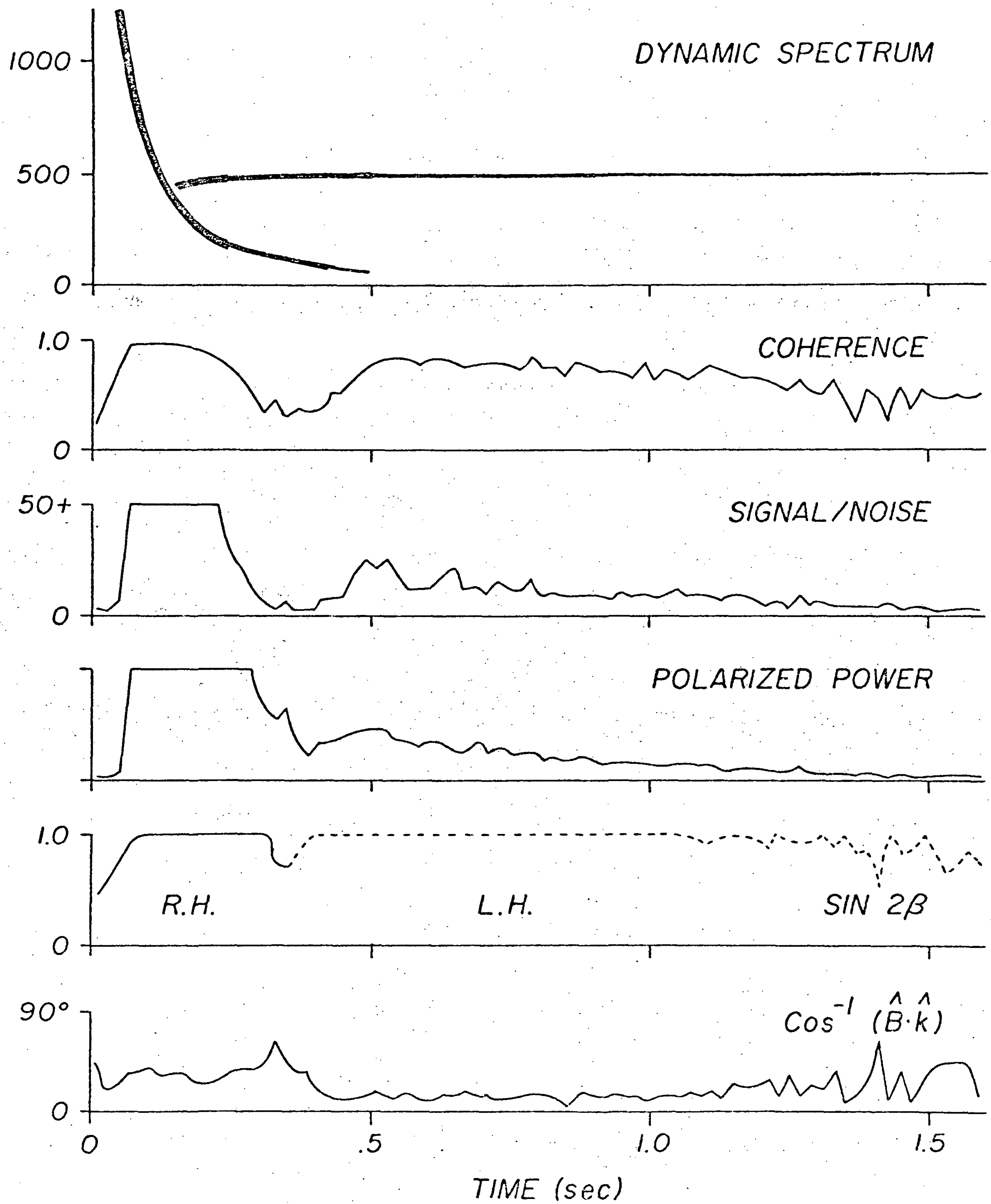
Figure 1

Functional block diagram of an analog circuit for determining the wave normal vector.

Figure 2

Polarization analysis of a proton whistler detected by the OGO-6 search coil magnetometer. The polarized power is given in arbitrary units. The dotted portion of the sin 2β curve corresponds to left-hand polarization.


Input x(t)
Modulator
Band-pass Filter
Multiplier
Averaging Circuit
Ixz
Input y(t)
Modulator
Band-pass Filter
90-degree Phase-shift
Multiplier
Averaging Circuit
Ixy
Local Oscillator
Input z(t)
Modulator
Band-pass Filter
90-degree Phase-shift
Multiplier
Averaging Circuit
Iyz



DYNAMIC SPECTRUM
1000
500
0
COHERENCE
1.0
0
SIGNAL/NOISE
50+
0
POLARIZED POWER
1.0
R.H.
L.H.
SIN 2β
0
Cos^-1 (B̂·k̂)
90°
0
0
.5
1.0
1.5
TIME (sec)