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Abstract

Given a linear time invariant multivariable system $\dot{\underline{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$, $\underline{y} = \hat{C}\underline{x}$ with m inputs and p outputs, Davison [6] has shown that p closed loop poles of the system can be preassigned arbitrarily using constant gain output feedback provided $[\hat{A}, \hat{B}]$ is controllable. This paper shows that if $[\hat{A}, \hat{B}, \hat{C}]$ is controllable and observable, and Rank $\hat{B} = m$, Rank $\hat{C} = p$, then max (m,p) poles of the system can be assigned arbitrarily using constant gain output feedback. Further, it is shown that in some cases more than max (m,p) poles can be arbitrarily assigned. A least square design technique is outlined to approximate the desired pole locations when it is not possible to place all the poles.

Introduction

The design of linear multivariable control systems using output feedback has attracted the attention of several authors. [1-3]. There are two ways of approaching this problem. The first method consists of estimating the states of the system using an observer and use these states in the subsequent design. In the second approach, either static or dynamic feedback of the output is used directly in the control problem and this view is adopted here.

Consider a linear time-invariant multivariable system

$$\underline{\dot{x}} = \hat{A}\underline{x} + \hat{B}\underline{u}$$
$$\underline{y} = \hat{C}\underline{x} \tag{1}$$

where $\underline{x}$ is an n vector of states, $\underline{u}$ is an m vector of inputs and $\underline{y}$ is a vector of outputs. It is well-known that the problem of pole assignment using state feedback is equivalent to the controllability of the pair $(\hat{A},\hat{B})$ [4]. Pole-shifting techniques for multivariable systems using static feedback has been studied by Retallak [5], Davison [6] and others. It has been shown by Davison that if $(\hat{A}, \hat{B})$ is controllable, and if Rank $\hat{C} = p$, then p poles of the system can be arbitrarily placed using output feedback. This paper shows that given $(\hat{A}, \hat{B}, \hat{C})$ controllable and observable, Rank $\hat{B} = m$ and Rank $\hat{C} = p$, then at least max (m,p) poles of the closed loop system can be arbitrarily placed using output feedback.

The above result may be used in designing systems for high integrity in the event of failure of transducers and/or acuators [7]. Due to failure there may be loss of inputs and/or outputs in the system. In such an event this design takes advantage of the unequal number of inputs and outputs so as to assure no loss in pole assignability.

Theorem

Given the system (1) with Rank $\hat{B} = m \leq n$ and Rank $\hat{C} = p \leq n$, then a linear feedback of the output $\underline{u} = K\underline{y}$, where K is a (mxp) constant gain matrix, can always be found such that max(m,p) eigenvalues of the closed loop system can be be made to take preassigned (complex eigenvalues occurring in conjugate pairs) values.

Proof:

Let $(\lambda_1, \lambda_2, ---, \lambda_n)$ and $(\rho_1, \rho_2, --- , \rho_n)$ be the eigenvalues of the open-loop and closed-loop system respectively.

We have

open loop characteristic polynomial $= |sI-\hat{A}| = (s-\lambda_1)(s-\lambda_2) --- (s-\lambda_n)$ (2)

and closed loop characteristic polynomial $= |sI-\hat{A}+\hat{B}K\hat{C}| = (s-\rho_1)(s-\rho_2)---(s-\rho_n)$ (3)

Then

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = \det [I + \hat{B}K\hat{C} (sI-\hat{A})^{-1}]. \quad (4)$$

Choosing $K = fd^T$ where f is a mx1 (column) vector and d^T is a 1xp (row) vector, and using the identity det [I + MN] = det [I + NM], equation (4) becomes

$$\begin{aligned} \frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} &= 1 + d^T\hat{C} (sI-\hat{A})^{-1} \hat{B}f \\ &= 1 + d^T\hat{C}T (sI-T^{-1}\hat{A}T)^{-1} T^{-1}\hat{B}f \\ &= 1 + d^TC (sI-A)^{-1} Bf \end{aligned} \quad (5)$$

where $C = \hat{C}T$, $A = T^{-1}\hat{A}T$, $B= T^{-1}\hat{B}$ and T is a nxn nonsingular matrix.

For clarity, the theorem is initially proved for the case of distinct eigenvalues of $\hat{A}$ and the multiple eigenvalues of $\hat{A}$ are considered in the latter half of the proof.

Distinct Eigenvalues

In this case equation (5) gives

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = 1 + \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \tag{6}$$

The value of α_i depends on the closed loop eigenvalues $(\rho_1, ---, \rho_n)$.

From (5) and (6),

$$d^T C \ (sI-A)^{-1} \ Bf = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \tag{7}$$

Choosing T as a modal matrix equation (7) becomes

$$d^T C \ (sI-\Lambda)^{-1} \ Bf = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \tag{8}$$

where $\Lambda = \text{diag.}\ (\lambda_1, \lambda_2, ---, \lambda_n)$.

Let c^i be the i^{th} column of C and b_i be the i^{th} row of B. Then,

$$\alpha_i = d^T c^i b_i f \quad - \qquad i = 1,\ 2,\ ---,\ n. \tag{9}$$

Case (i)

Let p>m i.e. more outputs than inputs. Choose f_i such that $b_i f = \delta_i \neq 0$ $i = 1,2,---, n$. This can always be done since $b_i \neq 0$, for controllability.

Hence, $d^T c^i = \alpha_i/\delta_i \qquad i = 1, ---, n.$ (10)

This gives $C^T d = \underline{\alpha}$ (11)

where $\underline{\alpha} = \text{col}\ [\alpha_1/\delta_1,\ \alpha_2/\delta_2,\ ---,\ \alpha_n/\delta_n]$.

Now, let C_p be the matrix made of the p independent rows of C^T and $\underline{\alpha}_p$ the corresponding subset of $\underline{\alpha}$. Then,

$$d = C_p^{-1}\ \underline{\alpha}_p \tag{12}$$

Thus $(d_1, d_2, ---, d_p)$ can be chosen corresponding to the p desired pole locations. Once this is done the remaining (n-p) poles are fixed automatically.

Case (ii)

Let $m > p$ i.e. more inputs than outputs.

Choose d_i such that $d^T c^i = \gamma_i \neq 0$ $\quad i = 1, 2, ---, n.$

This can always be done since $c^i \neq 0$, for observability.

Hence, $b_i f = \alpha_i/\gamma_i$ $\quad i = 1, ---, n.$ (13)

This gives $Bf = \underline{\alpha}$ (14)

where $\underline{\alpha} = \text{col}\,[\alpha_1/\delta_1, \alpha_2/\gamma_2, ---, \alpha_n/\gamma_n].$

Since the rank of B is m, there are m independent rows of B, B_m, such that

$$B_m f = \underline{\alpha}_m$$

where $\underline{\alpha}_m$ is the corresponding subset of $\underline{\alpha}$.

$$f = B_m^{-1}\,\underline{\alpha}_m \qquad (15)$$

Thus $(f_1, f_2, ---, f_m)$ can be chosen corresponding to the m desired pole locations and the remaining (n-m) poles are located automatically. From case (i) and case (ii) it is evident that at least max (m,p) poles of the system can be assigned arbitrarily.

Multiple Eigenvalues

Let the eigenvalues of matrix $\hat{A}$ be $\lambda_1, \lambda_2, ---, \lambda_\omega$ with multiplicity $n_1, n_2, ---, n_\omega$ respectively. Choose T such that $A = T^{-1}\hat{A}T$ has the Jordan canonical form with ω blocks of respective sizes $n_1, n_2, ---, n_\omega$ and $\lambda_1, \lambda_2, ---, \lambda_\omega$ the corresponding eigenvalues.

Now, we have

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = \frac{(s-\rho_1)(s-\rho_2)\ ---\ (s-\rho_n)}{(s-\lambda_1)^{n_1}(s-\lambda_2)^{n_2}---(s-\lambda_\omega)^{n_\omega}} \qquad (16)$$

where $\sum_{i=1}^{\omega} n_i = n.$

Equation (16) can be re-written as

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-\hat{A}|} = 1 + \sum_{i=1}^{n_1} \frac{\alpha_i^1}{(s-\lambda_1)^i} + --- + \sum_{i=1}^{n_\omega} \frac{\alpha_i^\omega}{(s-\lambda_\omega)^i} \tag{18}$$

The value of α_i^j ($i=1, ---n_j$, $j=1,---\omega$) depends on the closed loop poles (ρ_1, ρ_2, $--\rho_n$).

From equations (5) and (18), we get

$$d^T C(sI-A)^{-1} Bf = \sum_{i=1}^{n_1} \frac{\alpha_i^1}{(s-\lambda_1)^i} + --- + \sum^{n_\omega} \frac{\alpha_i^\omega}{(s-\lambda_\omega)^i} \tag{19}$$

$(sI-A)^{-1}$ has the quasi-diagonal form diag $[J_1, J_2, ---, J_\omega]$ where J_i is a $n_i \times n_i$ matrix of the form

$$\begin{bmatrix} \frac{1}{(s-\lambda_i)} & \frac{1}{(s-\lambda_i)^2} & ---- & \frac{1}{(s-\lambda_i)^{n_i}} \\ 0 & \frac{1}{(s-\lambda_i)} & ---- & \frac{1}{(s-\lambda_i)^{n_i-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & & \frac{1}{(s-\lambda_i)} & \frac{1}{(s-\lambda_i)^2} \\ \vdots & & & \\ 0 & 0 & ---- & \frac{1}{(s-\lambda_i)} \end{bmatrix} \tag{20}$$

Let $C = [C^1, C^2 ---, C^\omega]$ and $B = B^1, B^2, --- B^\omega]^T$ where C^j is a $p \times n_j$ matrix and B^j is a $n_j \times m$ matrix. Then it can be easily seen that

$$d^T C^j J_j B^j f = \sum_{i=1}^{n_j} \frac{\alpha_i^j}{(s-\lambda_j)^i} \qquad j=1,2,---\omega \tag{21}$$

Further it can be shown that

$$\begin{aligned}
\alpha_1^j &= d^T[C_1^j B_1^j + C_2^j B_2^j + --- + C_{n_j}^j B_{n_j}^j]f \\
\alpha_2^j &= d^T[C_1^j B_2^j + --- + C_{n_j-1}^j B_{n_j}^j]f \\
&\;\vdots \qquad\qquad\qquad\qquad\qquad\qquad j=1,2,---\omega \\
\alpha_{n_j}^j &= d^T C_1^j B_{n_j}^j f.
\end{aligned} \tag{22}$$

Where C_i^j is the i^{th} column of C^j and B_i^j is the i^{th} row of B^j.

In the matrix form equation (22) can be written as

$$d^T \begin{bmatrix}
C_1^1 B_1^1 + C_2^1 B_2^1 + - - - + \quad C_{n_1}^1 B_{n_1}^1 \\
C_1^1 B_2^1 + C_2^1 B_3^1 + - - - + C_{n_1-1}^1 B_{n_1} \\
\vdots \\
C_1^1 B_{n_1-1}^1 + C_2^1 B_{n_1} \\
C_1^1 B_{n_1} \\
\hline
\vdots \\
\hline
C_1^\omega B_1^\omega + C_2^\omega B_2^\omega + - - - + \quad C_{n_\omega}^\omega B_{n_\omega}^\omega \\
C_1^\omega B_2^\omega + C_2^\omega B_3^\omega + - - - + C_{n_\omega-1}^\omega B_{n_\omega}^\omega \\
\vdots \\
C_1^\omega B_{n_\omega-1}^\omega + C_2^\omega B_{n_\omega}^\omega \\
C_1^\omega B_{n_\omega}^\omega
\end{bmatrix} f = \begin{bmatrix}
\alpha_1^1 \\
\alpha_2^1 \\
\vdots \\
\alpha_{n_1-1}^1 \\
\alpha_{n_1}^1 \\
\hline
\vdots \\
\hline
\alpha_1^\omega \\
\alpha_2^\omega \\
\vdots \\
\alpha_{n_\omega-1}^\omega \\
\alpha_{n_\omega}^\omega
\end{bmatrix} \tag{23}$$

Case (i)

Let p>m i.e. more outputs than inputs and $B_i^j f = \delta_i^j$, $i = 1,---,n_j$ and $j=1,---\omega$. For controllability, every row of B corresponding to the last row of each Jordan block of A is linearly independent [8] i.e., $B_{n_j}^j \neq 0$, $j=1,2,---\omega$, are linearly independent.

Now, we can choose $(f_1,f_2,---,f_m)$ such that

$$B_{n_j}^j f = \delta_{n_j}^j \neq 0, \quad j=1,---\omega. \tag{24}$$

Substituting this in equation (23) we get

$$\begin{bmatrix} \delta_1^1 C_1^{1^T} + \delta_2^1 C_2^{1^T} + ---- + \delta_{n_1}^1 C_{n_1}^{1^T} \\ \delta_2^1 C_1^{1^T} + \delta_3^1 C_2^{1^T} + ---- + \delta_{n_1}^1 C_{n_1-1}^{1^T} \\ \vdots \\ \delta_{n_1}^1 C_1^{1^T} \\ \hline \vdots \\ \hline \delta_1^\omega C_1^{\omega^T} + --- \quad ---+ \delta_{n_\omega}^\omega C_{n_\omega}^{\omega^T} \\ \delta_2^\omega C_1^{\omega^T} + ------- + \delta_{n_\omega}^\omega C_{n_\omega-1}^{\omega^T} \\ \vdots \\ \delta_{n_\omega}^\omega C_{n_\omega}^{\omega^T} \end{bmatrix} d = \underline{\alpha}$$

Where $\underline{\alpha} = \text{col}\,[\alpha_1^1,\alpha_2^1 --- \alpha_{n_1}^1,\ \alpha_1^2 --- \alpha_{n_2}^2,\ ---,\ ---\ \alpha_{n_\omega}^\omega]$.

Define a quasi-diagonal matrix M,

$$M \triangleq \text{diag}[M_1, M_2, --- M_\omega]$$

where M_j is given by

$$M_j = \begin{bmatrix} \delta_1^j & \delta_2^j & \cdots & \delta_{n_j}^j \\ \delta_2^j & \delta_3^j & \delta_{n_j}^j & 0 \\ \vdots & \cdot & & \vdots \\ \delta_{n_j}^j & 0 & \cdots & 0 \end{bmatrix}$$

we have $MC^T d = \underline{\alpha}$ (25)

M is a nxn non-singular matrix since $\det M = \prod_{i=1}^{\omega} (\delta_{n_i}^i)^{n_i} \neq 0$ by (24). Hence, Rank M=n and Rank $MC^T = p$. Let C_p be the p independent rows of MC^T and let $\underline{\alpha}_p$ be the corresponding subset of $\underline{\alpha}$. This gives

$$C_p d = \underline{\alpha}_p$$

or

$$d = C_p^{-1} \underline{\alpha}_p \quad (26)$$

Equation (26) is similar to equation (12) and the rest of the proof follows as in the Case (i) of distinct eigenvalues.

Case (ii): m>p i.e. more inputs than outputs.

Let $d^T C_i^j = \delta_i^j \quad i=1,2,---,n_j, \quad j = 1,2,---\omega.$

For observability, every column of C corresponding to the first column of each Jordan block of A is linearly independent [8] i.e. $C_1^j \neq 0$, $j=1,2,---\omega$. are linearly independent.

Now, we can choose $(d_1, d_2, --- d_p)$ such that $d^T C_1^j = \delta_1^j \neq 0$, $j = 1,2,---\omega$. (27)

Substituting this in equation (23) and defining a quasi-diagonal matrix N,

$$N \triangleq \mathrm{diag}[N_1, N_2, --- N_\omega]$$

where N_j is given by

$$\begin{bmatrix} \delta_1^j & \delta_2^j & --- & \delta_{n_j}^j \\ 0 & \delta_1^j & & \delta_{n_j-1}^j \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \delta_1^j \end{bmatrix}$$

It is seen that

$$NBf = \underline{\alpha}.$$

N is a nxn non-singular matrix since det N= $\prod_{i=1}^{\omega} (\delta_1^j)^{n_j} \neq 0$ by (27).

Hence Rank NB=m and let B_m be the m independent rows of NB and let $\underline{\alpha}_m$ be the corresponding subset of $\underline{\alpha}$. This gives

$$B_m f = \underline{\alpha}_m$$

or $$f = B_m^{-1} \underline{\alpha}_m. \tag{28}$$

Equation (28) is similar to equation (15) and the rest of the proof follows as in the case (ii) of distinct eigenvalues.

This completes the proof in the case of multiple eigenvalues.

<u>Nature of The Design Equation (23):</u>

In general, the output feedback gain matrix $K = fd^T$ is obtained by solving the set of n non-linear simultaneous equations in (m+p) variables $(d_1, d_2, ---, d_p, f_1, f_2, --- f_m)$. However, in the proof of the theorem either $(d_1, d_2, --- d_p)$ or $(f_1, f_2, --- f_m)$ are selected arbitrarily and (23) is reduced to a set of linear equations and this assures at least max(m,p) poles can be placed arbitrarily.

In certain cases the non-linear nature of (23) can be exploited to assign more than max(m,p) poles of the closed loop system.

Complex eigenvalues of the matrix $\hat{A}$ present an interesting situation. The Jordan canonical form $A = T^{-1}\hat{A}T$ and the matrices $B=T^{-1}\hat{B}$ and $C = \hat{C}T$ will then be complex matrices. However, K will be real since the complex columns of T^{-1} and elements of $\underline{\alpha}$ occur in conjugate pairs.

In designing the control system using the theorem of this paper, max (m,p) poles of the system can be assigned as desired and this fixes the location of the remaining [n-max (m,p)] poles of the system. Let us call these poles the "dependent poles", $\underline{\beta}$, where $\underline{\beta} = (\rho_{max(m,p)+1}, ---, \rho_{n-1}, \rho_n)$. In some cases, by taking advantage of the non-linear nature of (23) more than max(m,p) poles can be arbitrarily assigned and this reduces the number of dependent poles. However, nothing can be said a priori about the location of these dependent poles. Now, the α_i would be a function of $\underline{\beta}$. By minimizing a least square error criterion of the form

$$J = \sum_{i=1}^{n} q_i \, [d^T C^I B_i f - \alpha_i(\underline{\beta})]^2$$

subject to the constraints $g(\underline{\beta}) \geq 0$ an approximate set of the desired closed loop poles can be realized. The weighting coefficients q_i can be used to control the error between a pole in the desired set and its corresponding pole in the approximate set. A somewhat similar approach has been suggested by Jameson [9] and Fallside [10].

If no satisfactory set of poles results from the least square error approach, e.g., closed loop poles unstable, then a dynamic compensator [3] would be necessary for pole-placement. It should be noted that some systems which need a dynamic compensator for pole placement using [3] can be made to attain any desired closed loop poles using only constant output feedback with the above design method.

Example 1:

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \underline{x}$$

The system is controllable and observable with two unstable poles at 1 and 2. Also, m = 3 and p = 2. According to Davison two poles can be placed arbitrarily. However, according to the theorem stated in this paper three poles can be placed arbitrarily. Pearson would need a first order compensator to control all the poles. Here, it will be shown that by solving the equation (23) in its non-linear form all the four poles of the system can be placed arbitrarily. We have

$d^T C^i B_i f = \alpha_i \qquad i = 1,2,4,4.$

so,

$$f_1 d_1 = \alpha_1$$

$$f_2 d_1 = \alpha_2$$

$$f_3 d_2 = \alpha_3$$

$$(f_1+f_2+f_3)d_2 = \alpha_4.$$

solving these equations with $d_1 = 1$, we get

$$d_2 = \frac{\alpha_4-\alpha_3}{\alpha_1+\alpha_2},\ f_1 = \alpha_1,\ f_2 = \alpha_2 \text{ and } f_3 = \frac{\alpha_3(\alpha_1+\alpha_2)}{\alpha_4-\alpha_3}.$$

$$K = fd^T = \begin{bmatrix} \alpha_1 & (\alpha_4-\alpha_3)/(\alpha_1+\alpha_2) \\ \alpha_2 & \alpha_2(\alpha_4-\alpha_3)/(\alpha_1+\alpha_2) \\ \alpha_3(\alpha_1+\alpha_2)/(\alpha_4-\alpha_3) & \alpha_3 \end{bmatrix}$$

with this choice of K all the closed loop poles can be placed at the desired locations.

If the closed loop poles are desired at -1,-2,-3 and -5, then α_1=-7.2, α_2 = 14, α_3 = 0 and α_4 = 0.2. This gives f_1 = -7.2, f_2 = 14, f_3 = 0, d_1 = 1, d_2 = 1/34.

and
$$K = \begin{bmatrix} -7.2 & -7.2/34 \\ 14 & 14/34 \\ 0 & 0 \end{bmatrix}.$$

Example 2:

$$\underline{\dot{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \underline{x}$$

This problem illustrates the nature of (23) when A has complex open loop poles. The open loop poles are at 1 and $-\frac{1}{2} \pm j\,\sqrt{3}/2$. If the modal matrix T and its inverse are chosen to be .

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + j\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} & -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix} \;\&\; T^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -j\frac{1}{2\sqrt{3}} - \frac{1}{6} & j\frac{1}{2\sqrt{3}} - \frac{1}{6} \\ \frac{1}{3} & j\frac{1}{2\sqrt{3}} - \frac{1}{6} & -j\frac{1}{2\sqrt{3}} - \frac{1}{6} \end{bmatrix}$$

Then A, B and C are given by

$$A = T^{-1}\hat{A}T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} + j\frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & -\frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}, \quad B = T^{-1}\hat{B} = \frac{1}{3}\begin{bmatrix} 1 \\ -\frac{1}{2} - j\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + j\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\text{and } C = \hat{C}T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2} + j\frac{\sqrt{3}}{2} & \frac{1}{2} - j\frac{\sqrt{3}}{2} \end{bmatrix}.$$

If the closed loop poles are chosen to be at $\pm j$ and 1, then we have the set of equations

$$\frac{1}{3}[d_1 + 2d_2]f = \alpha_1 \tag{29}$$

$$\frac{1}{6}[-d_1 + d_2] - j\frac{1}{2\sqrt{3}}[d_1 + d_2] = \alpha_2 \tag{30}$$

$$\frac{1}{6}[-d_1 + d_2] + j\frac{1}{2\sqrt{3}}[d_1 + d_2] = \alpha_3 \tag{31}$$

where $\alpha_1 = 0$, $\alpha_2 = -\frac{1}{2} - \frac{j}{2\sqrt{3}}$ and $\alpha_3 = -\frac{1}{2} + \frac{j}{2\sqrt{3}}$ for the desired pole assignment. Equation (30) and (31) are complex conjugates and give the same set of equations in (d_1, d_2, f). From equation (29) and the real and imaginary parts of (30), we get

$$\begin{aligned} (d_1 + 2d_2)f &= 0 \\ (d_1 - d_2)f &= 3 \\ (d_1 + d_2)f &= 1. \end{aligned} \tag{32}$$

Solving (32) with $f = 1$, gives $d_1 = 2$, $d_2 = -1$ and $K = [2\ -1]$. This choice of K gives the desired pole-placement.

Conclusions:

It is shown that for a controllable, observable system at least max (m,p) poles of the system can be arbitrarily assigned if Rank $\hat{B}$=m and Rank $\hat{C}$=p. Further, it is shown that in some cases more than max (m,p) poles can be arbitrarily assigned. A least square design technique is outlined to get an approximate set for the desired pole placement when it is not possible to place all the poles. The application of this technique for high integrity design of Sikorsky SH-3D Sea King helicopter [11, 12] is under study.

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