Stability of Certain Periodic Solutions of a Forced System with Hysteresis*

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Abstract: Many physical systems exhibit the phenomenon of nonlinear hysteresis when the amplitude of oscillation exceeds some small limit [3]. It has been shown in [2] that under certain conditions there are periodic motions of such a system with small forcing which are near the largest periodic motion of the corresponding unforced system. In this paper, conditions for the stability of these periodic motions are derived.

INTRODUCTION

Consider the vibrations of a unit mass mounted on a spring which exhibits nonlinear hysteresis. A large class of such nonlinear hysteretic behavior can be described in the following manner where, for simplicity of notation, we assume completely elastic behavior for amplitudes of oscillation less than or equal to one unit. Let g be an odd increasing differentiable function on $(-\infty, \infty)$ and let f be an odd differentiable function on $(-\infty,\infty)$, increasing on $(-1, 1)$ with $f(y) = f(1)\mathrm{sgn}(y)$ for $|y| > 1$. The restoring force is then expressed as $-f(x) - g(y)$ where $x(t)$ denotes the position of the system and $y(t)$ is an auxilliary variable with the property that $y'(t) = x'(t)$ for all t except for times $\hat{t}$ such that both $x'(\hat{t}) = 0$ and $\lim_{t\uparrow\hat{t}} |y(t)| > 1$. At such times $\hat{t}$, let $y(t)$ have a jump discontinuity

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with $y(t)$ continuous from the right at $\hat{t}$ and $y(\hat{t}) = -\operatorname{sgn} x''(\hat{t})$.

With the value of y varying as described above, the position of the unforced system with respect to its initial point of equilibrium satisfies

$$x'' + g(x) + f(y) = 0. \tag{1}$$

Any periodic motion of the system must have an amplitude less than or equal to unity. Let $x_o(t)$ be the solution of (1) which satisfies

$$x_o(0) = 1,\ y_o(0) = 1 \text{ and } x_o'(0) = 0.$$

The solution $x_o(t)$ is an even, periodic, odd harmonic function with amplitude one. (To say $x_o(t)$ is odd harmonic means $x_o(t + \tau/2) = -x_o(t)$, where τ denotes the period of $x_o(t)$.)

It has been shown in [2] that if we perturb (1) by a small forcing term $\varepsilon h(t + \eta)$, yielding the differential equation

$$x'' + g(x) + f(y) = \varepsilon h(t + \eta), \tag{2}$$

where h is a differentiable, τ periodic, odd harmonic function, then under certain conditions, given any sufficiently small ε, we can find a $\delta \geq 0$ and an η such that the initial conditions $x(0) = 1 + \delta$, $x'(0) = 0$, $y(0) = 1$ will yield a τ periodic, odd harmonic solution to (2).

Let $x(t, \delta, \eta, \varepsilon)$ denote such a τ periodic solution of (2) with initial conditions $x(0, \delta, \eta, \varepsilon) = 1 + \delta$, $x'(0, \delta, \eta, \varepsilon) = 0$, $y(0) = 1$. If the values of δ, η and ε are understood, we will simply write $x(t)$ instead of $x(t, \delta, \eta, \varepsilon)$.

When discussing the solution of (2) which has its initial conditions perturbed from those of $x(t)$ and $y(t)$, we will use $z(t)$ and $w(t)$. We would like the solution $z(t)$ to (2) with initial conditions

$$z(0) = 1 + \delta + \alpha, \ z'(0) = \beta$$

to be such that for small α and β

$$(3) \quad \lim_{t\to\infty} |z(t) - x(t)| = 0 \text{ and } \lim_{t\to\infty} |z'(t) - x'(t)| = 0 .$$

However, in order to maintain the hysteretic nature of our system, we want $w(\Theta) = 1$ when $z'(\Theta) = 0$ where Θ is small. If $\beta < 0$, this means that we need to find Θ and ξ such that if $z'' + g(z) + f(z-\delta-\xi) = \varepsilon h(t+\eta)$, $z(\Theta) = 1 + \delta + \xi$, $z'(\Theta) = 0$ and $w(\Theta) = 1$, then $z(0) = 1 + \delta + \alpha$ and $z'(0) = \beta$. If $\beta > 0$, the differential equation we are working with changes significantly. In any case there is the possibility that Θ and ξ might not be unique.

These difficulties have prompted the following formulation of the stability problem. We want a solution $z(t)$ to (2) with initial conditions $z(\Theta) = 1 + \delta + \xi$, $z'(\Theta) = 0$ and $w(\Theta) = 1$ to be such that (3) holds for ξ and Θ small.

STABILITY OF $x(t, \delta, \eta, \varepsilon)$

Lemma 1: Let $\phi(t)$ and $\psi(t)$ be solutions of

$$x'' + (g'(x_o(t)) + f'(x_o(t)))x = 0$$

with $\phi(0) = 1$, $\phi'(0) = 0$, $\psi(0) = 0$, $\psi'(0) = 1$. Let $D = f(1) + g(1)$.

Then $\phi(t)\psi'(t) - \phi'(t)\psi(t) \equiv 1$;

$\psi(t) = \frac{-1}{D} x_o'(t)$ and $\psi(t)$ is odd, τ periodic and odd harmonic;

$\phi(t)$ is even and $\phi(t) = Kt\psi(t) + r(t)$ where $r(t)$ is even,

τ periodic and odd harmonic.

Moreover, we have the following table of functional values.

TABLE 1.

Proof: See [2].

Since $x(t)$ is known to exist under certain conditions with η and δ determined as functions of ε, let us write $x(t) = x_o(t) + \varepsilon x_1(t) + o(\varepsilon)$.

Lemma 2: Let $E = 1 + g(1)/f(1)$. $x_1(t)$ is a τ periodic, odd harmonic function such that $x_1(0) = \frac{d\delta}{d\varepsilon}(0) = -\frac{E}{2}\int_o^{\tau/2}\psi(s)h(s + \eta_o)ds$ and $x_1'(0) = 0$, where $\eta_o = \eta(0)$.

Proof: See [2]

Theorem: The solution $x(t, \delta(\varepsilon), \eta(\varepsilon), \varepsilon)$ of (2) with $x(0) = 1 + \delta(\varepsilon) > 1$, $x'(0) = 0$ and $y(0) = 1$ is stable if

$$\varepsilon \int_o^{\tau/2} \{4\phi(s) - K\tau\,(E + 1)\psi(s)\}\, h'(s + \eta_o)ds < 0$$

and is unstable if it is greater than zero.

Proof of Theorem:

Let t_2 be the smallest $t > \Theta$ such that $z'(t) = 0$. Let t_3 be the smallest $t > t_2$ such that $z'(t) = 0$. Consider the transformation

$$T(\Theta,\xi) = (t_2 - \tau/2, -z(t_2) - 1 - \delta).$$

If we can show that $T(\Theta,\xi)$ is near (0,0) when (Θ,ξ) is near (0, 0) we can state that

$$(t_3 - \tau,\ z(t_3) - 1 - \delta) = T^2(\Theta,\xi).$$

If in addition we can show that $\lim_{n\to\infty} T^n(\Theta,\xi) = (0, 0)$ for (Θ,ξ) near (0, 0), we can conclude that (3) holds. We can show that the hypotheses of the preceding two statements are true by proving that the eigenvalues of the linear part of $T(\Theta,\xi)$ are both less than one in absolute value. Also, if one of the eigenvalues exceeds one in absolute value, we will know that $T^n(\Theta,\xi)$ does not approach (0, 0) and therefore that (3) does not hold.

We need an expression for the eigenvalues, λ_1 and λ_2, of the linear part of T. For this purpose we write

$$t_2 - \frac{\tau}{2} = a_o + a_1\Theta + a_2\xi + a_3\varepsilon\Theta + a_4\varepsilon\xi + o(\varepsilon)\Theta + o(\varepsilon)\xi + \text{higher order terms in } \Theta \text{ and } \xi.$$

$$-z(t_2) - 1 - \delta = b_o + b_1\Theta + b_2\xi + b_3\varepsilon\Theta + b_4\varepsilon\xi + o(\varepsilon)\Theta + o(\varepsilon)\xi + \text{higher order terms in } \Theta \text{ and } \xi.$$

We easily see $a_o = b_o = 0$ by considering $\Theta = \xi = 0$. The linear part of T is expressed by the matrix

$$\begin{bmatrix} a_1 + a_3\varepsilon + o(\varepsilon) & a_2 + a_4\varepsilon + o(\varepsilon) \\ b_1 + b_3\varepsilon + o(\varepsilon) & b_2 + b_4\varepsilon + o(\varepsilon) \end{bmatrix}$$

The eigenvalues of this matrix are given by

$$(4)\quad \frac{1}{2}\{a_1+a_3\varepsilon+b_2+b_4\varepsilon\pm((a_1+a_3\varepsilon-b_2-b_4\varepsilon)^2 + 4(a_2+a_4\varepsilon)(b_1+b_3\varepsilon))^{1/2}\}+o(\varepsilon).$$

A preliminary calculation shows that $a_1 = 1$, $b_1 = 0$ and $|b_2| = |(1 - \frac{2}{E})| < 1$. Using this information, (4) can be written as

$$\frac{1}{2}\{1+a_3\varepsilon+b_2+b_4\varepsilon\pm(1+a_3\varepsilon-b_2-b_4\varepsilon+\frac{1}{2}(1+a_3\varepsilon-b_2-b_4\varepsilon)^{-1}4(a_2+a_4\varepsilon)b_3\varepsilon)\} + o(\varepsilon).$$

$$\lambda_1 = 1 + \frac{a_3(1-b_2) + a_2b_3}{1-b_2}\varepsilon + o(\varepsilon)$$

$$\lambda_2 = b_2 + \frac{b_4(1-b_2)-a_2b_3}{1-b_2}\varepsilon + o(\varepsilon)$$

For small ε, $\lambda_1 > |\lambda_2|$ and $|\lambda_2| < 1$. Hence to show $T^n(\Theta,\xi) \to (0, 0)$, we must show $\lambda_1 < 1$ for small ε.

We will now make an extended calculation to determine a_1, a_2, a_3, b_1, b_2, and b_3 so that the critical quantity

$$(a_3(1-b_2) + a_2b_3)/(1-b_2)$$

may be calculated. The preliminary calculation mentioned above is a simple case of this calculation.

Let $z(t, \Theta, \xi, \varepsilon)$ be the solution of (2) which satisfies $z(\Theta, \Theta, \xi, \varepsilon) = 1 + \delta(\varepsilon) + \xi$, $z'(\Theta, \Theta, \xi, \varepsilon) = 0$, $w(\Theta) = 1$. Letting t_2 be the smallest $t > \Theta$ such that $x'(t, \Theta, \xi, \varepsilon) = 0$, we have

$$z'' + g(z) + f(z-\delta-\xi) = \varepsilon h(t+\eta) \text{ for } \Theta \leq t \leq t_2.$$

The quantity t_2 is a function of Θ, ξ, and ε. For convenience we let $\hat{\xi} = -z(t_2, \Theta, \xi, \varepsilon) - 1 - \delta$ and $\hat{\Theta} = t_2 - \tau/2$. In order to find a_1, a_2, a_3, b_1, b_2 and b_3, we need to determine the partial derivatives of $\hat{\xi}$ and $\hat{\Theta}$ with respect to ξ and Θ at $\xi = \Theta = 0$. First we will find $\partial\hat{\Theta}/\partial\Theta$. Since $z'(t_2, \Theta, \xi, \varepsilon) = 0$ and $\partial\hat{\Theta}/\partial\Theta = \partial t_2/\partial\Theta$, we have

$$(5)\qquad \frac{\partial z'}{\partial t_2}\cdot\frac{\partial t_2}{\partial\Theta}+\frac{\partial z'}{\partial\Theta}=0 \text{ and } \frac{\partial\hat{\Theta}}{\partial\Theta}=-\frac{\partial z'}{\partial\Theta}\Big/\frac{\partial z'}{\partial t_2}\,.$$

We now need $\partial z'/\partial t_2$ and $\partial z'/\partial\Theta$ at $\xi = \Theta = 0$.

$$\frac{\partial z'}{\partial t_2} = z''(t_2, 0, 0, \varepsilon) = z''(\tau/2, 0, 0, \varepsilon) =$$

$$-g(-1-\delta) - f(-1-2\delta) + \varepsilon h(\tau/2 + \eta) = g(1 + \delta) + f(1) - \varepsilon h(\eta).$$

Since, by page 27 of [1], $\partial z/\partial\Theta$ satisfies $u'' + g'(z)u + f'(z-\delta-\xi)u = 0$ with $u(\Theta) = 0$ and $u'(\Theta) = g(1+\delta+\xi) + f(1) - \varepsilon h(\Theta+\eta)$, we have that at $\Theta = \xi = 0$, $\partial z/\partial\Theta$ satisfies

$$u'' + g'(x)u + f'(x-\delta)u = 0,\ u(0) = 0,\ u'(0) = g(1+\delta) + f(1) - \varepsilon h(\eta).$$

We must now find $u'(\tau/2)$ since

$$\frac{\partial z'}{\partial\Theta}\ (t_2, 0, 0, \varepsilon) = \frac{\partial z'}{\partial\Theta}\ (\tfrac{\tau}{2}, 0, 0, \varepsilon) = u'(\tau/2).$$

Using the fact that $x(t) = x_o(t) + \varepsilon x_1(t) + o(\varepsilon)$, we have

$$u'' + \{g'(x_o) + f'(x_o) + \varepsilon x_1(g''(x_o) + f''(x_o)) - \delta f''(x_o)\}\ u+o(\varepsilon) = 0.$$

Recalling $\phi(s)$ and $\psi(s)$ from Lemma 1, we write

$$(6)\qquad u(t) = u(0)\phi(t) + u'(0)\psi(t) - \int_o^t\{\phi(s)\psi(t) - \phi(t)\psi(s)\}\{\varepsilon x_1(s)(g''(x_o(s)) + f''(x_o(s))) - \delta f''(x_o(s))\}\ u(s)ds + o(\varepsilon).$$

We may replace δ by $\gamma\ \varepsilon$ in (6) since, by Lemma 2, $\delta = \gamma\ \varepsilon + o(\varepsilon)$, where $\gamma = x_1(0)$. We may also replace $u(s)$ in the right hand side of (6) by $u'(0)\psi(s)$ since $u(t) = u'(0)\psi(t) + o(1)$. Using these modifications, we find $u'(\tau/2) = u'(0)[-1 + \varepsilon\int_o^{\tau/2}\{\phi(s) - \frac{K\tau}{2}\ \psi(s)\}$.

$$\{x_1(s)(g''(x_o(s)) + f''(x_o(s))) - \gamma f''(x_o(s))\}\psi(s)ds] + o(\varepsilon).$$

Since $u'(0) = \partial z'/\partial t_2$ when $\xi = \Theta = 0$, line (5) yields

$$\partial\hat{\Theta}/\partial\Theta = -u'(\tau/2)/u'(0)$$

which, after considerable computation, leads to

$$\frac{\partial\hat{\Theta}}{\partial\Theta} = 1 - \frac{\varepsilon}{D}\int_0^{\tau/2} \{\phi'(s) - \frac{K\tau}{2}\psi'(s)\}h(s+\eta_o)ds + o(\varepsilon).$$

Now we will find $\partial\hat{\Theta}/\partial\xi$ at $\Theta = \xi = 0$. Since $z'(t_2, \Theta, \xi, \varepsilon) = 0$ and $\partial\hat{\Theta}/\partial\xi = \partial t_2/\partial\xi$, we have

(7) $$\frac{\partial z'}{\partial t_2} \cdot \frac{\partial t_2}{\partial\xi} + \frac{\partial z'}{\partial\xi} = 0 \text{ and } \frac{\partial\hat{\Theta}}{\partial\xi} = -\frac{\partial z'}{\partial\xi} \Big/ \frac{\partial z'}{\partial t_2} .$$

$\partial z/\partial\xi$ satisfies $u'' + g'(z)u + f'(z-\delta-\xi)(u-1) = 0$ with $u(\Theta) = 1$, $u'(\Theta) = 0$. At $\varepsilon = \Theta = \xi = 0$, $\frac{\partial z}{\partial\xi}$ satisfies

(8) $u'' + g'(x_o)u + f'(x_o)u = f'(x_o)$ with $u(0) = 1$, $u'(0) = 0$.

We must now find $u'(\tau/2)$ since

$$\frac{\partial z'}{\partial\xi}(t_2, 0, 0, 0) = \frac{\partial z'}{\partial\xi}(\tfrac{\tau}{2}, 0, 0, 0) = u'(\tau/2).$$

$$u'(\tfrac{\tau}{2}) = u(0)\phi'(\tfrac{\tau}{2}) + u'(0)\psi'(\tfrac{\tau}{2}) + \int_o^{\tau/2}\{\phi(s)\psi'(\tfrac{\tau}{2}) - \phi'(\tfrac{\tau}{2})\psi(s)\}f'(x_o(s))ds$$

$$= -\frac{K\tau}{2} - \int_o^{\tau/2}(Ks\psi(s)+r(s))f'(x_o(s))ds - \frac{K\tau}{2D}\int_o^{\tau/2}f'(x_o(s))x_o'(s)ds = \frac{K\tau}{2E}(1-E)$$

Thus, we have $\frac{\partial z'}{\partial\xi}(t_2, 0, 0, \varepsilon) = \frac{\partial z'}{\partial\xi}(t_2, 0, 0, 0) + o(1) = \frac{K\tau}{2E}(1-E) + o(1)$.

Since $\partial z'/\partial t_2 = g(1+\delta) + f(1) - \varepsilon h(\eta) = D + o(1)$ when $\xi = \Theta = 0$, line (7) yields

$$\frac{\partial\hat{\Theta}}{\partial\xi} = \frac{-\frac{K}{2E}(1-E) + o(1)}{D + o(1)} = \frac{K\tau}{2DE}(E-1) + o(1).$$

Now we will find $\partial\hat{\xi}/\partial\Theta$ at $\Theta = \xi = 0$. Since $\frac{\partial\hat{\xi}}{\partial\Theta} = \frac{-\partial z}{\partial\Theta}(t_2, 0, 0, \varepsilon) = -u(\tau/2)$ in (6), we have

$$\frac{\partial\hat{\xi}}{\partial\Theta} = -u'(0)\psi(\tau/2) + \int_o^{\tau/2}\{\phi(s)\psi(\tfrac{\tau}{2}) - \phi(\tfrac{\tau}{2})\psi(s)\}.$$
$$\{\varepsilon x_1(s)(g''(x_o(s))+f''(x_o(s))) - \gamma\varepsilon f''(x_o(s))\}u'(0)\psi(s)ds + o(\varepsilon)$$

which reduces to

$$\frac{\partial\hat{\xi}}{\partial\Theta} = \varepsilon\int_0^{\tau/2}\psi'(s)h(s+\eta_0)ds + o(\varepsilon)$$

Finally we will find $\partial\hat{\xi}/\partial\xi$ at $\Theta = \xi = 0$. Since

$$\frac{\partial\hat{\xi}}{\partial\xi} = \frac{-\partial z}{\partial\xi}(t_2, 0, 0, \varepsilon) = \frac{-\partial z}{\partial\xi}(t_2, 0, 0, 0) + o(1) = -u(\tfrac{\tau}{2}) + o(1),$$

where u(t) is specified in (8), we have

$$\frac{\partial\hat{\xi}}{\partial\xi} = -u(0)\phi(\tfrac{\tau}{2}) - \int_0^{\tau/2}\{\phi(s)\psi(\tfrac{\tau}{2})-\phi(\tfrac{\tau}{2})\psi(s)\}f'(x_0(s))ds + o(1)$$

$$= 1 + \frac{1}{D}\int_0^{\tau/2}f'(x_0(s))x_0'(s)ds + o(1) = 1 - \frac{2}{E} + o(1).$$

We can now write an explicit matrix expressing the linear part of the transformation $T(\Theta,\xi) = (\hat{\Theta}, \hat{\xi})$.

$$\begin{bmatrix} a_1 + a_3\varepsilon + o(\varepsilon) & a_2 + o(1) \\ b_1 + b_3\varepsilon + o(\varepsilon) & b_2 + o(1) \end{bmatrix} = \begin{bmatrix} \partial\hat{\Theta}/\partial\Theta & \partial\hat{\Theta}/\partial\xi \\ \partial\hat{\xi}/\partial\Theta & \partial\hat{\xi}/\partial\xi \end{bmatrix} =$$

$$\begin{bmatrix} 1 - \frac{\varepsilon}{D}\int_0^{\tau/2}\{\phi'(s) - \frac{K\tau}{2}\psi'(s)\}h(s+\eta_0)ds + o(\varepsilon) & \frac{K\tau}{2DE}(E-1)+o(1) \\ \varepsilon\int_0^{\tau/2}\psi'(s)h(s+\eta_0)ds + o(\varepsilon) & 1 - \frac{2}{E} + o(1) \end{bmatrix}$$

Using this result we find

$$(a_3(1-b_2) + a_2b_3)/(1-b_2) = + \frac{1}{D}\int_0^{\tau/2}\{-\phi'(s) + \frac{K\tau}{4}(E+1)\psi'(s)\}h(s+\eta_0)ds$$

$$= \frac{1}{D}\int_0^{\tau/2}\{\phi(s) - \frac{K\tau}{4}(E+1)\psi(s)\}h'(s+\eta_0)ds.$$

This completes the proof of the stability theorem.

BIBLIOGRAPHY

1. Coddington, E. A., and Levinson, N., "Theory of Ordinary Differential Equations," McGraw-Hill, New York, 1955.

2. Drew, J. H., "Periodic Solutions of a Forced System with Hysteresis," International Journal of Nonlinear Mechanics, Vol. 7, 1972, pp. 93-99.

3. Pisarenko, G. S., "Vibrations of Elastic Systems Taking Account of Energy Dissipation in the Material", WADD Technical Documentary Report No. 60-582, 1962.

TABLE 1.

t	ψ	ψ'	ϕ	ϕ'	r	r'
0	0	1	1	0	1	0
$\tau/2$	0	-1	-1	$\frac{-K\tau}{2}$	-1	0
τ	0	1	1	$K\tau$	1	0