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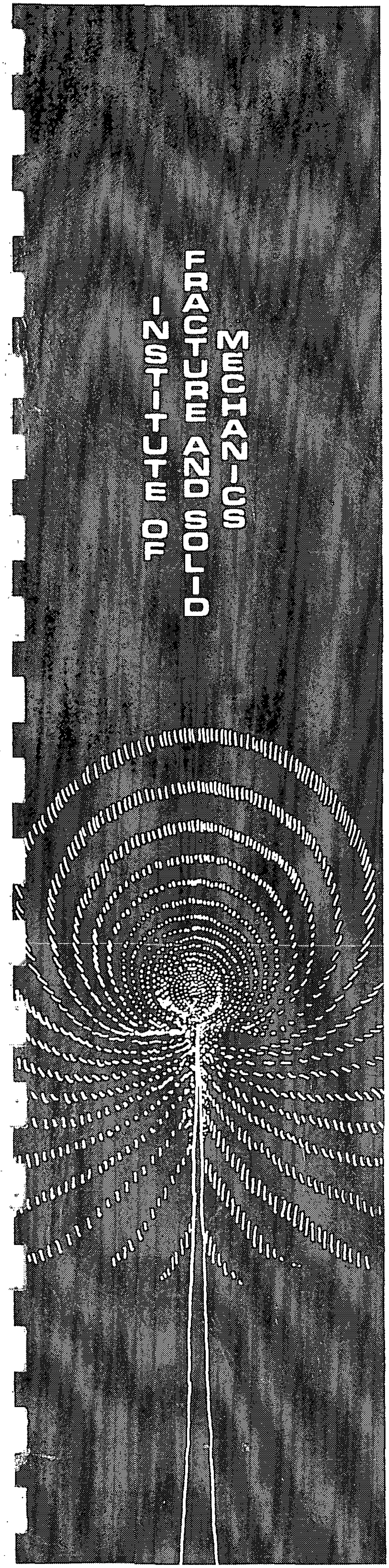

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ABSTRACT

A semi-infinite strip held rigidly on its short end is considered. Loads in the strip at infinity (far away from the fixed end) are prescribed. Integral transform technique is used to provide an exact formulation of the problem in terms of a singular integral equation. Stress singularity at the strip corner is obtained from the singular integral equation which is then solved numerically. Stresses along the rigid end are determined and the effect of the material properties on the stress intensity factor is presented.

The method can also be applied to the problem of a laminate composite with a flat inclusion normal to the interfaces.

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INTRODUCTION

In the past there has been considerable interest in the strip problems from both mechanics and mathematical points of view. A few good solutions exist [1-6]; however, none of these methods appears to provide a solution which can directly give the correct behavior of the stresses at the corners and does not present any convergence difficulties in the evaluation of the contact stresses. In [1-4], the stress singularities at the corners were ignored. Benthem [5] used the Laplace transform technique to formulate the problem but required a knowledge of stress singularity by alternate means [7]. In [6], Vorovich and Kopasenko considered a problem of a semi-infinite strip loaded symmetrically on its longitudinal sides and reduced the problem to a singular integral equation. Again stress singularities were obtained by separate analysis and the integral equation was solved using a very cumbersome method. With the recent advances in the analysis and solution of such singular integral equations [8], it seems possible to extract the corner singularities and solve the derived integral equation in [6]. The method presented here reduces the problem to a singular integral equation which contains the stress singularity and can be solved numerically giving very satisfactory results.

To solve the problem of a strip with fixed end we will first consider the plane elastostatic problem for a flat inclusion situated centrally in an infinite strip with sides free of tractions (Figure 1). A similar problem with the inclusion lying parallel to the interface was considered in [9] where stress field around the inclusion tip was developed. Since the problem considered here is symmetric with respect to the axis containing the inclusion, a rigid inclusion has the same effect as an inextensible inclusion with zero bending rigidity. The case of an elastic inclusion can also be treated without any added difficulty [9]. In this paper, we will concentrate on the case when the inclusion extends to the surfaces, thus reducing the problem to that of a semi-infinite strip with fixed end. Numerical results will be presented only for the strip problem.

The method can easily be applied to the case where the strip containing the inclusion is bonded to two similar half planes with different elastic properties than the strip. This problem makes the equations and the algebra much more involved, but presents no conceptual difficulty.

FORMULATION OF PROBLEM

Consider a strip of width 2h in plane strain condition with a rigid or inextensible inclusion of length 2a ($a \le h$) located centrally along the x-axis (Figure 1a). The shear modulus and Poisson's ratio of the strip are μ and ν respectively. Let the strip be uniformly loaded at both ends far away from the inclusion. Figure 1b shows the main problem of interest which is obtained from Figure 1a by extending the inclusion to the surfaces $x = \pm h$, or letting $a = h$. Note that the chosen problem has symmetry with respect to both x and y axes.

The problem described above can be recovered by the superposition of two subproblems I and II as shown in Figure 2. In I, we simply have a uniformly loaded strip without any inclusion. Solution of I is given as

$$\begin{aligned}
&\sigma_{yy}{}^{I}(x,y) = \sigma_{yy}{}^{I}(x,\infty) = p_o \\
&\sigma_{xy}{}^{I}(x,y) = \sigma_{xx}{}^{I}(x,y) = 0 \\
&u_I(x,0) = -\varepsilon_o x \ ; \qquad \varepsilon_o = \frac{(3-\kappa)}{8\mu}\, p_o \\
&v_I(x,0) = 0
\end{aligned} \tag{1}$$

where $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

Problem II is the disturbance problem where the input function must be the displacement in x-direction at $y = 0$ plane, equal to the negative of that in I. Hence the boundary conditions for II become

$$\begin{aligned}
&\sigma_{xx}(h,y) = \sigma_{xy}(h,y) = 0 \ , \\
&v(x,0) = 0 \ , \qquad |x| < h,
\end{aligned} \tag{2}$$

$$\begin{aligned}
&u(x,0) = \varepsilon_o x \ , \qquad |x| < a, \\
&\sigma_{xy}(x,0) = 0 \ , \qquad a < |x| < h.
\end{aligned} \tag{3}$$

Due to the symmetry of the problem, it is sufficient to consider only one quarter of the problem. Note that $y=0$, $|x|<a$ is a singular surface across which the displacements are continuous and the stress vector suffers a discontinuity. The displacement and stress fields for the strip can be written as a superposition of two transform solutions. One is the solution for an infinite strip ($|x|\le h$, $|y|<\infty$) in plane strain with symmetry about $x=0$ and $y=0$ planes, and the other is the half plane solution with $x=0$ as the plane of symmetry. Expressing the solution as

$$u(x,y) = -\frac{2}{\pi}\int_0^\infty \{\frac{1}{\eta}[f(\eta) - \frac{\kappa-1}{2}\,g(\eta)]\sinh(\eta x)$$

$$+ xg(\eta)\cosh(\eta x)\}\cos\eta y\,d\eta - \frac{2}{\pi}\int_0^\infty \frac{\phi(\xi)}{\xi}\,(\kappa-\xi y)e^{-\xi y}\sin\xi x\,d\xi$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \{\frac{1}{\eta}[f(\eta) + \frac{\kappa+1}{2}\,g(\eta)]\cosh(\eta x)$$

$$+ xg(\eta)\sinh(\eta x)\}\sin\eta y\,d\eta + \frac{2}{\pi}\int_0^\infty y\phi(\xi)e^{-\xi y}\cos\xi x\,d\xi$$

$$\frac{\sigma_{xx}(x,y)}{2\mu} = -\frac{2}{\pi}\int_0^\infty [f(\eta)\cosh(\eta x) + \eta x g(\eta)\sinh(\eta x)]\cos\eta y\,d\eta$$

$$-\frac{2}{\pi}\int_0^\infty \phi(\xi)[\frac{\kappa+3}{2} - \xi y]e^{-\xi y}\cos\xi x\,d\xi \tag{4}$$

$$\frac{\sigma_{yy}(x,y)}{2\mu} = \frac{2}{\pi}\int_0^\infty \{[f(\eta) + 2g(\eta)]\cosh(\eta x)$$

$$+ \eta x g(\eta)\sinh(\eta x)\}\cos\eta y\,d\eta + \frac{2}{\pi}\int_0^\infty \phi(\xi)[\frac{\kappa-1}{2} - \xi y]e^{-\xi y}\cos\xi x\,d\xi$$

$$\frac{\sigma_{xy}(x,y)}{2\mu} = \frac{2}{\pi}\int_0^\infty \{[f(\eta) + g(\eta)]\sinh(\eta x)$$

$$+ \eta x g(\eta)\cosh(\eta x)\}\sin\eta y d\eta + \frac{2}{\pi}\int_0^\infty \phi(\xi)[\frac{\kappa+1}{2} - \xi y]e^{-\xi y}\sin\xi x d\xi$$

it may be seen that this solution identically satisfies the third condition $v(x,0) = 0$ of (2). The three unknowns f, g and ϕ must then be determined by using the first two conditions of (2) and the mixed boundary conditions (3). The first two conditions of (2) may be written as

$$f(\eta)\cosh(\eta h) + \eta h g(\eta)\sinh(\eta h)$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\xi\phi(\xi)}{\eta^2+\xi^2}[\frac{\kappa+1}{2} + \frac{2\eta^2}{\eta^2+\xi^2}]\cos\xi h d\xi$$

$$f(\eta)\sinh(\eta h) + g(\eta)[\sinh(\eta h) + \eta h\cosh(\eta h)] \tag{5}$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\eta\phi(\xi)}{\eta^2+\xi^2}[\frac{\kappa+1}{2} - \frac{2\xi^2}{\eta^2+\xi^2}]\sin\xi h d\xi$$

Mixed boundary conditions (3) can be expressed as follows:

$$\frac{\partial u(x,0)}{\partial x} = -\frac{2}{\pi}\lim_{y\to 0^+}\int_0^\infty (\kappa-\xi y)\phi(\xi)e^{-\xi y}\cos\xi x d\xi$$

$$-\frac{2}{\pi}\int_0^\infty \{[f(\eta) - \frac{\kappa-3}{2}g(\eta)]\cosh(\eta x) + \eta x g(\eta)\sinh(\eta x)\}d\eta$$

$$= \varepsilon_0 , \qquad |x| < a \tag{6}$$

$$\frac{\sigma_{xy}(x,0^+)}{2\mu} = \frac{\kappa+1}{\pi}\int_0^\infty \phi(\xi)\sin\xi x d\xi = 0 , \qquad a < |x| < h \tag{7}$$

Note that displacement derivative is used in (6) instead of displacement in order to have a dimensional consistency in (6) and (7).

A new unknown function G(x) is now defined as the shear stress along the line y = +0 which is to be distinguished from the shear stress at y = -0 as the stress vector suffers a discontinuity across this boundary. Due to symmetry, the two values should be numerically equal and negative of each other. The function G(x) is written as

$$\sigma_{xy}(x,0^{+}) = G(x) \;, \qquad |x| < h \tag{8}$$

From (7), $G(x) = 0$, $a < |x| < h$. Inverting the integral obtained from (7) and (8) we have

$$\phi(\xi) = -\frac{1}{\mu(\kappa+1)} \int_0^a G(t)\sin\xi t\,dt \tag{9}$$

Solving (5) simultaneously we obtain

$$\begin{aligned}
\frac{1}{2} f(\eta) &= \frac{D_1(\eta)[\sinh(\eta h) + \eta h\cosh(\eta h)] - D_2(\eta)\eta h\sinh(\eta h)}{\sinh(2\eta h) + 2\eta h} \\
\frac{1}{2} g(\eta) &= \frac{-D_1(\eta)\sinh(\eta h) + D_2(\eta)\cosh(\eta h)}{\sinh(2\eta h) + 2\eta h}
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
D_1(\eta) &= -\frac{2}{\pi}\int_0^\infty \frac{\xi\phi(\xi)}{\eta^2+\xi^2}\left[\frac{\kappa+1}{2} + \frac{2\eta^2}{\eta^2+\xi^2}\right]\cos\xi h\,d\xi \\
D_2(\eta) &= -\frac{2}{\pi}\int_0^\infty \frac{\eta\phi(\xi)}{\eta^2+\xi^2}\left[\frac{\kappa+1}{2} - \frac{2\xi^2}{\eta^2+\xi^2}\right]\sin\xi h\,d\xi
\end{aligned} \tag{11}$$

In order to reduce (6) to a singular integral equation f and g must be substituted in it from (10) and then (9) should be used to obtain the equation in terms of G(x). In this symmetric problem shear stress is an odd function in x; therefore, $G(x) = -G(-x)$. Using this property, the first integral in (6) when combined with (9) yields a Cauchy kernel:

$$\lim_{y\to 0^+} \int_{-a}^{a} G(t)dt \int_0^\infty e^{-\xi y} \sin\xi(t-x)d\xi = \int_{-a}^{a} \frac{G(t)}{t-x} dt \tag{12}$$

Now substituting ϕ from (9) into (11) and changing the order of integration gives

$$\begin{aligned} &2\mu(\kappa+1)D_1(\eta) \\ &\quad = \frac{2}{\pi} \int_{-a}^{a} G(t)dt \int_0^\infty \frac{\xi}{\eta^2+\xi^2} \left[\frac{\kappa+1}{2} + \frac{2\eta^2}{\eta^2+\xi^2}\right]\sin\xi(h-t)d\xi \\ &2\mu(\kappa+1)D_2(\eta) \\ &\quad = -\frac{2}{\pi} \int_{-a}^{a} G(t)dt \int_0^\infty \frac{\eta}{\eta^2+\xi^2} \left[\frac{\kappa+1}{2} - \frac{2\xi^2}{\eta^2+\xi^2}\right]\cos\xi(h-t)d\xi \end{aligned} \tag{13}$$

From the tables of Fourier Transform in [10] we have

$$\begin{aligned} &\int_0^\infty \frac{\cos\xi z}{\eta^2+\xi^2} d\xi = \frac{\pi}{2\eta} e^{-\eta z} \\ &\int_0^\infty \frac{\xi}{(\eta^2+\xi^2)^2} \sin\xi z d\xi = \frac{\pi}{4\eta} z e^{-\eta z} \end{aligned} \tag{14}$$

By differentiating (14) with respect to z we obtain

$$\int_0^\infty \frac{\xi}{\eta^2+\xi^2} \sin\xi z d\xi = \frac{\pi}{2} e^{-\eta z}$$

$$\int_0^\infty \frac{\xi^2}{(\eta^2+\xi^2)^2} \cos\xi z d\xi = \frac{\pi}{4\eta} [1 - \eta z]e^{-\eta z} \tag{15}$$

Using (14) and (15) and replacing z by h-t, (13) can now be written as

$$2\mu(\kappa+1)D_1(\eta) = \int_{-a}^{a} G(t)[\frac{\kappa+1}{2} + \eta(h-t)]e^{-\eta(h-t)}dt$$

$$2\mu(\kappa+1)D_2(\eta) = - \int_{-a}^{a} G(t)[\frac{\kappa-1}{2} + \eta(h-t)]e^{-\eta(h-t)}dt \tag{16}$$

The singular integral equation obtained from (6) using relations (12), (10) and (16) can now be expressed as

$$\kappa \int_{-a}^{a} \frac{G(t)}{t-x} dt + \int_{-a}^{a} G(t)K(t,x)dt = - \mu(\kappa+1)\pi\varepsilon_0 , \quad |x| < a \tag{17}$$

where

$$K(t,x) = \int_0^\infty k(t,x,\eta)e^{-\eta(h-t)}d\eta$$

$$k(t,x,\eta) = \frac{e^{-\eta h}}{1 + 4\eta h e^{-2\eta h} - e^{-4\eta h}} \Big[\cosh(\eta x)\{1 + [\kappa + 2\eta(h-t)][\kappa - 2 + 2\eta h] + 2e^{-2\eta h}(1 - \kappa + \eta t)\} - 2\eta x \sinh(\eta x)\{\kappa + 2\eta(h-t) - e^{-2\eta h}\}\Big] \tag{18}$$

The kernel K(t,x) is bounded for all values of t and x in (-a,a) if a < h. The integral equation (17) should be solved subject to the following equilibrium condition:

$$\int_{-a}^{a} G(t)dt = 0 \tag{19}$$

For $a < h$, the solution of (17) is rather straightforward [8], and will not be considered in this paper. The actual problem of interest (Figure 1b) is the case when $a = h$. For this case the kernel $K(t,x)$ is no longer bounded for all values of t and x, and contains point singularities at $t = h$ and $x = \pm h$. These singularities can be extracted by using the asymptotic value of the integrand $k(t,x,\eta)$ as $\eta \to \infty$ in (18). Let

$$K_s(t,x) = \int_0^\infty k_\infty(t,x,\eta)e^{-\eta(h-t)}d\eta \tag{20}$$

where $K_s(t,x)$ is the singular part of the kernel. From (18) it follows that

$$k_\infty(t,x,\eta) = 2e^{-\eta h}\Big[\cosh(\eta x)\{\frac{(\kappa-1)^2}{2} + \eta[\kappa h + (\kappa-2)(h-t)] + 2\eta^2 h(h-t)\} - \eta x \sinh(\eta x)\{\kappa + 2\eta(h-t)\}\Big] \tag{21}$$

Using the following result [10]

$$\int_0^\infty \eta^m e^{-\eta(2h-t)} \begin{Bmatrix} \sinh(\eta x) \\ \cosh(\eta x) \end{Bmatrix} d\eta = \frac{d^m}{dt^m} \int_0^\infty e^{-\eta(2h-t)} \begin{Bmatrix} \sinh(\eta x) \\ \cosh(\eta x) \end{Bmatrix} d\eta$$

$$= \frac{d^m}{dt^m}\left[\frac{1}{(2h-t)^2 - x^2}\begin{Bmatrix} x \\ 2h-t \end{Bmatrix}\right] \tag{22}$$

the singular kernel $K_s(t,x)$ now becomes

$$K_s(t,x) = \frac{(\kappa-1)^2(2h-t)}{[(2h-t)^2 - x^2]}$$

$$+ \frac{2}{[(2h-t)^2 - x^2]^2} \Big[\{\kappa h + (\kappa-2)(h-t)\}\{(2h-t)^2 + x^2\}$$

$$- 2\kappa(2h-t)x^2 + \frac{4(h-t)}{(2h-t)^2 - x^2} \{h(2h-t)[(2h-t)^2 + 3x^2]$$

$$- x^2[3(2h-t)^2 + x^2]\Big] \tag{23}$$

Also, we observe that $k(t,x,\eta) \to \infty$ near $\eta = 0$. This can be easily handled by isolating the value of the integral near $\eta = 0$. We have

$$K_0(t,x) = \int_0^\infty k_0(t,x,\eta)e^{-\eta(h-t)}d\eta \tag{24}$$

where

$$k_0(t,x,\eta) = \frac{(3-\kappa)(1-\kappa)}{8\eta h}$$

Therefore,

$$K_0(t,x) = -\frac{(3-\kappa)(1-\kappa)}{8h} \ln(h-t) \tag{25}$$

Note that $K_0(t,x)$ is independent of x and has a weak singularity in the sense that it is square integrable, hence does not need any special attention.

To analyze the behavior of the unknown function $G(t)$ near the end points, dominant part of the equation consisting of the Cauchy kernel and the singular kernel $K_s(t,x)$ must be considered. We can express these dominant terms as

$$\frac{1}{\pi}\int_{-h}^{h} G(t)\left[\frac{\kappa}{t-x} + \frac{1}{2}\{\kappa^2 - 3 + 12(h-x)\frac{d}{dx} - 4(h-x)^2\frac{d^2}{dx^2}\}\frac{1}{2h-t-x}\right.$$

$$\left. + \frac{1}{2}\{\kappa^2 - 3 - 12(h+x)\frac{d}{dx} - 4(h+x)^2\frac{d^2}{dx^2}\}\frac{1}{2h-t+x}\right]dt$$

$$= -\mu(\kappa+1)\varepsilon_0 + A(x) , \qquad |x| < h \tag{26}$$

where A(x) is a bounded function containing the terms coming from the Fredholm kernel in (17), i.e.,

$$A(x) = -\int_{-h}^{h} [K(t,x) - K_s(t,x)]G(t)dt$$

G(t) is assumed to have integrable singularities at $t = \pm h$ and, following [11], may be expressed as

$$G(t) = \frac{H(t)}{(h^2-t^2)^{\alpha}} = \frac{H(t)e^{\pi i\alpha}}{(t-h)^{\alpha}(t+h)^{\alpha}} , \qquad |t| < h \tag{27}$$

where $0 < Re(\alpha) < 1$ and H(t) satisfies a Hölder condition in the closed interval $-h \le t \le h$. Procedure for determining α requires studying (26) as in [11, Chapter 4] and has been presented in detail in [8].

Considering the following sectionally holomorphic function

$$\phi(z) = \frac{1}{\pi}\int_{-h}^{h}\frac{G(t)}{t-z}dt = \frac{1}{\pi}\int_{-h}^{h}\frac{H(t)e^{\pi i\alpha}}{(t-h)^{\alpha}(t+h)^{\alpha}(t-z)}dt \tag{28}$$

according to [8, Chapter 4]

$$\phi(z) = \frac{H(-h)}{(2h)^{\alpha}}\frac{e^{\pi i\alpha}}{\sin\pi\alpha}\frac{1}{(z+h)^{\alpha}} - \frac{H(h)}{(2h)^{\alpha}}\frac{1}{\sin\pi\alpha(z-h)^{\alpha}} + \phi_0(z) \tag{29}$$

where $\phi_0(z)$ is bounded everywhere except possibly at the end points $\pm h$ where it may have the following behavior:

$$|\phi_0(z)| < \frac{H_0(\pm h)}{(z\pm h)^{\alpha_0}}\,, \qquad \mathrm{Re}(\alpha_0) < \mathrm{Re}(\alpha) \tag{30}$$

Now taking the limit of (29) as $z \to x$, $2h+x$ and $2h-x$, we find

$$\phi(x) = \frac{H(-h)\cot\pi\alpha}{(2h)^\alpha(h+x)^\alpha} - \frac{H(h)\cot\pi\alpha}{(2h)^\alpha(h-x)^\alpha} + \phi^*(x)\,, \qquad |x| < h$$

$$\phi(2h+x) = -\frac{H(h)}{(2h)^\alpha\sin\pi\alpha}\,\frac{1}{(h+x)^\alpha} + \phi_1^*(x)\,, \qquad h < 2h+x < 3h$$

$$\phi(2h-x) = -\frac{H(h)}{(2h)^\alpha\sin\pi\alpha}\,\frac{1}{(h-x)^\alpha} + \phi_2^*(x)\,, \qquad h < 2h-x < 3h$$

(31)

where around the end points ϕ^*, ϕ_1^*, and ϕ_2^* have behavior similar to that of $\phi_0(z)$.

Substituting (31) into (26) we obtain

$$\frac{1}{(2h)^\alpha\sin\pi\alpha(h+x)^\alpha}\left[\kappa H(-h)\cos\pi\alpha + \frac{H(h)}{2}\{(\kappa^2-3) + 12\alpha - 4\alpha(\alpha+1)\}\right]$$

$$- \frac{H(h)}{(2h)^\alpha\sin\pi\alpha(h-x)^\alpha}\left[\kappa\cos\pi\alpha - \frac{1}{2}\{(\kappa^2-3) + 12\alpha - 4\alpha(\alpha+1)\}\right]$$

$$= P(x) \tag{32}$$

where $P(x)$ contains all the bounded functions.

Since $G(t)$ is an odd function of t, hence $H(t) = -H(-t)$, and since $H(\pm h) \neq 0$, equation (32) can only be satisfied if

$$2\kappa\cos\pi\alpha = \kappa^2 + 1 - 4(\alpha-1)^2 \tag{33}$$

which is the characteristic equation to determine α. The equation is identical to the one derived in [7] for a wedge with stress-displacement boundary conditions. It depends only on the Poisson's ratio of the strip and gives a real value of α for any material, i.e., for $0 \leq \nu \leq 0.5$. If $a < h$ as in Figure 1a, only singular kernel is a Cauchy kernel and the characteristic equation becomes

$$\cot\pi\alpha = 0 \ , \qquad \alpha = 1/2 \tag{34}$$

which is the well known singularity at the tip of a flat inclusion [9]. For the strip problem, the singular integral equation can be written as

$$\kappa \int_{-h}^{h} \frac{G(t)}{t-x}\, dt + \int_{-h}^{h} G(t)[K_s(t,x) + K_0(t,x)]dt$$

$$+ \int_{-h}^{h} G(t)K_F(t,x)dt = -\mu(\kappa+1)\pi\varepsilon_0 \ , \qquad |x| < h \tag{35}$$

$$K_F(t,x) = \int_{0}^{\infty} [k(t,x,\eta) - k_\infty(t,x,\eta) - k_0(t,x,\eta)]e^{-\eta(h-t)}dt$$

$K_s(t,x)$, $K_0(t,x)$, $k_\infty(t,x,\eta)$ and $k_0(t,x,\eta)$ are given by equations (23), (25), (21) and (24) respectively. $K_F(t,x)$ is a Fredholm kernel for $a \leq h$.

SOLUTION OF THE INTEGRAL EQUATION

To solve (35) we first normalize the dimensions with respect to h by the following transformations:

$$\tau = \frac{t}{h}, \qquad y = \frac{x}{h}, \qquad G(t) = G(h\tau) = \phi(\tau) \tag{36}$$

Hence (35) can be expressed as

$$\int_{-1}^{1} \phi(\tau)\left[\frac{1}{\tau-y} + \frac{h}{\kappa}\{K_s(h\tau,hy) + K_0(h\tau,hy) + K_F(h\tau,hy)\}\right]d\tau$$

$$= -\frac{\mu(\kappa+1)\pi\varepsilon_0}{\kappa}, \qquad |y| < 1 \tag{37}$$

and (27) becomes

$$\phi(\tau) = \frac{\psi(\tau)}{(1-\tau^2)^{\alpha}} \tag{38}$$

where α is given by (33). Equation (37) can now be solved by using Gauss-Jacobi Integration formula. The method has been previously used and described in [8]. Equation (37) must be solved subject to the additional condition (19). We obtain a set of N×N simultaneous algebraic equations given as

$$\sum_{j=1}^{N} A_j\psi(\tau_j)\left[\frac{1}{\tau_j-y_i} + \frac{h}{\kappa}\{K_s(h\tau_j,hy_i) + K_0(h\tau_j,hy_i)\right.$$

$$\left. + K_F(h\tau_j,hy_i)\}\right] = -\frac{\mu(\kappa+1)\pi\varepsilon_0}{\kappa}$$

$$\sum_{j=1}^{N} A_j\psi(\tau_j) = 0 \tag{39}$$

where, noting that $(1-\tau^2)^{-\alpha}$ is the weight function of Jacobi polynomials $P_N^{(-\alpha,-\alpha)}(\tau)$, the constants τ_j and y_i were shown to be the roots of the following equations [9]:

$$P_N^{(-\alpha,-\alpha)}(\tau_j) = 0 \ , \qquad (j = 1,\ldots,N)$$

$$P_{N-1}^{(1-\alpha,1-\alpha)}(y_i) = 0 \ , \qquad (i = 1,\ldots,N-1)$$

and A_j's are the corresponding weighting constants [9]. $\psi(\tau_j)$ are numerically computed from (39). The shear stress $\sigma_{xy}(x,0)$ can then be expressed as

$$\sigma_{xy}(x,0) = G(x) = \frac{h^{2\alpha}\psi(\frac{x}{h})}{(h^2-x^2)^{\alpha}} \ , \qquad |x| < h \tag{40}$$

NORMAL STRESS $\sigma_{yy}(x,0)$ AND STRESS INTENSITY FACTOR

After solving for the shear stress for the disturbance problem, the only remaining important quantity of interest is the normal stress $\sigma_{yy}(x,0)$. To evaluate σ_{yy}, we start with the fourth equation in (4) and use (9), (10), (12) and (16). For $y = 0$ we find

$$\sigma_{yy}(x,0) = \frac{\kappa-1}{\kappa+1}\frac{1}{\pi}\int_{-h}^{h}\frac{G(t)}{t-x}\,dt + \frac{2}{\kappa+1}\frac{1}{\pi}\int_{-h}^{h} G(t)[K_{10}(t,x)$$

$$+ K_{1s}(t,x) + K_{1F}(t,x)]dt \ , \qquad |x| < h \tag{41}$$

where

$$K_{1F}(t,x) =$$

$$\int_0^{\infty}[k_1(t,x,\eta) - k_{1\infty}(t,x,\eta) - k_{10}(t,x,\eta)]e^{-\eta(h-t)}d\eta$$

and

$$K_{10}(t,x) = -\frac{1-\kappa}{2h}\log(h-t)$$

$$K_{1s}(t,x) = \frac{(1-3\kappa)(2h-t)}{(2h-t)^2 - x^2}$$

$$+ \frac{2}{[(2h-t)^2 - x^2]^2}\Big[\{\kappa h - 3(h-t)\}\{(2h-t)^2 + x^2\}$$

$$- 2\kappa(2h-t)x^2 + \frac{4(h-t)}{(2h-t)^2 - x^2}\{h(2h-t)[(2h-t)^2 + 3x^2]$$

$$- x^2[3(2h-t)^2 + x^2]\}\Big] \tag{42}$$

$$k_1(t,x,\eta) = \frac{e^{-\eta h}}{1 + 4\eta h e^{-2\eta h} - e^{-4\eta h}}\Big[\cosh(\eta x)\{1$$

$$- [\kappa + 2\eta(h-t)][3 - 2\eta h] + e^{-2\eta h}(3 - \kappa + 2\eta t)\}$$

$$- 2\eta x \sinh(\eta x)\{\kappa + 2\eta(h-t) - e^{-2\eta h}\}\Big]$$

$$k_{1\infty}(t,x,\eta) = e^{-\eta h}\Big[\cosh(\eta x)\{1 - [\kappa + 2\eta(h-t)](3 - 2\eta h)$$

$$- 2\eta x \sinh(\eta x)\{\kappa + 2\eta(h-t)\}\Big]$$

$$k_{10}(t,x,\eta) = \frac{1-\kappa}{2\eta h}$$

To determine the behavior of $\sigma_{yy}(x,0)$ near the corner points, we must consider the dominant part of the equation (41) which can be written as

$$(\kappa+1)\sigma_{yy}(x,0) = \frac{1}{\pi}\int_{-h}^{h} G(t)\Big[\frac{\kappa-1}{t-x}$$

$$+ \{-(3\kappa+5) + 2(\kappa+7)(h-x)\frac{d}{dx} - 4(h-x)^2\frac{d^2}{dx^2}\}\frac{1}{2h-t-x}$$

$$+ \{-(3\kappa+5) - 2(\kappa+7)(h+x)\frac{d}{dx} - 4(h+x)^2\frac{d^2}{dx^2}\}\frac{1}{2h-t+x}\Big]dt \tag{43}$$

Substituting (27) in (43) and using the relations (31), the dominant part of the normal stress becomes

$$(\kappa+1)\sigma_{yy}(x,0) = \frac{1}{(2h)^{\alpha}\sin\pi\alpha}[(\kappa-1)(\cos\pi\alpha + 1) - 2(\kappa+1)(\alpha-1)$$

$$+ 4(\alpha-1)^2][\frac{H(-h)}{(h+x)^{\alpha}} - \frac{H(h)}{(h-x)^{\alpha}}] , \qquad x \to \pm h \tag{44}$$

Stress intensity factors K_1, K_2 can be defined as

$$K_1 = \lim_{x\to h} \sqrt{2}\,(h-x)^{\alpha}\sigma_{yy}(x,0)$$

$$K_2 = \lim_{x\to h} \sqrt{2}\,(h-x)^{\alpha}\sigma_{xy}(x,0) \tag{45}$$

Using (27) and (44), these can be rewritten as

$$K_1 = -\frac{\sqrt{2}\,H(h)}{(\kappa+1)(2h)^{\alpha}\sin\pi\alpha}[(\kappa-1)(\cos\pi\alpha + 1) - 2(\kappa+1)(\alpha-1)$$

$$+ 4(\alpha-1)^2]$$

$$K_2 = \sqrt{2}\,\frac{H(h)}{(2h)^{\alpha}} \tag{46}$$

NUMERICAL RESULTS AND DISCUSSION

The total solution of the problem shown in Figure 1b is now the sum of the two problems I and II. Hence,

$$\sigma_{yy}{}^{T}(x,0) = \sigma_{yy}{}^{I}(x,0) + \sigma_{yy}(x,0) = p_0 + \sigma_{yy}(x,0)$$

and (47)

$$\sigma_{xy}{}^{T}(x,0) = \sigma_{xy}{}^{I}(x,0) + \sigma_{xy}(x,0) = \sigma_{xy}(x,0)$$

Since the problem of main interest here is the semi-infinite strip constrained at $y = 0$, the results only for the case of $(a = h)$ are presented. Figures 3 and 4 show the variations of normal and shear stresses, respectively, along the fixed end for various values of the Poisson's ratio. When the Poisson's ratio is zero, we do not have a disturbance problem and the solution of the total problem is identical to that of problem I (Figure 2). Thus, for $\nu = 0$

$$\begin{aligned} \sigma_{xy}{}^{T}(x,0) &= 0 \\ \sigma_{yy}{}^{T}(x,0) &= p_0 \end{aligned} \tag{48}$$

As the Poisson's ratio of the strip increases (to a maximum value 0.5), effect of the disturbance problem (II in Figure 2) increases as shown in these figures. Also, for higher value of the Poisson's ratio, a larger value of the power α of the stress singularity is obtained. This effect is depicted clearly from graphs 3 and 4 by the behavior of stresses near $x = h$.

Figure 5 shows the variation of the stress intensity factor K_2 (as defined in (46)) with respect to the Poisson's ratio of the strip. As seen from (46), the stress intensity factor K_1 depends on K_2 and their ratio K_2/K_1 is a function of the power of the stress singularity α and the Poisson's ratio. From (46)

$$\frac{K_2}{K_1} = -\frac{(\kappa+1)\sin\pi\alpha}{[(\kappa-1)(\cos\pi\alpha + 1) - 2(\kappa+1)(\alpha-1) + 4(\alpha-1)^2]} \tag{49}$$

Figure 5 also shows a variation of this ratio K_2/K_1 with respect to ν. Negative sign in (49) comes from the fact that we have defined the stress intensity factors near $x = h$ (45) where we have negative shear stress and positive normal stress. The physical significance of the ratio K_2/K_1 is clear in the case of an elastic strip pressing against a much stiffer body (i.e., p_0 is negative). In this problem if the coefficient of friction, f, is greater than K_2/K_1, it may be assumed that no sliding would occur between the strip and the adjoining body, i.e., the contact condition is that of perfect adhesion, and the solution given in this paper would be valid. If $f < K_2/K_1$, the problem becomes that of an elastic punch on a rigid half space with friction. From Figure 5 it is seen that, in testing whether the end condition is that of perfect adhesion or sliding for a given strip under compression, as a first approximation one may assume that $K_2/K_1 = \nu$.

REFERENCES

[1] Horvay, G., "The End Problem of Rectangular Strips", J. Appl. Mech., Vol. 20, No. 1, pp. 87-94, 1953.

[2] Horvay, G., "Biharmonic Eigenvalue Problem of the Semi-Infinite Strip", Quart. Appl. Math., Vol. 15, No. 1, pp. 65-81, 1957.

[3] Zorski, H., "A Semi-Infinite Strip with Discontinuous Boundary Conditions", Arch. Mech. Stosowanej, Vol. 10, No. 3, p. 371, 1958.

[4] Theocaris, P. S., "The Stress Distribution in a Semi-Infinite Strip Subjected to a Concentrated Load", Trans. J. Appl. Mech., Vol. 26, No. 3, pp. 401-406, 1959.

[5] Benthem, J. P., "A Laplace Transform Method for the Solution of Semi-Infinite and Finite Strip Problems in Stress Analysis", Quart. J. Mech. and Appl. Math., Vol. 16, No. 4, pp. 413-429, 1963.

[6] Vorovich, I. I., and Kopasenko, V. V., "Some Problems in the Theory of Elasticity for a Semi-Infinite Strip", PMM, Vol. 30, No. 1, pp. 109-115, 1966.

[7] Williams, M. L., "Stress Singularities Resulting from Various Boundary Conditions in Angular Corners of Plates in Extension", J. Appl. Mech., Vol. 19, No. 4, p. 526, 1952.

[8] Erdogan, F., Gupta, G. D., and Cook, T. S., "The Numerical Solutions of Singular Integral Equations", METHODS OF ANALYSIS AND SOLUTIONS TO CRACK PROBLEMS, ed., G. C. Sih, Walters-Noordhoff Publishing, 1972.

[9] Erdogan, F., and Gupta, G. D., "Stresses Near a Flat Inclusion in Bonded Dissimilar Materials", Int. J. Solids Structures, Vol. 8, pp. 533-547, 1972.

[10] Erdelyi, A. (ed.), TABLES OF INTEGRAL TRANSFORMS, Vol. 1, McGraw-Hill, New York, 1953.

[11] Muskhelishvili, N. I., SINGULAR INTEGRAL EQUATIONS, P. Noordhoff Ltd., Groningen, Holland, 1953.

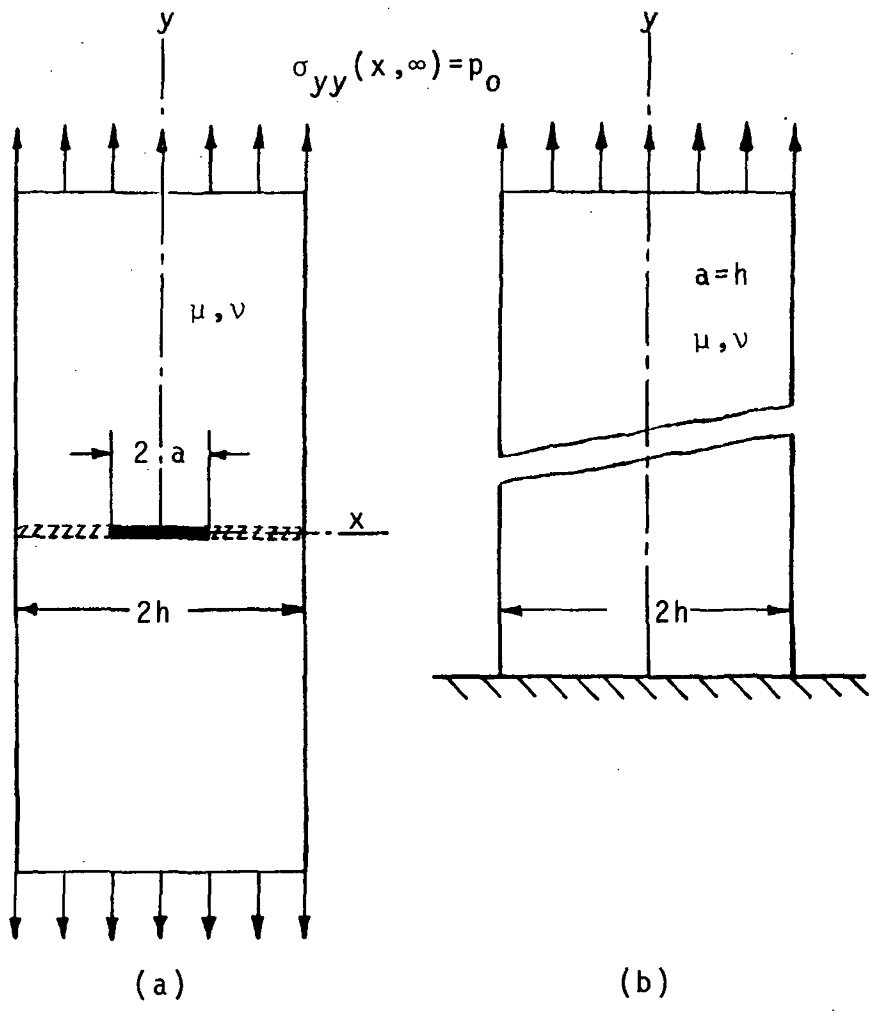

Figure 1. Geometry of Infinite Strip with a Flat Inclusion and a Semi-Infinite Strip (a=h).

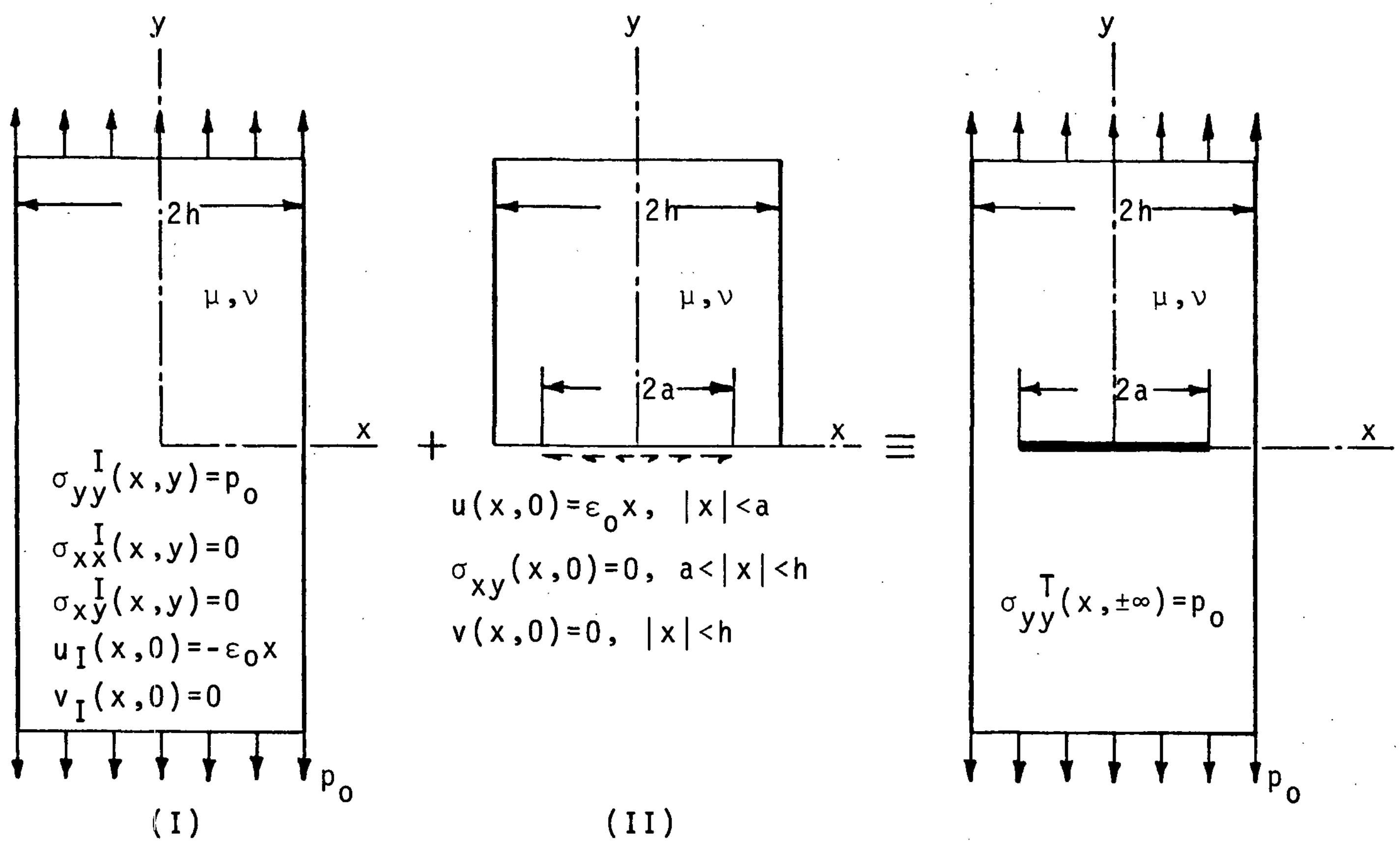

Figure 2. Superposition of Two Solutions to Give the Total Solution.

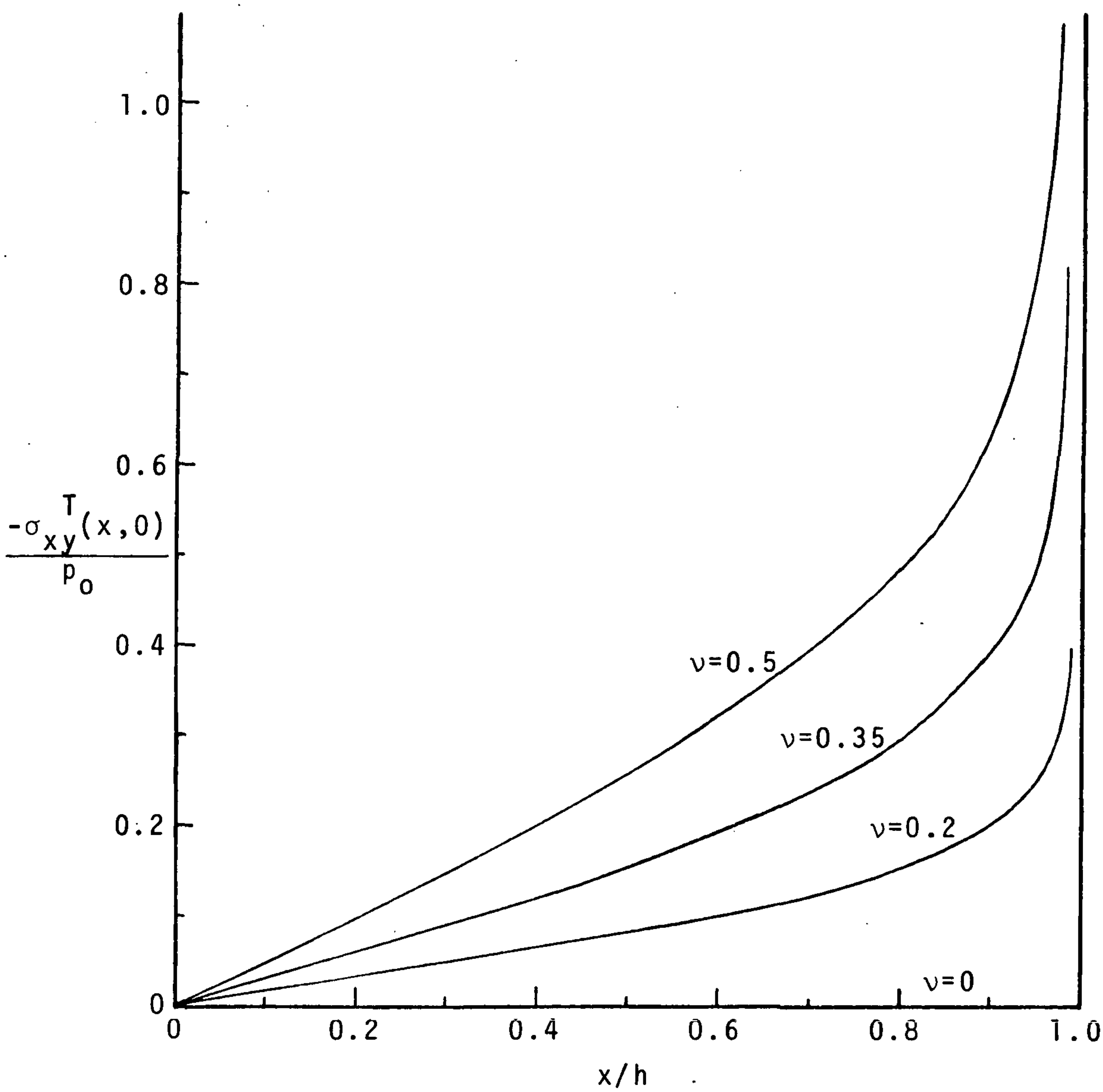

Figure 3. Shear Stress vs. Poisson's Ratio for the Semi-Infinite Strip.

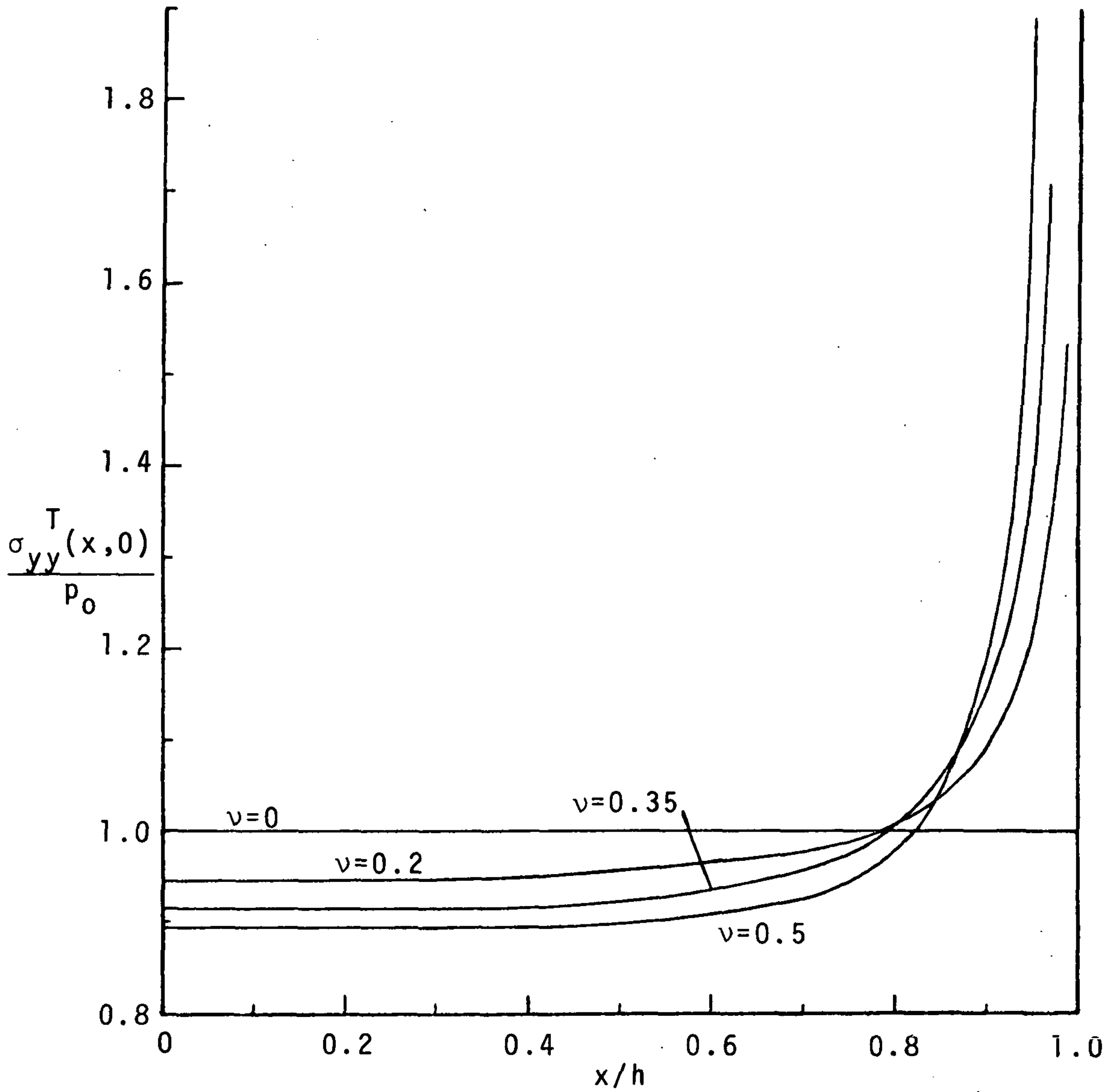

Figure 4. Normal Stress vs. Poisson's Ratio for the Semi-Infinite Strip.

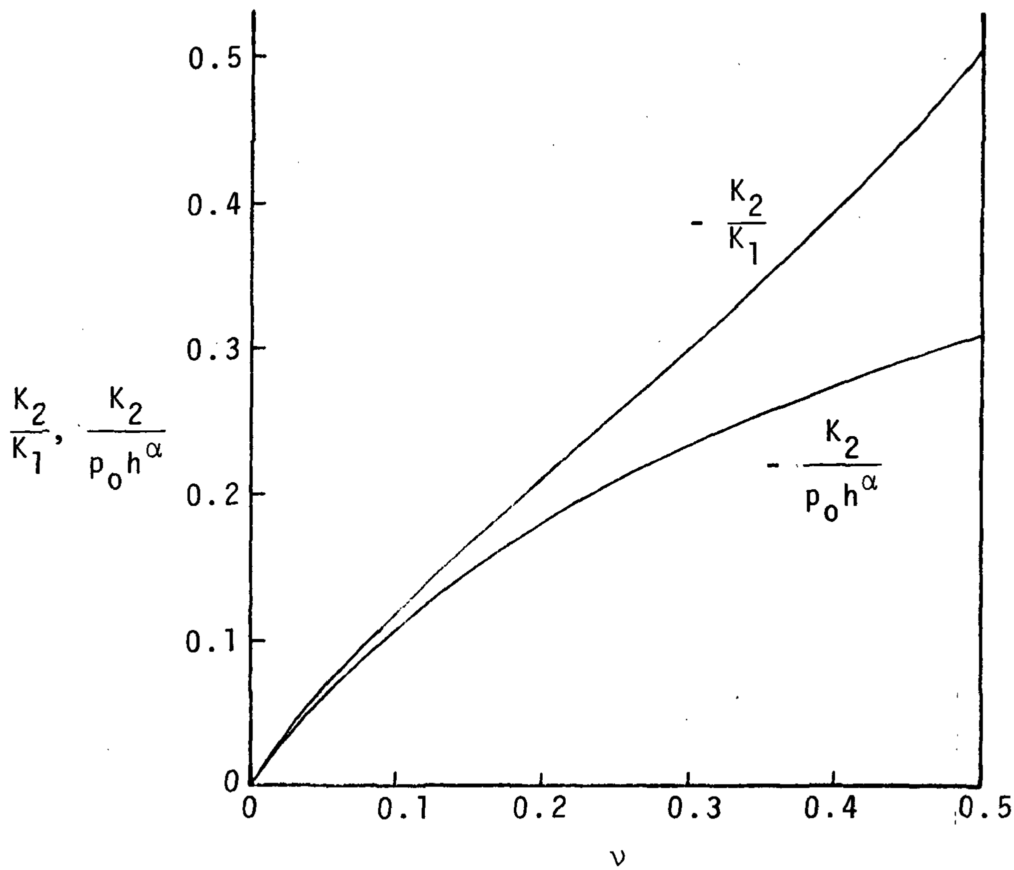

Figure 5. Stress Intensity Factor $\frac{K_2}{p_o h^\alpha}$ and K_2/K_1 vs. ν.