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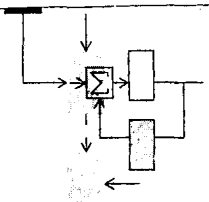

ADAPTIVE CONTROL OF STOCHASTIC LINEAR SYSTEMS WITH UNKNOWN PARAMETERS

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This report is based on the unaltered thesis of Richard Tse-Min Ku submitted in partial fulfillment of the requirements for the degrees of Master of Science and Electrical Engineer at the Massachusetts Institute of Technology in May, 1972. This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory with partial support provided by the Air Force Office of Scientific Research under grants AFOSR 70-1941 and 72-2273 (DSR Project 73935) and by the NASA Ames Research Center under grant NGL-22-009-(124) (DSR Project 76265).

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Chairman, Departmental Committee on Graduate Students

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Submitted to the Department of Electrical Engineering on May 12, 1972, in partial fulfillment of the requirements for the degrees of Master of Science and Electrical Engineer.

ABSTRACT

This thesis considers the problem of optimal control of linear discrete-time stochastic dynamical system with unknown and, possibly, stochastically varying parameters on the basis of noisy measurements. It is desired to minimize the expected value of a quadratic cost functional. Since the simultaneous estimation of the state and plant parameters is a nonlinear filtering problem, the extended Kalman filter algorithm is used. The open-loop feedback optimal control technique is investigated as a computationally feasible solution to the adaptive stochastic control problem. The open-loop feedback optimal control system adaptive gains depend on the current and future uncertainty of the parameters estimation. Thus, the standard Separation Theorem does not hold in this problem. Suboptimal control system in which Separation Theorem is arbitrarily enforced is also considered. The identifier is the same as that of the open-loop feedback optimal control system. Several qualitative and asymptotic properties of the open-loop feedback optimal control and the enforced separation scheme are discussed. Simulation results via Monte Carlo method show that, in terms of the performance measure, for stable systems the open-loop feedback optimal control system is slightly better than the enforced separation scheme, while for unstable systems the latter scheme is far better.

Thesis Supervisor: Michael Athans
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CHAPTER 1

INTRODUCTION

1.1 Brief Historical Review

The theory of stochastic optimal control of linear systems with known dynamics with respect to quadratic performance criterion is fairly well developed [1],[2],[3],[4]. The uncertainties arise from the plant and observation disturbances and initial states of the system. The statistical laws of these uncertainties are assumed to be known. A class of such problems has been considered in discrete time by Joseph and Tou [5] and Gunckel and Franklin [6], and in continuous time by Wonham [7]. Under fairly general assumptions of the Gaussian noise structure, the Separation Theorem or the Certainty-Equivalence Principle [1],[2] is constructive in studying the optimal control problem of purely stochastic systems. The optimal closed-loop stochastic control can be obtained by combining the solution to two separate problems — optimal linear estimation of the states and optimal feedback control of the corresponding deterministic system. The results have been extended to more general performance criteria by Striebel [8] and Wonham [9].

However, in many practical control systems, the dynamics of the system are not completely known. Such problems occur in a variety of engineering designs of aerospace and process control systems. In the most general setting, the plant and observation parameters, the various noises, the initial conditions and/or description of the inputs are imperfectly known. We assume that we know the structure of the dynamics. In our approach it is assumed that all the uncertainties are random processes with known statistics.

The control of linear systems with unknown plant dynamics has to be parameter adaptive. An adaptive control system should have an identifier that generates the estimates of the states, the system parameters (adaptive filtering) and their levels of uncertainty. Based on the identification of the plant, the adaptive controller makes a decision followed by modification or activation.

If the plant is imperfectly known because of random time-varying parameters, then the initial identification, decision, and modification procedures must be done continuously. This constant self-organization of the system is characteristic of all adaptive systems. The controller should reflect (1) the initial uncertainty about the system and the desire to minimize that uncertainty; (2) the dependence upon current estimates and the confidence one can attach to these values, and (3) the ultimate and basic objective to minimize the cost functional. Thus the control input must be used for the identification of parameters and for attaining the desired system response. The "dual" nature of adaptive control is clearly emphasized in this philosophy of adaptive systems. The feedback estimation controller subsystems approach implies that the Separation Theorem will not hold in adaptive systems. We shall find that the adaptive control gains depend upon the parameter estimation accuracy.

The problem of controlling a system with imperfectly known parameters operating in a stochastic environment has been considered by several people. An optimal solution of this problem may be obtained based on Feldbaum's "dual" control approach [10]. Practical on-line

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computation of the optimal closed-loop control is not currently feasible, however, due to computer limitations. Rigorous close approximations to the optimal solution such as the "parameter-adaptive self-organizing" approach proposed by Stein and Saridis [24] contain computational algorithms too complicated for practical implementation. Therefore, different suboptimal but practical solution methods were proposed in the literature. Lee [30] has suggested an arbitrary separation of the parameter identification, state estimation, and control. Identification algorithms based on stochastic approximation [39] or maximum likelihood [40] preclude the use of dynamic feedback control depending on the current estimates of the unknown parameters. Farison <u>et al</u> [23] have considered a suboptimal closed-loop adaptive control scheme for unknown systems with perfect measurements using conditional quadratic cost function to force separation of identification and control. Florentin [22] and Murphy [25] considered the stochastic optimal control of linear systems with unknown but constant gain in discrete time by formulating the identification as a linear estimation problem and approximating the predictor equations and, using suboptimal approach, reduced it to a two-point-boundary value problem. Gorman and Zaborszky [26] considered the problem in continuous time and arrived at results similar to Murphy's [25]. Assuming separation, Saridis and Lobbia [27] considered an online stochastic approximation algorithm for parameter identification and showed that the per-interval feedback controller gave better performance than the overall optimal feedback controller. Schmidt [28] worked out a linear perturbation-controller for systems with unknown parameters by finding the optimal open-loop control minimizing the cost functional which was expanded in a power series. Aoki [4], Spang [11], Bar-Shalom and Sivan [12], Dreyfus [13],[14], and Curry [15] all used the optimal open-loop feedback approach for linear stochastic discrete-time systems to obtain a suboptimal closed-loop control. Tse and Athans [16],[29] showed that the use of this design concept leads to a stochastic control system that is adaptive and computationally feasible for on-line implementation.

In this thesis, the open-loop feedback optimal technique is used to consider the problem of adaptive control of linear discrete-time systems whose poles and zeros are unknown based on inaccurate measurements, with respect to a quadratic cost functional. The unknown parameter values may be time-varying and random. The disturbances in the state and measurement equations are assumed to be additive, white Gaussian stationary noise sequences. All the uncertainties are assumed to have known statistical laws.

The simultaneous estimation of state and plant parameters is a nonlinear filtering problem. Since the truly optimal nonlinear estimator cannot be implemented exactly with current digital computers, one is forced to use a suboptimal estimation algorithm such as the extended Kalman filter [17],[18],[19],[20]. The approximate expressions for the conditional means and error covariance matrix are summarized in Section 2.3.

1.2 Structure of the Thesis

The structure of the thesis is as follows. In Chapter 2 we describe the adaptive stochastic control problem under consideration and give the statistical assumptions. We discuss the philosophy of control based on the open-loop feedback doctrine. We state the solution to the resulting deterministic optimal control problem. We define all the variables and summarize the equations of the open-loop feedback optimal control algorithm. In Chapter 3 we shall first modify the original cost functional into a deterministic cost functional. The original problem is thus reformulated as a completely deterministic optimal control problem. Appendix A contains the details of the reformulation. We then derive the optimal open-loop control law via dynamic programming. We shall derive the "conditional open-loop optimal cost-to-go" to describe the performance of the optimal open-loop control sequence. We then interpret the derived solution in a feedback sense.

In Chapter 4 we present the results on the uniqueness and existence of the open-loop feedback optimal control. We shall consider the asymptotic behavior of the identifier and the overall adaptive control system as the time index $k \to \infty$. We define the enforced separation scheme, in which the actual parameter values are replaced by their current estimates, and discuss its asymptotic properties. In Chapter 5 we discuss the open-loop feedback optimal approach and the qualitative properties of the results obtained. The O.L.F.O. equations given in Chapter 2 and derived in Chapter 3 are given further heuristic interpretations.

In Chapter 6 we present the simulation results on first-order linear time-invariant (stable and unstable) single-input single-output systems. The actual plant parameter values are, thus, unknown constants. Simulation results will compare the response of the resultant stochastic control system when 1) the actual parameters are known 2) OLFO design is employed, and 3) enforced separation is employed. The identifier used in 2) and 3) is the extended Kalman filter and reduces to the optimal filter in 1). In Chapter 7 we discuss the results of the simulation studies in light of the general qualitative properties given in Chapters 4 and 5. We consider, finally, the computational aspects of the two sub-optimal closed-loop control systems. In Chapter 8 we summarize the theoretical and simulation results on the adaptive control systems. Further research in this area are also discussed.

1.3 Contribution of the Thesis

This thesis extends the previous work by Tse and Athans [29] to a larger class of practical control problems, which involve imperfectly known system dynamics as well as input gains. We solve this nonlinear stochastic control problem using the open-loop feedback optimal control via dynamic programming. The main analytical result is the development of the open-loop feedback optimal control structure and equations. The precise variation of the open-loop feedback optimal control adaptive gains as a function of the future expected uncertainty of the parameters is derived. Application of the OLFO adaptive gain plus correction term control to the N-stage state error plus control effort stochastic linear regulator problem with unknown but constant parameters is compared with the design in which the Separation Theorem is arbitrarily enforced [23] in terms of the performance measure. The system resulting from the arbitrary use of the current estimates for the actual (but unknown) parameters is much simpler to implement since the propagation of the covariance matrices is not needed. It was expected that the enforced separation scheme would be worse than that obtained by OLFO design.

However, simulation results via Monte Carlo method showed that for stable systems the OLFO control system is slightly better than the enforced separation,while for unstable systems, the latter design is far better.

CHAPTER 2

PROBLEM DEFINITION AND SOLUTION

2.1 Problem Statement

In this section we shall state the problem of interest — the control of discrete-time linear stochastic dynamical system with unknown parameters based on noisy observations of its output. The dimension of the system is assumed to be known, but the pole and zero locations may not be completely known and they may vary in a stochastic manner. Basically then, we assume a canonical structure for the state-space model of the system, but the actual plant parameter values are not completely specified. The particular class of problems that we shall examine has both the plant time constants and input gain vector imperfectly known. The performance criterion is chosen so as to minimize the expectation of a quadratic form in the state and control variables over a fixed interval of time.

Suppose we have a discrete-time n-dimensional linear dynamical system, with an imperfectly known initial state with noise disturbances entering the plant equation and the output measurement, governed by the following vector stochastic difference equations (integer k is the time index)

S1:

<u>Plant</u>: $\underline{x}(k+1) = \underline{A}(k)\underline{x}(k)+\underline{b}(k)u(k)+\underline{\xi}(k) \quad k=0,1,\ldots,N-1$ (2.1.1)

<u>Measurement</u>: $\underline{z}(k)=\underline{C}\,\underline{x}(k)+\underline{\theta}(k) \quad k=0,1,\ldots,N$ (2.1.2)

where we assume that $\underline{x}(k)$, $\underline{b}(k)$ and $\underline{\xi}(k)\varepsilon R^n$, $\underline{z}(k)$ and $\underline{\theta}(k)\varepsilon R^m$, $\underline{C}$ is a known constant $m \times n$ matrix, and $u(k)$ is an unconstrained <u>scalar</u> input. Furthermore, we assume that the actual (but unknown) $n \times n$ plant matrix

$\underline{A}(k)$ has the following canonical representation or is convertible to this equivalent system by a linear nonsingular transformation.[30]

$$\underline{A}(k)=\left[\begin{array}{c:c} \underline{0} & \underline{I}_{n-1} \\ \hdashline \multicolumn{2}{c}{\leftarrow \underline{a}'(k) \rightarrow} \end{array}\right], \quad \underline{a}(k)=\begin{bmatrix} a_1(k) \\ a_2(k) \\ \vdots \\ a_n(k) \end{bmatrix} \varepsilon R^n \tag{2.1.3}$$

The matrix $\underline{A}(k)$ is called a companion matrix. The unknown plant parameter vector $\underline{a}(k)$ and gain vector $\underline{b}(k)$ are assumed to satisfy the difference equations

$$\underline{a}(k+1) = \underline{a}(k) + \underline{\delta}(k) \tag{2.1.4}$$

$$\underline{b}(k+1) = \underline{b}(k) + \underline{\gamma}(k) \tag{2.1.5}$$

respectively, where $\underline{\delta}(k)$, $\underline{\gamma}(k)\varepsilon R^n$. We shall assume also that the triplet $[\underline{A}(k),\underline{b}(k),\underline{C}]$ of Eqs. (2.1.1) and (2.1.2) represents a completely controllable and completely observable system [31].

We shall assume that the stationary random sequences $\{\underline{\xi}(\cdot),\underline{\theta}(\cdot),\underline{\delta}(\cdot),\underline{\gamma}(\cdot)\}$ are white Gaussian, zero-mean, and uncorrelated with each other and with the Gaussian random vectors $\{\underline{x}(0),\underline{a}(0),\underline{b}(0)\}$. The statistical laws of the underlying random vectors are assumed known for all k. The symbol $N[\underline{m},\underline{\Sigma}]$ denotes normal distribution with mean equals to $\underline{m}$ and finite covariance matrix equals to $\underline{\Sigma}$. Thus, let

$$\underline{x}(0) \sim N[\underline{x}_o,\underline{\Sigma}_{xo}] \tag{2.1.6}$$

$$\underline{a}(0) \sim N[\underline{a}_o,\underline{\Sigma}_{ao}] \tag{2.1.7}$$

$$\underline{b}(0) \sim N[\underline{b}_o,\underline{\Sigma}_{bo}] \tag{2.1.8}$$

$$\underline{\xi}(k) \sim N[\underline{0},\underline{\Xi}(k)] \tag{2.1.9}$$

$$\underline{\theta}(k) \sim N[\underline{0},\underline{\Theta}(k)] \tag{2.1.10}$$

$$\underline{\delta}(k) \sim N[\underline{0},\underline{\Delta}(k)] \tag{2.1.11}$$

$$\underline{\gamma}(k) \sim N[\underline{0},\underline{\Gamma}(k)] \tag{2.1.12}$$

where the covariance matrices have the properties

$$\underline{\Sigma}_{xo} = \underline{\Sigma}'_{xo} \geq \underline{0} \tag{2.1.13}$$

$$\underline{\Sigma}_{ao} = \underline{\Sigma}'_{ao} \geq \underline{0} \tag{2.1.14}$$

$$\underline{\Sigma}_{bo} = \underline{\Sigma}'_{bo} \geq \underline{0} \tag{2.1.15}$$

$$\underline{\Xi}(k) = \underline{\Xi}'(k) \geq \underline{0} \tag{2.1.16}$$

$$\underline{\Theta}(k) = \underline{\Theta}'(k) > \underline{0} \tag{2.1.17}$$

$$\underline{\Delta}(k) = \underline{\Delta}'(k) \geq \underline{0} \tag{2.1.18}$$

$$\underline{\Gamma}(k) = \underline{\Gamma}'(k) \geq \underline{0} \tag{2.1.19}$$

Thus, additive discrete white plant driving noise $\underline{\xi}(k)$, observation noise $\underline{\theta}(k)$, and parameter noises $\underline{\delta}(k)$ and $\underline{\gamma}(k)$ are used to model the uncertainties in the state evolution $\underline{x}(k)$, the measurements $\underline{z}(k)$ and the parameter vectors $\underline{a}(k)$ and $\underline{b}(k)$, respectively.

We are also given a quadratic cost functional $J(u)$ which is given by

$$J(u) = \frac{1}{2}\,\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \frac{1}{2}\sum_{k=0}^{N-1}\{\underline{x}'(k)\underline{Q}(k)\underline{x}(k) + r(k)u^2(k)\} \tag{2.1.20}$$

with N fixed and finite. The objective of the problem is to find the optimal control sequence that minimizes in some sense Eq. (2.1.20). We will want to make use of all the available information in the computation of the optimal control sequence. Thus, the control input at each time k will, in general, be a function of the measurements and all <u>a priori</u> information up to and including time k. We are seeking, therefore, a physically realizable control.

Since the system S1 is operating in a stochastic environment, both the state trajectory $\underline{x}(\cdot)$ and the input $u(\cdot)$ are random sequences. A suitable performance criterion to choose then is the scalar real-valued cost functional

$$\bar{J}(u) \stackrel{\Delta}{=} \frac{1}{2} E\{\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \sum_{k=0}^{N-1} \underline{x}'(k)\underline{Q}(k)\underline{x}(k) + r(k)u^2(k)\} \qquad (2.1.21)$$

where E denotes expected value. The expectation is taken over all the underlying random processes, $\underline{a}(0)$, $\underline{b}(0)$, $\underline{x}(0)$, $\underline{\xi}(\cdot)$, $\underline{\theta}(\cdot)$, $\underline{\delta}(\cdot)$, and $\underline{\gamma}(\cdot)$. Thus, given the noise-corrupted unknown linear dynamic system, the objective is to find the <u>admissible</u> control sequence $U(0,N-1) \stackrel{\Delta}{=} \{u(j)\}_{j=0}^{N-1}$ which performs best "on the average" such that it minimizes the performance measure $\bar{J}(u)$ of Eq. (2.1.21) subject to the system equations (2.1.1) and (2.1.2).

Physically, the optimum control sequence will "on the average" drive the state $\underline{x}(k)$ of the system S1 to zero without excessive expenditure of control energy. We shall assume in Eq. (2.1.21) that $\underline{Q}(\cdot)$ is a positive semidefinite symmetric matrix and $r(\cdot)$ is a positive scalar, and that the initial and final times are fixed and finite ($N < \infty$).

To complete the problem statement we must now define what we mean by <u>admissible</u> control sequence. For a stochastic optimal control problem it is very important to specify precisely that data or information pattern which is available for determining the control input.

Depending on what measurements the computation of the control sequence $U(0,N-1)$ is based on exactly, different formulations of the optimal stochastic control problem are possible. In the general case when the parameters are <u>not independent</u> random variables, one cannot obtain explicit solutions to the optimal closed-loop controller, so that one has to resort to a reasonable and feasible controller. We shall, therefore, restrict our consideration to the deterministic open-loop controls, and derive the <u>open-loop feedback optimal controller</u>.

The <u>open-loop feedback optimal control policy</u> can be interpreted as follows. [4], [12], [13], [15] At time k $(k=0,1,\ldots,N-1)$ we are to control a system based on measurements taken up to time k; we assume that no observations will be made in the future. Under this assumption one generates an optimal control sequence $\{u^o(j|k)\}_{j-k}^{N-1}$ according to the open-loop policy. Only the first element of this sequence $u^*(k) \overset{\Delta}{=} u^o(k|k)$ is actually used. The applied control changes the probabilistic information provided by the estimator at time k+1. At time $k+1$, an additional measurement becomes available. The next control input $u^*(k+1)$ is again computed according to the open-loop policy, but based on all the information at time k+1. <u>Thus, we shall recompute the open-loop optimal deterministic control after new information becomes available at each time instant</u>. This technique applies in general, and includes the case when the parameters are random. It turns out that this open-loop control can be expressed as a function of the state and parameter statistics; hence, the name open-loop feedback optimal (O.L.F.O.) is used. The form of the open-loop feedback controller is shown in Fig. 2.1. The control at each time k is an explicit

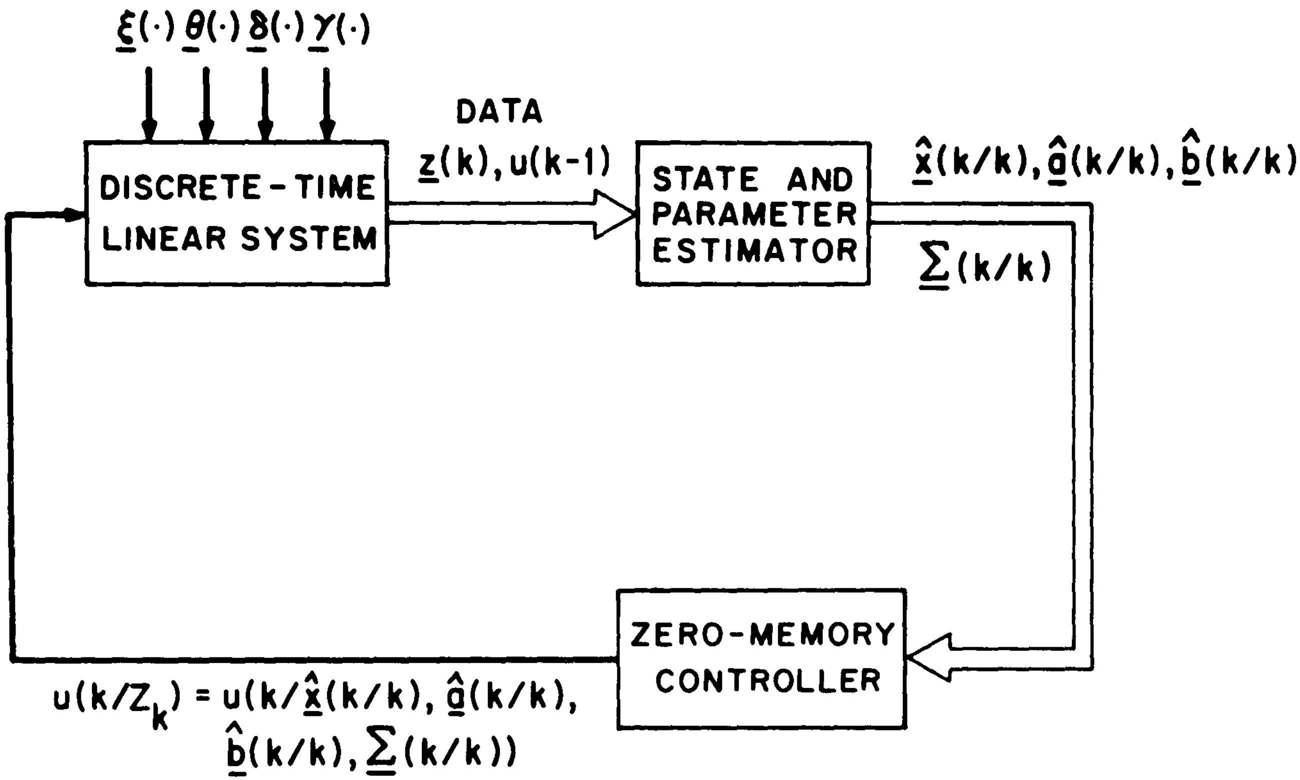

Fig. 2.1 Block Diagram of the Open-Loop Feedback Control System

function of state and parameter estimates and their level of uncertainty. The controller is to be designed from a deterministic system, once the estimator and the predictor are obtained. We shall see that the open-loop feedback optimal control sequence is in some sense "adaptive".

2.2 Definitions of Variables

Let the present time be indexed by k. Denote the accumulative observation statistic, that is, the (random) observation outputs or the available information at time k by

$$Z_k \triangleq \{\underline{z}(0), \underline{z}(1), \ldots, \underline{z}(k)\} \tag{2.2.1}$$

Let us also assume that the optimal control sequence $U^*(0,k-1) \triangleq \{u^*(j)\}_{j=0}^{k-1}$ has been applied to the system. Let us then define for $j \geq k$ the conditional expectations

$$\hat{\underline{x}}(j|k) \triangleq E\{\underline{x}(j)|Z_k\} \tag{2.2.2}$$

$$\hat{\underline{a}}(j|k) \triangleq E\{\underline{a}(j)|Z_k\} \tag{2.2.3}$$

$$\hat{\underline{b}}(j|k) \triangleq E\{\underline{b}(j)|Z_k\} \tag{2.2.4}$$

the error vectors

$$\underline{e}_x(j|k) \triangleq \hat{\underline{x}}(j|k) - \underline{x}(j) \tag{2.2.5}$$

$$\underline{e}_a(j|k) \triangleq \hat{\underline{a}}(j|k) - \underline{a}(j) \tag{2.2.6}$$

$$\underline{e}_b(j|k) \triangleq \hat{\underline{b}}(j|k) - \underline{b}(j) \tag{2.2.7}$$

and the conditional error covariance matrix

$$\underline{\Sigma}(j|k) \triangleq E\left\{\begin{bmatrix}\underline{e}_x(j|k)\\ \cdots\cdots\\ \underline{e}_a(j|k)\\ \cdots\cdots\\ \underline{e}_b(j|k)\end{bmatrix}[\underline{e}_x'(j|k) \vdots \underline{e}_a'(j|k) \vdots \underline{e}_b'(j|k)] \Big| Z_k\right\} \tag{2.2.8}$$

where $\underline{\Sigma}(j|k)$ can be decomposed into its submatrices:

$$\underline{\Sigma}(j|k) = \begin{bmatrix}\underline{\Sigma}_{xx}(j|k) & \underline{\Sigma}_{xa}(j|k) & \underline{\Sigma}_{xb}(j|k)\\ \underline{\Sigma}_{ax}(j|k) & \underline{\Sigma}_{aa}(j|k) & \underline{\Sigma}_{ab}(j|k)\\ \underline{\Sigma}_{bx}(j|k) & \underline{\Sigma}_{ba}(j|k) & \underline{\Sigma}_{bb}(j|k)\end{bmatrix} \tag{2.2.9}$$

Let us also define for $k \leq j \leq N-1$ the $3n \times 3n$ Jacobian matrix of the nonlinear augmented state vector system

$$S2: \begin{bmatrix}\underline{x}(i+1)\\ \underline{a}(i+1)\\ \underline{b}(i+1)\end{bmatrix} = \underline{\tilde{A}}(i,\underline{a}(i),u^*(i))\begin{bmatrix}\underline{x}(i)\\ \underline{a}(i)\\ \underline{b}(i)\end{bmatrix} + \begin{bmatrix}\underline{\xi}(i)\\ \underline{\delta}(i)\\ \underline{\gamma}(i)\end{bmatrix}$$

$$\underline{z}(i) = \underline{\tilde{C}}\begin{bmatrix}\underline{x}(i)\\ \underline{a}(i)\\ \underline{b}(i)\end{bmatrix} + \underline{\theta}(i); \quad i=0,1,\ldots,k-1 \tag{2.2.10}$$

where

$$\underline{\tilde{A}}(i,\underline{a}(i),u^*(i)) \triangleq \begin{bmatrix}\underline{A}(i) & \underline{0} & u^*(i)\underline{I}_n\\ \underline{0} & \underline{I}_n & \underline{0}\\ \underline{0} & \underline{0} & \underline{I}_n\end{bmatrix}, \tag{2.2.11}$$

$$\underline{\tilde{C}} \triangleq [\underline{C} \vdots \underline{0} \vdots \underline{0}]$$

and $\underline{A}(i)$ is given by Eq. (2.1.3)

Hence, the 3n x 3n Jacobian matrix $\hat{\underline{F}}(i,\hat{\underline{a}}(i|i),\hat{\underline{x}}(i|i),u^*(i))$ of Eq. (2.2.10) is given by

$$\hat{\underline{F}}(i,\hat{\underline{a}}(i|i),\hat{\underline{x}}(i|i),u^*(i)) \triangleq \begin{bmatrix} \hat{\underline{A}}(i|i) & \vdots & \hat{\underline{X}}(i|i) & \vdots & u^*(i)\underline{I}_n \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{I}_n & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{I}_n \end{bmatrix} \tag{2.2.12}$$

where

$$\hat{\underline{A}}(i|i) \triangleq \begin{bmatrix} \underline{0} & \vdots & \underline{I}_{n-1} \\ \cdots & \cdots & \cdots \\ \leftarrow & \hat{\underline{a}}'(i|i) & \rightarrow \end{bmatrix} \tag{2.2.13}$$

and

$$\hat{\underline{X}}(i|i) \triangleq \begin{bmatrix} \underline{0} & \vdots & \underline{0}_{n-1} \\ \cdots & \vdots & \cdots \\ \leftarrow & \hat{\underline{x}}'(i|i) & \rightarrow \end{bmatrix} \tag{2.2.14}$$

2.3 Structure of the Open-Loop Feedback Optimal Control

We shall state in this section the solution to the deterministic optimal control problem to be formulated in Section 3.2. The detailed derivation via dynamic programming is given in Section 3.3. The symbol $u^o(j|k)$ denotes the optimal open-loop control conditioned on the observations up to and including time k, and is given for $j \geq k$ below. It should be stressed that $\hat{\underline{x}}(j|k)$ and $\underline{\Sigma}(j|k)$ are not the exact conditional means and error covariance matrix.

$$u^{o}(j|k) = -\Big\{[\tilde{r}(j|k) + \underline{\tilde{b}}'(j|k)\,\underline{\tilde{K}}(j+1|k)\,\underline{\tilde{b}}(j|k)]^{-1}\underline{\tilde{b}}'(j|k)\underline{\tilde{K}}(j+1|k)\,\underline{\Phi}(j|k) + \tilde{r}^{-1}(j|k)\underline{d}'(j+1|k)\Big\}\begin{bmatrix}\underline{\hat{x}}(j|k)\\ \underline{\sigma}(j|k)\\ \underline{\rho}(j|k)\end{bmatrix} \qquad (2.3.1)$$

where the $n(2n+1) \times n(2n+1)$ symmetric matrix $\underline{\tilde{K}}(j+1|k)$ is the unique solution of the nonlinear matrix Riccati difference equation for $k+1 \leq j \leq N-1$

$$\underline{\tilde{K}}(j|k) = \underline{\Phi}'(j|k)[\underline{\tilde{K}}(j+1|k) - \underline{\tilde{K}}(j+1|k)\,\underline{\tilde{b}}(j|k)\,(\tilde{r}(j|k) + \underline{\tilde{b}}'(j|\tilde{k})\underline{\tilde{K}}(j+1|k)\underline{\tilde{b}}(j|k))^{-1}\underline{\tilde{b}}'(j|k)\underline{\tilde{K}}(j+1|k)]\,\underline{\Phi}(j|k) + \underline{V}(j|k) \qquad (2.3.2)$$

satisfying the boundary condition.

$$\underline{\tilde{K}}(N|k) = \begin{bmatrix}\underline{Q}(N) & \underline{0} & \underline{0}\\ \underline{0} & \underline{0}_{n^2} & \underline{0}\\ \underline{0} & \underline{0} & \underline{0}_{n^2}\end{bmatrix} \qquad (2.3.3)$$

where the parameters $\tilde{r}(j|k)$, $\underline{\tilde{b}}(j|k)$, $\underline{\Phi}(j|k)$, $\underline{d}(j+1|k)$, and $\underline{V}(j|k)$ will be defined below.

The open-loop feedback optimal control actually applied at time k is given by

$$u^{*}(k) = u^{o}(k|k) \qquad (2.3.4)$$

To find the open-loop feedback optimal control sequence, we have to solve the open-loop control problem for $k=0,1,\ldots,N-1$. We shall show in Section 3.4 that we can write the open-loop feedback optimal control as

$$u^{*}(k) = \underline{\phi}'(k)\underline{\hat{x}}(k|k) + u_{c}(k) \qquad (2.3.5)$$

where the 1 x n (row) vector

$$\underline{\phi}'(k) \triangleq -\left\{\left[\tilde{r}(k|k)+\tilde{\underline{b}}'(k|k)\,\tilde{\underline{K}}(k+1|k)\tilde{\underline{b}}(k|k)\right]^{-1}\tilde{\underline{b}}'(k|k)\tilde{\underline{K}}(k+1|k)\underline{\Phi}(k|k) + \tilde{r}^{-1}(k|k)\underline{d}'(k+1|k)\right\}\begin{bmatrix}\underline{I}_n\\ \underline{0}_{n^2}\\ \underline{0}_{n^2}\end{bmatrix} \tag{2.3.6}$$

is defined as the optimal open-loop feedback adaptive gain. We shall call the scalar

$$u_c(k) \triangleq -\left\{\left[\tilde{r}(k|k)+\tilde{\underline{b}}'(k|k)\tilde{\underline{K}}(k+1|k)\tilde{\underline{b}}(k|k)\right]^{-1}\tilde{\underline{b}}'(k|k)\tilde{\underline{K}}(k+1|k)\underline{\Phi}(k|k) + r^{-1}(k|k)\underline{d}'(k+1|k)\right\}\begin{bmatrix}\underline{0} & \underline{0} & \underline{0}\\ \underline{0} & \underline{I}_{n^2} & \underline{0}\\ \underline{0} & \underline{0} & \underline{I}_{n^2}\end{bmatrix}\begin{bmatrix}\hat{\underline{x}}(k|k)\\ \underline{\sigma}(k|k)\\ \underline{\rho}(k|k)\end{bmatrix} \tag{2.3.7}$$

the adaptive control correction term.

The structure of the overall open-loop feedback optimal control system is given in Fig. 2.2. The digital computer implementation of the open-loop feedback optimal control algorithm is straightforward. In Fig. 2.3, we give a flow chart description for on-line computation of the open-loop feedback optimal control. We summarize below all the equations needed for the computation of the optimal open-loop control sequence.

a. Identification Equations

Since the simultaneous state and parameter estimation is a non-linear filtering problem, we shall only give the approximate expressions for the conditional expectations of the state and parameter estimates and their error covariance matrices as generated by the extended Kalman filter algorithm [17], [33], [18]. The structure of the extended Kalman

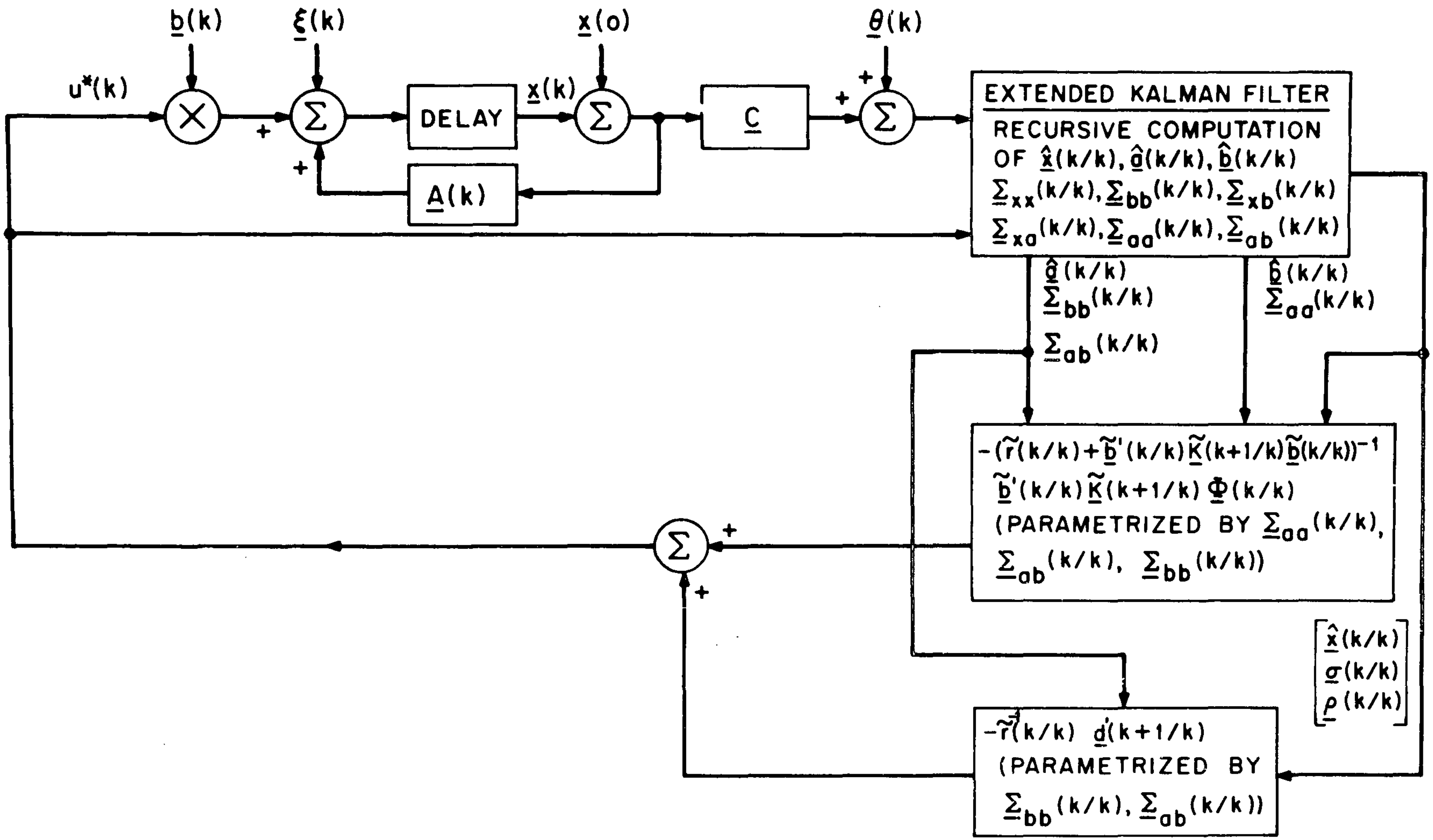

Fig. 2.2 Structure of the Open-Loop Feedback Optimal Control System

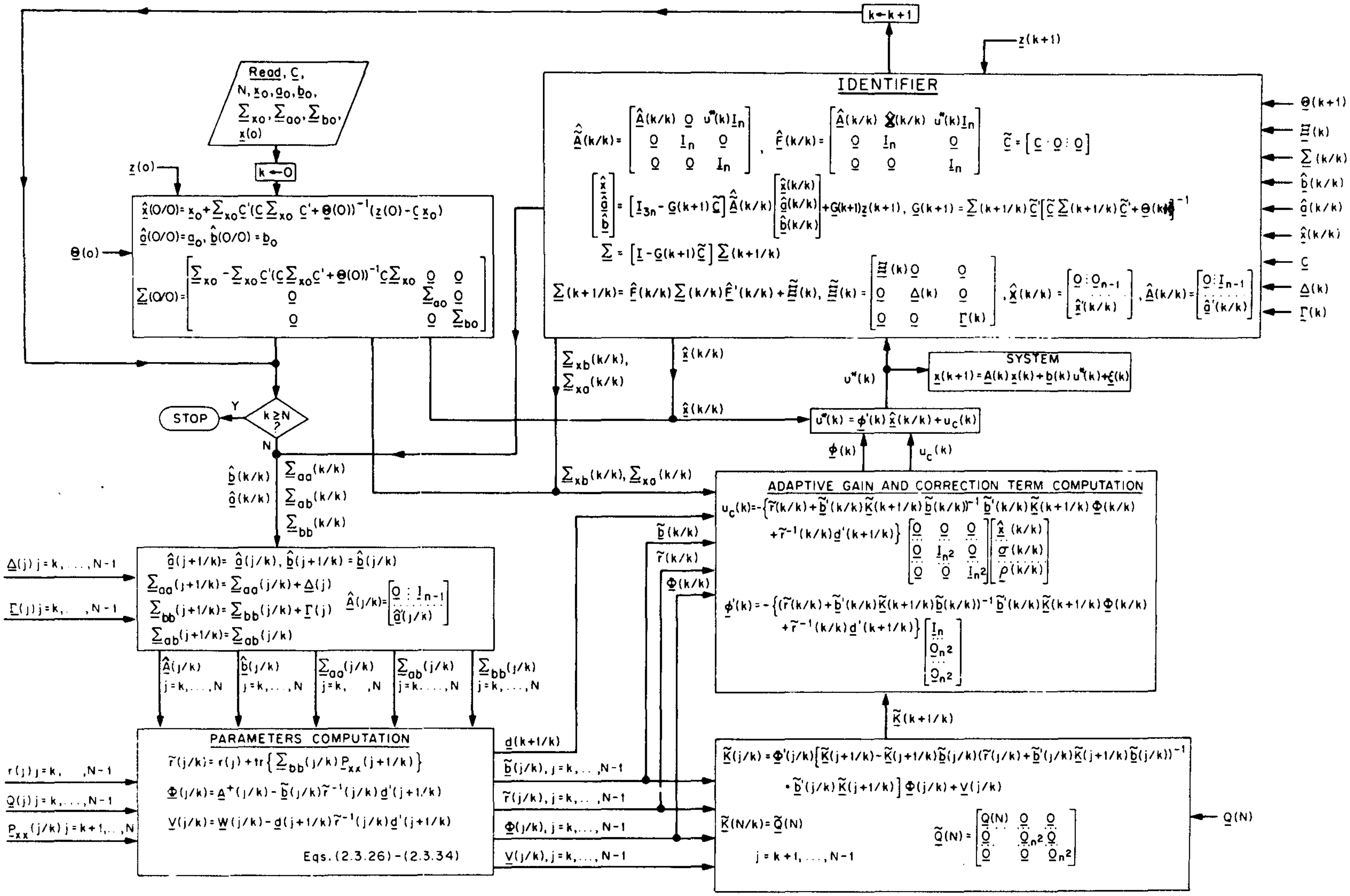

Fig. 2.3 Flowchart of the On-Line Open-Loop Feedback Optimal Control Computation Algorithm

filter is given in Fig. 2.4. We assume that $U^*(0,k-1)$ has been chosen and that Z_k is available. Then the estimates of the augmented state vector of S2 Eq. (2.2.10) are generated via the equations

$$\begin{bmatrix} \hat{\underline{x}}(i+1|i+1) \\ \hat{\underline{a}}(i+1|i+1) \\ \hat{\underline{b}}(i+1|i+1) \end{bmatrix} = [\underline{I}_{3n} - \underline{G}(i+1)\tilde{\underline{C}}]\hat{\tilde{\underline{A}}}(i|i) \begin{bmatrix} \hat{\underline{x}}(i|i) \\ \hat{\underline{a}}(i|i) \\ \hat{\underline{b}}(i|i) \end{bmatrix} + \underline{G}(i+1)\underline{z}(i+1) \qquad (2.3.8)$$

$$i = 0,1,\ldots, k-1$$

where

$$\hat{\tilde{\underline{A}}}(i|i) \stackrel{\Delta}{=} \begin{bmatrix} \hat{\underline{A}}(i|i) & \underline{0} & u^*(i)\underline{I}_n \\ \underline{0} & \underline{I}_n & \underline{0} \\ \underline{0} & \underline{0} & \underline{I}_n \end{bmatrix} \qquad (2.3.9)$$

with the intial estimation vector given by

$$\begin{bmatrix} \hat{\underline{x}}(0|0) \\ \hat{\underline{a}}(0|0) \\ \hat{\underline{b}}(0|0) \end{bmatrix} \stackrel{\Delta}{=} \begin{bmatrix} \underline{x}_o + \underline{\Sigma}_{xo}\underline{C}'(\underline{C}\underline{\Sigma}_{xo}\underline{C}'+\underline{\Theta}(0))^{-1}(\underline{z}(0)-\underline{C}\,\underline{x}_o) \\ \underline{a}_o \\ \underline{b}_o \end{bmatrix} \qquad (2.3.10)$$

The <u>filter gain</u> matrix $\underline{G}(i+1)$ in (2.3.8) satisfies the relation

$$\underline{G}(i+1) = \underline{\Sigma}(i+1|i)\tilde{\underline{C}}'[\tilde{\underline{C}}\underline{\Sigma}(i+1|i)\tilde{\underline{C}}' + \underline{\Theta}(i+1)]^{-1} \quad i=0,1,\ldots,k-1 \qquad (2.3.11)$$

$$= \underline{\Sigma}(i+1|i+1)\tilde{\underline{C}}'\underline{\Theta}^{-1}(i+1) \qquad (2.3.12)$$

provided the indicated inverse exists. The extended Kalman filter gain matrix cannot be precomputed since from Eq. (2.2.12)

$$\underline{\Sigma}(i+1|i) = \hat{\underline{F}}(i;\hat{\underline{a}}(i|i),\hat{\underline{x}}(i|i),u^*(i))\underline{\Sigma}(i|i)\hat{\underline{F}}'(i;\hat{\underline{a}}(i|i),\hat{\underline{x}}(i|i),u^*(i)) + \tilde{\underline{\Xi}}(i)$$

$$i=0,1,\ldots,k-1 \qquad (2.3.13)$$

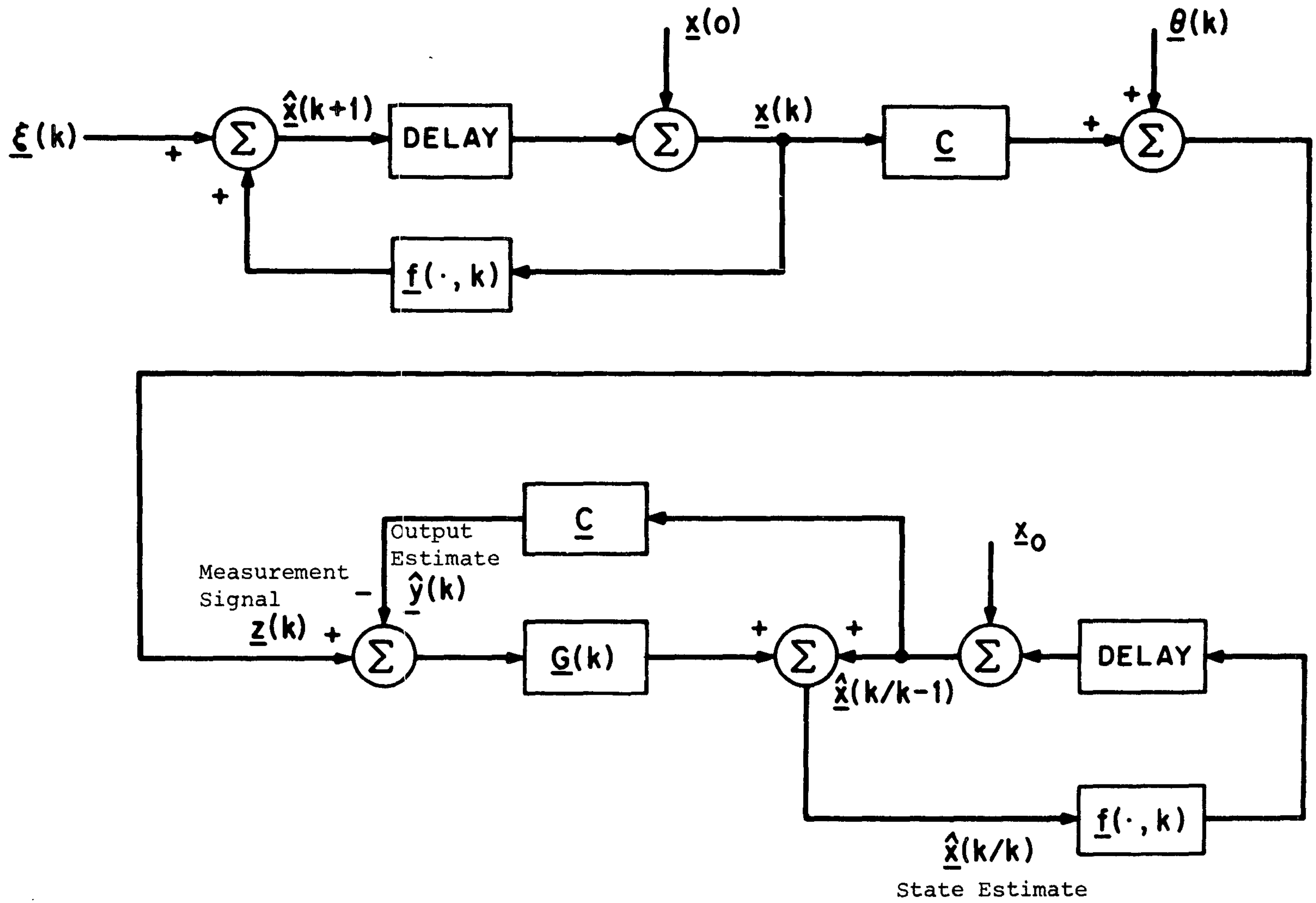

Fig. 2.4 Structure of the Extended Kalman Filter

is a function of the current estimates of the state $\underline{x}(i)$ and parameter $\underline{a}(i)$, and hence, dependent on the observations. The matrix $\tilde{\underline{\Xi}}(i)$ is given by

$$\tilde{\underline{\Xi}}(i) \triangleq \begin{bmatrix} \underline{\Xi}(i) & \underline{0} & \underline{0} \\ \underline{0} & \underline{\Delta}(i) & \underline{0} \\ \underline{0} & \underline{0} & \underline{\Gamma}(i) \end{bmatrix} \quad i = 0, 1, \ldots, k-1 \tag{2.3.14}$$

$\underline{\Sigma}(i+1|i+1)$ is a 3n x 3n symmetric positive semi-definite matrix given by

$$\begin{aligned} \underline{\Sigma}(i+1|i+1) &= \underline{\Sigma}(i+1|i) - \underline{G}(i+1)\tilde{\underline{C}}\underline{\Sigma}(i+1|i) \qquad i = 0,1,\ldots,k-1 \\ &= [\underline{I}-\underline{G}(i+1)\tilde{\underline{C}}]\underline{\Sigma}(i+1|i)[\underline{I}-\underline{G}(i+1)\tilde{\underline{C}}]' + \underline{G}(i+1)\underline{\Theta}(i+1)\underline{G}'(i+1) \end{aligned} \tag{2.3.15}$$

with the initial condition

$$\underline{\Sigma}(0|0) \triangleq \begin{bmatrix} \underline{\Sigma}_{xo} - \underline{\Sigma}_{xo}\underline{C}'(\underline{C}\underline{\Sigma}_{xo}\underline{C}' + \underline{\Theta}(0))^{-1}\underline{C}\underline{\Sigma}_{xo} & \underline{0} & \underline{0} \\ \underline{0} & \underline{\Sigma}_{ao} & \underline{0} \\ \underline{0} & \underline{0} & \underline{\Sigma}_{bo} \end{bmatrix} \tag{2.3.16}$$

b. Predictor Equations

It will be shown in Appendix A that for the open-loop control policy we have the following deterministic dynamic equations for $j \geq k$

$$\hat{\underline{x}}(j+1|k) = \hat{\underline{A}}(j|k)\hat{\underline{x}}(j|k) + \hat{\underline{b}}(j|k)u(j) \tag{2.3.17}$$

$$\hat{\underline{a}}(j+1|k) = \hat{\underline{a}}(j|k) \tag{2.3.18}$$

$$\hat{\underline{b}}(j+1|k) = \hat{\underline{b}}(j|k) \tag{2.3.19}$$

given the initial conditions $\hat{\underline{x}}(k|k)$, $\hat{\underline{a}}(k|k)$, $\hat{\underline{b}}(k|k)$ from Eq. (2.3.8).

The submatrices of $\underline{\Sigma}(j+1|k)$ are given by

$$\begin{aligned}\underline{\Sigma}_{xx}(j+1|k) = {} & \hat{\underline{A}}(j|k)\underline{\Sigma}_{xx}(j|k)\hat{\underline{A}}'(j|k)+\hat{\underline{A}}(j|k)\underline{\Sigma}_{xa}(j|k)\hat{\underline{X}}'(j|k)+\hat{\underline{X}}(j|k)\underline{\Sigma}'_{xa}(j|k)\hat{\underline{A}}'(j|k)\\ & +u(j)\hat{\underline{A}}(j|k)\underline{\Sigma}_{xb}(j|k)+u(j)\underline{\Sigma}'_{xb}(j|k)\hat{\underline{A}}'(j|k)+u(j)\hat{\underline{X}}(j|k)\underline{\Sigma}_{ab}(j|k)\\ & +u(j)\underline{\Sigma}'_{ab}(j|k)\hat{\underline{X}}'(j|k)+\hat{\underline{X}}(j|k)\underline{\Sigma}_{aa}(j|k)\hat{\underline{X}}'(j|k)+u^2(j)\underline{\Sigma}_{bb}(j|k)+\underline{\Xi}(j)\end{aligned}$$

where

$$\hat{\underline{A}}(j|k) \triangleq \begin{bmatrix} \underline{0} & \vdots & \underline{I}_{n-1} \\ \cdots & \vdots & \cdots \\ & \hat{\underline{a}}'(j|k) & \end{bmatrix} \quad \hat{\underline{X}}(j|k) \triangleq \begin{bmatrix} \underline{0} & \vdots & \underline{0}_{n-1} \\ \cdots & \vdots & \cdots \\ & \hat{\underline{x}}'(j|k) & \end{bmatrix} \tag{2.3.20}$$

$$\underline{\Sigma}_{xa}(j+1|k) = \hat{\underline{A}}(j|k)\underline{\Sigma}_{xa}(j|k)+\hat{\underline{X}}(j|k)\underline{\Sigma}_{aa}(j|k)+u(j|k)\underline{\Sigma}'_{ab}(j|k) \tag{2.3.21}$$

$$\underline{\Sigma}_{xb}(j+1|k) = \hat{\underline{A}}(j|k)\underline{\Sigma}_{xb}(j|k)+\hat{\underline{X}}(j|k)\underline{\Sigma}_{ab}(j|k)+u(j|k)\underline{\Sigma}_{bb}(j|k) \tag{2.3.22}$$

$$\underline{\Sigma}_{aa}(j+1|k) = \underline{\Sigma}_{aa}(j|k) + \underline{\Delta}(j) \tag{2.2.23}$$

$$\underline{\Sigma}_{ab}(j+1|k) = \underline{\Sigma}_{ab}(j|k) \tag{2.3.24}$$

$$\underline{\Sigma}_{bb}(j+1|k) = \underline{\Sigma}_{bb}(j|k) + \underline{\Gamma}(j) \tag{2.3.25}$$

given the initial conditions $\underline{\Sigma}_{aa}(k|k)$, $\underline{\Sigma}_{ab}(k|k)$, $\underline{\Sigma}_{bb}(k|k)$, $\underline{\Sigma}_{xx}(k|k)$, $\underline{\Sigma}_{xa}(k|k)$ and $\underline{\Sigma}_{xb}(k|k)$ from Eq. (2.3.15).

c. Parameter Equations

To compute the open-loop feedback optimal control we need to determine the parameters $\underline{\Phi}(j|k)$, $\underline{V}(j|k)$, $\underline{d}(j+1|k)$, $\tilde{\underline{b}}(j|k)$, $\tilde{r}(j|k)$ for $j=k$, $k+1,\ldots, N-1$ using the equations below,* (2.3.26) - (2.3.34).

---

* $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ denote the natural basis vectors in R^n

$$\underline{\sigma}(j|k) \triangleq \begin{bmatrix} \underline{\Sigma}_{xb}(j|k)\underline{e}_1 \\ \vdots \\ \underline{\Sigma}_{xb}(j|k)\underline{e}_n \end{bmatrix} \varepsilon R^{n^2}$$

$$\underline{\rho}(j|k) \triangleq \begin{bmatrix} \underline{\Sigma}_{xa}(j|k)\underline{e}_1 \\ \vdots \\ \underline{\Sigma}_{xa}(j|k)\underline{e}_n \end{bmatrix} \varepsilon R^{n^2} \tag{2.3.26}$$

$$\tilde{\underline{b}}(j|k) \triangleq \begin{bmatrix} \hat{\underline{b}}(j|k) \\ \underline{\Sigma}_{bb}(j|k)\underline{e}_1 \\ \vdots \\ \underline{\Sigma}_{bb}(j|k)\underline{e}_n \\ \underline{\Sigma}'_{ab}(j|k)\underline{e}_1 \\ \vdots \\ \underline{\Sigma}'_{ab}(j|k)\underline{e}_n \end{bmatrix} \varepsilon R^{n(2n+1)} \tag{2.3.27}$$

$$\underline{d}(j+1|k) \triangleq \begin{bmatrix} \underline{\Sigma}_{ab}(j|k)\underline{P}_{xx}(j+1|k)\underline{e}_n \\ \hat{\underline{A}}'(j|k)\underline{P}_{xx}(j+1|k)\underline{e}_1 \\ \vdots \\ \hat{\underline{A}}'(j|k)\underline{P}_{xx}(j+1|k)\underline{e}_n \\ \underline{0} \\ \vdots \\ \underline{0} \end{bmatrix} \varepsilon R^{n(2n+1)} \tag{2.3.28}$$

$$\underline{A}^{\dagger}(j|k) \stackrel{\Delta}{=} \begin{bmatrix} \hat{\underline{A}}(j|k) & \underline{0} & \cdots & \underline{0} \\ \underline{e}_n\underline{e}_1'\underline{\Sigma}_{ab}'(j|k) & \hat{\underline{A}}(j|k) & \cdots & \underline{0} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{e}_n\underline{e}_n'\underline{\Sigma}_{ab}'(j|k) & \underline{0} & \cdots & \underline{0} \\ \underline{e}_n\underline{e}_1'\underline{\Sigma}_{aa}(j|k) & \underline{0} & \cdots & \underline{0} \\ \cdots & \underline{0} & \cdots & \underline{0} \\ \underline{e}_n\underline{e}_n'\underline{\Sigma}_{aa}(j|k) & \underline{0} & \cdots & \hat{\underline{A}}(j|k) \end{bmatrix} (n(2n+1)\,x\;n\,(2n+1)) \qquad (2.3.29)$$

$$\underline{\Phi}(j|k) \stackrel{\Delta}{=} \underline{A}^{\dagger}(j|k) - \tilde{\underline{b}}(j|k)\tilde{r}^{-1}(j|k)\underline{d}'(j+1|k) \qquad (2.3.30)$$

$$\underline{W}(j|k) = \begin{bmatrix} \underline{Q}(j)+p_{nn}(j+1|k)\underline{\Sigma}_{aa}(j|k) & \underline{0} & \cdots & \underline{0} & \underline{e}_1\underline{e}_n'\underline{P}_{xx}(j+1|k)\hat{\underline{A}}'(j|k) & \cdots & \underline{e}_n\underline{e}_n'\underline{P}_{xx}(j+1|k)\hat{\underline{A}}'(j|k) \\ \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{0} & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \hat{\underline{A}}(j|k)\underline{P}_{xx}(j+1|k)\underline{e}_n\underline{e}_1' & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \underline{0} \\ \hat{\underline{A}}(j|k)\underline{P}_{xx}(j+1|k)\underline{e}_n\underline{e}_n' & \underline{0} & \cdots & \underline{0} & \underline{0} & \cdots & \underline{0} \end{bmatrix}$$

$$(n(2n+1)xn(2n+1)) \qquad (2.3.31)$$

where

$$p_{nn}(j+1|k)=\underline{e}_n'\underline{P}_{xx}(j+1|k)\underline{e}_n$$

$$\underline{V}(j|k)=\underline{W}(j|k)-\underline{d}(j+1|k)\tilde{r}^{-1}(j|k)\underline{d}'(j+1|k) \tag{2.3.32}$$

Since we are given $r(j)>0$, $\underline{Q}(j)\underline{\geq}\underline{0}$, and $\underline{\Sigma}_{bb}(j|k)$, $\underline{P}_{xx}(j|k)\underline{\geq}\ \underline{0}$ from Eq. (2.3.24) and

$$\underline{P}_{xx}(j|k)=\hat{\underline{A}}'(j|k)\underline{P}_{xx}(j+1|k)\hat{\underline{A}}(j|k)+\underline{Q}(j),\underline{P}_{xx}(N|k)=\underline{Q}(N)$$

$$j=k,\ldots,N-1 \tag{2.3.33}$$

we can define the "modified control weighting" to be

$$\tilde{r}(j|k) = r(j) + \mathrm{tr}\underline{\Sigma}_{bb}(j|k)\underline{P}_{xx}(j+1|k)>0 \tag{2.3.34}$$

and it is positive definite. We remark that the matrix $\underline{P}_{xx}(j|k)$ is dependent upon observation, and, thus, cannot be precomputed.

CHAPTER 3

METHOD OF SOLUTION

3.1 Cost Transformation

Suppose at $j = k$, $k = 0, 1, \ldots, N-1$ the system is at $\underline{x}(k)$, then we can rewrite the main cost functional Eq. (2.1.21) as

$$\bar{J}(u) = \frac{1}{2} E\{\sum_{j=k}^{N-1} \underline{x}^{*\prime}(j)\underline{Q}(j)\underline{x}^{*}(j) + r(j)(u^{*}(j))^{2}\}$$

$$+ \frac{1}{2} E\{\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \sum_{j=k}^{N-1} \underline{x}'(j)\underline{Q}(j)\underline{x}(j) + r(j)u^{2}(j)\}$$

$$k = 0,1,\ldots,N-1 \qquad (3.1.1)$$

Using Bellman's principle of optimality, we have the equivalent minimization problem at time k.

$$\bar{J}_k \triangleq J(U^{*}(0,k-1),U(k,N-1))$$

$$= \frac{1}{2} E\{\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \sum_{j=k}^{N-1} \underline{x}'(j)\underline{Q}(j)\underline{x}(j) + r(j)u^{2}(j)\}$$

$$k = 0,1,\ldots,N-1 \qquad (3.1.2)$$

The optimal closed-loop control policy uses the a priori known statistics of the future measurements. The a priori probability density functions are those at time $j = k$.

The suboptimal closed-loop control policy we are solving consists of replacing the closed-loop controls with open-loop controls for $j \geq k$. It does not take into account the knowledge that future measurements will be made. Therefore, the objective of the problem becomes to find a future control sequence such that the average value of the cost-to-go given by

$$\bar{J}_k = \frac{1}{2}E\{\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \sum_{j=k}^{N-1}\underline{x}'(j)\underline{Q}(j)\underline{x}(j)|Z_k\} + \frac{1}{2}\sum_{j=k}^{N-1} r(j)u^2(j) \qquad k = 0,1,\ldots,N-1 \tag{3.1.3}$$

conditioned on the total available data at the present time k, is minimized (open-loop) subject to the constraint equations (2.1.1)-(2.1.2).

We can take u(j) out of the expectation in (3.1.3) since the future control sequence $\{u(j)\}_{j=k}^{N-1}$ is assumed to be deterministic, and the optimal open-loop control sequence $\{u^o(j|k)\}_{j=k}^{N-1}$ is only based upon the measurement statistic Z_k. It is possible now to formulate exactly the stochastic control problem Eqs. (2.1.1),(2.1.2), and (2.1.21) as a completely deterministic optimization problem.

Using Eqs. (2.2.2)-(2.2.9) we then obtain the conditional cost (3.1.3) as

$$\bar{J}_k = \frac{1}{2}\hat{\underline{x}}'(N|k)\underline{Q}(N)\hat{\underline{x}}(N|k) + \frac{1}{2}\operatorname{tr}[\underline{Q}(N)\underline{\Sigma}_{xx}(N|k)] + \frac{1}{2}\sum_{j=k}^{N-1}\{\hat{\underline{x}}'(j|k)\underline{Q}(j)\hat{\underline{x}}(j|k) + \operatorname{tr}[\underline{Q}(j)\underline{\Sigma}_{xx}(j|k)] + r(j)u^2(j)\}$$

$$k = 0,1,\ldots,N-1 \tag{3.1.4}$$

since for any matrix $\underline{M}$

$$E\{\underline{x}'(j)\underline{M}\,\underline{x}(j)|Z_k\} = \hat{\underline{x}}'(j|k)\underline{M}\,\hat{\underline{x}}(j|k) + E\{\underline{e}_x'(j|k)\underline{M}\,\underline{e}_x(j|k)\} \tag{3.1.5}$$

We have obtained, therefore, in Eq. (3.1.4) the deterministic form of the expected value of the cost-to-go.

3.2 Open-Loop Control Problem Definition

To complete the formulation of the deterministic open-loop control problem, we will need the deterministic dynamical equations satisfied by $\hat{\underline{x}}(j|k)$ and $\underline{\Sigma}_{xx}(j|k)$ $(j \geq k)$, the respective estimates of the state vector and its error covariance matrix conditioned upon the output measurements Z_k and the past control history $U^*(0,k-1)$. We shall, however, develop only the approximate expressions for $\hat{\underline{x}}(j|k)$ and $\underline{\Sigma}_{xx}(j|k)$ in Appendix A. These expressions and the identification equations (2.3.8)-(2.3.16) are then used to define a completely deterministic optimal control problem for the kth step, whose solution would yield the optimal open-loop controls for $k \leq j \leq N-1$.

$$\text{Given:}\quad \hat{\underline{x}}(j+1|k) = \hat{\underline{A}}(j|k)\hat{\underline{x}}(j|k) + \hat{\underline{b}}(j|k)u(j) \tag{3.2.1}$$

where

$$\hat{\underline{a}}(j+1|k) = \hat{\underline{a}}(j|k) \tag{3.2.2}$$

$$\hat{\underline{b}}(j+1|k) = \hat{\underline{b}}(j|k) \tag{3.2.3}$$

$$\begin{aligned}\underline{\Sigma}(j+1|k) = {} & \hat{\underline{F}}(j,\hat{\underline{a}}(j|k),\hat{\underline{x}}(j|k),u(j))\underline{\Sigma}(j|k)\hat{\underline{F}}'(j,\hat{\underline{a}}(j|k), \\ & \hat{\underline{x}}(j|k),u(j)) + \tilde{\underline{\Xi}}(j)\end{aligned} \tag{3.2.4}$$

where

$$\hat{\underline{F}}(j,\hat{\underline{a}}(j|k),\hat{\underline{x}}(j|k),u(j)) \triangleq \left(\begin{array}{c:c:c} \hat{\underline{A}}(j|k) & \hat{\underline{X}}(j|k) & u(j)\underline{I}_n \\ \hdashline \underline{0} & \underline{I}_n & \underline{0} \\ \hdashline \underline{0} & \underline{0} & \underline{I}_n \end{array}\right) \tag{3.2.5}$$

$$\underline{\tilde{\Xi}}(j) \triangleq \begin{bmatrix} \underline{\Xi}(j) & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{\Delta}(j) & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{\Gamma}(j) \end{bmatrix} \tag{3.2.6}$$

The initial conditions are $\underline{\hat{x}}(k|k)$, $\underline{\hat{a}}(k|k)$, $\underline{\hat{b}}(k|k)$, and $\underline{\Sigma}(k|k)$ specified by the extended Kalman filter, (2.3.8)-(2.3.16).

The aim is to find a deterministic control sequence $\{u(k),\ldots,u(N-1)\}$ such that it minimizes in (N-k) steps the average value cost-to-go

$$\bar{J}_k = \frac{1}{2}\{\underline{\hat{x}}'(N|k)\underline{Q}(N)\underline{\hat{x}}(N|k) + tr[\underline{\tilde{Q}}(N)\underline{\Sigma}(N|k)] + \sum_{j=k}^{N-1} \underline{\hat{x}}'(j|k)\underline{Q}(j)\underline{\hat{x}}(j|k)$$

$$+ tr[\underline{\tilde{Q}}(j)\underline{\Sigma}(j|k)] + r(j)u^2(j)\} \tag{3.2.7}$$

where

$$\underline{\tilde{Q}}(N) = \begin{bmatrix} \underline{Q}(N) & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0}_n & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{0}_n \end{bmatrix}, \quad \underline{\tilde{Q}}(j) = \begin{bmatrix} \underline{Q}(j) & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0}_n & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{0}_n \end{bmatrix} \tag{3.2.8}$$

and $\underline{\tilde{Q}}(N)$, $\underline{\tilde{Q}}(j) \geq \underline{0}$ and $r(j) > 0$ subject to the constraint equations (3.2.1)-(3.2.4). We shall also assume that $\underline{Q}(j)$ and $\underline{Q}(N)$ are not both the zero matrix. The terminal states $\underline{\hat{x}}(N|k)$ and $\underline{\Sigma}(N|k)$ are not specified.

We see from (3.2.7) that the conditional cost-to-go depends on the estimate $\underline{\hat{x}}(j|k)$ and the error covariance matrix $\underline{\Sigma}_{xx}(j|k)$ generated by the predictor equations (3.2.1)-(3.2.4), and u(j), an arbitrary deterministic input for $k \leq j \leq N-1$. The difference equation (3.2.1) for $\underline{\hat{x}}(j|k)$ involves $\underline{\hat{a}}(j|k)$, $\underline{\hat{b}}(j|k)$ and u(j). The difference equation (3.2.4) for $\underline{\Sigma}_{xx}(j|k)$ involves the error covariances $\underline{\Sigma}_{bb}(j|k)$, $\underline{\Sigma}_{xb}(j|k)$, $\underline{\Sigma}_{xa}(j|k)$, $\underline{\Sigma}_{aa}(j|k)$, $\underline{\Sigma}_{ab}(j|k)$ and the estimates $\underline{\hat{a}}(j|k)$, $\underline{\hat{x}}(j|k)$, and the control u(j).

We have then a well-posed deterministic optimal control problem. The deterministic formulation allows us to use the discrete Minimum Principle [34] to derive the necessary conditions for optimality (in the open-loop feedback sense). We shall, however, present an alternative method of solution via dynamic programming in the next section [35].

3.3 Open-Loop Control Problem Solution

In this section we shall derive the result stated in Section 2.3, using the dynamic programming method for the discrete optimal control problem defined by Eqs.(3.2.1)-(3.2.4) and the cost functional (3.2.7) to be minimized is

$$\bar{J}_k = \frac{1}{2}\{\underline{\hat{x}}'(N|k)\underline{Q}(N)\underline{\hat{x}}(N|k) + \text{tr}\,\underline{\tilde{Q}}(N)\underline{\Sigma}(N|k)\}$$
$$+ \sum_{j=k}^{N-1} L(\underline{\hat{x}}(j|k),\underline{\Sigma}(j|k),u(j),j) \qquad (3.3.1)$$

where

$$L(\underline{\hat{x}}(j|k),\underline{\Sigma}(j|k),u(j),j) \triangleq \frac{1}{2}\{\underline{\hat{x}}'(j|k)\underline{Q}(j)\underline{\hat{x}}(j|k) + \text{tr}\,\underline{\tilde{Q}}(j)\underline{\Sigma}(j|k)$$
$$+ r(j)u^2(j)\} \qquad (3.3.2)$$

To use the standard dynamic programming algorithm, we shall define the "conditional open-loop optimal cost-to-go" at $j = i$, for $i\varepsilon[k,N-1]$. The superscript circle denotes open-loop optimal. We shall define $J^o_{i|k}(\underline{\hat{x}}(i|k),\underline{\Sigma}(i|k))$ as the minimum cost remaining along the optimal trajectory starting at time i and the initial "states" $\underline{\hat{x}}(i|k)$ and $\underline{\Sigma}(i|k)$ given by (3.2.1)-(3.2.4). Hence we have the functional equation

$$\bar{J}^{o}_{i|k}(\hat{\underline{x}}(i|k),\underline{\Sigma}(i|k)) = \min_{\substack{u(j)\\ j=i,i+1,\ldots,N-1}} \{\frac{1}{2}\{\hat{\underline{x}}'(N|k)\underline{Q}(N)\hat{\underline{x}}(N|k) + \mathrm{tr}\,\tilde{\underline{Q}}(N)\underline{\Sigma}(N|k)\}$$

$$+ \sum_{j=i}^{N-1} L(\hat{\underline{x}}(j|k),\underline{\Sigma}(j|k),u(j),j)\}$$

$$i=k,k+1,\ldots,N-1 \qquad (3.3.3)$$

$$\bar{J}^{o}_{i|k}(\hat{\underline{x}}(i|k),\underline{\Sigma}(i|k)) = \min_{u(i)} \{L(\hat{\underline{x}}(i|k),\underline{\Sigma}(i|k),u(i),i)$$

$$+ \bar{J}^{o}_{i+1|k}(\hat{\underline{x}}^{o}(i+1|k),\underline{\Sigma}^{o}(i+1|k))\}$$

$$i = k,k+1,\ldots,N-1 \qquad (3.3.4)$$

using Eq. (3.3.2). The method of dynamic programming yields necessary conditions based on the optimality principle or condition Eq. (3.3.4).

We remark that $\bar{J}^{o}_{i|k}(\cdot,\cdot)$ is defined as a function on $M_{(3nx3n)} \times R^{n}$. From the error covariance equation (3.2.4) $\underline{\Sigma}(j|k) \geq \underline{0}$, $j=i,i+1,\ldots,N-1$ if and only if $\underline{\Sigma}(i|k) \geq \underline{0}$, $i = k, k+1,\ldots, N-1$. Thus, from Eq. (3.3.3) we obtain that

$$\bar{J}^{o}_{i|k}(\hat{\underline{x}},\underline{\Sigma}) \geq 0, \text{ if } \underline{\Sigma} \geq \underline{0} \qquad (3.3.5)$$

Hence, if $\underline{Q}(i) \geq \underline{0}$ and $r(i) > 0$, then $\bar{J}^{o}_{i|k} \geq 0$ and $\underline{\Sigma} \geq \underline{0}$ in Eqs. (3.3.3) and (3.2.4).

The terminal time is N. Let us define

$\bar{J}^{o}_{N-1|k}(\hat{\underline{x}}(N-1|k),\underline{\Sigma}(N-1|k)) \triangleq$ optimal value of $\bar{J}$ for one-stage control process starting at i=N-1 and using an optimal u(N-1).

Using (3.3.5) the conditional optimal cost-to-go is given by

$$\bar{J}^{o}_{N-1|k}(\hat{\underline{x}}(N-1|k),\underline{\Sigma}(N-1|k)) = \min_{u(N-1)} \{R(\hat{\underline{x}}(N-1|k),\underline{\Sigma}(N-1|k),u(N-1))\} \qquad (3.3.6)$$

where

$$R(\underline{\hat{x}}(N-1|k),\underline{\Sigma}(N-1|k),u(N-1)) \triangleq L(\underline{\hat{x}}(N-1|k),\underline{\Sigma}(N-1|k),u(N-1),N-1) + \bar{J}^o_{N|k}(\underline{\hat{x}}^o(N|k),\underline{\Sigma}^o(N|k)) \qquad (3.3.7)$$

But,

$$\bar{J}^o_{N|k}(\underline{\hat{x}}^o(N|k),\underline{\Sigma}^o(N|k)) = \frac{1}{2}\,\underline{\hat{x}}^{o\prime}(N|k)\underline{Q}(N)\underline{\hat{x}}^o(N|k) + \frac{1}{2}\,\mathrm{tr}\underline{Q}(N)\underline{\Sigma}^o_{xx}(N|k) \qquad (3.3.8)$$

Hence we obtain*

$$R(\underline{\hat{x}}(N-1|k),\underline{\Sigma}(N-1|k),u(N-1)) = \underline{\hat{x}}'(N-1|k)\underline{Q}(N-1)\underline{\hat{x}}(N-1|k) + \mathrm{tr}\,\underline{Q}(N-1)\underline{\Sigma}_{xx}(N-1|k) + r(N-1)u^2(N-1) + \underline{\hat{x}}^{o\prime}(N|k)\underline{Q}(N)\underline{\hat{x}}^o(N|k) + \mathrm{tr}\underline{Q}(N)\underline{\Sigma}^o_{xx}(N|k) \qquad (3.3.9)$$

Using the system equations (3.2.1)-(3.2.6) we get

$$\begin{aligned}
R(\underline{\hat{x}}(N-1|k),\underline{\Sigma}(N-1|k),u(N-1)) = {} & \underline{\hat{x}}'(N-1|k)\underline{Q}(N-1)\underline{\hat{x}}(N-1) + \mathrm{tr}\underline{Q}(N-1)\underline{\Sigma}_{xx}(N-1|k) \\
& + r(N-1)u^2(N-1) + \underline{\hat{x}}'(N|k)\underline{\hat{A}}'(N-1|k)\underline{Q}(N)\underline{\hat{A}}(N-1|k)\underline{\hat{x}}(N-1|k) \\
& + \underline{\hat{x}}'(N-1|k)\underline{\hat{A}}'(N-1|k)\underline{Q}(N)\underline{\hat{b}}(N-1|k)u(N-1) \\
& + u(N-1)\underline{\hat{b}}'(N-1|k)\underline{Q}(N)\underline{\hat{A}}(N-1|k)\underline{\hat{x}}(N-1|k) \\
& + u^2(N-1)\underline{\hat{b}}'(N-1|k)\underline{Q}(N)\underline{\hat{b}}(N-1|k) \\
& + \mathrm{tr}\{\underline{Q}(N)\,[\underline{\hat{A}}(N-1|k)\underline{\Sigma}_{xx}(N-1|k)\underline{\hat{A}}'(N-1|k) \\
& + 2\underline{\hat{A}}(N-1|k)\underline{\Sigma}_{xa}(N-1|k)\underline{\hat{X}}'(N-1|k) + 2\underline{\hat{A}}(N-1|k)\underline{\Sigma}_{xb}(N-1|k)u(N-1) \\
& + \underline{\hat{X}}(N-1|k)\underline{\Sigma}_{aa}(N-1|k)\underline{\hat{X}}'(N-1|k) + 2\underline{\hat{X}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k)u(N-1) \\
& + u^2(N-1)\underline{\Sigma}_{bb}(N-1|k) + \underline{\Xi}(N-1)]\} \qquad (3.3.10)
\end{aligned}$$

---

* We shall drop the factor of 1/2 in R(·,·,·) for notational convenience.

If we consider $\hat{\underline{x}}(N-1|k)$ as fixed, then we can minimize Eq. (3.3.6) with respect to $u(N-1)$ only to yield $u^o(N-1|k)$. Note that the optimal open-loop control will depend on the state $\hat{\underline{x}}(N-1|k)$. Performing the minimization of R gives

$$\frac{\partial R}{\partial u(N-1)} = 2r(N-1)u(N-1) + 2\hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k)$$
$$+ 2u(N-1)\hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{b}}(N-1|k) + 2\mathrm{tr}[\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xb}(N-1|k)$$
$$+ \underline{Q}(N)\hat{\underline{X}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k) + u(N-1)\underline{Q}(N)\underline{\Sigma}_{bb}(N-1|k)]$$

$$\left.\frac{\partial R}{\partial u(N-1)}\right|_{u^o(N-1|k)} = 0 = [r(N-1) + \mathrm{tr}\underline{Q}(N)\underline{\Sigma}_{bb}(N-1|k) + \hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{b}}(N-1|k)]u^o(N-1|k)$$
$$+ \hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k)$$
$$+ \mathrm{tr}[\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xb}(N-1|k) + \underline{Q}(N)\hat{\underline{X}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k)] \tag{3.3.11}$$

Therefore the optimal open-loop control is given by

$$u^o(N-1|k) = -Z_{uu}^{-1}(N-1|k)\{\hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k)$$
$$+ \mathrm{tr}[\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xb}(N-1|k) + \underline{Q}(N)\hat{\underline{x}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k)]\} \tag{3.3.12}$$

provided the indicated inverse exists, and

$$Z_{uu}(N-1|k) \triangleq r(N-1) + \hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{b}}(N-1|k) + \mathrm{tr}\underline{Q}(N)\underline{\Sigma}_{bb}(N-1|k) \tag{3.3.13}$$

It can be shown that we can write (3.3.12) in the form of Eq. (2.3.1) using Eqs. (2.3.26)-(2.3.34)

$$u^o(N-1|k) = -(\tilde{r}(N-1|k) + \tilde{\underline{b}}'(N-1|k)\tilde{\underline{K}}(N|k)\tilde{\underline{b}}(N-1|k))^{-1}\tilde{\underline{b}}'(N-1|k)$$
$$\tilde{\underline{K}}(N|k)\underline{\Phi}(N-1|k)\begin{pmatrix}\hat{\underline{x}}(N-1|k)\\ \cdots\cdots\\ \underline{\sigma}(N-1|k)\\ \cdots\cdots\\ \underline{\rho}(N-1|k)\end{pmatrix} - \tilde{r}^{-1}(N-1|k)\underline{d}'(N|k)\begin{pmatrix}\hat{\underline{x}}(N-1|k)\\ \cdots\cdots\\ \underline{\sigma}(N-1|k)\\ \cdots\cdots\\ \underline{\rho}(N-1|k)\end{pmatrix} \tag{3.3.14}$$

where $\tilde{\underline{K}}(N|k)$ is given by Eq. (2.3.3). Substituting this value of $u^o(N-1|k)$ Eq. (3.3.12) in Eqs.(3.3.6)-(3.3.10), and do some manipulating we obtain the conditional optimal cost-to-go

$$
\begin{aligned}
2\cdot\bar{J}^o_{N-1|k}(\hat{\underline{x}}(N-1|k),\underline{\Sigma}(N-1|k)) = {} & \hat{\underline{x}}'(N-1|k)\underline{Q}(N-1)\hat{\underline{x}}(N-1|k)+\mathrm{tr}\underline{Q}(N-1)\underline{\Sigma}_{xx}(N-1|k) \\
& - Z^{-1}_{uu}(N-1|k)\{\hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k) \\
& + \mathrm{tr}[\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xb}(N-1|k)+\underline{Q}(N)\hat{\underline{X}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k)\}^2 \\
& + \hat{\underline{x}}'(N-1|k)\hat{\underline{A}}'(N-1|k)\underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k) \\
& + \mathrm{tr}\{\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xx}(N-1|k)\hat{\underline{A}}'(N-1|k) \\
& + 2\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xa}(N-1|k)\hat{\underline{X}}'(N-1|k) \\
& + \underline{Q}(N)\hat{\underline{X}}'(N-1|k)\underline{\Sigma}_{aa}(N-1|k)\hat{\underline{X}}'(N-1|k)+\underline{Q}(N)\underline{\Xi}(N-1)\}
\end{aligned}
\tag{3.3.15}
$$

$$
\begin{aligned}
= {} & \hat{\underline{x}}'(N-1|k)\underline{Q}(N-1)\hat{\underline{x}}(N-1|k)+\mathrm{tr}[\underline{\Sigma}_{xx}(N-1|k)(\hat{\underline{A}}'(N-1|k) \\
& \underline{Q}(N)\hat{\underline{A}}(N-1|k)+\underline{Q}(N-1))]+\mathrm{tr}[\underline{Q}(N)\hat{\underline{X}}'(N-1|k) \\
& \underline{\Sigma}_{aa}(N-1|k)\hat{\underline{X}}'(N-1|k)+2\underline{Q}(N)\hat{\underline{A}}(N-1|k)\underline{\Sigma}_{xa}(N-1|k) \\
& \hat{\underline{X}}'(N-1|k)+\underline{Q}(N)\underline{\Xi}(N-1)]+\hat{\underline{x}}'(N-1|k)\hat{\underline{A}}'(N-1|k) \\
& \underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k)-Z^{-1}_{uu}(N-1|k)\{\hat{\underline{b}}'(N-1|k) \\
& \underline{Q}(N)\hat{\underline{A}}(N-1|k)\hat{\underline{x}}(N-1|k)+\mathrm{tr}[\underline{Q}(N)\hat{\underline{A}}(N-1|k) \\
& \underline{\Sigma}_{xb}(N-1|k)+\underline{Q}(N)\hat{\underline{X}}(N-1|k)\underline{\Sigma}_{ab}(N-1|k)]\}^2
\end{aligned}
\tag{3.3.16}
$$

which has the closed form

$$\bar{J}^{o}_{N-1|k}(\underline{\hat{x}}(N-1|k),\underline{\Sigma}(N-1|k)) = \frac{1}{2} < \begin{pmatrix} \underline{\hat{x}}(N-1|k) \\ \cdots\cdots \\ \underline{\sigma}(N-1|k) \\ \cdots\cdots \\ \underline{\rho}(N-1|k) \end{pmatrix}, \ \underline{\tilde{K}}(N-1|k) \begin{pmatrix} \underline{\hat{x}}(N-1|k) \\ \cdots\cdots \\ \underline{\sigma}(N-1|k) \\ \cdots\cdots \\ \underline{\rho}(N-1|k) \end{pmatrix} >$$

$$+ \frac{1}{2} \operatorname{tr} [\underline{\Sigma}_{xx}(N-1|k)(\underline{\hat{A}}'(N-1|k)\underline{P}_{xx}(N|k)\underline{\hat{A}}(N-1|k) + \underline{Q}(N-1))]$$

$$+ \frac{1}{2} \operatorname{tr} \underline{Q}(N)\underline{\Xi}(N-1) \ ; \ \underline{P}_{xx}(N|k) = \underline{Q}(N) \qquad (3.3.17)$$

if we choose $\underline{\tilde{K}}(N-1|k)$ such that

$$\underline{\tilde{K}}(N-1|k) = \underline{\Phi}'(N-1|k)\underline{\tilde{K}}(N|k)\underline{\Phi}(N-1|k) + \underline{V}(N-1|k) - \underline{\Phi}'(N-1|k)$$

$$\underline{\tilde{K}}(N|k)\underline{\tilde{b}}(N-1|k)(\tilde{r}(N-1|k) + \underline{\tilde{b}}'(N-1|k)\underline{\tilde{K}}(N|k)\underline{\tilde{b}}(N-1|k))^{-1}$$

$$\underline{\tilde{b}}'(N-1|k)\underline{\tilde{K}}(N|k)\underline{\Phi}(N-1|k) \qquad (3.3.18)$$

where $\underline{\tilde{K}}(N|k)$ is given by (2.3.3) and the parameters are computed from Eqs. (2.3.26)-(2.3.34). We have thus far assumed that $Z_{uu}(N-1|k)$ is nonzero. In fact, in order that the control law (3.3.14) minimizes the cost-to-go, $Z_{uu}(N-1|k)$ must be positive definite.

In Equation (3.3.17) we have determined the expression for the conditional optimal cost-to-go at step N-1. Now using Equation (3.3.5) we have

$$\bar{J}^{o}_{N-2|k}(\underline{\hat{x}}(N-2|k),\underline{\Sigma}(N-2|k)) = \min_{u(N-2)} \{L(\underline{\hat{x}}(N-2|k),\underline{\Sigma}(N-2|k),u(n-2),N-2)$$

$$+ \bar{J}^{o}_{N-1|k}(\underline{\hat{x}}^{o}(N-1|k),\underline{\Sigma}^{o}(N-1|k))\} \qquad (3.3.19)$$

Using Equation (3.3.17), this reduces to a form exactly identical to Eq. (3.3.6) except for the indices. Thus we have, comparing Equation (3.3.9)

$$R(\underline{\hat{x}}(N-2|k),\underline{\Sigma}(N-2|k)) = \underline{\hat{x}}'(N-2|k)\underline{Q}(N-2)\underline{\hat{x}}(N-2|k)$$

$$+ \operatorname{tr} \underline{Q}(N-2)\underline{\Sigma}_{xx}(N-2|k) + r(N-2)u^2(N-2)$$

$$+ < \begin{pmatrix} \underline{\hat{x}}^o(N-1|k) \\ \cdots\cdots\cdots \\ \underline{\sigma}^o(N-1|k) \\ \cdots\cdots\cdots \\ \underline{\rho}^o(N-1|k) \end{pmatrix}, \underline{\tilde{K}}(N-1|k) \begin{pmatrix} \underline{\hat{x}}^o(N-1|k) \\ \cdots\cdots\cdots \\ \underline{\sigma}^o(N-1|k) \\ \cdots\cdots\cdots \\ \underline{\rho}^o(N-1|k) \end{pmatrix} > + s(N-1) \tag{3.3.20}$$

where the scalar

$$s(N-1) = \operatorname{tr}[\underline{\Sigma}^o_{xx}(N-1|k)\underline{P}^o_{xx}(N-1|k) + \underline{P}^o_{xx}(N|k)\underline{\Xi}(N-1)] \tag{3.3.21}$$

and

$$\underline{P}^o_{xx}(N-1|k) = \underline{\hat{A}}^{o\prime}(N-1|k)\underline{Q}(N)\underline{\hat{A}}^o(N-1|k) + \underline{Q}(N-1) \tag{3.3.22}$$

and $\underline{\tilde{K}}(N-1|k)$ is given by (3.3.18). The cycle now repeats. Thus, by induction on i we have shown that the optimal open-loop control sequence is given by

$$u^o(i|k) = -\{[\tilde{r}(i|k) + \underline{\tilde{b}}'(i|k)\underline{\tilde{K}}(i+1|k)\underline{\tilde{b}}(i|k)]^{-1}\underline{\tilde{b}}'(i|k)\underline{\tilde{K}}(i+1|k)\underline{\Phi}(i|k)$$

$$+ \tilde{r}^{-1}(i|k)\underline{d}'(i+1|k)\} \begin{pmatrix} \underline{\hat{x}}(i|k) \\ \cdots\cdots \\ \underline{\sigma}(i|k) \\ \cdots\cdots \\ \underline{\rho}(i|k) \end{pmatrix} \qquad i = k,k+1,\ldots,N-1 \tag{3.3.23}$$

and the conditional optimal (minimum) open-loop cost-to-go is given by

$$\overline{J}^o_{i|k}(\underline{\hat{x}}(i|k),\underline{\Sigma}(i|k)) = \frac{1}{2} < \begin{pmatrix} \underline{\hat{x}}(i|k) \\ \cdots\cdots \\ \underline{\sigma}(i|k) \\ \cdots\cdots \\ \underline{\rho}(i|k) \end{pmatrix}, \underline{\tilde{K}}(i|k) \begin{pmatrix} \underline{\hat{x}}(i|k) \\ \cdots\cdots \\ \underline{\sigma}(i|k) \\ \cdots\cdots \\ \underline{\rho}(i|k) \end{pmatrix} >$$

$$+ \frac{1}{2}s(i) \qquad i = k,k+1,\ldots,N-1 \tag{3.3.24}$$

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where

$$s(i) = \mathrm{tr}\{\underline{\Sigma}_{xx}(i|k)\underline{P}_{xx}(i|k) + \sum_{j=i}^{N-1}\underline{P}_{xx}(j+1|k)\underline{\Xi}(j)\} \tag{3.3.25}$$

and $\underline{\tilde{K}}(i|k)$ satisfies the matrix difference Equation (2.3.2) and the parameters are computed from Equations (2.3.26)-(2.3.34).

The matrix $\underline{\tilde{K}}(i|k)$ is a symmetric matrix. Taking the transpose of both sides of Equation (2.3.2) we find that

$$\underline{\tilde{K}}'(i|k) = \underline{\Phi}'(i|k)\{\underline{\tilde{K}}'(i+1|k) - \underline{\tilde{K}}'(i+1|k)\underline{\tilde{b}}(i|k)(\tilde{r}(i|k) + \underline{\tilde{b}}'(i|k)$$
$$\underline{\tilde{K}}'(i+1|k)\underline{\tilde{b}}(i|k))^{-1}\underline{\tilde{b}}'(i|k)\underline{\tilde{K}}'(i+1|k)\}\underline{\Phi}(i|k) + \underline{V}(i|k)$$
$$i = k, k+1, \ldots, N-1 \tag{3.3.26}$$

since $\underline{V}(i|k)$ is symmetric. Comparing this with Equation (2.3.2), we observe that both $\underline{\tilde{K}}(i|k)$ and $\underline{\tilde{K}}'(i|k)$ are solutions of the same difference equation. At $i = N$, we have the boundary condition $\underline{\tilde{K}}'(N|k) = \underline{\tilde{Q}}(N)$. Since $\underline{\tilde{Q}}(N)$ is symmetric, $\underline{\tilde{Q}}(N) = \underline{\tilde{Q}}'(N)$, we conclude that

$$\underline{\tilde{K}}(N|k) = \underline{\tilde{K}}'(N|k) = \underline{\tilde{Q}}(N) \tag{3.3.27}$$

Since $\underline{\tilde{K}}(i|k)$ and $\underline{\tilde{K}}'(i|k)$ are solutions of the same difference equation with the same boundary conditions, we conclude $\underline{\tilde{K}}(i|k) = \underline{\tilde{K}}'(i|k)$ from the uniqueness of solutions of difference equations.

We note that in the conditional optimal cost-to-go equation (3.3.24), that it depends on the initial estimation of the state $\underline{\hat{x}}(i|k)$ and $\underline{\Sigma}(i|k)$, and the random disturbance $\underline{\xi}(j)$ which is forcing the system. The presence of the plant noise ($\underline{\Xi}(j) \neq \underline{0}$), therefore, increases the cost-to-go on the average, since $\mathrm{tr}[\underline{P}_{xx}(j+1|k)\underline{\Xi}(j)]$ is nonnegative if $\underline{P}_{xx}(j+1|k)$ and $\underline{\Xi}(j)$ are positive semidefinite.

3.4 Feedback Interpretation

In this section we will show that the open-loop feedback optimal sequence applied is

$$u^*(k) = \underline{\phi}'(k)\hat{\underline{x}}(k|k) + u_c(k) \quad ; \quad k = 0,1,\dots,N-1 \tag{3.4.1}$$

Let us rewrite Eq. (3.3.23) at i = k.

$$\begin{aligned} u^o(k|k) = &-\{(\tilde{r}(k|k) + \underline{\tilde{b}}'(k|k)\underline{\tilde{K}}(k+1|k)\underline{\tilde{b}}(k|k))^{-1}\underline{\tilde{b}}'(k|k)\underline{\tilde{K}}(k+1|k)\underline{\Phi}(k|k) \\ &+ \tilde{r}^{-1}(k|k)\underline{d}'(k+1|k)\}\begin{pmatrix} \underline{I}_n & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{0} \end{pmatrix}\begin{pmatrix} \hat{\underline{x}}(k|k) \\ \cdots \\ \underline{\sigma}(k|k) \\ \cdots \\ \underline{\rho}(k|k) \end{pmatrix} \\ &- \{(\tilde{r}(k|k) + \underline{\tilde{b}}'(k|k)\underline{\tilde{K}}(k+1|k)\underline{\tilde{b}}(k|k))^{-1}\underline{\tilde{b}}'(k|k)\underline{\tilde{K}}(k+1|k)\underline{\Phi}(k|k) \\ &+ \tilde{r}^{-1}(k|k)\underline{d}'(k+1|k)\}\begin{pmatrix} \underline{0} & \vdots & \underline{0} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{I}_{n^2} & \vdots & \underline{0} \\ \cdots & & \cdots & & \cdots \\ \underline{0} & \vdots & \underline{0} & \vdots & \underline{I}_{n^2} \end{pmatrix}\begin{pmatrix} \hat{\underline{x}}(k|k) \\ \cdots \\ \underline{\sigma}(k|k) \\ \cdots \\ \underline{\rho}(k|k) \end{pmatrix} \end{aligned} \tag{3.4.2}$$

$$k = 0,1,\dots,N-1$$

By definitions (2.3.6) and (2.3.7) Eq. (3.4.1) is immediately verified.

The adaptive control gain $\underline{\phi}'(k)$ is, by definition, independent of the current estimate of the state vector $\hat{\underline{x}}(k|k)$. It depends on $\hat{\underline{b}}(k|k)$ and $\hat{\underline{b}}(j|k)$, $\hat{\underline{a}}(k|k)$ and $\hat{\underline{a}}(j|k)$, $\underline{\Sigma}_{bb}(k|k)$ and $\underline{\Sigma}_{bb}(j|k)$, $\underline{\Sigma}_{ab}(k|k)$ and $\underline{\Sigma}_{ab}(j|k)$, and $\underline{\Sigma}_{aa}(k|k)$ and $\underline{\Sigma}_{aa}(j|k)$ evaluated along the open-loop feedback optimal trajectory for $k \leq j \leq N-1$.

The control correction term $u_c(k)$ is independent of $\hat{\underline{x}}(k|k)$. It depends on $\hat{\underline{b}}(k|k)$ and $\hat{\underline{b}}(j|k)$, $\hat{\underline{a}}(k|k)$ and $\hat{\underline{a}}(j|k)$, $\underline{\Sigma}_{xb}(k|k)$ and $\underline{\Sigma}_{xb}(j|k)$ $\underline{\Sigma}_{xa}(k|k)$ and $\underline{\Sigma}_{xa}(j|k)$, $\underline{\Sigma}_{bb}(k|k)$ and $\underline{\Sigma}_{bb}(j|k)$, and $\underline{\Sigma}_{ab}(k|k)$ and $\underline{\Sigma}_{ab}(j|k)$ all evaluated along the open-loop feedback optimal control trajectory for

$k \leq j \leq N-1$. If the cross error covariances $\underline{\Sigma}_{xb}(k|k)$, $\underline{\Sigma}_{xa}(k|k)$, and $\underline{\Sigma}_{ab}(k|k)$ are zero, then the adaptive control gain $u_c(k) = 0$ in Eq. (3.4.1).

In Eqs. (3.4.1) and (3.4.2) we have the explicit variation of the adaptive gain as a function of the future expected uncertainty of the parameters. The OLFO control correction term is affected by the estimation accuracy of the $\underline{a}$ and $\underline{b}$ vector through $\underline{\Sigma}_{xa}(\cdot|k)$, $\underline{\Sigma}_{xb}(\cdot|k)$, $\underline{\Sigma}_{ab}(\cdot|k)$, $\underline{\Sigma}_{aa}(\cdot|k)$, and $\underline{\Sigma}_{bb}(\cdot|k)$. We note that the uncertainty in the state vector $\underline{x}$ given by $\underline{\Sigma}_{xx}(\cdot|k)$ does not affect the OLFO control calculation, and, hence, consistant with the results of the standard Separation Theorem.

If $\underline{\Sigma}_{bb}(k|k) = \underline{0}$ and $\underline{\Sigma}_{aa}(k|k) = \underline{0}$, then we have "identified" $\underline{b}$ and $\underline{a}$, that is, $\hat{\underline{b}}(k|k) = \underline{b}(k)$ and $\hat{\underline{a}}(k|k) = \underline{a}(k)$. We also have then $\underline{\Sigma}_{xb}(k|k) = \underline{0}$, $\underline{\Sigma}_{xa}(k|k) = \underline{0}$, and $\underline{\Sigma}_{ab}(k|k) = \underline{0}$. It can be shown by induction from Eq. (2.3.2) and Eqs. (2.3.26)-(2.3.34) that if we define in this case

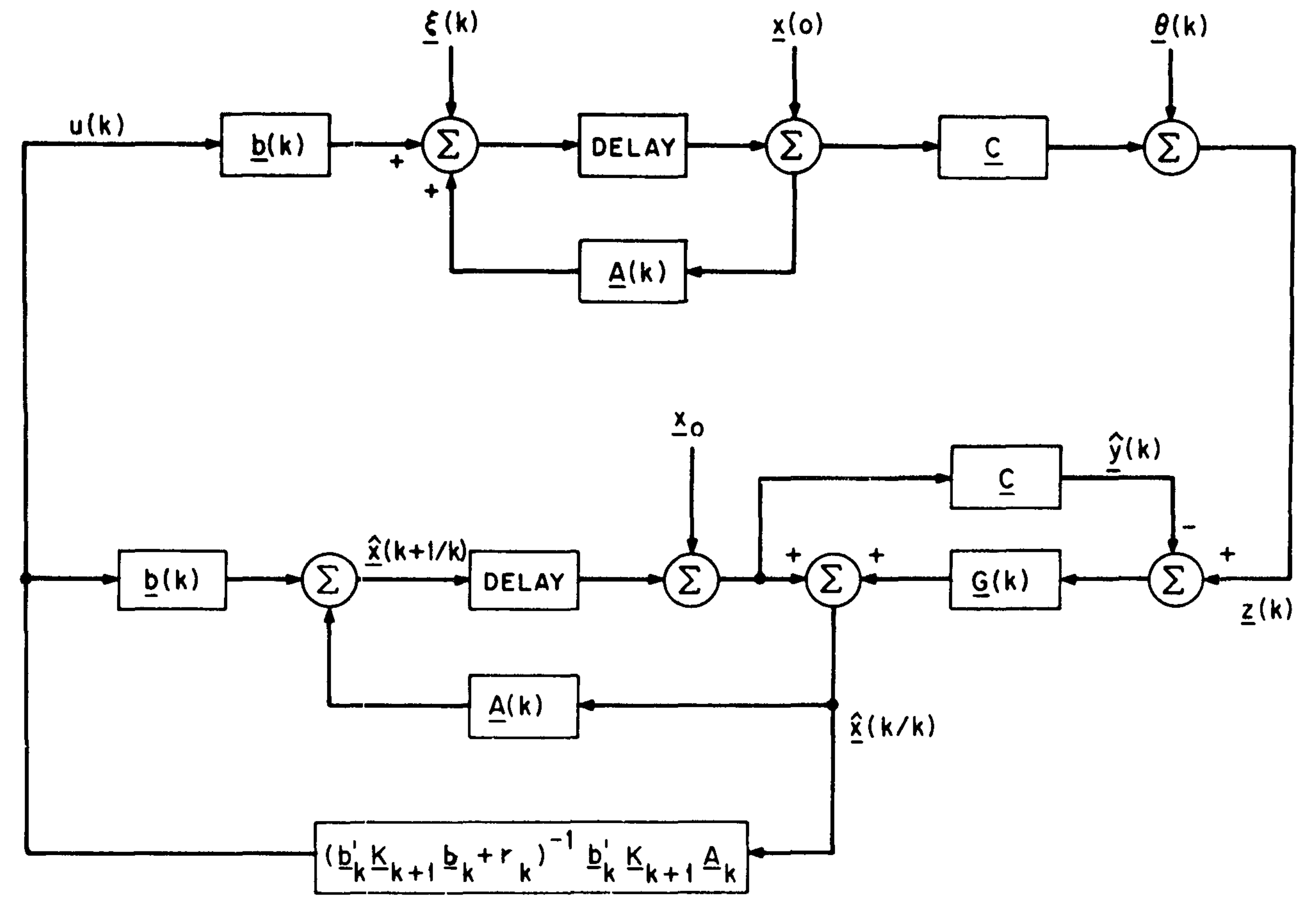
$$\tilde{\underline{K}}(k|k) \triangleq \left[\begin{array}{c:c:c} \underline{K}_{11}(k) & \underline{K}_{12}(k) & \underline{K}_{13}(k) \\ \hdashline \underline{K}_{21}(k) & \underline{K}_{22}(k) & \underline{K}_{23}(k) \\ \hdashline \underline{K}_{31}(k) & \underline{K}_{32}(k) & \underline{K}_{33}(k) \end{array}\right] \tag{3.4.3}$$

then $\underline{K}_{11}(k)$ satisfies the matrix Riccati equation

$$\begin{aligned} \underline{K}_{11}(k) &= \underline{A}'(k)[\underline{K}_{11}(k+1) - \underline{K}_{11}(k+1)\underline{b}(k)(r(k) \\ &\quad + \underline{b}'(k)\underline{K}_{11}(k+1)\underline{b}(k))^{-1}\underline{b}'(k)\underline{K}_{11}(k+1)]\underline{A}(k) + \underline{Q}(k), \\ \underline{K}_{11}(N) &= \underline{Q}(N) \end{aligned} \tag{3.4.4}$$

The optimal open-loop feedback adaptive gain is then effectively

$$\begin{aligned} \underline{\phi}'(k) &= -[r(k) + \underline{b}'(k)\underline{K}_{11}(k+1)\underline{b}(k)]^{-1}\underline{b}'(k)\underline{K}_{11}(k+1)\underline{A}(k) \\ &= \underline{\phi}^{*\prime}(k) \end{aligned} \tag{3.4.5}$$



$$\bar{J}(u) = E\left\{ \underline{x}'(N)\underline{Q}(N)\,\underline{x}(N) + \sum_{k=0}^{N-1} (\underline{x}'(k)\,\underline{Q}(k)\,\underline{x}(k) + r(k)\,u^2(k)) \right\}$$

$$\underline{K}(k) = \underline{A}'(k)\left[\underline{K}(k+1) - \underline{K}(k+1)\,\underline{b}(k)(r(k)+\underline{b}'(k)\underline{K}(k+1)\underline{b}(k))^{-1}\underline{b}'(k)\underline{K}(k+1)\right]\underline{A}(k) + \underline{Q}(k)$$

Fig. 3.1 Combined Estimation and Control of Linear Stochastic System with Known Dynamics which is the truly optimal gain for the linear-quadratic-Gaussian problem given by the Separation Theorem [1]. Under these assumptions, the adaptive control correction term $u_c(k) = 0$, and the stochastic optimal control is, therefore,

$$u^*(k) = -[r(k) + \underline{b}'(k)\underline{K}_{11}(k+1)\underline{b}(k)]^{-1}\underline{b}'(k)\underline{K}_{11}(k+1)\underline{A}(k)\hat{\underline{x}}(k|k) \quad (3.4.6)$$

The structure of the truly optimal stochastic control is given in Fig. 3.1. Therefore, if at time k, the identification of the parameters $\underline{a}(k)$ and $\underline{b}(k)$ has a very high level of confidence, i.e., $\underline{\Sigma}_{bb}(k|k) \cong \underline{0}$, and $\underline{\Sigma}_{aa}(k|k) \cong \underline{0}$, then the optimal open-loop feedback control will act nearly optimal, and use the generated estimates $\hat{\underline{a}}(k|k)$ and $\hat{\underline{b}}(k|k)$ as if they were the correct parameter values.

CHAPTER 4

FURTHER PROPERTIES

4.1 Existence and Uniqueness of the O.L.F.O. Solution

In obtaining the dynamic programming solution to the deterministic open-loop control problem Eq. (3.2.1)-(3.2.6), the convexity condition (3.3.13)

$$Z_{uu}(i|k) > 0 \tag{4.1.1}$$

where

$$Z_{uu}(i|k) = r(i|k) + \mathrm{tr}\underline{\Sigma}_{bb}(i|k)\underline{P}_{xx}(i+1|k) + \underline{\tilde{b}}'(i|k)\underline{\tilde{K}}(i+1|k)\underline{\tilde{b}}(i|k) \tag{4.1.2}$$

is required. If $Z_{uu}(i|k) < 0$, then a bounded optimal solution does not exist. Also, if $Z_{uu}(i|k)$ is singular, the optimal solution will not be unique. If we assume that neither of the foregoing cases hold, then we have the following theorem.

Theorem 4.1 (Uniqueness and Sufficiency):

If $Z_{uu}(i|k) > 0$; $i = k,k+1,\ldots,N-1$

then the optimal open-loop control of the deterministic problem exists, is unique, and is given by Eq. (3.3.23).

Proof: The proof follows directly from the derivation of the above dynamic programming equations.

We shall use Eqs. (3.3.24) and (3.3.5) to show the existence and uniqueness of $\underline{\tilde{K}}(j|k)$, $j = k,k+1,\ldots,N-1$; $k = 0,1,\ldots,N-1$ in Eq. (2.3.2). We will show by induction the following Lemma.

Lemma 4.1.1:

$$\tilde{r}(j|k) + \tilde{\underline{b}}'(j|k)\tilde{\underline{K}}(j+1|k)\tilde{\underline{b}}(j|k) > 0; \quad j = k,k+1,\ldots,N-1 \qquad (4.1.3)$$

Proof: We have $\underline{Q}(N) \geq \underline{0}$, $r(j) > 0$ for all j. In particular, $r(N-1) > 0$.

Using Eq. (2.3.34) we obtain

$$\tilde{r}(N-1|k) + \tilde{\underline{b}}'(N-1|k)\tilde{\underline{K}}(N|k)\tilde{\underline{b}}(N-1|k)$$

$$= r(N-1) + \mathrm{tr}\underline{\Sigma}_{bb}(N-1|k)\underline{P}_{xx}(N|k) + \tilde{\underline{b}}'(N-1|k)\tilde{\underline{Q}}(N)\tilde{\underline{b}}(N-1|k)$$

$$= r(N-1) + \mathrm{tr}\underline{\Sigma}_{bb}(N-1|k)\underline{Q}(N) + \hat{\underline{b}}'(N-1|k)\underline{Q}(N)\hat{\underline{b}}(N-1|k) > 0$$

$$(4.1.4)$$

Assume

$$\tilde{r}(\ell|k) + \tilde{\underline{b}}'(\ell|k)\tilde{\underline{K}}(\ell+1|k)\tilde{\underline{b}}(\ell|k) > 0; \quad \begin{array}{l}\ell = i,i+1,\ldots,N-1\\ i = k,k+1,\ldots,N-1\end{array} \qquad (4.1.5)$$

Let $\underline{\Xi}(j) = \underline{0}$, $j = k,\ldots,N-1$. Then, by the induction hypothesis, Eq. (3.3.24) and Eq. (3.3.5) imply that

$$\overline{J}^{o}_{i/k}(\hat{\underline{b}}(i-1|k),\hat{\underline{\Sigma}}(i-1|k)) = \frac{1}{2}\,\mathrm{tr}[\underline{\Sigma}_{bb}(i-1|k)\underline{P}_{xx}(i|k)]$$

$$+ \frac{1}{2}\,\tilde{\underline{b}}'(i-1|k)\tilde{\underline{K}}(i|k)\tilde{\underline{b}}(i-1|k) \geq 0$$

$$(4.1.6)$$

if we choose

$$\hat{\underline{\Sigma}}(i-1|k) = \begin{pmatrix} \underline{\Sigma}_{bb}(i-1|k) & \underline{\Sigma}_{ba}(i-1|k) & \underline{\Sigma}_{bb}(i-1|k) \\ \underline{\Sigma}_{ab}(i-1|k) & \underline{\Sigma}_{aa}(i-1|k) & \underline{\Sigma}_{ab}(i-1|k) \\ \underline{\Sigma}_{bb}(i-1|k) & \underline{\Sigma}_{ba}(i-1|k) & \underline{\Sigma}_{bb}(i-1|k) \end{pmatrix} \qquad (4.1.7)$$

and since $r(i-1) > 0$, we have then from Eq. (3.3.24)

$$\tilde{r}(i-1|k) + \tilde{\underline{b}}'(i-1|k)\tilde{\underline{K}}(i|k)\tilde{\underline{b}}(i-1|k)$$

$$= r(i-1) + 2\overline{J}^{o}_{i/k}(\hat{\underline{b}}(i-1|k),\hat{\underline{\Sigma}}(i-1|k)) > 0 \qquad (4.1.8)$$

Thus assertion in Eq. (4.1.3) is proved by induction.

By Lemma 4.1.1, the optimal open-loop control $\{u^o(j|k)\}_{j=k}^{N-1}$ exists and is unique for all $k = 0,1,\ldots,N-1$, and, therefore, the open-loop feedback optimal control $\{u^*(k)\}_{k=0}^{N-1}$ exists and is unique.

4.2 Asymptotic Behavior

In this section we study the asymptotic properties of the overall system by considering the behavior of $\underline{\Sigma}_{bb}(k|k)$ and $\underline{\Sigma}_{aa}(k|k)$ as $k \to \infty$. We assume that the corresponding deterministic system of S1 is completely controllable and completely observable. If the nonlinear filter worked properly then the estimates given by the extended Kalman filter would approach the actual values of the parameters, and the observation noise would loose its effect on the estimates as time increases. The filter gain in Eq. (2.3.12) would, therefore, decrease, since the predicted error covariance is expected to go to zero if there are no plant disturbances. However, since the filter is only an approximation to the optimal Bayesian filter, a bias error may occur. The extended Kalman filter behaves as if this error did not exist. The propagated error becomes smaller than the real errors. To tell the filter that we have nonlinearities, we can introduce fictitious driving noises into the augmented state equation, and hence, make use of all the measurements.

Lemma 4.2.1: Let $\underline{\delta}(k) = \underline{0}$, that is, there is no stochastic variation in parameter vector $\underline{a}(k)$ and the $\underline{a}(k)$ vector is constant. Then, given any control sequence, the error covariance $\underline{\Sigma}_{aa}$ is monotonically decreasing.*

$$\underline{\Sigma}_{aa}(k+1|k+1) \leq \underline{\Sigma}_{aa}(k|k) \tag{4.2.1}$$

Proof: From Eq. (2.3.15)

$$\underline{\Sigma}_{aa}(k+1|k+1) = \underline{\Sigma}_{aa}(k|k) - \left(\underline{0} \vdots \underline{I}_n \vdots \underline{0}\right)\underline{G}(k+1)\,[\underline{\tilde{C}}\,\underline{\Sigma}(k+1|k)\underline{\tilde{C}}' + \underline{\Theta}(k+1]\underline{G}'(k+1)\begin{pmatrix}\underline{0}\\ \cdots\\ \underline{I}_n\\ \cdots\\ \underline{0}\end{pmatrix} \tag{4.2.2}$$

where $\underline{G}(k+1)$ is given by Eq. (2.3.12). The Lemma then follows immediately. Intuitively, since

$$\underline{a}(k+1) = \underline{a}(k)$$

the uncertainty in $\underline{a}(k)$ cannot grow.

As a result of the Lemma 4.2.1, there exists then a $\underline{\Sigma}_{aa}$ such that

$$\lim_{k\to\infty} \underline{\Sigma}_{aa}(k|k) = \underline{\Sigma}_{aa} \tag{4.2.3}$$

Lemma 4.2.2: For $\underline{b}(k+1) = \underline{b}(k)$, given any control sequence we have from Eq. (2.3.15)

$$\underline{\Sigma}_{bb}(k+1|k+1) \leq \underline{\Sigma}_{bb}(k|k) \tag{4.2.4}$$

Proof: The proof is similar to that for Lemma 4.2.1. There exists then a $\underline{\Sigma}_{bb}$ such that

$$\lim_{k\to\infty} \underline{\Sigma}_{bb}(k|k) = \underline{\Sigma}_{bb} \tag{4.2.5}$$

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* The matrix $\underline{P}$ is said to be small than $\underline{M}$ if for all nonzero vectors, the scalar quantity $\underline{x}'\underline{P}\,\underline{x} < \underline{x}'\underline{M}\,\underline{x}$.

In analogy with the deterministic case, we can thus say that the parameters $\underline{a}$ and $\underline{b}$ are observable since the variance of the estimation error of $\underline{a}$ and $\underline{b}$ can be decreased by operation on $\underline{z}$.

It can be shown that if $\underline{\delta}(k) = \underline{0}$, $\underline{\gamma}(k) = \underline{0}$, and the system completely observable, then for any bounded but nonzero control u(k), k = 0,1,... [16], [29]

$$\lim_{k\to\infty} \underline{\Sigma}_{aa}(k|k) = \underline{0} \tag{4.2.6}$$

$$\lim_{k\to\infty} \underline{\Sigma}_{bb}(k|k) = \underline{0} \tag{4.2.7}$$

Since $\underline{\Sigma}(k|k) \geq \underline{0}$, this result implies that $\lim_{k\to\infty} \underline{\Sigma}_{xb}(k|k) \to \underline{0}$ $\lim_{k\to\infty} \underline{\Sigma}_{xa}(k|k) \to \underline{0}$, and $\lim_{k\to\infty} \underline{\Sigma}_{ab}(k|k) \to \underline{0}$.

Hence, we can design a reasonable controller for a completely observable system with unknown parameters.

$$\underline{a}(k+1) = \underline{a}(k)$$

$$\underline{b}(k+1) = \underline{b}(k)$$

using an ad-hoc control law $\zeta_k(\hat{\underline{x}}(k|k), \hat{\underline{a}}(k|k), \hat{\underline{b}}(k|k), \underline{\Sigma}_{aa}(k|k), \underline{\Sigma}_{bb}(k|k))$ for $k \geq 0$ given by (Fig. 4.1)

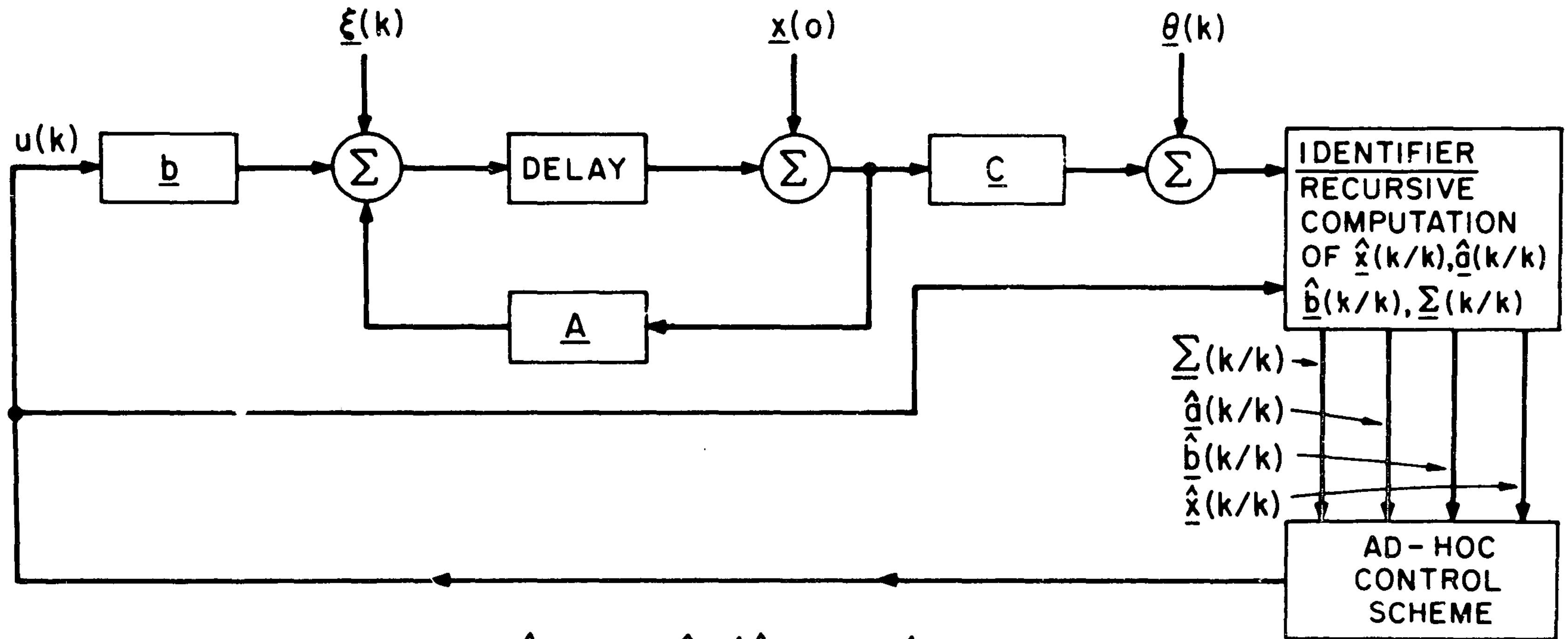
(1) $\zeta_k(\underline{x},\underline{a},\underline{b},\underline{\Sigma}_{aa},\underline{\Sigma}_{bb}) : R^n \times R^n \times R^n \times M_{nn} \times M_{nn} \to R$

$\underline{x} \varepsilon R^n$, $\underline{a} \varepsilon R^n$, $\underline{b} \varepsilon R^n$, $\underline{\Sigma}_{aa} \varepsilon M_{nn}$, $\underline{\Sigma}_{bb} \varepsilon M_{nn}$

(2) $\zeta_k(\underline{x},\underline{a},\underline{b},\underline{\Sigma}_{aa},\underline{\Sigma}_{bb}) \neq 0$, $\underline{\Sigma}_{bb} \neq \underline{0}$, $\underline{\Sigma}_{aa} \neq \underline{0}$, $\underline{x} \neq \underline{0}$

(3) $\zeta_k(\underline{x},\underline{a},\underline{b},\underline{0},\underline{0}) = -(r(k) + \underline{b}'\underline{K}(k+1)\underline{b})^{-1}\underline{b}'\underline{K}(k+1)\underline{A}\,\underline{x}$

Condition 2 satisfies Eqs. (4.2.6) and (4.2.7) and Condition 3 implies that the ad-hoc control will converge to the optimal control when



$$u(k) = -(r(k) + \hat{\underline{b}}'\,\underline{K}(k+1)\,\hat{\underline{b}})^{-1}\,\hat{\underline{b}}'\,\underline{K}(k+1)\,\hat{\underline{A}}\,\hat{\underline{x}}$$

Fig. 4.1 Block Diagram of the ad-hoc Control Scheme

$\underline{a}$ and $\underline{b}$ become known. Hence, the ad-hoc control scheme for system with unknown parameters can provide reasonable simulation results.

Let us now assume an observable system S1, where

$$\underline{a}(k+1) = \underline{a}(k), \ \underline{b}(k+1) = \underline{b}(k)$$

that is, the parameters are constant and not growing. We want to control the system over a finite interval N. In the beginning when the uncertainty in $\underline{b}(k)$ is large ($\underline{\Sigma}_{bb}$ big) $\tilde{r}(j|k)$ is big the adaptive control gain $\underline{\phi}(k)$ is small for both the stable and unstable systems. The trajectory of the overall control system then approximates that of the input-free trajectory of system S1. The initial guess on $\underline{b}(k)$ is not changed since little control input is applied. For the not exponentially stable system $\underline{\phi}(k) \sim \underline{0}$ at the beginning. The low control magnitude nevertheless starts the identification of $\underline{a}(k)$. It is expected that the resulting in stability of the system will cause large inputs to be applied which results in the identification of $\underline{b}(k)$. The high control magnitude will be used mainly for identification of the parameters. The control will be nonzero as long as the identification process is not completed. We, therefore, expect that for the unstable system, the estimate of $\underline{b}(k)$ will be identified before the control magnitude goes to zero.

In the exponentially stable systems, we have nonzero control in the beginning. Small control (even zero) may be utilized to bring the state toward zero. Hence, little energy is needed to keep the trajectory near zero, and identification of $\underline{b}(k)$ is expected to be slow. Identification of $\underline{a}(k)$ is expected to be reasonable, since the entire output will be due to mainly the product $\underline{A}(k)\underline{x}(k)$, although as $k \to \infty$ the observation approaches white noise. Hence, one may end up with good control but bad

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estimates, since little control effort is spent for identification and control purposes. We remark that the above interpretation of the derived equations assumes that the nonlinear estimator is working satisfactorily, that the deviations of the estimates from the actual values are not large.

Finally, we shall consider the problem of controlling the time-invariant system S1 with unknown parameters over an infinite time period ($N \to \infty$). Assume that the system is controllable and observable, then the window-shifting approach suggested in [16] can be employed. By Lemmas 4.2.1 and 4.2.2, ($\underline{\gamma}(k) = \underline{0}$ and $\underline{\delta}(k) = \underline{0}$) the estimates in $\underline{a}$ and $\underline{b}$ will converge asymptotically and hence, the time-varying adaptive control system tends to a time-invariant control system.

4.3 Enforced Separation Scheme

In this section we shall discuss the ad-hoc control design in which the Separation Theorem is arbitrarily enforced. This controller is a direct approximation of the stochastic optimal controller resulting from Separation Theorem, which in the absence of uncertainty about the plant parameters of S1 minimizes the cost functional

$$\bar{J} = \frac{1}{2} E\{\underline{x}'(N)\underline{Q}(N)\underline{x}(N) + \sum_{j=0}^{N-1} \underline{x}'(j)\underline{Q}(j)\underline{x}(j) + r(j)u^2(j)\} \quad (4.3.1)$$

where the expectation is taken over the underlying random variables $\underline{x}(0)$, $\underline{\xi}(\cdot)$, and $\underline{\theta}(\cdot)$. The approximate minimization is based on the present estimates of the system parameters $\hat{\underline{a}}(k|k)$ and $\hat{\underline{b}}(k|k)$. We assume the parameter values to be known ($\underline{\Sigma}_{aa}(\cdot|k) = \underline{\Sigma}_{bb}(\cdot|k) = \underline{0}$). Thus at each time step k, $0 \leq k \leq N$ we have the conditional cost given by Eq. (3.1.4) to be minimized subject to the dynamics Eqs. (3.2.1) - (3.2.4). Since

$\underline{\Sigma}_{xx}(\cdot|k)$ is now independent of $u(\cdot)$, the equivalent functional minimization is given by

$$\bar{J} = \frac{1}{2}\hat{\underline{x}}'(N|k)\underline{Q}(N)\hat{\underline{x}}(N|k) + \frac{1}{2}\sum_{j=k}^{N-1}\hat{\underline{x}}'(j|k)\underline{Q}(j)\hat{\underline{x}}(j|k) + r(j)u^2(j) \quad (4.3.2)$$

The structure of the resulting control law has the form

$$u(k|k) = -\underline{g}'(k|k)\hat{\underline{x}}(k|k) \quad (4.3.3)$$

where $\underline{g}'(k|k)$ is the optimal deterministic gain which arbitrarily uses the current parameter estimates for the unknown system parameter values,

$$\underline{g}(k|k) = (r + \hat{\underline{b}}'(k|k)\underline{K}(k+1|k)\hat{\underline{b}}(k|k))^{-1}\hat{\underline{b}}'(k|k)\underline{K}(k+1|k)\hat{\underline{A}}(k|k) \quad (4.3.4)$$

$$\underline{K}(k|k) = \hat{\underline{A}}'(k|k)[\underline{K}(k+1|k) - \underline{K}(k+1|k)\hat{\underline{b}}(k|k)(r + \hat{\underline{b}}'(k|k)\underline{K}(k+1|k)$$
$$\hat{\underline{b}}(k|k))^{-1}\hat{\underline{b}}'(k|k)\underline{K}(k+1|k)]\hat{\underline{A}}(k|k) + \underline{Q}(k)$$
$$\underline{K}(N|k) = \underline{Q}(N) \quad (4.3.5)$$

We shall denote this suboptimal open-loop feedback design the enforced separation scheme. The control law is identical with the linear-quadratic-Gaussian case with known parameters, except that the actual parameter values of $\underline{a}(k)$ and $\underline{b}(k)$ are arbitrarily replaced by their updated estimates. We shall use the extended Kalman filter to generate the estimates. Optimality of the cascaded form feedback controller, thus, cannot be claimed any more.

As in the open-loop feedback optimal design, the enforced separation scheme controller is time-varying, because the feedback control gains must be recomputed to make use of the updated estimates. At each time step k a Riccati-type equation needs to be solved backwards in time, since $\hat{\underline{a}}(k|k) \neq \hat{\underline{a}}(k+1|k+1)$ and $\hat{\underline{b}}(k|k) \neq \hat{\underline{b}}(k+1|k+1)$. The enforced separation scheme differs from the open-loop feedback optimal design, essentially, in that the control does not depend explicitly on the goodness of the parameter estimation process. Since arbitrarily enforcing Separation Theorem does not require the propagation of the error covariance matrices and predictor equations of $\hat{\underline{a}}(j|k)$ and $\hat{\underline{b}}(j|k)$, the computational requirements are that much simpler compared with the open-loop feedback approach. We emphasize that both designs are suboptimal closed-loop control systems. In the limit as the parameter estimates converges to the time parameter vectors $\underline{a}$ and $\underline{b}$, from Eq. (4.2.6), so will each suboptimal control policy asymptotically approach the truly optimal solution when the actual parameters are known.

CHAPTER 5

INTERPRETATION OF THE RESULTS

In this chapter we shall discuss further the O.L.F.O. approach and the results on adaptive systems. The adaptive control problem we stated in Section 2.1 and solved is one of optimal nonlinear stochastic control. In the formulation of computationally feasible solution to the optimal control problem, we have, in essence, forced separation in the control action. The separation in the open-loop feedback approach yields a suboptimum controller. Restricting the form to a reasonable closed-loop stochastic control system of Fig. 2.1, the open-loop feedback optimal control system can be viewed as a separation of the overall control system into an estimator and a zero-memory controller. The first subsystem is the learning device where the states and the parameters are identified in real time and can be designed independently of the controller objectives. The controller subsystem computes the on-line optimal control for a deterministic system and is parameter adaptive.

The results we derived in Section 3.3 do not correspond to the strict separation of optimum estimation and optimum deterministic control as is in the case of uncertain linear systems with known parameters and quadratic cost criteria. [6],[5] We expect that the Separation Theorem does not hold in the adaptive control problem. The value of the optimal open-loop feedback control depends on both the estimate (by examining the adaptive control gain vector $\underline{\phi}'(k)$) of the parameter and their error covariance matrices. The effect of identification error is, therefore, taken into account in the deterministic control problem we have formulated and solved. The design bears the adaptive properties and violates the pure Separation Theorem.

The system S1 contains an input signal $u(k), k = 0,1,\ldots,N-1$. It is expected that the choice of this input will affect how well the system identification can be performed. Let $\underline{\xi}(k), \underline{\delta}(k), \underline{\gamma}(k) = \underline{0}$, then we obtain for the time-invariant system

$$\underline{x}(k+1) = \underline{A}^{k+1}\underline{x}(0) + \sum_{i=0}^{k} \underline{A}^{k-i}\underline{b}\,u(i) \tag{5.1}$$

$$\underline{z}(k+1) = \underline{C}\,\underline{A}^{k+1}\underline{x}(0) + \underline{C}\,\underline{b} \sum_{i=0}^{k} \underline{A}^{k-i}u(i) + \underline{\theta}(k+1) \tag{5.2}$$

For example, if $\underline{x}(0) = \underline{0}$, then a nonzero $u(k)$ is required for the system to identify the parameters. From Eq. (2.1.1), it is obvious that the larger the control input $u(k)$, the larger the contribution to the state trajectory at $\underline{x}(k+1)$. This implies that the observation will contain a large amount of information about the gain parameter $\underline{b}(k)$. Large values of control input will, therefore, help in the identification of $\underline{b}(k)$. The control signal can also be used to regulate the signal-to-noise ratio at the sensor. But, large input is discouraged in our quaratic control-penalty term in the cost functional. Therefore, the dual nature of the control input is clearly emphasized in our formulation. A reasonable adaptive control sequence must then be a compromise between the desire to get accurate estimates and the desire to minimize the cost.

Let us consider further the open-loop feedback optimal solution. The original state weighting matrix was $\underline{Q}(j)$, and the original control weighting was $r(j)$. The effect of the parameter uncertainties in the open-loop feedback optimal control problem is to transform it, heuristically speaking, to a linear quadratic tracking problem with modified weightings, where $u^o(j|k)$ is the optimal control for the following control system [37]

$$\underline{\tilde{x}}(j+1|k) = \underline{A}^{\dagger}(j|k)\underline{\tilde{x}}(j|k) + \underline{\tilde{b}}(j|k)u(j|k) \tag{5.3}$$

$$\underline{\tilde{x}}(j|k) \triangleq \begin{pmatrix} \underline{\hat{x}}(j|k) \\ \cdots\cdots \\ \underline{\sigma}(j|k) \\ \cdots\cdots \\ \underline{\rho}(j|k) \end{pmatrix} \tag{5.4}$$

with the cost functional

$$J = \underline{\tilde{x}}'(N|k)\underline{\tilde{Q}}(N)\underline{\tilde{x}}(N|k) + \sum_{j=k}^{N-1} \{\underline{\tilde{x}}'(j|k)\underline{W}(j|k)\underline{\tilde{x}}(j|k) + \tilde{r}(j|k)u^2(j|k)$$

$$+ 2\underline{\tilde{x}}'(j|k)\underline{d}(j+1|k)u(j|k)\} \tag{5.5}$$

Comparing this problem with the original linear quadratic state-regulator problem, we can call $\tilde{r}(j|k)$ the modified relative weighting on the control.

We remark that the scalar weighting $\tilde{r}(j|k)$ in Eq. (2.3.34) is related directly to the conditional error covariance $\underline{\Sigma}_{bb}(j|k)$. The modified relative weighting on the control $\tilde{r}(j|k)$ indicates that a low level of confidence in the estimate of the gain parameter $\underline{\hat{b}}(j|k)$ or that $\underline{\Sigma}_{bb}(j|k)$ is big will weight the control heavily in Eq. (5.5) so that little energy will be expended. Hence, the more the gain parameter uncertainty (now and in the future) the larger $\tilde{r}(j|k)$, which implies the smaller the value of the adaptive gain $\underline{\phi}(k)$ in Eq. (2.3.6). The control gain is, therefore, adjusted by the level of uncertainty of the estimation of $\underline{b}(k)$. We note that the parameter uncertainty $\underline{\Sigma}_{bb}(j|k)$ is modified by the matrix $\underline{P}_{xx}(j+1|k)$ in the computation of $\tilde{r}(j|k)$ in Eq. (2.3.34).

In Eq. (2.3.33) if $\|\hat{\underline{A}}(j|k)\| < 1$, then $\|\underline{P}_{xx}(j|k)\| \approx \|\underline{Q}(j)\|$ for $j \geq k$. If, however, $\|\hat{\underline{A}}(j|k)\| > 1$, then $\|\underline{P}_{xx}(j|k)\| >> \|\underline{Q}(j)\|$, and, thus there is much more contribution from the trace term to the value of $\tilde{r}(j|k)$ in Eq. (2.3.34). We remark that at the same level of parameter uncertainty, the more unstable the system, the larger $\underline{P}_{xx}(j|k)$, and hence the larger the value of the $\tilde{r}(j|k)$. The result is the smaller the adaptive control gain $\underline{\phi}(k)$ at the initial stages the more the system model is unstable.

In the discussion thus far we have not examined the effect of uncertainty in $\underline{a}(k)$ ($\underline{\Sigma}_{aa}(k|k) \neq \underline{0}$). Recall that from Eqs. (2.3.1)-(2.3.3), (2.3.31)-(2.3.32) the computation of the O.L.F.O. adaptive control is affected not only by $\underline{\Sigma}_{bb}(j|k)$, but also by $\underline{\Sigma}_{aa}(j|k)$. The original state weighting matrix $\underline{Q}(j)$ is modified in Eq. (2.3.31). If the system is asymptotically stable, then $\|\underline{Q}(j)\| + \|p_{nn}(j+1|k)\underline{\Sigma}_{aa}(j|k)\| \cong \|\underline{Q}(j)\|$. The state weighting matrix is not affected by the uncertainty in $\underline{a}(k)$. Hence, the O.L.F.O. will be cautious in applying the control to the system in the beginning due to uncertainty in $\underline{b}$. Intuitively, this is reasonable, since the system is stable, and will decay to zero asymptotically. If the system is not asymptotically stable, then the state weighting matrix is modified by $\|\underline{Q}(j)\| + \|p_{nn}(j+1|k)\underline{\Sigma}_{aa}(j|k)\| >> \|\underline{Q}(j)\|$. The net effect is that the O.L.F.O. will act with large control magnitudes to regulate the system. This is not easily seen heuristically from the equations. The interaction between the uncertainty in $\underline{b}(k)$ and $\underline{a}(k)$ is not completely predictable from the equations (2.3.31)-(2.3.32), (2.3.2) for the adaptive gain, since $\underline{P}_{xx}(j+1|k)$ is observation dependent.

Finally, we want to discuss the optimization of the (deterministic) conditional cost functional (3.1.4). The open-loop covariance matrix should be a function of the conditional probability density functions. The reformulation would have been exact. Since $\underline{A}(k)$ is unknown, the estimation problem is an infinite dimensional one. We are forced by computational limitation to approximate the conditional expectations $\hat{\underline{x}}(k|k)$ and $\hat{\underline{x}}(j|k)$ and $\underline{\Sigma}_{xx}(k|k)$ and $\underline{\Sigma}_{xx}(j|k)$ using the extended Kalman filter. The optimal open-loop control sequence we have derived in Chapter 3 is at best an approximation to the optimal conditional open-loop control law. If $\underline{A}(k)$ is known, then the exact conditional means and error covariances can be generated by the optimal linear filter [16]. Hence, in the absence of rigorous analytical results on the goodness of the approximation we have developed, we shall turn to simulation experiments to provide further clues to the adaptive control problem under consideration.

CHAPTER 6

SIMULATION RESULTS

We have derived the open-loop feedback optimal control sequence in Chapter 3 via dynamic programming for the problem stated in Section 2.1. Based on the equations (2.3.8)-(2.3.16) for the identifier and Eqs.(2.3.5)-(2.3.7) for the feedback gain plus correction term controller we discussed qualitatively the asymptotic behavior of the identifier in Chapter 4 and of the overall control system in Chapter 5. In this chapter we shall report the results of simulated experimentation made on some dynamical systems. The main purpose of the simulation studies is to verify the qualitative properties that we predicted and to provide the quantitative measures on the convergence rate of both the O.L.F.O. control and the enforced separation schemes to the truly optimal stochastic control when the parameters are known. The simulation studies will compare the performance measure of (1) the truly optimal stochastic control system when the full dynamics are known, (2) the O.L.F.O. adaptive control system, and (3) the enforced separation scheme.

We shall consider specifically the first-order linear dynamical system described by the stochastic difference equation

$$x(k+1) = ax(k) + bu(k) + \xi(k) \tag{6.1}$$

with noisy measurements given by

$$z(k) = cx(k) + \theta(k) \tag{6.2}$$

We assume that the unknown parameters are constant.

The initial values $x(0)$, $a(0)$, and $b(0)$ are assumed to be Gaussian random variables with known statistics

$$E\{x(0)\} = x_o , \quad cov[x(0),x(0)] = \Sigma_{xo} \tag{6.3}$$

$$E\{a(0)\} = a_o , \quad cov[a(0),a(0)] = \Sigma_{ao} \tag{6.4}$$

$$E\{b(0)\} = b_o , \quad cov[b(0),b(0)] = \Sigma_{bo} \tag{6.5}$$

where Σ_{xo}, Σ_{ao}, and Σ_{bo} are positive semidefinite. The scalar zero-mean white Gaussian driving noise sequence has the known statistics

$$E\{\xi(k)\} = 0, \quad cov[\xi(j),\xi(k)] = \Xi(k)\delta_{jk} \tag{6.6}$$

and the scalar zero-mean white Gaussian observation noise has the known statistics

$$E\{\theta(k)\} = 0, \quad cov[\theta(j),\theta(k)] = \Theta(k)\delta_{jk} \tag{6.7}$$

where $\Xi(\cdot) \geq 0$ and $\Theta(\cdot) > 0$. The random variables $\{x(0),a(0),b(0),\xi(\cdot),\theta(\cdot)\}$ are mutually independent.

The objective of the problem is to find an optimal control sequence such that the expected cost functional

$$\bar{J} = E\{\frac{1}{2} q(N)\, x^2(N) + \frac{1}{2} \sum_{k=0}^{N-1} q(k)\, x^2(k) + r(k)u^2(k)\} \tag{6.8}$$

is minimized based on some information set.

We can now apply the theoretical results of the open-loop feedback optimal control obtained in Chapters 2 and 3 to Eqs. (6.1)-(6.8). A digital computer program was written, in which the identification equations (2.3.8)-(2.3.16), and the O.L.F.O. parameters equations (2.3.17)-(2.3.19), (2.3.23)-(2.3.25), (2.3.27)-(2.3.34), and adaptive control equations (2.3.1)-(2.3.2) are programmed as individual subroutines. The computer program listing is contained in Appendix B. In the computer program, if we let $a_o = a$, $b_o = b$, and $\Sigma_{ao} = \Sigma_{bo} = 0$, then the simulation results would correspond to 1) the optimal closed-loop stochastic control for the system defined by Eqs. (6.1)-(6.8) with known parameters. If we now at each time step k assume that $\Sigma_{aa}(k|k) = \Sigma_{bb}(k|k) = 0$, and use the instantaneous estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ of parameters a(k) and b(k) to compute the suboptimal feedback control, Eqs. (3.4.4)-(3.4.6) the simulation results in this case would correspond to the system 3) in which the Separation Theorem is arbitrarily enforced. Hence, the algorithm we have is a general stochastic control simulation program that readily accomodates changes in the simulation model and initial conditions, and can be trivially modified for n-dimensional systems.

Using a plotting subroutine, we then plotted for a series of sample runs the resulting trajectory for 1) the stochastic control system with known parameters (where Separation Theorem holds), 2) the open-loop feedback optimal control and 3) the enforced separation scheme; the changes in the estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ in the open-loop feedback optimal and in the enforced separation schemes; the changes in the optimal feedback gain $\phi^*(k)$, the adaptive gain $\phi(k)$, and the ad-hoc gain, and lastly, the applied control sequence in the three different stochastic control systems. In each sample run, to evaluate the cost functional for the three separate stochastic control systems, we computed the quadratic terms in x(k) and u(k), and these are also plotted versus k.

We note that we do not present here a comparison between the suboptimal control schemes and the true closed-loop stochastic control for systems with unknown parameters. The latter can be obtained only if we are able to compute the discrete probability distributions. The comparison of the system responses reported here will only indicate the

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Table 6.1

Summary of the Monte Carlo Simulation (Sampling)

Sample size = 20, $c = 1$, $b_o = 1$, $x_o = 0$,
$\Sigma_{xo} = 3$, $\Xi = 0.04$, $r = 1$

Simulation	a_o	Σ_{ao}	Σ_{bo}	Θ	q	Table
U1	1.2	0.0049	0.25	1.0	10	C.1
U2	1.2	0.0009	0.25	1.0	10	C.2
U3	1.2	0.0009	0.25	1.0	2	C.3
U4	1.2	0.0049	0.25	1.0	2	C.4
U5	1.2	0.0009	0.25	4.0	10	C.5
U6	1.2	0.0009	0.25	9.0	10	C.6
U7	1.2	0.0049	0.0	1.0	10	C.7
U8	1.2	0.0	0.25	1.0	10	C.8
S1	0.8	0.0049	0.25	1.0	10	C.9
S2	0.8	0.0049	0.25	4.0	10	C.10

loss in performance from the control viewpoint when simultaneous parameter identification is necessary. By examining these simulated time responses, we can obtain clues to the interaction between identification and control in an actual system. By evaluating the cost functional for the resulting trajectory and control sequence, we can numerically compare the performance of the open-loop feedback optimal design with the enforced separation design (in the closed-loop sense).

Since only first-order systems will be considered, the most number of unknowns we can have is three (x, a, and b). The uncertainty in b corresponds to uncertainty in the plant d. c. gain, while the uncertainty in a corresponds to uncertainty in the plant time constant. Not knowing a exactly is equivalent to not knowing the pole location of the system function. The only information about the system comes from the observation of the state trajectory obtained in the presence of a random disturbance. For each control system design, we used the same sample random sequence in the simulation via the Monte Carlo method, which required the establishment of statistical population for the uncertain quantities, repetitive calculation of performance using random samples from these populations, and averaging over the ensemble of results. We repeat (consecutively) the entire simulation run 20 times with the same initial conditions. A summary of the simulation experiments is given in Table 6.1. The comparison for the expected costs using controls 1), 2), and 3) is given in Appendix C.

The simulation results of U1 using the crude Monte Carlo method are shown in Figs. 6.1 - 6.5. We assume that the <u>a priori</u> distribution of a(0) is given by

$a(0) \sim N[1.2,\ 0.0049]$

From the plot of state trajectories in Fig. 6.1, we see that the open-loop feedback optimal trajectory has an overshoot at $k = 1$ due to the large control $u(0)$. The large control magnitude is used for identification and control purposes. The average of the parameter estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ are shown in Figs. 6.2-6.3. They approach nearly the initial mean values. The identification of b was better using the O.L.F.O. control than the enforced separation scheme. In the plot of the feedback gains, Fig. 6.4 we have the experimental result that the open-loop feedback initial adaptive gains are nonzero and, surprisingly, large compared to the truly optimal feedback gain. The open-loop feedback optimal control sequence proved to be more "aggressive" on the average than the enforced separation design as seen from Fig. 6.5. Both the open-loop feedback optimal and the enforced separation control sequence were able to stabilize the system, although the identification was not exact. Not shown here is the O.L.F.O. correction term which does go to zero as $k \to N$ in all the simulation runs.

In Figs. 6.6 - 6.7 we plot the quadratic terms in $x(k)$ and $u(k)$ vs. k averaged over 20 sample runs. They indicate the large contribution to the O.L.F.O. average cost occurs at $k = 1$. A detailed examination of the computer data also showed that the sample variance of the cost is huge, thus implying that possibly a small number of statistically bad runs contributed to making the sample mean large. The detailed computations of the sample run costs and cumulative average costs are given in Table C.1.

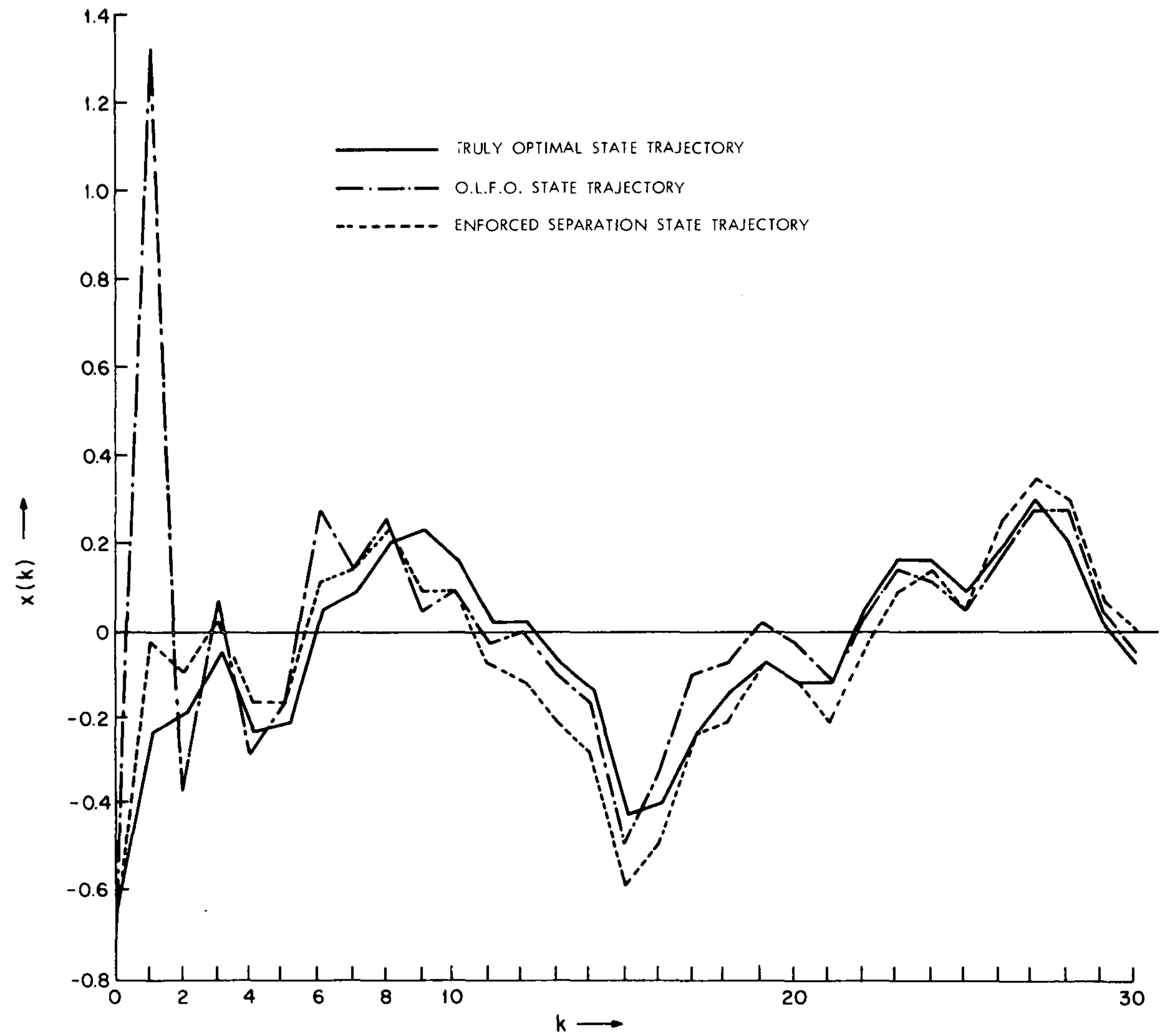

Fig. 6.1 Comparison of the average response of the unstable systems UI when the parameters a(k) and b(k) are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme). The sample noise sequence was the same. Sample size = 20.

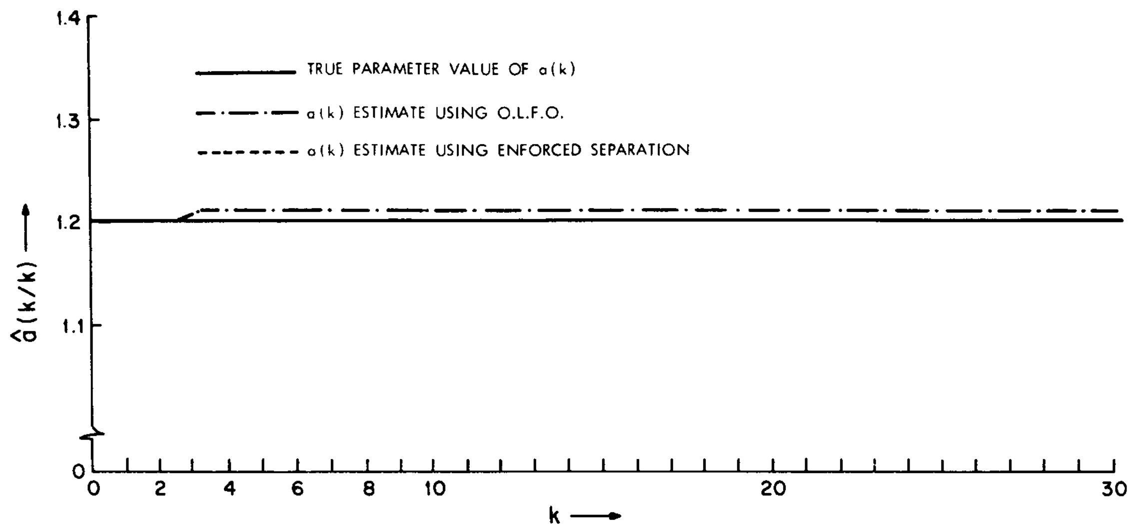

Fig. 6.2 Average Behavior of the Estimate of a(k) for the Unstable Systems UI

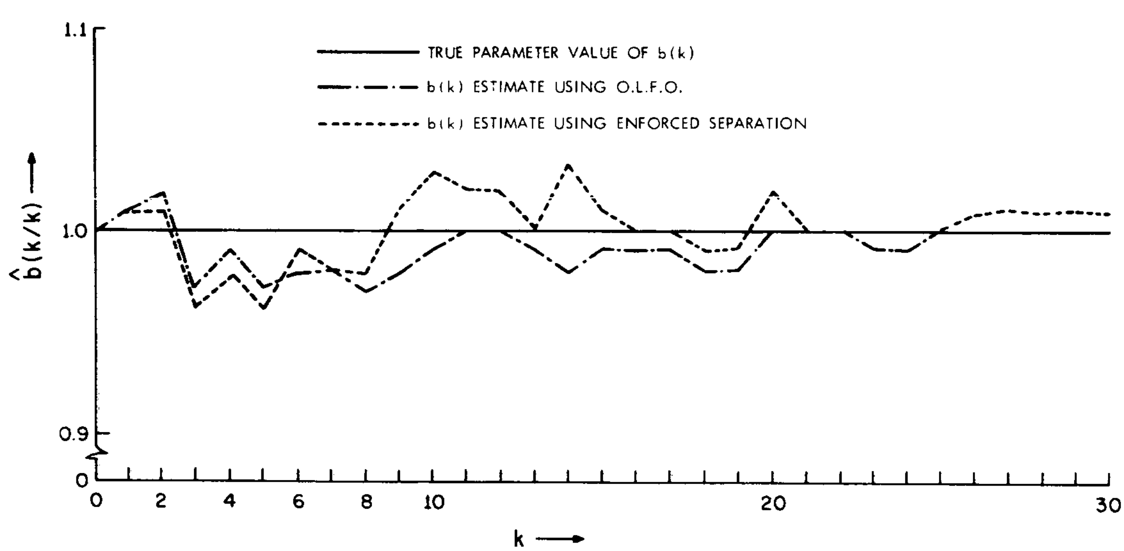

Fig. 6.3 Average Behavior of the Estimate of b(k) for the Unstable Systems UI

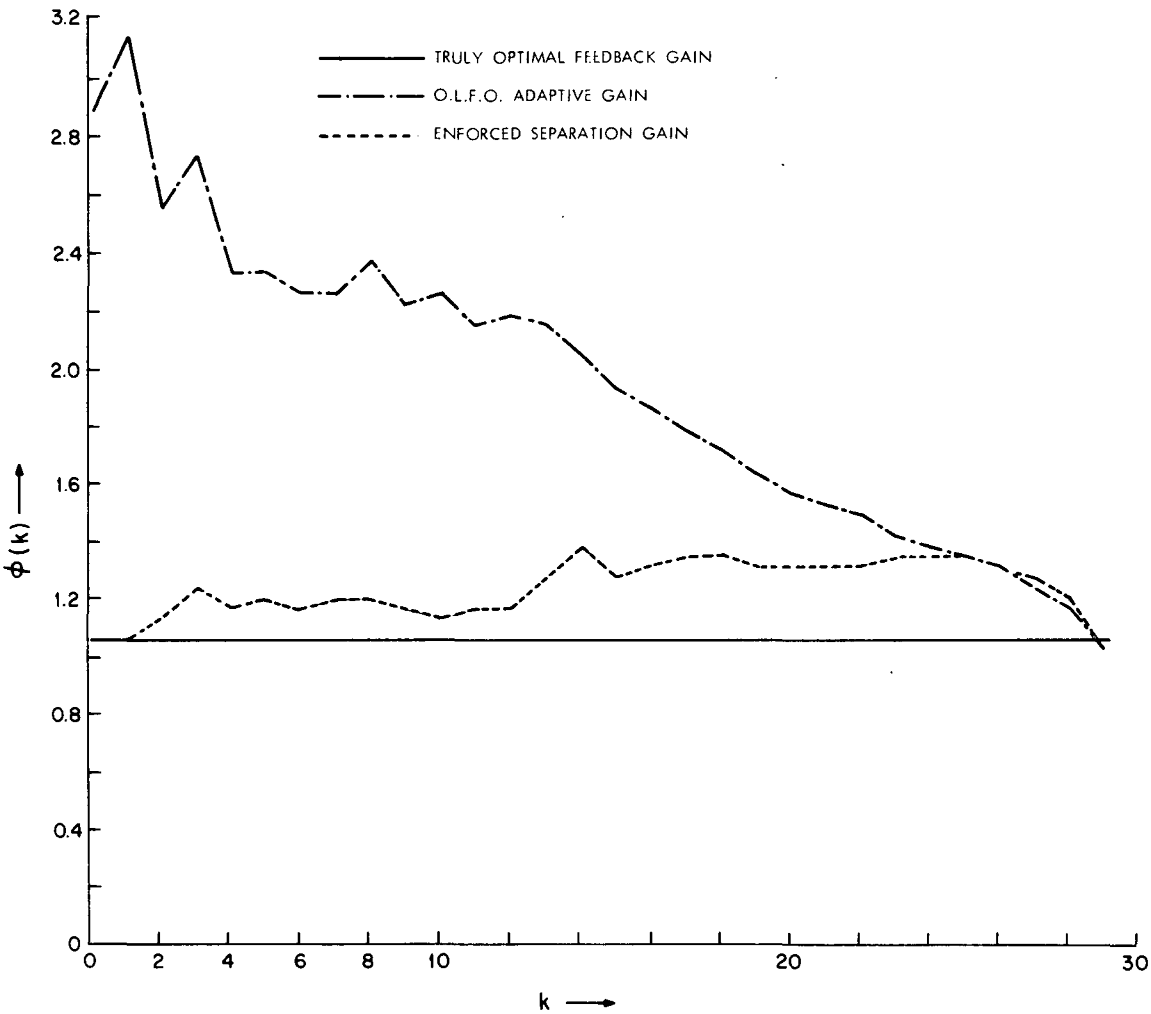

Fig. 6.4 Comparison between the Average Behavior of the Optimal Feedback Gain (when the parameters are known) and the two Suboptimal Feedback Gains (when the parameters are unknown) for the Unstable Systems UI

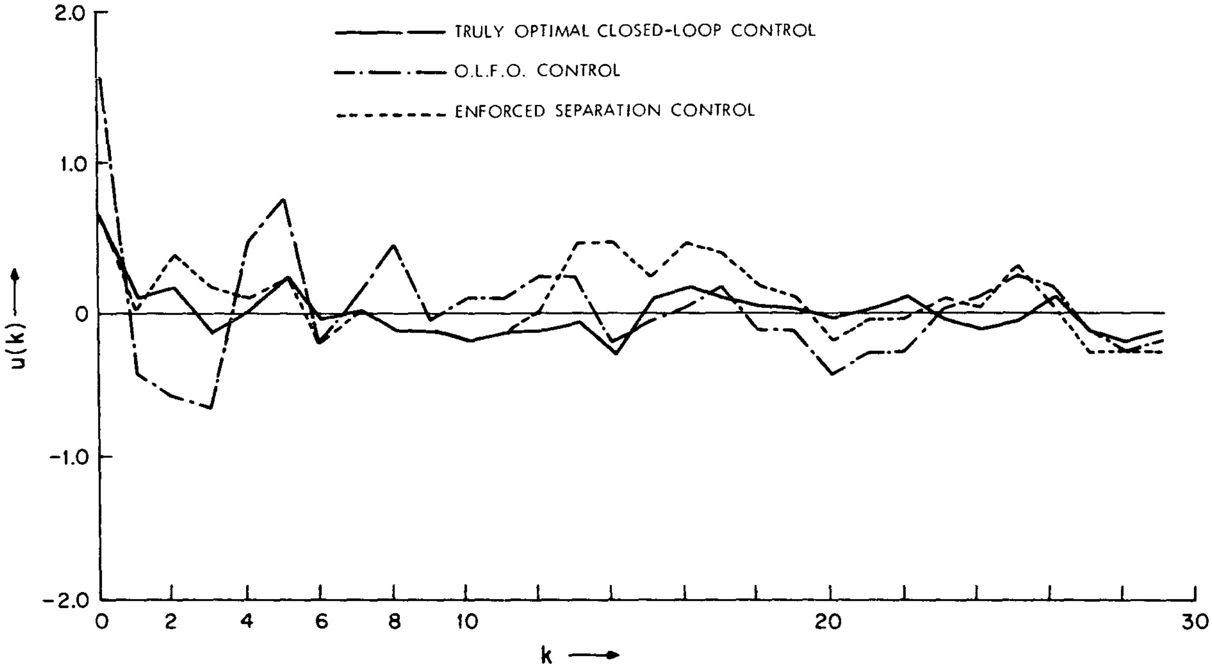

Fig. 6.5 Comparison between the Average Behavior of the Optimal Stochastic Control (when the parameters are known) and the two Suboptimal Stochastic Controls (when the parameters are unknown) for the Unstable Systems UI

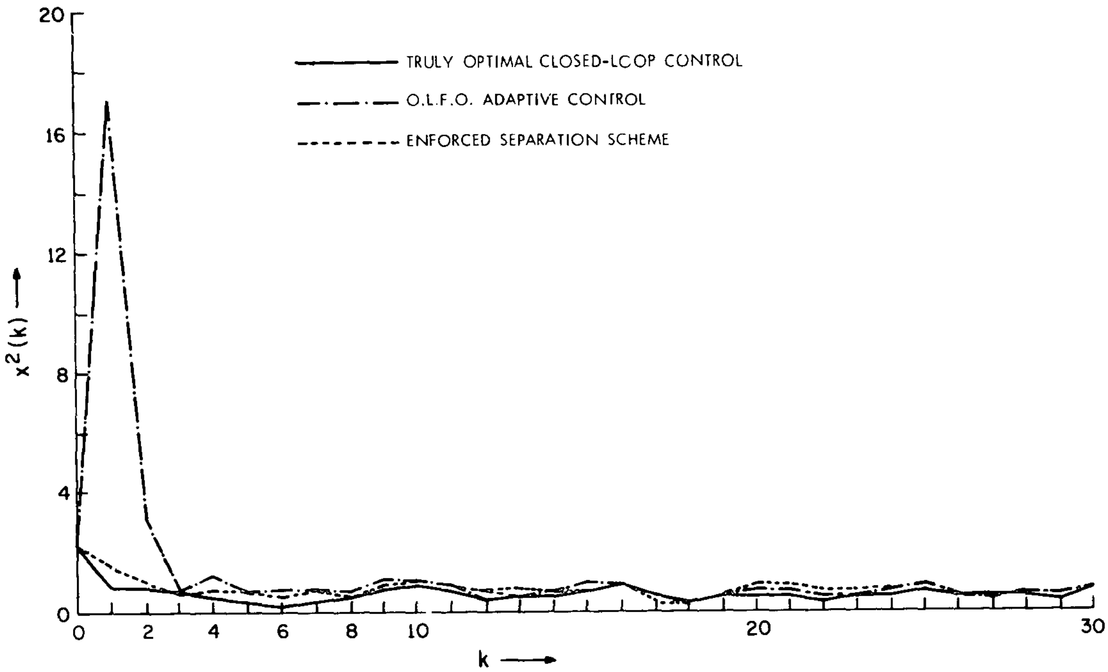

Fig. 6.6 Comparison of the Average Behavior of $x^2(k)$ for the Unstable Systems UI

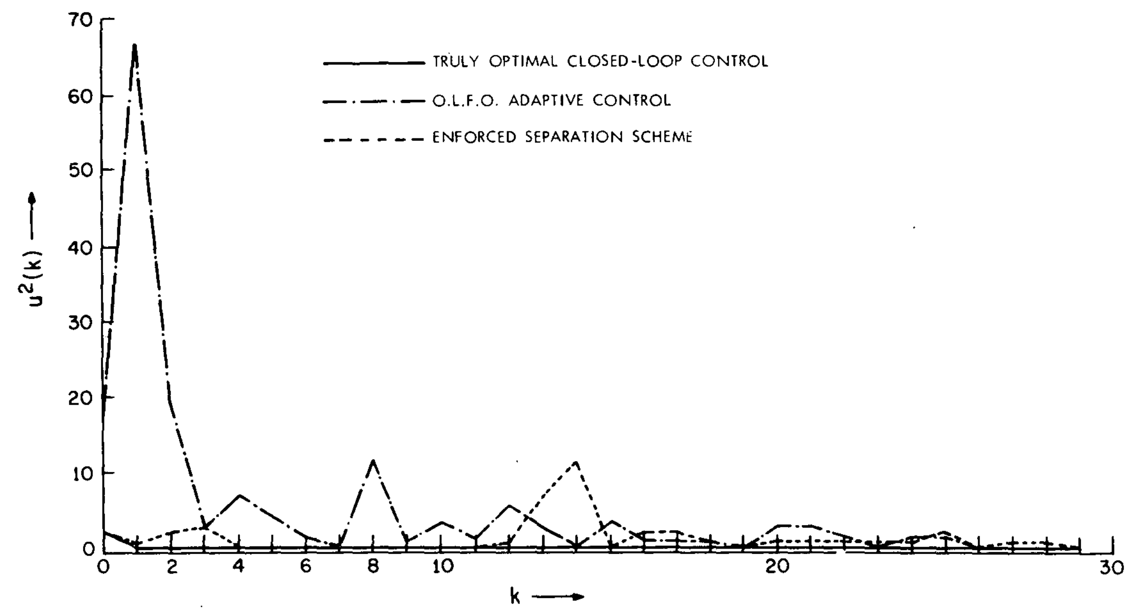

Fig. 6.7 Comparison of the Average Behavior of $u^2(k)$ for the Unstable Systems UI

In simulation run U2 the initial uncertainty in a_o was reduced (σ_{ao} = 0.07 in U1 and σ_{ao} = 0.03 in U2). When Σ_{ao} = 0, the estimation problem becomes linear, so decreasing Σ_{ao} would tend to reduce the non-linearity in the system. Simulation results showed that the identifier and the controller exhibited similar average behavior in all three cases as in U1. The open-loop feedback control gains and controls are a little larger than they were in U1. This tells us to look at the effect of $\Sigma_{aa}(j|k)$ in Eq. (2.3.31). The initial overshoot in x(k) is also little larger as a result.

Since the open-loop feedback optimal controller seemed too anxious to reduce the covariance in x, we ran simulations U3 and U4 with state weightings decreased to 2. The only noticeable effect is that the control magnitudes are smaller on the average than in U2 and U1. The average costs for the open-loop feedback optimal control remained greater than the enforced separation scheme.

By using Eq. (6.8) to evaluate the average costs, we are then comparing the open-loop feedback optimal and the enforced separation scheme in terms of performance index in the optimal closed-loop sense. Hence, in experiments U5 and U6 the measurement noise covariance was increased to 4 and then 9. We know that the open-loop feedback optimal design optimalizes the open-loop cost functional in the open-loop feed-back sense. By increasing the measurement uncertainty, the estimation process would tend to ignore the data in the beginning and rely on the predicted observation. The system will then act more according to its average behavior (ignoring the zero-mean random processes) and, hence, more in the open-loop sense. We expect that the controls are not going to be good when the noise covariances become large. This is seen from the larger costs incurred in U5 and U6 as compared to U2.

In simulation U5, the controller is more cautious on the average. The initial control is smaller than it was in U2,but takes on bigger values at $k = 2$ and $k = 3$ as the controller notices the divergence phenomenon in the state $x(k)$. The rate of convergence of the estimates was slower in this system. As Θ was increased to 9, the initial measurements are used with even less confidence. The system does not respond to the divergence phenomenon until later, and thus, the initial large overshoots due to large control magnitudes are removed. The trajectory in $x(k)$ showed instability in the beginning and larger variations than previous simulation runs. The biggest control occurred at $k = 3$, which produced the largest deviation in x at $k = 4$.

The simulation plots of S1 for stable systems using the crude Monte Carlo method are given in Figs. 6.8 - 6.12. We assume that the *a priori* distribution for $a(0)$ is

$$a(0) \sim N[0.8, 0.0049]$$

From Fig. 6.8, we see that the state trajectory $x(k)$ is essentially input-free. The initial open-loop feedback optimal control is nonzero, but small as expected since the modified control weighting Eq. (2.3.34) is large due to the initial uncertainty in b_o. The systems are asymptotically stable, hence, the control magnitudes are kept small. The estimates of parameters a and b are further off from the true values than in the unstable systems. But, the suboptimal controls do converge very fast to the truly optimal control. Again, the exact identification

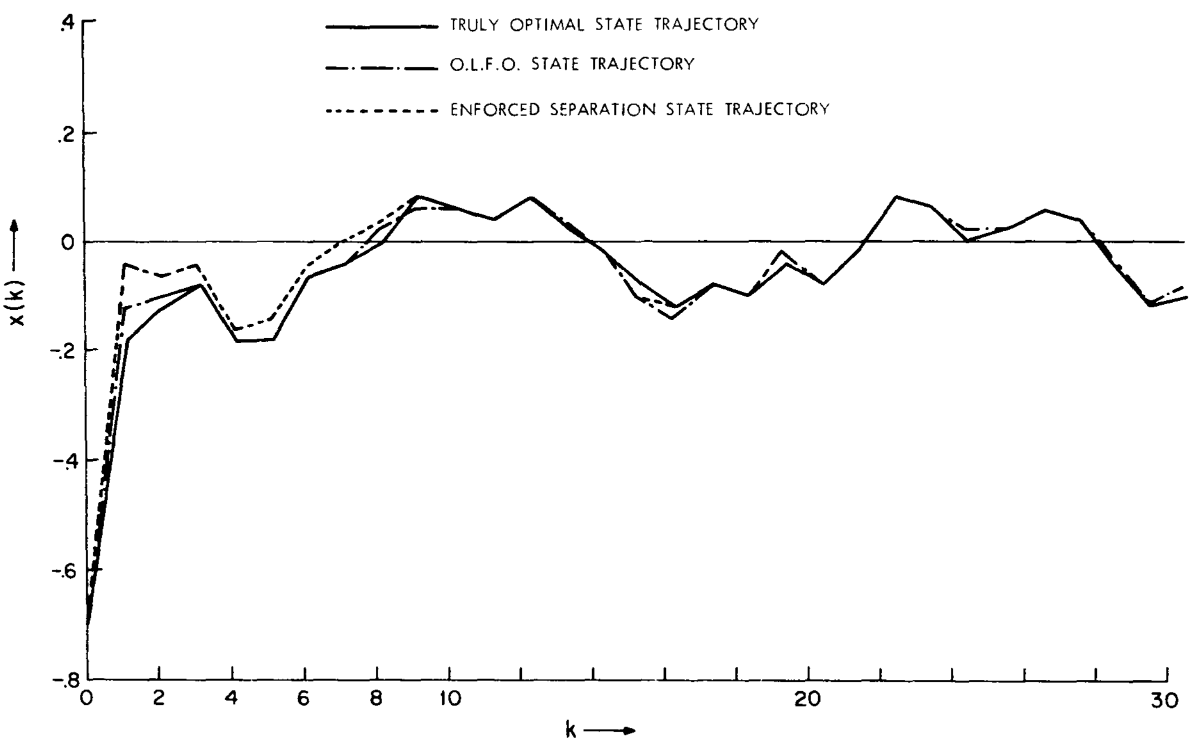

Fig. 6.8 Comparison of the average response of the stable systems S I when the parameters a(k) and b(k) are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme). The sample noise sequence was the same. Sample size = 20.

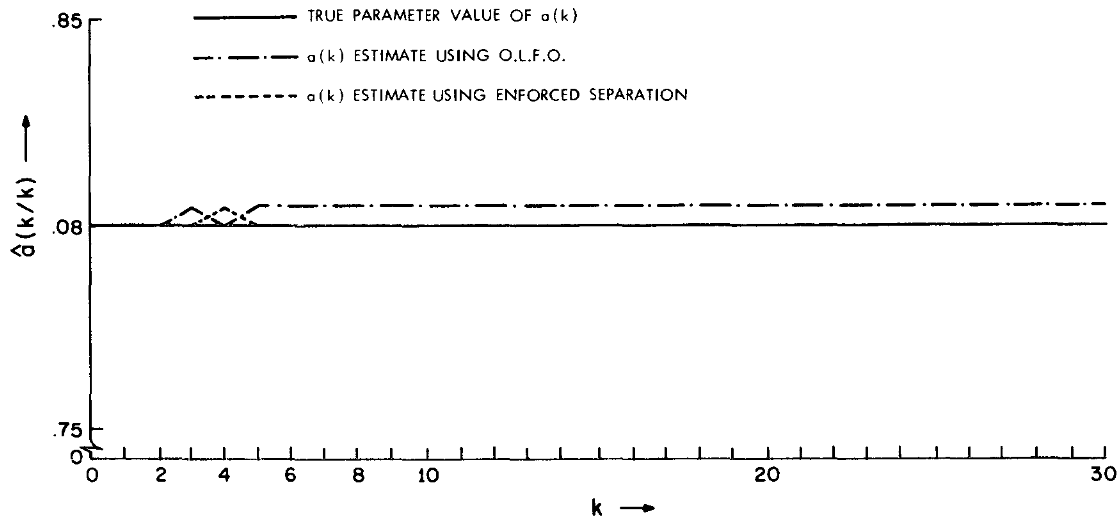

Fig. 6.9 Average Behavior of the Estimate of a(k) for the Stable Systems SI

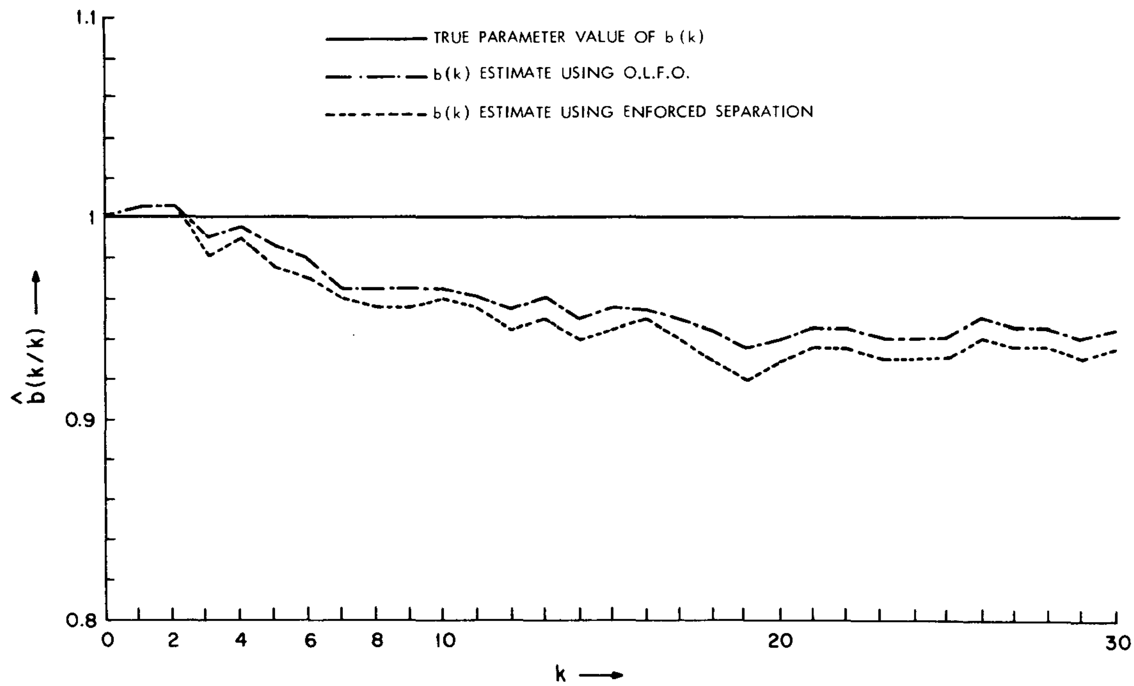

Fig. 6.10 Average Behavior of the Estimate of b(k) for the Stable Systems SI

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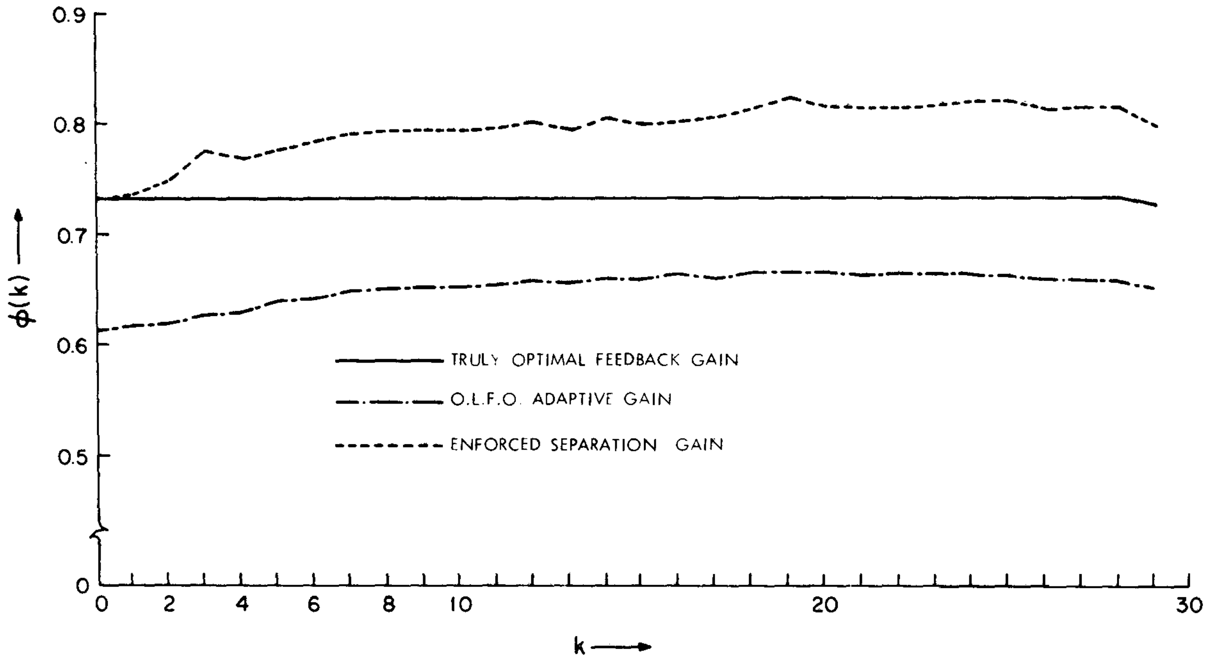

Fig. 6.11 Comparison between the Average Behavior of the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for the Stable Systems S1

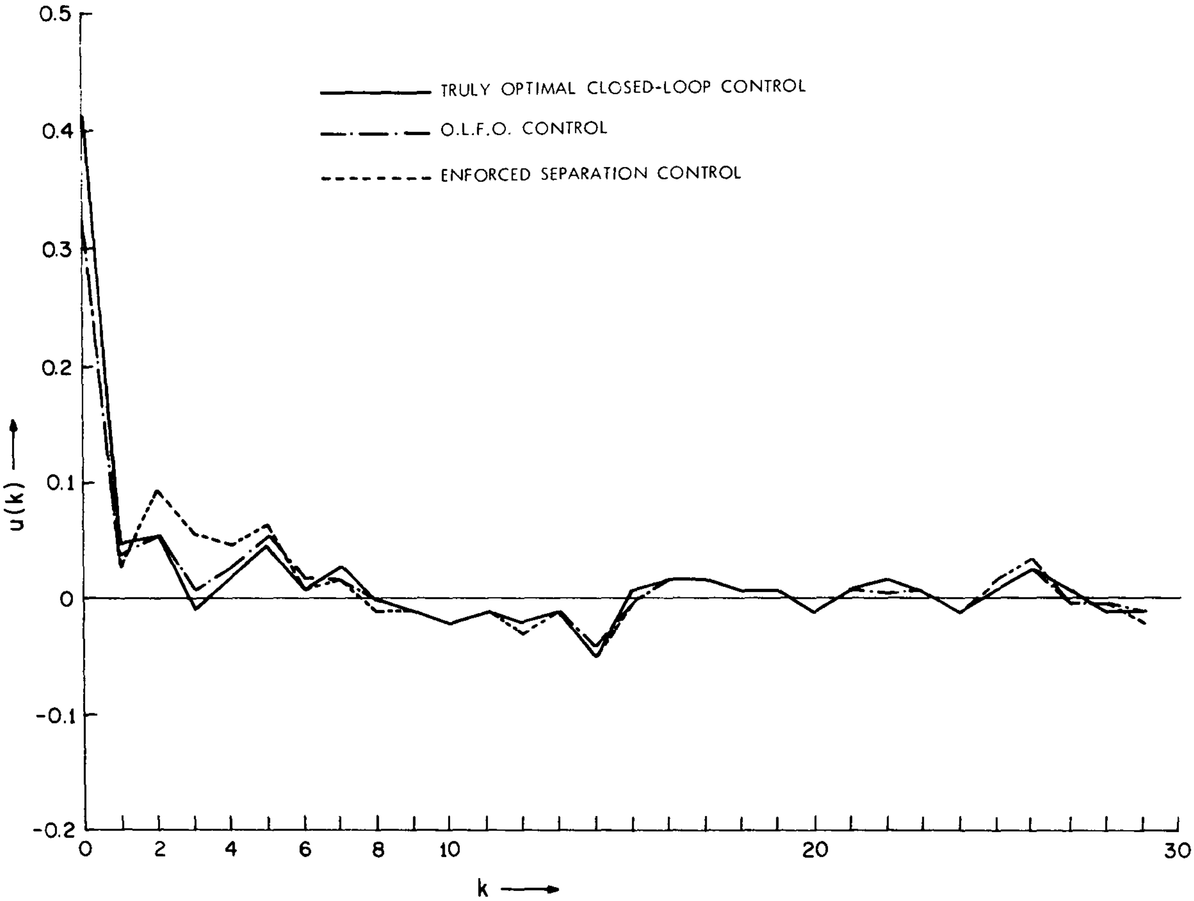

Fig. 6.12 Comparison between the Average Behavior of the Optimal Stochastic Control (When the parameters are known) and the two Suboptimal Stochastic Controls (When the parameters are unknown) for the Stable Systems SI

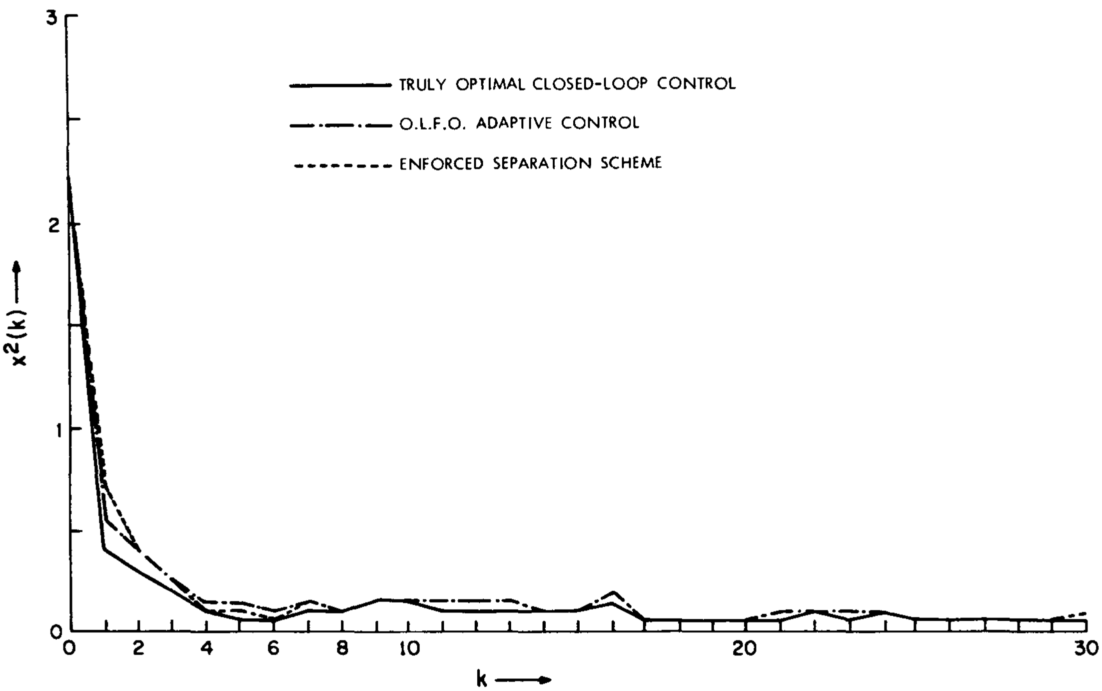

Fig. 6.13 Comparison of the Average Behavior of $x^2(k)$ for the Stable Systems SI

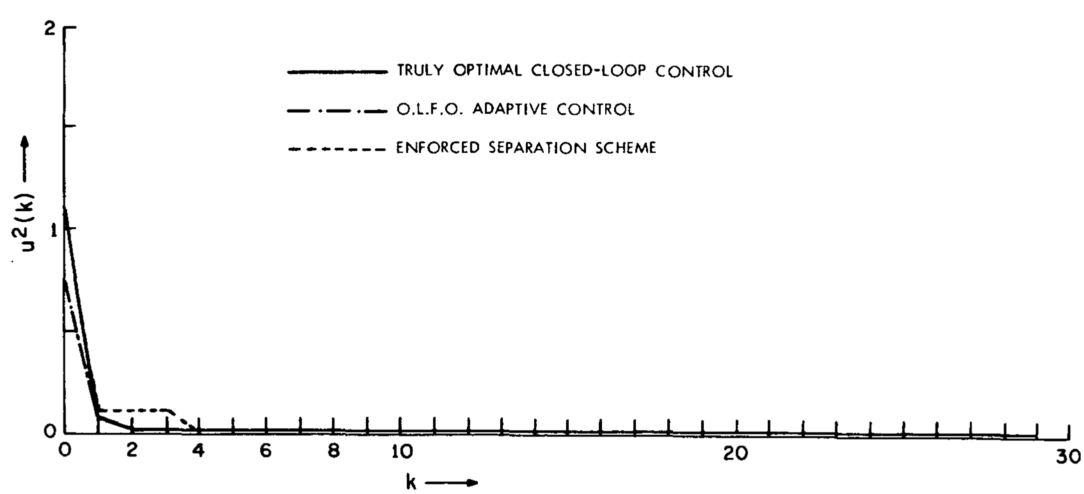

Fig. 6.14 Comparison of the Average Behavior of $u^2(k)$ for the Stable Systems SI

of the unknown parameters is not necessary for good control. The averaged estimates in Figs. 6.9 - 6.10 are not good, since small magnitude input sequence is used, and, thus, little identification is accomplished. In Figs. 6.13 - 6.14, the quadratic terms in x(k) and u(k) are plotted. The greater part of the open-loop costs comes from the initial state deviations and large controls at the beginning.

By increasing the measurement noise covariance in S1 from 1.0 to 4.0, the aggregate and average costs increased for all three control strategies. The system was made to act more open-loop, as little confidence is placed on the initial data. The average behavior of the identifier and the controller was not changed greatly since the systems are asymptotically stable. From simulations S1 and S2, we have a comparisons of the average costs. The results show that O.L.F.O. control incurred a little smaller cost than the enforced separation scheme.

Finally, in Figs. 6.15 - 6.19 we present a set of single run plots for an unstable system and in Figs. 6.20 - 6.24 plots for a typical stable system.

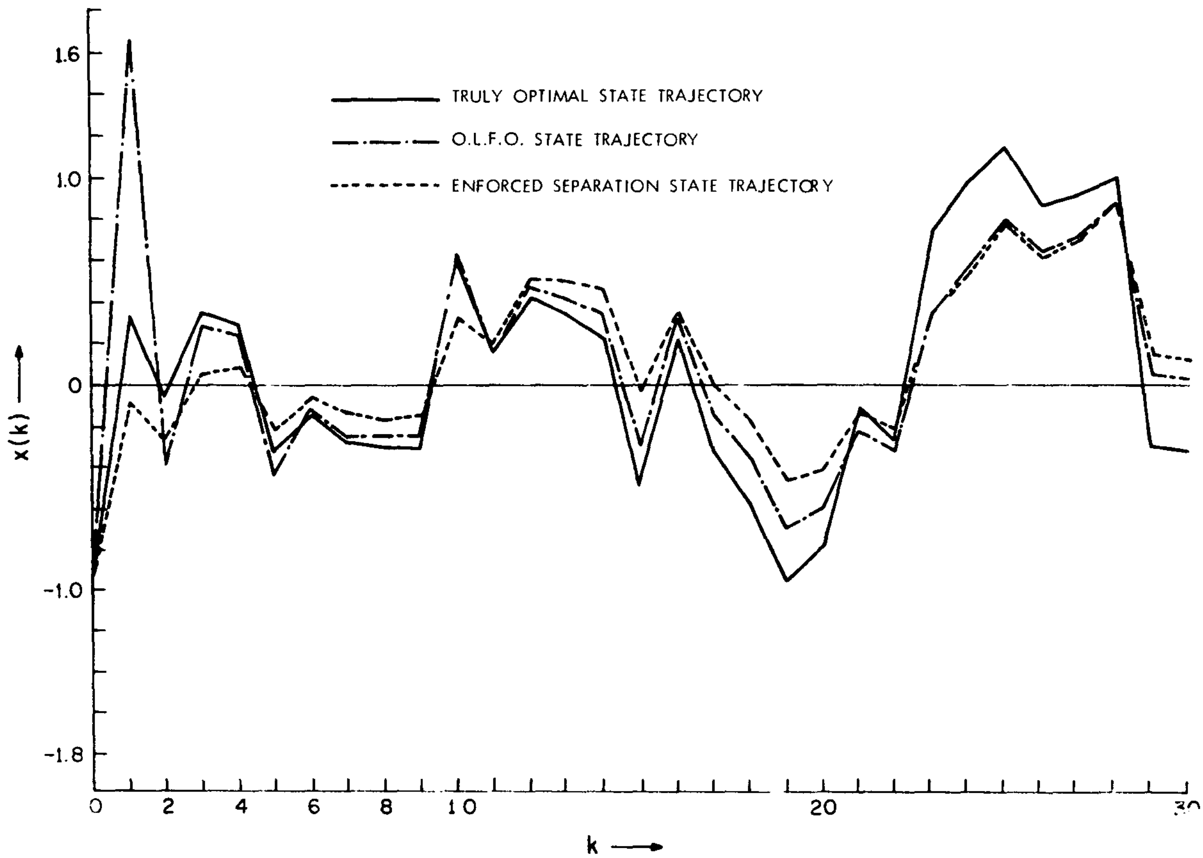

Fig. 6.15 Comparison of the typical response of the unstable system when the parameters a(k) and b(k) are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme)

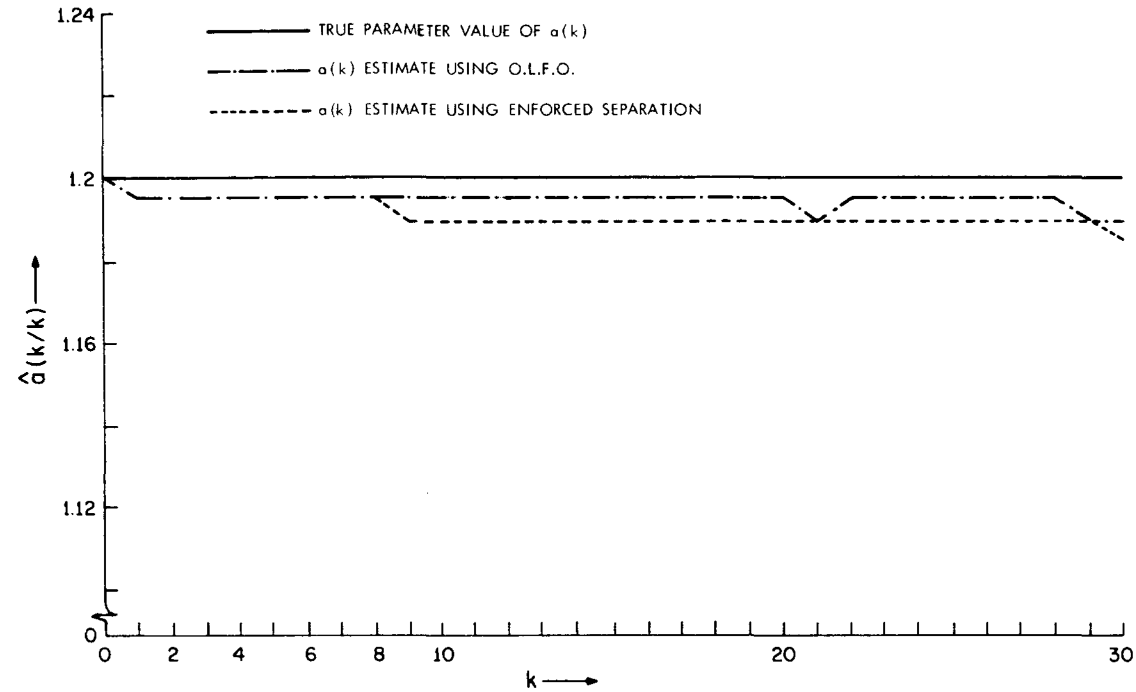

Fig. 6.16 Estimate of a(k) for an Unstable System

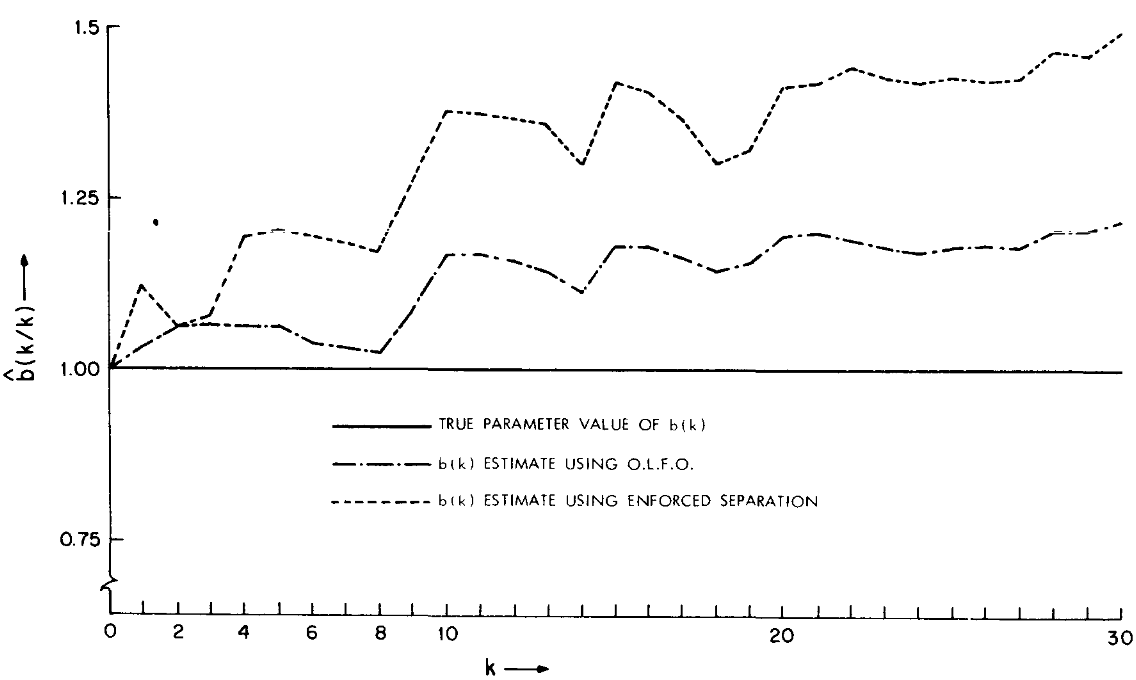

Fig. 6.17 Estimate of b(k) for an Unstable System

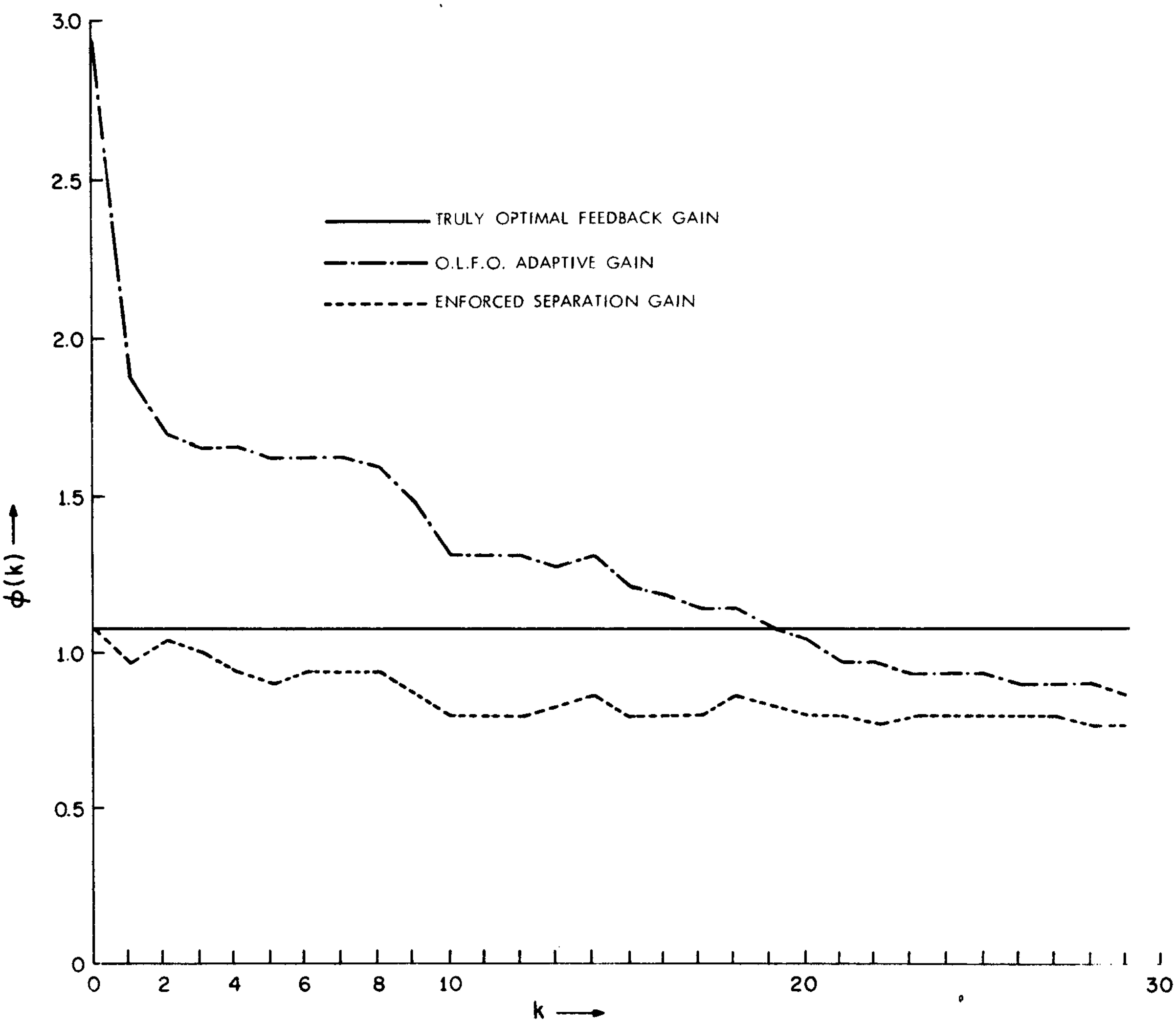

Fig. 6.18 Comparison between the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for an Unstable System

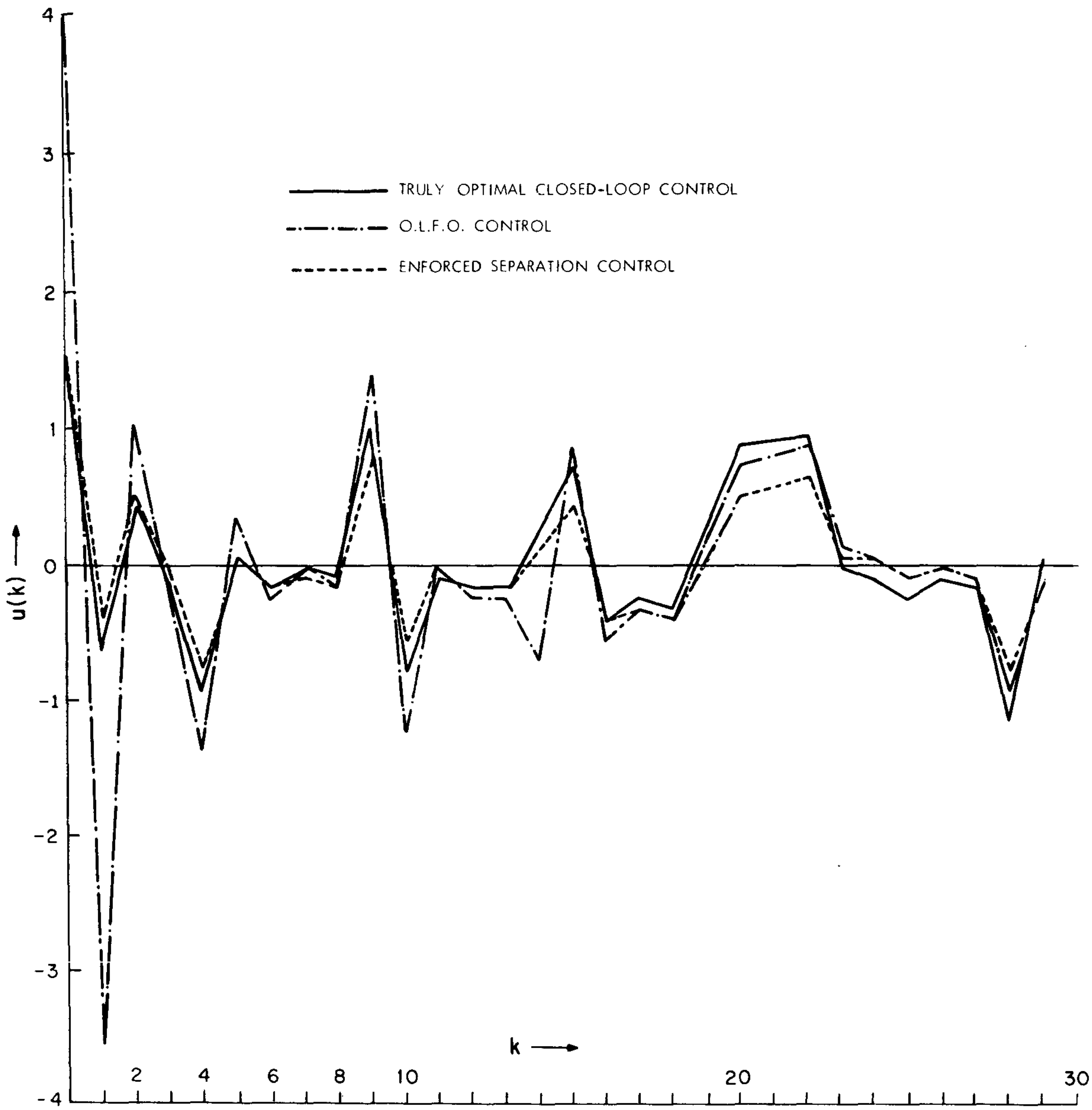

Fig. 6.19 Comparison between the Optimal Stochastic Control (When the parameters are known) and the Suboptimal Stochastic Controls (When the parameters are unknown) for an Unstable System

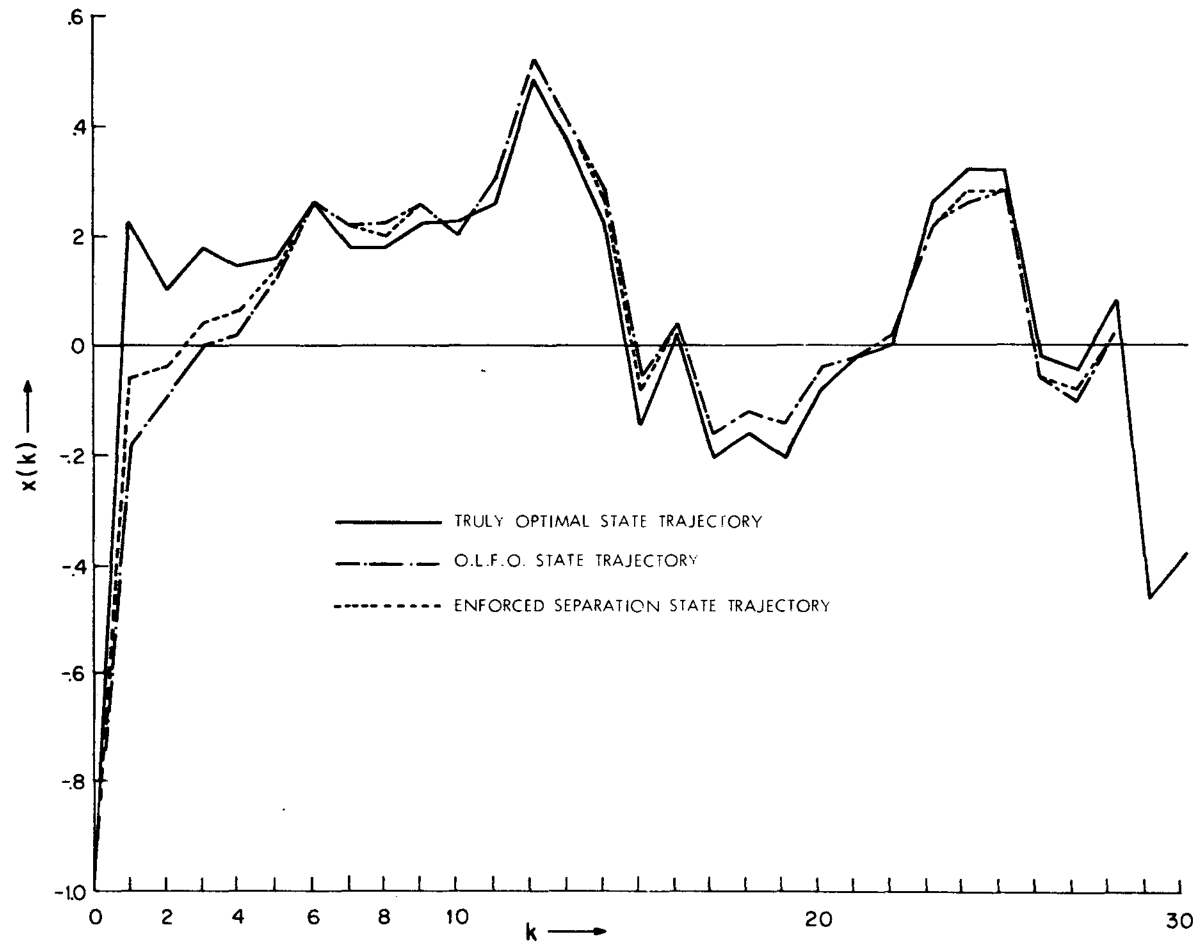

Fig. 6.20 Comparison of the typical response of the stable system when the parameters a(k) and b(k) are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme)

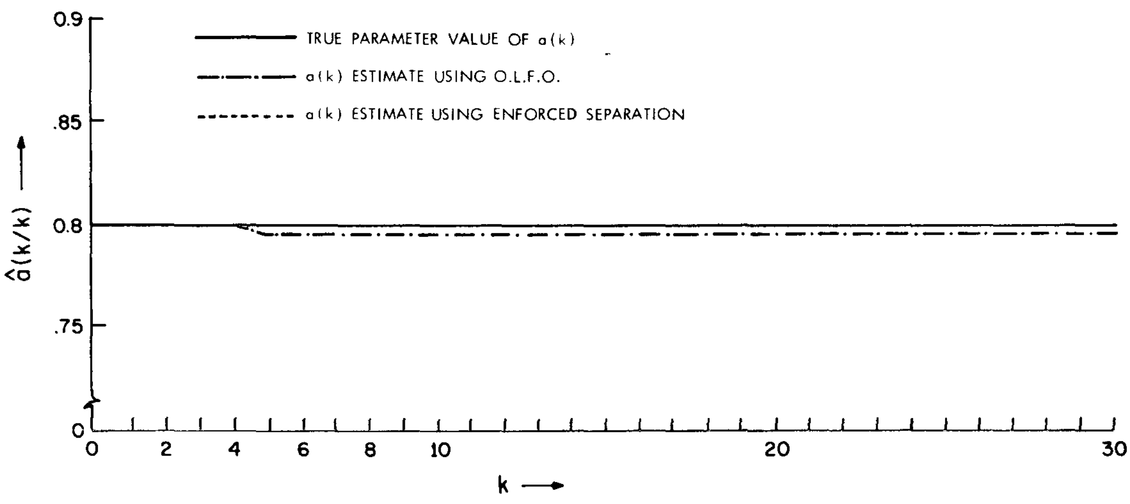

Fig. 6.21 Estimate of a(k) for a Stable System

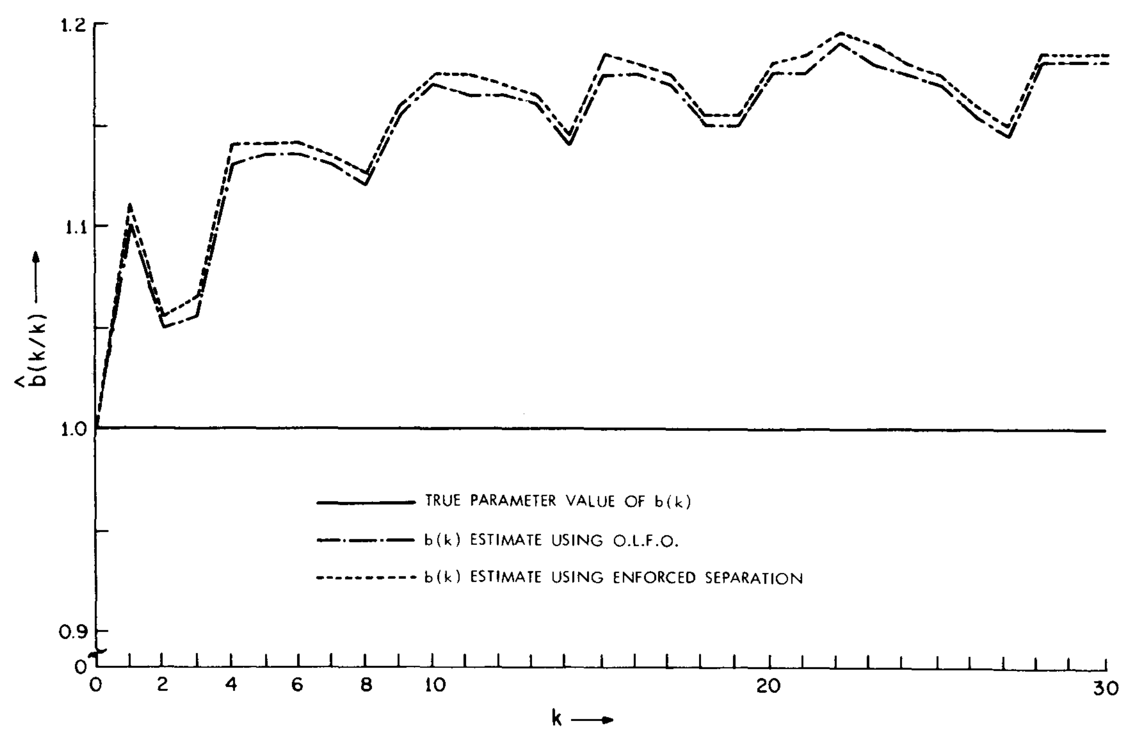

Fig. 6.22 Estimate of b(k) for a Stable System

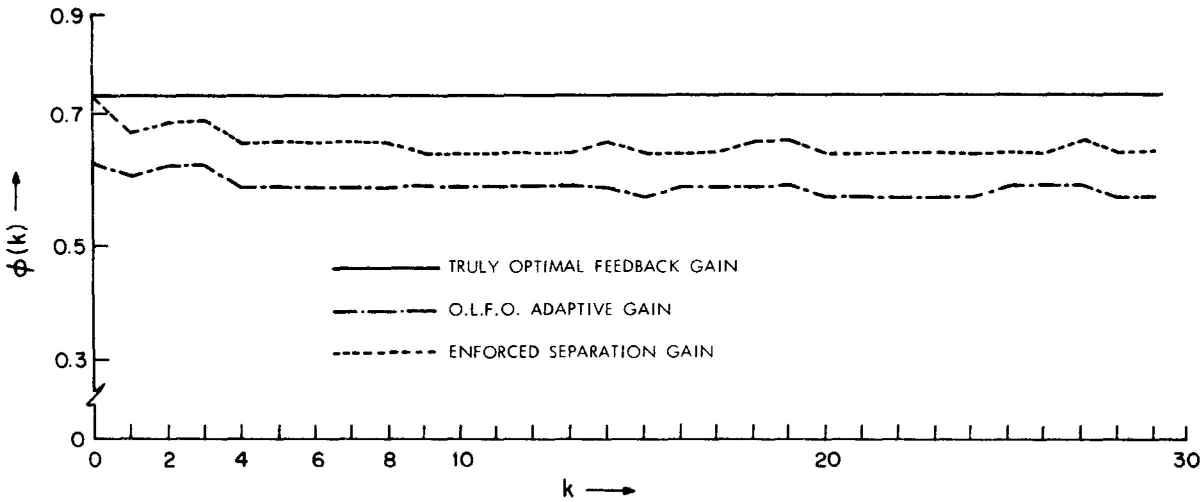

Fig. 6.23 Comparison between the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for a Stable System

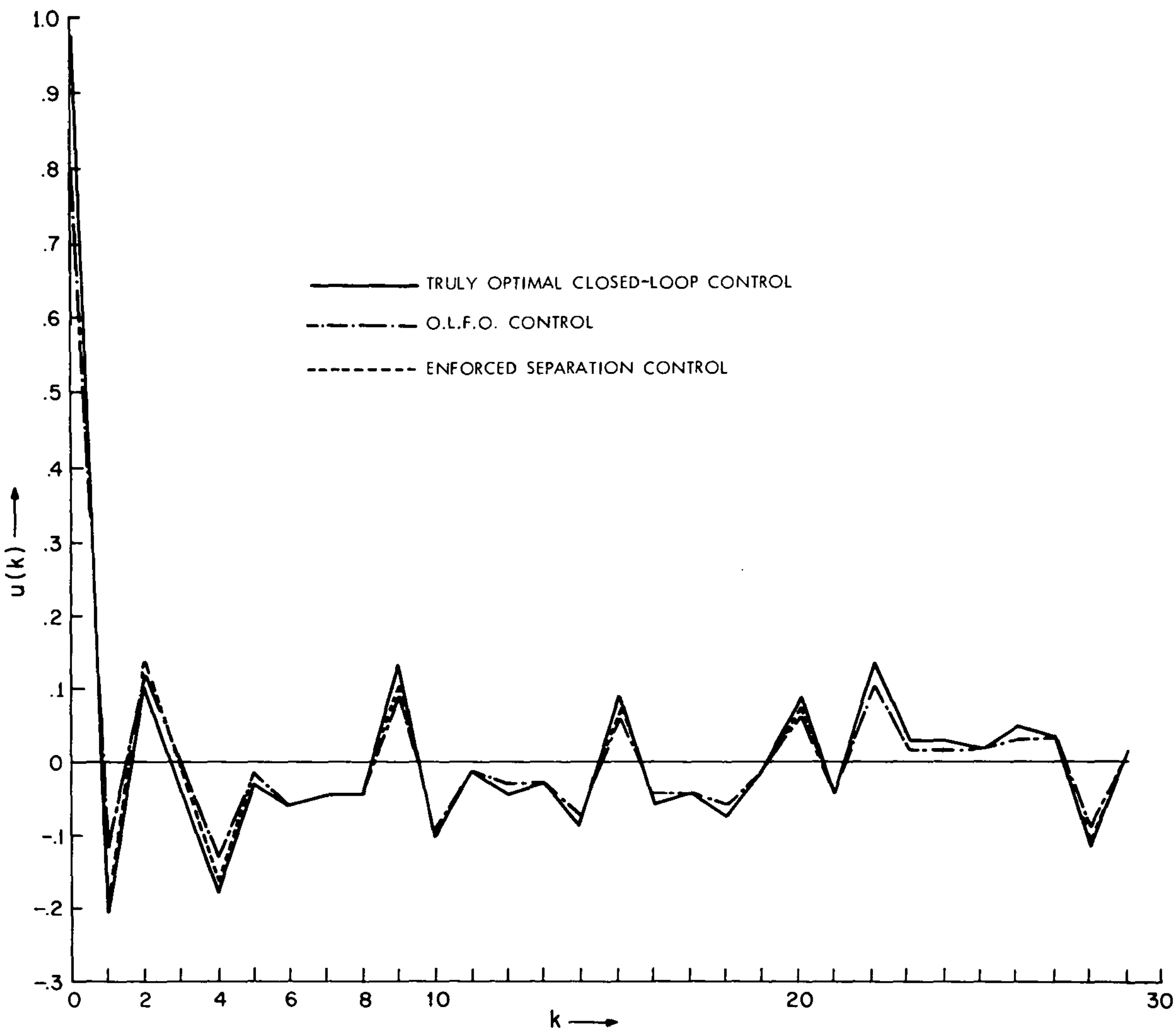

Fig. 6.24 Comparison between the Optimal Stochastic Control (When the parameters are known) and the Suboptimal Stochastic Controls (When the parameters are unknown) for a Stable System

CHAPTER 7

DISCUSSION

In Chapter 6 we described the digital computer simulation results in detail. In this Chapter we will interpret the results in light of the qualitative properties discussed in Chapters 4 and 5. From the simulations we found that nonzero control were applied at the beginning in both the stable and unstable systems using the open-loop feedback approach. The control input is used for identification and control purposes. This high-gain nature of the adaptive control is the response of the open-loop feedback approach to systems with more unknown parameters.

The rate of convergence seems to be dependent upon the stability of the system. The more unstable the system, the faster the estimates tend to the true parameter values. Further, the rate of convergence also depends on the initial guess $\underline{\hat{b}}(0|0)$ and $\underline{\hat{a}}(0|0)$, more true in the case of stable systems than the unstable systems. The initial guess on $\underline{b}(k)$ and $\underline{a}(k)$ hence affect the OLFO trajectory of the stable systems more than the unstable systems.

Large controls help in the identification of the unknown parameters. From Eqs. (2.3.15) and (2.3.31) - (2.3.33), we note that the larger the control $u^*(k)$, the faster $\underline{\Sigma}_{aa}(k|k)$ and $\underline{\Sigma}_{bb}(k|k)$ decrease. The estimates $\underline{\hat{a}}(k|k)$ and $\underline{\hat{b}}(k|k)$ of the parameter vectors $\underline{a}(k)$ and $\underline{b}(k)$ themselves will depend on the particular control law from Eqs. (2.3.8) - (2.3.14), since the recursive filter contains $u^*(k)$ as a parameter. This is verified by the simulation results. Conversely, the goodness of the estimates $\underline{\hat{x}}(k|k)$, $\underline{\hat{a}}(k|k)$, and $\underline{\hat{b}}(k|k)$ will affect the control law actually used.

For both the unstable and stable systems we remark that exact identification of $\underline{a}(k)$ and $\underline{b}(k)$ is not necessary from the control viewpoint. This was shown experimentally to be more so in the case of stable systems. Simulation results showed that the open-loop feedback optimal control systems can work well even if the parameter estimates are bad. The use of feedback also reduces the effect of parameter variations or the system's sensitivity to parameter inaccuracy. We recall that our objective functional rewards the system for good control performance, but not for good estimation of parameters.

Simulation results via Monte Carlo method compare the average cost incurred from combined identification and control, using, first, the open-loop feedback technique and, second, the enforced separation scheme. Since the sample size is arbitrarily chosen, we cannot make any precise conclusions. From statistical theory, we can say with probability 0.99 for sample size 20 independent of distributions that 7/10 of the values will fall in the range of the sample. The experimental results seem to indicate that in the stable systems, the open-loop feedback optimal method on the average incurred a smaller performance index than the ad-hoc scheme. For unstable systems, the results seem to indicate that the enforced separation scheme will incur smaller average cost in doing the job than the open-loop feedback optimal control design. This imbalance seems to originate in the large control magnitude that the open-loop feedback optimal technique uses to probe the parameters and stabilize the system in the beginning.

Finally, we shall discuss the computation feasibility of the OLFO control algorithm in real time. In the estimator Eqs. (2.3.8) - (2.3.16) at each time step k, k=0,1,...,N-1, we solve forward in time a 1-step 3n vector difference equation to generate the current state and parameter estimates, and a 1-step 3nx3n matrix difference equation to propagate the covariance matrices, which must be computed on-line since they depend on the measurements. Next we compute the parameters $\underline{\Phi}(j|k)$, $\underline{A}^{\dagger}(j|k)$, $\underline{V}(j|k)$, $\underline{d}(j|k)$, $\underline{\mathcal{B}}(j|k)$, and $\mathcal{Y}(j|k)$ which involve some 1-step computations, Eqs. (2.3.27) - (2.3.32) and solve forward in time two (N-k)-steps n-vector difference equations (2.3.18) - (2.3.19), three (N-k)-steps nxn matrix difference equations (2.3.23) - (2.3.25), and lastly both a (2n+1)nx(2n+1)n matrix difference equation (2.3.2) and a matrix difference equation (2.3.33) which will have to be computed on-line backward in (N-k) steps. The O.L.F.O. control is then obtained from Eqs. (2.3.1) and (2.3.4) using Eqs. (2.3.26) - (2.3.27). For the scalar system we have simulated and reported in Chapter 6, the actual computation of the O.L.F.O. control sequence for N=30 was about 0.06 second. The digital computer used was IBM 370/155. The total storage requirement corresponds to storing on-line $\hat{\underline{x}}(k|k)$, $\hat{\underline{a}}(k|k)$, $\hat{\underline{b}}(k|k)$ and $\underline{\Sigma}(k|k)$ which require a total of $(3n+(\frac{3n+1}{2})3n)$ memory locations, since $\underline{\Sigma}(k|k)$ is symmetric. We conclude that the O.L.F.O. control algorithm is recursive and computationally easy to implement.

The computational requirement of the enforced separation scheme will be less. Since we assume that at each time step, the parameter values are exact, there is no need to propagate the error covariance matrices associated with the unknown parameters. This translates into a saving both in memory and execution time for on-line adaptive control implementation.

CHAPTER 8

CONCLUSIONS

We have considered in detail both analytically and experimentally, the problem of controlling a discrete linear system S1, subject to stochastic disturbances, on the basis of noisy measurements. In addition, the system has a number of unknown parameters which may also vary in a stochastic manner. We assumed that the structure of the dynamical system is known. All of the underlying uncertain quantities are modeled as random variables with known statistics. It is assumed that the controller has perfect recall. The aim is to control the system such that the expected value of a quadratic cost functional of the state and control is minimized. Since the truly optimal closed-loop adaptive control solution given by Bellman's equation cannot be easily implemented because of the "curse of dimensionality", we sought to use suboptimal but simple and computationally feasible adaptive control algorithms.

The results of previous work on the control of a linear discrete-time stochastic system with unknown and stochastically varying control gains by Tse and Athans [16],[29] are extended to a larger class of problems where the poles are also unknown. We have investigated this class of problems in great detail, both from a theoretical and a simulation point of view. We adopted the open-loop feedback control philosophy, techniques, and theory in considering this broader class of problems. The O.L.F.O. approach led to a feedback controller design that has the desired adaptive properties. The analytical results showed that the gains of the O.L.F.O.

adaptive control system were "modulated" by the current and future uncertainty (error covariance matrices) of the parameter estimation. Therefore, the standard Separation Theorem did not hold in this class of problems.

The O.L.F.O. control system consists of an identifier and a feedback gain plus correction term controller. Since this adaptive control problem involves nonlinear estimation, the exact solution cannot be realized by a finite dimensional system. Thus, we used a nonlinear extended Kalman filter to generate the approximate conditional means and the associated error covariances of the state and unknown parameters based on noisy data. The identification technique has to be on-line. It was shown that the identifier can be designed apart from control purposes. We have, therefore, in the O.L.F.O. approach forced some kind of a one-way separation [7],[42], although the identification and control aspects of the problem are not necessarily independent. Any interaction between identification and control is explicitly exhibited in the form of constraints imposed by the identification technique on the optimal control problem.

From the equations we derived, qualitative properties of the overall O.L.F.O. control system were discussed. The derived theoretical results are in a form that can be easily programmed on a digital computer for on-line applications. Simulation studies on some constant first-order dynamic systems were made to obtain some quantitative measures on the convergence and the aggregate performance characteristics of the overall O.L.F.O. control system. It was found experimentally that the average behavior of the adaptive control system was drastically different for stable and unstable systems. For the unstable system, the O.L.F.O. control converged greater to the true stochastic optimal closed-loop system than the stable system. However, excellent control can result even if the identification algorithm did not identify the parameters accurately. In optimizing control systems, the goodness of identification is not rewarded.

In Chapter 4 we also developed a second approximation to the optimal closed-loop solution in the form of a cascade of the extended Kalman filter with a deterministic actuator. This arbitrary use of the Separation Theorem led to a feedback controller that performed on the average little inferior than the O.L.F.O. control for the stable systems, but was much superior than the O.L.F.O. control for the unstable systems. The performance comparison was based on the evaluation of the original cost functional provided by the Monte Carlo technique for lack of analytical tools. The O.L.F.O. adaptive control system was found to be somewhat high gain. The evaluation of the cost functional also provided some quantitative measures on the loss in performance due to identification and control when compared with the truly optimal control system. It must be stressed that simulation is inherently an imprecise technique and can only provide numerical data about the performance of the system, and is useful for lack of more satisfactory techniques of analysis.

In the following we discuss some possible extensions of the research related to the class of problems we considered.

(A) Different Cost Criteria

We have considered only the quadratic cost criteria. In our reformulation of the stochastic control problem into a deterministic control problem, the quadratic criteria is preserved, and we can thus derive explicit results. Theoretically, we can extend the O.L.F.O. approach to the more general case where the cost criteria is not necessarily quadratic, e.g., exponential. The identification equations will remain unchanged, but the open-loop control problem to be formulated will be different from Eqs. (3.2.1) - (3.2.6).

We note that in general the performance indexes used specified the cost of the system operation in terms of error and energy, but do not give us information about the transient-response characteristics of the system. Therefore, to assure satisfactory transient characteristics, we need secondary criteria relating to response characteristics in order to influence the choice of cost weighting elements.

(B) Vector Control

The O.L.F.O. approach can be directly extended to $\underline{u}(k) \in R^r$ and $\underline{B}(k)$ is a nxr input gain matrix. First, the identifier equations need to be derived which will generate the current estimate of the state and unknown matrices $\underline{A}(k)$ and $\underline{B}(k)$ and their associate error covariance matrices. An open-loop control problem can be formulated and solved as in Chapters 2 and 3. The results should be similar to those we derived, but the equations will be more complicated.

(C) Convergence Rate

We have not analyzed in detail the convergence rate of the sub-optimal O.L.F.O. control system to the optimal system. We have turned to simulation results to provide the basis for quantitative estimates about the convergence rate for stable and unstable systems. An error bound for the identifier and the controller would prove necessary to analyze and compare the expected value of the cost functional incurred by using the open-loop feedback optimal versus the enforced separation scheme.

(D) Control with Unknown Parameters and Imperfectly Known Distrubances

We have the matrices $\underline{A}(k)$ and $\underline{b}(k)$, $k = 0,1,\ldots$ partially known but satisfying some difference equations. The noise vectors $\underline{\xi}(k)$, $\underline{\theta}(k)$, $\underline{\delta}(k)$, and $\underline{\gamma}(k)$; $k = 0,1,\ldots$ are independent Gaussian vectors with unknown means and / or covariances. It is necessary then to perform adaptive filtering to recover the true means and covariances of the noise processes.

(E) Second-Order Filter

Since the identifier can be designed independently of the feed-back controller, we can use the second-order filter to generate the approximate conditional estimates and covariance matrices. The second-order filter will in general remove the bias error contained in the extended Kalman filter due to multiplicative effects of nonlinearities in the plant equation and in the level of the driving noise.

APPENDIX A

FORMULATION OF THE OPEN-LOOP CONTROL PROBLEM

In this appendix we shall derive the dynamic equations satisfied by $\hat{\underline{x}}(j|k)$ and $\underline{\Sigma}_{xx}(j|k)$. We will then have completed the formulation of a deterministic optimal control problem from the stochastic control problem with unknown parameters. We index the present time by k. Since all the noise sequences are assumed to be white and uncorrelated, then for $j \geq k$.

$$E\{\underline{\xi}(j)|Z_k\}=\underline{0},\ E\{\underline{\delta}(j)|Z_k\}=\underline{0},\ E\{\underline{\gamma}(j)|Z_k\}=\underline{0} \qquad (A.1)$$

The future control sequence is restricted to be deterministic and we assumed that there are no more observations, Eqs. (A.1) and (2.1.) imply that for $j \geq k$ prediction beyond time k:

$$\hat{\underline{x}}(j+1|k)=\hat{\underline{A}}(\hat{\underline{a}}(j|k))\hat{\underline{x}}(j|k)+\hat{\underline{b}}(j|k)u(j) \qquad (A.2)$$

where the parameters are constant

$$\hat{\underline{a}}(j+1|k) = \hat{\underline{a}}(j|k) \qquad (A.3)$$

$$\hat{\underline{b}}(j+1|k) = \hat{\underline{b}}(j|k) \qquad (A.4)$$

with the initial values defined at j=k, $\hat{\underline{x}}(k|k)$, $\hat{\underline{a}}(k|k)$, $\hat{\underline{b}}(k|k)$ which is to be obtained from the extended Kalman filter for the nonlinear augmented system S2. Since it is assumed that the control sequence $U^*(0,k-1)$ has been applied to the system, the parameters in Eq. (2.2.10) are known.

The error vectors for $j \geq k$ are given by

$$\underline{e}_x(j+1|k) = \underline{\hat{A}}(\underline{\hat{a}}(j|k))\underline{\hat{x}}(j|k) + \underline{\hat{b}}(j|k)u(j) - \underline{A}(j)\underline{x}(j) - \underline{b}(j)u(j) - \underline{\xi}(j) \qquad (A.5)$$

To the first order approximation we can write (A.5) using Eq. (2.2.1) - (2.2.7)

$$\underline{e}_x(j+1|k) = \underline{\hat{A}}(\underline{\hat{a}}(j|k))\underline{e}_a(j|k) + \underline{\hat{X}}(\underline{\hat{x}}(j|k))\underline{e}_x(j|k) + u(j)\underline{e}_b(j|k) - \underline{\xi}(j) \qquad (A.6)$$

where

$$\underline{e}_a(j+1|k) = \underline{e}_a(j|k) - \underline{\delta}(j) \qquad (A.7)$$

$$\underline{e}_b(j+1|k) = \underline{e}_b(j|k) - \underline{\gamma}(j) \qquad (A.8)$$

The initial error at time j = k depends, however, only on $\{\underline{\xi}(i), \underline{\delta}(i), \underline{\gamma}(i),\ i \leq k-1\}$ and $\{\underline{\theta}(i),\ i \leq k\}$, but not on $\{\underline{\xi}(i), \underline{\delta}(i), \underline{\gamma}(i),\ i \geq k\}$. Assuming all the noise sequences to be zero-mean, Gaussian, white, and uncorrelated with Z_{k-1} for $j \geq k$, the noise covariances are

$$E\{\underline{\xi}(j)\underline{\xi}'(j)|Z_k\} = \underline{\Xi}(j) \qquad (A.9)$$

$$E\{\underline{\delta}(j)\underline{\delta}'(j)|Z_k\} = \underline{\Delta}(j) \qquad (A.10)$$

$$E\{\underline{\gamma}(j)\underline{\gamma}'(j)|Z_k\} = \underline{\Gamma}(j) \qquad (A.11)$$

Since the initial error and the future noise sequence are independent, from Eq. (A.6)-A.11) and Eq. (2.2.8) we have for $j \geq k$

$$\underline{\Sigma}(j+1|k) = \underline{\hat{F}}(j|k)\underline{\Sigma}(j|k)\underline{\hat{F}}'(j|k) + \underline{\tilde{\Xi}}(j) \qquad (A.12)$$

where $\underline{\hat{F}}(j|k)$ and $\underline{\tilde{\Xi}}(j)$ are defined in Eqs. (3.2.5) - (3.2.6). The initial conditions are those given at j = k.

We should note that $\hat{\underline{x}}(k|k)$, $\hat{\underline{a}}(k|k)$ and $\hat{\underline{b}}(k|k)$ are the <u>approximate</u> conditional means of $\underline{x}(k)$, $\underline{a}(k)$, and $\underline{b}(k)$ respectively, while $\underline{\Sigma}(k|k)$ is the approximate conditional error covariance of the augmented state vector $\begin{bmatrix} \hat{\underline{x}}(k|k) \\ \hat{\underline{a}}(k|k) \\ \hat{\underline{b}}(k|k) \end{bmatrix}$. These approximate conditional estimates and error covariances are generated by the extended Kalman filter [17], [20] given the past control $U^*(0,k-1)$ has been chosen. The identification equations represent an optimum first-order estimator for the augmented states system S2, and are summarized in Eqs. (2.3.8)- (2.3.16).

Assuming that the state and parameter estimates and their associated error covariances are known along with the past control sequence, we can then formulate a entirely deterministic (open-loop) control problem at time k, k=0,1,..., N-1. We have then the deterministic dynamic system given by Eqs. (A.2)- (A.4) and (A.12).

APPENDIX B

FORTRAN SOURCE PROGRAM LISTING

The entire computer simulation program is included for the sake of completeness. It consists of the FORTRAN language routines (modules) — MAIN, PLOT1, GNOISE, RAND, PJ, OLFO, PARAM, ESTIM, GRAPH, and STAT. Each routine is documented with a descriptive preamble.

The compiler used is FORTRAN IV G LEVEL 20 of the M.I.T. Information Processing Center.

```
C*************************************************************************
C        ADAPTIVE CONTROL OF A SCALAR DISCRETE TIME-INVARIANT
C        LINEAR SYSTEM (A,B,C) WITH UNKNOWN DYNAMICS DRIVEN BY
C        ADDITIVE WHITE NOISE. THE MEASUREMENTS ARE NOISY.
C        PLANT NOISE IS XI AND HAS STATISTICS N(O,Q)
C        MEASUREMENT NOISE IS THETA AND HAS STATISTICS N(O,R)
C
C        ICODE= CODE FOR THE THREE CASES UNDER CONSIDERATION
C        1. SYSTEM WITH KNOWN DYNAMICS - SET SAA(0)=0, SBB(0)=0
C        BHAT=B,AHAT=A
C        2. SYSTEM WITH UNKNOWN DYNAMICS- O.L.F.O. APPROACH
C        3. SYSTEM WITH UNKNOWN DYNAMICS- 'SEPARATION PRINCIPLE'
C     FOR K .GT. 0
C        COMPUTE THE ESTIMATES-XHAT, AHAT,BHAT
C        COMPUTE THE ERROR COVARIANCE MATRICES
C             SXX,SXA,SXB,SAA,SAB,SBB
C        COMPUTE THE FORWARD DIFFERENCE EQUATIONS
C        COMPUTE THE PARAMETERS-- PHI,BT,RT,V
C        COMPUTE THE BACKWARD DIFFERENCE EQUATIONS
C
C        SUBROUTINES CALLED---
C        ESTIM, PARAM,OLFO
C
C*************************************************************************
C     MAIN PROGRAM
C     ALLOWS MONTE CARLO SIMULATION
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 KK
      INTEGER T,T1
      COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
      COMMON /WHK/ SXX,SXA,SXB,SAA,SAB,SBB
      COMMON /HMK/ SXXP
      COMMON /JPE/ KT,ICODE,MC
      COMMON /SWK/ QJ,RJ
C     FOR DIFFERENT FINAL TIME THEDIMENSION STMENT HAVE TO CHANGE
      DIMENSION CK(20,31)
```

```
      DIMENSION XK(20,31)
      DIMENSION COST(3)
      DIMENSION XIK(20,30), THEK(20,31)
C     T IS THE FINAL TIME
      READ (5,101) T
      T1=T+1
C     SET THE INDICATOR FOR NOISE DATA  - 1 YES, 2 NO
      NSYS=2
   95 CONTINUE
      READ (5,101) MC
C     MC IS THE NUMBER OF SAMPLE RUNS
C     F,QJ ARE THE STATE WEIGHTINGS,RJ IS THE CONTROL WEIGHTING
      READ (5,100) F,QJ,RJ
      READ (5,100) C
      READ (5,100) Q,R
   91 CONTINUE
      CALL PLOT1(T,T1)
   96 CONTINUE
C     PARAMETERS FOR SEPARATE CASE RUNS
      READ (5,101) ICODE
      IF (ICODE .EQ. T) GO TO 95
C     A PRIORI PROBABILITY DISTRIBUTIONS
      READ (5,100) X0,SXX0
      READ (5,100) A0,SAA0
      READ (5,100) B0,SBB0
  100 FORMAT (6F10.5)
  101 FORMAT (I10)
  104 FORMAT (' A=', F15.6)
  102 FORMAT (' B=', F15.6)
  103 FORMAT (' C=', F15.6)
  105 FORMAT (' Q=',F15.6  )
  106 FORMAT (' R=', F15.6)
  107 FORMAT (' X0=',F15.6)
  108 FORMAT (' A0=',F15.6)
  109 FORMAT (' B0=',F15.6)
  110 FORMAT  (' SXX0=',F15.6, ' SAA0=',F15.6, ' SBB0=',F15.6)
```

```
  111 FORMAT (' X(0)=', F15.6)                                                    MAIN0073
  112 FORMAT (' Q(N)=', F15.6, ' Q(J)=', F15.6,    ' R(J)=', F15.6)               MAIN0074
  113 FORMAT ('1',I10)                                                            MAIN0075
  114 FORMAT (' COST 1=',F15.6, 'COST 2=',F15.6,' COST 3=',F15.6)                 MAIN0076
  115 FORMAT (8F10.5)                                                             MAIN0077
  116 FORMAT (' EXPECTED COST =', F15.6)                                          MAIN0078
  117 FORMAT(' COST', I5,' =', F15.6)                                             MAIN0079
  118 FORMAT ('1')                                                                MAIN0080
  119 FORMAT (' INCREMENTAL COSTS')                                               MAIN0081
  120 FORMAT (10 F12.3)                                                           MAIN0082
  121 FORMAT (' ')                                                                MAIN0083
    5 CONTINUE                                                                    MAIN0084
      WRITE (6,113) ICODE                                                         MAIN0085
      WRITE (6,112) F,QJ,RJ                                                       MAIN0086
      WRITE (6,103) C                                                             MAIN0087
      WRITE (6,105) Q                                                             MAIN0088
      WRITE (6,106) R                                                             MAIN0089
      WRITE (6,107) X0                                                            MAIN0090
      WRITE (6,108) A0                                                            MAIN0091
      WRITE (6,109)  B0                                                           MAIN0092
      WRITE (6,110) SXX0, SAA0, SBB0                                              MAIN0093
      COST(ICODE)=0.                                                              MAIN0094
      COSTL=0.                                                                    MAIN0095
      SIGMAA=DSQRT(SAA0)                                                          MAIN0096
      SIGMAB=DSQRT(SBB0)                                                          MAIN0097
      SIGMAX=DSQRT(SXX0)                                                          MAIN0098
      SIGMAV=DSQRT(Q)                                                             MAIN0099
      SIGMAW=DSQRT(R)                                                             MAIN0100
C     SAME SAMPLE NOISE FOR EACH CASE                                             MAIN0101
      IXV1=100001                                                                 MAIN0102
      IXW1=10000001                                                               MAIN0103
      IA=100001                                                                   MAIN0104
      IB=1000001                                                                  MAIN0105
      IX=111111                                                                   MAIN0106
C     SET THE RUN INDICATOR                                                       MAIN0107
      KRUN=0                                                                      MAIN0108
```

```
   93 CONTINUE                                                         MAIN0109
C     RUN NEXT WITH DIFFERENT NOISE SEQUENCE                           MAIN0110
      IXV=IXV1                                                         MAIN0111
      IXW=IXW1                                                         MAIN0112
C     SYSTEM PARAMETERS                                                MAIN0113
      CALL GNOISE(IA,SIGMAA,A0,A)                                      MAIN0114
      CALL GNOISE(IB,SIGMAB,B0,B)                                      MAIN0115
      CALL GNOISE(IX,SIGMAX,X0,X0)                                     MAIN0116
C     X0 IS THE STARTING TRAJECTORY                                    MAIN0117
      WRITE (6,104) A                                                  MAIN0118
      WRITE (6,102) B                                                  MAIN0119
      WRITE (6,111) X0                                                 MAIN0120
C     SET THE TIME STEP INDICATOR                                      MAIN0121
      KT=0                                                             MAIN0122
C     COMPUTE Y(0)                                                     MAIN0123
      X=X0                                                             MAIN0124
      CALL GNOISE(IXW,SIGMAW,0.0,THETA)                                MAIN0125
      THEK(KRUN+1,KT+1)=THETA                                          MAIN0126
      Y=C*X+THETA                                                      MAIN0127
C     COMPUTE THE ESTIMATES AT (0/0)                                   MAIN0128
      GAINX=SXX0*C/(C*SXX0*C+R)                                        MAIN0129
      GAINA=0                                                          MAIN0130
      GAINB=0                                                          MAIN0131
       XHAT=X0+GAINX*(Y-C*X0)                                          MAIN0132
      AHAT=A0                                                          MAIN0133
      BHAT=B0                                                          MAIN0134
C     SET THE CROSS COVARIANCES TO ZERO                                MAIN0135
      SXX=SXX0-GAINX*C*SXX0                                            MAIN0136
      SXA=0.                                                           MAIN0137
      SXB=0.                                                           MAIN0138
      SAA=SAA0                                                         MAIN0139
      SAB=0.                                                           MAIN0140
      SBB=SBB0                                                         MAIN0141
   98 CONTINUE                                                         MAIN0142
      XK(KRUN+1,KT+1)=X                                                MAIN0143
      CK(KRUN+1,KT+1)=X*X*QJ/2.                                        MAIN0144
```

```
C     FORM THE OUTPUT PLOT MATRIX
      CALL PLOT2(X,Y)
C
C     ACTUAL COST COMPUTATION
      COST (ICODE)=COST(ICODE)+X*X*QJ/2.
C
      IF (KT .GE. T) GO TO 99
C     COMPUTE THE 'OPTIMUM' CONTROL GAIN
      CALL  PJ(KT,T,AHAT,QJ,PXX,F)
      NAMELIST/BUG2/PXX
      CALL OLFO(QJ,RJ,F,GAIN,UC,KT,T,ICODE,&94)
C     COMPUTE THE ADAPTIVE CONTROL
      U=GAIN*XHAT+UC
C     PLOT THE CONTROL
      CALL PLOT3(U,UC,GAIN)
C
      CK(KRUN+1,KT+1)=CK(KRUN+1,KT+1)+U*U*RJ/2.
      COST (ICODE)=COST(ICODE)+U*U*RJ/2.
C
      NAMELIST /BUG1/ KT, THETA,X, XI,Y,XHAT,AHAT,BHAT,SXX,SXA,SXB,SAA,
     * SAB,SBB,U ,A,B,X0
C
C     RECORD X(K+1)
      CALL GNOISE(IXV,SIGMAV,0.0,XI)
      XIK(KRUN+1,KT+1)=XI
      X=A*X+B*U+XI
      KT=KT+1
C     DATA Y(K+1)
      CALL GNOISE(IXW,SIGMAW,0.0,THETA)
      THEK(KRUN+1,KT+1)=THETA
      Y=C*X+THETA
C     COMPUTE THE ESTIMATES OF STATE AND PARAMETERS AT (K/K)
C     COMPUTE THE ERROR COVARIANCE MATRIX
      CALL ESTIM(U,Y,C,Q,R)
      GO TO 98
   99 CONTINUE
```

```
      KRUN=KRUN+1
C
      COST1=COST(ICODE)-COSTL
      WRITE (6,117) KRUN,COST1
      COSTL=COST(ICODE)
      IF (KRUN .GE. MC) GO TO 82
C
      IXV1=IXV
      IXW1=IXW
C
      GO TO 93
   82 CONTINUE
C
C     AVERAGING OVER THE SAMPLE RUNS
      COST(ICODE)=COST(ICODE)/MC
      WRITE(6,113)
      WRITE (6,119)
      DO 60 I=1,MC
      WRITE (6,121)
      WRITE (6,120)   (CK(I,J),J=1,T1)
   60 CONTINUE
      CALL PLOT5(XK,CK)
      IF (ICODE .NE. 3) GO TO 96
      WRITE (6,114) COST(1),COST(2),COST(3)
C     OUTPUTS ARE THE SET OF PLOTS
      CALL PLOT4(SXXO,SAAO,SBBO)
      IF (NSYS .GT. 1) GO TO 91
      CALL STAT(XIK, 30,MC)
      CALL STAT(THEK,31,MC)
      NSYS=2
C     SIMULATION WITH DIFFERENT INITIAL CONDITIONS OR ANOTHER SYSTEM
      GO TO 91
   94 WRITE (6,BUG1)
      WRITE (6,101) KRUN
C
      END
```

```
      SUBROUTINE PLOT1(T,T1)
C************************************************************
C         PLOTTING GRAPHS SUBPROGRAM
C         SUBROUTINE CALLED-
C             GRAPH(MODIFIED VERSION OF PRTPLT OF I.P.C.)
C************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
      COMMON /WHK/ SXX,SXA,SXB,SAA,SAB,SBB
      COMMON /JPE/ KT,ICODE,MC
      COMMON /SWK/ QJ,RJ
      INTEGER T,T1
      DIMENSION XS(31,4), US(30,4)
      DIMENSION XK(20,31), XV(31,4)
      DIMENSION XPLOT(31,4),APLOT(31,4),BPLOT(31,4)
      DIMENSION UPLOT(30,4)
      DIMENSION GPLOT(30,4),UCPLOT(30,4)
      DIMENSION CV(31,4),CK(20,31)
      DIMENSION CPLOT(31,4)
      DIMENSION XHPLOT(31,4),YPLOT(31,4)
      DIMENSION GXPLOT(101,4),GAPLOT(101,4),GBPLOT(101,4)
      DIMENSION SXPLOT(31,4),SAPLOT(31,4),SBPLOT(31,4)
      DIMENSION EXPLOT(31,4)
C     INITIALIZING THE MATRICES TO ZERO
      DO 1 I=1,T1
      DO 1 J=1,4
      XPLOT(I,J)=0
      APLOT(I,J)=0
      BPLOT(I,J)=0
      XS(I,J)=0.
      SXPLOT(I,J)=0.
      SAPLOT(I,J)=0.
      SBPLOT(I,J)=0.
      XHPLOT(I,J)=0.
      CPLOT(I,J)=0.
      EXPLOT(I,J)=0.
```

```
    1 CONTINUE                                                          PLOT0037
      DO 2 I=1,T                                                        PLOT0038
      DO 2 J=1,4                                                        PLOT0039
      UPLOT(I,J)=0                                                      PLOT0040
      UCPLOT(I,J)=0.                                                    PLOT0041
      GPLOT(I,J)=0.                                                     PLOT0042
      US(I,J)=0.                                                        PLOT0043
    2 CONTINUE                                                          PLOT0044
      RETURN                                                            PLOT0045
      ENTRY PLOT2(X,Y)                                                  PLOT0046
C     TO PLOT THE TRAJECTORY FOR THE THREE CASES                        PLOT0047
      XPLOT(KT+1,1)=KT                                                  PLOT0048
      XPLOT(KT+1,ICODE+1)=X    +XPLOT(KT+1,ICODE+1)                     PLOT0049
      XS(KT+1,1)=KT                                                     PLOT0050
      XS(KT+1,ICODE+1)=X*X+XS(KT+1,ICODE+1)                             PLOT0051
C     TO PLOT THE A ESTIMATE FOR THE THREE CASES                        PLOT0052
      APLOT(KT+1,1)=KT                                                  PLOT0053
      APLOT(KT+1,ICODE+1)=AHAT +APLOT(KT+1,ICODE+1)                     PLOT0054
C     TO PLOT THE B ESTIMATES                                           PLOT0055
      BPLOT(KT+1,1)=KT                                                  PLOT0056
      BPLOT(KT+1,ICODE+1)=BHAT +BPLOT(KT+1,ICODE+1)                     PLOT0057
      YPLOT(KT+1,1)=KT                                                  PLOT0058
      YPLOT(KT+1,ICODE+1)=Y                                             PLOT0059
      XHPLOT(KT+1,1)=KT                                                 PLOT0060
      XHPLOT(KT+1,ICODE+1)=XHAT+XHPLOT(KT+1,ICODE+1)                    PLOT0061
      SXPLOT(KT+1,1)=KT                                                 PLOT0062
      SAPLOT(KT+1,1)=KT                                                 PLOT0063
      SBPLOT(KT+1,1)=KT                                                 PLOT0064
      SXPLOT(KT+1,ICODE+1)=SXX +SXPLOT(KT+1,ICODE+1)                    PLOT0065
      SAPLOT(KT+1,ICODE+1)=SAA+SAPLOT(KT+1,ICODE+1)                     PLOT0066
      SBPLOT(KT+1,ICODE+1)=SBB+SBPLOT(KT+1,ICODE+1)                     PLOT0067
      GXPLOT(KT+1,1)=KT                                                 PLOT0068
      GAPLOT(KT+1,1)=KT                                                 PLOT0069
      GBPLOT(KT+1,1)=KT                                                 PLOT0070
      GXPLOT(KT+1,ICODE+1)=GAINX                                        PLOT0071
      GAPLOT(KT+1,ICODE+1)=GAINA                                        PLOT0072
```

```
      GBPLOT(KT+1,ICODE+1)=GAINB                                        PLOT0073
      CPLOT(KT+1,1)=KT                                                  PLOT0074
      CPLOT(KT+1,ICODE+1)=X*X*QJ/2.+CPLOT(KT+1,ICODE+1)                 PLOT0075
      EXPLOT(KT+1,1)=KT                                                 PLOT0076
      EXPLOT(KT+1,ICODE+1)=(XHAT-X)**2+EXPLOT(KT+1,ICODE+1)             PLOT0077
      RETURN                                                            PLOT0078
      ENTRY PLOT3(U,UC,GAIN)                                            PLOT0079
C     TO PLOT THE OPTIMAL FEEDBACK OR ADAPTIVE GAIN                     PLOT0080
      GPLOT(KT+1,1)=KT                                                  PLOT0081
      GPLOT(KT+1,ICODE+1)=-GAIN+GPLOT(KT+1,ICODE+1)                     PLOT0082
      UCPLOT(KT+1,1)=KT                                                 PLOT0083
      UCPLOT(KT+1,ICODE+1)=UC   +UCPLOT(KT+1,ICODE+1)                   PLOT0084
      UPLOT(KT+1,1)=KT                                                  PLOT0085
      UPLOT(KT+1,ICODE+1)=U    +UPLOT(KT+1,ICODE+1)                     PLOT0086
      US(KT+1,1)=KT                                                     PLOT0087
      US(KT+1,ICODE+1)=U*U+US(KT+1,ICODE+1)                             PLOT0088
      CPLOT(KT+1,ICODE+1)=U*U*RJ/2. +CPLOT(KT+1,ICODE+1)                PLOT0089
      RETURN                                                            PLOT0090
      ENTRY PLOT5(XK,CK)                                                PLOT0091
      J=ICODE+1                                                         PLOT0092
      DO 92 I=1,T1                                                      PLOT0093
      XPLOT(I,J)=XPLOT(I,J)/MC                                          PLOT0094
      APLOT(I,J)=APLOT(I,J)/MC                                          PLOT0095
      BPLOT(I,J)=BPLOT(I,J)/MC                                          PLOT0096
      XS(I,J)=XS(I,J)/MC                                                PLOT0097
      XHPLOT(I,J)=XHPLOT(I,J)/MC                                        PLOT0098
      SXPLOT(I,J)=SXPLOT(I,J)/MC                                        PLOT0099
      SAPLOT(I,J)=SAPLOT(I,J)/MC                                        PLOT0100
      SBPLOT(I,J)=SBPLOT(I,J)/MC                                        PLOT0101
      CPLOT(I,J)=CPLOT(I,J)/MC                                          PLOT0102
      EXPLOT(I,J)=EXPLOT(I,J)/MC                                        PLOT0103
   92 CONTINUE                                                          PLOT0104
      DO 80 K=1,T1                                                      PLOT0105
      CV(K,1)=K-1                                                       PLOT0106
      XV(K,1)=K-1                                                       PLOT0107
      CV(K,J)=0.                                                        PLOT0108
```

```
      XV(K,J)=0.0
      DO 81 I=1,MC
      CV(K,J)=CV(K,J)+(CK(I,K)-CPLOT(K,J))**2
      XV(K,J)=XV(K,J)+(XK(I,K)-XPLOT(K,J))**2
   81 CONTINUE
      CV(K,J)=CV(K,J)/MC
      XV(K,J)=XV(K,J)/MC
   80 CONTINUE
      DO 90 I=1,T
      GPLOT(I,J)=GPLOT(I,J)/MC
      UCPLOT(I ,J)=UCPLOT(I,J)/MC
      UPLOT(I,J)=UPLOT(I,J)/MC
      US(I,J)=US(I,J)/MC
   90 CONTINUE
      RETURN
      ENTRY PLOT4(SXXO,SAAO,SBBO)
      NL1=2*T+1
      NL2=2*T-1
      YMAX=1.
      YMIN=-1.
      CALL GRAPH(1,XPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
      XMAX=YMAX
      XMIN=YMIN
      YMAX=1.25
      YMIN=0.75
      CALL GRAPH(2,APLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
      YMAX=1.25
      YMIN=0.75
      CALL GRAPH(3,BPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
      YMAX=1.5
      YMIN=-.5
      CALL GRAPH(4,GPLOT,T,4,NL2,0,T,4,YMAX,YMIN)
      YMAX=0.45
      YMIN=-0.5
      CALL GRAPH(5,UPLOT,T,4,NL2,0,T,4,YMAX,YMIN)
      YMAX=.04
```

```
      YMIN=-.06                                                         PLOT0145
      CALL GRAPH(6,UCPLOT,T,4,NL2,0,T,4,YMAX,YMIN)                      PLOT0146
      YMAX=SXXO                                                         PLOT0147
      YMIN=0.0                                                          PLOT0148
      CALL GRAPH(7,XV,T1,4,NL1,0,T1,4,YMAX,YMIN)                        PLOT0149
      YMAX=SXXO+.0000001                                                PLOT0150
      YMIN=0.                                                           PLOT0151
      CALL GRAPH(8,SXPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                    PLOT0152
      YMAX=SAAO+.0000001                                                PLOT0153
      YMIN=0.                                                           PLOT0154
      CALL GRAPH(9,SAPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                    PLOT0155
      YMAX=SBBO+.0000001                                                PLOT0156
      YMIN=0.                                                           PLOT0157
      CALL GRAPH(10,SBPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                   PLOT0158
      YMAX=XMAX                                                         PLOT0159
      YMIN=XMIN                                                         PLOT0160
      CALL GRAPH (11,XHPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                  PLOT0161
      YMAX=SXXO                                                         PLOT0162
      YMIN=0.                                                           PLOT0163
      CALL GRAPH(12,EXPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                   PLOT0164
      YMAX=10.                                                          PLOT0165
      YMIN=0.                                                           PLOT0166
      CALL GRAPH(13,CV,T1,4,NL1,0,T1,4,YMAX,YMIN)                       PLOT0167
      YMAX=10.                                                          PLOT0168
      YMIN=0.                                                           PLOT0169
      CALL GRAPH(14,CPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)                    PLOT0170
      YMAX=5.                                                           PLOT0171
      YMIN=0.                                                           PLOT0172
      CALL GRAPH(20,XS,T1,4,NL1,0,T1,4,YMAX,YMIN)                       PLOT0173
      YMAX=5.                                                           PLOT0174
      YMIN=0.                                                           PLOT0175
      CALL GRAPH(21,US,T,4,NL2,0,T,4,YMAX,YMIN)                         PLOT0176
      RETURN                                                            PLOT0177
      END                                                               PLOT0178
```

C3

```
      SUBROUTINE GNOISE(IX,S,AM,V)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*4 Y
      A=0.0
      DO 50 I=1,12
      CALL RAND(IX,IY,Y)
      IX=IY
   50 A=A+Y
      V=(A-6.)*S+AM
      RETURN
      END
```

```
      SUBROUTINE RAND(IX,IY,YFL)                                        RAND0001
      IY=IX*65539                                                       RAND0002
      IF (IY) 5,6,6                                                     RAND0003
    5 IY=IY+2147483647+1                                                RAND0004
    6 YFL=IY                                                            RAND0005
      YFL=YFL*.4656613E-9                                               RAND0006
      RETURN                                                            RAND0007
      END                                                               RAND0008
```

```
      SUBROUTINE  PJ(KT,N,AHAT,QJ,PXX,F)                                PXXJ0001
C***********************************************************************PXXJ0002
C          SOLVING A BACKWARD DIFFERENCE EQUATION NAMED PXX             PXXJ0003
C        COMPUTES PXX(N/K) BACKWARDS TO PXX(J+1/K)                      PXXJ0004
C          F IS THE TERMINAL STATE WEIGHTING                            PXXJ0005
C          QJ IS THE STATE WEIGHTING MATRIX                             PXXJ0006
C          RJ IS THE CONTROL WEIGHTING                                  PXXJ0007
C***********************************************************************PXXJ0008
      IMPLICIT REAL*8 (A-H,O-Z)                                         PXXJ0009
      DIMENSION PXX(N)                                                  PXXJ0010
      PXX(N)=F/2.                                                       PXXJ0011
      J=N-1                                                             PXXJ0012
      IF (J .EQ. KT) GO TO 2                                            PXXJ0013
    1 CONTINUE                                                          PXXJ0014
      PXX(J)=AHAT*PXX(J+1)*AHAT+QJ/2.                                   PXXJ0015
      IF (J .EQ. (KT+1)) GO TO 2                                        PXXJ0016
      J=J-1                                                             PXXJ0017
      GO TO 1                                                           PXXJ0018
    2 CONTINUE                                                          PXXJ0019
      RETURN                                                            PXXJ0020
      END                                                               PXXJ0021
```

```
      SUBROUTINE OLFO(QJ,RJ,F,GAIN,UC,KT,N,ICODE,*)
C*****************************************************************
C         SOLVE THE MATRIX DIFFERENCE EQUATION FOR K BACKWARDS IN TIME
C         GAIN IS THE ADAPTIVE CONTROL GAIN
C         UC IS THE CONTROL CORRECTION TERM
C         KT IS THE PRESENT TIME INDEX
C         N IS THE FINAL TIME INDEX
C*****************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
      COMMON /WHK/ SXX,SXA,SXB,SAA,SAB,SBB
      REAL*8 K,KBBK,K1,K2
      DIMENSION D(3),BK(3),KBBK(3,3)
      DIMENSION BKP(3)
      DIMENSION K(3,3),K1(3,3),K2(3,3)
      DIMENSION BT(3),PHI(3,3),V(3,3)
C     K IS THE RICCATI MATRIX
C     THE MATRIX DIFFERENCE EQUATION CAN BE PRECOMPUTED IN CASE 1
      ZERO=0.
      JT=N
      DO 1 L=1,3
      DO 1 M=1,3
      K(L,M)=0
    1 CONTINUE
      K(1,1)=F
      IF (KT .EQ. (N-1)) GO TO 8
C     SOLVING A TIME-BACKWARD DIFFERENCE EQUATION
    9 CONTINUE
      JT=JT-1
      IF (ICODE .EQ. 3) GO TO 15
      CALL PARAM(QJ,RJ,  JT,BT,RT,PHI,V,D,SAA,SAB,SBB)
      GO TO 14
   15 CONTINUE
      CALL PARAM(QJ,RJ,  JT,BT,RT,PHI,V,D,ZERO,ZERO,ZERO)
   14 CONTINUE
      DO 2 M=1,3
```

```
      BK(M)=0.                                                          OLF00037
      DO 2 I=1,3                                                        OLF00038
      BK(M)=BT(I)*K(I,M)+3K(M)                                          OLF00039
    2 CONTINUE                                                          OLF00040
      BKB=RT                                                            OLF00041
      DO 3 M=1,3                                                        OLF00042
    3 BKB=BK(M)*BT(M)+BKB                                               OLF00043
C     CHECK THE INVERSE                                                 OLF00044
      IF (BKB) 16,16,13                                                 OLF00045
   13 CONTINUE                                                          OLF00046
      DO 4 L=1,3                                                        OLF00047
      DO 4 M=1,3                                                        OLF00048
      KBBK(L,M)=(BK(L )*BK(M))/BKB                                      OLF00049
    4 CONTINUE                                                          OLF00050
      DO 5 L=1,3                                                        OLF00051
      DO 5 M=1,3                                                        OLF00052
      K1(L,M)=K(L,M)-KBBK(L,M)                                          OLF00053
    5 CONTINUE                                                          OLF00054
      DO 6 L=1,3                                                        OLF00055
      DO 6 M=1,3                                                        OLF00056
      K2(L,M)=0                                                         OLF00057
       DO 6 I=1,3                                                       OLF00058
       K2(L,M)=PHI(I,L)*K1(I,M)+K2(L,M)                                 OLF00059
    6 CONTINUE                                                          OLF00060
      DO 7 L=1,3                                                        OLF00061
      DO 7 M=1,3                                                        OLF00062
      K(L,M)=V(L,M)                                                     OLF00063
      DO 7 I=1,3                                                        OLF00064
      K(L,M)=K2(L,I)*PHI(I,M)+K(L,M)                                    OLF00065
    7 CONTINUE                                                          OLF00066
      NAMELIST /BUG2/ PHI,JT,BT,RT,V,K,D,SXA,SXB,BK,BKB,KBBK,K1,K2,     OLF00067
     1 AHAT,BHAT,SAA,SAB,SBB                                            OLF00068
      IF (JT .GT. (KT+1)) GO TO 9                                       OLF00069
    8 CONTINUE                                                          OLF00070
C     OBTAINED K(K+1/K)                                                 OLF00071
      JT=JT-1                                                           OLF00072
```

```
      IF (ICODE .EQ. 3) GO TO 17
      CALL PARAM(QJ,RJ,  JT,BT,RT,PHI,V,D,SAA,SAB,SBB)
      GO TO 18
   17 CONTINUE
      CALL PARAM(QJ,RJ,  JT,BT,RT,PHI,V,D,ZERO,ZERO,ZERO)
C     DATA ARE RT(K/K),BT(K/K),PHI(K/K),SXB(K/K),SXA(K/K) , D(K+1)
   18 CONTINUE
      DO 10 M=1,3
      BK(M)=0
      DO 10 I=1,3
      BK(M)=BT(I)*K(I,M)+BK(M)
   10 CONTINUE
      BKB=RT
      DO 11 M=1,3
   11 BKB=BK(M)*BT(M)+BKB
      IF (BKB) 16,16,21
   21 CONTINUE
      DO 12 M=1,3
      BKP(M)=0
      DO 12 I=1,3
      BKP(M)=BKP(M)+BK(I)*PHI(I,M)
   12 CONTINUE
C     CONTROL GAIN
      GAIN =-BKP(1)/BKB-D(1)/RT
      IF (ICODE .NE. 2) GO TO 19
C     ADAPTIVE CONTROL CORRECTION TERM
      UC =-(BKP(2)/BK3+D(2)/RT)*SXB-(BKP(3)/BKB)*SXA
      GO TO 20
   19 CONTINUE
      UC=0.
   20 CONTINUE
      NAMELIST /BUG3/EKP
      RETURN
   16 WRITE (6,BUG2)
      RETURN 1
      END
```

```
      SUBROUTINE PARAM(QJ,RJ,   JT,BT,RT,PHI,V,D,SAA,SAB,SBB)
C**********************************************************************
C        COMPUTATION OF PARAMETERS AT (J/K).
C        FOR THE TIME-INVARIANT CASE THERE IS NO DYNAMICS IN THE
C        PREDICTOR EQUATIONS
C        DEFINITION OF BT,RT,PHI,V,D
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
      DIMENSION        PHI(3,3),V(3,3),W(3,3),AT(3,3),BT(3)   ,D(3)
C     FORWARD DIFFERENCE EQUATIONS
      SBBJ=SBB
      SAAJ=SAA
      SABJ=SAB
      AJ=AHAT
      BJ=BHAT
C     DEFINITION OF VARIABLES
      BT(1)=BJ
      BT(2)=SBBJ
      BT(3)=SABJ
      D(1)=2.*SABJ*PXX(JT+1)
      D(2)=2.*AJ*PXX(JT+1)
      D(3)=0.
      RT=RJ+SBBJ*PXX(JT+1)*2.
      AT(1,1)=AJ
      AT(1,2)=0
      AT(1,3)=0
      AT(2,1)=SABJ
      AT(2,2)=AJ
      AT(2,3)=0
      AT(3,1)=SAAJ
      AT(3,2)=0
      AT(3,3)=AJ
      DO 1 I=1,3
      DO 1 J=1,3
      PHI(I,J)=AT(I,J)-((BT(I)*D(J))/RT)
```

```
1 CONTINUE                                                          PARM0037
  W(1,1)=QJ+2.*SAAJ*PXX(JT+1)                                       PARM0038
  W(1,2)=0                                                          PARM0039
  W(1,3)= 2.*AJ*PXX(JT+1)                                           PARM0040
  W(2,1)=0                                                          PARM0041
  W(2,2)=0                                                          PARM0042
  W(2,3)=0                                                          PARM0043
  W(3,1)=2.*AJ*PXX(JT+1)                                            PARM0044
  W(3,2)=0.                                                         PARM0045
  W(3,3)=0                                                          PARM0046
  DO 2 I=1,3                                                        PARM0047
  DO 2 J=1,3                                                        PARM0048
  V(I,J)=W(I,J)-((D(I)*D(J))/RT)                                    PARM0049
2 CONTINUE                                                          PARM0050
  NAMELIST /BUG/ XHAT,AHAT,BHAT,SXX,SXA,SXB,W,AT                    PARM0051
  RETURN                                                            PARM0052
  END                                                               PARM0053
```

```
      SUBROUTINE ESTIM(U,Y,C,Q,R)
C**********************************************************************
C         IDENTIFICATION EQUATIONS FOR A LINEAR DISCRETE TIME SCALAR
C         SYSTEM WITH UNKNOWN PARAMETERS DRIVEN BY ADDITIVE WHITE NOISES
C         EXTENDED KALMAN FILTER ALGORITHM
C         PREDICTED ERROR COVARIANCES-- SXXP,SXAP,SXBP,SAAP,SABP,SBBP
C         UPDATED ESTIMATES-- XHAT,AHAT,BHAT
C         UPDATED ERROR COVARIANCES-- SXX,SXA,SXB,SAA,SAB,SBB
C         U IS THE KNOWN INPUT
C         Y IS THE NOISY MEASUREMENT
C         Q IS THE PLANT NOISE COVARIANCE MATRIX
C         R IS THE OBSERVATION NOISE COVARIANCE MATRIX
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
      COMMON /WHK/ SXX,SXA,SXB,SAA,SAB,SBB
      COMMON /HMK/ SXXP
      SXXP=AHAT*SXX*AHAT+2.*(AHAT*SXA*XHAT+AHAT*SXB*U+XHAT*SAB*U)+
     * XHAT*SAA*XHAT+U*U*SBB+Q
      SXAP=AHAT*SXA+XHAT*SAA+U*SAB
      SXBP=AHAT*SXB+XHAT*SAB+U*SBB
      SAAP=SAA
      SABP=SAB
      SBBP=SBB
C     COMPUTE THE GAIN VECTOR
      GAINX=SXXP*C/(C*SXXP*C+R)
      GAINA=SXAP*C/(C*SXXP*C+R)
      GAINB=SXBP*C/(C*SXXP*C+R)
C     COMPUTE THE RESIDUAL
      RES=Y-C*(AHAT*XHAT+BHAT*U)
C     COMPUTE THE CURRENT ESTIMATES
      XHAT=AHAT*XHAT+BHAT*U+GAINX*RES
      AHAT=AHAT+GAINA*RES
      BHAT=BHAT+GAINB*RES
C     COMPUTE THE ERROR COVARIANCE MATRIX
      SXX=SXXP-GAINX*C*SXXP
```

```
      SXA=SXAP-GAINX*C*SXAP                                             ESTI0037
      SXB=SXBP-GAINX*C*SXBP                                             ESTI0038
      SAA=SAAP-GAINA*C*SXAP                                             ESTI0039
      SAB=SABP-GAINA*C*SXBP                                             ESTI0040
      SBB=SBBP-GAINB*C*SXBP                                             ESTI0041
      NAMELIST /BUG/ GAINX,GAINA,GAINB,SXXP,SXAP,SXBP,SAAP,SABP,SBBP,RES ESTI0042
      RETURN                                                            ESTI0043
      END                                                               ESTI0044
```

```
      SUBROUTINE GRAPH (NO,B,N,M,NL,NS,KX,JX,YMAX,YMIN)                 GRAF0001
C                                                                       GRAF0002
C     ..................................................................GRAF0003
C                                                                       GRAF0004
C                                                                       GRAF0005
C        PURPOSE                                                        GRAF0006
C           PLOT SEVERAL CROSS-VARIABLES VERSUS A BASE VARIABLE         GRAF0007
C                                                                       GRAF0008
C        USAGE                                                          GRAF0009
C           CALL PRTPLT(NO,B,N,M,NL,NS,KX,JX)                           GRAF0010
C                                                                       GRAF0011
C        DESCRIPTION OF PARAMETERS                                      GRAF0012
C           NO - CHART NUMBER (3 DIGITS MAXIMUM)                        GRAF0013
C           B  - MATRIX OF DATA TO BE PLOTTED. FIRST COLUMN REPRESENTS  GRAF0014
C                BASE VARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSS-    GRAF0015
C                VARIABLES (MAXIMUM IS 9).                              GRAF0016
C           N  - NUMBER OF ROWS IN MATRIX B                             GRAF0017
C           M  - NUMBER OF COLUMNS IN MATRIX B (EQUAL TO THE TOTAL      GRAF0018
C                NUMBER OF VARIABLES). MAXIMUM IS 10.                   GRAF0019
C           NL - NUMBER OF LINES IN THE PLOT. IF 0 IS SPECIFIED, 50     GRAF0020
C                LINES ARE USED. THE NUMBER OF LINES MUST BE EQUAL TO   GRAF0021
C                OR GREATER THAN N                                      GRAF0022
C           NS - CODE FOR SORTING THE BASE VARIABLE DATA IN ASCENDING   GRAF0023
C                ORDER                                                  GRAF0024
C                  0  SORTING IS NOT NECESSARY (ALREADY IN ASCENDING    GRAF0025
C                     ORDER).                                           GRAF0026
C                  1  SORTING IS NECESSARY.                             GRAF0027
C                                                                       GRAF0028
C            KX- DIMENSION OF B MATRIX FROM DIMENSION STATEMENT.        GRAF0029
C                IT MUST BE OF THE FORM B(KX,JX)                        GRAF0030
C                                                                       GRAF0031
C            JX- DIMENSION OF B MATRIX FROM DIMENSION STATEMENT.        GRAF0032
C                IT MUST BE OF THE FORM B(KX,JX)                        GRAF0033
C                                                                       GRAF0034
C                                                                       GRAF0035
C     ..................................................................GRAF0036
```

RC

```
C                                                                       GRAF0037
      IMPLICIT REAL*8 (A-H,O-Z)                                         GRAF0038
      DIMENSION OUT(101),YPR(11),IANG(9),A(10000),B(KX,JX)              GRAF0039
      INTEGER IDUM /'1'/,IANG /'.','*','+','-','X','Y','#','O','&'/     GRAF0040
      INTEGER OUT                                                       GRAF0041
      DATA  BLANK /' '/                                                 GRAF0042
C                                                                       GRAF0043
  998 FORMAT (1H1,//)                                                   GRAF0044
  999 FORMAT (8E15.6)                                                   GRAF0045
      WRITE (6,998)                                                     GRAF0046
      WRITE (6,999) ((B(I,J),J=1,M),I=1,N)                              GRAF0047
C                                                                       GRAF0048
C     STORE COLUMNWISE                                                  GRAF0049
      I=1                                                               GRAF0050
      DO 39 J=1,M                                                       GRAF0051
      DO 39 K=1,N                                                       GRAF0052
      A(I)=B(K,J)                                                       GRAF0053
      I=I+1                                                             GRAF0054
39    CONTINUE                                                          GRAF0055
C                                                                       GRAF0056
    1 FORMAT(1H1,60X,7H CHART ,I3,//)                                   GRAF0057
    2 FORMAT(1H ,F11.4,5X,101A1)                                        GRAF0058
    3 FORMAT(1H )                                                       GRAF0059
  4   FORMAT (1H ,E15.8,1X,101A1)                                       GRAF0060
    7 FORMAT(1H ,16X,101HI         I         I         I         I      GRAF0061
     1    I         I         I         I         I         I)          GRAF0062
  8   FORMAT (1H ,9X,5(G12.4,8X),G12.4)                                 GRAF0063
  9   FORMAT(1H ,19X,4(G12.4,8X),G12.4)                                 GRAF0064
C                                                                       GRAF0065
C     ..................................................................GRAF0066
C                                                                       GRAF0067
      NLL=NL                                                            GRAF0068
C                                                                       GRAF0069
C                                                                       GRAF0070
C        SORT BASE VARIABLE DATA IN ASCENDING ORDER                     GRAF0071
C                                                                       GRAF0072
```

```
C
C        TEST NLL
C
C
C        PRINT TITLE
C
   20 WRITE(6,1)NO
C
C        DEVELOP BLANK AND DIGITS FOR PRINTING
C
C
C        FIND SCALE FOR BASE VARIABLE
C
      XSCAL=(A(N)-A(1))/(FLOAT(NLL-1))
C
C        FIND SCALE FOR CROSS-VARIABLES
C
C     SCALING DONE ON THE MAX OR MIN OF THE TWO
      M1=N+1
      YYMIN=A(M1)
      YYMAX=YYMIN
      M2=M*N
      DO 40 J=M1,M2
      IF (A(J) .GT. YYMAX) YYMAX=A(J)
      IF (A(J) .LT. YYMIN) YYMIN=A(J)
   40 CONTINUE
      YMAX=DMAX1(YYMAX,YMAX)
      YMIN=DMIN1(YYMIN,YMIN)
      YSCAL=(YMAX-YMIN)/100.0
C
C        FIND BASE VARIABLE PRINT POSITION
C
      XB=A(1)
      L=1
      MY=M-1
      DO 8C I=1,NLL
```

```
      F=I-1
      XPR=XB+F*XSCAL
      NSIZE = 2
      IF(XPR .GE. (-99999.).AND.XPR.LE.999999.) NSIZE = 1
      IF(A(L)-XPR-XSCAL*0.5) 50,50,70
C
C        FIND CROSS-VARIABLES
C
   50 DO 55 IX=1,101
   55 OUT(IX)=BLANK
   57 CONTINUE
      DO 60 J=1,MY
      JJ = MY + 1 - J
      LL = L + JJ*N
      JP=((A(LL)-YMIN)/YSCAL)+1.0
      OUT (JP) = IANG(JJ)
   60 CONTINUE
C
C        PRINT LINE AND CLEAR, OR SKIP
C
      IF(L.EQ.N)  GO TO 61
      L=L+1
      IF(A(L)-XPR-XSCAL*0.5)  57,57,61
 61   GO TO (62, 63), NSIZE
 62   WRITE(6,2) XPR, (OUT(IZ), IZ = 1, 101)
      GO TO 80
 63   WRITE(6,4) XPR, (OUT(IZ), IZ = 1, 101)
      GO TO 80
   70 WRITE(6,3)
   80 CONTINUE
C
C        PRINT CROSS-VARIABLES NUMBERS
C
      WRITE(6,7)
      YPR(1)=YMIN
      DO 90 KN=1,9
```

```
90 YPR(KN+1)=YPR(KN)+YSCAL*10.0
   YPR(11)=YMAX
   WRITE(6,8) (YPR (IP), IP=1,11,2)
   WRITE(6,9) (YPR(IP), IP=2, 10, 2)
   RETURN
   END
```

```
      SUBROUTINE STAT(X, NS,MC)
C**********************************************************************
C         THIS SUBROUTINE COMPUTES THE STATISTICS OF THE PSEUDO-RANDOM
C         VARIABLE USED IN THE MONTE CARLO SIMULATION.
C         MATRIX X CONTAINS MC SAMPLE RUNS OF THE RANDOM SEQUENCE OF
C         LENGTH NS.
C         COR IS THE CORRELATION MATRIX
C         COV IS THE COVARIANCE MATRIX
C         X MEAN IS THE SAMPLE MEAN VECTOR
C**********************************************************************
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION XMEAN(31),COR(31,31),COV(31,31)
      DIMENSION X(MC,NS)
  100 FORMAT (8F15.6)
  101 FORMAT (' AVERAGE VALUES OF THE NOISE AT TIME K')
  102 FORMAT (' CORRELATION MATRIX')
  103 FORMAT (' COVARIANCE MATRIX')
  104 FORMAT ('1')
  106 FORMAT ('0')
      DO 7 I=1,MC
      WRITE (6,106)
      WRITE (6,100) (X(I,J),J=1,NS)
    7 CONTINUE
      DO 1 J=1,NS
      XMEAN(J)=0.
      DO 1 I=1,MC
      XMEAN(J)=X(I,J)/MC+XMEAN(J)
    1 CONTINUE
      DO 2 I=1,NS
      DO 2 J=I,NS
      COR(I,J)=0.
      DO 2 N=1,MC
      COR(I,J)=COR(I,J)+X(N,I)*X(N,J)/MC
    2 CONTINUE
      DO 3 I=1,NS
      DO 3 J=I,NS
```

```
      COV(I,J)=COR(I,J)-XMEAN(I)* XMEAN(J)                              STAT0037
    3 CONTINUE                                                          STAT0038
      WRITE (6,104)                                                     STAT0039
      WRITE (6,101)                                                     STAT0040
      WRITE (6,100) (XMEAN(J),J=1,NS)                                   STAT0041
      WRITE (6,102)                                                     STAT0042
      DO 5 I=1,NS                                                       STAT0043
      WRITE (6,100) (COR(I,J), J=I,NS)                                  STAT0044
    5 CONTINUE                                                          STAT0045
      WRITE (6,104)                                                     STAT0046
      WRITE (6,103)                                                     STAT0047
      DO 6 I=1,NS                                                       STAT0048
      WRITE (6,100) (COV(I,J),J=I,NS)                                   STAT0049
    6 CONTINUE                                                          STAT0050
      RETURN                                                            STAT0051
      END                                                               STAT0052
```

APPENDIX C

MONTE CARLO SIMULATION TABLES

J_{OLFO} = performance index using the open-loop feedback optimal approach

J_{SEP} = performance index using the enforced separation scheme

$\bar{J}_{OLFO}$ = cumulative average of J_{OLFO}

$\bar{J}_{SEP}$ = cumulative average of J_{SEP}

J_{OPT} = performance index using the truly optimal stochastic control

Table C.1

Monte Carlo Simulation U1

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.212	0.725	-0.973	67.246	30.144	67.25	30.14	56.265
2	1.245	0.784	-1.426	119.567	93.316	93.4	61.73	73.966
3	1.197	1.762	-3.539	679.93	165.84	288.91	96.43	174.181
4	1.208	1.694	-0.581	425.476	161.465	323.05	112.69	100.537
5	1.271	0.724	-1.227	157.568	64.105	289.95	102.97	63.176
6	1.257	0.734	-0.839	146.208	107.446	265.99	103.72	80.342
7	1.213	0.886	1.680	86.562	64.747	240.36	98.15	58.763
8	1.221	1.561	0.348	131.404	80.022	226.74	95.89	67.055
9	1.243	1.708	0.540	404.466	262.9	246.49	114.44	215.156
10	1.169	1.106	-1.218	170.405	91.891	238.88	112.19	94.885
11	1.269	0.518	0.097	318.554	336.31	246.12	132.56	233.382
12	1.298	1.266	0.187	143.63	64.425	237.58	126.88	42.202
13	1.202	0.200	-1.762	969.215	292.344	293.86	139.611	129.966
14	1.185	0.085	0.516	631.017	381.924	318.37	157.35	127.891
15	1.118	0.883	-2.529	91.163	78.764	303.22	152.11	85.895
16	1.216	1.948	-2.583	349.022	180.342	306.09	153.87	181.578
17	1.143	0.626	-0.918	113.232	71.204	294.74	149.01	89.851
18	1.213	1.262	-0.024	158.854	113.322	287.20	147.03	94.709
19	1.201	1.610	-0.982	707.907	179.586	309.34	148.74	137.411
20	1.229	0.651	2.095	161.136	139.322	301.93	148.27	91.440
Average Cost				301.93	148.27			109.93

Table C.2

Monte Carlo Simulation U2

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.205	0.725	-0.973	69.074	29.864	69.07	29.86	56.265
2	1.219	0.784	-1.426	110.867	82.737	89.97	56.30	73.966
3	1.198	1.762	-3.539	737.160	167.274	305.70	93.29	174.181
4	1.203	1.694	-0.581	446.164	160.766	340.82	110.16	100.537
5	1.230	0.724	-1.227	163.426	58.810	305.34	99.89	63.176
6	1.224	0.734	-0.839	122.513	95.318	274.87	99.13	80.342
7	1.205	0.886	1.680	85.127	62.871	247.76	93.95	58.763
8	1.209	1.561	0.348	131.635	78.527	233.25	92.02	67.055
9	1.218	1.708	0.540	410.976	244.943	252.99	109.01	215.156
10	1.186	1.106	-1.218	178.816	96.080	245.58	107.72	94.885
11	1.229	0.518	0.097	300.744	314.394	250.59	126.51	233.382
12	1.242	1.266	0.187	129.774	57.956	240.52	120.80	42.202
13	1.201	0.200	-1.762	1000.099	291.394	298.95	133.92	129.966
14	1.193	0.085	0.516	602.795	425.495	320.65	154.74	127.891
15	1.165	0.883	-2.529	90.647	84.280	305.32	150.05	85.895
16	1.207	1.948	-2.583	369.494	177.491	309.33	151.76	181.578
17	1.175	0.626	-0.918	122.203	81.956	298.32	147.66	89.851
18	1.205	1.262	-0.024	154.300	110.445	290.32	145.59	94.709
19	1.200	1.610	-0.982	758.806	177.905	314.98	147.29	137.411
20	1.212	0.651	2.095	161.938	131.846	307.33	146.52	91.440
Average Cost				307.33	146.52			109.93

Table C.3

Monte Carlo Simulation U3

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.205	0.725	-0.973	20.100	7.293	20.10	7.29	12.498
2	1.219	0.784	-1.426	28.144	18.484	24.12	12.89	16.006
3	1.198	1.762	-3.539	135.127	35.473	61.12	20.42	41.542
4	1.203	1.694	-0.581	89.465	32.027	68.21	23.32	23.062
5	1.230	0.724	-1.227	48.283	15.784	64.22	21.81	14.226
6	1.224	0.734	-0.839	31.644	22.475	58.79	21.92	17.867
7	1.205	0.886	1.680	20.748	14.637	53.36	20.88	13.202
8	1.209	1.561	0.348	27.407	16.845	50.11	20.38	15.135
9	1.218	1.708	0.540	82.444	51.901	53.71	23.88	48.439
10	1.186	1.106	-1.218	40.858	20.887	52.42	23.58	20.826
11	1.229	0.518	0.097	91.201	85.950	55.95	29.25	52.800
12	1.242	1.266	0.187	29.973	12.212	53.78	27.83	9.786
13	1.201	0.200	-1.762	301.462	117.551	72.84	34.73	30.360
14	1.193	0.085	0.516	395.297	291.287	95.87	53.06	29.339
15	1.165	0.883	-2.529	24.691	21.561	91.12	50.96	21.803
16	1.207	1.948	-2.583	65.716	36.872	89.54	50.08	42.741
17	1.175	0.626	-0.918	30.312	20.746	86.05	48.35	20.025
18	1.205	1.262	-0.024	34.765	24.142	83.20	47.01	21.262
19	1.200	1.610	-0.982	147.008	35.524	86.56	46.40	31.045
20	1.212	0.651	2.095	51.519	37.683	84.81	45.97	22.471
Average Cost				84.81	45.97			25.22

Table C.4

Monte Carlo Simulation U4

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.212	0.725	-0.973	19.693	7.291	19.69	7.29	12.498
2	1.245	0.784	-1.426	30.930	21.107	25.31	14.20	16.006
3	1.197	1.762	-3.539	125.604	35.155	58.74	21.18	41.542
4	1.208	1.694	-0.581	86.506	32.060	65.68	23.90	23.062
5	1.271	0.724	-1.227	46.881	18.671	61.92	22.86	14.226
6	1.257	0.734	-0.839	38.813	25.801	58.07	23.35	17.867
7	1.213	0.886	1.680	21.198	15.116	52.80	22.17	13.202
8	1.221	1.561	0.348	27.228	17.183	49.60	21.55	15.135
9	1.243	1.708	0.540	81.925	56.215	53.19	25.40	48.439
10	1.169	1.106	-1.218	39.303	19.932	51.81	24.85	20.826
11	1.269	0.518	0.097	99.595	97.435	56.15	31.45	52.800
12	1.298	1.266	0.187	34.086	13.500	54.31	29.96	9.786
13	1.202	0.200	-1.762	300.745	118.467	73.27	36.76	30.360
14	1.185	0.085	0.516	410.689	265.029	97.37	53.07	29.339
15	1.118	0.883	-2.529	25.196	19.814	92.56	50.85	21.803
16	1.216	1.948	-2.583	62.872	37.544	90.70	50.02	42.741
17	1.143	0.626	-0.918	28.050	17.574	87.02	48.11	20.025
18	1.213	1.262	-0.024	35.998	24.857	84.18	46.82	21.262
19	1.201	1.610	-0.982	138.125	35.919	87.02	46.25	31.045
20	1.229	0.651	2.095	51.669	40.260	85.26	45.95	22.471
Average Cost				85.26	45.95			25.22

Table C.5

Monte Carlo Simulation U5

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.205	0.725	-0.973	111.715	80.726	111.72	80.72	172.362
2	1.219	0.784	-1.426	410.635	352.109	261.18	216.41	285.854
3	1.198	1.762	-3.539	706.219	479.799	409.52	304.21	515.761
4	1.203	1.694	-0.581	796.118	412.957	506.17	331.40	256.066
5	1.230	0.724	-1.227	214.843	134.544	447.91	292.03	132.889
6	1.224	0.734	-0.839	386.692	233.277	437.70	282.23	227.986
7	1.205	0.886	1.680	220.483	188.011	406.67	268.77	169.176
8	1.209	1.561	0.348	392.105	229.877	404.85	263.91	183.818
9	1.218	1.708	0.540	962.742	828.595	466.84	326.65	691.488
10	1.186	1.106	-1.218	413.583	336.777	461.51	327.69	279.181
11	1.229	0.518	0.097	1130.142	1061.054	522.30	394.36	793.478
12	1.242	1.266	0.187	396.185	178.950	511.79	376.41	137.766
13	1.201	0.200	-1.762	3269.505	877.106	723.92	414.92	430.654
14	1.193	0.085	0.516	3508.175	820.996	922.80	443.93	393.552
15	1.165	0.883	-2.529	223.028	183.496	876.14	426.56	204.595
16	1.207	1.948	-2.583	808.132	608.232	871.90	437.92	564.735
17	1.175	0.626	-0.918	332.343	247.908	840.16	426.74	292.189
18	1.205	1.262	-0.024	537.145	429.057	823.32	426.87	372.793
19	1.200	1.610	-0.982	901.496	390.079	827.44	424.93	335.091
20	1.212	0.651	2.095	328.367	341.459	802.48	420.75	231.623
Average Cost				802.48	420.75			333.55

Table C.6

Monte Carlo Simulation U6

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.205	0.725	-0.973	263.893	270.026	263.89	270.03	373.682
2	1.219	0.784	-1.426	943.310	790.982	603.60	530.50	655.758
3	1.198	1.762	-3.539	1927.855	1037.637	1045.02	699.55	1093.491
4	1.203	1.694	-0.581	1450.612	690.182	1146.42	697.21	457.420
5	1.230	0.724	-1.227	336.973	329.998	984.53	623.77	253.131
6	1.224	0.734	-0.839	884.016	446.696	967.78	594.25	466.717
7	1.205	0.886	1.680	431.998	380.801	891.24	563.76	339.075
8	1.209	1.561	0.348	764.472	457.306	875.40	550.45	360.579
9	1.218	1.708	0.540	1955.645	1728.216	995.42	681.32	1412.076
10	1.186	1.106	-1.218	1006.145	688.350	996.49	682.02	538.284
11	1.229	0.518	0.097	2551.711	2252.933	1137.86	824.83	1651.491
12	1.242	1.266	0.187	800.986	387.971	1109.80	788.42	298.200
13	1.201	0.200	-1.762	2602.600	2112.281	1224.63	890.26	920.533
14	1.193	0.085	0.516	11046.683	1257.690	1926.21	916.51	775.195
15	1.165	0.883	-2.529	403.048	331.811	1824.66	877.53	388.504
16	1.207	1.948	-2.583	1795.522	1281.303	1822.84	902.76	1087.797
17	1.175	0.626	-0.918	641.414	494.509	1753.35	818.75	591.436
18	1.205	1.262	-0.024	1111.463	956.722	1717.69	883.08	826.138
19	1.200	1.610	-0.982	1004.562	654.970	1680.15	871.07	560.563
20	1.212	0.651	2.095	705.120	696.304	1631.40	862.33	472.159
Average Cost				1631.40	862.33			676.11

Table C.7

Monte Carlo Simulation U7

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.212	1.0	-0.973	69.045	53.939	69.05	53.94	56.265
2	1.245	1.0	-1.426	93.451	84.331	81.25	69.14	73.966
3	1.197	1.0	-3.539	177.063	174.022	113.19	104.10	174.181
4	1.208	1.0	-0.581	132.679	100.506	118.06	103.20	100.537
5	1.271	1.0	-1.227	89.521	64.201	112.35	95.40	63.176
6	1.257	1.0	-0.839	101.449	90.722	110.53	94.62	80.342
7	1.213	1.0	1.680	64.36	60.679	103.94	89.77	58.763
8	1.221	1.0	0.348	74.417	69.132	100.25	87.19	67.055
9	1.243	1.0	0.540	240.460	264.459	115.83	106.89	215.156
10	1.169	1.0	-1.218	111.012	89.680	115.35	105.17	94.885
11	1.269	1.0	0.097	326.004	350.993	134.50	127.51	233.382
12	1.298	1.0	0.187	63.648	48.082	128.59	120.90	42.202
13	1.202	1.0	-1.762	136.168	131.324	129.18	121.70	129.966
14	1.185	1.0	0.516	124.166	125.052	128.82	121.94	127.891
15	1.118	1.0	-2.529	73.401	74.045	125.12	118.75	85.895
16	1.216	1.0	-2.583	173.536	187.504	128.15	123.04	181.578
17	1.143	1.0	-0.918	91.204	77.940	125.98	120.40	89.851
18	1.213	1.0	-0.024	110.348	99.460	125.11	119.23	94.709
19	1.201	1.0	-0.982	159.215	141.599	126.90	120.46	137.411
20	1.229	1.0	2.095	109.958	99.740	126.05	119.37	91.440
Average Cost				126.05	119.37			109.93

Table C.8

Monte Carlo Simulation U8

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	1.2	0.725	-0.973	26.682	29.625	26.68	29.63	56.265
2	1.2	0.784	-1.426	61.401	75.964	44.04	52.79	73.966
3	1.2	1.762	-3.539	161.962	167.790	83.35	91.13	174.181
4	1.2	1.694	-0.581	132.130	160.043	95.54	108.36	100.537
5	1.2	0.724	-1.227	59.371	56.366	88.31	97.96	63.176
6	1.2	0.734	-0.839	84.159	87.822	87.62	96.27	80.342
7	1.2	0.886	1.680	65.571	61.573	84.47	91.31	58.763
8	1.2	1.561	0.348	71.640	77.287	82.86	89.56	67.055
9	1.2	1.708	0.540	219.980	232.843	98.10	105.48	215.156
10	1.2	1.106	-1.218	85.611	99.233	96.85	104.85	94.885
11	1.2	0.518	0.097	306.562	269.154	115.92	119.79	233.382
12	1.2	1.266	0.187	42.051	53.972	109.76	114.31	42.202
13	1.2	0.200	-1.762	398.525	289.713	131.97	127.80	129.966
14	1.2	0.085	0.516	627.885	454.03	167.40	151.10	127.891
15	1.2	0.883	-2.529	99.838	89.136	162.89	146.97	85.895
16	1.2	1.948	-2.583	172.321	175.380	163.48	148.75	181.578
17	1.2	0.626	-0.918	89.551	91.454	159.13	145.38	89.851
18	1.2	1.262	-0.024	98.902	108.328	155.79	143.32	94.709
19	1.2	1.610	-0.982	154.027	177.376	155.69	145.11	137.411
20	1.2	0.651	2.095	143.634	126.561	155.09	144.18	91.440
Average Cost				155.09	144.18			109.93

Table C.9

Monte Carlo Simulation S1

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	0.812	0.725	-0.973	13.123	13.219	13.12	13.21	13.628
2	0.845	0.784	-1.426	16.966	15.972	15.04	14.6	16.094
3	0.797	1.762	-3.539	71.876	72.995	33.98	34.06	78.966
4	0.808	1.694	-0.581	39.543	48.787	35.37	37.74	27.221
5	0.871	0.724	-1.227	33.445	32.313	34.99	36.66	28.036
6	0.857	0.734	-0.839	36.947	35.587	35.31	36.48	29.167
7	0.813	0.886	1.680	24.131	23.528	33.71	34.63	23.089
8	0.821	1.561	0.348	27.717	28.361	32.96	33.85	26.267
9	0.843	1.708	0.540	36.689	40.805	33.38	34.62	27.338
10	0.769	1.106	-1.218	18.126	19.586	31.85	33.12	19.135
11	0.869	0.518	0.097	38.811	40.311	32.48	33.77	35.303
12	0.898	1.266	0.187	23.014	22.20	31.70	32.81	14.290
13	0.802	0.200	-1.762	59.689	52.1	33.85	34.29	30.304
14	0.785	0.085	0.516	19.556	19.658	32.83	33.24	27.937
15	0.718	0.883	-2.529	49.977	46.426	33.97	34.12	52.109
16	0.816	1.948	-2.583	54.568	57.771	35.26	35.60	55.494
17	0.743	0.626	-0.918	13.333	12.92	33.97	34.27	13.263
18	0.813	1.262	-0.024	10.362	11.535	32.66	33.00	9.901
19	0.801	1.610	-0.982	50.230	61.218	33.58	34.48	39.835
20	0.829	0.651	2.095	61.816	57.706	34.99	35.65	43.637
Average Cost				34.99	35.65			30.55

Table C.10

Monte Carlo Simulation S2

Run No.	a(0)	b(0)	x(0)	J_{OLFO}	J_{SEP}	$\bar{J}_{OLFO}$	$\bar{J}_{SEP}$	J_{OPT}
1	0.812	0.725	-0.973	13.373	13.258	13.37	13.26	13.028
2	0.845	0.784	-1.426	20.973	18.924	17.17	16.09	16.971
3	0.797	1.762	-3.539	87.171	81.426	40.51	37.87	102.918
4	0.808	1.694	-0.581	52.253	70.731	43.44	46.08	32.456
5	0.871	0.724	-1.227	37.966	36.244	42.35	44.12	29.187
6	0.857	0.734	-0.839	46.384	46.084	43.02	44.44	37.261
7	0.813	0.886	1.680	29.117	28.603	41.03	42.18	27.676
8	0.821	1.561	0.348	31.422	33.180	39.83	41.06	29.023
9	0.843	1.708	0.540	37.986	42.890	39.63	41.26	27.004
10	0.769	1.106	-1.218	17.820	19.051	37.45	39.04	18.906
11	0.869	0.518	0.097	38.387	40.050	37.53	39.13	37.944
12	0.898	1.266	0.187	39.597	40.749	37.70	39.27	19.706
13	0.802	0.200	-1.762	65.662	61.738	39.85	40.99	34.741
14	0.785	0.085	0.516	20.268	20.655	38.46	39.54	39.313
15	0.718	0.883	-2.529	65.256	61.736	40.24	41.02	75.397
16	0.816	1.948	-2.583	72.620	81.117	42.27	43.53	76.036
17	0.743	0.626	-0.918	14.617	13.994	40.64	41.79	14.139
18	0.813	1.262	-0.024	10.349	10.921	38.96	40.08	10.127
19	0.801	1.610	-0.982	85.745	81.523	40.37	42.26	47.329
20	0.829	0.651	2.095	81.378	76.628	42.41	43.97	55.058
Average Cost				42.41	43.97			37.21

REFERENCES

[1] A.E. Bryson and Y.C. Ho, Applied Optimal Control, Waltham, Mass., Ginn and Company, 1969.

[2] J.S. Meditch, Stochastic Optimal Linear Estimation and Control, New York, McGraw-Hill, 1969.

[3] K.J. Astrom, Introduction to Stochastic Control, New York, Academic Press, 1971.

[4] M. Aoki, Optimization of Stochastic Systems, New York, Academic Press, 1967.

[5] P.D. Joseph and J. Tou, "On Linear Control Theory," AIEE Transactions (Applications and Industry), Pt. II, vol. 80, p. 193, 1961.

[6] T.L. Gunckel and G.F. Franklin, "A General Solution for Linear Sampled-Data Control," Transactions ASME, Journal of Basic Engineering, vol. 85-D, pp. 197-203, June 1963.

[7] W.M. Wonham, "On the Separation Theorem of Stochastic Control," SIAM Journal on Control, vol. 6, p. 312, 1968.

[8] C.T. Striebel, "Sufficient Statistics in the Optimal Control of Stochastic Systems," Journal of Mathematical Analysis and Applications, vol. 12, p. 576, 1965.

[9] W.M. Wonham, "Random Differential Equations in Control Theory," in A.T. Bharucha-Reid, ed. Probabilistic Methods in Applied Mathematics, vol. II, New York, Academic Press, 1969.

[10] A.A. Feldbaum, "Dual Control Theory," Automation and Remote Control, Pt. I, vol. 21, pp. 874-880, April 1961, Pt. II, vol. 21, pp. 1033-1046, Pt. III, vol. 22, pp. 1-12, Pt. IV, vol. 22, pp. 109-121.

[11] H.A. Spang, "Optimum Control of an Unknown Linear Plant Using Bayesian Estimation of the Error," IEEE Transactions on Automatic Control, vol. AC-10, pp. 80-83, January 1965.

[12] Y. Bar-Shalom and R. Sivan, "The Optimal Control of Discrete Time Linear Systems with Random Parameters," IEEE Trans - on Automatic Control, vol. AC-14, pp. 3-8, February 1969.

[13] S.E. Dreyfus, Dynamic Programming and the Calculus of Variations, New York, Academic Press, 1965.

[14] S.E. Dreyfus, "Some Types of Optimal Control of Stochastic Systems," SIAM Journal on Control, vol.2, No.1, p.120-134,1964.

[15] R.E. Curry, Estimation and Control with Quantized Measurements, Cambridge, Mass., MIT Press, 1970.

[16] E. Tse, "On the Optimal Control of Linear Systems with Incomplete Information," Massachusetts Institute of Technology, Cambridge, Mass., Report ESL-R-412, January 1970.

[17] A.H. Jazwinski, Stochastic Process and Filtering Theory, New York, Academic Press 1970.

[18] R.P. Wishner, J.A. Tabaczynski, and M. Athans, "A Comparison of three nonlinear filters," Automatica, vol. 5, pp. 487-496, 1969.

[19] R.K. Mehra, "A Comparison of Several Nonlinear Filters for Reentry Vehicle Tracking," IEEE Transactions on Automatic Control, vol. AC-16, p.307, 1971.

[20] M. Athans, R.P. Wishner, and A. Bertolini, "Suboptimal State Estimation for Continuous Time Nonlinear Systems from Discrete Noisy Measurements," IEEE Transactions on Automatic Control, vol. AC-13, pp.504-514, October 1968.

[21] H.J. Kushner, "Approximations to Optimal Nonlinear Filters," IEEE Transactions on Automatic Control, vol. AC-12, p.546, 1967.

[22] J.J. Florentin, "Optimal Probing, Adaptive Control of a Simple Bayesian System," Journal of Electronics and Control, vol.II, p.165, 1962.

[23] J. Farison, R.E. Graham and R.C. Shelton, "Identification and Control of Linear Discrete Systems," IEEE Transactions on Automatic Control, vol. AC-12, pp. 438-442, August 1967.

[24] G. Stein and G.N. Saridis, "A Parameter-adaptive Control Technique," Automatica, vol. 5, November 1969.

[25] W.J Murphy, "Optimal Stochastic Control of Discrete-Linear Systems with Unknown Gain," IEEE Transactions on Automatic Control, AC-13, pp.338-344, August 1968.

[26] D. Gorman and J. Zaborszky, "Stochastic Optimal Control of Continuous Time System with Unknown Gain," IEEE Transactions on Automatic Control, vol. AC-13, pp. 630-638, December, 1968.

[27] G.N. Saridis and R.N. Lobbia, "Parameter Identification and Control of Linear Discrete Time Systems," in 1971, JACC Preprints, pp. 724-730.

[28] G.T. Schmidt, "A New Technique for Identification and Control of System with Unknown Parameters," Massachusetts Institute of Technology, Cambridge, Mass., Ph.D. Thesis, October 1970.

[29] E. Tse and M. Athans, "Adaptive Stochastic Control for Linear Systems," Pt. I and Pt. II in Proceedings IEEE Symposium, Adaptive Processes, Decision and Control, 1970.

[30] R.C.K. Lee, Optimal Estimation, Identification and Control, Cambridge, Mass., MIT Press, 1964.

[31] M. Athans and P. Falb, Optimal Control, New York, McGraw-Hill, 1966.

[32] F.C. Schweppe, Massachusetts Institute of Technology, notes for Modelling, Estimation, and Identification.

[33] N.E. Nahi, Estimation Theory and Applications, New York, John Wiley and Sons, 1969.

[34] M. Athans, "The Matrix Minimum Principle," Information and Control, vol. II, pp. 592-606, 1967.

[35] P. Dyer and S.R. McReynolds, The Computation and Theory of Optimal Control, New York, Academic Press, 1970.

[36] D.L. Kleinman and M. Athans, "The Discrete Minimum Principle with Application to the Linear Regulator Problem," Massachusetts Institute of Technology, Cambridge, Mass., Report ESL-R-260, February 1966.

[37] A. Levis, "On the Optimal Sampled-data Control of Linear Processes," Massachusetts Institute of Technology, Ph.D. Thesis in M.E., 1968.

[38] G.R. Tait and P.R Belanger, "A Comparison of Some Parameter Identification Schemes Using First and Second Order Extended Kalman-Bucy Filters and Sensitivity Functions," in Identification and Process Parameter Estimation, 2nd Prague IFAC Symposium, June 1970.

[39] G.N. Saridis and G. Stein, "Stochastic Approximation Algorithms for Linear Discrete-time System Identification," IEEE Transactions on Automatic Control, vol. AC-13, pp. 515-523, October 1968.

[40] R.L. Kashyap, "Maximum Likelihood Identification of Stochastic Linear Systems," IEEE Transactions on Automatic Control, vol. AC-15, p.25, 1970.

[41] H. Freeman, Discrete-Time Systems, New York, John Wiley and Sons, 1965.

[42] L.S. Kramer, "On Stochastic Control and Optimal Measurement Strategies," Ph.D. Thesis, E.E. MIT, October 1971.