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72917 and 76265
NSF Grant GK-25781 and
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OPTIMAL CONTROL OF FIRST-ORDER DISTRIBUTED SYSTEMS

Timothy L. Johnson

Electronic Systems Laboratory *Decision and Control Sciences Group*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Department of Electrical Engineering

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for

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Page 116 — This page is missing in some copies of the above publication. Attached is page 116 for your copy.

Electronic Systems Laboratory
Department of Electrical Engineering
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

May, 1972 Report ESL-R-482

OPTIMAL CONTROL OF FIRST-ORDER DISTRIBUTED SYSTEMS

by

Timothy Lee Johnson

This report is based on the unaltered thesis of Timothy Lee Johnson submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Massachusetts Institute of Technology in May, 1972. This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory, with partial support extended by the National Science Foundation under grant NSF-GK-25781 and by NASA/AMES under grant NGL-22-009(124).

Electronic Systems Laboratory
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OPTIMAL CONTROL OF FIRST-ORDER DISTRIBUTED SYSTEMS

by

TIMOTHY LEE JOHNSON

B.S., Massachusetts Institute of Technology
(1969)

M.S., Massachusetts Institute of Technology
(1969)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
May, 1972

Signature of Author ...Timothy L. Johnson...............
Department of Electrical Engineering, May 19, 1972

Certified by ...Michael Athans...............
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee on Graduate Students

OPTIMAL CONTROL OF FIRST-ORDER

DISTRIBUTED SYSTEMS

BY

Timothy Lee Johnson

Submitted to the Department of Electrical Engineering on 5/19/1972 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

ABSTRACT

The problem of characterizing optimal controls for a class of distributed-parameter systems is considered. The system dynamics are characterized mathematically by a finite number (n_x) of coupled partial differential equations involving first-order time and space derivatives of the state variables, which are constrained at the boundary by a finite number of algebraic relations ($n_q \leq n_x$). Multiple control inputs, extending over the entire spatial region occupied by the system ("distributed controls") are to be designed so that the response of the system is optimal (as measured by a scalar performance functional of the state and control variables).

The question of existence and uniqueness of solutions to the state equations is discussed for both linear and nonlinear systems, and examples of well-posed types of systems are given prior to treatment of problems in control.

For nonlinear systems, a Minimum Principle giving necessary conditions for optimality is stated and proved, using variational methods. Under additional assumptions, it is proven (Decomposition Theorem) that such optimal controls may be synthesized by solving a nonlinear static ("steady-state") optimization problem and a linearized dynamic optimal control problem with a quadratic performance functional.

For well-posed linear systems of the class studies by Kreiss [23] and Rauch [35], the optimal boundary control problem is solved using the method of variational inequalities as developed by Lions [25]. An operator Riccati equation characterizing the optimal feedback control law is also derived.

A major example involving boundary control of an unstable low-density plasma is developed from physical laws.

Optimal control problems for the steady state (a nonlinear problem) and the linearized dynamic equations are formulated, and conditions for optimality are derived using the above theories.

THESIS SUPERVISOR: Michael Athans

TITLE: Associate Professor of Electrical Engineering

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Professor Michael Athans, as friend and advisor, has played a major role in my graduate education. His inspired suggestions, enthusiasm, and encouragement of individualism have greatly contributed to this research. I particularly wish to thank him for his guidance and constructive comments during the development of the Minimum Principle of Chapter 1.

Professor Sanjoy K. Mitter has given very generously of his time and extensive background in distributed systems during our numerous technical discussions over the past two years. In addition to introducing me to the works of Lions and others, he has contributed immeasurably by clarifying many of the technical issues discussed in Chapter 2.

I am indebted to Professor Ronald Parker for inspiring and commenting on the early development of the plasma example of Chapter 3. Professor Gilbert Strang of the Mathematics Department brought to my attention the important works by H. O. Kreiss and J. Rauch used in Chapter 2.

For acting as readers and for providing knowledgeable ctiticism, I also wish to thank Professor Jan Willems of the Electrical Engineering Department, and Professor Steven A. Orszag of the Mathematics Department. I further wish to acknowledge the interest of Professors Gould and Smullin in this research.

Mrs. Saba Simsek is to be congratulated for her accuracy, skill, and speed in typing both the draft and final copies of the thesis, and for her phenomenal attention to detail.

Finally, I acknowledge the nontechnical but nevertheless inestimable contribution of my wife, Nancy, and the sacrifices she has made "for the sake of science" over the last three years.

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DEDICATED

TO MY PARENTS

WALTER K. JOHNSON

&

DOROTHY C. JOHNSON

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INTRODUCTION

The purpose of this research is to characterize distributed and boundary controls which are optimal with respect to scalar performance indices for the class of first-order distributed systems. Considerable motivation for such an investigation exists on both mathematical and engineering grounds. Mathematically, first-order systems[(1)] may be considered as a <u>canonical form</u> for systems of partial differential equations in the sense that higher order equations or systems may be reduced formally to first-order systems by taking all but the highest-order space and time derivatives as "state variables". Admittedly, this is a purely formal artifice which will not in itself yield any mathematical advantage, but the value of adopting a canonical form should not be under-estimated, as a great deal of insight may be gained through simplified notation. From an engineering viewpoint as well, mathematical models of distributed physical systems are usually developed from basic physical principles (e.g., the Schrodinger wave equation, Maxwell's equations, fluid mechanics, etc.). Most of these equations are naturally written in first-order form; when several physical properties of a single medium are modelled,

---

(1) Of a more general class than those studied here, however.

the equation(s) describing each property are coupled, resulting in a system of equations. These observations led the author to attempt to formulate first-order control problems in a general setting.

A significant feature of control problem formulation, from an engineering viewpoint, is the inclusion of boundary controls. Distributed controls are usually difficult, if not impossible, to implement (or even approximate) in practice; boundary control leads to a reduction in controller hardware as well as being a more "natural" means of influencing the system in most applications. Mathematically, boundary control of first-order systems is complicated by the fact that in general, only certain linear combinations of state variables (not the whole state!) may be constrained in a given boundary region for the state equations to possess solutions. Fortunately, it is possible to cast boundary conditions in a sufficiently general framework that the optimization theory may be carried through (formally) without a general mathematical existence theory (not yet available for first-order systems). This is <u>not</u> to say that well-posedness is unimportant for control problems!

The Minimum Principle, giving necessary conditions for optimality in the nonlinear case, is developed in Chapter 1 using local extremal conditions of differential calculus in several variables. The novel features of the proof are (1) use of the Divergence Theorem to eliminate variations of the spatial derivatives of the state, (2) use of the theory of pseudoinverse operators in characterizing boundary controls. The problem could be formulated in greater generality, and more sophisticated variational theory could be applied, but the two features above, in the author's estimation, would be crucial ingredients of any minimum principle for first-order systems. The objective herein is to treat the most commonly-occurring type of problem. Derivation of second-order necessary conditions for optimality with boundary controls (Appendix A) also requires some nontrivial development; these conditions are often useful for computational algorithms.

The problem of designing "steady-state" controls, which may be regarded as a parameter optimization problem, has long been of interest to the author. The performance improvement due to optimal "operating conditions" in applications is expected to be greater than the improvement afforded by optimal regulation. The Decomposition Theorem, (Chapter 1, Section 5), interestingly, gives a means for insuring that regulator design is consistent with steady-state performance objectives and for assigning relative weight to regulation versus steady-state performance (for instance, weighting the regulator problem just enough to achieve satisfactory stability). Furthermore, the regulation problem is a linear one and hence may be properly solved using Laplace-Fourier transform or modal analysis techniques.

The objective of Chapter 2 is to present a rigorous treatment of an optimal control problem involving a particular type of linear first-order system. Essentially, this requires a demonstration that the existence theory of Rauch [36] for first-order hyperbolic systems on half-spaces is adequate for the application of Lions' method [23] of solving optimal control problems. The original contributions of this application lie in the technical details of the proofs and in the inclusion of boundary controls. Such systems are not only important in their own right for a number of engineering problems, but also give a clear indication of the considerations involved in treating more general linear problems (for which an adequate mathematical existence theory is not yet available).

In order to illustrate the relevance of these theories to an engineering problem, boundary control of a plasma instability is considered. The derivation of a mathematical model is carried through in detail in order to demonstrate the relatively minor violence done to the physics in applying the mathematics. The development of a state-variable model for this system requires introduction of a viewpoint which is quite divergent from that used in conventional analysis of the problem, and is hence of value for its own sake. It is emphasized, however, that the calculations given herein should not be taken in the sense of a "solution" to a particular practical problem, but rather as an illustration of the potential usefulness of the foregoing theories.

The author wishes to recognize the works of Lions [26], [27], Rauch [36], and Penfield and Haus [33]. The contributions of Friedrichs and Lax [15], [13], [14], Kreiss [24], Garabedian [17], and Sage [43] are also acknowledged. Other mathematical theories relevant to first-order control problems include those of Butkovskiy [7] and Egorov [12] (extension of Pontryagin's results for finite-dimensional systems), Wang [47,48,49] (dynamic programming on function spaces) and Lurie [31] (use of integral equations rather than partial differential equations) for nonlinear problems; and Balakrishnan [2] (optimal control on Banach spaces), Lukes and Russell [30,42], Phillips [35] and Kato [23] (semigroup methods), and Gould [18], Russell [41], and Egorov [12] (modal analysis), for linear problems. Of course a large number of other authors may be cited. Extensive bibliographies may be found in [2], [48], [26], [7], and [39].

NOTATION

Independent Variables (Geometry):

The relevant dynamic behavior of the system is assumed to occur within a finite time interval $T = [0,t_f]$; where t_f is called the terminal time; the time variable is usually denoted by "t" or "τ".

Spatially, the system is confined to a closed, bounded, simply-connected region, Z, of R^m with non-empty interior. The boundary, B, of Z is assumed to be a C^∞ surface in R^m, with unit outward normal $\underline{n} = (n_1 \ldots n_m)$. The space variables are denoted by the m-vector $\underline{z}$, which refers to a single (i.e., not local) coordinate system throughout. Reference will frequently be made to the spatiotemporal domain, $D = Z \times T$, and to the lateral boundary of that domain, $L = B \times T$. Figure 1 illustrates this terminology for $Z \subset R^2$.

The significant geometrical assumptions are thus

(1) finite time interval,

(2) bounded spatial domain, and

(3) time-invariant boundaries.

The first two assumptions are relaxed with relative ease, while the third may sometimes be relaxed, with considerable theoretical difficulty, provided the rate of boundary motion is sufficiently small. These extensions will not be discussed in detail.

GEOMETRY FOR $m = 2$

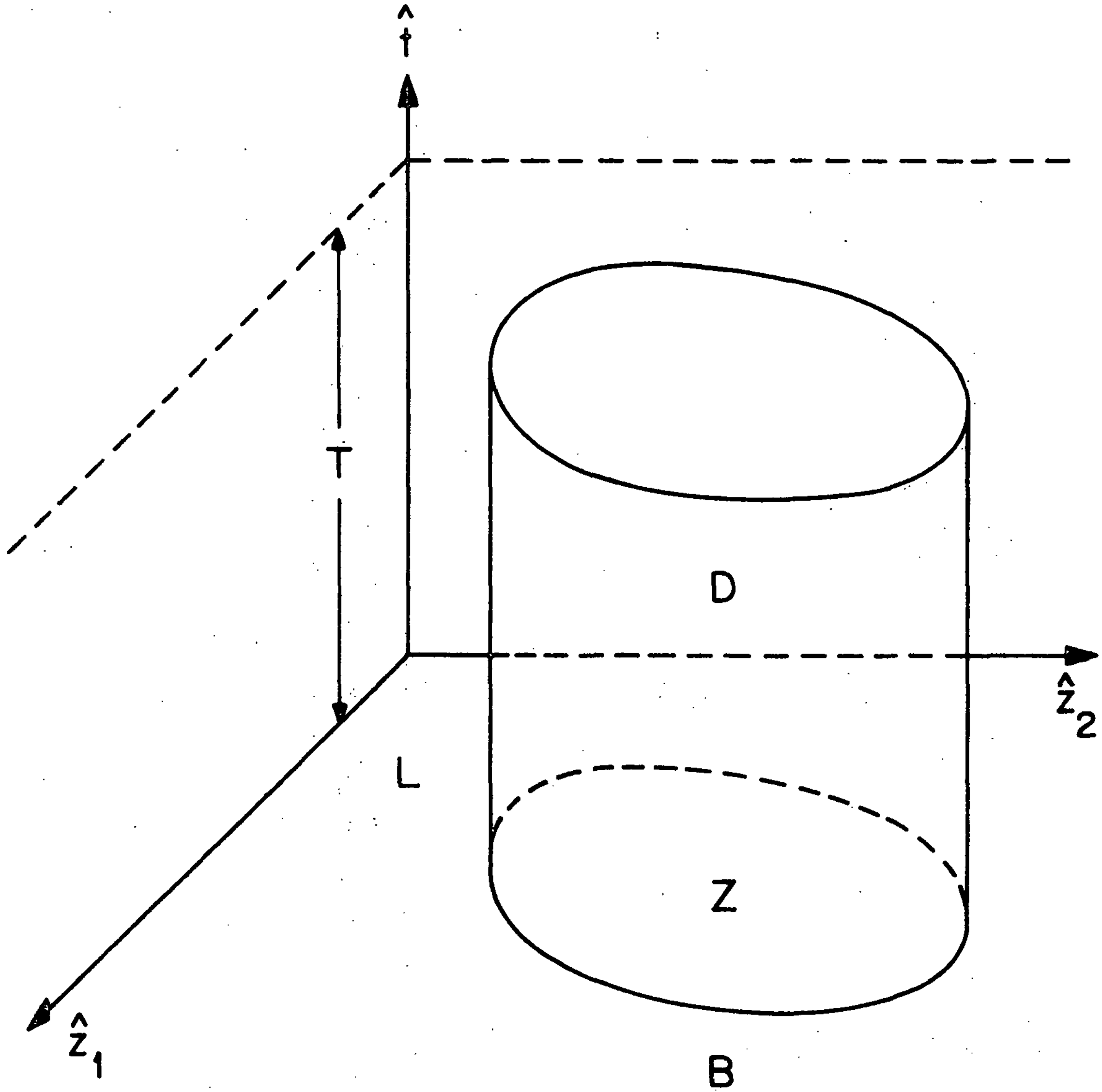

Figure 1

<u>Dependent Variables (State, Observation, Controls):</u>

At each fixed time, t, the system is characterized by a function on Z taking values in R^{n_x}, termed the <u>state at time t</u>, and, denoted $\underline{x}(t)$ or $\underline{x}(t,\underline{z})$. This function evolves according to dynamic and boundary constraints of the system; in this sense, the <u>state</u> is viewed as a collection of functions parameterized by $t \in T$. Alternatively, the state is sometimes considered as a function on the spatiotemporal domain, D. The former viewpoint is taken in Chapter II, while the latter predominates in Chapter I.

The <u>response</u> $\underline{y}(t)$ at each fixed time $t \in T$ is a function on $D \times R^{n_x}$ taking values in R^{n_y}, depending on the current value of the state function.

The evolution of the state is determined by <u>distributed forces</u> $\underline{f}$ taking values in R^{n_x} and <u>boundary forces</u> $\underline{g}$ taking values in R^{n_g} $(n_g \leq n_x)$. Control over the system is exercised by <u>distributed controls</u>, $\underline{u}(t)$, with values in R^{n_u}, and <u>boundary controls</u>, $\underline{v}(t)$, with values in R^{n_v} $(n_v \leq n_g$, normally). The function $\underline{f}$ depends on the current value of the function $\underline{u}$, and $\underline{g}$ depends on $\underline{v}$, with further details specified in the text.(1)

(1) Including the function spaces to which $\underline{x}$, $\underline{y}$, $\underline{f}$, $\underline{g}$, $\underline{u}$, $\underline{v}$, etc., are assumed to belong.

Summary of Notation

Independent Variables

$T = [0,t_f]$	Time domain
t, τ	Time variables
$Z \subset R^m$	Spatial domain (closed, bounded)
$\underline{z}$	Space variables
B	Boundary of Z
$D = T \times Z$	Spatiotemporal Domain
$L = T \times B$	Lateral boundary of D
$\underline{n}$	Unit outward normal to L

Dependent Variables

$\underline{x}$	System State (n_x-vector)
$\underline{y}$	Response (n_y-vector)
$\underline{f}$	Distributed force (n_x-vector)
$\underline{g}$	Boundary force (n_g-vector)
$\underline{u}$	Distributed control (n_u-vector)
$\underline{v}$	Boundary control (n_v-vector)

I. OPTIMAL CONTROL OF NONLINEAR FIRST-ORDER SYSTEMS

Necessary conditions for optimality are derived for "smooth" nonlinear systems having well-posed strict solutions, where the performance index is a scalar functional of the state and controls. For time-invariant systems, it is shown that these conditions may be "decomposed" into conditions for a nonlinear static problem and a linearized dynamic problem. A specific application to the subclass of quasilinear analytic systems is discussed.

1.1 Problem Statement

System Constraints:

The evolution of the state is governed by a system of partial differential equations, subject to initial conditions (IC) and boundary conditions (BC):

$$\partial \underline{x}(t,\underline{z})/\partial t = \underline{f}(\underline{x}(t,\underline{z}),\ \underline{x}_z(t,\underline{z}),\ \underline{u}(t,\underline{z}),\ t,\underline{z}) \qquad (1.1)$$

$$\underline{f}:\ R^{n_x} \times (R^{n_x})^m \times R^{n_u} \times D \to R^{n_x}$$

$$\text{IC: } \underline{x}(0,\underline{z}) = \underline{x}_o(\underline{z}) \qquad \underline{z} \in Z \qquad (1.2)$$

$$\underline{x}_o:\ Z \to R^{n_x}$$

$$\text{BC: } \underline{g}(\underline{x}_L(t,\underline{z}),\ \underline{v}(t,\underline{z}),\ t,\ \underline{z}) = \underline{0} \qquad (1.3)$$

$$\underline{g}:\ R^{n_x} \times R^{n_v} \times L \to R^{n_g}$$

where $\underline{x}_z$ is an abbreviation for the set of functions

$(\partial \underline{x}/\partial z_1, \partial \underline{x}/\partial z_2 \ldots \partial \underline{x}/\partial z_m)$ and $\underline{x}_L(t,\underline{z})$ denotes the function $\underline{x}(t,\underline{z})|_L$ on L.

Assumptions:

$$\underline{f} \in \{\underline{\tilde{f}} \in C(R^{(n_x)(m.n_x)(n_u)(m+1)};R^{n_x}) \mid \underline{\tilde{f}} \text{ is } C^2 \text{ in } \underline{x},\underline{x}_z,\underline{u}; \; C^o \text{ in } t; \; \& \; C^1 \text{ in } \underline{Z}\}$$

$$\underline{g} \in \{\underline{\tilde{g}} \in C(R^{(n_x)(n_v)(m)};R^{n_g}) \mid \underline{\tilde{g}} \text{ is } C^2 \text{ in } \underline{x}_L, \underline{v} \; \& \; C^\infty \text{ in } t, \underline{z}\}$$

$$\underline{x}_o \in X_o \subset C(Z;R^{n_x}) \tag{1.4}$$

$$\underline{u} \in U(T) \subset C(D;R^{n_u}) \tag{1.5}$$

$$\underline{v} \in V(T) \subset C(L;R^{n_v}) \tag{1.6}$$

Performance Index:

A scalar index of performance is specified:

$$J(\underline{u},\underline{v}) = \int_Z j_f(\underline{x}(t_f,\underline{z}),\underline{z})dZ + \int_L j_L(\underline{x}_L(t,\underline{z}),\underline{v}(t,\underline{z}),t,\underline{z})dL + \int_D j_D(\underline{x}(t,\underline{z}),\underline{x}_z(t,\underline{z}),\underline{u}(t,\underline{z}),t,\underline{z})dD \tag{1.7}$$

where

$$j_f: R^{n_x} \times Z \rightarrow R$$

$$j_L: R^{n_x} \times R^{n_v} \times L \rightarrow R$$

$$j_D: R^{n_x} \times (R^{n_x})^m \times R^{n_u} \times D \rightarrow R$$

Assumptions:

$j_f \in \{\tilde{j}_f \in C(R^{(n_x)(m)};R) \mid \tilde{j}_f$ is C^2 in $\underline{x}_f$ and C^0 in $\underline{z}\}$

$j_L \in \{\tilde{j}_L \in C(R^{(n_x)(n_v)(m)};R) \mid \tilde{j}_L$ is C^2 in $\underline{x}_L$, $\underline{v}$, C^∞ in t, $\underline{z}\}$

$j_D \in \{\tilde{j}_D \in C(R^{(n_x)(m,n_x)(n_u)(m+1)};R) \mid \tilde{j}_D$ is C^2 in $\underline{x}$, $\underline{x}_z$, $\underline{u}$; C^0 in t; & C^1 in $\underline{z}\}$

The state in (1.7) is also implicitly a function of $\underline{u}$ and $\underline{v}$, by virtue of the system constraints (1.1)-(1.3).

Optimal Controls:

The optimal controls $\underline{u}^* \in U(T)$, $\underline{v}^* \in V(T)$ are defined by the condition

$$J(\underline{u}^*,\underline{v}^*) \leq J(\underline{u},\underline{v}) \qquad \underline{u} \in U(T),\ \underline{v} \in V(T) \tag{1.8}$$

1.2 Well-posedness of Solutions

Definition: The system (1.1)-(1.3) is termed well-posed (or well-set) if there exists a function space X(T) such that given any $\underline{x}_o \in X_o$, $\underline{u}(t) \in U(T)$, $\underline{v}(t) \in V(T)$, there exists a unique solution (in general, a weak solution) $\underline{x}(t, \underline{x}_o, \underline{u}(t), \underline{v}(t)) \in X(T)$, and the mapping

$$(\underline{x}_o, \underline{u}, \underline{v}) \to \underline{x}$$

$$X_o \times U(T) \times V(T) \to X(T) \tag{2.1}$$

is continuous (for specified topologies on X_o, U(T), V(T) and X(T).

Definition: A function $\underline{x}(t) \in C^1(D;R^{n_x})$ and which satisfies (1.1)-(1.3) is said to be a strict solution.

Hypothesis 2/1: Problem (1.1)-(1.3) is well-posed with $X(T) \subseteq C^1(D;R^{n_x})$, i.e., with strict solutions.

Remark 2-1: Unfortunately, this hypothesis is quite strong, (even for linear systems!) and it represents one of the major (often unstated) shortcomings of most available results on optimal control of nonlinear distributed systems. Furthermore, it is not generally possible to guarantee the hypothesis (as it is for finite-dimensional systems) by merely assuming smoothness of X_o, U(T), V(T) and the functions $\underline{f}$ and $\underline{g}$. In general, well-posedness will require in addition restrictions on the functional forms of $\underline{f}$ and $\underline{g}$, and on the initial data, $\underline{x}_o$.

Remark 2-2: A literature search has not revealed any general well-posedness theorems for nonlinear first-order systems, although progress has been reported in special cases (see [10], [40], [47]). A well-posedness proof for the class of quasilinear analytic systems is sketched in Appendix B. Finite element methods (e.g., quantization in space and time, Galerkin's method, etc.) appear to offer the most promise for systems of higher dimension (n_x) and/or greater nonlinearity when analytical solutions are not known to exist. []

One additional hypothesis is required in order to justify a variational approach to the derivation of necessary conditions of optimality (Section 1.3, following) - well-posedness of the costate equations. Although the costate equations and associated boundary conditions for the nonlinear optimal control problem are linear (considering $\underline{x}_o(\underline{z})$, $\underline{u}(t,\underline{z})$, $\underline{v}(t,\underline{z})$ and $\underline{x}(t,\underline{z})$ to be fixed), it is <u>not</u> possible to conclude from this fact, as it is in the finite-dimensional case, that there exists a unique solution to the costate system. The costate equations are (see (3.5)-(3.7)):

$$\partial\underline{p}/\partial t = \sum_{i=1}^{m} (\partial\underline{f}/\partial\underline{x}_{z_i})' \, \partial\underline{p}/\partial z_i + [\sum_{i=1}^{m} \partial(\partial\underline{f}/\partial\underline{x}_{z_i})'/\partial z_i - (\partial\underline{f}/\partial\underline{x})']\underline{p}$$

$$+ [\sum_{i=1}^{m} \partial(\partial j_D/\partial\underline{x}_{z_i})'/\partial z_i - (\partial j_D/\partial\underline{x})'] \tag{2.2}$$

$$\underline{FC}\text{: } \underline{p}(t_f,\underline{z}) = \partial j_f/\partial\underline{x} \tag{2.3}$$

$$\text{BC: } [\underline{T}^{\perp}\underline{T}^{\perp\prime}]^{-1} \, \underline{T}^{\perp} \{(\partial j_L/\partial\underline{x})'$$

$$+ \sum_{i=1}^{m} n_i (\partial j_D/\partial\underline{x}_{z_i})' + (\sum_{i=1}^{m} \partial\underline{f}/\partial\underline{x}_{z_i})'\underline{p}_L\} = \underline{0} \tag{2.4}$$

where $\underline{T} = [\partial\underline{g}/\partial\underline{x}]'$ and all coefficients are evaluated at $\underline{x}$, $\underline{u}$, $\underline{v}$, t, $\underline{z}$. Note that j_f, j_L, and j_D contribute only non-autonomous driving terms and do not affect the essential systems of higher dimension (n_x) and/or greater nonlinearity when analytical solutions are not known to exist. []

One additional hypothesis is required in order to justify a variational approach to the derivation of necessary conditions of optimality (Section 1.3, following) - well-posedness of the costate equations. Although the costate equations and associated boundary conditions for the nonlinear optimal control problem are linear (considering $\underline{x}_o(\underline{z})$, $\underline{u}(t,\underline{z})$, $\underline{v}(t,\underline{z})$ and $\underline{x}(t,\underline{z})$ to be fixed), it is not possible to conclude from this fact, as it is in the finite-dimensional case, that there exists a unique solution to the costate system. The costate equations are (see (3.5)-(3.7)):

$$\partial \underline{p}/\partial t = \sum_{i=1}^{m} (\partial \underline{f}/\partial \underline{x}_{z_i})' \, \partial \underline{p}/\partial z_i + [\sum_{i=1}^{m} \partial(\partial \underline{f}/\partial \underline{x}_{z_i})'/\partial z_i - (\partial \underline{f}/\partial \underline{x})']\underline{p}$$

$$+ [\sum_{i=1}^{m} \partial(\partial j_D/\partial \underline{x}_{z_i})'/\partial z_i - (\partial j_D/\partial \underline{x})'] \qquad (2.2)$$

$$\underline{F}\text{C: } \underline{p}(t_f,\underline{z}) = \partial j_f/\partial \underline{x} \qquad (2.3)$$

$$\text{BC: } [\underline{T}^{\perp}\underline{T}^{\perp\prime}]^{-1}\, \underline{T}^{\perp}\{(\partial j_L/\partial \underline{x})'$$

$$+ \sum_{i=1}^{m} n_i (\partial j_D/\partial \underline{x}_{z_i})' + (\sum_{i=1}^{m} \partial \underline{f}/\partial \underline{x}_{z_i})'\underline{p}_L\} = \underline{0} \qquad (2.4)$$

where $\underline{T} = [\partial \underline{g}/\partial \underline{x}]'$ and all coefficients are evaluated at $\underline{x}$, $\underline{u}$, $\underline{v}$, t, $\underline{z}$. Note that j_f, j_L, and j_D contribute only non-autonomous driving terms and do not affect the essential structure of the dynamic constraints. While it is conceivable that specific assumptions on $\underline{f}$ and $\underline{g}$ which satisfy Hypothesis 2/1 may automatically imply well-posedness of solutions to (2.2)-(2.4), at least in special cases, not enough is known of either nonlinear or linear existence theories for such systems to guarantee this result. In the linear example of Chapter II, this possibility is realized. Until a general existence theory becomes available, it is necessary to postulate, however:

Hypothesis 2/2: Hypothesis 2/1 is satisfied. Given any $\underline{x}_o \in X_o$, $\underline{u} \in U(T)$, $\underline{v} \in V(T)$ and corresponding unique solution of (1.1)-(1.3), $\underline{x} \in X(T)$, the coefficients of (2.2)-(2.4) may be evaluated; then it is hypothesized that there exists a unique strict solution $\underline{p} \in C^1(D;R^{n_x})$ to this problem.

Remark 2-3: Hypothesis 2/2 does not necessarily imply a solution of (2.2)-(2.4) for all $\underline{x} \in X(T)$. Still, the hypothesis is rather restrictive.

1.3 Necessary Conditions for Optimality

Minimum Principle:

Consider the optimal control problem defined in Section 1.1, and assume Hypothesis 2/1 and 2/2 to be valid. Let $\underline{p}(t)$ denote an n_x-vector valued function on Z for each $t \in T$. Define the scalar Hamiltonian, h, by:

$$h(\underline{x},\underline{x}_z,\underline{p},\underline{u},t,\underline{z}) = j_D(\underline{x},\underline{x}_z,\underline{u},t,\underline{z}) + \underline{p}'\underline{f}(\underline{x},\underline{x}_z,\underline{u},t,\underline{z})$$

$$h: R^{n_x} \times (R^{n_x})^m \times R^{n_x} \times R^{n_u} \times D \to R \tag{3.1}$$

(where $(\cdot)'$ denotes transpose throughout). It is assumed for simplicity that $(\frac{\partial \underline{g}}{\partial \underline{x}})^*$ and $(\frac{\partial \underline{g}}{\partial \underline{v}})^*$ are of full rank, $n_g \leq n_x$. Then in order that the interior control $\underline{u}^*(t)$ and the boundary control $\underline{v}^*(t)$ be optimal, it is necessary that:(1)

(State Equations)

$$\partial \underline{x}^*/\partial t = \partial h/\partial \underline{p}\ (\underline{x}^*,\underline{x}_z^*,\underline{p}^*,\underline{u}^*,t,\underline{z}) \tag{3.2}$$

$$\text{IC: } \underline{x}^*(0,\underline{z}) = \underline{x}_o(\underline{z}) \tag{3.3}$$

$$\text{BC: } \underline{g}\ (\underline{x}_L^*,\underline{v}^*,t,\underline{z}) = \underline{0} \tag{3.4}$$

(Costate Equations)

$$\partial \underline{p}^*/\partial t = \sum_{i=1}^{m} \frac{\partial}{\partial z_i} (\partial h/\partial \underline{x}_{z_i})^* - \partial h/\partial \underline{x}(\underline{x}^*,\underline{x}_z^*,\underline{p}^*,t,\underline{z}) \tag{3.5}$$

$$\text{FC: } \underline{p}^*(t_f,\underline{z}) = \partial j_f/\partial \underline{x}\ (\underline{x}_f^*,\underline{z}) \tag{3.6}$$

$$\text{BC: } [(\partial \underline{g}/\partial \underline{x})^{\perp'}(\partial \underline{g}/\partial \underline{x})^{\perp}]^{*-1}(\partial \underline{g}/\partial \underline{x})^{*\perp'}[(\frac{\partial j_L}{\partial \underline{x}})^* + \sum_{i=1}^{m} n_i(\partial h/\partial \underline{x}_{z_i})^*] = \underline{0} \tag{3.7}$$

---

(1) $\partial(\cdot)/\partial x$ is the column vector (or matrix, if $(\cdot)$ is a vector)

$$\begin{bmatrix} \partial(\cdot)/\ x_1 \\ \partial(\cdot)/\ x_2 \\ \cdots \\ \partial(\cdot)/\ x_{n_x} \end{bmatrix}$$

and $(\cdot)^*$ denotes evaluation at $\underline{x}^*$, $\underline{u}^*$, $\underline{v}^*$, $\underline{g}^*$.

(Hamiltonian Minimization)

$$\partial h/\partial \underline{u}\ (\underline{x}^*, \underline{x}_z^*, \underline{p}^*, \underline{u}^*, t, \underline{z}) = \underline{0} \tag{3.8}$$

$$(\partial j_L/\partial \underline{v})^* + (\partial \underline{g}/\partial \underline{v})^*[(\partial \underline{g}/\partial \underline{x})'(\partial \underline{g}/\partial \underline{x})]^{*-1}(\partial \underline{g}/\partial \underline{x})^{*'}[(\frac{\partial j_L}{\partial \underline{x}})^*$$

$$+ \sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^*] = \underline{0} \tag{3.9}$$

Proof: Let $\underline{u}^*$ and $\underline{v}^*$ denote a set of optimal controls (see 1.8).(2) Consider the perturbed controls

$$\underline{u} = \underline{u}^* + \varepsilon\ \delta\underline{u} \tag{3.10}$$

$$\underline{v} = \underline{v}^* + \varepsilon\ \delta\underline{v} \tag{3.11}$$

where ε is a sufficiently small constant. Let $\underline{x}^*$ denote the state corresponding to $\underline{u}^*$, $\underline{v}^*$, and let $\underline{x}$ denote the perturbed state (i.e., strict solution of (1.1)-(1.3)) corresponding to $\underline{u}$, $\underline{v}$. Define $\delta\underline{x}$, the state perturbation by

$$\varepsilon\ \delta\ \underline{x} = \underline{x}^* - \underline{x} \tag{3.12}$$

Then $\delta\underline{x}$ satisfies the equation:

$$\partial(\delta\underline{x})/\partial t = \frac{1}{\varepsilon}[\underline{f}(\underline{x}, \underline{x}_z, \underline{u}, t, \underline{z}) - \underline{f}(\underline{x}^*, \underline{x}_z^*, \underline{u}^*, t, \underline{z})]$$

$$= (\partial \underline{f}/\partial \underline{x})^{*'}\delta\underline{x} + \sum_{i=1}^{m} (\partial \underline{f}/\partial \underline{x}_{z_i})^{*'}\delta(\underline{x}_{z_i})$$

$$+ (\partial \underline{f}/\partial \underline{u})^{*'}\delta\underline{u} + 0_1(\varepsilon)/\varepsilon \tag{3.13}$$

---

(2) Dependences on t and $\underline{z}$ are supressed in the following manipulations.

$$\text{IC: } \delta\underline{x}(0,\underline{z}) = \underline{x}^*(0,\underline{z}) - \underline{x}(0,\underline{z}) = \underline{0} \tag{3.14}$$

$$\text{BC: } (\partial\underline{g}/\partial\underline{x})^{*'}\,\delta\underline{x}_L + (\partial\underline{g}/\partial\underline{v})^{*'}\,\delta\underline{v} + 0_2(\varepsilon)/\varepsilon = \underline{0} \tag{3.15}$$

where

$$\delta(\underline{x}_{z_i}) = \frac{1}{\varepsilon}\,[\partial\underline{x}^*/\partial z_i - \partial\underline{x}/\partial z_i] = \frac{1}{\varepsilon}\,\frac{\partial}{\partial z_i}\,(\varepsilon\delta\underline{x}) = \partial(\delta\underline{x})/\partial z_i$$

and $\lim_{\varepsilon\to 0} \underline{0}_i(\varepsilon)/\varepsilon = 0$ (guaranteed by continuity assumptions on $\underline{f}$ and $\underline{g}$). The performance index (1.7) is modified by the addition a Lagrange multiplier term:(3)

$$\tilde{J}(\underline{u},\underline{v}) = \int_Z j_f(\underline{x}_f,\underline{z})\,dZ + \int_L j_L(\underline{x}_L,\underline{v},t,\underline{z})\,dL + \int_D j_D(\underline{x},\underline{x}_z,\underline{u},t,\underline{z})\,dD$$

$$+ \int_D \underline{p}'\,[f(\underline{x},\underline{x}_z,\underline{u},t,\underline{z}) - \partial\underline{x}/\partial t]\,dD \tag{3.16}$$

Defining

$$\varepsilon\delta\tilde{J} = \tilde{J}(\underline{u},\underline{v}) - \tilde{J}(\underline{u}^*,\underline{v}^*) \tag{3.17}$$

it is required to evaluate the first-order necessary condition for a minimum

$$\delta\tilde{J}_o = \lim_{\varepsilon\to 0}\,\delta\tilde{J} = 0 \tag{3.18}$$

This is accomplished using the Taylor series expansions

$$j_f(\underline{x}_f,\underline{z}) = j_f(\underline{x}_f^*,\underline{z}) + \varepsilon\frac{\partial j_f}{\partial\underline{x}}\,(\underline{x}_f^*,\underline{z})'\,\delta\underline{x}_f + 0_3(\varepsilon) \tag{3.19}$$

(3) The assumptions guarantee sufficient differentiability to validate a LaGrangian variational procedure.

$$h(\underline{x},\underline{x}_z,\underline{p},\underline{u},t,\underline{z}) = \underline{h}(\underline{x}^*,\underline{x}^*_z,\underline{p},\underline{u}^*,t,\underline{z})$$

$$+ \varepsilon(\partial h/\partial\underline{x})^{*'}\delta\underline{x} + \varepsilon\sum_{i=1}^{m}(\partial h/\partial\underline{x}_{z_i})^{*})' \, \partial(\delta\underline{x})/\partial z_i$$

$$+ \varepsilon(\partial h/\partial\underline{u})^{*'}\delta\underline{u} + 0_4(\varepsilon) \tag{3.20}$$

and

$$j_L(\underline{x}_L,\underline{v}\ ,t,\underline{z}) = j_L(\underline{x}^*_L,\underline{v}^*,t,\underline{z}) + \varepsilon(\partial j_L/\partial\underline{x})^{*'}\delta\underline{x}_L$$

$$+ \varepsilon(\partial j_L/\partial\underline{v})^{*'}\delta\underline{v} + 0_5(\varepsilon) \tag{3.21}$$

where again, differentiability assumptions guarantee that $\lim_{\varepsilon\to 0} 0_i(\varepsilon)/\varepsilon = 0$. Together, these results imply

$$\delta\tilde{J} = \int_Z (\partial j_f/\partial\underline{x})^{*'}\delta\underline{x}_f \, dZ$$

$$+ \int_L [(\partial j_L/\partial\underline{x})^{*'} \delta\underline{x}_L + (\partial j_L/\partial\underline{v})^{*'}\delta\underline{v}] \, dL$$

$$+ \int_D [(\partial h/\partial\underline{x})^{*'}\delta\underline{x} + \sum_{i=1}^{m}(\partial h/\partial\underline{x}_{z_i})^{*'}\partial(\delta\underline{x})/\partial_{z_i}$$

$$+ (\partial h/\partial\underline{u})^{*'}\delta\underline{u} - \underline{p}'\partial(\delta\underline{x})/\partial t] \, dD$$

$$+ \frac{1}{\varepsilon}\{\int_Z 0_3(\varepsilon)\, dZ + \int_L 0_5(\varepsilon)\, dL + \int_D 0_4(\varepsilon)\, dD\} \tag{3.22}$$

where as $\varepsilon \to 0$ the last term vanishes - since D is bounded. The continuity assumptions on $\underline{p}$ (Hypothesis 2/2), j_D and $\underline{f}$ imply:

$$\frac{\partial}{\partial t}(\underline{p}'\delta\underline{x}) = \underline{p}'\,\partial(\delta\underline{x})/\partial t + (\partial\underline{p}'/\partial t)\delta\underline{x} \tag{3.23}$$

and

$$\frac{\partial}{\partial z_i}[(\partial h/\partial\underline{x}_{z_i})^{*'}\,\delta\underline{x}] = (\partial h/\partial\underline{x}_{z_i})^{*'}\,\partial(\delta\underline{x})/\partial z_i + [\frac{\partial}{\partial z_i}(\partial h/\partial\underline{x}_{z_i})^{*'}]\,\delta\underline{x} \tag{3.24}$$

Using these in (3.22), as $\varepsilon \to 0$,

$$\begin{aligned}\delta\tilde{J}_o = &\int_Z (\partial j_f/\partial\underline{x})^{*'}\,\delta\underline{x}_f + \int_L [(\partial j_L/\partial\underline{x})\delta\underline{x}_L + (\partial j_L/\partial\underline{v})^*\,\delta\underline{v}]\,dL \\ &+ \int_D \{[\partial\underline{p}'/\partial t - \sum_{i=1}^{m}\frac{\partial}{\partial z_i}(\partial h/\partial\underline{x}_{z_i})^{*'} + (\partial h/\partial\underline{x})^*]\,\delta\underline{x} \\ &\quad + (\partial h/\partial\underline{u})^{*'}\,\delta\underline{u}\}\,dD \\ &+ \int_D \{-\frac{\partial}{\partial t}(\underline{p}'\delta\underline{x}) + \sum_{i=1}^{m}\frac{\partial}{\partial z_i}[(\partial h/\partial\underline{x}_{z_i})^{*'}\,\delta\underline{x}]\}\,dD\end{aligned} \tag{3.25}$$

The last term may be evaluated by means of the following

Lemma 1 (Divergence Theorem [21], p.292.)

Let D be a closed bounded region in R^{m+1}, with a boundary surface, ∂D, which is piecewise smooth. Let $w_0, w_1 \ldots w_m$ be (scalar) continuously differentiable functions on D and let $(n_0, n_1 \ldots n_m)$ denote the unit outward normal to ∂D (which is a function of the boundary point and is uniquely defined a.e. on ∂D). Then

$$\int_D [\sum_{i=0}^{m} \partial w_i/\partial z_i] dD = \int_{\partial D} [\sum_{i=0}^{m} n_i w_i] d(\partial D) \tag{3.26}$$

The continuity conditions on B and on $\underline{p}$ (Hypothesis 2/2, $\underline{x}$ (Hypothesis 2/1), j_D and $\underline{f}$ are sufficient for application of the theorem with

$$w_o = -\underline{p}' \, \delta\underline{x} \tag{3.27}$$

$$w_i = (\partial h/\partial \underline{x}_{z_i})^{*\prime} \, \delta\underline{x}; \quad i = 1, 2, \ldots, m \tag{3.28}$$

Thus the last term of (3.25) becomes

$$\int_D [-\frac{\partial}{\partial t} (\underline{p}'\delta\underline{x}) + \sum_{i=1}^{m} \frac{\partial}{\partial z_i} ((\partial h/\partial \underline{x}_{z_i})^{*\prime}\delta\underline{x})] \, dD$$

$$= \int_{\partial D} [-n_o\underline{p}' + \sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^{*\prime}] \delta\underline{x} \, d(\partial D) \tag{3.29}$$

Now $\partial D = L \cup (Z \times \{0\}) \cup (Z \times \{t_f\})$. On L, $n_o = 0$ and $\delta\underline{x} = \delta\underline{x}_L$; at $t = 0, t_f$, $n_i = 0$ $(i = 1, 2, \ldots m)$, $n_o = \pm 1$, with $\delta\underline{x}(0) = \underline{0}$ - see (3.12) - and $\delta\underline{x}(t_f) = \delta\underline{x}_f$. So

$$\int_D [-\frac{\partial}{\partial t} (\underline{p}'\delta\underline{x}) + \sum_{i=1}^{m} \frac{\partial}{\partial z_i} ((\partial h/\partial \underline{x}_{z_i})^{*\prime}\delta\underline{x})] \, dD$$

$$= \int_Z -\underline{p}_f'\delta\underline{x}_f \, dZ + \int_L [\sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^{*\prime}] \delta\underline{x}_L \, dL \tag{3.30}$$

Referring to (3.25)

$$\delta\tilde{J}_o = \int_Z [(\partial j_f/\partial \underline{x})^{*\prime} - \underline{p}_f'] \, \underline{x}_f \, dZ$$

$$+ \int_L \{[(\partial j_L/\partial \underline{x})^{*\prime} + \sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^{*\prime}]\delta \underline{x}_L$$

$$+ (\partial j_L/\partial \underline{v})^{*\prime}\delta \underline{v}\} \, dL$$

$$+ \int_D \{[\frac{\partial \underline{p}'}{\partial t} - \sum_{i=1}^{m} \frac{\partial}{\partial z_i} (\partial h/\partial \underline{x}_{z_i})^{*\prime} + (\partial h/\partial \underline{x})^{*\prime}]\delta \underline{x}$$

$$+ (\partial h/\partial \underline{u})^{*\prime}\delta \underline{u}\} \, dD \qquad (3.31)$$

The costate function $\underline{p}$ has remained unsepcified to this point. In order to guarantee (3.18) for arbitrary $\delta\underline{u}$, $\delta\underline{v}$, it is necessary to define the optimal costate $\underline{p}^*$ by (3.5), (3.6). The hypothesis 2/2 is required in order to guarantee that these equations, with boundary conditions (3.7), admit a solution.

The boundary condition (3.7) requires additional manipulation, since the boundary variations $\underline{x}_L$ and $\underline{v}$ are partially constrained by (3.15). Noting that $(\frac{\partial \underline{g}}{\partial \underline{x}})^{*\prime}$ was assumed to be of full rank $n_g \le n_x$, it is possible using standard results of linear algebra to complete (at each point in D) a basis by selecting vectors orthogonal to the rows of $(\frac{\partial \underline{g}}{\partial \underline{x}})^{*\prime}$. Letting $\underline{w}_1' \ldots \underline{w}_{n_g}'$ denote the rows, select $\underline{w}_{n_g+1} \ldots \underline{w}_{n_x}$ such that $\underline{w}_i'\underline{w}_j = 0$; $i = 1, 2, \ldots n_g$; $j = n_g+1, n_x$, and $\underline{w}_i' \ldots \underline{w}_{n_x}'$ are linearly independent. Let $[(\frac{\partial \underline{g}}{\partial \underline{x}})^{*\prime}]^{\perp}$ denote the matrix formed by the row vectors $\underline{w}_{n_g+1}'; \ldots \underline{w}_{n_x}'$. Then

$$[(\partial \underline{g}/\partial \underline{x})^{*\prime}]^{\perp} (\partial \underline{g}/\partial \underline{x})^{*} = \underline{0} \tag{3.32}$$

In the following, let $T = \left(\frac{\partial g}{\partial \underline{x}} \right)^{*\prime}$ and let

$$\underline{r}' = [(\partial j_L/\partial \underline{x})^{*\prime} + \sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^{*\prime}] \tag{3.33}$$

Now

$$\underline{r}' = \underline{\alpha}' \, \underline{T} + \underline{\beta}' \, \underline{T}^{\perp} \tag{3.34}$$

where $\underline{\alpha}$ is an n_g-vector-valued function on D, and $\underline{\beta}$ is an (n_x-n_g) vector valued function on D (nonunique,in general). Thus

$$\underline{r}'\delta \underline{x}_L = \underline{\alpha}' \, \underline{T}\delta \underline{x}_L + \underline{\beta}' \, \underline{T}^{\perp}\delta \underline{x}_L$$

$$= - \underline{\alpha}' \, (\partial \underline{g}/\partial \underline{v})^{*\prime} \, \delta \underline{v} + \underline{\beta}' \, \underline{T}^{\perp} \, \delta \underline{x}_L \tag{3.35}$$

by (3.15). Since $\underline{T}^{\perp}\delta \underline{x}_L$ is <u>un</u>constrained, require

$$\underline{\beta} = \underline{0} \tag{3.36}$$

(this leads to boundary conditions for $\underline{p}$), and from the second term of (3.31), obtain the condition (see 3.18):

$$\int_L \delta \underline{v}' \, [(\partial j_L/\partial \underline{v})^{*} - (\partial \underline{g}/\partial \underline{v})^{*}\underline{\alpha}] \, dL = 0 \tag{3.37}$$

For $\underline{p} = \underline{p}^{*}$, (3.31) becomes (see Remark 3-1 below):

$$\delta\tilde{J}_o = \int_L \delta\underline{v}' \{(\partial j_L/\partial\underline{v})^* -$$

$$(\partial\underline{g}/\partial\underline{v})[(\partial\underline{g}/\partial\underline{x})^{*'}(\partial\underline{g}/\partial\underline{x})^*]^{-1}(\partial\underline{g}/\partial\underline{x})^{*'}[(\partial j_L/\partial\underline{x})^*$$

$$+ \sum_{i=1}^{m} n_i (\partial h/\partial x_{z_i})^*]\}dL$$

$$+ \int_D \delta\underline{u}'(\partial h/\partial\underline{u})^* \, dD \tag{3.38}$$

The remaining conditions (3.8), (3.9) are a consequence of:

Lemma 2: Let S be a closed bounded simply-connected subset of R^k of the form $T \times \tilde{S}$ for $T = [0,t_f]$ and $\tilde{S}$ having a smooth (C^∞) boundary. If the continuously - differentiable function $\underline{w}$ defined on S satisfies

$$\int_S \underline{f}' \, \underline{w} \, d\,S = 0 \tag{3.39}$$

for all continuous functions $\underline{f}$ on S, then

$$\underline{w} \equiv 0. \tag{3.40}$$

Proof: Consider the case where f and w are scalar-valued functions on S; the extension to vector-valued functions is a consequence of Tychonov's theorem [52] and arguments analogous to those of [1], p. 260. Since S is closed and w is continuous, w is bounded and takes on its minimum and maximum values, denoted w^- and w^+, respectively, in S. It is clear (construct a simple counter example) that $w^+ \geq 0$. The point w^+ is a compact subset of R; hence its inverse image under w, denoted K^+, is a compact subset of S[4] (K^+ is therefore closed). The inverse image of the open interval $(0,\infty)$ under w is an open set P; the complement of P relative to S, P^c, is therefore closed. The sets P^c and K^+ are either disjoint or not. If they are not disjoint, then by the above comment, $w^+ = 0$. This implies $w \leq 0$, and $|w| = -w$. But choosing $f \equiv 1$, (3.39) becomes

$$\int_S w \, dS = -\int_S |w| \, dS = 0 \Rightarrow w \equiv 0 \quad \text{a.e. on } S$$

By continuity, $w \equiv 0$ on S, in this case. On the other hand, if P^c and K^+ are disjoint, it is possible to construct (Urysohn's Theorem, [52], p.7) a continuous real-valued function f on S such that $0 \leq f(\underline{z}) \leq 1$ on S, $f \equiv 0$ on P^c and $f \equiv 1$ on K^+. Since P^c consists of precisely those points where $w \leq 0$, $fw \geq 0$ on S. Suppose $w^+ > 0$ and let $\underline{z}^+$ be a point of K^+, i.e., $w(\underline{z}^+) = w^+ > 0$. Then for $\varepsilon > 0$ sufficiently small there exists a ball $B(z^+,\varepsilon)$ such that $(B(z^+,\varepsilon) \cap S$ is open relative to S, and):

(i) $m(B(z^+,\varepsilon) \cap S) = b > 0$ (a consequence of the assumptions on S)

(ii) $f \geq (1-\delta)$, $0 < \delta < 1$, on $B(z^+,\varepsilon) \cap S$ (continuity of f at z^+)

(iii) $w > (w^+ - \delta)$ on $B(z^+,\varepsilon) \cap S$ (continuity of w at z^+)

But this implies

$$\int_S f \, w \, dS \geq b(1-\delta)(w^+-\delta) > 0,$$

---

(4) See [22], for example.

contradicting (3.39). Therefore $w^{+} = 0$, and thus (by the argument above), $w = 0$. Q.E.D.

This completes the proof of the Minimum Principle.

<u>Remark 3-1:</u> It is important to verify that (1) $\underline{\alpha}$ can be chosen independently of how $\underline{T}^{\perp}$ is selected, and (2) $\underline{\beta}$ depends only on the range of $\underline{T}^{\perp}$, and not the particular way in which it is chosen. A verification of these facts, as well as explicit formulas for $\underline{\alpha}$ and $\underline{\beta}$, is provided by reference to the theory of pseudoinverse operators, (see [29], p.166). It is found that $\underline{\alpha}$ may be chosen as

$$\underline{\alpha} = \underline{T}'^{\dagger}\underline{r} = [\underline{T}\,\underline{T}']^{-1}\,\underline{T}\,\underline{r} \tag{3.41}$$

which is independent of $\underline{T}^{\perp}$ as desired. Then $\underline{\beta}$ (from (3.35)) is determined by

$$\underline{T}^{\perp'}\,\underline{\beta} = [\underline{I} - \underline{T}'(\underline{T}\,\underline{T}')^{-1}\underline{T}]\,\underline{r} \tag{3.42}$$

whence

$$\begin{aligned} \underline{\beta} &= (\underline{T}^{\perp'})^{\dagger}[\underline{I} - \underline{T}'(\underline{T}\,\underline{T}')^{-1}\underline{T}]\,\underline{r} \\ &= (\underline{T}^{\perp}\underline{T}^{\perp'})^{-1}\underline{T}^{\perp}\,[\underline{I} - \underline{T}'(\underline{T}\,\underline{T}')^{-1}\,\underline{T}]\,\underline{r} \qquad (3.43) \\ &= (\underline{T}^{\perp}\,\underline{T}^{\perp'})^{-1}\,\underline{T}^{\perp}\,\underline{r} \end{aligned}$$

using (3.32). With this choice, the condition (3.36) depends only on the range space of $\underline{T}$.

Remark 3-2: The second-variation condition for a minimum is derived in Appendix A.

A few comments concerning the method of proof and its relationship to prior and future works may be made. The method of proof has certain points in common with the methods of Krotov (see [7], pp.49-59 - i.e., use of the divergence theorem), and Sage ([43], pp. 137-144, variational formulation). Wang [48] has also derived a maximum principle via dynamic programming. These works are unclear in their treatment of continuity assumptions and well-posedness (e.g. appropriate boundary conditions). Particularly noteworthy herein is the treatment of boundary conditions having less than full rank ($n_g \leq n_x$) and the incorporation of boundary controls which were not included in previous works. The frequently used assumption of homogeneous boundary conditions ($\underline{x}_L = \underline{0}$) in these prior works, for example, may lead to problems which are "almost never" well-posed, particularly in the hyperbolic case. The use of the Divergence Theorem was discovered independently of Krotov's work. The incorporation of the variational constraints at the boundary (3.15) appears to be novel, as are the resulting boundary controls.

Much future work remains to be done in order to achieve a distributed maximum principle of comparable elegance and generality to that available for lumped systems. The continuity assumptions could be relaxed, for instance, by defining weak solutions of the state and costate equations.

The entire problem should ultimately be cast in a distributional framework and the appropriate generalizations of the two lemmas used above could probably still be applied. A second major improvement would be the use of global (rather than local) control perturbations to obtain global (rather than local) necessary conditions. Lions [26] has already made considerable contributions in this area. Finally, there are the generalizations to movable boundaries, control constraints, free terminal time and fixed terminal state (etc.) problems. All of these improvements are considered to be presently feasible. It appears that it might ultimately be possible to prove well-posedness of the costate equation (perhaps under additional assumptions) in a weak sense, using well-posedness of the state equation, but this result is probably some years away.

1.4 Quasilinear Analytic Systems

This special case of the above theory is considered because (1) a large number of (nonlinear) systems occurring in practice are of this type, (2) a concrete means of proving well-posedness is available, (3) meaningful "quasi-quadratic" performance indices can be found (e.g., energy functionals), (4) the conditions of the minimum principle may be written out explicitly and serve as a guide for comparison with the linear case.

Problem Statement:

In (1.1)-(1.7), make the specific additional assumptions:

$$\underline{f}(\underline{x},\underline{x}_z,\underline{u},t,\underline{z}) = \sum_{i=1}^{m} \underline{A}_i(\underline{x},t,\underline{z})\ \partial\underline{x}/\partial z_i + \underline{K}(\underline{x},t,\underline{z})\ \underline{x}$$

$$+ \underline{B}(\underline{x},t,\underline{z})\ \underline{u} + \underline{\tilde{f}}(t,\underline{z}) \tag{4.1}$$

$$\underline{g}(\underline{x}_L,\underline{v},t,\underline{z}) = \underline{T}(t,\underline{z})\ \underline{x}_L - \underline{G}(t,\underline{z})\underline{v} \tag{4.2}$$

$$U(T) = \{\underline{u}(t)\,|\,\underline{u} \text{ continuous on } T, \text{ analytic on } Z\} \tag{4.3}$$

$$V(T) = \{\underline{v}(t)\,|\,\underline{v} \text{ continuous on } T, \text{ analytic on } B\} \tag{4.4}$$

$$X_o = \{\underline{x}\ |\ \underline{x} \text{ analytic on } Z\} \tag{4.5}$$

$$j_f(\underline{x}_f\underline{z}) = \underline{x}_f'\ \underline{S}(\underline{x}_f,\underline{z})\ \underline{x}_f \tag{4.6}$$

$$j_L(\underline{x}_L,\underline{v},t,\underline{z}) = \underline{v}'\underline{R}_v(t,\underline{z})\underline{v} + \underline{x}_L'\ \underline{F}(t,\underline{z})\underline{x}_L \tag{4.7}$$

$$j_D(\underline{x},\underline{x}_z,\underline{u},t,\underline{z}) = \underline{x}'\underline{Q}_o(\underline{x},t,\underline{z})\underline{x} + \underline{u}'\underline{R}_u(t,\underline{z})\underline{u}$$

$$+ \sum_{i=1}^{m} \underline{x}_{z_i}'\ \underline{Q}_i(t,\underline{z})\underline{x}_{z_i} \tag{4.8}$$

Assume all coefficient matrices to be continuous in t, analytic in $\underline{z}$ for $t \in T$, and (where $\underline{x}$ is indicated as an argument) affine in $\underline{x}$. Assume $\underline{R}_u$, $\underline{R}_v > \underline{0}$ symmetric, and $\underline{S}$, $\underline{F}$, $\underline{Q}_i \geq 0$, symmetric, where $i = 0, 1 \ldots m$.

Well-Posedness of Solutions:

Theorem: For sufficiently small t_f and appropriate $\underline{T}(t,\underline{z})$, the problem (1.1)-(1.6), (4.1)-(4.8) is well posed with

$$X(T) = \{\underline{x}(t) \,|\, \underline{x} \text{ continuous on } T, \text{ analytic on } Z\} \qquad (4.9)$$

Proof: See Appendix B.

Compared with the conditions of the preceding section, the continuity conditions of this theorem are somewhat weaker in time (continuity vs. continuous differentiability) and stronger in space (analycity vs. continuous differentiability). The proof of the theorem (for the nonlinear case) hinges on the fact that the method of characteristics extends to nonlinear systems; the details of this extension yield "natural" boundary conditions of the form specified by (4.2). In general, it is necessary to limit t_f as solutions may grow unbounded, at least for certain initial conditions. Assumptions similar to these would have to be made for any other method of proof.

Remark 4-1: The method of proof is readily extended to the costate equations in this case.

Necessary Conditions for Optimality:

Consider the optimal control problem (1.1)-(1.8) subject to assumptions (4.1)-(4.8) and assume that the hypotheses of the preceding theorem are satisfied. Consider the subclass of systems which have coefficients continuously differentiable in time (and hence solutions with the same property). Then the Minimum Principle (Section 1.3) applies, and the equations (3.2)-(3.9) may be written explicitly.

(State Equations)

$$\partial\underline{x}^*/\partial t = \sum_{i=1}^{m} \underline{A}_i(\underline{x}^*,t,\underline{z})\ \partial\underline{x}^*/\partial z_i + \underline{K}(\underline{x}^*,t,\underline{z})\ \underline{x}^* + \underline{B}(\underline{x}^*,t,\underline{z})\ \underline{u}^* + \underline{\tilde{f}}(t,\underline{z}) \tag{4.10}$$

$$\text{IC: } \underline{x}^*(0,\underline{z}) = \underline{x}_o(\underline{z}) \tag{4.11}$$

$$\text{BC: } \underline{T}(t,\underline{z})\ \underline{x}^*_L = \underline{G}(t,\underline{z})\ \underline{v}^* \tag{4.12}$$

$$\begin{aligned}\partial\underline{p}^*/\partial t = {} & \sum_{i=1}^{m} \frac{\partial}{\partial z_i} [\underline{A}_i'(\underline{x}^*,t,\underline{z})\underline{p}^* + 2\underline{Q}_i(t,\underline{z})\ \partial\underline{x}^*/\partial z_i] \\ & -[\underline{K}'(\underline{x}^*,t,\underline{z}) + \sum_{j=1}^{n_x} x_j^*(\partial\underline{K}'(\underline{x}^*,t,\underline{z})/\partial x_j) \\ & \qquad + \underline{u}^*\partial\underline{B}'/\partial x_j(\underline{x}^*,t,\underline{z})]\underline{p}^* \\ & -[2\underline{Q}_o(\underline{x}^*,t,\underline{z}) + \sum_{j=1}^{n_x} x_j^*\ \partial\underline{Q}_o^*(\underline{x}^*,t,\underline{z})/\partial x_j]\underline{x}^*\end{aligned}$$

(Costate Equations) (4.13)

$$\text{FC: } \underline{p}^*(t_f,\underline{z}) = 2\underline{S}(\underline{x}_f^*,\underline{z})\underline{x}^*_f + [\sum_{j=1}^{n_x} x^*_{f_j}\ \partial\underline{S}(\underline{x}_f^*,\underline{z})/\partial x_{f_j}]\underline{x}_f^* \tag{4.14}$$

$$\text{BC:}\quad (\underline{T}^{\perp'}\underline{T}^{\perp})^{-1}(\underline{T}^{\perp'})\,[\underline{F}'\,\underline{x}_L^* + \sum_{i=1}^{m} n_i\,(\underline{A}_i'(\underline{x}^*,t,\underline{z})\,\underline{p}_L^*$$

$$+\ 2\underline{Q}_i(t,\underline{z})\,\partial\underline{x}_L^*/\partial z_i)] = \underline{0} \tag{4.15}$$

$$2\underline{R}_u\,\underline{u}^* + \underline{B}'(\underline{x}^*,t,\underline{z})\ \underline{p}^* = \underline{0} \tag{4.16}$$

$$2\underline{R}_v\underline{v}^* - \underline{G}'(\underline{T}'\underline{T})^{-1}\underline{T}'\,[\underline{F}'\underline{x}_L^* + \sum_{i=1}^{m} n_i\,(\underline{A}_i'(\underline{x}^*,t,\underline{z})\,\underline{p}_L^*$$

$$+\ 2\underline{Q}_i(t,\underline{z})\ \ \partial\underline{x}_L^*/\partial z_i)] = \underline{0} \tag{4.17}$$

Remark 4-2: The costate equation (4.13) contains first-order spatial derivatives of the costate and second-order spatial derivatives of the state. These latter terms disappear if derivativesof the state are not penalized in the performance index; in this case, if $\underline{B}$ is independent of $\underline{x}$, the costate equation belongs to the same class as the state equation (viz., quasilinear, first order).

Remark 4-3: The optimal distributed control (4.16) depends on the costate, whereas the boundary control (4.17) depends on those linear combinations of costate variables which are not constrained by the boundary conditions (4.15).

1.5 Steady State Analysis and Linearization of Time-Invariant Systems

Assume that the system equations (1.1)-(1.3) and performance index (1.7) are time-invariant, i.e., that $\underline{f}$, $\underline{g}$, j_D and j_L do not depend explicitly on t. All continuity assumptions of Section 1.1 and the well-posedness hypotheses of Section 1.2 are retained, so that the Minimum Principle of Section 1.3 applies.

In engineering applications, the behavior of such systems is usually analyzed by examining the static solution(s) and then linearizing the state about the equilibrium point(s) to reveal the dynamic evolution of (small) initial perturbations from equilibrium. Let $(\cdot)^s$ denote steady-state quantities and $(\cdot)^t$ denote perturbed quantities. The static equations are then defined to be

$$\underline{f}(\underline{x}^s(\underline{z}),\ \underline{x}_z^s(\underline{z}),\ \underline{u}^s(\underline{z}),\underline{z}) = \underline{0} \qquad \underline{z} \in Z \tag{5.1}$$

$$\text{BC:}\ \underline{g}(\underline{x}_B^s(\underline{z}),\ \underline{v}^s(\underline{z}),\ \underline{z}) = \underline{0} \qquad \underline{z} \in B \tag{5.2}$$

and the linearized perturbation equations are

$$\partial\underline{x}^t/\partial t = (\underline{f}_{\underline{x}})^s\ \underline{x}^t + \sum_{i=1}^{m} (\underline{f}_{\underline{x}_{z_i}})^s \underline{x}_{z_i}^t + (\underline{f}_{\underline{u}})^s\ \underline{u}^t$$

$$(t,\underline{z}) \in D \tag{5.3}$$

$$\text{IC:}\ \underline{x}^t(0,\underline{z}) = \underline{x}_o^t(\underline{z}) \qquad \underline{z} \in Z \tag{5.4}$$

$$\text{BC:}\ (\underline{g}_{\underline{x}})^s\ \underline{x}^t + (\underline{g}_{\underline{v}})^s\ \underline{v}^t = \underline{0} \quad (t,\underline{z}) \in L \tag{5.5}$$

The solution of the nonlinear boundary value problem (5.1), (5.2) is used in evaluation of the coefficients of the linear initial boundary value problem (5.3)-(5.5). Provided that the dynamic perturbations $\underline{u}^t$, $\underline{v}^t$ and the initial perturbation $\underline{x}_o^t$ are sufficiently small, and provided that the linearized system is sufficiently "stable" ($||\underline{x}^t(t,\underline{z})|| \leq c||\underline{x}_o^t(\underline{z})||$ for $t \in T$, $c \leq 1$, for example), it is reasonable to expect the solution

$$\underline{x}(t,\underline{z}) = \underline{x}^s(\underline{z}) + \varepsilon\underline{x}^t(t,\underline{z}) \tag{5.6}$$

for initial conditions

$$\underline{x}_o(\underline{z}) = \underline{x}^s(\underline{z}) + \varepsilon\underline{x}_o^t(\underline{z}) \tag{5.7}$$

and controls

$$\underline{u}(t,\underline{z}) = \underline{u}^s(\underline{z}) + \varepsilon\underline{u}^t(t,\underline{z}) \tag{5.8}$$

$$\underline{v}(t,\underline{z}) = \underline{v}^s(\underline{z}) + \varepsilon\underline{v}^t(t,\underline{z}) \tag{5.9}$$

for sufficiently small ε.

The question which naturally arises is: Can this philosophy be extended to optimal control problems? Certainly $\underline{u}^s$ and $\underline{v}^s$ may be taken as solutions of a static optimization problem, and $\underline{u}^t$ and $\underline{v}^t$ may be found by solving a linear optimal control problem; but which problem? The following theorem establishes that under certain conditions, the solution to the nonlinear dynamic optimal control problem (1.1)-(1.3), (1.7) may be approximated as the sum of the solution to a static boundary-value optimization problem and the solution to a linear-quadratic dynamic optimal control problem.

Theorem (Decomposition Theorem):

Assume $\underline{f}$, $\underline{g}$, j_f, j_D and j_L are time-invariant and satisfy the conditions of the Minimum Principle (Sections 1.1, 1.3). Let

$$\underline{u}^s = \underline{u}^s(\underline{z}) \qquad \underline{z} \in Z$$

$$\underline{v}^s = \underline{v}^s(\underline{z}) \qquad \underline{z} \in B$$

$$\underline{x}^s = \underline{x}^s(\underline{z}) \qquad \underline{z} \in Z$$

denote a solution (assumed to exist) of the static nonlinear boundary-value optimization problem:

$$\underline{f}(\underline{x},\underline{x}_z,\underline{u},\underline{z}) = \underline{0} \qquad \underline{z} \in Z \tag{5.10}$$

$$\text{BC: } \underline{g}(\underline{x},\underline{v},\underline{z}) = \underline{0} \qquad \underline{z} \in B \tag{5.11}$$

$$J^s(\underline{u},\underline{v}) = \int_B j_L(\underline{x}_B,\underline{v},\underline{z}) + \int_Z j_D(\underline{x},\underline{x}_z,\underline{u},\underline{z}) \; dZ \tag{5.12}$$

(i.e., $J^s(\underline{u}^s,\underline{v}^s) \leq J^s(\underline{u},\underline{v})$, $\forall$ $\underline{u} \in U$, $\underline{v} \in V$)

Furthermore, let

$$\underline{u}^t = \underline{u}^t(t,\underline{z}) \qquad (t,\underline{z}) \in D$$

$$\underline{v}^t = \underline{v}^t(t,\underline{z}) \qquad (t,\underline{z}) \in L$$

$$\underline{x}^t = \underline{x}^t(t,\underline{z}) \qquad (t,\underline{z}) \in D$$

denote a solution (assumed to exist) of the linear-quadratic dynamic optimal control problem:

$$\partial\underline{x}/\partial t = (\underline{f}_{\underline{x}})^s \underline{x} + \sum_{i=1}^{m} (\underline{f}_{\underline{x}_{z_i}})^s \underline{x}_{z_i} + (\underline{f}_{\underline{u}})^s \underline{u} \tag{5.13}$$

$$\text{IC: } \underline{x}(0,\underline{z}) = \frac{1}{\varepsilon} [\underline{x}_o(\underline{z}) - \underline{x}^s(\underline{z})] \tag{5.14}$$

$$\text{BC: } (\underline{g}_{\underline{x}})^s \underline{x}_L + (\underline{g}_{\underline{v}}^s) \underline{v} = \underline{0} \tag{5.15}$$

$$\begin{aligned} J^t(\underline{u},\underline{v}) = & \int_Z \{\tfrac{1}{2}\underline{x}_f'(j_{f_{\underline{xx}}})^s \underline{x}_f + \tfrac{1}{\varepsilon}\,\underline{x}_f'[(j_f)_{\underline{x}}^s - \underline{p}^s]\}\, dZ \\ & + \int_L \frac{1}{2} [\underline{x}_L' \; \underline{v}'] \begin{bmatrix} j_{L_{\underline{xx}}} & j_{L_{\underline{xv}}} \\ j_{L_{\underline{vx}}} & j_{L_{\underline{vv}}} \end{bmatrix}^s \begin{bmatrix} \underline{x}_L \\ \underline{v} \end{bmatrix} dL \\ & + \int_D \frac{1}{2} [\underline{x}' \; \underline{x}_z' \; \underline{u}'] \begin{bmatrix} h_{\underline{xx}} & h_{\underline{xx}_z} & h_{\underline{xu}} \\ h_{\underline{x}_z\underline{x}} & h_{\underline{x}_z\underline{x}_z} & h_{\underline{x}_z\underline{u}} \\ h_{\underline{ux}} & h_{\underline{ux}_z} & h_{\underline{uu}} \end{bmatrix}^s \begin{bmatrix} \underline{x} \\ \underline{x}_z \\ \underline{u} \end{bmatrix} dD \end{aligned} \tag{5.16}$$

where $\underline{x}_z = (\underline{x}_{z_1}, \underline{x}_{z_2}, \ldots, \underline{x}_{z_m})$ in (5.16) is an nm-vector and $(h_{\underline{xx}})^s$, for instance, is an $(n_x)\mathrm{x}(n_x)$ matrix function with (i,j)th element

$$[(h_{\underline{xx}})^s]_{ij} = (\partial^2 h/\partial x_i \partial x_j)(\underline{x}^s, \underline{x}_z^s, \underline{p}^s, \underline{u}^s, \underline{z})$$

for $h = j_D(\underline{x}, \underline{x}_z, \underline{u}, \underline{z}) + \underline{p}'\underline{f}(\underline{x}, \underline{x}_z, \underline{u}, \underline{z})$

where $\underline{f}$ and j_D are as in (5.10) and (5.12), and $\underline{p}^s$ is the costate for the static optimization problem.

Then, provided

$$||(\underline{x}_o(z) - \underline{x}^s(z))|| < C_1(\varepsilon); \quad \underline{z} \in Z \tag{5.17}$$

$$||((jf_{\underline{x}})^s - \underline{p}^s)|| < C_2(\varepsilon) \tag{5.18}$$

the solution to the time-invariant nonlinear optimal control problem (1.1)-(1.3), (1.7), $\underline{u}^*(t,\underline{z})$, $\underline{v}^*(t,\underline{z})$, $\underline{x}^*(t,\underline{z})$, with costate $\underline{p}^*(t,\underline{z})$ satisfying the minimum principle exists and is given by

$$\underline{u}^* = \underline{u}^s + \varepsilon\underline{u}^t + 0(\varepsilon) \tag{5.19}$$

$$\underline{v}^* = \underline{v}^s + \varepsilon\underline{v}^t + 0(\varepsilon) \tag{5.20}$$

$$\underline{x}^* = \underline{x}^s + \varepsilon\underline{x}^t + 0(\varepsilon) \tag{5.21}$$

$$\underline{p}^* = \underline{p}^s + \varepsilon\underline{p}^t + 0(\varepsilon) \tag{5.22}$$

Proof: See Appendix C.

Remark 5-1: The proof contains necessary conditions for the solution of the static problem (5.10)-(5.12).

Remark 5-2: The conditions (5.17)-(5.18) are required to insure that the initial state is sufficiently close to the equilibrium solution and that the desired terminal state is also sufficiently close to equilibrium; these conditions are essential for any (optimal or non-optimal) steady state dynamic perturbation analysis.

Remark 5-3: The admissable range of ε for approximate validity of the linearized analysis is likely to be larger for closed-loop systems of this type than for open-loop systems, because the state perturbation $\underline{x}^t$ will be (for "reasonable" j_f, j_L, j_D) more closely regulated by the optimal controls.

Remark 5-4: The theorem may be extended to systems with static constraint equations, e.g., first-order system representations of higher order partial differential equations (see Chapter 3).

Remark 5-5: In practice, the boundary value optimization problem (5.10)-(5.12) may be degenerate and/or relatively easy to solve. Furthermore, if the matrices of (5.16) happen to be block-diagonal (and provided the linear problem is well-posed), the linear theory of Chapter 2 may be used to derive explicit solutions to the linear problem (5.13)-(5.16). Hence, the decomposed problem may be far easier to solve in practice than the original nonlinear dynamic distributed two-point-boundary value problem resulting from the minimum principle.

Remark 5-6: This theorem may be viewed as an extension of the Quadratic Equivalence Theorem of G. B. Skelton [45] to distributed optimal control problems, although our motivation is different. Viewed in the sense of Skelton's results, a means of choosing quadratic weights for non-quadratic cost functionals has been derived.

Remark 5-7: From the standpoint of controller implementation, the static controls $\underline{u}^s$, $\underline{v}^s$ would usually be included in the basic design of the system (e.g., choice of geometry); while the dynamic controls would involve specialized dynamic control hardware with gains designed on the basis of the static analysis.

II. OPTIMAL CONTROL OF LINEAR FIRST-ORDER SYSTEMS

The results of Lions ([23], pp. 100-160) characterizing optimal controls for certain linear distributed systems with quadratic performance functionals are applied to a class of linear hyperbolic first-order systems studied by Hersh [20], Kreiss [24], and Rauch [36]. Both distributed and boundary controls are considered. Necessary conditions for the extension of this theory to a larger class of first-order systems are discussed with reference to current mathematical literature. A simple example is given in Appendix E.

2.1 Hyperbolic Systems on Half-Spaces

Results of Hersh [20], Kreiss [24], and Rauch [36] pertaining to existence of solutions for a class of initial boundary value problems involving linear hyperbolic first-order systems on a half-space are summarized.

The spatial domain is taken to be an open half-space:

$$Z = \{\underline{z} \in R^m \mid z_1 > 0\} \tag{1.1}$$

with boundary

$$B = \{\underline{z} \in R^m \mid z_1 = 0\} \tag{1.2}$$

The time interval is $T = [0,t_f]$, or $T_\infty = R$, for certain of the following results. The space-time domain is $D = T \times Z$, with lateral boundary $L = T \times B$ (resp. D_∞, L_∞).

The main results of these authors are concerned with existence and uniqueness of solutions to the initial boundary value problem:

$$(\partial/\partial t - A)\underline{x} = \underline{f}(t,\underline{z}) \qquad (t,\underline{z}) \in D \tag{1.3}$$

$$\text{IC: } \underline{x}(0,\underline{z}) = \underline{x}_o(\underline{z}) \qquad \underline{z} \in Z \tag{1.4}$$

$$\text{BC: } \underline{T}(t,\underline{z})\underline{x}(t,\underline{z}) = \underline{g}(t,\underline{z}) \qquad (t,\underline{z}) \in L \tag{1.5}$$

where the linear differential operator A is defined by[1]

$$A(\cdot) = \sum_{i=1}^{m} \underline{A}_i(t,\underline{z})\partial(\cdot)/\partial z_i + \underline{K}(t,\underline{z})(\cdot) \tag{1.6}$$

with smoothness assumptions on the coefficients:

$$\underline{A}_i \in C_S^\infty(D_\infty;\ R^{n_x^2}) \qquad i = 1,2,\ldots m$$

$$\underline{K} \in C_S^\infty(D_\infty;\ R^{n_x^2})$$

$$\underline{T} \in C_E^\infty(L_\infty;\ R^{n_g n_x})$$

where for S, E compact subsets of D_∞, L_∞ respectively,

$$C_S^\infty(D_\infty;\ R^{n_x^2}) = \{\underline{M} \in C^\infty(D_\infty;\ R^{n_x^2}) \mid \underline{M}(t,\underline{z}) \equiv \text{constant for } (t,\underline{z}) \in D_\infty - S\}$$

$$C_E^\infty(L_\infty;\ R^{n_g n_x}) = \{\underline{N} \in C^\infty(L_\infty;\ R^{n_x n_g}) \mid \underline{N}(t,\underline{z}) \equiv \text{constant for } (t,\underline{z}) \in L_\infty - E\}$$

(1) See Remark 2-1.

Additional assumptions are:

Assumption 1-1: (Strict hyperbolicity) The equation

$$|\lambda_o \underline{I} - \sum_{i=1}^{m} \lambda_i \underline{A}_i(t,\underline{z})| = 0$$

has n_x distinct real roots for all $[\lambda_1 \ldots \lambda_m] \varepsilon R^m - \{\underline{0}\}$ and for all $(t,\underline{z}) \ \varepsilon \ \overline{D}_\infty$.

Assumption 1-2: (Noncharacteristic boundary)

$$|\underline{A}_1(t,\underline{z})| \neq 0$$

for all $(t,\underline{z}) \ \varepsilon \ L_\infty$.

Assumption 1-3: (Determinate boundary conditions) The matrix $\underline{A}_1$ is assumed to have Jordan form:

$$\underline{A}_1 = \begin{vmatrix} \underline{A}_1^- & \underline{0} \\ \underline{0} & \underline{A}_1^+ \end{vmatrix} \qquad \begin{matrix} \underline{A}_1^- = \text{diag } (a_1 \ldots a_r) \\ \underline{A}_1^+ = \text{diag } (a_{r+1} \ldots a_{n_x}) \end{matrix}$$

with $a_i < 0$, $i = 1,2\ldots,r$; $a_i > 0$, $i = r+1,\ldots,n_x$; this is no restriction. Then for all $(t,\underline{z}) \ \varepsilon \ L_\infty$, $\underline{T}$ is assumed to be of full rank, and

$$n_g = \text{rank } \underline{T} = r$$

Hersh [20] gives necessary and sufficient conditions for the constant-coefficient problem with $\underline{K} \equiv \underline{0}$ and $\underline{f} = \underline{0}$ to have unique smooth solutions (well-posedness in the sense of Hadamard). Results are obtained by LaPlace transformation in t and Fourier transformation in $\underline{z}$, and are merely summarized here. Consider the matrix function

$$\underline{M}(s,\underline{k}) = \underline{A}_1^{-1}[s\underline{I} - i \sum_{j=2}^{m} k_j \underline{A}_j] \qquad (1.7)$$

$$\underline{k} = (k_2,\ldots k_m) \ \varepsilon \ R^{m-1} \qquad s \ \varepsilon \ C$$

with associated eigenvalue problem (given s, $\underline{k}$):

$$\underline{e}'(s,\underline{k})[\underline{M}(s,\underline{k}) - \lambda(s,\underline{k})\underline{I}] = \underline{0} \qquad (1.8)$$

There will be n_x (complex) eigenvectors $\underline{e}^j(s,\underline{k})$ and corresponding eigenvalues $\lambda^j(s,\underline{k})$, which will be distinct by virtue of Assumption 1-1 (Hersh more generally admits multiple roots). It is then shown that

Theorem 1-1: (Hersh [20]) Consider the constant-coefficient version of (1.3)-(1.6) with $\underline{f} = 0$, $\underline{K} = 0$. Assume(2) $\underline{x}_o \ \varepsilon \ C^\infty(Z, R^{n_x})$, $\underline{g} \ \varepsilon \ C^\infty(L, R^{n_g})$. Then a necessary and sufficient condition for well-posedness in the sense of Hadamard is (under Assumptions 1-1 to 1-3 above): For all $\underline{k} \ \varepsilon \ R^{m-1}$, for all s with Re(s) > 0, the rows of $\underline{T}$ are linearly independent of all eigenvectors $\underline{e}^{j'}(s,\underline{k})$ corresponding to eigenvalues $\lambda^j(s,\underline{k})$ with $\mathrm{Re}(\lambda^j(s,\underline{k})) > 0$.

(2) Hersh is unclear reading continuity conditions imposed on $\underline{x}_o$, $\underline{g}$; existence of the inverse Fourier-Laplace transforms appears to be the basic requirement - hence the continuity assumptions above may be relaxed. Rauch ([36],p.4) adds a consistency condition that $\underline{x}_o$ and $\underline{g}$ vanish in a neighborhood of {t=0, z_1=0}, but this condition was not originally imposed by Hersh.

Hersh gives a counterexample due to Agmon which demonstrates that Assumption 1-3 alone cannot guarantee continuity of solutions in the initial data. His condition, furthermore, does not represent a straightforward extension of the much-studied special cases $n_x = 1,2$, which are misleading. Hersh also studies continuity of solutions in Sobolev spaces, but his results are superceded by the more recent work of Rauch (following).

Kreiss [24] attacks the same class of initial boundary value problem, but with variable coefficients meeting the smoothness conditions following (1.6), and $\underline{K} \neq \underline{0}$, $\underline{f} \neq 0$. Using symmetrization theory of Friedrichs and Lax [15], and canonical transformations of the equations, he derives an L^2 bound on the solution(s) of this problem in terms of initial and boundary data. His main theorems may be stated as follows:(3)

<u>Theorem 1-2 (Kreiss [24]):</u> Considering the variable-coefficient problem, assume that for each boundary point $(t,\underline{z}) \in L$, the constant coefficient problem obtained by freezing the coefficients of $\underline{A}_i$ $(i = 1,2...m)$ and $\underline{T}$, and setting $\underline{K} = 0$, $\underline{f} = \underline{0}$, $\underset{\sim}{\underline{x}}_o = \underline{0}$, satisfies Theorem 1-1 for all $n_g.n_x$ matrices $\underline{\tilde{T}}$ in a neighborhood of $\underline{T}(t,\underline{z})$. Then (for $\underline{K}$, $\underline{f}$ nonzero, $\underline{x}_o = 0$), the following estimate holds:(4)

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(3) This restatement is due to Rauch [36].

(4) Where $(\bar{\cdot})$ denotes complex conjugation and c_{t_f} is a constant depending only on t_f.

$$\int_L \underline{\bar{x}}_L' \, \underline{x}_L \, dL + \int_D \underline{\bar{x}}' \, \underline{x} \, dD \leq c_{t_f} \left[\int_L \underline{\bar{g}}' \, \underline{g} \, dL + \int_D \underline{\bar{f}}' \, \underline{f} \, dD\right] \qquad (1.9)$$

Kreiss also considers additional problems of existence and uniqueness. Of primary interest here, however, are the generalizations of his results by Rauch, which require the conditions of Theorems 1-1 and 1-2.

In order to present these results, it is necessary to define weighted Sobolev spaces of integral order and the notion of a strong solution. For an open set $S \subset R^n$, $C^\infty_{(o)}(\bar{S};R^q)$ denotes the set of real q-vector valued $C^\infty_o(R^n)$ functions, restricted to $\bar{S}$.

Definition 1-1:[5] For $S \subset R^{m+1}$ of the form (time interval) × (spatial domain) and nonnegative integer s, $H^s_\alpha(S;R^q)$ is the completion of $C^\infty_{(o)}(\bar{S};R^q)$ in the norm

$$||\underline{f}||_{H^s_\alpha(S;R^q)} = \sum_{k<s} \int_S ||D^k\underline{f}||^2 \, e^{-2\alpha t} \, dS \qquad (1.10)^{(6)}$$

This is termed the weighted s-th order Sobolev space of q-vector valued functions on S.

Definition 1-2: A strong solution of the IBVP (1.3)-(1.5) is a function $\underline{x} \in L^2(D;R^{n_x})$ such that there exist a function $\underline{x}_L \in L^2(L;R^{n_x})$ and a sequence of functions $\{\underline{x}^n\}$, $\underline{x}^n \in C^\infty_{(o)}(\bar{D})$, satisfying:

---

(5) The argument R^q will be omitted when it is obvious from the context.

(6) $D^k\underline{f}$ is conventional shorthand for the k^{th} degree partial derivatives of $\underline{f}$; if the time interval is degenerate to a point, partial derivatives with respect to t are not included.

$$\lim_{n\to\infty} ||\underline{x}^n - \underline{x}||_{L^2(D)} = 0$$

$$\lim_{n\to\infty} ||\underline{x}_L^n - \underline{x}_L||_{L^2(L)} = 0$$

$$\lim_{n\to\infty} ||(\partial/\partial t - A)\underline{x}^n - \underline{f}||_{L^2(D)} = 0$$

$$\lim_{n\to\infty} ||\underline{x}^n(0,\underline{z}) - \underline{x}_o(\underline{z})||_{L^2(Z)} = 0$$

with $\underline{T}\ \underline{x}_L = \underline{g}$. The function $\underline{x}_L$ is termed the <u>strong boundary value of $\underline{x}$ on L.</u>

With these definitions, the main results of Rauch may be stated as follows:(7)

<u>Theorem 1-3 (Rauch [36]):</u> For any $t_f > 0$, $\underline{f} \in L^2(D;R^{n_x})$, $\underline{g} \in L^2(L;R^{n_g})$, $\underline{x}_o \in L^2(Z;R^{n_x})$, problem (1.3)-(.5) has a unique strong solution $\underline{x}$. For s=0, the following estimate applies to $\underline{x}$ and its (strong) boundary value $\underline{x}_L$:

$$||\underline{x}(t)||_{H^s_\alpha(Z)} + \sqrt{\alpha}||\underline{x}||_{H^s_\alpha(D)} + ||\underline{x}_L||_{H^s_\alpha(L)} \leq$$

$$c_s[||\underline{x}_o||_{H^s_\alpha(Z)} + (\sqrt{\alpha})^{-1}||\underline{f}||_{H^s_\alpha(D)} + ||\underline{g}||_{H^s_\alpha(L)}] \tag{1.11}$$

with c_s a constant, independent of t, $\underline{x}_o$, $\underline{g}$, $\underline{f}$ or α, and α sufficiently large.

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(7) Assumptions 1-1 to 1-3, and conditions of Theorems 1-1 and 1-2 in force.

Theorem 1-4 (Rauch [36]): Suppose there exist sequences $\{\underline{f}^n\}$, $\{\underline{g}^n\}$, $\{\underline{x}_o{}^n\}$ such that

$$\underline{f}^n \in C^{\infty}_{(o)}(\bar{D}_{\infty};R^{n_x}), \qquad \underline{f}^n \equiv \underline{0} \quad \text{for} \quad t \leq 0 \tag{1.12}$$

$$\underline{g}^n \in C^{\infty}_{o}(L_{\infty};R^{n_g}), \qquad \underline{g}^n \equiv \underline{0} \quad \text{for} \quad t \leq 0 \tag{1.13}$$

$$\underline{x}_o{}^n \in C^{\infty}_{o}(Z;R^{n_x}) \tag{1.14}$$

and that in the limit $n \to \infty$,

$$\underline{f}^n \to \underline{f} \quad \underline{\text{in}} \quad H^s(D;R^{n_x}) \tag{1.15}$$

$$\underline{g}^n \to \underline{g} \quad \underline{\text{in}} \quad H^s(L;R^{n_g}) \tag{1.16}$$

$$\underline{x}_o^n \to \underline{x}_o \quad \underline{\text{in}} \quad H^s(Z;R^{n_x}) \tag{1.17}$$

Then Theorem 1-3 holds with $\underline{x} \in H^s(D,R^{n_x})$ and (1.11) holds for this value of s with α sufficiently large.

The equations (1.12)-(1.14) represent compatibility conditions on the data at $t=0$, $z_1=0$. These results guarantee unique smooth strong solutions for smooth initial data, boundary data, and forcing functions.

Remark 1-1: For the following optimal control problem, Theorem 1-4 with $s=1$ is required. To summarize, for

$$\underline{x}_o \in H^1(Z;R^{n_x}) \tag{1.18}$$

$$\underline{f} \in H^1(D;R^{n_x}) \tag{1.19}$$

$$\underline{g} \in H^1(L;R^{n_g}) \tag{1.20}$$

satisfying (1.12)-(1.17), there exists a unique strong solution $\underline{x} \in H^1(D;R^{n_x})$ with strong boundary value $\underline{x}_L \in H^1(L;R^{n_x})$, and the mappings

$$(\underline{x}_o,\underline{f},\underline{g}) \mapsto \underline{x} \tag{1.21}$$

$$(\underline{x}_o,\underline{f},\underline{g}) \mapsto \underline{x}_L \tag{1.22}$$

are continuous from $H^1(Z;R^{n_x}) \times H^1(D;R^{n_x}) \times H^1(L;R^{n_g})$ to $H^1(D;R^{n_x})$ and $H^1(L;R^{n_x})$, respectively, in the weighted first-order Sobolev norm, for α sufficiently large, by (1.11).

Remark 1-2: Of crucial importance is the fact (Rauch, [36], p.11) that when Theorem 1-2 applies to the state equations, it is automatically guaranteed to apply to the adjoint system equations. This in turn permits the application of Theorems (1-3) and (1-4) to the adjoint system, as developed in the following section.

Remark 1-3: Rauch, [36], p.3, notes that certain more general spatial domains may be cast in this form by use of local coordinate transformations.

2.2 An Optimal Control Problem

A rigorous formulation of optimal control problem involving first-order hyperbolic systems on half-spaces with distributed and boundary controls is given. The problem treated here is perhaps the most "conventional" case; there are many variations on the theme.

<u>Notation:</u> The main function spaces used herein are defined as follows. Let S denote a simply-connected subset of R^k.

$C^\infty(S;R^n)$ = space of infinitely differentiable functions on S taking values in R^n.

$L^2(S;R^n)$ = space of square-integrable functions on S taking values in R^n; this is a Hilbert space with inner product

$$\langle \underline{x},\underline{y}\rangle_{L^2(S;R^n)} = \int_S \underline{x}'\underline{y}\, dS \tag{2.1}$$

$H^1(S,R^n) = \{\underline{x} \in L^2(S,R^n) \mid \partial\underline{x}/\partial z_i \in L^2(S,R^n),\ i=1,\dots,k\}$ which is also a Hilbert space with inner product[8]

$$\langle \underline{x},\underline{y}\rangle_{H'(S;R^n)} = \int_S [\underline{x}'\underline{y} + \sum_{i=1}^{k} (\partial\underline{x}'/\partial z_i)(\partial\underline{y}/\partial z_i)]\, dS \tag{2.2}$$

The second argument (R^n) will be dropped when n is obvious from the context.

---

(8) z_i may represent "time" in (2.2). If $\mathbb{C}^n$ replaces R^n, the first function $\underline{x}$ is also conjugated in (2.1)-(2.2).

Spatiotemporal Domain: Z is a half-space in R^m (1.1); B is the plane $z_1 = 0$ (1.2); $T = [0,t_f]$ is the time domain ($t_f < \infty$).

System Constraints and Assumptions: The evolution of the state $\underline{x}$ is governed by the IBVP:

$$\partial\underline{x}/\partial t = \sum_{i=1}^{m} \underline{A}_i(t,\underline{z})\ \partial\underline{x}/\partial z_i + \underline{K}(t,\underline{z})\underline{x} + \underline{B}(t,\underline{z})\underline{u} + \underline{f}(t,\underline{z})$$

$$(t,\underline{z}) \in D \qquad (2.3)$$

$$\text{IC: } \underline{x}(0,\underline{z}) = \underline{x}_o(\underline{z}) \qquad (\underline{z}) \in Z \qquad (2.4)$$

$$\text{BC: } \underline{T}(t,\underline{z})\underline{x}_L = \underline{G}(t,\underline{z})\underline{v} \qquad (t,\underline{z}) \in L \qquad (2.5)$$

The coefficients $\underline{A}_i$, $\underline{K}$, $\underline{B}$, $\underline{T}$, and $\underline{G}$ are restrictions to $\bar{D}$ and $\bar{L}$, respectively, of the functions

$$\tilde{\underline{A}}_i \in C_S^\infty(D_\infty;\ R^{n_x^2}) \qquad (2.6)$$

$$\tilde{\underline{K}} \in C_S^\infty(D_\infty, R^{n_x^2}) \qquad (2.7)$$

$$\tilde{\underline{B}} \in C_S^\infty(D_\infty, R^{n_u n_x}) \qquad (2.8)$$

$$\tilde{\underline{T}} \in C_E^\infty(L_\infty, R^{n_g n_x}) \qquad (2.9)$$

$$\tilde{\underline{G}} \in C_E^\infty(L_\infty, R^{n_g n_v}) \qquad (2.10)$$

where the spaces C_S^∞, C_E^∞ are defined following (1.6). The distributed control $\underline{u}$, the autonomous force $\underline{f}$, the initial state $\underline{x}_o$ and the boundary control $\underline{v}$ are elements of the following spaces

$$\underline{u} \in U(D,R^{n_u}) = \{\underline{u} \in H^1(D,R^{n_u}) \mid \exists \{\underline{u}^n\},\ \underline{u}^n \in C^\infty_{(o)}(\overline{D}_\infty,R^{n_u}),$$

$$\underline{u}^n \equiv 0,\ t \leq 0 \ni \lim_{n\to\infty} \underline{u}^n = \underline{u}\} \tag{2.11}$$

$$\underline{f} \in X(D,R^{n_x}) = \{\underline{f} \in H^1(D,R^{n_x}) \mid \ (1.12),\ (1.15) \text{ for } s = 1\} \tag{2.12}$$

$$\underline{x}_o \in X_o(Z,R^{n_x}) = \{\underline{x}_o \in H^1(Z,R^{n_x}) \mid \ (1.14),\ (1.17) \text{ for } s = 1\} \tag{2.13}$$

$$\underline{v} \in V(L,R^{n_v}) = \{\underline{v} \in H^1(L,R^{n_v}) \mid \exists \{\underline{v}^n\},\ \underline{v}^n \in C^\infty_o(L_\infty,R^{n_v}),$$

$$\underline{v}^n \equiv 0,\ t \leq 0 \ni \lim_{n\to\infty} \underline{v}^n = \underline{v}\} \tag{2.14}$$

Referring to Remark 1-1, the following existence theorem is obtained as a consequence of Theorem 1-4.

<u>Theorem 2-1:</u> If the IBVP (2.3)-(2.10) satisfies Assumptions (1-1)-(1-3) and Kreiss' condition (Theorem 1-2), then given any $\underline{u} \in U$, $\underline{f} \in X$, $\underline{x}_o \in X_o$ and $\underline{v} \in V$ there exists a unique strong solution $\underline{x} \in H^1(D;R^{n_x})$ with unique strong boundary value $\underline{x}_L \in H^1(L;R^{n_x})$, and for α sufficiently large

$$||\underline{x}(t)||_{H^1_\alpha(Z)} + \sqrt{\alpha}||\underline{x}||_{H^1_\alpha(D)} + ||\underline{x}_L||_{H^1_\alpha(L)} \leq c\ [||\underline{x}_o||_{H^1_\alpha(Z)}$$

$$+ (\sqrt{\alpha})^{-1}(||\underline{u}||_{H^1_\alpha(D)} + ||\underline{f}||_{H^1_\alpha(D)}) + ||\underline{v}||_{H^1_\alpha(L)}] \tag{2.15}$$

where c is a constant and $t \in T$

In the control problem, $\underline{x}_o \in X_o$ and $\underline{f} \in X$ are assumed to be prespecified. Given this existence theorem, it is obvious that $\underline{x}$ and $\underline{x}_L$ are affine linear functionals of the controls $\underline{u} \in U$, $\underline{v} \in V$; that dependence is made explicit by the notations

$$\underline{x} = \underline{x}(\underline{u},\underline{v})\,;\; \underline{x}_L = \underline{x}_L(\underline{u},\underline{v}) \tag{2.16}$$

which are convenient in the following sections. Thus

$$\underline{x}(\underline{u},\underline{v}) - \underline{x}(\underline{0},\underline{0}) \in \mathcal{L}(U(D;R^{n_u}) \times V(L;R^{n_v});\; H^1(D;R^{n_x})) \tag{2.17}$$

$$\underline{x}_L(\underline{u},\underline{v}) - \underline{x}_L(\underline{0},\underline{0}) \in \mathcal{L}(U(D;R^{n_u}) \times V(L;R^{n_v});\; H^1(L;R^{n_x})) \tag{2.18}$$

<u>Responses:</u> The responses are linear functionals of the state variables, of the form

$$\underline{y}(t,\underline{z}) = \underline{C}(t,\underline{z})\; \underline{x}\;(t,\underline{z}) \qquad (t,\underline{z}) \in D \tag{2.19}$$

where $\underline{C}$ is assumed to be the restriction to $\overline{D}$ of a matrix-valued function

$$\tilde{\underline{C}} \in C_S^{\infty}(D_{\infty}, R^{n_y n_x}) \tag{2.20}$$

Considered as a linear operator,

$$\underline{C} \; \varepsilon \; \mathcal{L}(H^1(D;R^{n_x});\; H^1(D;R^{n_y}))$$

And considered as a function of the controls,

$$\underline{y}(\underline{u},\underline{v}) - \underline{y}(\underline{0},\underline{0}) = \underline{y}(\underline{x}(\underline{u},\underline{v})) - \underline{y}(\underline{x}(\underline{0},\underline{0}))$$

$$\varepsilon \; \mathcal{L}(U(D;R^{n_u}) \times V(L,R^{n_v});\; H^1(D;R^{n_y}))$$

The desired response is specified as

$$\underline{d} \; \varepsilon \; Y(D;R^{n_y}) = \{\underline{d} \; \varepsilon \; H'(D;R^{n_y}) \mid \exists \{\underline{d}^n\},$$

$$\underline{d}^n \; \varepsilon \; C^{\infty}_{(o)}(D_{\infty},R^{n_y}),\; \underline{d}^n \equiv 0,\; t \leq 0$$

$$\ni \lim_{n\to\infty} \underline{d}^n = \underline{d}\} \tag{2.21}$$

Performance Index: A scalar valued measure of performance,

$$J:\; U \times V \to R \tag{2.22}$$

is specified indirectly (i.e., via the response mapping $\underline{y}(\underline{u},\underline{v})$) by the formula

$$J(\underline{u},\underline{v}) = \int_D \{(\underline{y}(\underline{u},\underline{v}) - \underline{d})'\, \underline{Q}(\underline{y}(\underline{u},\underline{v}) - \underline{d}) + \underline{u}'\underline{R}_u\, \underline{u}\} \;dD$$

$$+ \int_L \underline{v}'\underline{R}_v\, \underline{v} \; dL \tag{2.23}$$

where $\underline{Q}$, $\underline{R}_u$, $\underline{R}_v$ are restrictions to D, L, respectively, of

$$\underset{\sim}{Q} \in C_S^\infty(D_\infty; R^{n_y^2}) \tag{2.24}$$

$$\underset{\sim}{R}_u \in C_S^\infty(D_\infty; R^{n_u^2}) \tag{2.25}$$

$$\underset{\sim}{R}_v \in C_E^\infty(L_\infty; R^{n_v^2}) \tag{2.26}$$

The matrix valued functions $\underline{Q}$, $\underline{R}_u$, $\underline{R}_v$ are all assumed to be positive definite symmetric for all $(t,\underline{z}) \in D$, $(t,\underline{z}) \in L$, respectively. Thus J is non-negative valued.

<u>Control Problem:</u> The space of admissable controls is

$$W = U \times V \tag{2.27}$$

The control problem is to find the optimal control pair(s) $\underline{w}^* = (\underline{u}^*,\underline{v}^*) \in W$ such that

$$J(\underline{u}^*,\underline{v}^*) = \operatorname*{Inf}_{(\underline{u},\underline{v})\in W} J(\underline{u},\underline{v}) \tag{2.28}$$

In preparation for the solution of problem (2.28) it is appropriate to digress at this point on an independent study of the adjoint system to (2.3)-(2.5), in order to obtain an existence theorem analogous to Theorem 2-1.

<u>Adjoint System:</u> Let $\underline{x}$ (with boundary value $\underline{x}_L$) denote the unique strong solution of Theorem 2-1 to the homogeneous IBVP (2.3)-(2.5), viz., $\underline{u} = \underline{0}$, $\underline{f} = \underline{0}$, $\underline{x}_o = \underline{0}$, $\underline{v} = \underline{0}$. Thus there exists a sequence $\{\underline{x}^n\}$, $\underline{x}^n \in C_{(o)}^\infty(\overline{D}, R^{n_x})$ such that

$$\lim_{n\to\infty} \int_D (\underline{x}^n - \underline{x})'(\underline{x}^n - \underline{x})\, dD = 0 \tag{2.29}$$

$$\lim_{n\to\infty} \int_Z \underline{x}^{n'}(0,\underline{z})\underline{x}^n(0,\underline{z})\, dZ = 0 \tag{2.30}$$

$$\lim_{n\to\infty} \int_L (\underline{x}_L^n - \underline{x}_L)'(\underline{x}_L^n - \underline{x}_L)\, dL = 0 \tag{2.31}$$

(see Definition 1-2), with $\underline{T}\,\underline{x}_L = \underline{0}$

Defining the operator

$$M = (\partial/\partial t - A) \tag{2.32}$$

with initial conditions (2.4) and boundary conditions (2.5),

$$\lim_{n\to\infty} \int_D (M\,\underline{x}^n)'(M\,\underline{x}^n)\, dD = 0 \tag{2.33}$$

Given a sequence $\underline{p}^n \in C^\infty_{(o)}(\bar{D};R^{n_x})$ converging as in (2.29)-(2.31), define $M^{(*)}$, the adjoint of M, to be the operator such that

$$\lim_{n\to\infty} \int_D (M\underline{x}^n)'\,\underline{p}^n\, dD = \lim_{n\to\infty} \int_D \underline{x}^{n'}(M^{(*)}\underline{p}^n)\, dD \tag{2.34}$$

Now[9]

---

(9) (2.35) is a consequence of the Divergence Theorem (Chapter 1, (3.26)). The definition (2.34) of $M^{(*)}$ is required because this theorem cannot be applied directly to functions $\underline{x}$, $\underline{p}$ in $H^1(D;R^{n_x})$. Note also that the functions $\underline{x}^n$ are only required to satisfy the initial and boundary conditions in the limit $n \to \infty$.

$$\int_D (\underline{M}\underline{x}^n)'\underline{p}^n \, dD = \int_D (\partial\underline{x}^n/\partial t - \sum_{i=1}^m \underline{A}_i \; \partial\underline{x}^n/\partial z_i - \underline{K}\underline{x}^n)'\underline{p}^n \, dD$$

$$= \int_D \underline{x}^{n'}(-\partial\underline{p}^n/\partial t + \sum_{i=1}^m \partial(\underline{A}_i'\underline{p}^n)/\partial z_i - \underline{K}'\underline{p}^n) \, dD$$

$$+ \int_Z [\underline{x}^{n'}(t_f,\underline{z})\underline{p}^n(t_f,\underline{z}) - \underline{x}^{n'}(0,\underline{z})\underline{p}^n(0,\underline{z})] \, dZ$$

$$+ \int_L \underline{x}_L^{n'}\underline{A}_1'\,\underline{p}_L^n \, dL \qquad (2.35)$$

The operator $M^{(*)}$ is thus identified by[10]

$$M^{(*)}\underline{p} = -\partial\underline{p}/\partial t + \sum_{i=1}^m \underline{A}_i' \; \partial\underline{p}/\partial z_i + (\sum_{i=1}^m \partial\underline{A}_i/\partial z_i - \underline{K}')\,\underline{p} \qquad (2.36)$$

with <u>final</u> conditions

$$\underline{p}(t_f) = \underline{0} \qquad (2.37)$$

The adjoint boundary conditions require somewhat closer scrutiny. Let $\underline{T}$ be partitioned in accordance with Assumption 1-3, viz.

$$\underline{T} = [\underline{T}^-, \underline{T}^+] \qquad (2.38)$$

with $\underline{T}^-$ invertible; partition $\underline{x}_L$ and $\underline{p}_L$ similarly. Then as $n \to \infty$, since $\underline{A}_1 = \underline{A}_1'$,

$$\int_L \underline{x}_L'\,\underline{A}_1'\,\underline{p}_L \, dL = \int_L (\underline{x}_L^{-'}\underline{A}_1^-\,\underline{p}_L^- + \underline{x}_L^{+'}\,\underline{A}_1^+\,\underline{p}_L^+) \, dL \qquad (2.39)$$

---

(10) There is a sign error in the last term of (2.36) in the original version of Rauch's paper.

By (2.5),

$$\underline{T}^{-}\ \underline{x}_L^{-} + \underline{T}^{+}\ \underline{x}_L^{+} = \underline{0} \tag{2.40}$$

It is readily verified that the relation

$$\underline{p}_L^{+} = (\underline{A}_1^{+})^{-1}\ \underline{T}^{+'}\ [\underline{T}^{-'}]^{-1}\ \underline{A}_1^{-}\ \underline{p}_L^{-} \tag{2.41}$$

causes (2.39) to vanish. Thus the adjoint boundary condition associated with $M^{(*)}$ is

$$\underline{T}^{(*)}\ \underline{p}_L = \underline{0} \tag{2.42}$$

with

$$\underline{T}^{(*)} = [\underline{T}^{+'}\ (\underline{T}^{-'})^{-1}\ \underline{A}_1^{-},\ \underline{A}_1^{+}] \tag{2.43}$$

In summary, the adjoint system may be written:

$$-\partial p/\partial t = -\sum_{i=1}^{m} \underline{A}_i'\ \partial \underline{p}/\partial z_i + [\underline{K}' - \sum_{i=1}^{m} \partial \underline{A}_i'/\partial z_i]\underline{p} + \underline{h} \tag{2.44}$$

$$\text{FC: } \underline{p}(t_f,\underline{z}) = \underline{0} \tag{2.45}$$

$$\text{BC: } \underline{T}^{(*)}\ \underline{p}_L = \underline{0} \tag{2.46}$$

with $\underline{h} = 0$.

Recalling Remark 1-2, Theorem 1-2 is guaranteed and hence Theorems 1-3 and 1-4 may be applied immediately to the nonhomogeneous problem ($\underline{h} \neq 0$), with the change of variable $\tau = t_f - t$. These results are summarized as follows:

Theorem 2-2: Let the conditions of Theorem 2-1 be satisfied. Then given any $\underline{h} \in X$, there exists a unique strong solution $\underline{p} \in H^1(D;R^{n_x})$ with unique strong boundary value $p_L \in H^1(L;R^{n_x})$ for the adjoint system (2.44)-(2.46), and for α sufficiently large,

$$||\underline{p}(\tau)||_{H^1_\alpha(Z)} + \sqrt{\alpha}||\underline{p}||_{H^1_\alpha(D)} + ||\underline{p}_L||_{H^1_\alpha(L)}$$

$$\leq c\ (\sqrt{\alpha})^{-1}\ ||\underline{h}||_{H^1_\alpha(D)} \tag{2.47}$$

where c is a constant and $\tau \in T$.

Remark 2-1: The operation of matrix multiplication (matrices are underlined) is implied by the notation; for each matrix valued function, there is a corresponding linear operator.

Remark 2-2: The performance functional (2.23) actually includes symmetric positive semidefinite quadratic forms in the state, of the form $\underline{x}'\ \underline{\tilde{Q}}\ \underline{x}$ $\underline{\tilde{Q}} \geq 0$; pick $\underline{C}$ = row space $[\underline{\tilde{Q}}^{1/2}]$ and $\underline{Q} = \underline{I}$.

2.3 Necessary and Sufficient Conditions of Optimality

Lions' results [26] are shown to guarantee existence and uniqueness of optimal controls $\underline{u}^*$, $\underline{v}^*$ satisfying (2.28); necessary and sufficient conditions for an infimum are given by a variational inequality, which is shown to be equivalent to the solution of a two-point boundary value problem involving partial differential equations.

Preliminaries: For the sake of convenience, a transformation of the response-weighting term of the performance index (2.23) is made. The matrix-valued function $\tilde{\underline{Q}}$ of (2.24) possesses a unique positive-definite (symmetric) square root $\tilde{\underline{Q}}^{1/2} \varepsilon C_S^{\infty}(D_{\infty}, R^{n_y^2})$, which uniquely defines $\underline{Q}^{1/2}$. The cost functional with

$$\underline{Q}^{1/2}\ \underline{C} \rightarrow \underline{C} \tag{3.1}$$

$$\underline{Q}^{1/2}\ \underline{d} \rightarrow \underline{d} \tag{3.2}$$

$$\underline{Q} \rightarrow \underline{I} \tag{3.3}$$

is precisely equivalent to (2.23) and satisfies the conditions (2.20) and (2.21); hence in the sequel there is no loss of generality in taking $\underline{Q} = \underline{I}$ (the identity matrix) in (2.23) and reversing (3.1)-(3.3) at the end.

With this substitution, (2.23) may be expressed as

$$\begin{aligned} J(\underline{u},\underline{v}) = &< \underline{C}\ \underline{x}(\underline{u},\underline{v})-\underline{d},\ \underline{C}\ \underline{x}(\underline{u},\underline{v})-\underline{d} >_{L^2(D;R^{n_y})} \\ &+ <\underline{R}_u\ \underline{u},\ \underline{u}>_{L^2(D;R^{n_u})} \\ &+ <\underline{R}_v\ \underline{v},\ \underline{v}>_{L^2(L;R^{n_v})} \end{aligned} \tag{3.4}$$

This functional may be written as the difference of a bilinear quadratic form $\mathfrak{Q}$ and a linear functional $\mathcal{L}$ by noting that

$$\underline{C}\ \underline{x}\ (\underline{u},\underline{v})-\underline{d} = \underline{C}[\underline{x}(\underline{u},\underline{v})-\underline{x}(\underline{0},\underline{0})] + [\underline{C}\ \underline{x}(\underline{0},\underline{0})-\underline{d}] \tag{3.5}$$

and defining

$$\mathcal{Q}(\underline{w},\tilde{\underline{w}}) = \langle \underline{C}[\underline{x}(\underline{w})-\underline{x}(\underline{0})],\ \underline{C}[\underline{x}(\tilde{\underline{w}})-\underline{x}(\underline{0})]\rangle_{L^2(D;R^{n_y})}$$

$$+ \langle \underline{R}_u\ \underline{u},\ \tilde{\underline{u}}\rangle_{L^2(D;R^{n_u})} + \langle \underline{R}_v\ \underline{v},\tilde{\underline{v}}\rangle_{L^2(L;R^{n_v})} \tag{3.6}$$

$$\mathcal{L}(\tilde{\underline{w}}) = \langle \underline{d}-\underline{C}\ \underline{x}(\underline{0}),\ \underline{C}[\underline{x}(\tilde{\underline{w}})-\underline{x}(\underline{0})]\rangle_{L^2(D;R^{n_y})} \tag{3.7}$$

where $\underline{w} = (\underline{u},\underline{v}) \varepsilon W$ (see (2.27)). By completing the square in (3.4), obtain

$$J(\underline{w}) = \mathcal{Q}(\underline{w},\underline{w}) - 2\mathcal{L}(\underline{w}) + ||\underline{d}-\underline{C}\ \underline{x}(\underline{0})||^2_{L^2(D;R^{n_y})} \tag{3.8}$$

Since $\underline{x}(\underline{0})$ represents the state of the uncontrolled system, with $\underline{f} \varepsilon X$, $\underline{x}_o \varepsilon X_o$ and $\underline{d} \varepsilon Y$ prespecified, the last term of (3.8) is an irreducible positive constant which does not affect the optimization problem. Hence it is completely equivalent to consider the performance index

$$\tilde{J}(\underline{w}) = \mathcal{Q}(\underline{w},\underline{w}) - 2\mathcal{L}(\underline{w}) \tag{3.9}$$

One remaining preliminary difficulty is encountered due to the fact that the space of admissable controls, W, is not complete in the norm

$$||(\underline{u},\underline{v})||^2_{L^2(D;R^{n_u})\times L^2(L;R^{n_v})} = ||\underline{u}||^2_{L^2(D;R^{n_u})} + ||\underline{v}||^2_{L^2(L;R^{n_v})} \quad (3.10)$$

of the space

$$\tilde{W} = L^2(D;R^{n_u}) \times L^2(L;R^{n_v}) \quad (3.11)$$

which is the natural function space for the minimization problem. The following lemma, however, will suffice:

Lemma 3-1: W is a closed, convex subset of $\tilde{W}$.

Proof: $W \subset \tilde{W}$, since $H^1(D;R^{n_u}) \subset L^2(D;R^{n_u})$ and $H^1(L;R^{n_v}) \subset L^2(L;R^{n_v})$. W is convex, since for $\underline{w}_1 = (\underline{u}_1,\underline{v}_1)$ and $\underline{w}_2 = (\underline{u}_2,\underline{v}_2)$ in W there exist Cauchy sequences $\{\underline{w}_1^n\}$, $\{\underline{w}_2^n\}$ of C^∞ functions defined by (2.11), (2.14) with $\underline{w}_1{}^n \to \underline{w}_1$, $\underline{w}_2{}^n \to \underline{w}_2$ in W. But then the Cauchy sequence $\{\beta\, \underline{w}_1^n + (1-\beta)\underline{w}_2^n\}$ converges to $\beta\underline{w}_1 + (1-\beta)\underline{w}_2 \in W$; $0 \leq \beta \leq 1$. W is closed, because its elements are defined as the limit points of Cauchy sequences; i.e., given a Cauchy sequence $\{\underline{w}_k \in W\}$ it is readily shown that $\lim_{k\to\infty} \underline{w}_k \in W$ by selecting a convergent sequence $\{\underline{w}_k^n\}$ of C^∞ functions composed of subsequences of the C^∞ sequences converging to each $\underline{w}_k$. The fact that $\lim_{n\to\infty} \underline{w}_k \in H^1(D;R^{n_u}) \times H^1(L;R^{n_v})$ is a consequence of the closedness of this space.

The existence and uniqueness of optimal controls is then guaranteed by a theorem of Lions.

Theorem 3-1 [Lions]: Let $\mathcal{Z}(w,\tilde{w})$ be a continuous symmetric bilinear form on $\tilde{W}$ which satisfies

$$\mathcal{Z}(\tilde{\underline{w}},\tilde{\underline{w}}) \geq c\ ||\tilde{\underline{w}}||^2_{\tilde{W}} \quad \forall\ \tilde{\underline{w}} \in \tilde{W} \tag{3.12}$$

Let W be a closed convex subset of $\tilde{W}$. Then there exists a unique element $\underline{w}^* \in W$ such that

$$\tilde{J}(\underline{w}^*) = \operatorname*{Inf}_{\underline{w} \in W} \tilde{J}(\underline{w}) \tag{3.13}$$

Proof: See Lions, [26], pp.7-8.

Corollary 3-1: There exists a unique optimal control pair $(\underline{u}^*,\underline{v}^*) \in W$ satisfying (2.28), for the problem of Section 2.2.

Proof: From the preliminaries and Lemma 3-1, it suffices merely to demonstrate that $\mathcal{Z}(\underline{w},\underline{w})$ defined by (3.6) satisfies the conditions of Theorem 3-1. (i) $\mathcal{Z}$ is continuous on $\tilde{W}$ defined by (3.11). Continuity of the last two terms of (3.6) is obvious. Continuity of the first term requires continuity of solutions $\underline{x}$ for $\underline{u} \in L^2(D;R^{n_x})$, $\underline{v} \in L^2(L;R^{n_v})$; this result is a consequence of Theorem 1-3 (not Theorem 2-1). (ii) is symmetric, i.e., $\mathcal{Z}(\tilde{w}_1,\tilde{w}_2) = \mathcal{Z}(\tilde{w}_2,\tilde{w}_1)$; this follows from symmetry of $\underline{R}_u$, $\underline{R}_v$. (iii) $\mathcal{Z}$ is bilinear. Bilinearity of the last two terms of $\mathcal{Z}$ in (3.6) is obvious (note, however, that linearity is with respect to $\underline{w}$, i.e., one considers changing $\underline{u}$ and $\underline{v}$ by the same proportions). Bilinearity of the first term follows from the fact that $\underline{x}(\tilde{\underline{w}})-\underline{x}(\underline{0})$, for instance, is the solution for $\underline{f} \equiv \underline{0}$, $\underline{x}_o \equiv \underline{0}$ with controls $\tilde{\underline{u}} \in L^2(D;R^{n_u})$, $\tilde{\underline{v}} \in L^2(L;R^{n_v})$, which is linear (by Theorem 1-3) when considered as a function of the control pair $(\tilde{\underline{u}},\tilde{\underline{v}}) = \tilde{\underline{w}}$. (iv) Coercivity (3.12) follows from non-negativity of the first term of (3.6) and the positive definiteness of $\underline{R}_u$, $\underline{R}_v$:

$$\mathcal{Q}(\tilde{\underline{w}},\tilde{\underline{w}}) \geq \min[||\underline{R}_u||,\ ||\underline{R}_v||]\ ||\tilde{\underline{w}}||^2_{\tilde{W}} \tag{3.14}$$

where (3.10) defines the norm on $\tilde{W}$. q.e.d.

Furthermore, Lions has demonstrated that the minimizing element $\underline{w}^*$ of Theorem 3-1 is characterized by a variational inequality:

Theorem 3-2 [Lions]: Under the conditions of Theorem 3-1, $\underline{w}^* \in W$ is characterized by

$$\mathcal{Q}(\underline{w}^*,\underline{w}-\underline{w}^*) \geq \mathcal{L}(\underline{w}-\underline{w}^*) \quad \text{for all} \quad \underline{w} \in W \tag{3.15}$$

Proof: Lions, [26], p.9.

Corollary 3-2: The unique optimal controls $(\underline{u}^*,\underline{v}^*) \in W$ of Corollary 3-1 are characterized by

$$\langle \underline{C}\ \underline{x}(\underline{u}^*,\underline{v}^*) - \underline{d},\ \underline{C}[\underline{x}(\underline{u},\underline{v}) - \underline{x}(\underline{u}^*,\underline{v}^*)] \rangle_{L^2(D;R^{n_Y})}$$

$$+ \langle \underline{R}_u\ \underline{u}^*,\ \underline{u}-\underline{u}^* \rangle_{L^2(D;R^{n_u})} + \langle \underline{R}_v \underline{v}^*,\ \underline{v}-\underline{v}^* \rangle_{L^2(L;R^{n_v})} \geq 0 \tag{3.16}$$

for all $(\underline{u},\underline{v}) \in W$.

Proof: Theorem 3-2 is directly applicable. Using (3.6), (3.7) and writing

$$\mathcal{Q}(\underline{w}^*,\underline{w}-\underline{w}^*) - \mathcal{L}(\underline{w}-\underline{w}^*) \geq 0$$

yields (3.16). q.e.d.

The main conclusion of this section, a two-point boundary value problem characterizing the optimal controls, is a relatively straightforward consequence of (3.16).

Theorem 3-3: Consider the system (2.3)-(2.5) with response (2.17) and performance index (2.23). Let the conditions of Theorem 2-1 be satisfied. Then the optimal controls $(\underline{u}^*,\underline{v}^*) \in W$ defined by (2.28) exist, are unique, and are characterized by the solution of the TPBVP:

$$\partial\underline{x}^*/\partial t = \sum_{i=1}^{m} \underline{A}_i\ \partial\underline{x}^*/\partial z_i + \underline{K}\ \underline{x}^* + \underline{B}\ \underline{u}^* + \underline{f} \qquad (t,\underline{z}) \in D \tag{3.17}$$

$$\text{IC: } \underline{x}^*(0,\underline{z}) = \underline{x}_o(\underline{z}) \qquad \underline{z} \in Z \tag{3.18}$$

$$\text{BC: } \underline{T}\ \underline{x}_L^* = \underline{G}\ \underline{v}^* \qquad (t,\underline{z}) \in L \tag{3.19}$$

$$-\partial\underline{p}^*/\partial t = -\sum_{i=1}^{m} \underline{A}_i'\ \partial\underline{p}^*/\partial z_i + [\underline{K}' - \sum_{i=1}^{m} \partial\underline{A}_i'/\partial z_i]\underline{p}^* + \underline{C}'\underline{Q}[\underline{C}\ \underline{x}^* - \underline{d}] \qquad (t,\underline{z}) \in D \tag{3.20}$$

$$\text{FC: } \underline{p}^*(t_f,\underline{z}) = \underline{0} \qquad \underline{z} \in Z \tag{3.21}$$

$$\text{BC: } \underline{T}^{(*)}\underline{p}_L^* = \underline{0} \qquad (t,\underline{z}) \in L \tag{3.22}$$

with $\underline{x}^* \in X$, $\underline{p}^* \in X$,

where

$$\underline{u}^* = -\underline{R}_u^{-1}\, \underline{B}'\, \underline{p}^* \qquad (t,\underline{z}) \in D \tag{3.23}$$

$$\underline{v}^* = +\underline{R}_v^{-1}\, \underline{G}'\, (\underline{T}^{-'})^{-1}\, \underline{A}_1^-\, \underline{p}_L^{*-} \qquad (t,\underline{z}) \in L \tag{3.24}$$

Proof: Existence and uniqueness are guaranteed by Corollary 3-1. To obtain (3.17)-(3.24), Corollary 3-2 is applied. Consider the first term of (3.16):

$$< \underline{C}\, \underline{x}(\underline{w}^*) - \underline{d},\ \underline{C}[\underline{x}(\underline{w}) - \underline{x}(\underline{w}^*)] >_{L^2(D;R^{n_Y})}$$

$$= < \underline{C}'(\underline{C}\, \underline{x}(\underline{w}^*) - \underline{d}),\ \underline{x}(\underline{w}) - \underline{x}(\underline{w}^*) >_{L^2(D;R^{n_x})} \tag{3.25}$$

since $\underline{x} \in X \subset L^2(D;R^{n_x})$ is guaranteed by Theorem 2-1. Next, Theorem 2-1. Next, Theorem 2-2, with

$$\underline{h} = \underline{C}'(\underline{C}\, \underline{x}(\underline{w}^*) - \underline{d}) \in X$$

is applied, guaranteeing existence of $\underline{p}^*$ satisfying (3.20)-(3.22); note that the remainder of the adjoint system was derived completely independently of the optimal control problem. Then from (2.36) and (2.44), (3.25) becomes

$$< \underline{C}'(\underline{C}\, \underline{x}(\underline{w}^*) - \underline{d}),\ \underline{x}(\underline{w}) - \underline{x}(\underline{w}^*) >_{L^2(D;R^{n_x})}$$

$$= < M^{(*)}\, \underline{p}(\underline{w}^*),\ \underline{x}(\underline{w}) - \underline{x}(\underline{w}^*) >_{L^2(D;R^{n_x})}$$

$$= \langle \underline{p}(\underline{w}^*),\ M(\underline{x}(\underline{w}) - \underline{x}(\underline{w}^*)\rangle_{L^2(D;R^{n_x})}$$

$$- \langle \underline{p}_L(\underline{w}^*),\ \underline{A}_1(\underline{x}_L(\underline{w}) - \underline{x}_L(\underline{w}^*))\rangle_{L^2(L;R^{n_x})}$$

$$- \langle \underline{p}(t_f;\underline{w}^*),\ \underline{x}(t_f,\underline{w}) - x(t_f;\underline{w}^*)\rangle_{L^2(Z;R^{n_x})}$$

$$+ \langle \underline{p}(0;\underline{w}^*),\ \underline{x}(0;\underline{w}) - \underline{x}(0;\underline{w}^*)\rangle_{L^2(Z;R^{n_x})} \qquad (3.26)$$

The last step requires considerationssimilar to those leading to (2.35), viz., application of the divergence theorem to sequences in C^∞ converging to $\underline{u}$, $\underline{v}$, $\underline{x}$, $\underline{p}$, etc. The last two terms of (3.26) vanish because $\underline{p}(t_f;\underline{w}^*) = \underline{0}$ and $\underline{x}(0;\underline{w}) = \underline{x}(0;\underline{w}^*) = \underline{x}_o$. Using the state equation (2.3), the first term becomes

$$\langle \underline{p}(\underline{w}^*),\ M(\underline{x}(\underline{w}) - \underline{x}(\underline{w}^*))\rangle_{L^2(D;R^{n_x})} = \langle \underline{p}(\underline{w}^*),\ \underline{B}(\underline{u}-\underline{u}^*)\rangle_{L^2(D;R^{n_x})} \qquad (3.27)$$

(see (2.32)). Using the boundary conditions on the state (2.40) and costate (2.41), the second term can be written:

$$- \langle \underline{p}_L(\underline{w}^*),\ \underline{A}_1(\underline{x}_L(\underline{w}) - \underline{x}_L(\underline{w}^*))\rangle_{L^2(L;R^{n_x})}$$

$$= - \langle \underline{p}_L^-(\underline{w}^*),\ \underline{A}_1^-(\underline{x}_L^-(\underline{w}) - \underline{x}_L^-(\underline{w}^*))\rangle_{L^2(L;R^{n_g})}$$

$$- \langle \underline{p}_L^+(\underline{w}^*),\ \underline{A}_1^+(\underline{x}_L^+(\underline{w}) - \underline{x}_L^+(\underline{w}^*))\rangle_{L^2(L;R^{n_x-n_g})}$$

$$= -\langle \underline{p}_L^-(\underline{w}^*),\ \underline{A}_1^-([\underline{x}_L^-(\underline{w}) + (\underline{T}^-)^{-1}\underline{T}^+\ \underline{x}_L^+(\underline{w})]$$

$$- [x_L^-(\underline{w}^*) + (\underline{T}^-)^{-1}\underline{T}^+\ \underline{x}_L^+(\underline{w}^*)])\rangle_{L^2(L;R^{n_g})}$$

$$= -\langle p_L^-(\underline{w}^*),\ \underline{A}_1^-(\underline{T}^-)^{-1}\ \underline{G}(\underline{v}-\underline{v}^*)\rangle_{L^2(L;R^{n_g})} \tag{3.28}$$

Returning to the optimality condition (3.16), in view of (3.25)-(3.28),

$$\langle \underline{R}_u\ \underline{u}^* + \underline{B}'\ \underline{p}(\underline{w}^*),\ \underline{u}-\underline{u}^*\rangle_{L^2(D;R^{n_u})}$$

$$+\langle \underline{R}_v\ \underline{v}^* - G'(\underline{T}^{-'})^{-1}\ \underline{A}_1^-\ \underline{p}_L^-(\underline{w}^*),\ \underline{v}-\underline{v}^*\rangle_{L^2(L;R^{n_v})} \geq 0 \tag{3.29}$$

for all $(\underline{u},\underline{v}) \in W$. It is readily seen that the optimal control laws (3.23) and (3.24) guarantee this condition. The factor of $\underline{Q}$ in (3.20) is re-introduced by reversing the preliminary substitution (3.1)-(3.3). To summarize, it has been shown that the solution to (3.17)-(3.22) exists for any $(\underline{u},\underline{v}) \in W$. The optimal control laws (3.23), (3.24) satisfy the necessary and sufficient conditions of optimality (3.16); hence the solution of (3.17)-(3.24) exists (by Corollary 3-1) and is unique. q.e.d.

<u>Remark 3-1:</u> The method of proof is very similar to that used by Lions.

Remark 3-2: The continuity restrictions on the data and solution spaces are undoubtedly conservative. The question of how far these conditions may be relaxed is still under investigation. It is certain, however, that $X = L^2(D;R^{n_x})$ will not be achieved, since $\underline{x} \in L^2(D;R^{n_x})$ is not guaranteed to have a well-defined boundary value, $\underline{x}_L$. Use of strong solutions in $L^2(D;R^{n_x})$, giving $X \subset L^2(D;R^{n_x})$, may be feasible; potential difficulties involve use of the divergence theorem and guaranteeing that the space of admissable controls thus defined is closed in $L^2(D;R^{n_u}) \times L^2(L;R^{n_v})$.

2.4 Decoupling

The optimal control laws (3.23), (3.24) may be realized in feedback form, since the costate is a linear (affine) transformation of the state in (3.17)-(3.22). The results of Lions [26], pp.133-136 are shown to be valid for the present problem in order to establish this result. An integrodifferential equation of Riccati type is formally derived using the Schwartz-Kernel Theorem in order to characterize the feedback operator explicitly; a rigorous justification for this procedure is still under investigation, but appears to be quite likely.

Coincidentally, the numbering of the following lemmas and corollaries is identical to that used by Lions, facillitating cross-reference; the proofs differ in certain details, however. It is also noted that the problem statement (Section 2.2) of this example automatically satisfies all of the preliminary conditions imposed by Lions. A straightforward but essential observation is the following:[11]

Lemma 4.1: Under the hypotheses of Theorem 3-3, the following system admits a unique solution.

$$\partial \underline{x}/\partial t = A\,\underline{x} - \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\underline{p} + \underline{f} \tag{4.1}$$

$$\text{IC: } \underline{x}(t_o,\underline{z}) = \tilde{\underline{x}}; \qquad t_o \in (0,t_f), \quad \tilde{\underline{x}} \in X_o \tag{4.2}$$

$$\text{BC: } \underline{T}\,\underline{x}_L = \underline{G}\,\underline{R}_v^{-1}\,\underline{G}'\,(\underline{T}^{-'})^{-1}\,\underline{A}_1^{-}\,\underline{p}_L^{-} \tag{4.3}$$

$$-\partial \underline{p}/\partial t = A^{(*)}\,\underline{p} + \underline{C}'\,\underline{Q}\,\underline{C}\,\underline{x} - \underline{C}'\,\underline{Q}\,\underline{d} \tag{4.4}$$

$$\text{FC: } \underline{p}(t_f,\underline{z}) = \underline{0} \tag{4.5}$$

$$\text{BC: } \underline{T}^{(*)}\,\underline{p}_L = \underline{0} \tag{4.6}$$

with $t \in T_o = (t_o, t_f)$, $\underline{z} \in Z$.

Proof: Equations (4.1)-(4.6) characterize the solution of the following optimal control problem on $D_o = T_o \times Z$:

$$\partial \underline{x}/\partial t = A\,\underline{x} + \underline{B}\,\underline{u} + \underline{f} \qquad (t,\underline{z}) \in D_o \tag{4.7}$$

$$\text{IC: } \underline{x}(t_o,\underline{z}) = \tilde{\underline{x}} \qquad \underline{z} \in Z \tag{4.8}$$

$$\text{BC: } \underline{T}\,\underline{x}_L = \underline{G}\,\underline{v} \qquad (t,\underline{z}) \in L_o \tag{4.9}$$

---

(11) The technique used is similar to invariant embedding, but the conceptual approach is different and guarantees more satisfactory results.

for $(\underline{u},\underline{v}) \in U_o \times V_o = W_o$, with performance index,

$$\tilde{J}_o(\underline{u},\underline{v}) = \langle \underline{C}\ \underline{x} - \underline{d},\ \underline{C}\ \underline{x} - \underline{d} \rangle_{L^2(D_o;R^{n_y})}$$

$$+ \langle \underline{R}_u\ \underline{u},\underline{u} \rangle_{L^2(D_o;R^{n_u})} + \langle \underline{R}_v\ \underline{v},\underline{v} \rangle_{L^2(L_o;R^{n_v})} \quad (4.10)$$

The coefficients of (4.7)-(4.10) are assumed to be restrictions to T_o of the functions defined in Section 2.3. Existence and uniqueness of solutions to (4.1)-(4.6) are a direct consequence of Theorem 3-3. The presence of the costate in (4.1), (4.3) is explained by inserting the optimal control laws (see (3.23), (3.24)) into (4.7), (4.9). The control and solution spaces are obtained by replacing T by T_o in Theorem 3-3.[12] q.e.d.

<u>Lemma 4.2:</u> Let $(\underline{x},\ \underline{p})$ denote the solution of (4.1)-(4.6). The mapping $\tilde{\underline{x}} \to (\underline{x},\ \underline{p})$ is a continuous affine mapping of $X_o \to X_{(o)} \times X_{(o)}$.

<u>Proof:</u> The linear part of the mapping corresponds to the case $\underline{f} \equiv \underline{0}$, $\underline{d} \equiv \underline{0}$. Let $\underline{x}^n(\underline{w})$ denote the state of the system (4.7)-(4.9) for $\underline{w} = (\underline{u},\underline{v})$, and $\tilde{\underline{x}} = \tilde{\underline{x}}^n$. Fix $\underline{w} \in W_o$ and let $\{\tilde{\underline{x}}^n\}$ denote any sequence in X_o converging to $\tilde{\underline{x}}$. Then

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(12) $\tilde{\underline{x}} \in X_o$, $\underline{x} \in X_{(o)}$, $\underline{p} \in X_{(o)}$. X_o is given by (2.13); $X_{(o)} \doteq H^1(D_o;\ R^{n_x})$.

$$\underline{x}^n(\underline{w}) \to \underline{x}(\underline{w}) \quad \text{in} \quad X_{(o)} \tag{4.11}$$

(see Footnote (12)). Let $\underline{w}^n$ denote the optimal controls and $\tilde{J}_o^n$ denote the performance index for problem (4.7)-(4.10) with $\tilde{\underline{x}} = \tilde{\underline{x}}^n$. Then

$$\tilde{J}_o^n(\underline{w}^n) = \operatorname*{Inf}_{w\,\varepsilon\,W_o} \tilde{J}_o^n(\underline{w}) \le \tilde{J}_o(\underline{w}^*) \tag{4.12}$$

where $\underline{w}^* = (\underline{u}^*, \underline{v}^*)$ are the optimal controls (defined by (3.23), (3.24)) corresponding to $\tilde{\underline{x}}$. Furthermore, referring to (4.10), (4.11), with $\underline{w} = \underline{w}^*$,

$$\tilde{J}_o^n(\underline{w}^*) \to \tilde{J}_o(\underline{w}^*) \tag{4.13}$$

Thus

$$\limsup \{\tilde{J}_o^n(\underline{w}^n)\} \le \tilde{J}_o(\underline{w}^*) = \operatorname*{Inf}_{\underline{w}\,\varepsilon\,W_o} \tilde{J}_o(\underline{w}) \tag{4.14}$$

But $\tilde{J}_o^n(\underline{w}^n) \ge \min[||\underline{R}_u||, ||\underline{R}_v||] \; ||\underline{w}^n||^2_{W_o}$ and hence $||\underline{w}^n||_{W_o} < \infty$, i.e., $\underline{w}^n$ ranges in a bounded subset of W_o as $\tilde{\underline{x}}^n \to \tilde{\underline{x}}$. Since W_o is a closed convex subset of $L^2(D_o;R^{n_u}) \times L^2(L_o;R^{n_v})$, it is possible to extract a subsequence $\{\underline{w}^k\}$ such that $\underline{w}^k \to \underline{w}^*$ weakly in W_o.(13) Hence $\underline{x}^k(\underline{x}^k) \to \underline{x}^k(\tilde{\underline{w}})$ weakly in $X_{(o)}$, and

$$\liminf \{\tilde{J}_o^k(\underline{w}^k)\} \ge \tilde{J}_o(\tilde{\underline{w}}) \tag{4.15}$$

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(13) See Lemma 3-1. This is the first significant modification of Lions' method.

But (4.14), (4.15) then imply $\tilde{J}_o(\tilde{\underline{w}}) \leq \tilde{J}_o(\underline{w}^*)$, and so $\tilde{\underline{w}} = \underline{w}^*$. It has been established that

$$\underline{w}^n \rightarrow \underline{w}^* \text{ weakly in } W_o;\ \tilde{J}_o^n(\underline{w}^n) \rightarrow \tilde{J}_o(\underline{w}^*) \tag{4.16}$$

and

$$\underline{x}^n(\underline{w}^n) \rightarrow \underline{x}(\underline{w}^*);\ \underline{p}^n(\underline{w}^n) \rightarrow \underline{p}(\underline{w}^*) \text{ weakly in } X_{(o)} \tag{4.17}$$

Thus the linear part of the map $\tilde{\underline{x}} \rightarrow (\underline{x},\underline{p})$, is continuous in the strong topology of X_o and the weak topology of $X_{(o)} \times X_{(o)}$. q.e.d.

<u>Corollary 4-1:</u> Let $(\underline{x},\underline{p})$ denote the solution of (4.1)-(4.6) for $\tilde{\underline{x}} \in X_o$. The mapping $\tilde{\underline{x}} \rightarrow \underline{p}(t_o)$ is a continuous affine mapping of $X_o \rightarrow X_o$.

<u>Proof:</u> The mapping $\tilde{\underline{x}} \rightarrow \underline{p}(t_o)$ is a composition of the mapping $\tilde{\underline{x}} \rightarrow (\underline{x},\underline{p})$ and the mapping $(\underline{x},\underline{p}) \rightarrow \underline{p}(t_o)$ (sic!). The continuity of this map of $X_{(o)} \times X_{(o)} \rightarrow X_o$ may be demonstrated in several ways:[14] (1) $H^1(D_o;R^{n_x})$ is a subset of the space $W(0,T)$ used by Lions (then apply his Theorem 1.1), or (2) guarantee $p(t_o) \in H^{1/2}(Z;R^{n_x})$ using the Trace Theorem of Lions (p.22), and use the fact that mappings of $H^{1/2}(Z;R^{n_x}) \rightarrow H^1(Z;R^{n_x})$ are continuous, or (3) refer to the definition of the strong solution $\underline{p}$ guaranteed by Theorem 2-2. q.e.d.

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(14) A second modification.

Corollary 4-2: The mapping $\underline{\tilde{x}} \to \underline{p}(t_o)$ may be written uniquely as

$$\underline{p}(t_o) = P(t_o)\underline{\tilde{x}} + \underline{r}(t_o) \tag{4.18}$$

where $P(t_o) \in \mathcal{L}(X_o;X_o)$, $\underline{r}(t_o) \in X_o$.

Proof: Representation theorem for linear affine mappings.

Lemma 4-3: Let $\{\underline{x}^*, \underline{p}^*\}$ be a solution of (3.17)-(3.24) on the interval T. Then

$$\underline{p}^*(t) = P(t)\ \underline{x}^*(t) + \underline{r}(t) \qquad t \in T \tag{4.19}$$

where the operator $P(t_o)$ and the function $\underline{r}(t_o)$ are determined as follows, for $t_o \in T$:

$P(t_o)\underline{\tilde{x}} = \underline{p}^1(t_o)$, where

$$\partial \underline{x}^1/\partial t = A\underline{x}^1 - \underline{B}\ \underline{R}_u^{-1}\ \underline{B}'\ \underline{p}^1 \tag{4.20}$$

$$\text{IC:}\ \underline{x}^1(t_o,\underline{z}) = \underline{\tilde{x}}(\underline{z}) \tag{4.21}$$

$$\text{BC:}\ \underline{T}\ \underline{x}_L^1 = \underline{G}\ \underline{R}_v^{-1}\ \underline{G}'\ (T^{-\prime})^{-1}\ A_1^-\ p_L^{1-} \tag{4.22}$$

$$-\partial \underline{p}^1/\partial t = A^{(*)}\ \underline{p}^1 + \underline{C}'\ \underline{Q}\ \underline{C}\ \underline{x}^1 \tag{4.23}$$

$$\text{FC:}\ \underline{p}^1(t_f,\underline{z}) = \underline{0} \tag{4.24}$$

$$\text{BC:}\ \underline{T}^{(*)}\ \underline{p}_L^1 = 0 \tag{4.25}$$

and $\underline{r}(t_o) = \underline{p}^2(t_o)$, where

$$\partial \underline{x}^2/\partial t = A\ \underline{x}^2 - \underline{B}\ \underline{R}_u^{-1}\ \underline{B}'\ \underline{p}^2 + \underline{f} \tag{4.26}$$

$$\text{IC: } \underline{x}^2(t_o,\underline{z}) = \underline{0} \tag{4.27}$$

$$\text{BC: } \underline{T}\ \underline{x}_L^2 = \underline{G}\ \underline{R}_v^{-1}\ \underline{G}'\ (\underline{T}^{-'})^{-1}\ \underline{A}_1^-\ \underline{p}_L^{2-} \tag{4.28}$$

$$-\partial \underline{p}^2/\partial t = A^{(*)}\ \underline{p}^2 + \underline{C}'\ \underline{Q}\ \underline{C}\ \underline{x}^2 - \underline{C}'\ \underline{Q}\ \underline{d} \tag{4.29}$$

$$\text{FC: } \underline{p}^2(t_f,\underline{z}) = \underline{0} \tag{4.30}$$

$$\text{BC: } \underline{T}^{(*)}\ \underline{p}_L^2 = \underline{0} \tag{4.31}$$

<u>Proof:</u> Equations (4.20)-(4.31) are merely an application of the representation theorem giving Corollary 4-2 (4.18). To prove (4.19), consider (4.1)-(4.6) with $\underline{\tilde{x}} = \underline{x}^*(t_o)$ and let $(\underline{\tilde{x}}, \underline{\tilde{p}})$ be the restrictions of $(\underline{x}^*, \underline{p}^*)$ to T_o; since $(\underline{\tilde{x}}, \underline{\tilde{p}})$ satisfy (4.1)-(4.6), $\underline{\tilde{x}} = \underline{x}$ and $\underline{\tilde{p}} = \underline{p}$. Hence $p(t_o) = \underline{\tilde{p}}(t_o) = \underline{p}^*(t_o)$ and (4.19) follows from Corollary 4-2, since t_o is fixed but arbitrary. q.e.d.

<u>Remark 4-1:</u> Applying (4.19) to (3.23), (3.24), the optimal controls are given by the <u>feedback</u> laws:

$$\underline{u}^*(t) = -\underline{R}_u^{-1}(t)\ \underline{B}'(t)\ [P(t)\ \underline{x}^*(t) + \underline{r}(t)] \tag{4.32}$$

$$\underline{v}^*(t) = \underline{R}_v^{-1}(t)\ \underline{G}'(t)\ (\underline{T}^{-'})^{-1}\ \underline{A}_1^-\ [P(t)\underline{x}^*(t) + \underline{r}(t)]_L^- \tag{4.33}$$

The symmetry of the operator $P(t)$ will be required in the sequel:

<u>Lemma 4-4:</u>[15] $\quad P(t)^{(*)} = P(t)$ (4.34)

<u>Proof:</u> Using (4.23), for $\hat{\underline{x}} \in X_{(o)}$ solving (4.20)-(4-25) with $\tilde{\underline{x}} = \hat{\underline{x}}_o \in X_o$,

$$\underline{0} = \langle(-\partial/\partial t - A^{(*)})\underline{p}^1 - \underline{C}'\underline{Q}\,\underline{C}\,\underline{x}^1,\ \hat{\underline{x}}\rangle_{L^2(D_o;R^{n_x})}$$

$$= \langle\underline{p}^1(t_o),\ \hat{\underline{x}}_o\rangle_{L^2(Z;R^{n_x})}$$

$$- \langle\underline{C}'\underline{Q}\,\underline{C}\,\underline{x}^1,\ \hat{\underline{x}}\rangle_{L^2(D_o;R^{n_x})}$$

$$+ \langle\underline{p}^1,\ (\partial/\partial t - A)\,\hat{\underline{x}}\rangle_{L^2(D_o;R^{n_x})}$$

$$- \langle\underline{A}_1\,\underline{p}^1_L,\ \hat{\underline{x}}_L\rangle_{L^2(L_o;R^{n_x})}$$

$$= \langle p^1(t_o), \hat{\underline{x}}_o\rangle_{L^2(Z;R^{n_x})} - \langle\underline{C}'\underline{Q}\,\underline{C}\,\underline{x}^1,\ \hat{\underline{x}}\rangle_{L^2(D_o;R^{n_x})}$$

$$- \langle\underline{p}^1,\ \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\hat{\underline{p}}\rangle_{L^2(D_o;R^{n_x})}$$

$$- \langle\underline{A}_1\,\underline{p}^1_L,\ \hat{\underline{x}}_L\rangle_{L^2(L_o;R^{n_x})} \tag{4.35}$$

---

(15) Continuity of $P(t)$ will not be proven here; the calculations are quite involved and require no new ideas. The boundary term in this proof, however, is a third modification of Lions' results.

Therefore since $\underline{A}_1$, $\underline{B}\ \underline{R}_u^{-1}\ \underline{B}'$ and $\underline{C}'\underline{Q}\ \underline{C}$ are symmetric,

$$<P(t_o)\tilde{\underline{x}},\hat{\underline{x}}_o>_{L^2(Z,R^{n_x})} = <\underline{C}'\underline{Q}\ \underline{C}\ \underline{x}^1,\ \hat{\underline{x}}>_{L^2(D_o;R^{n_x})}$$

$$+ <\underline{p}^1,\ \underline{B}\ \underline{R}_u^{-1}\ \underline{B}'\ \hat{\underline{p}}>_{L^2(D_o;R^{n_x})}$$

$$+ <\underline{p}_L^{1-},\underline{A}_1^-(\underline{T}^-)^{-1}\underline{G}\ \underline{R}_v^{-1}\underline{G}'(\underline{T}^{-'})^{-1}\underline{A}_1^-\hat{\underline{p}}_L^->$$

(4.36)

where the last term is a consequence of the boundary conditions (4.22) on $\hat{\underline{p}}$ and (4.25) on $\underline{p}^1$. The symmetry of this expression establishes (4.34). q.e.d.

<u>Lemma 4-5:</u> There exists a constant, c, such that

$$||P(t_o)\tilde{\underline{x}}||_{L^2(Z;R^{n_x})} < c\ ||\tilde{\underline{x}}||_{L^2(Z;R^{n_x})} \tag{4.37}$$

for all $\tilde{\underline{x}} \in X_o$, $t_o \in T$.

Proof: The solution $(\underline{x}^1,\ \underline{p}^1)$ of (4.20)-(4.25) with $\underline{Q} \equiv \underline{I}$ corresponds to the solution of an optimal control problem with state equation (4.7)-(4.9) and cost function (4.10) with $\underline{f} = \underline{0}$, $\underline{d} = \underline{0}$. Let $\tilde{J}_o(\underline{w})$ denote the value of this cost function for controls $(\underline{u},\underline{v}) = \underline{w}$. If $\underline{w}^*$ denotes the optimal controls, then

$$\tilde{J}_o(\underline{w}^*) \le \tilde{J}_o(\underline{0}) \le (T - t_o)||\underline{C}'\underline{C}||\ ||\tilde{\underline{x}}||^2_{L^2(Z;R^{n_x})} \tag{4.38}$$

since $\underline{x} \equiv \underline{\tilde{x}}$ is a solution of (4.7)-(4.9)[16] for $\underline{w} = \underline{0}$.
From (4.20)-(4.25),

$$\tilde{J}_o(\underline{w}^*) - \langle \underline{R}_u\ \underline{u}^*, \underline{u}^* \rangle_{L^2(D_o;R^{n_u})} - \langle \underline{R}_v\ \underline{v}^*, \underline{v}^* \rangle_{L^2(L_o;R^{n_v})}$$

$$= \langle \underline{C}\ \underline{x}^1, \underline{C}\ \underline{x}^1 \rangle_{L^2(D_o;R^{n_y})}$$

$$= \langle \underline{C}'\underline{C}\ \underline{x}^1, \underline{x}^1 \rangle_{L^2(D_o;R^{n_x})}$$

$$= \langle (-\partial/\partial t - A^{(*)})\underline{p}^1, \underline{x}^1 \rangle_{L^2(D_o;R^{n_x})}$$

$$= \langle \underline{p}^1(t_o), \underline{\tilde{x}} \rangle_{L^2(Z;R^{n_x})}$$

$$+ \langle \underline{p}^1, (\partial/\partial t + A)\underline{x}^1 \rangle_{L^2(D_o;R^{n_x})}$$

$$- \langle \underline{p}_L^1, \underline{A}_1\ \underline{x}_L^1 \rangle_{L^2(L_o;R^{n_x})}$$

$$= - \langle \underline{p}^1, \underline{B}\ \underline{R}_u^{-1}\underline{B}'\underline{p}^1 \rangle_{L^2(D_o;R^{n_x})} + P(t_o)\underline{\tilde{x}}, \underline{\tilde{x}} \rangle_{L^2(Z;R^{n_x})}$$

$$- \langle \underline{p}_L^{1-}, \underline{A}_1^-(\underline{T}^-)^{-1}\underline{G}\ \underline{R}_v^{-1}\underline{G}'(\underline{T}^{-\prime})^{-1}\underline{A}_1^-\underline{p}_L^{1-} \rangle_{L^2(L_o;R^{n_g})}$$

(4.39)

by reference to (2.35) and the last term of (4.35)-(4.36).
But the optimal controls are given by

---

(16) Note that $\tilde{x}$ satisfies the boundary condition (4.9) - by virtue of (1.17) in the definition of X_o!

$$\underline{u}^* = - \underline{R}_u^{-1} \underline{B}' \underline{p}^1 \tag{4.40}$$

$$\underline{v}^* = \underline{R}_v^{-1} \underline{G}' (\underline{T}^{-'})^{-1} \underline{A}_1^- \underline{p}_L^{1-} \tag{4.41}$$

and hence

$$\langle \underline{R}_u \underline{u}^*, \underline{u}^* \rangle_{L^2(D_o;R^{n_u})} + \langle \underline{R}_v \underline{v}^*, \underline{v}^* \rangle_{L^2(L_o;R^{n_v})} \tag{4.42}$$

$$= \langle \underline{p}^1, \underline{B} \underline{R}_u^{-1} \underline{B}' \underline{p}^1 \rangle_{L^2(D_o;R^{n_x})}$$

$$+ \langle \underline{p}_L^{1-}, \underline{A}_1^- (\underline{T}^-)^{-1} \underline{G} \underline{R}_v^{-1} \underline{G}' (\underline{T}^{-'})^{-1} \underline{A}_1^- \underline{p}_L^{1-} \rangle_{L^2(L_o;R^{n_g})}$$

From (4.39), (4.42) obtain the compact result

$$\tilde{J}_o(\underline{w}^*) = \langle P(t_o) \tilde{\underline{x}}, \tilde{\underline{x}} \rangle_{L^2(Z;R^{n_x})} \tag{4.43}$$

Then (4.37) follows directly from (4.38). q.e.d.

<u>Lemma 4-6:</u> For all $\tilde{\underline{x}} \in X_o$, $t_o \in T$

$$\langle P(t_o) \tilde{\underline{x}}, \tilde{\underline{x}} \rangle_{L^2(Z;R^{n_x})} \geq 0 \tag{4.44}$$

<u>Proof:</u> See (4.43). q.e.d.

Thus it has been established that the feedback operator $P(t)$ is a linear bounded positive self-adjoint operator. Continuity may be also be established precisely in the manner of Lions [26], pp.137-140. Any linear continuous bounded operator $P(t) \varepsilon \ (X_o, X_o)$ may be represented via the Schwartz Kernel Theorem ([44], Theorem 1.2) as

$$(P(t)\underline{x})\ (\underline{z}) = \int_Z \underline{P}\ (t,\underline{z},\underline{s})\ \underline{x}(\underline{s})\ d\underline{s} \tag{4.45}$$

for all $\underline{x} \ \varepsilon \ \mathcal{D}(Z;R^{n_x})$, the space of n_x-vector valued distributions on Z. The kernel

$$\underline{P}(t,\underline{z},\underline{s}) \ \varepsilon \ \mathcal{D}(Z \times Z;\ R^{n_x^2}) \tag{4.46}$$

defines (or is defined by) the operator $P(t)$. When $P(t)$ is self-adjoint, positive, the matrix $\underline{P}(t,\underline{z},\underline{s})$ is self-adjoint, positive, on $R^{n_x^2}$ and the arguments $\underline{z}$, $\underline{s}$ are interchangable:

$$\underline{P}(t,\underline{z},\underline{s}) = \underline{P}(t,\underline{s},\underline{z}) \tag{4.47}$$

The remainder of this section is devoted to a formal[17] derivation of the Riccati integro-differential equation characterizing the kernel of the feedback operator, $P(t)$, for the TPBVP of Theorem 3-3. In the sequel, $<\cdot>$ denotes the scalar product on $L^2(Z)$ and $<\cdot>_B$ denotes the scalar product on $L^2(B)$; the superscript $(\cdot)^*$ is supressed.

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(17) A rigorous derivation requires proving regularity properties of P(t); this problem is technically quite involved and is still under investigation. The conditions of Theorem 3-3 are probably sufficient. (For parabolic systems, Lions requires stronger conditions see [26], Theorem 5.2, p.160.)

Two-Point Boundary Value Problem: Equations (3.17)-(3.24) may be rewritten as

$$\dot{\underline{x}} = A\,\underline{x} - \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\underline{p} + \underline{f} \tag{4.48}$$

$$\text{IC: } \underline{x}(0) = \underline{x}_o \tag{4.49}$$

$$\text{BC: } \underline{T}\,\underline{x}_L = \underline{G}\,\underline{R}_V^{-1}\,\underline{G}'\,(\underline{T}^{-'})^{-1}\,\underline{A}_1^{-}\,\underline{p}_L^{-} \tag{4.50}$$

$$-\dot{\underline{p}} = A^{(*)}\,\underline{p} + \underline{C}'\,\underline{Q}\,\underline{C}\,\underline{x} - \underline{C}'\,\underline{Q}\,\underline{d} \tag{4.51}$$

$$\text{FC: } \underline{p}(t_f) = \underline{0} \tag{4.52}$$

$$\text{BC: } \underline{T}^{(*)}\,\underline{p}_L = \underline{0} \tag{4.53}$$

where the operators A, $A^{(*)}$ are evident from (3.17), (3.20).

Variational Formulation: The boundary conditions on the Riccati integrodifferential equation are most readily calculated by considering a variational formulation of the problem (4.48)-(4.53) - after Lions [26], p.162. Consider $t \in T$ fixed; then for any $\underline{h}$, $\underline{k} \in X$ the solution $(\underline{x},\underline{p}) = (\underline{x}(t),\underline{p}(t))$ of this problem satisfies:

$$\begin{aligned}\langle \dot{\underline{x}},\underline{h} \;\; &= \langle \underline{x},\, A^{(*)}\underline{h}\rangle - \langle \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\underline{p},\underline{h}\rangle + \langle \underline{f},\, \underline{h}\rangle \\ &\quad - [\langle \underline{M}\,\underline{x},\underline{h}\rangle + \langle \underline{N}\,\underline{p},\, \underline{h}\rangle\,]_B \end{aligned}\tag{4.54}$$

$$-\langle \dot{\underline{p}},\underline{k}\rangle = \langle \underline{p}, A\ \underline{k}\rangle + \langle \underline{C}'\underline{Q}\ \underline{C}\ \underline{x},\underline{k}\rangle - \langle \underline{C}'\underline{Q}\ \underline{d},\underline{k}\rangle$$

$$- \langle \underline{E}\ \underline{p},\underline{k}\rangle_B \tag{4.55}$$

where (following the partitioning of $\underline{A}_1$)

$$\underline{M} = \begin{bmatrix} 0 & -\underline{A}_1^- \ (\underline{T}^-)^{-1}\ \underline{T}^+ \\ 0 & \underline{A}_1^+ \end{bmatrix} \tag{4.56}$$

(not to be confused with the operator M of Section 2.2), and

$$\underline{N} = \begin{bmatrix} \underline{A}_1^-(\underline{T}^-)^{-1}\ \underline{G}\ \underline{R}_v^{-1}\ \underline{G}'\ (\underline{T}^{-'})^{-1}\ \underline{A}_1^- & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} \tag{4.57}$$

$$\underline{E} = \begin{bmatrix} -\underline{A}_1^- & \underline{0} \\ \underline{T}^{+'}(\underline{T}^{-'})^{-1}\ \underline{A}_1^- & \underline{0} \end{bmatrix} \tag{4.58}$$

The variational equalities (4.54), (4.55) are derived via the method of transposition (Lions [26], p.181, 186). A formal verification is made by noting that

$$\langle \underline{x},\ A^{(*)}\underline{h}\rangle = \langle A\ \underline{x},\underline{h}\rangle + \langle \underline{A}_1\underline{x},\underline{h}\rangle_B \tag{4.59}$$

$$\langle \underline{p},\ A\ \underline{k} = \langle A^{(*)}\underline{p},\underline{k}\rangle - \langle \underline{A}_1\ \underline{p},\underline{k}\rangle_B \tag{4.60}$$

and that the boundary conditions (4.50), (4.53) imply

$$\langle \underline{A}_1\ \underline{x},\underline{h}\rangle_B = \langle \underline{M}\ \underline{x},\underline{h}\rangle_B + \langle \underline{N}\ \underline{p},\underline{h}\rangle_B \tag{4.61}$$

$$\langle A_1 \underline{p},\underline{k}\rangle_B = -\langle \underline{E}\,\underline{p},\underline{k}\rangle_B \tag{4.62}$$

Decoupling: Applying Lemma 4-3, equation (4.19) to (4.55), and assuming the existence (in an appropriate sense)[18] of an operator $\dot{P} = \partial P/\partial t$:

$$-\langle \dot{P}\,\underline{x} + P\,\dot{\underline{x}} + \dot{\underline{r}},\ \underline{k}\rangle = \langle P\,\underline{x} + \underline{r},\ A\,\underline{k}\rangle + \langle \underline{C}'\underline{Q}\,\underline{C}\,\underline{x},\underline{k}\rangle$$

$$-\langle \underline{C}'\underline{Q}\,\underline{d},\underline{k}\rangle - \langle \underline{E}(P_{\underline{x}} + \underline{r}),\underline{k}\rangle_B \tag{4.63}$$

Since $P = P^{(*)}$ by Lemma 4-4,

$$\langle P\,\dot{\underline{x}},\underline{k}\rangle = \langle \dot{\underline{x}},\ P\,\underline{k}\rangle \tag{4.64}$$

Applying (4.54) with $\underline{h} = P\,\underline{k}$ to this term of (4.63), obtain

$$-\langle \dot{P}\,\underline{x} + \dot{\underline{r}},\underline{k}\rangle = \langle \underline{x},A^{(*)}P\,\underline{k}\rangle - \langle \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\underline{p},\ P\,\underline{k}\rangle$$

$$+\langle P\,\underline{x} + \underline{r},A\,\underline{k}\rangle + \langle \underline{C}'\underline{Q}\,\underline{C}\,\underline{x},\underline{k}\rangle$$

$$+\langle \underline{f},\ P\,\underline{k}\rangle - \langle \underline{C}'\underline{Q}\,\underline{d},\ \underline{k}\rangle$$

$$-\left[\langle \underline{M}\,\underline{x},P\,\underline{k}\rangle + \langle \underline{N}\,\underline{p},P\,\underline{k}\rangle + \langle \underline{E}(P\,\underline{x} + \underline{r}),\underline{k}\rangle\right]_B \tag{4.65}$$

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(18) Lions considers finite-dimensional approximations of the problem.

Applying (4.19) once more to the terms involving $\underline{p}$ in (4.65), and separating terms involving $\underline{r}$ and $\underline{x}$, one derives two variational equalities characterizing P and $\underline{r}$:

$$-\langle \dot{P}\,\underline{x},\underline{k}\rangle = \langle \underline{x},\, A^{(*)}P\,\underline{k}\rangle + \langle P\,\underline{x},\, A\,\underline{k}\rangle$$
$$-\langle \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,P\,\underline{x},\, P\,\underline{k}\rangle + \langle \underline{C}'\underline{Q}\,\underline{C}\,\underline{x},\,\underline{k}\rangle$$
$$-\,[\langle M\,\underline{x},\,P\,\underline{k}\rangle + \langle \underline{E}\,P\,\underline{x},\underline{k}\rangle + \langle \underline{N}\,P\,\underline{x},\,P\,\underline{k}\rangle\,]_B \tag{4.66}$$

$$-\langle \dot{\underline{r}},\underline{k}\rangle = \langle \underline{r},\,A\,\underline{k}\rangle - \langle \underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,\underline{r},\,P\,\underline{k}\rangle + \langle \underline{f},\,P\,\underline{k}\rangle$$
$$-\langle \underline{C}'\underline{Q}\,\underline{d},\underline{k}\rangle - [\langle \underline{N}\,\underline{r},P\,\underline{k}\rangle + \langle E\,\underline{r},\,\underline{k}\rangle]_B \tag{4.67}$$

Applying (4.59)-(4.60) to (4.66) results in

$$-\langle P\,\underline{x},\underline{k}\rangle = \langle P\,A\,\underline{x},\underline{k}\rangle + \langle A^{(*)}P\,\underline{x},\underline{k}\rangle$$
$$-\langle P\,\underline{B}\,\underline{R}_u^{-1}\,\underline{B}'\,P\,\underline{x},\underline{k}\rangle + \langle \underline{C}'\underline{Q}\,\underline{C}\,\underline{x},\underline{k}\rangle$$
$$+\langle P\,\underline{N}\,P\,\underline{x},\underline{k}\rangle_B$$
$$+\,[+\langle P\,\underline{A}_1\,\underline{x},\underline{k}\rangle - \langle P\,\underline{M}\,\underline{x},\underline{k}\rangle - \langle \underline{A}_1\,P\,\underline{x},\underline{k}\rangle$$
$$-\langle \underline{E}\,\underline{P}\,\underline{x},\underline{k}\rangle]_B \tag{4.68}$$

Using the kernel representation (4.45) and noting that the kernel of $P\,A$ is $A^{(*)}(t,\underline{s})\underline{P}(t,\underline{z},\underline{s})$, it is readily verified that $\underline{P}(t,\underline{z},\underline{s})$ satisfies the following integro-differential equation in $\underline{z}$ and $\underline{s}$:

$$\frac{\partial}{\partial t}\,\underline{P}(t,\underline{z},\underline{s}) = [A^{(*)}(t,\underline{s}) + A^{(*)}(t,\underline{z})]\underline{P}(t,\underline{z},\underline{s})$$

$$- \{ \int_Z \underline{P}(t,\underline{z},\tilde{\underline{z}})\,[\underline{B}\ \underline{R}_u^{-1}\ \underline{B}'](t,\tilde{\underline{z}})\underline{P}(t,\tilde{\underline{z}},\underline{s})\,d\tilde{\underline{z}}$$

$$+ \int_B \underline{P}(t,\underline{z},\tilde{\underline{z}})\underline{N}(t,\tilde{\underline{z}})\underline{P}(t,\tilde{\underline{z}},\underline{s})\,d\tilde{\underline{z}}\}$$

$$+ \delta(\underline{z}-\underline{s})\,[\underline{C}'\underline{Q}\ \underline{C}](t,\underline{s}) \qquad (4.69)$$

where $\delta(\cdot)$ denotes the Dirac delta function on Z. The associated boundary conditions are

$$-[\underline{E} - \underline{A}_1](t,\underline{z})\underline{P}(t,\underline{z},\underline{s}) = \underline{0} \qquad \underline{z} \in B, \qquad \underline{s} \in Z \qquad (4.70)$$

$$-\underline{P}(t,\underline{z},\underline{s})\,[\underline{M} - \underline{A}_1](t,\underline{s}) = \underline{0} \qquad \underline{z} \in Z, \qquad \underline{s} \in B \qquad (4.71)$$

Given the solution of this equation, $\underline{r}$ is seen to satisfy the linear integrodifferential equation (from (4.60), (4.67)):

$$-\frac{\partial}{\partial t}\,\underline{r}(t,\underline{z}) = A^{(*)}(t,\underline{z})\ \underline{r}\ (t,\underline{z}) - [\underline{C}'\underline{Q}\ \underline{d}](t,\underline{z})$$

$$- \int_Z \underline{P}(t,\underline{z},\underline{s})\,[\underline{B}\ \underline{R}_u^{-1}\ \underline{B}'](t,\underline{s})\underline{r}(t,\underline{s})\,d\underline{s}$$

$$+ \int_{\underline{Z}} \underline{P}(t,\underline{z},\underline{s})\underline{f}(t,\underline{s})d\underline{s}$$

$$+ \int_{B} \underline{P}(t,\underline{z},\underline{s})\underline{N}(t,\underline{s})\underline{r}(t,\underline{s})d\underline{s} \qquad (4.72)$$

with boundary condition

$$\underline{T}^{+'}(\underline{T}^{-'})^{-1}\underline{A}_1^{-}\ \underline{r}^{-} - \underline{A}_1^{+}\ \underline{r}^{+} = \underline{0} \qquad \underline{z} \in B \qquad (4.73)$$

These equations run backwards in time from the final conditions:

$$\underline{P}(t_f,\underline{z},\underline{s}) = \underline{0} \qquad (4.74)$$

$$\underline{r}(t_f,\underline{z}) = \underline{0} \qquad (4.75)$$

which are derived from (4.19) and (3.21). The potential advantage of this formulation is that the two-point boundary value problem (3.17)-(3.24) has been converted into a problem with only terminal conditions.

Remark 4-2: The boundary conditions (4.70), (4.71) are indeed consistent with the fact that P(t) is self-adjoint.

2.5 Generalizations

This chapter concludes with a brief discussion of related research, with particular emphasis on well-posedness of first-order systems. From the preceding sections, it is clear that the extension of the results of Theorems 1-3 and 1-4 to more general types of linear first-order systems would (more than) suffice for the application of the optimization theory of Lions. Regarding optimization theory itself for linear first-order systems with quadratic cost functionals, the approach of Lions demonstrates a number of significant advantages. First, the conditions for application of the theory are almost as weak as could be expected. For general linear first-order systems the state at each fixed time must lie in a Hilbert space V on Z such that the boundary conditions make sense[(19)]; in addition, the state must belong to $L^2(T;V)$ with (distributional) derivative $\partial \underline{x}/\partial t \in L^2(T;V')$ where V' is the dual of V and T denotes the time interval (notation following Lions). Coercivity of the spatial operator, A, is not essential. Thus one has certain well-defined criteria for the existence theory of linear first-order systems. Secondly, this method guarantees existence and uniqueness of optimal controls, and of the two-point boundary value problem characterizing them, provided that the costate equations permit a unique solution in $L^2(T;V)$. Furthermore, under slightly stronger hypotheses, one has a feedback realization and a Riccati operator equation (existence and uniqueness guaranteed!) characterizing the feedback operator. Thirdly, the theory readily extends to nonlinearities such as magnitude constraints on the controls, unilateral problems, etc.

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(19) $H^1(Z;R^{n_x})$ with boundary values in $H^{1/2}(B;R^{n_x})$ will suffice for weak solutions; for the strong solutions considered by Rauch $L^2(Z;R^{n_x})$ may possibly suffice. Note, however, that these solutions require C^∞ coefficients whereas C° or C^1 coefficients may suffice for weak solutions.

In particular, Lions remarks on the fact that the conditions of this theory are satisfied for operators A having the semigroup property. Thus optimal control theories for such systems developed by Balakushnan [2], Lukes and Russell [30], and others may be seen as special cases of Lions' results. Regarding such operators, the papers by Phillips [34] and Barston [5] treating dissipative hyperbolic first-order systems are noteworthy. Phillips (see below) demonstrates the semigroup property for such systems in the symmetric time-invariant case for a single spatial variable having a bounded spatial domain, with dissipative boundary conditions at the endpoints. Boundary control of such systems thus constitutes a second example where complete results analogous to those of Sections 2.2 - 2.4 may be derived.

Returning to the topic of existence theories for first-order linear systems, surprisingly little research appears to have been published on solutions of the initial boundary value problem.

K. O. Friedrichs [13], [14], [15] was perhaps the first contemporary mathematician to devote major attention to first-order systems. He sought $C^{\infty}(Z)$ solutions to symmetric homogeneous boundary value problems of the form

$$A(\underline{z})\ \underline{x}(\underline{z}) + \underline{f} = \underline{0} \qquad \underline{z} \in Z \tag{5.1}$$

$$\text{BC: } \underline{T}(\underline{z}_B)\ \underline{x}(\underline{z}_B) = 0 \qquad \underline{z}_B \in B \tag{5.2}$$

where the matrices $\underline{A}_i(\underline{z})$, $i=1,\ldots,m$ and $\underline{K}(\underline{z})$ of the operator $A(\underline{z})$ (as in (1.6) but independent of t) are symmetric and $\underline{K}$ is sufficiently positive. Provided $\left|\sum_{i=1}^{m} n_i(z_B)\underline{A}_i(z_B)\right| \neq 0$, $\underline{z}_B \in B$, the range space of $\underline{T}$ can be chosen ("maximally-nonnegative" boundary conditions) to guarantee existence and uniqueness of solutions, with

$$\mathcal{D}_O(A) = \{\underline{x} \in C^\infty(Z) \mid \underline{T}\,\underline{x}_B = \underline{0}\}$$

$$\mathcal{R}(A) \subseteq L^2(Z) \qquad \text{(dense)}.$$

The operator A was shown to be bounded with bounded inverse to establish this result.

About the same time, R. S. Phillips [34] considered time-invariant symmetric hyperbolic systems (1.3)-(1.6) in one spatial variable. For $(\underline{K} + \underline{K}') < 0$ sufficiently negative definite and a certain class of homogeneous "dissipative" boundary conditions (again, this involved choosing the range of $\underline{T}$, at the two boundary points), he proved that A is the infinitesimal generator of a strongly continuous contraction semigroup. A variation-of-constants formula for the solution of such problems was given. More recent works by Kato [23] and Phillips [35] allow these results to be extended to time-varying systems and conservative (e.g., wave-propagating) systems. Unfortunately, the proof of the above results relies on transformation to a canonical form which cannot be achieved in higher spatial dimensions than one.[20] The analysis is quite closely related to the classical method of characteristics.

More recently, Friedrichs in collaboration with Lax [15] attempted to generalize his earlier work to non-symmetric boundary value problems which are "symmetrized" by a non-singular pseudodifferential operator. Again there emerge similar conditions on $\underline{K}$ and the range of $\underline{T}$; these are quite difficult to interpret, however, due to their added dependence on the pseudodifferential operator. Unfortunately, the technique only succeeds for existence but not uniqueness of solutions; whether the latter is an inherent difficulty in the problem formulation or a deficiency of the technique is not clear, but existence and uniqueness of optimal controls as in Section 2.3 will not hold without uniqueness of solutions to the state equation.[21]

Yet another approach to the existence problem was suggested by Lions' application [26] of the Lax-Milgram Lemma [51, p.92] to scalar parabolic equations. C. Bardos [3] demonstrates the semigroup property for scalar equations in higher space dimensions (with nonhomogeneous data!) under very weak assumptions, and then is able to define weak solutions in the Lions manner, for hyperbolic equations. Carroll [9] has recently considered a scalar system in one space dimension and used a modified definition of coercivity

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(20) Except for $m = 2$, $n_x = 2$.

(21) Incidentally, the IBVP (2.3)-(2.5) on TxZ may be formulated as a BVP (5.1)-(5.2) on D - hence the interest in BVP's.

to demonstrate existence of solutions for smooth initial data and homogeneous boundary data. He is unable to prove uniqueness in this manner, however. Attempts to apply the Lax-Milgram Lemma to more general cases indicate a trade-off between continuity (needed for uniqueness) and coercivity (needed for existence) in that both cannot be achieved simultaneously.

In summary, these diverse but related approaches illustrate that existence theories may be derived in certain special cases. There is as yet no counter-example known to the author which necessitates either symmetry or hyperbolicity assumptions, although these are required at present. On the other hand, the spectrum of A, the "shape" of the boundary B, and the range of $\underline{T}$ appear to interact essentially in defining well-posed problems. A generalization of the results of Friedrichs and Lax [15] and Rauch [36] will undoubtedly be obtained shortly by one of these authors, and this appears to be the most fruitful direction for future effort.

III. AN EXAMPLE(1)

A problem of plasma control was chosen to indicate the applicability (and the short-comings!) of the theory developed in Chapters I and II. Doubtless, the plasma physicist will be disappointed at some oversimplifications of the model, but then so will the purist who is satisfied with control of rods and beams! Hopefully, the former will recognize the potentialities of modern control theory, while the latter will recognize some pitfalls of tidy theorems.

3.1 Physical Description of a Plasma Control Problem

Figure 2 illustrates a crude confinement scheme for a low-density low-temperature plasma. The device consists of an evacuated cylinder containing initially a rarefied gas (e.g., helium, neon) surrounded by charged conducting sheets and magnetic coils. A high-voltage electrical discharge (or other excitation) causes the gas to ionize, creating a plasma consisting of free electrons and positive ions. A time-invariant z-directed magnetic field produced by the external coils causes these charged particles to rotate about the z-axis with a characteristic "cyclotron"

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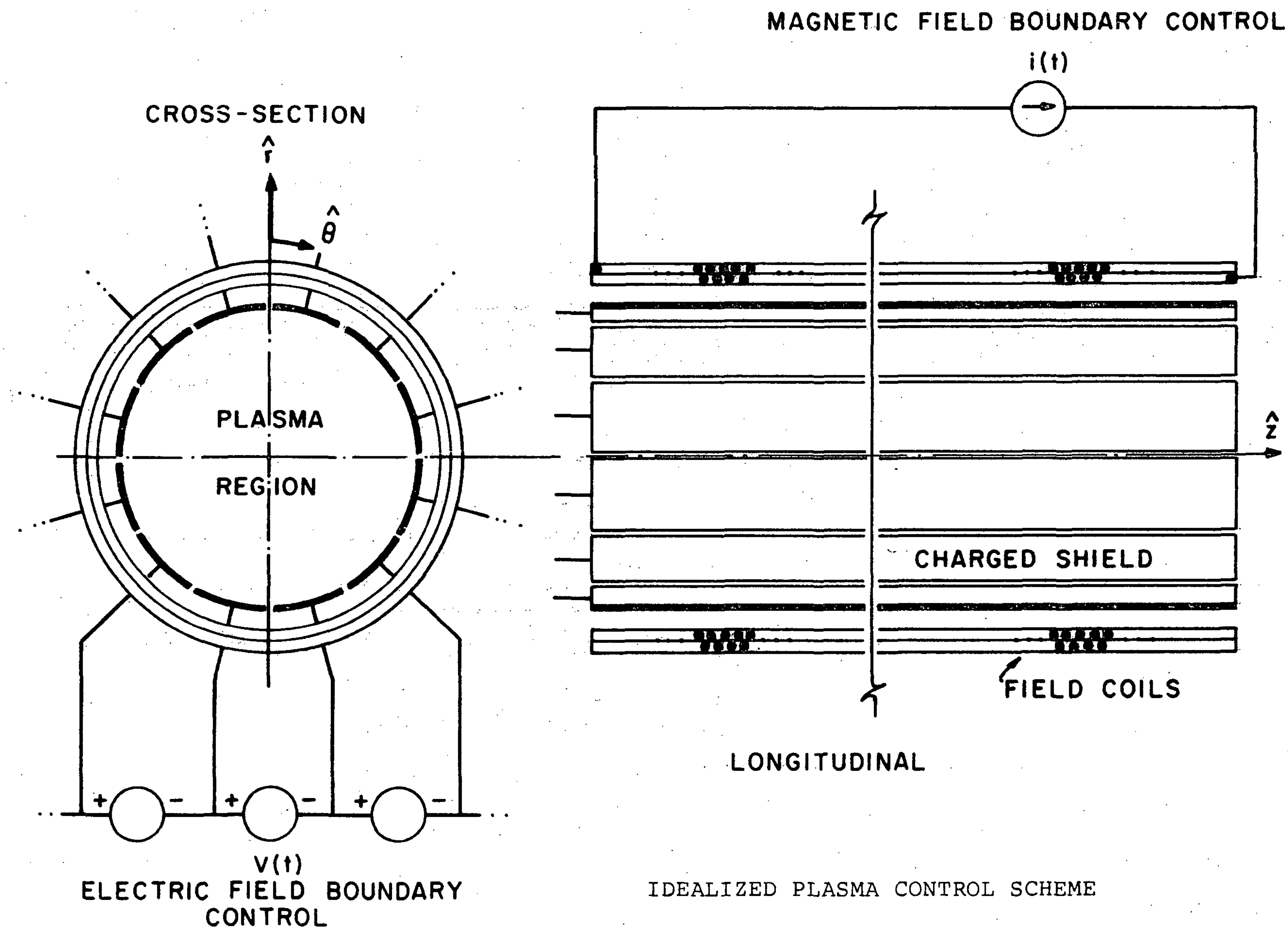
(1) This example is modified from the diocotron instability studied by Briggs [6], Levy [25], and Parker [32] using classical electromagnetic theory. A somewhat analogous electron beam instability is analyzed in [10]. Development of the mathematical model is original, however.



IDEALIZED PLASMA CONTROL SCHEME

Figure 2

frequency, determined by particle charge and imposed field. This closed cyclotron rotation constitutes a steady state, which is considered to be the plasma confinement objective. The particle density in this steady state varies as a function of radius and depends on the initializing discharge, the magnetic field distribution and the charge on the conducting sheets at the boundary of the cylinder.

The closed cyclotron motion, however, is dynamically unstable. Instabilities appear as rapid periodic space-charge oscillations about the steady state rotation; there are several modes of instability (radial, rotational, and longitudinal), each having a characteristic frequency. Only one instability, the diocotron instability, will be examined in this analysis (see Figure 3);[(2)] longitudinal oscillations are not considered.

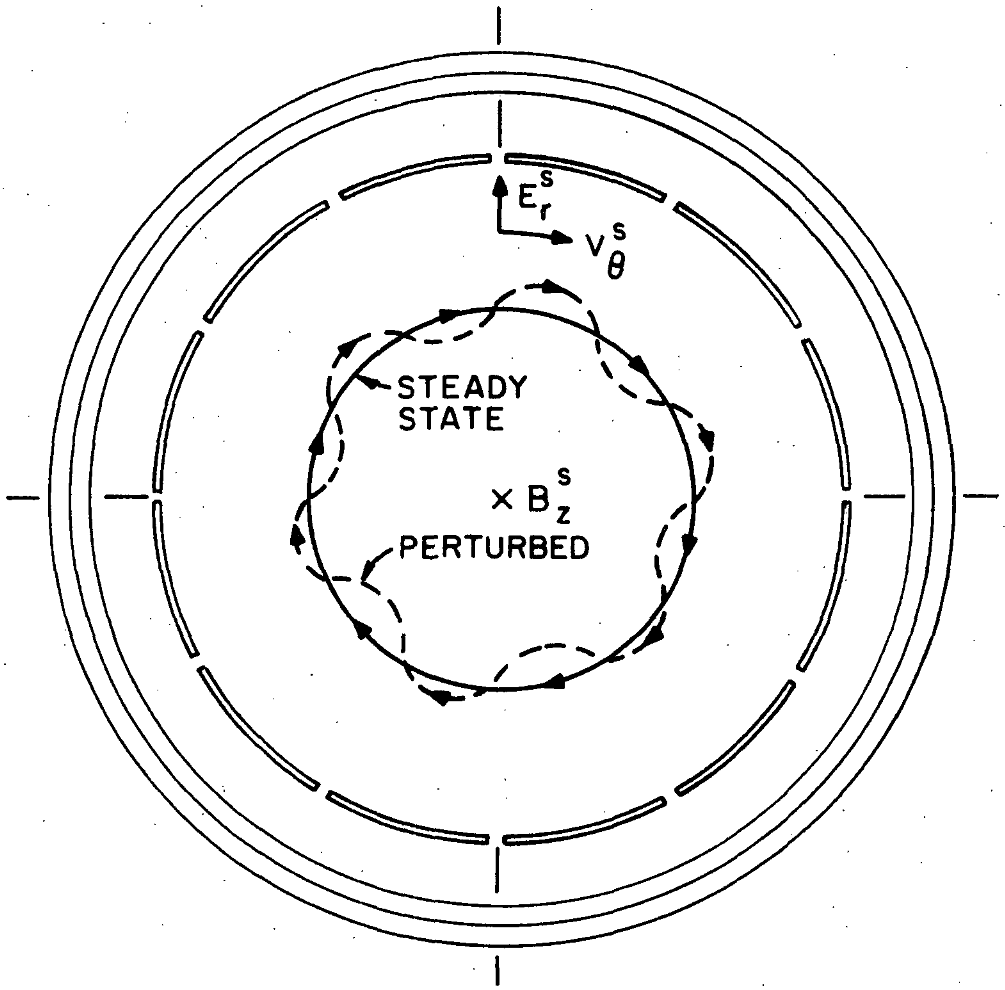
Boundary control of this instability is postulated via two variables: perturbation of the z-directed magnetic-field intensity and perturbation of the surface charge density on the outer conductors as a function of azimuth. While implementation of such controls is not fully "practical", it is at least possible to visualize the hardware required.[(3)]

---

(2) It should be evident that the same methodology is equally applicable to other instabilities <u>or</u> combinations of instabilities, however.

(3) Vary field coil current; apply voltages to strips of outer conducting shield. "Actuator dynamics" of these methods are not considered, but again could be included.

STEADY STATE AND DIOCOTRON INSTABILITY



NOTE: AMPLITUDE OF PERTURBED (DIOCOTRON) MOTION GROWS WITH TIME

Figure 3

The static (state-state, or "set-point") controls are designed to (1) maximize rotational kinetic energy, (2) minimize radiation losses, and (3) minimize control energy, while maintaining a cyclotron rotation of plasma particles. The dynamic controls are chosen to regulate plasma motion by minimizing (1) kinetic energy, (2) electromagnetic field energy of the perturbed motion, and (3) control energy. A great variety of other performance objectives could also be postulated.

3.2 State-Variable Formulations of Plasma Control Problems

The equations governing the plasma are developed from physical principlesin Appendix D, and simplifying assumptions are made in order to obtain a reduced model (equations (D.38)-(D.77)). Performance objectives for steady-state and linearinzed dynamic models are proposed to quantify the preceding discussion of Section 3.1.

On the basis of these physical models, two mathematical optimal control problems (steady-state and linearized dynamic cases) are formulated. State and control variables are identified as follows.

Notation

State-Space Model	Physical Model	Description
t	t	time
z_1	r	radial coordinate
z_2	θ	azimuthal coordinate

x_1	v_{ir}	mean radial ion velocity
x_2	$v_{i\theta}$	mean azimuthal ion velocity
x_3	n_i	ion particle density
x_4	v_{er}	mean radial electron velocity
x_5	$v_{e\theta}$	mean azimuthal electron velocity
x_6	n_e	electron particle density
x_7	E_r	radial electric field
x_8	E_θ	azimuthal electric field
x_9	B_z	longitudinal magnetic field
v_1	B_z^E	applied magnetic field at r=0
v_2	E_r^E	applied electric field at r=R
v_{3-11}	$(x_{1-8})^0$	dummy control variables at $\theta=2\pi$

The MKS system of measurement is implied throughout. Physical constants are as defined in Appendix D. Remaining state-variable notation is consistent with Chapters I and II.

Problem Formulation: Nonlinear Steady State Control

System Constraints: The equations are time-invariant and are actually of the special quasilinear form discussed in Section 1.4, though not analytic at $z_1 = 0$. Furthermore, the Decomposition Theorem of Section 1.5 is presumed to apply, and only the steady-state problem is considered here.[4] The steady-state equations thus take the form

$$\underline{0} = \underline{f}(\underline{x}^s, \underline{x}_z^s, \underline{z}) = \underline{A}_1(\underline{x}^s)\partial\underline{x}^s/\partial z_1 + \underline{K}(\underline{x}^s, \underline{z})\underline{x}^s \tag{2.1}$$

$$\text{BC: } \underline{g}(\underline{x}_B^s, \underline{v}^s) = \underline{T}(\underline{x}_B^s)\underline{x}_B^s - \underline{G}\,\underline{v}^s = \underline{0} \tag{2.2}$$

where the boundary $B = B_1^- \cup B_1^+ \cup B_2^- \cup B_2^+$ (see equation (D.23)), and

$$\underline{A}_1(\underline{x}^s, \underline{z}) = \begin{bmatrix} -x_1^s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_1^s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_3^s & 0 & -x_1^s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_4^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_4^s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x_6^s & 0 & -x_4^s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\varepsilon_o\mu_o)^{-1} \end{bmatrix} \tag{2.3}$$

---

(4) The nonlinear dynamic problem is so complicated as to be unenlightening.

$$\underline{K}(\underline{x}^s,\underline{z}) =$$

$$\begin{bmatrix}
0 & (q_i x_9^s/m_i) & 0 & 0 & 0 & 0 & (q_i/m_i) & 0 & 0 \\
(-q_i x_9^s/m_i) & 0 & 0 & 0 & 0 & 0 & 0 & (q_i/m_i) & 0 \\
0 & 0 & (-x_1^s/z_1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (q_e x_9^s/m_e) & 0 & (q_e/m_e) & 0 & 0 \\
0 & 0 & 0 & (-q_e x_9^s/m_e) & 0 & 0 & 0 & (q_e/m_e) & 0 \\
0 & 0 & 0 & 0 & 0 & (-x_4^s/z_1) & 0 & 0 & 0 \\
0 & 0 & (-q_i/\varepsilon_o) & 0 & 0 & (-q_e/\varepsilon_o) & (z_1)^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (z_1)^{-1} & 0 \\
0 & 0 & (-q_i x_2^s/\varepsilon_o) & 0 & 0 & (-q_e x_5^s/\varepsilon_o) & 0 & 0 & 0
\end{bmatrix} \tag{2.4}$$

which follows directly from (D.38)-(D.43), (D.47), (D.46), (D.45), respectively. The remaining equations (D.44), (D.48) prove to be redundant. At the boundaries

$$\underline{T}(\underline{x}^s_{B_1}-) = \begin{bmatrix}
m_i x_3^s & 0 & 0 & m_e x_6^s & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q_i x_1^s & 0 & 0 & q_e x_4^s & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{2.5}$$

$$\underline{T}(\underline{x}^s_{B_1}+) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \tag{2.6}$$

$$\underline{T}(\underline{x}^s_{B_2}-) = \underline{0} \tag{2.7}$$

$$\underline{T}(\underline{x}^s_{B_2}+) = \underline{I} \tag{2.8}$$

and

$$\underline{G}\ \underline{v}^s\Big|_{B_1^-} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_1^s \end{bmatrix} \quad \underline{G}\ \underline{v}^s\Big|_{B_1^+} = \begin{bmatrix} 0 \\ 0 \\ v_2^s \end{bmatrix} \quad \underline{G}\ \underline{v}^s\Big|_{B_2^-} = [\underline{0}] \quad \underline{G}\ \underline{v}^s\Big|_{B_2^+} = \begin{bmatrix} v_3^s \\ v_4^s \\ \cdots \\ v_{10}^s \\ v_{11}^s \end{bmatrix} \tag{2.9}$$

which follow from (D.49)-(D.58).

Performance Index: The performance index proposed in Appendix D (D.78) is of the "quasi-quadratic" class described in Section 1.4 (4.7,4.8). However, only the corresponding static problem of Section 1.5 is considered. Equation (5.12) becomes

$$J^s(\underline{v}^s) = \int_{B_1^-} j_{B_1^-}(\underline{x}^s_{B_1}-, \underline{v}^s)\, dB_1^- + \int_{B_1^+} j_{B_1^+}(\underline{x}^s_{B_1}+, \underline{v}^s)\, dB_1^+$$

$$+ \int_Z j_Z\ (\underline{x}^s, \underline{z})\ dZ \tag{2.10}$$

where from (D. 8),

$$j_{B_1^-}\ (\underline{x}^s_{B_1^-},\ \underline{v}^s) = \mu_o v_1^{\ 2} \tag{2.11}$$

$$j_{B_1^+}\ (\underline{x}^s_{B_1^+},\ \underline{v}^s) = \varepsilon_o v_2^{\ 2} + x_8^s\ x_9^s \tag{2.12}$$

$$j_Z\ (\underline{x}^s, \underline{z}) = -z_1^2 [m_i (x_2^s)^2 x_3^s + m_e (x_5^s)^2\ x_6^s] \tag{2.13}$$

The problem is to find a control $\underline{v}^{s*}$ such that

$$J^s(\underline{v}^{s*}) = \inf_{\underline{v}^s}\ J^s(\underline{v}^s) \tag{2.14}$$

Problem Formulation: Linearized Dynamics and Control

System Constraints: Linearized dynamics may be derived for any (not necessarily optimal) steady state solution of (2.1)-(2.2); for steady-state cyclotron rotation, $x_1^s = x_4^s = x_8^s = 0$. The model below applies to any cyclotron motion $\underline{x}^s(\underline{z})$ of the system and is developed independently of the preceding problem. The state equations are

$$\partial \underline{x}^t/\partial t = \sum_{i=1}^{2} \underline{A}_i(\underline{z})\ \partial \underline{x}^t/\partial z_i + \underline{K}(\underline{z})\underline{x}^t \qquad (t,\underline{z})\ \varepsilon\ D \tag{2.15}$$

$$\text{IC: } \underline{x}^t(0,\underline{z}) = \underline{x}_o^t(\underline{z}) \qquad \underline{z}\ \varepsilon\ Z \tag{2.16}$$

$$\text{BC: } \underline{T}(\underline{z}_B)\ \underline{x}_B^t(t) = \underline{G}(\underline{z}_B)\ \underline{v}^t(t) \qquad (t,\underline{z}_B)\ \varepsilon\ L \tag{2.17}$$

where

$$
\underline{A}_1(\underline{z}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_3^s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_6^s & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\varepsilon_o\mu_o)^{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\tag{2.18}
$$

$$
\underline{A}_2(\underline{z}) = \begin{bmatrix}
-x_2^s/z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_2^s/z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x_3^s/z_1 & -x_2^s/z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_5^s/z_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_5^s/z_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_6^s/z_1 & -x_5^s/z_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (z_1)^{-1} & 0 & 0
\end{bmatrix}
\tag{2.19}
$$

$$\underline{K}(\underline{z}) = \begin{bmatrix} 0 & q_i x_9^s/m_i & 0 & 0 & 0 & 0 & q_i/m_i & 0 & q_i x_2^s/m_i \\ -(x_2^s)_{z_1} - q_i x_9^s/m_i & 0 & 0 & 0 & 0 & 0 & 0 & q_i/m_i & 0 \\ -(x_3^s)_{z_1} - x_3^s/z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_e x_9^s/m_e & 0 & q_e/m_e & 0 & q_e x_5^s/m_e \\ 0 & 0 & 0 & -(x_5^s)_{z_1} - q_e x_9^s/m_e & 0 & 0 & 0 & q_e/m_e & 0 \\ 0 & 0 & 0 & -(x_6^s)_{z_1} - x_6^s/z_1 & 0 & 0 & 0 & 0 & 0 \\ -q_i x_3^s/\varepsilon_o & 0 & 0 & -q_e x_6^s/\varepsilon_o & 0 & 0 & 0 & 0 & 0 \\ 0 & -q_i x_3^s/\varepsilon_o & -q_i x_2^s/\varepsilon_o & 0 & -q_e x_6^s/\varepsilon_o & -q_e x_5^s/\varepsilon_o & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(z_1)^{-1} & 0 \end{bmatrix} \tag{2.20}$$

which follows from (D.59)-(D.67). At the boundaries

$$\underline{T}(0,z_2)=\begin{bmatrix} m_i x_3^s & 0 & 0 & m_e x_6^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_i x_3^s & 0 & 0 & q_e x_6^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.21}$$

$$\underline{T}(R,z_2)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{2.22}$$

$$\underline{T}(z_1,0) \;=\; \underline{0} \tag{2.23}$$

$$T(z_1,2\pi) \;=\; \underline{I} \tag{2.24}$$

The control term, $\underline{G}\ \underline{v}$ in (2.17) has the same form as in (2.9). These results follow from (D.68)-(D.77).

<u>Performance Index:</u> The performance index proposed in Appendix D (D.79) is quadratic in the response

$$\underline{y} = \underline{C}\ \underline{x} \tag{2.25}$$

where

$$\underline{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.26)$$

It may be written

$$J^t(\underline{v}^t) = \int_{L_1^-} r_v^- v_1^2 \, dL_1^- + \int_{L_1^+} r_v^+ \, v_2^2 \, dL_1^+ + \int_D \underline{y}' \, \underline{Q}(\underline{z}) \, \underline{y} \, dD \quad (2.27)$$

where

$$r_v^- = \mu_o \; ; \qquad r_v^+ = \varepsilon_o \quad (2.28)$$

and

$$\underline{Q}(\underline{z}) = \begin{bmatrix} m_i x_3^s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i x_3^s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_e x_6^s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_e x_6^s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_o & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_o & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_o \end{bmatrix}$$

(2.29)

The problem is to find a control $\underline{v}^{t*}$ such that

$$J^t(\underline{v}^{t*}) = \inf_{\underline{v}^t} J^t(\underline{v}^t) \tag{2.30}$$

3.3 Well Posedness of Solutions to State Equations

Steady State Equations:

Equation (2.1) is actually a nonlinear system of ordinary differential equations in z_1. Cyclotron rotation ($x_1^s = x_4^s = 0$) represents a singular solution of these equations, however, which simplifies the algebra and complicates the mathematics. Assuming cyclotron motion, algebraic manipulation of (2.1) yields

$$\frac{\partial}{\partial z_1} \begin{bmatrix} x_7^s \\ \\ x_9^s \end{bmatrix} = \begin{bmatrix} -(z_1)^{-1} x_7^s + (\varepsilon_o)^{-1}(q_i x_3^s + q_e x_6^s) \\ \\ (\mu_o x_7^s / x_9^s)(q_i x_3^s + q_e x_6^s) \end{bmatrix} \tag{3.1}$$

with
$$\begin{aligned} x_1^s &= 0 \\ x_2^s &= -x_7^s/x_9^s \\ x_4^s &= 0 \\ x_5^s &= -x_7^s/x_9^s \\ x_8^s &= 0 \end{aligned}$$

the particle densities x_3^s, x_6^s are indeterminate due to the singularity of the equations, but given these quantities, such that (2.2) is satisfied, (3.1) can be integrated to yield a solution for any set of boundary controls, v_1^s, v_2^s.

A rigorous mathematical treatment of well-posedness theory for this system of equations is beyond the scope of this research, but would necessarily include (1) treatment of the allowable boundary values for x_3^s, x_6^s, x_7^s, x_9^s, (2) development of a formalism to handle the indeterminacy of x_3^s, x_6^s, and (3) consideration of the singularity of (3.1) at $z_1 = 0$. The reader is referred to [37], [46] for a mathematical treatment of similar singular problems. It is noted that the continuity assumptions of the Minimum Principle are satisfied except at $z_1 = 0$.

Linearized Dynamic Equations:

Proof of well-posedness of the initial boundary value problem (2.15)-(2.17) is beyond the scope of this research. It is remarked that the system is not strictly hyperbolic and that the boundary conditions are not imposed on a half-space; hence the theory of Sections 2.1-2.4, due to Rauch, does not apply. Nevertheless, there is reason to believe that the system may be well-posed (at least for certain steady states) in the sense of Section 2.1. The hyperbolicity condition (see Chapter 2, Assumption 1-1

$$|\lambda_o \underline{I} - \lambda_1 \underline{A}_1(\underline{z}) - \lambda_2 \underline{A}_2(\underline{z})|$$

$$= (\lambda_o + \lambda_2 x_2^s/z_1)^3 (\lambda_o + \lambda_2 x_5^s/z_1)^3 \lambda_o (\lambda_o + \lambda_1)(\lambda_o + \lambda_1 (\mu_o \varepsilon_o)^{-1})$$

$$= 0 \tag{3.2}$$

reveals multiple real roots

$$\lambda_o = -(x_2^s(z_1)/z_1)\lambda_2 \qquad \text{(multiplicity 3)} \tag{3.3}$$

$$\lambda_o = -(x_5^s(z_1)/z_1)\lambda_2 \qquad \text{(multiplicity 3)}$$

$$\lambda_o = 0 \qquad \text{(multiplicity 1)}$$

$$\lambda_o = -\lambda_1 \qquad \text{(multiplicity 1)}$$

$$\lambda_o = -(\mu_o\varepsilon_o)^{-1}\lambda_1 \qquad \text{(multiplicity 1)}$$

The system may be split into coupled hyperbolic systems by integrating the ordinary linear differential equation for $\partial x_7^t/\partial t$ (eliminating the root $\lambda_o = 0$) and considering the equations for $\partial x_8^t/\partial t$, $\partial x_9^t/\partial t$ as partial differential equations in z_1 with z_2 as a parameter. these hyperbolic subsystems may then be analyzed by the method of characteristics as developed in Appendix B, equations (B.14)ff, without using analytic continuation. It is assumed that the conditions of Ch.II,Th.3-3 are valid in the following derivations (this assumption is untrue; it is postulated a priori in order to reveal "what goes wrong").

3.4 Optimality Conditions

Nonlinear Steady State Control

The static optimization theory developed in Appendix C

(Proof of Decomposition Theorem) is used in evaluating necessary conditions for the solution of the steady-state boundary control problem (2.1), (2.2), (2.10). The Hamiltonian is (see (2.1), (2.10), (C.5))

$$h(\underline{x}^s, \underline{x}_z^s, \underline{p}^s, \underline{z}) = j_z(\underline{x}^s, \underline{z}) + \underline{p}^{s'} \underline{f}(\underline{x}^s, \underline{x}_z^s, \underline{z})$$

$$= j_z(\underline{x}^s, \underline{z}) + \underline{p}^{s'} [\underline{A}_1(\underline{x}^s, \underline{z}) \partial \underline{x}^s / \partial z_1 + \underline{K}(\underline{x}^s, \underline{z}) \underline{x}^s] \tag{4.1}$$

Letting ( )* denote optimal quantities, necessary conditions are (see (C.18)-(C.21)):

$$\underline{f}^s(\underline{x}^*, \underline{x}_z^*, \underline{z}) = \underline{0} \tag{4.2}$$

$$\text{BC: } \underline{g}^s(\underline{x}_B^*, \underline{v}^*) = \underline{0} \tag{4.3}$$

$$\partial(\partial h/\partial \underline{x}_{z_1})^*/\partial z_1 - (\partial h/\partial \underline{x})^* = \underline{0} \tag{4.4}$$

BC:

$$([\underline{g}_{\underline{x}}^{\perp'} \underline{g}_{\underline{x}}^{\perp}]_{B_1^-}^*)^{-1} \underline{g}_{\underline{x}}^{\perp *'}{}_{B_1^-} [-\partial h/\partial \underline{x}_{z_1}]_{B_1^-}^* = \underline{0} \tag{4.5}$$

$$([\underline{g}_{\underline{x}}^{\perp'} \underline{g}_{\underline{x}}^{\perp}]_{B_1^+}^*)^{-1} \underline{g}_{\underline{x}}^{\perp *'}{}_{B_1^+} [\partial j_{B_1^+}/\partial \underline{x} + \partial h/\partial \underline{x}_{z_1}]_{B_1^+}^* = \underline{0} \tag{4.6}$$

and

$$\partial j_{B_1^-}^*/\partial v_1 - \partial \underline{g}_{B_1^-}^*/\partial v_1 ([\underline{g}_{\underline{x}}' \underline{g}_{\underline{x}}]_{B_1^-}^*)^{-1} \underline{g}_{\underline{x}}^{*'}{}_{B_1^-} [-\partial h/\partial \underline{x}_{z_1}]_{B_1^-}^* = \underline{0} \tag{4.7}$$

$$\partial j_{B_1^+}^*/\partial v_2 - \partial \underline{g}_{B_1^+}^*/\partial v_2 ([\underline{g}_{\underline{x}}' \underline{g}_{\underline{x}}]_{B_1^+}^*)^{-1} \underline{g}_{\underline{x}}^{*'}{}_{B_1^+} [\partial j_{B_1^+}/\partial \underline{x} + \partial h/\partial \underline{x}_{z_1}]_{B_1^+}^* = 0 \tag{4.8}$$

The required quantities are

$$\partial h/\partial \underline{x} = \partial j_Z/\partial \underline{x} + (\partial [\underline{K}(\underline{x}^s, \underline{z})\underline{x}^s]'/\partial \underline{x}\ \underline{p}^s) \tag{4.9}$$

$$\partial h/\partial \underline{x}_{z_1} = \underline{A}_1'(\underline{x}^s, \underline{z})\ \underline{p}^s \tag{4.10}$$

$$(\partial \underline{g}/\partial \underline{x}^{\perp})'_{B_1^-} = \begin{bmatrix} x_1^s & 0 & -x_3^s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4^s & 0 & -x_6^s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{4.11}$$

$$(\partial \underline{g}/\partial \underline{x}^{\perp})'_{B_1^+} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.12}$$

$$[\partial j_{B_1^+}/\partial \underline{x}]' = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad x_9^s \quad x_8^s] \tag{4.13}$$

$$\partial j_{B_1^-}/\partial v_1 = 2\mu_o\ v_1 \tag{4.14a}$$

$$\partial j_{B_1^+}/\partial v_2 = 2\varepsilon_o\ v_2 \tag{4.14b}$$

$$\partial \underline{g}_{B_1^-}/\partial v_1 = -[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \tag{4.15}$$

$$\partial \underline{g}_{B_1^+}/\partial v_2 = -[0 \quad 0 \quad 1] \tag{4.16}$$

The costate equations (4.4)-(4.6) may thus be rewritten

$$\underline{0} = -\,\underline{A}_1'(\underline{x}^*,\underline{z})\,\partial\underline{p}^*/\partial z_1$$

$$+\begin{bmatrix} 0 & -q_i x_9^*/m_i & -x_3^*/z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_i x_9^*/m_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_i x_3^*/\varepsilon_o \\ 0 & 0 & -x_1^*/z_1 & 0 & 0 & 0 & -q_i/\varepsilon_o & 0 & -q_i x_2^*/\varepsilon_o \\ 0 & 0 & 0 & 0 & -q_e x_9^*/m_e & -x_6^*/z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_e x_9^*/m_e & 0 & 0 & 0 & 0 & -q_e x_6^*/\varepsilon_o \\ 0 & 0 & 0 & 0 & 0 & -x_4^*/z_1 & -q_e/\varepsilon_o & 0 & -q_e x_5^*/\varepsilon_o \\ q_i/m_i & 0 & 0 & q_e/m_e & 0 & 0 & 1/z_1 & 0 & 0 \\ 0 & q_i/m_i & 0 & 0 & q_e/m_e & 0 & 0 & 1/z_1 & 0 \\ -q_i x_2^*/m_i & -q_i x_1^*/m_i & 0 & -q_e x_5^*/m_e & -q_e x_4^*/m_e & 0 & 0 & 0 & 0 \end{bmatrix} \underline{p}^* + \begin{bmatrix} 0 \\ -2z_1^2 m_i x_2^* x_3^* \\ -z_1^2 m_i (x_2^*)^2 \\ 0 \\ -2z_1^2 m_e x_5^* x_6^* \\ -z_1^2 m_e (x_5^*)^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(4.17)

With boundary conditions (only on $B_1^{\pm}$, none on $B_2^{\pm}$)

$$\begin{bmatrix} (x_1^{*2} + x_3^{*2})^{-1} \; x_1^{*2} \; p_1^* \\ (x_4^{*2} + x_6^{*2})^{-1} \; x_4^{*2} \; p_4^* \\ -p_7^* \end{bmatrix}_{B_1^-} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.18}$$

$$\begin{bmatrix} -x_1^* \; p_2^* \\ -x_1^* \; p_3^* \\ -x_4^* \; p_5^* \\ -x_4^* \; p_6^* \\ p_8^* + x_9^* \\ -(\varepsilon_o\mu_o)^{-1} p_9^* + x_8^* \end{bmatrix}_{B_1^+} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.19}$$

And boundary controls

$$v_1{}^* = -(2\varepsilon_o\mu_o{}^2)^{-1} \; p_9{}^* \qquad (\text{on } B_1^-) \tag{4.20}$$

$$v_2{}^* = +(2\varepsilon_o)^{-1} \; p_7{}^* \qquad (\text{on } B_1^+) \tag{4.21}$$

<u>Remark 4-1:</u> The necessary conditions leave the dummy control variables v_{3-11} on B_2^+ indeterminate, and these have no effect on the conditions of optimality, which is consistent with intuition. By reference to (D.33)-(D.37) and (2.9),

recall that these variables are chosen equal to the state variables on B_2^- due to geometric constraints.

Remark 4-2: The structure of the costate equations is the same as the structure of the state equations, except for the autonomous driving term of (4.17); again these equations are nonlinear ordinary differential equations in z_1. Standard methods of algebra may be used to determine whether the necessary conditions (4.2)-(4.8) are sufficient to uniquely specify the optimal controls. Such computational techniques are beyond the scope of the present research.

Linear Dynamic Control

The linear control theory developed in Section 2.3 is used in characterizing the optimal controls for the present boundary control example, under the assumption that the linearized dynamics constitute a well-posed system in the sense discussed in Section 2.5. Applying equations (3.2)-(3.9) of Chapter I to the problem (2.15)-(2.17), (2.25), (2.27), the following system is obtained:(5)

$$\partial \underline{x}^*/\partial t = \sum_{i=1}^{2} \underline{A}_i(\underline{z})\ \partial \underline{x}^*/\partial z_i + \underline{K}(\underline{z})\underline{x}^* \qquad (t,\underline{z}) \ \varepsilon\ D \tag{4.22}$$

$$\text{IC: } \underline{x}^*(0) = \underline{x}_o(\underline{z}) \qquad \underline{z}\ \varepsilon\ Z \tag{4.23}$$

$$\text{BC: } \underline{T}(\underline{z})\underline{x}^*_L = \underline{G}\ \underline{v}^* \qquad (t,\underline{z})\ \varepsilon\ L \tag{4.24}$$

(5) The superscript $(\ )^t$ is supressed.

$$-\partial \underline{p}^*/\partial t = -\sum_{i=1}^{2} \underline{A}_i'(\underline{z})\partial \underline{p}^*/\partial z_i + [\underline{K}'(\underline{z}) - \sum_{i=1}^{2} \partial \underline{A}_i'(\underline{z})/\partial z_i]\underline{p}^*$$

$$+ \underline{C}'(\underline{z})\ \underline{Q}(\underline{z})\ \underline{C}(\underline{z})\underline{x}^* \qquad (t,\underline{z}) \in D \tag{4.25}$$

$$\text{FC: } \underline{p}^*(t_f) = \underline{0} \qquad \underline{z} \in Z \tag{4.26}$$

$$\text{BC: } (\underline{T}\ (\underline{z})\ \underline{T}\ (\underline{z})')^{-1}\underline{T}\ (\underline{z})[\sum_{i=1}^{2} n_i \underline{A}_i'(\underline{z})]\ \underline{p}_L^* = \underline{0} \quad (t,\underline{z}) \in L \tag{4.27}$$

$$\underline{v}^* = -\underline{R}_v^{-1}\ \underline{G}'(\underline{T}(z)\underline{T}'(z))^{-1}\underline{T}(\underline{z})[\sum_{i=1}^{2} n_i \underline{A}_i'(\underline{z})]\ \underline{p}_L^* \tag{4.28}$$

The meaning of these equations is clear, except the boundary conditions (4.24), (4.27) and (4.28), which may be written more explicitly as follows. On the boundary B_1^-,

$$m_i x_3^s(0,z_2)x_1^*(0,z_2) + m_e x_6^s(0,z_2)x_4^*(0,z_2) = 0 \tag*{(4.24a)$_1$}$$

$$x_2^*(0,z_2) = 0 \tag*{(4.24b)$_1$}$$

$$q_i x_3^s(0,z_2)x_1^*(0,z_2) + q_e x_6^s(0,z_2)x_4^*(0,z_2) = 0 \tag*{(4.24c)$_1$}$$

$$x_5^*(0,z_2) = 0 \tag*{(4.24d)$_1$}$$

$$x_8^*(0,z_2) = 0 \tag*{(4.24e)$_1$}$$

$$x_9^*(0,z_2) = v_1^* \tag*{(4.24f)$_1$}$$

$$0 = 0 \qquad (4.27a)_1$$

(No costate constraints!) $$0 = 0 \qquad (4.27b)_1$$

$$0 = 0 \qquad (4.27c)_1$$

$$v_1^* = -(\mu_o^2 \varepsilon_o)^{-1} \quad p_8^*(0, z_2) \qquad (4.28a)_1$$

On the boundary B_1^+,

$$x_1{}^*(R, z_2) = 0 \qquad (4.24g)_1$$

$$x_4{}^*(R, z_2) = 0 \qquad (4.24h)_1$$

$$x_7{}^*(R, z_2) = v_2{}^* \qquad (4.24i)_1$$

(No costate constraints!)

$$0 = 0 \qquad (4.27d)_1$$

$$0 = 0 \qquad (4.27e)_1$$

$$0 = 0 \qquad (4.27f)_1$$

$$0 = 0 \qquad (4.27g)_1$$

$$-p_9{}^*(R, z_2) = 0 \qquad (4.27h)_1$$

$$-(\mu_o \varepsilon_o)^{-1} p_8{}^*(R, z_2) = 0 \qquad (4.27i)_1$$

$$v_2{}^* = 0 \qquad (4.28b)_1$$

On the boundary B_2^-,

(No state constraints) $$0 = 0 \qquad (4.24)_2$$

$$(z_1)^{-1} x_2^s(z_1, 0) \; p_1{}^*(z_1, 0) = 0 \qquad (4.27a)_2$$

$$(z_1)^{-1} x_2^s(z_1, 0) \; p_2{}^*(z_1, 0) = 0 \qquad (4.27b)_2$$

$$(z_1)^{-1}[x_3^s(z_1,0)p_2^*(z_1,0) + x_2^s(z_1,0)p_3^*(z_1,0)] = 0 \qquad (4.27c)_2$$

$$(z_1)^{-1}\ x_5^s(z_1,0)\ p_4^*(z_1,0) = 0 \qquad (4.27d)_2$$

$$(z_1)^{-1}\ x_5^s(z_1,0)\ p_5^*(z_1,0) = 0 \qquad (4.27c)_2$$

$$(z_1)^{-1}[x_6^s(z_1,0)p_5^*(z_1,0) + x_5^s(z_1,0)p_6^*(z_1,0)] = 0 \qquad (4.27f)_2$$

$$(z_1)^{-1}\ p_9^*(z_1,0) = 0 \qquad (4.27g)_2$$

$$0 = 0 \qquad (4.27h)_2$$

(No costate constraints)

$$0 = 0 \qquad (4.27i)_2$$

On the boundary B_2^+,

$$x_{1-9}^*(z_1,2\pi) = v_{3-11}^* \qquad (4.24a\text{-}i)_2$$

(No costate constraints) $\quad 0 = 0 \qquad (4.27)_2$

$$v_{3-11}^*\ :\ \text{indeterminate} \qquad (4.28)_2$$

<u>Remark 4-3:</u> The fact that $v_2^* = 0$, $(4.28b)_1$, provides for an interesting commentary on the importance of existence theory (Section 2.5) in the solution of optimal control problems. The preceding results were derived under the assumption of well-posedness of linearized state and costate equations; although the state equations are likely to be well-posed (see Section 3.4), the costate equations, <u>as defined by</u> (4.25)-(4.27) may not be well-posed. This fact is due to the high degree of singularity of $\underline{A}_1(\underline{z})$, which in turn is due to the singularity of the solution of the steady-state solution

representing cyclotron motion (viz. $x_1^s = x_4^s = 0$). The boundaries $B_1^{\pm}$ are not "non-characteristic" in the sense of Friedrichs [15], and hence violate an (implicit) condition, used in deriving the costate boundary conditions (4.27). It is possible to modify the derivations of Section 2.3 by redefining the costate boundary conditions in order to remove this difficulty. This modification, however, does not affect the form of the optimal control law for v_1* as given by $(4.28a)_1$.

Remark 4-4: By examining the costate equations it is possible to verify that $p_8{}^*(0,z_2)$ depends (linearly) only on the perturbed particle densities $x_1{}^*$, $x_3{}^*$; hence the optimal control law could be implemented by sensing these two quantities. This might be accomplished by measuring absorbtion or scattering of electromagnetic radiation (low intensity) passed longitudinally through the plasma.

Remark 4-5: It is possible to apply the decoupling theory of Section 2.4, but by virtue of the structure of the equations (see Remark 4-4), this is not necessary to evaluate the control law (for $v_1{}^*$, at least) in this example.

CONCLUSIONS and SUGGESTIONS for FUTURE RESEARCH

Optimal control of first-order distributed systems constitutes a theoretically rich and potentially applicable research topic. It is possible to identify particularly important control problems (e.g., boundary control) as well as very challenging mathematical difficulties (e.g., uniqueness of solutions), and a variety of physical applications which are characteristic of first-order systems. The class of systems is simple enough, however, that concrete solutions may be obtained (with diligence) to at least some of these questions. More general abstract optimization theories may be given concrete meaning, considering first-order systems as a special case. The major contributions of this research are:

(1) Formulation of the boundary control problem.

(2) A minimum principle for nonlinear systems.

(3) A decomposition theorem.

(4) Verification and extension of Lions' results to boundary control of first-order systems.

(5) Development of a realistic example involving control of a plasma.

These results constitute a very modestfirst approach to the general theory of control for first-order systems. Among the many topics worthy of future research are:

(1) Use of first-order systems as a canonical form; development of a minimum principle with state space constraints.

(2) Definition of weak solutions to the nonlinear problem; formulation of a minimum principle for such solutions.

(3) Development of a global minimum principle using finite (rather than differential) control perturbations.

(4) Properties of "steady-state" controls; definition of relevant concepts (e.g., controllability).

(5) Computational algorithms for the Riccati integro-differential equation.

(6) Concrete characterization and explanation of well set initial boundary value problems for hyperbolic systems with simple boundaries, with particular attention to characterizing the range of the boundary operator $\underline{T}$.

(7) Extension and/or application of the results of Carroll & Mazumdar on generalized concepts of coercivity. Definition of weak solutions.

(8) Study of examples which are not hyperbolic but are well-posed in the sense of Friedrichs and Lax.

(9) Study of degenerate boundary conditions, such as arose in the linearized plasma example.

(10) First-order system representations for fundamental laws of science.

(11) Controller structure simplification; investigation of modal control.

(12) Effect of coordinate transformations and local geometries on optimal control laws.

And many more!

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APPENDIX A: SECOND VARIATION CONDITIONS FOR A MINIMUM

Consider control variations $\delta\underline{u}$, $\delta\underline{v}$ as given in Chapter 1, (3.10), (3.11). The state perturbation is now carried to second order in ε:

$$\underline{x} = \underline{x}^* + \varepsilon\delta\underline{x} + \varepsilon^2\delta^2\underline{x} + 0_1(\varepsilon^2) \tag{A.1}$$

which replaces (3.12). $\delta\underline{x}$ is now <u>defined</u> as the solution of (3.13)-(3.15) with the terms $0_1(\varepsilon)/\varepsilon$ and $0_2(\varepsilon)/\varepsilon$ omitted.(1) The second-order variation, $\delta^2\underline{x}$ is subject to:

$$\text{IC} : \delta^2\underline{x}\ (0,\underline{z}) = \underline{0} \tag{A.2}$$

$$\text{BC} : \tag{A.3}$$

$$(\underline{g}_{\underline{x}})^{*\prime}\delta^2\underline{x}_L + \frac{1}{2}[\delta\underline{x}_L'\ \ \delta\underline{v}_L'] \begin{bmatrix} \underline{g}^*_{\underline{xx}} & \underline{g}^*_{\underline{xv}} \\ \underline{g}^*_{\underline{vx}} & \underline{g}^*_{\underline{vv}} \end{bmatrix} \begin{bmatrix} \delta\underline{x}_L \\ \delta\underline{v} \end{bmatrix} = \underline{0}$$

where for instance

$$[\delta\underline{x}_L'\ \underline{g}^*_{\underline{xx}}]_{ij} = \sum_{k=1}^{n_x} [\partial(\partial\underline{g}/\partial\underline{x})_{ij}/\partial x_k]^*(\delta\underline{x}_L)_k$$

$$i = 1, 2, \ldots, n_g \tag{A.4}$$

$$j = 1, 2, \ldots, n_x$$

---

(1) First and second variational quantities should not be confused with those used in Chapter 1 unless the relationship is clearly indicated.

defines the shorthand notation for the second variations (tensors) of $\underline{g}$.

The performance index is modified by the addition of a LaGrange multiplier, yielding (3.16). The modified index $\tilde{J}(\underline{u},\underline{v})$ is now expanded through second order in ε:

$$\tilde{J}(\underline{u},\underline{v}) = \tilde{J}(\underline{u}^*,\underline{v}^*) + \varepsilon\delta\tilde{J}(\underline{u}^*,\underline{v}^*) + \varepsilon^2\delta^2\tilde{J}(\underline{u}^*,\underline{v}^*) + 0(\varepsilon^2) \quad (A.5)$$

The condition

$$\delta\tilde{J}(\underline{u}^*,\underline{v}^*) = 0 \quad \forall \delta\underline{u},\ \delta\underline{v} \quad (A.6)$$

which replaces (3.18) leads to the first-order conditions of the Minimum Principle given in Chapter 1. It is now desired to evaluate the second variation condition for a minimum,

$$\delta^2\tilde{J}(\underline{u}^*,\underline{v}^*) > 0 \quad \forall \delta\underline{u},\ \delta\underline{v} \quad (A.7)$$

explicitly. This quantity depends on $\delta\underline{u}$ and $\delta\underline{v}$, and on the solution of the (linear) equations for $\delta\underline{x}$, but the terms involving $\delta^2\underline{x}$ can be eliminated by using (A.3), (A.4), and (A.6), as follows:

Again, Taylor series expansions of j_f, j_L and h about the optimum may be used, and disappearance of the higher-order terms (in the limit $\varepsilon \to 0$) is guaranteed by the differentiability assumptions of the Theorem. The series are:

$$j_f = j_f^* + \varepsilon(\partial j_f/\partial\underline{x})^{*\prime}\delta\underline{x}_f + \varepsilon^2[(\frac{\partial j_f}{\partial\underline{x}})^{*\prime}\delta^2\underline{x}_f + \frac{1}{2}\,\delta\underline{x}_f'(\frac{\partial^2 j_f}{\partial\underline{x}^2})^*\delta\underline{x}_f] + 0(\varepsilon^2) \quad (A.8)$$

where $\partial^2 j_f/\partial \underline{x}^2$ is the matrix with elements

$$[\partial^2 j_f/\partial \underline{x}^2]^*_{ij} = \partial^2 j_f/\partial x_i \partial x_j \; (\underline{x}^*_f, \underline{z}) \qquad i,j = 1,2,\ldots,n_x \tag{A.9}$$

and the superscript (*) is an abbreviation for evaluation at $\underline{u}^*$, $\underline{v}^*$, $\underline{x}^*$, $\underline{p}^*$, etc. as defined in the statement of the Theorem (Chapter 1). The fact that j_f is twice continuously differentiable in $\underline{x}$ guarantees that

$$\lim_{\varepsilon \to 0} \theta(\varepsilon^2)/\varepsilon^2 = 0 \tag{A.10}$$

Similar comments apply to the remaining expansions.

$$j_L = j_L^* + \varepsilon[(\partial j_L/\partial \underline{x})^{*\prime} \, \delta \underline{x}_L + (\partial j_L/\partial \underline{v})^{*\prime} \, \delta \underline{v}]$$

$$+ \varepsilon^2\{(\partial j_L/\partial \underline{x})^{*\prime} \, \delta \underline{x}_L + \frac{1}{2} [\delta \underline{x}_L' \;\; \delta \underline{v}'](\delta^2 j_L)^* \begin{bmatrix} \delta x_L \\ \delta \underline{v} \end{bmatrix} \} + 0(\varepsilon^2)$$

$$\tag{A.11}$$

where

$$(\delta^2 j_L)^* = \begin{bmatrix} j^*_{L_{\underline{xx}}} & j^*_{L_{\underline{xv}}} \\ j^*_{L_{\underline{vx}}} & j^*_{L_{\underline{vv}}} \end{bmatrix} \tag{A.12}$$

is an $(n_x + n_v) \times (n_x + n_v)$ matrix.

$$h = h^* + \varepsilon[(\partial h/\partial \underline{x})^{*'}\ \delta\underline{x} + (\partial h/\partial \underline{x}_z)^{*'}\ \delta\underline{x}_z + (\partial h/\partial \underline{u})^{*'}\ \delta\underline{u}]$$

$$+ \varepsilon^2\{(\partial h/\partial \underline{x})^{*'}\ \delta^2\underline{x} + (\partial h/\partial \underline{x}_z)^{*'}\ \delta^2\underline{x}_z$$

$$+ \frac{1}{2}\ [\delta\underline{x}'\ \ \delta\underline{x}_z'\ \ \delta\underline{u}'](\delta^2 h)^* \begin{bmatrix} \delta\underline{x} \\ \delta\underline{x}_z \\ \delta\underline{u} \end{bmatrix}\ \}$$

$$+ 0(\varepsilon^2) \qquad \text{(A.13)}$$

where

$$(\partial h/\partial \underline{x}_z)^{*'} = [(\partial h/\partial \underline{x}_{z_1})^{*'}\ (\partial h/\partial \underline{x}_{z_2})^{*'}\ \ldots\ (\partial h/\partial \underline{x}_{z_m})^{*'}] \qquad \text{(A.14)}$$

is an mn_x-dimensional row vector, and

$$(\delta^2\underline{h})^* = \begin{bmatrix} (h_{\underline{xx}})^* & (h_{\underline{xx}_z})^* & (h_{\underline{xu}})^* \\ (h_{\underline{x}_z\underline{x}})^* & (h_{\underline{x}_z\underline{x}_z})^* & (h_{\underline{x}_z\underline{u}})^* \\ (h_{\underline{ux}})^* & (h_{\underline{ux}_z})^* & (h_{\underline{uu}})^* \end{bmatrix} \qquad \text{(A.15)}$$

is an $(n_x+mn_x+n_u) \times (u_x+mn_x+n_u)$ matrix. The relations (3.23) and (3.24) apply to the first-order quantities. In addition there is a second-order term

$$\varepsilon^2\underline{p}'\partial(\delta^2\underline{x})/\partial t = \varepsilon^2[\partial(\underline{p}'\delta^2\underline{x})/\partial t - (\partial\underline{p}'/\partial t)\delta^2\underline{x}] \qquad \text{(A.16)}$$

and the relation

$$\varepsilon^2 (\partial h/\partial \underline{x}_{z_i})^{*'} \partial(\delta^2 \underline{x})/\partial z_i = \varepsilon^2 [\frac{\partial}{\partial z_i} \{(\partial h/\partial \underline{x}_{z_i})^{*'} \delta^2 \underline{x}\}$$

$$- \{\frac{\partial}{\partial z_i} (\partial h/\partial \underline{x}_{z_i})^{*'}\} \delta^2 \underline{x}] \qquad \text{(A.17)}$$

which may be applied to (A.13). These results combine to yield

$$\delta \tilde{J}(\underline{u}^*, \underline{v}^*) = \int_Z (\partial j_f/\partial \underline{x})^{*'} \delta \underline{x}_f dZ + \int_L [(\partial j_L/\partial \underline{x})^{*'} \delta \underline{x}_L + (\partial j_L/\partial \underline{v})^{*'} \delta \underline{v}] dL$$

$$+ \int_D \{[\partial \underline{p}'/\partial t + (\partial h/\partial \underline{x})^{*'}] \delta \underline{x} - (\partial h/\partial \underline{x}_z)^{*'} \delta \underline{x}_z$$

$$+ (\partial h/\partial \underline{u})^{*'} \partial \underline{u}\} \, dD$$

$$+ \int_D [-\frac{\partial}{\partial t} (\underline{p}' \delta \underline{x}) + \sum_{i=1}^{m} \frac{\partial}{\partial z_i} ((\partial h/\partial \underline{x}_{z_i})^{*'} \delta \underline{x})] dD \qquad \text{(A.18)}$$

which is analogous to (3.25) and yields the same first-order necessary conditions. The second variation becomes

$$\delta^2\tilde{J}(\underline{u}^*,\underline{v}^*) = \frac{1}{2}\Big\{\int_Z \delta\underline{x}_f'(\partial^2 j_f/\partial\underline{x}^2)^*\ \delta\underline{x}_f\ dZ$$

$$+ \int_L [\delta\underline{x}_L'\ \ \delta\underline{v}']\,\delta^2 j_L^* \begin{bmatrix}\delta\underline{x}_L\\ \delta\underline{v}\end{bmatrix} dL$$

$$+ \int_D [\delta\underline{x}'\ \ \delta\underline{x}_z'\ \ \delta\underline{u}']\ \delta^2 h^* \begin{bmatrix}\delta\underline{x}\\ \delta\underline{x}_z\\ \delta\underline{u}\end{bmatrix} dD\Big\}$$

$$+ \Big\{\int_Z (\partial j_f/\partial\underline{x})^{*\prime}\ \delta^2\underline{x}_f dZ + \int_L (\partial j_L/\partial\underline{x})^{*\prime}\ \delta^2\underline{x}_L dL$$

$$+ \int_D [\partial\underline{p}'/\partial t - \sum_{i=1}^{m}\partial(\partial h/\partial\underline{x}_{z_i})^{*\prime}/\partial z_i + (\partial h/\partial\underline{x})^{*\prime}]\,\delta^2\underline{x} dD\Big\}$$

$$+ \int_D \{-\partial(\underline{p}'\delta^2\underline{x})/\partial t + \sum_{i=1}^{m}\partial[(\partial h/\partial\underline{x}_{z_i})^{*\prime}\delta^2\underline{x}]/\partial z_i\}\ dD$$

$$\text{(A.19)}$$

The next-to-last term of $\delta^2\tilde{J}(\underline{u}^*,\underline{v}^*)$ is zero by virtue of the costate equation. Using the divergence theorem, the last term may be written:

$$\int_D \{-\partial[\underline{p}'\delta^2\underline{x}]/\partial t + \sum_{i=1}^{m}\partial[(\partial h/\partial\underline{x}_{z_i})^{*\prime}\ \delta^2\underline{x}]/\partial z_i\}\ dD$$

$$= \int_{\partial D} [-n_o\underline{p}' + \sum_{i=1}^{m} n_i(\partial h/\partial\underline{x}_{z_i})^{*\prime}]\,\delta^2\underline{x}\, d(\partial D)$$

$$= \int_Z -\underline{p}_f'\ \delta^2\underline{x}_f\ dD + \int_L [\sum_{i=1}^{m} n_i(\partial h/\partial\underline{x}_{z_i})^{*\prime}]\ \delta^2\underline{x}_L\ dL \qquad \text{(A.20)}$$

(see (3.29)-(3.30).) The first term of (A.20) cancels the fourth term of (A.19), by virtue of the terminal conditions on $\underline{p}^*$. The term

$$\int_L [(\partial j_L/\partial \underline{x})^{*'} + \sum_{i=1}^{m} n_i (\partial h/\partial \underline{x}_{z_i})^{*'}] \, \delta^2 \underline{x}_L \, dL$$

remains. The coefficient of $\delta^2 \underline{x}_L$ is the vector $\underline{r}'$ defined by (3.33). In view of (3.34)-(3.37) and the boundary condition (A.3) on $\delta^2 \underline{x}_L$, this term is simply

$$\frac{1}{2} \int_L - \underline{\alpha}' \; [\delta \underline{x}_L' \delta \underline{v}'] \; \delta^2 \underline{g} \begin{bmatrix} \delta \underline{x}_L \\ \delta \underline{v} \end{bmatrix} dL \tag{A.21}$$

where $\delta^2 \underline{g}$ is given by (A.3) and $\underline{\alpha}$ is defined by (3.41). Thus the second variation condition is given by

$$\begin{aligned} 2\delta^2 \tilde{J}(\underline{u}^*, \underline{v}^*) = & \int_Z \delta \underline{x}_f' (\partial^2 j_f/\partial \underline{x}^2)^* \, \delta \underline{x}_f \, dZ \\ & + \int_L [\delta \underline{x}_L' \delta \underline{v}'] [\delta^2 j_L - \underline{\alpha}' \delta^2 \underline{g}] \begin{bmatrix} \delta \underline{x}_L \\ \delta \underline{v} \end{bmatrix} dL \\ & + \int_D [\delta \underline{x}' \;\; \delta \underline{x}_z' \;\; \delta \underline{u}'] \delta^2 h \begin{bmatrix} \delta \underline{x} \\ \delta \underline{x}_z \\ \delta \underline{u} \end{bmatrix} dD > 0 \end{aligned}$$

$$\forall \; \delta \underline{u}, \; \delta \underline{v} \tag{A.22}$$

APPENDIX B: WELL-POSEDNESS OF QUASILINEAR ANALYTIC SYSTEMS

Proof:(1) Continue the space variables $\underline{z}$ into the complex domain, letting

$$\underline{z} = \underline{r} + i\underline{w} \tag{B.1}$$

By the assumption of analycity, all coefficients remain well-defined. By (4.9), the spatial derivative $\underline{x}_{z_j}$ may be written

$$\partial\underline{x}/\partial z_j = \frac{1}{2}\left[\partial\underline{x}/\partial r_j - i\ \partial\underline{x}/\partial w_j\right] \tag{B.2}$$

The Cauchy-Riemann equations (which are satisfied in view of (4.9)) become:

$$\partial\underline{x}/\overline{\partial z_j} = \frac{1}{2}\left[\partial\underline{x}/\partial r_j + i\ \partial\underline{x}/\partial w_j\right] = \underline{0} \tag{B.3}$$

where $(\bar{\cdot})$ denotes complex conjugation. Left-multiplying (B.3) by $\overline{\underline{A}_j}'$, summing over j, and adding this to (1.1), obtain

$$\partial\underline{x}/\partial t = \sum_{j=1}^{m} (\underline{A}_j\ \partial\underline{x}/\partial z_j + \overline{\underline{A}}_j{}'\partial\underline{x}/\partial\overline{z}_j) + \underline{K}\ \underline{x} + \underline{B}\ \underline{u} + \underline{f} \tag{B.4}$$

Using (B.1), (B.2):

$$\partial\underline{x}/\partial t = \sum_{j=1}^{m} (\underline{A}_{r_j}\ \partial\underline{x}/\partial r_j + \underline{A}w_j\ \partial\underline{x}/\partial_{w_j}) + \underline{K}\ \underline{x} + \underline{B}\ \underline{u} + \underline{f} \tag{B.5}$$

---

(1) The method of continuation leading to (B.12)-(B.14) is due to Garabedian [17], who also describes the general method of characteristics developed,here qualitatively.

where

$$\underline{A}_{r_j} = \frac{1}{2}(\underline{A}_j+\underline{A}_j') \; ; \; \underline{A}_{w_j} = \frac{i}{2}(\underline{\bar{A}}_j'-\underline{A}_j) \tag{B.6}$$

are Hermitian. Separating real and imaginary parts of $\underline{x}$ in (B.5):

$$\begin{aligned}\partial \mathrm{Re}(\underline{x})/\partial t = \sum_{j=1}^{m} [&\mathrm{Re}(\underline{A}_{r_j})\partial \; \mathrm{Re}(\underline{x})/\partial r_j - \mathrm{Im}(\underline{A}_{r_j})\partial \mathrm{Im}(\underline{x})/\partial w_j] \\ &+ \mathrm{Re}(\underline{A}_{w_j}) \; \partial \mathrm{Re}(\underline{x})/\partial w_j - \mathrm{Im}(\underline{A}_{r_j})\partial \mathrm{Im}(\underline{x})/\partial w_j \\ &+ \mathrm{Re}(\underline{K})\mathrm{Re}(\underline{x}) - \mathrm{Im}(\underline{K})\mathrm{Im}(\underline{x}) \\ &+ \mathrm{Re}(\underline{B})\mathrm{Re}(\underline{x}) - \mathrm{Im}(\underline{B})\mathrm{Im}(\underline{x}) \\ &+ \mathrm{Re}(\underline{f})\end{aligned} \tag{B.7}$$

and similarly for $\partial \mathrm{Im}(\underline{x})/\partial t$. The initial conditions are

$$\mathrm{Re}[\underline{x}(0,\underline{r},\underline{w})] = \mathrm{Re}[\underline{x}_o(\underline{r},\underline{w})] \tag{B.8}$$

$$\mathrm{Im}[\underline{x}(0,\underline{r},\underline{w})] = \mathrm{Im}[\underline{x}_o(\underline{r},\underline{w})] \tag{B.9}$$

with boundary conditions on the "cylinder" $(\underline{r},\underline{w}) \in B \times R^m$, $t \in T$:

$$\mathrm{Re}(\underline{T})\mathrm{Re}(\underline{x}_L) - \mathrm{Im}(\underline{T})\mathrm{Im}(\underline{x}_L) = \mathrm{Re}(\underline{G})\mathrm{Re}(\underline{v}) - \mathrm{Im}(\underline{G})\mathrm{Im}(\underline{v}) \tag{B.10}$$

$$\mathrm{Im}(\underline{T})\mathrm{Re}(\underline{x}_L) + \mathrm{Re}(\underline{T})\mathrm{Im}(\underline{x}_L) = \mathrm{Im}(\underline{G})\mathrm{Re}(\underline{v}) + \mathrm{Re}(\underline{G})\mathrm{Im}(\underline{v}) \tag{B.11}$$

The boundary conditions as $\underline{w} \to \infty$, that $\underline{\bar{x}}_L'\underline{x}_L < \infty$, are guaranteed by analycity of $\underline{x}$. In summary, one obtains a real symmetric (hyperbolic) system of $2n_x$ equations in $2m+1$ independent variables, with symmetric boundary conditions, of the form:

$$\partial\tilde{\underline{x}}/\partial t = \sum_{j=1}^{2m} \tilde{\underline{A}}_j \, \partial\tilde{\underline{x}}/\partial\tilde{\underline{z}} + \tilde{\underline{K}}\,\tilde{\underline{x}} + \tilde{\underline{B}}\,\tilde{\underline{u}} + \tilde{\underline{f}} \tag{B.12}$$

$$\text{IC:} \quad \tilde{\underline{x}}(0,\tilde{z}) = \tilde{\underline{x}}_o \tag{B.13}$$

$$\text{BC:} \quad \tilde{\underline{T}}\,\tilde{\underline{x}}_{\tilde{L}} = \tilde{\underline{G}}\,\tilde{\underline{v}} \tag{B.14}$$

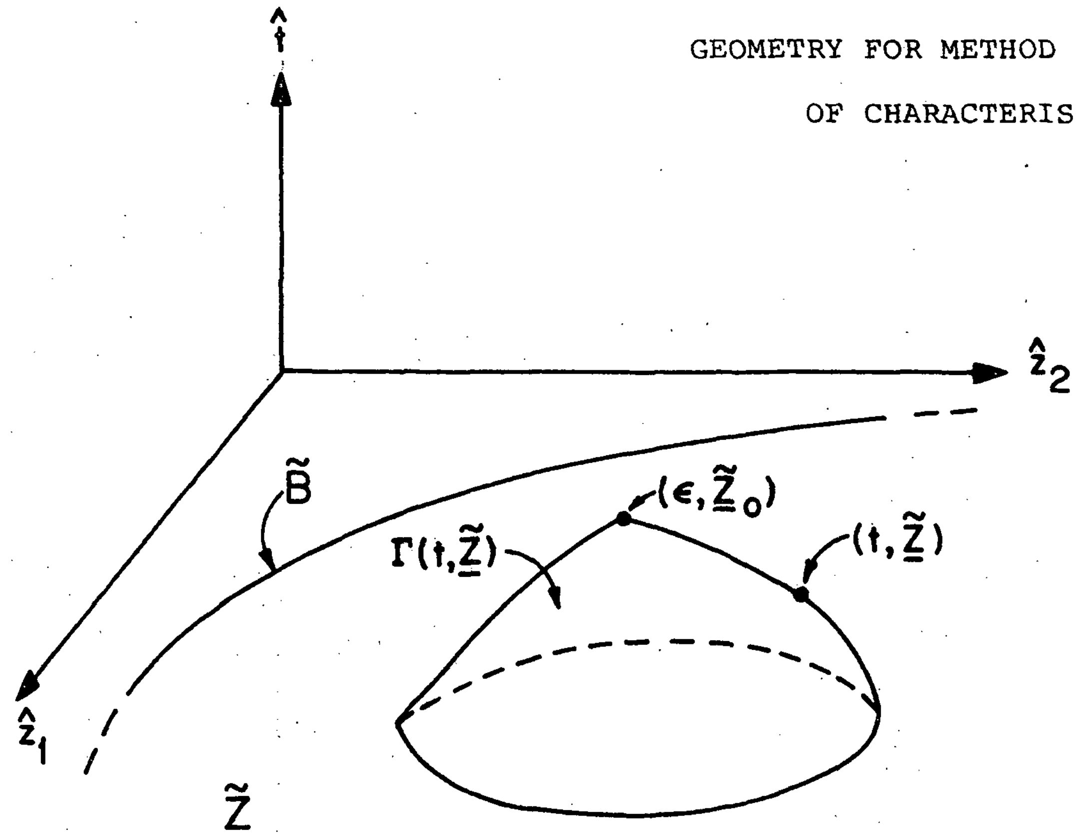
(with $\tilde{\underline{A}}_j$, $\tilde{\underline{K}}$ and $\tilde{\underline{T}}$ symmetric positive definite matrix functions). This system is still quasilinear. The method of characteristics (in its general form) is valid for this class of systems and may be used to determine solutions (see Figure 4).

Consider the problem of determining $\tilde{\underline{x}}(\varepsilon,\tilde{\underline{z}}_o)$ for $t=\varepsilon$ sufficiently small, where $\tilde{\underline{z}}_o \in \tilde{Z}$ (interior point). The characteristic conoid Γ $(\tilde{\underline{z}},t)$ with vertex at $(\varepsilon,\tilde{\underline{z}}_o)$ is determined by the condition that the directional derivative of $\tilde{\underline{x}}$ in the direction of the outward normal $\underline{\nu}$ to Γ cannot be determined from the values of $\tilde{\underline{x}}$ interior to or on Γ. Considering a point $(t,\tilde{\underline{z}})$ on Γ , define a set of local coordinate directions $(\underline{\nu}, \underline{\gamma}_1 \cdots \underline{\gamma}_{2m})$, where $\underline{\gamma}_1, \cdots \underline{\gamma}_{2m}$ may be chosen tangent to Γ (nonuniquely) at $(t,\tilde{\underline{z}})$. A vector $\underline{e}_i$ along the $\tilde{z}_i$ axis may be written in terms of these local coordinates as

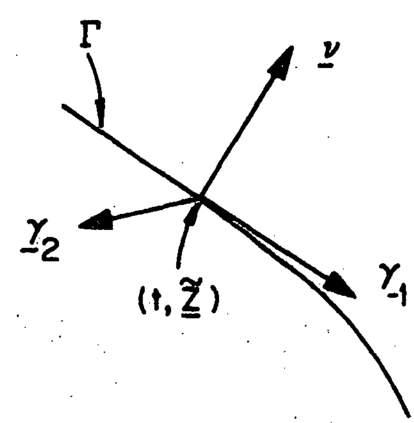
$$\underline{e}_i = e_{i\nu}\,\underline{\nu} + \sum_{j=1}^{2m} e_{i\gamma_j}\,\underline{\gamma}_j \tag{B.15}$$

GEOMETRY FOR METHOD

OF CHARACTERISTICS



A CHARACTERISTIC CONOID, $\Gamma(t, \tilde{\underline{Z}})$



LOCAL COORDINATES ON Γ

Figure 4

or along the t-axis as

$$\underline{t} = t_\nu \underline{\nu} + \sum_{j=1}^{2m} t\gamma_j \underline{\gamma}_j \tag{B.16}$$

where the scale of $\underline{\nu}$ can be chosen so that

$$e_{i\nu} = \partial\Gamma/\partial\tilde{z}_i(t,\tilde{\underline{z}}) \tag{B.17}$$

$$t_\nu = \partial\Gamma/\partial t\ (t,\tilde{\underline{z}}) \tag{B.18}$$

Applying the chain rule with $\tilde{\underline{x}}$ considered a function of the local coordinates,

$$\tilde{\underline{x}}_{z_i} = e_{i\nu}\,\tilde{\underline{x}}_\nu + \sum_{j=1}^{2m} e_{i\gamma_j}\tilde{\underline{x}}_{\gamma_j} \tag{B.19}$$

$$\tilde{\underline{x}}_t = t_\nu \tilde{\underline{x}}_\nu + \sum_{j=1}^{2m} t_{\gamma_j}\tilde{\underline{x}}_{\gamma_j} \tag{B.20}$$

Inserting these results in (B.12) yields

$$\begin{aligned}(t_\nu \underline{I} - \sum_{i=1}^{2m} e_{i\nu}\tilde{\underline{A}}_i)\tilde{\underline{x}}_\nu = {} & \sum_{j=1}^{2m} (-t_{\gamma_j}\underline{I} + \sum_{i=1}^{2m} e_{i\gamma_j}\tilde{\underline{A}}_i)\tilde{\underline{x}}_{\gamma_j} \\ & + \tilde{\underline{K}}(\tilde{\underline{x}}(\nu,\underline{\gamma}),\nu,\underline{\gamma})\,\tilde{\underline{x}}(\nu,\underline{\gamma}) \\ & + \tilde{\underline{B}}(\tilde{\underline{x}}(\nu,\underline{\gamma}),\nu,\underline{\gamma})\,\tilde{\underline{u}}(\nu,\underline{\gamma}) \\ & + \tilde{\underline{f}}(\nu,\underline{\gamma})\end{aligned} \tag{B.21}$$

Since $\underline{\Gamma}$ was assumed to be characteristic, $\tilde{\underline{x}}_\nu$ cannot be determined from $\tilde{\underline{x}}_{\gamma_j}$; j=1, 2...2m, i.e.

$$|t_\nu\ \underline{I} - \sum_{i=1}^{2m} e_{i\nu}\tilde{\underline{A}}_i| = 0$$

Or using (B.17), (B.18),

$$|\Gamma_t \underline{I} - \sum_{i=1}^{2m} \Gamma_{\tilde{z}_i} \underline{\tilde{A}}_i| = 0 \tag{B.22}$$

It is well known, however, that for symmetric hyperbolic systems of type (B.12), the equation

$$|\lambda_o \underline{I} - \sum_{i=1}^{2m} \lambda_i \underline{\tilde{A}}_i| = 0 \tag{B.23}$$

has $2n_x$ distinct real roots λ_o for all $(\underline{\tilde{x}},t,\underline{\tilde{z}})$ and real $(\lambda_1 \ldots \lambda_m)$. Thus, knowing the directional derivatives $\Gamma_{\tilde{z}_i}$ of Γ at a given time, Γ may be propogated backwards from the terminal point $(\hat{\varepsilon},\underline{\tilde{z}})$.

Actually the $2n_x$ roots of (B.23) give rise to $2n_x$ separate characteristic surfaces. Knowing the directional derivatives of $\underline{\tilde{x}}$ tangent to a characteristic Γ^j $(j=1,\ldots 2n_x)$ in the plane $(0,\underline{\tilde{z}})$ and the value of $\underline{\tilde{x}}(0,\underline{\tilde{z}})$ it is possible to determine a scalar linear functional $f^j(\underline{\tilde{x}}) = \sum_{i=1}^{m} f_k^j \underline{\tilde{x}}(\varepsilon,\underline{\tilde{z}}_o)$ of the state at $(\varepsilon,\underline{\tilde{z}}_o)$. Corresponding to the root λ_o^j of (B.23) which determines Γ^j , there is a distinct eigenvector $\underline{\tilde{w}}_j'$ such that

$$\underline{\tilde{w}}_j' \, [\lambda_o^j \underline{I} - \sum_{i=1}^{2m} \lambda_i \underline{\tilde{A}}_i] = 0 \tag{B.24}$$

Choose $\lambda_1 = \Gamma_{\tilde{z}_1}$ in the plane of $\underline{t}$ and $\underline{\nu}$, and $\lambda_1 = \Gamma_{\tilde{z}_i}$, $i=2\ldots 2m$ in the $(0,\underline{\tilde{z}})$ plane. Then multiplication of (B.21) on the left by $\underline{\tilde{w}}_j'$ yields:

$$f^j(\underline{\tilde{x}}) = \underline{\tilde{w}}_j' \sum_{k=2}^{2m} (-t_{\gamma_k}\underline{I} + \sum_{i=1}^{2m} e_{i\gamma_k}\underline{\tilde{A}}_k)\underline{\tilde{x}}_{\gamma_k}(0,\tilde{z})$$

$$+ \underline{\tilde{w}}_j'[\underline{\tilde{f}}(0,\underline{\tilde{z}}) + \underline{\tilde{K}}\ \underline{\tilde{x}}(0,\underline{\tilde{z}}) + \underline{\tilde{B}}\ \tilde{u}(0,\underline{\tilde{z}})] \tag{B.25}$$

where

$$(f_1^j, \ldots f_{2n_x}^j) = \underline{\tilde{w}}_j'[t_{\gamma_1}\underline{I} - \sum_{i=1}^{2m} e_{i\gamma_1}\underline{\tilde{A}}_i] \tag{B.26}$$

Proceeding in this manner on each conoid, $2n_x$ (independent) scalar linear functionals of $\underline{\tilde{x}}(\varepsilon,\underline{\tilde{z}}_o)$ are obtained; these uniquely determine $\underline{\tilde{x}}(\varepsilon,\underline{\tilde{z}}_o)$ as desired.

At the boundary, it will not in general be possible to specify all of the directional derivatives $\underline{\tilde{x}}_{\gamma_k}$ in (B.25); hence the matrix $\underline{\tilde{T}}$ must be chosen so that (1) the remaining $\underline{\tilde{x}}_{\gamma_k}$ are specified uniquely and (2) the available $\underline{\tilde{x}}_{\gamma_k}$ are not over-specified. These considerations serve to determine the allowable range and null spaces of $\underline{\tilde{T}}$ at each boundary point.

APPENDIX C: PROOF OF DECOMPOSITION THEOREM

Time-Invariant Nonlinear Optimal Control Problem:

$$\partial \underline{x}/\partial t = \underline{f}(\underline{x},\underline{x}_z,\underline{u},\underline{z}) \qquad (t,\underline{z}) \in D \tag{C.1}$$

(System Equations)

$$\text{IC: } \underline{x}(0,\underline{z}) = \underline{x}_o(\underline{z}) \qquad \underline{z} \in Z \tag{C.2}$$

$$\text{BC: } \underline{g}(\underline{x}_L,\underline{v},\underline{z}) = \underline{0} \qquad (t,\underline{z}) \in L \tag{C.3}$$

$$J(\underline{u},\underline{v}) = \int_Z F(\underline{x}_f,\underline{z})dZ + \int_L L(\underline{x}_L,\underline{v},\underline{z})dL + \int_D D(\underline{x},\underline{x}_z,\underline{u},\underline{z})dD \tag{C.4}$$

(Performance Index)

Necessary Conditions for Optimality: Let $\underline{u}^*$, $\underline{v}^*$ denote optimal controls and $\underline{x}^*$, $\underline{p}^*$ denote optimal state and costate; define the Hamiltonian

$$H(\underline{x},\underline{x}_z,\underline{p},\underline{u},\underline{z}= D(\underline{x},\underline{x}_z,\underline{u},\underline{z}) + \underline{p}'\underline{f}(\underline{x},\underline{x}_z,\underline{u},\underline{z}) \tag{C.5}$$

Then

$$\dot{\underline{x}}^* = \underline{f}^* \tag{C.6}$$

$$\text{IC: } \underline{x}^*(0) = \underline{x}_o$$

$$\text{BC: } \underline{g}^* = \underline{0}$$

$$\dot{\underline{p}}^* = \sum_{i=1}^{m} \partial(H^*_{\underline{x}_{z_i}})/\partial z_i - H^*_{\underline{x}} \tag{C.7}$$

$$\text{FC: } \underline{p}^*_f = F^*_{\underline{x}}$$

$$\text{BC: } (\underline{g}^{*1'}_{\underline{x}}\ \underline{g}^{*1}_{\underline{x}})^{-1}\ \underline{g}^{*1'}_{\underline{x}}\ [L^*_{\underline{x}} + \sum_{i=1}^{m} n_i\ H^*_{\underline{x}_{z_i}}] = \underline{0}$$

and

$$H^*_{\underline{u}} = \underline{0} \tag{C.8}$$

$$L^*_{\underline{v}} - \underline{g}^*_{\underline{v}}\ [\underline{g}^{*\prime}_{\underline{x}}\underline{g}^*_{\underline{x}}]^{-1}\ \underline{g}^*_{\underline{x}}\ [\underline{L}^*_{x} + \sum_{i=1}^{m} n_i\ H^*_{\underline{x}_{z_i}}] = \underline{0} \tag{C.9}$$

Conditions as sought which guarantee in the limit $\varepsilon \to 0$ that the optimal controls, state, and costate have the expansions

$$\underline{u}^*(t,\underline{z}) = \underline{u}^s(\underline{z}) + \varepsilon\underline{u}^t(t,\underline{z}) \tag{C.10}$$

$$\underline{v}^*(t,\underline{z}) = \underline{v}^s(\underline{z}) + \varepsilon\underline{v}^t(t,\underline{z}) \tag{C.11}$$

$$\underline{x}^*(t,\underline{z}) = \underline{x}^s(\underline{z}) + \varepsilon\underline{x}^t(t,\underline{z}) \tag{C.12}$$

$$\underline{p}^*(t,\underline{z}) = \underline{p}^s(\underline{z}) + \varepsilon\underline{p}^t(t,\underline{z}) \tag{C.13}$$

where higher-order terms in the expansions are $0(\varepsilon)$, $0(\varepsilon^2)$, etc., and are omitted for simplicity. Assuming for the moment that this is the case, it is permissable to linearize (C.6)-(C.9) about the steady-state quantities $\underline{u}^s$, $\underline{v}^s$, $\underline{x}^s$, $\underline{p}^s$:

$$\varepsilon\dot{\underline{x}}^t = \underline{f}^s + \varepsilon(\underline{f}^s_{\underline{x}}\,\underline{x}^t + \sum_{i=1}^{m} \underline{f}^s_{\underline{x}_{z_i}}\,\underline{x}^t_{z_i} + \underline{f}^s_{\underline{u}}\,\underline{u}^t) \tag{C.14}$$

$$\text{IC: } \underline{x}^t(0) = \frac{1}{\varepsilon}(\underline{x}_o - \underline{x}^s)$$

$$\text{BC: } \underline{g}^s + \varepsilon(\underline{g}^s_{\underline{x}}\,\underline{x}^t + \underline{g}^s_{\underline{v}}\,\underline{v}^t) = \underline{0}$$

$$\varepsilon\dot{\underline{p}}^t = \Big(\sum_{i=1}^{m} \partial(H^s_{\underline{x}_{z_i}})/\partial z_i - H^s_{\underline{x}}\Big) \tag{C.15}$$

$$+ \varepsilon\Big\{\sum_{i=1}^{m} \partial[H^s_{\underline{x}_{z_i}\underline{x}}\,\underline{x}^t + \sum_{j=1}^{m} H^s_{\underline{x}_{z_i}\underline{x}_{z_j}}\,\underline{x}^t_{z_j} + H^s_{\underline{x}_{z_i}\underline{p}}\,\underline{p}^t$$

$$+ H^s_{\underline{x}_{z_i}\underline{u}}\,\underline{u}^t]/\partial z_i$$

$$- [H^s_{\underline{xx}}\,\underline{x}^t + \sum_{j=1}^{m} H^s_{\underline{xx}_{z_j}}\,\underline{x}^t_{z_j} + H^s_{\underline{xp}}\,\underline{p}^t + H^s_{\underline{xu}}\,\underline{u}^t]\Big\}$$

$$\text{FC: } \underline{p}^t(t_f) = \frac{1}{\varepsilon}(F^s_{\underline{x}} - \underline{p}^s) + F^s_{\underline{xx}}\,\underline{x}^t(t_f)$$

$$\text{BC: } [\underline{T}^{s\perp}\underline{T}^{s\perp\prime}]^{-1}\,\underline{T}^{s\perp}\,\underline{r}^s + \varepsilon[(\underline{T}^{t\perp}\underline{T}^{t\perp\prime})^{-1}\,\underline{T}^{t\perp\prime}\,\underline{r}^t] = \underline{0}$$

where

$$\underline{T}^s = \underline{g}^{s\prime}_{\underline{x}}$$

$$\underline{r} = [L_{\underline{x}} + \sum_{i=1}^{m} n_i\,H_{\underline{x}_{z_i}}]$$

and

$$\underline{T}^t = [\underline{g}^s_{\underline{xx}}\ \underline{x}^t + \underline{g}^s_{\underline{xv}}\ \underline{v}^t]'$$

$$\underline{r}^t = [\underline{r}_{\underline{x}}\ \underline{x}^t + \underline{r}_{\underline{v}}\ \underline{v}^t + \underline{r}_{\underline{u}}\ \underline{u}^t]$$

$$H^s_{\underline{u}} + \varepsilon[H^s_{\underline{up}}\ \underline{p}^t + H^s_{\underline{ux}}\ \underline{x}^t + \sum_{j=1}^{m} H^s_{\underline{ux}_{z_j}}\ \underline{x}^t_{z_j} + H_{\underline{uu}}\ \underline{u}^t] = \underline{0} \qquad (C.16)$$

$$(L^s_{\underline{v}} - \underline{g}^s_{\underline{v}}[\underline{T}^s\underline{T}^{s'}]^{-1}\underline{T}^s\ \underline{r}^s) + \varepsilon(L^t_{\underline{v}}\underline{v}^t - \underline{g}^t_{\underline{v}}[\underline{T}^t\underline{T}^{t'}]^{-1}\underline{T}^t\ \underline{r}^t) = \underline{0} \quad (C.17)$$

where

$$\underline{L}^t_{\underline{v}} = \underline{L}^s_{\underline{vx}}\ \underline{x}^t + \underline{L}^s_{\underline{vv}}\ \underline{v}^t$$

$$\underline{g}^t_{\underline{v}} = \underline{g}^s_{\underline{vx}}\ \underline{x}^t + \underline{g}^s_{\underline{vv}}\ \underline{v}^t$$

The equalities indicated above are modulo terms involving ε^2, ε^3, etc., which become negligable relative to ε in the limit $\varepsilon \to 0$.

Separating steady-state and -order terms in (C.14)-(C.17), two sets of equations are obtained:

$$\text{(I)}\quad \underline{f}^s = \underline{0} \qquad (C.18)$$

$$\text{BC: } \underline{g}^s = \underline{0}$$

$$\sum_{i=1}^{m} \partial(H^s_{\underline{x}_{z_i}})/\partial z_i - H^s_{\underline{x}} = \underline{0} \qquad (C.19)$$

$$\text{BC: } (\underline{T}^{s\perp}\underline{T}^{s\perp'})^{-1}\ \underline{T}^{s\perp}\ \underline{r}^s = \underline{0}$$

$$\underline{H}^s_{\underline{u}} = \underline{0} \tag{C.20}$$

$$L^s_{\underline{v}} - \underline{g}^s_{\underline{v}} [\underline{T}^s \underline{T}^{s'}]^{-1} \underline{T}^s \underline{r}^s = \underline{0} \tag{C.21}$$

(II)
$$\dot{\underline{x}}^t = \underline{f}^s_{\underline{x}} \underline{x}^t + \sum_{j=1}^{m} \underline{f}^s_{\underline{x}_{z_j}} \underline{x}^t_{z_j} + \underline{f}^s_{\underline{u}} \underline{u}^t \tag{C.22}$$

$$\text{IC: } \underline{x}^t(0) = \frac{1}{\varepsilon} [\underline{x}_o - \underline{x}^s]$$

$$\text{BC: } \underline{g}^s_{\underline{x}} \underline{x}^t + \underline{g}^s_{\underline{v}} \underline{v}^t = \underline{0}$$

$$\begin{aligned} \dot{\underline{p}}^t = \sum_{i=1}^{m} \frac{\partial}{\partial z_i} \Big[& H^s_{\underline{x}_{z_i}\underline{x}} \underline{x}^t + \sum_{j=1}^{m} H^s_{\underline{x}_{z_i}\underline{x}_{z_j}} \underline{x}^t_{z_j} \\ & + H^s_{\underline{x}_{z_i}\underline{p}} \underline{p}^t + H^s_{\underline{x}_{z_i}\underline{u}} \underline{u}^t \Big] \\ & - \Big[H^s_{\underline{xx}} \underline{x}^t + \sum_{j=1}^{m} H^s_{\underline{xx}_{z_j}} \underline{x}^t_{z_j} + H^s_{\underline{xp}} \underline{p}^t + H^s_{\underline{xu}} \underline{u}^t \Big] \end{aligned} \tag{C.23}$$

$$\text{FC: } \underline{p}^t(t_f) = \frac{1}{\varepsilon} [F^s_{\underline{x}} - \underline{p}^s] + F^s_{\underline{xx}} \underline{x}^t(t_f)$$

$$\text{BC: } [\underline{T}^{t\perp} \underline{T}^{t\perp'}]^{-1} \underline{T}^{t\perp} \underline{r}^t = \underline{0}$$

$$H^s_{\underline{ux}} \underline{x}^t + \sum_{j=1}^{m} H^t_{\underline{ux}_{z_j}} + H^s_{\underline{uu}} \underline{u}^t + \underline{H}^s_{\underline{up}} \underline{p}^t = \underline{0} \tag{C.24}$$

$$L^t_{\underline{v}} \underline{v}^t - \underline{g}^t_{\underline{v}} [\underline{T}^t \underline{T}^{t'}]^{-1} \underline{T}^t \underline{r}^t = \underline{0} \tag{C.25}$$

The main point of the proof is to recognize these sets of equations as solutions of optimal control problems, i.e., to solve a pair of inverse problems. Then, assuming existence of solutions to (I) and (II), the above derivation demonstrates that for sufficiently small ε, a solution of the original nonlinear problem has been obtained.

By considering the proof of the Minimum Principle as in Section 1.3, but with $\partial \underline{x}/\partial t = \underline{0}$, it is readily verified that equations (I) are indeed necessary conditions for solution of problem (5.10)-(5.12). Equations (II) are obtained by applying the Minimum Principle directly to problem (5.13)-(5.16). Conditions (5.17)-(5.18) are of course required to assure that the linear problem makes sense as $\varepsilon \to 0$. The proof is thus completed.

Remark C-1: The boundary conditions on $\underline{p}^t$ and the auxiliary condition (C.25) are obtained by linearizing the original variational equations and constraints from which the boundary condition for (C.7) was derived in the proof of the Minimum Principle to give

$$(\underline{r}^s \delta \underline{x}_L + L^s_{\underline{v}} \delta \underline{v}) + \varepsilon (\underline{r}^t \delta \underline{x}_L + L^t_{\underline{v}} \delta \underline{v}) = 0 \tag{C.26}$$

subject to

$$(\underline{T}^s \delta \underline{x}_L + \underline{g}^s_{\underline{x}} \delta \underline{v}) + \varepsilon (\underline{T}^t \delta \underline{x}_L + \underline{g}^t_{\underline{v}} \delta \underline{v}) = \underline{0} \tag{C.27}$$

The reasoning of (3.34)-(3.37) in the proof may then be applied <u>separately</u> to the steady-state and the ε-order terms.

Remark C-2: In the definition of $\underline{T}^t$, the quantity $\underline{g}_{\underline{xx}}$ is a tensor; the shorthand expression $\underline{g}^s_{\underline{xx}}\ \underline{x}^t$, for instance, refers to a matrix with i-j th element

$$[\underline{g}^s_{\underline{xx}}\ \underline{x}^t]_{ij} = \sum_{k=1}^{n_x} \{\frac{\partial}{\partial x_k} (\partial g_i/\partial x_j)\}^s\ x_k^t$$

and hence $\underline{T}^t$ is a matrix.

Remark C-3: The matrices in (5.16) are guaranteed to be symmetric by the continuity assumptions on $\underline{f}$, j_L and j_D - viz. that these functions are twice continuously differentiable in $\underline{x}$, $\underline{x}_z$, $\underline{u}$ and/or $\underline{v}$.

APPENDIX D: NONLINEAR AND LINEAR MATHEMATICAL MODELS OF PLASMA DYNAMICS

For ease of interpretation, conventional notation (rather than state-variable notation) is used in developing and simplifying the mathematical model; state variables are assigned after the model is simplified (see Section 3.1).

Notation:

r	=	radial variable
θ	=	azimuthal variable
z	=	longitudinal variable
t	=	time variable
$\underline{v}_{i,e}$	=	local ion (i) or electron (e) velocity vector
$\underline{n}_{i,e}$	=	local ion (i) or electron (e) particle density (classical meaning)
$\underline{q}_{i,e}$	=	ion (i) or electron (e) charge (taken to be $\pm$ one electron charge unit)
$m_{i,e}$	=	ion (i) or electron (e) mass.
$\underline{E}$	=	total electric field vector (meaning may be modified by superscripts listed below)
$\underline{B}$	=	magnetic field vector (meaning may be modified by superscripts listed below)
$\underline{J}$	=	net particle current density
ρ	=	net particle charge density

ϵ_o, μ_o = permittivity & permeability of free space (MKS units are used throughout)

$(\cdot)^s$ = steady-state quantity (see Section 1.5)

$(\cdot)^t$ = time-varying perturbation quantity

$(\cdot)^I$ = internal or self-field quantity (assumed and hence supressed for any $(\cdot)_{i,e}$ quantity)

$(\cdot)^E$ = externally imposed field quantity

$\frac{d(\cdot)}{dt}$ = time derivative in particle reference frame

$\frac{\partial(\cdot)}{\partial t}$ = time derivative in lab reference frame

Fundamental Physics

The motion of the plasma particles is governed by three principles:

(i) Mechanics (Lorentz force law)

(ii) Thermodynamics (Second Law)

(iii) Electromagnetism (Maxwell's equations)

Basic simplifying assumptions throughout this discussion are:

Assumption (D/1): The plasma is of sufficiently low density and temperature as to be approximately collisionless.

Assumption (D/2): The mean particle velocities are sufficiently low to be nonrelativistic.

Assumption (D/1) considerably simplifies the thermodynamics, as it implies conservation of particle density following the plasma motion; it also leads to a deterministic (rather than stochastic) formulation of plasma dynamics. Assumption

(D/2) avoids relativistic transformation of particle and field quantities. Both assumptions are valid to good approximation for small-scale laboratory plasmas. The following equations then govern particle motions in the plasma:

(i) Lorentz Force Law

$$d(m_{i,e} n_{i,e} \underline{v}_{i,e})/dt = q_{i,e} n_{i,e} [\underline{E} + \underline{v}_{i,e} \times \underline{B}] \tag{D.1}$$

(ii) Particle Conservation (Second Law of Thermodynamics)

$$d(n_{i,e})/dt = 0 \tag{D.2}$$

(iii) Maxwell's Equations

$$\partial\underline{E}/\partial t = (\mu_o \varepsilon_o)^{-1} (\nabla \times \underline{B}) - (\varepsilon_o)^{-1} \underline{J} \tag{D.3}$$

$$\nabla \cdot \underline{E} = (\varepsilon_o)^{-1} \rho \tag{D.4}$$

$$\partial\underline{B}/\partial t = -(\nabla \times \underline{E}) \tag{D.5}$$

$$\nabla \cdot \underline{B} = \underline{0} \tag{D.6}$$

where

$$\underline{J} = q_i n_i \underline{v}_i + q_e n_e \underline{v}_e \tag{D.7}$$

$$\rho = q_i n_i + q_e n_e \tag{D.8}$$

and the transformation to lab frame variables is accomplished using the substantive derivatives

$$dn_{i,e}/dt = \partial n_{i,e}/\partial t + \nabla \cdot (n_{i,e} \underline{v}_{i,e}) \tag{D.9}$$

$$d\underline{v}_{i,e}/dt = \partial\underline{v}_{i,e}/\partial t + \underline{v}_{i,e}\cdot(\nabla\underline{v}_{i,e}) \tag{D.10}$$

Using (D.2) and (D.10), equations (D.1) are simplified and written in lab frame coordinates as

$$\partial\underline{v}_{i,e}/\partial t = -\underline{v}_{i,e}\cdot(\nabla\underline{v}_{i,e}) + (q_{i,e}/m_{i,e})[\underline{E} + \underline{v}_{i,e} \times \underline{B}] \tag{D.11}$$

while from (D.9), (D.2) becomes

$$\partial n_{i,e}/\partial t = -\nabla\cdot(n_{i,e}\,\underline{v}_{i,e}) \tag{D.12}$$

<u>Remark D-1:</u> Equations (D.11), (D.12), (D.3)-(D.6) constitute a first-order quasilinear system of 16 equations in 14 dependent variables ($\underline{v}_i$, n_i, $\underline{v}_e$, n_e, $\underline{E}$, $\underline{B}$); the (well-known) redundancy of Maxwell's equations may be (formally) removed by considering (D.4), (D.6) as static ("state-space") constraints on (D.3), (D.6), respectively. The control action is now extracted from Maxwell's equations.

<u>Externally-applied Fields (Controls):</u>

The field quantities in (D.1)-(D.6) refer to <u>total</u> fields, which are the sum of externally-applied fields (controls), and induced fields, due to the moving charged particles of the plasma itself. The applied fields are those which would be obtained in the absence of the plasma and are hence governed by Maxwell's equations in free space. To be more explicit,

$$\underline{E} = \underline{E}^E + \underline{E}^I \tag{D.13}$$

$$\underline{B} = \underline{B}^E + \underline{B}^I \tag{D.14}$$

where $\underline{E}^E$, $\underline{B}^E$ are governed by

$$\partial \underline{E}^E/\partial t = (\mu_o \varepsilon_o)^{-1}(\nabla\times\underline{B}^E) \quad (D.15)$$

$$\nabla\cdot\underline{E}^E = 0 \quad (D.16)$$

$$\partial \underline{B}^E/\partial t = -(\nabla\times\underline{E}^E) \quad (D.17)$$

$$\nabla\cdot\underline{B}^E = 0 \quad (D.18)$$

The true control action is applied through the <u>boundary conditions</u> on (D.15)-(D.16) at the charged conducting plates and on (D.17)-(D.18) at the field coils. In control engineering terminology, (D.16), (D.18) play the role of <u>control constraints</u> and (D.15), (D.17) play the role of <u>actuator dynamics</u>.(1)

In modelling, these equations should properly be appended to the other dynamic equations, yielding a boundary control problem with two "boundaries" and static state constraints, which is a more complicated type of system than that discussed in Section 1.1. However, since the purpose of this example is to illustrate the theories of Chapters 1 and 2, additional assumptions will be introduced which justify use of these results:(2)

<u>Assumption (D/3):</u> In the plasma confinement region $\underline{B}^E = (0,0,B_z^E(r,t))$.

<u>Assumption (D/4):</u> $\underline{B}^I \ll \underline{B}^E$.

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(1) There are in fact even more of these, viz., characteristics of voltage and current sources charging the plates and driving the field coils, and the capacitance and inductance of these elements, respectively.

(2) Author's license! The assumptions are physically reasonable but do not represent the author's preferred formulation of the problem. Consideration of the more general problem, however, is beyond the scope of this research.

Assumption (D/3) is false, since it overlooks (D.15)-(D.18) as constraints on the field $\underline{B}^E$; in fact, there is an innate tendency for confinement schemes as indicated by Figure 2 to allow escape of plasma from the "ends" of the cylinder precisely because of these constraints. Considering instabilities in a plane, z=constant, however, (which is the object of this example), the assumption may be approximately valid. On the basis of Assumption (D/3), $B_z^E(r=0,t)$ (rather than the field coil current) is considered to be a boundary control. On the basis of Assumption (D/4), the self-induced magnetic field of the plasma, $\underline{B}^I$, is ignored. For static fields (see below), this assumption is quite reasonable, although for dynamic perturbations it might be somewhat less accurate. The electric field, on the other hand, will <u>not</u> be split according to (D.13); the charged conductors will be considered to exert boundary control on the total field.[3]

<u>Physical Assumptions for Static and Dynamic Analysis:</u>

If the notion of steady-state developed in Section 1.5 is applied to equations (D.3)-(D.8), (D.11)-(D.12), and appropriate boundary conditions are supplied, the resulting boundary-value problem does not uniquely define a steady state motion. In order to restrict attention to the cyclotron motion indicated in Figure 3, the following assumption is made.

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(3) It is assumed that any longitudinally-invariant charge distribution may be synthesized instantly; i.e., capacitance of the conductors is ignored. (See also footnote (1)).

<u>Assumption (D/5):</u> $\partial(\cdot)^s/\partial z = 0$; $\partial(\cdot)^s/\partial\theta = 0$.

In particular, this requires $(\underline{B}^E)^s = (0,0,(B_z^E)^s)$ and $(\underline{E})^s = (\underline{E}_r^s,0,0)$. In steady state, z-components of particle velocities vanish, and the particle densities depend only on r.

The dynamic perturbation equations also define modes of instability of the cyclotron motion other than the diocotron instability (e.g., particles spiralling out the ends of the cylinder), which will not be considered in this example. To this end, it is necessary to institute

<u>Assumption (D/6):</u>(4) $\partial(\cdot)^t/\partial z = 0$

It is noted in particular that this is potentially in conflict with (D.17)-(D.18), which have been ignored in Assumption (D/3).

<u>Remark D-2:</u> The "static" constraint (D.4) is readily incorporated into the static analysis but proves to be redundant in the dynamic analysis, provided the initial conditions are required to satisfy equation (D.4). This may be seen as follows: $\partial(\text{D.4})/\partial t$ is already implied by (D.2) and (D.3). From equations (D.2), using (D.9), the right-hand side of $\partial(\text{D.4})/\partial t$ may be written

$$\begin{aligned}(\varepsilon_o)^{-1}\partial\rho/\partial t &= (\varepsilon_o)^{-1}(q_i\partial n_i/\partial t + q_e\partial n_e/\partial t)\\ &= -(\varepsilon_o)^{-1}\nabla\cdot(q_i n_i \underline{v}_i + q_e n_e \underline{v}_e) \qquad \text{(D.19)}\\ &= -(\varepsilon_o)^{-1}\nabla\cdot\underline{J}\end{aligned}$$

(4) This is <u>not</u> a consequence of (D/5).

On the other hand, the right-hand side may be independently evaluated from equation (D.3):

$$\begin{aligned}\partial(\nabla\cdot\underline{E})/\partial t &= \nabla\cdot(\partial\underline{E}/\partial t)\\ &= \nabla\cdot[(\mu_o\varepsilon_o)^{-1}(\nabla\times\underline{B}) - (\varepsilon_o)^{-1}\underline{J}] \qquad \text{(D.20)}\\ &= -(\varepsilon_o)^{-1}\nabla\cdot\underline{J}\end{aligned}$$

since $\nabla\cdot(\nabla\times\underline{B})=\underline{0}$, which is the same as (D.19). Therefore, nothing is lost by omitting (D.4) from the dynamic analysis, providing $\underline{E}^t(0)$ satisfies

$$\nabla\cdot\underline{E}^t(0) = (\varepsilon_o)^{-1}[q_i n_i^t(0) + q_e n_e^t(0)] \qquad \text{(D.21)}$$

This remark indicates (for this example) an added advantage of the steady-state/dynamic perturbation type of analysis, in model simplification. An analogous remark applies to the equation (D.6).

Boundary Conditions:

Under the preceding assumptions, the system becomes longitudinally invariant, and attention is restricted to a plane z=constant. The spatial domain under consideration is thus

$$(r,\theta) \in Z = [0,R] \times [0,2\pi] \subset R^2 \qquad \text{(D.22)}$$

which is rectangular in (r,θ); the boundary has four "sides":

$$\begin{aligned} B_1^- &: \quad r = 0 \\ B_1^+ &: \quad r = R \\ B_2^- &: \quad \theta = 0 \\ B_2^+ &: \quad \theta = 2\pi \end{aligned} \tag{D.23}$$

Boundary conditions at B_1^- may be determined from rotational symmetry:

$$m_r(0,\theta) = m_i n_i v_{ir} + m_e n_e v_{er} = 0 \tag{D.24}$$

where m_r denotes linear momentum, and

$$J_r(0,\theta) = q_i n_i v_{ir} + q_e n_e v_{er} = 0 \tag{D.25}$$

$$E_\theta(0,\theta) = 0 \tag{D.26}$$

$$B_z(0,\theta) = B_z^E \tag{D.27}$$

Since particles must have finite average angular momentum, it is also necessary to require

$$v_{i\theta}(0,\theta) = 0 \tag{D.28}$$

$$v_{e\theta}(0,\theta) = 0 \tag{D.29}$$

Boundary conditions at B_1^+ are difficult to derive, and depend on the materials used in constructing the plasma chamber and the conducting shields imposing the external electric field. To simplify the model, the following assumptions are added

<u>Assumption (D/7):</u> Particles experience elastic wall collisions.

<u>Assumption (D/8):</u> On the boundary B_1^+, any radial electric field distribution may be synthesized by an appropriate external surface charge distribution.

Assumption (D/8), like assumption (D/3), is unrealistic because it overlooks physical phenomena such as the surface currents on the shields induced by the applied magnetic field. Assumption (D/7) implies

$$v_{ir}(r,\theta) = 0 \qquad \text{(D.30)}$$

$$v_{er}(R,\theta) = 0 \qquad \text{(D.31)}$$

by virtue of conservation of (average) radial momentum in wall collisions; and assumption (D/8) implies the condition

$$E_r(R,\theta) = E_r^E(R,\theta) \qquad \text{(D.32)}$$

where $E^E(R,\theta)$ is regarded as a second boundary control.

Boundary conditions at $B_2^\pm$ are continuity of all variables, i.e.,

$$\underline{v}_i(r,0) = \underline{v}_i(r,\ 2\pi) \qquad \text{(D.33)}$$

$$\underline{v}_e(r,0) = \underline{v}_e(r,\ 2\pi) \qquad \text{(D.34)}$$

$$n_i(r,0) = n_i(r,\ 2\pi) \qquad \text{(D.35)}$$

$$n_e(r,0) = n_e(r,\ 2\pi) \qquad \text{(D.36)}$$

$$\underline{E}(r,0) = \underline{E}(r,\ 2\pi) \qquad \text{(D.37)}$$

which is obviously a consequence of the polar coordinate system. These conditions, unfortunately, are not of the form postulated in Chapter 1 (equation (1.3)). Formally, though, it is possible to cast them in this form by applying <u>no</u> boundary conditions on B_2^-, and by considering the left-hand sides of (D.33)-(D.37) to be dummy control variables on B_2^+ - i.e., by assuming boundary controls on <u>all</u> variables at this boundary. This approach is adopted below.

<u>Summary of Simplified Plasma Equations:</u>

(I) Steady State Equations

$$0 = -v_{ir}^s(\partial v_{ir}^s/\partial r) + (q_i/m_i)[E_r^s + v_{i\theta}^s B_z^s] \tag{D.38}$$

$$0 = -v_{ir}^s(\partial v_{i\theta}^s/\partial r) + (q_i/m_i)[E_\theta^s - v_{ir}^s B_z^s] \tag{D.39}$$

$$0 = -n_i^s(\partial v_{ir}^s/\partial r) - v_{ir}^s(\partial n_i^s/\partial r) - (r)^{-1} n_i^s v_{ir}^s \tag{D.40}$$

$$0 = -v_{er}^s(\partial v_{er}^s/\partial r) + (q_e/m_e)[E_r^s + v_{e\theta}^s B_z^s] \tag{D.41}$$

$$0 = -v_{er}^s(\partial v_{e\theta}^s/\partial r) + (q_e/m_e)[E_\theta^s - v_{er}^s B_z^s] \tag{D.42}$$

$$0 = -n_e^s(\partial v_{er}^s/\partial r) - v_{er}^s(\partial n_e^s/\partial r) - (r)^{-1} n_e^s v_{er}^s \tag{D.43}$$

$$0 = -(\varepsilon_o)^{-1} [q_i n_i^s v_{ir}^s + q_e n_e^s v_{er}^s] \tag{D.44}$$

$$0 = -(\varepsilon_o \mu_o)^{-1} \partial B_z^s/\partial r - (\varepsilon_o)^{-1}[q_i n_i^s v_{i\theta}^s + q_e n_e^s v_{e\theta}^s] \tag{D.45}$$

$$0 = \partial E_\theta^s/\partial r + (r)^{-1}E_\theta^s \tag{D.46}$$

$$0 = \partial E_r^s/\partial r - (\varepsilon_o)^{-1}[q_i n_i^s + q_e n_e^s] + (r)^{-1}E_r^s \tag{D.47}$$

$$0 = \partial B_z^s/\partial z \tag{D.48}$$

Boundary conditions on B_1^-:

$$m_i n_i^s v_{ir}^s + m_e n_e^s v_{er}^s = 0 \tag{D.49}$$

$$q_i n_i^s v_{ir}^s + q_e n_e^s v_{er}^s = 0 \tag{D.50}$$

$$v_{i\theta}^s = 0 \tag{D.51}$$

$$v_{e\theta}^s = 0 \tag{D.52}$$

$$E_\theta^s = 0 \tag{D.53}$$

$$B_z^s = B_z^{sE} \qquad \text{(Boundary Control)} \tag{D.54}$$

Boundary conditions on B_1^+:

$$v_{ir}^s = 0 \tag{D.55}$$

$$v_{er}^s = 0 \tag{D.56}$$

$$E_r^s = E_r^{sE} \qquad \text{(Boundary Control)} \tag{D.57}$$

Boundary conditions on B_2^-: None.

Boundary conditions on B_2^+:

$$x^s = (x^s)^o \tag{D.58}$$

where $x = v_{ir}, v_{i\theta}, n_i, v_{er}, v_{e\theta}, n_e, E_r, E_\theta, B_z$, respectively; and $(\ )^o$ denotes a dummy boundary control variable.

<u>Remark D-3:</u> These equations are sufficient to conclude that $v_{ir}^s \equiv v_{er}^s \equiv E_\theta^s \equiv 0$ on Z. This fact is used to eliminate some terms in the linearized equations below.

(II) Linearized Perturbation Equations

$$\partial v_{ir}^t/\partial t = -(r)^{-1} v_{i\theta}^s (\partial v_{ir}^t/\partial\theta) + (q_i/m_i)[E_r^t + B_z^s v_{i\theta}^t + v_{i\theta}^s B_z^t] \tag{D.59}$$

$$\begin{aligned}\partial v_{i\theta}^t/\partial t = &-(r)^{-1} v_{i\theta}^s (\partial v_{i\theta}^t/\partial\theta) - (\partial v_{i\theta}^s/\partial r) v_{ir}^t \\ &+ (q_i/m_i)[E_\theta^t - B_z^s v_{ir}^t]\end{aligned} \tag{D.60}$$

$$\begin{aligned}\partial n_i^t/\partial t = &-n_i^s (\partial v_{ir}^t/\partial r) - (r)^{-1} n_i^s (\partial v_{i\theta}^t/\partial\theta) - (r)^{-1} v_{i\theta}^s (\partial n_i^t/\partial\theta) \\ &- [(r)^{-1} n_i^s + (\partial n_i^s/\partial r)] v_{ir}^t\end{aligned} \tag{D.61}$$

$$\partial v_{er}^t/\partial t = -(r)^{-1} v_{e\theta}^s (\partial v_{er}^t/\partial\theta) + (q_e/m_e)[E_r^t + B_z^s v_{e\theta}^t + v_{e\theta}^s B_z^t] \tag{D.62}$$

$$\begin{aligned}\partial v_{e\theta}^t/\partial t = &-(r)^{-1} v_{e\theta}^s (\partial v_{e\theta}^t/\partial\theta) - (\partial v_{e\theta}^s/\partial r) v_{er}^t \\ &+ (q_e/m_e)[E_\theta^t - B_z^s v_{er}^t]\end{aligned} \tag{D.63}$$

$$\partial n_e^t/\partial t = -n_e^s(\partial v_{er}^t/\partial r) - (r)^{-1}n_e^s(\partial v_{e\theta}^t/\partial\theta) + (r)^{-1}v_{e\theta}^s(\partial n_e^t/\partial\theta) \quad \text{(D.64)}$$

$$- [(r)^{-1}n_e^s + (\partial n_e^s/\partial r)]v_{er}^s$$

$$\partial E_r^t/\partial t = -(\varepsilon_o)^{-1}[q_i n_i^s v_{ir}^t + q_e n_e^s v_{er}^t] \quad \text{(D.65)}$$

$$\partial E_\theta^t/\partial t = -(\mu_o\varepsilon_o)^{-1}\partial B_z^t/\partial r \quad \text{(D.66)}$$

$$- (\varepsilon_o)^{-1}[q_i(n_i^s v_{i\theta}^t + v_{i\theta}^t n_i^t) + q_e(n_e^s v_{e\theta}^t + v_{e\theta}^s n_e^t)]$$

$$\partial B_z^t/\partial t = -\partial E_\theta^t/\partial r + (r)^{-1}\partial E_r^t/\partial\theta - (r)^{-1}E_\theta^t \quad \text{(D.67)}$$

Boundary conditions on B_1^-:

$$0 = m_i n_i^s v_{ir}^t + m_e n_e^s v_{er}^t \quad \text{(D.68)}$$

$$0 = q_i n_i^s v_{ir}^t + q_e n_e^s v_{er}^t \quad \text{(D.69)}$$

$$0 = v_{i\theta}^t \quad \text{(D.70)}$$

$$0 = v_{e\theta}^t \quad \text{(D.71)}$$

$$0 = E_\theta^t \quad \text{(D.72)}$$

$$B_z^{tE} = B_z^t \quad \text{(Boundary Control)} \quad \text{(D.73)}$$

Boundary conditions on B_1^+:

$$v_{ir}^t = 0 \tag{D.74}$$

$$v_{er}^t = 0 \tag{D.75}$$

$$E_r^t = E_r^{tE} \qquad \text{(Boundary control)} \tag{D.76}$$

Boundary conditions on B_2^-: None.

Boundary conditions on B_2^+:

$$x^t = (x^t)^o \tag{D.77}$$

where x is as in (D.58).

Remark D-4: Note in both static and linear cases that the total number of independent boundary conditions on $B_1^{\pm}$, $B_2^{\pm}$ is equal to the number of equations. This situation is to be expected in most physical examples studied by the author; in fact, counting boundary conditions appears to be a good "rule of thumb" in checking out whether a first-order system is under-or overdetermined at the boundary. In the linear case, a major problem of well-posedness (with a smooth boundary, in general) is to find which linear combinations of state variables may be specified on what regions of the boundary.

Selection of Performance Index: Steady State

Steady-state design is based on a preconceived notion of what constitutes a desirable steady state; a considerable variety of performance objectives can be envisioned for this example. For purposes of illustration, it is supposed that these objectives are:

(i) Maximum rotational kinetic energy.

(ii) Minimum radiation loss.

(iii) Minimum applied fields (control energy).

A reasonable criterion for these objectives would be

$$J^s(B_z^{sE},E_r^{sE}) = \int_Z -[m_i n_i^s (v_{i\theta}^s)^2 + m_e n_e^s (v_{e\theta}^s)^2] r^2 \, dZ$$

$$+ \int_{B_1^-} \mu_o (B_z^{sE})^2 dB_1^- + \int_{B_1^+} [\varepsilon_o (E_r^{sE})^2 + E_\theta^s B_z^s] dB_1^+ \qquad (D.78)$$

Remark D-5: It is presumed that $n_i^s \geq 0$, $n_e^s \geq 0$ in Z.

Remark D-6: There is no a priori guarantee of the existence of optimal controls.

Remark D-7: The steady-state equations leave the distribution n_i^s (or alternatively n_e^s) indeterminate; hence it is interesting to choose a performance index which does not presuppose properties of this distribution.

Selection of Performance Index: Linearized Dynamics

Presuming that optimal controls exist for the steady-state problem and that the "closed-loop" steady state motion is a cyclotron-type rotation, it is reasonable to formulate the linearized problem as one of regulation about this steady state. Particular objectives might be

(i) Minimize kinetic energy of perturbed motions

(ii) Minimize energy of distorted fields

(iii) Minimize control energy.

Again, there are countless other possibilities. A reasonable performance index for these objectives would be

$$J^t(B_z^{tE},E_r^{tE}) = \int_0^{t_f} \int_Z \{m_i n_i^s[(v_{ir}^t)^2 + (rv_{i\theta}^t)^2]$$

$$+ m_e n_e^s[(v_{er}^t)^2 + (rv_{e\theta}^t)^2]$$

$$+ \varepsilon_o[(E_r^t)^2 + (E_\theta^t)^2] + \mu_o(B_z^t)^2\} \; dZ \; dt$$

$$+ \int_0^{t_f} [\int_{B_1^-} \mu_o(B_z^{tE})^2 dB_1^- + \int_{B_1^+} \varepsilon_o(E_r^{tE})^2 dB_1^+] dt \qquad (D.79)$$

<u>Remark D-8:</u> Penalizing energy dissipation due to perturbed motions would add cross-terms of the form $q_i n_i^s[v_{ir}^t E_r^t + rv_{i\theta}^t E_\theta^t]$.

<u>Remark D-9:</u> The static and dynamic indices given above are not related, as would be the case in applying the Decomposition Theorem of Section 1.5. This approach was not followed here in the interest of keeping nonlinear and linear applications separate and of clearly motivating the performance indices.

APPENDIX E: AT THE CIRCUS

Wishing to reduce their insurance costs, and concerned for the safety of their tightrope walkers, the Ringling Bros., having read this thesis, propose to build a "tightrope twitcher". This device wiggles the span at one end in such a manner as to insure that the rope is nearly stationary when the tightrope walker lands after bouncing on it.

The "twitcher" might be designed as follows, using the theory of Chapter 2. Let $T = [0,t_f]$ denote a time interval containing a "jump". Consider the rope to be infinitely long, $Z = [0,\infty)$, since control is exercised at one end only, the point $z = 0$. Let $x(t,\underline{z})$ denote the vertical deflection of the rope, whose motion is governed by the equation for a non-uniform vibrating string:

$$\rho(z)\ \partial x^2/\partial t^2 = \partial[\tau(z)\ \partial x/\partial z]/\partial z + f(t,z) \tag{E.1}$$

where $\rho(z)$ is the density and $\tau(z)$ is related to the tension on the rope (the more accurate case, $\tau = \tau(t,z)$, is also treated without difficulty). $f(t,z)$ is the force on rope by the tightrope walker, which is known from the sequence of the act. State variables may be chosen as

$$x_1 = \partial x/\partial t - (\tau/\rho)^{1/2}\ \partial x/\partial z \tag{E.2}$$

$$x_2 = \partial x/\partial t + (\tau/\rho)^{1/2}\ \partial x/\partial z \tag{E.3}$$

Equation (E.1) then takes the first-order form

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix} \frac{\partial}{\partial z}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ k & -k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad (E.4)$$

where

$$a = (\tau/\rho)^{1/2} \qquad (E.5)$$

$$k = [\tfrac{1}{2}\,(\tau_z/\rho) - \tfrac{1}{4}\,(\tau/\rho)_z]/a \qquad (E.6)$$

The conditions of Section 2.1 are "almost" satisfied:

(A-1) $|\lambda_o I - \lambda_1 \underline{A}| = \lambda_o^2 - \lambda_1^2 a^2$

$\lambda_o = \pm\lambda_1 a$ (distinct real roots)

(A-2) $|\underline{A}| = -a^2 \neq 0$ at $z = 0$

(A-3) $\underline{T}$ must have rank $n_g = 1$. $\underline{A}$ is in normal form.

Hersh's theorem (Theorem 1-1) applies, with

(1.7) $\underline{M}(s,\underline{k}) = \begin{bmatrix} -s/a & 0 \\ 0 & s/a \end{bmatrix}$

and for (1.8), $\lambda^1(s,\underline{k}) = -s/a$, $\lambda^2(s,\underline{k}) = +s/a$, giving

$\underline{e}^{1'} = [1, 0]$ $\qquad \underline{e}^{2'} = [0, 1]$

Since $\text{Re}\,\lambda^2 > 0$ for $\text{Re}\,s > 0$, the boundary operator $\underline{T}$, in addition to satisfying (A-3), must be independent of $\underline{e}^2$; this condition is satisfied by

$$\underline{T} = [1, 0] \tag{E.7}$$

Kreiss' condition (Theorem 1-2) is not quite satisfied since $\underline{T}_\varepsilon = [1,\varepsilon]$ is in a neighborhood of $\underline{T}$ but violates Hersh's theorem. This condition, however, is only a sufficient condition for well-posedness; it can be shown that in fact (E.7) gives rise to a well-posed problem. The system is then described by

$$\partial\underline{x}/\partial t = \underline{A}\ \partial\underline{x}/\partial z + \underline{K}\ \underline{x} + \underline{f} \tag{E.8}$$

$$\text{IC: } \underline{x}(0,z) = \underline{x}_o(z) \tag{E.9}$$

$$\text{BC: } \underline{T}\ \underline{x}(t,0) = v(t) \tag{E.10}$$

where $v(t)$ is the twitch (corresponds to shaking the rope at $z = 0$ - what else would you do?!).

The tightrope walker would like to land with the rope in its resting position, i.e., $\partial x/\partial t = 0$, $\partial x/\partial z = 0$. The response is thus

$$\underline{y} = \begin{bmatrix} x_t \\ x_z \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2a & 1/2a \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{C}\ \underline{x} \tag{E.11}$$

An appropriate performance index for the twitcher is

$$J(v) = \int_D \underline{y}'\ \underline{Q}(t,z)\ \underline{y}\ dD + \int_T v^2\ dT \tag{E.12}$$

where

$$\underline{Q}(t,z) = \begin{bmatrix} q_1(t,z) & 0 \\ 0 & q_2(t,z) \end{bmatrix} \tag{E.13}$$

with $q_i > 0$ chosen to be maximum at the point in time and space when the jump is complete.

With these choices, the optimal control (Chapter II, (3.17)-(3.24)) is determined by solving the TPBVP:

$$\partial \underline{x}^*/\partial t = \underline{A}\ \partial \underline{x}^*/\partial z + \underline{K}\ \underline{x}^* + \underline{f} \tag{E.14}$$

$$\text{IC: } \underline{x}^*(0,\underline{z}) = \underline{x}_o(\underline{z}) \tag{E.15}$$

$$\text{BC: } \underline{x}^*_1(t,0) = -a(0)p^*_1(t,0). \tag{E.16}$$

$$\partial \underline{p}^*/\partial t = -\underline{A}^*\ \partial \underline{p}^*/\partial z \tag{E.17}$$

$$+ [\underline{K}' - \partial \underline{A}/\partial z]\ \underline{p}^* + \underline{C}'\underline{Q}\ \underline{C}\ \underline{x}^*$$

$$\text{FC: } \underline{p}^*(t_f,z) = 0 \tag{E.18}$$

$$\text{BC: } a(0)\ p^*_2(t,0) = 0 \tag{E.19}$$

The optimal twitch is

$$v^*(t) = -a\ (0)\ p^*_1(t,0) \tag{E.20}$$

The sophisticated way to find $\underline{p}^*$ is by solving the equations (cf. Chapter II, (4.69)-(4.75)):

$$-\partial \underline{P}(t,z,s)/\partial t = -\underline{A}(s)\ \partial \underline{P}/\partial s - \underline{A}(z)\ \partial \underline{P}/\partial z$$

$$+ [\underline{K}'(s) + \underline{K}'(z) - \partial \underline{A}(s)/\partial s - \partial \underline{A}(z)/\partial z]\underline{P}$$

$$- \underline{P}_{11}(t,z,0)\ a^2(0)\ \underline{P}_{11}(t,0,s)$$

$$+ \delta(z-s)[\underline{C}'\underline{Q}\ \underline{C}]\ (t,s) \qquad (E.21)$$

$$\text{FC: } \underline{P}(t_f,z,s) = \underline{0} \qquad (E.22)$$

$$\text{BC: } \begin{bmatrix} 0 & 0 \\ 0 & -a(0) \end{bmatrix} \underline{P}(t,0,s) = \underline{0} \qquad (E.23)$$

$$\begin{bmatrix} -a(0) & 0 \\ 0 & 0 \end{bmatrix} \underline{P}(t,z,0) = \underline{0} \qquad (E.24)$$

and

$$-\partial \underline{r}(t,z)/\partial t = -\underline{A}(z)\partial \underline{r}/\partial z + [\underline{K}'(z) - \partial \underline{A}(z)/\partial z]\ \underline{r}$$

$$+ \int_0^\infty \underline{P}(t,z,s)\underline{f}(t,s)ds + a^2(0)\begin{bmatrix} \underline{P}_{11}(t,z,0)r_1(t,0) \\ \underline{P}_{21}(t,z,0)r_2(t,0) \end{bmatrix}$$

$$(E.25)$$

$$\text{FC: } \underline{r}(t_f,z) = \underline{0} \qquad (E.26)$$

$$\text{BC: } a(0)r_2(t,0) = 0 \qquad (E.27)$$

The best twitch is then

$$v^*(t) = -a(0)\left[\int_0^\infty (P_{11}(t,0,s)x_1^*(t,s) + P_{12}(t,0,s)x_2^*(t,s))ds + r_1(t,0)\right] \tag{E.28}$$

Mssrs. Ringling can build their tightrope twitcher if they know the position of the rope at each time.

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