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COMPOSITION DEPENDENCE OF ION DIFFUSION COEFFICIENTS  
IN GAS MIXTURES AT ARBITRARY FIELD STRENGTHS\*

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ABSTRACT

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Expressions for the diffusion coefficient of ions in gas mixtures are obtained from momentum-transfer theory, and are given in terms of the diffusion coefficients and drift velocities of the ions in the pure component gases. Blanc's law holds exactly at all field strengths if the mean free time between collisions is independent of velocity (Maxwell model), but otherwise there may be either positive or negative deviations from Blanc's law at high fields. Such deviations are of comparable magnitude for the diffusion coefficients and the mobility, but are not identical. Specific cases of inverse-power potentials are treated in further detail, and some numerical examples are given for rigid-sphere interactions.

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## I. INTRODUCTION

The diffusion coefficients of ions in gases are proportional to their mobilities at low electric field strengths, but at high fields the proportionality breaks down and the diffusion coefficients usually increase more rapidly than the mobilities.<sup>1</sup> Moreover, the diffusion coefficient becomes anisotropic at high fields, the rates of diffusion parallel and perpendicular to the field direction being different. The purpose of this paper is to find an expression for the composition dependence of ion diffusion coefficients in gas mixtures at arbitrary field strengths, in terms of the diffusion coefficients in the pure component gases. Although no data at present exist, measurements in mixtures introduce no new experimental difficulties, and the present work predicts interesting effects.

Since we seek only a composition dependence, we can use a simple momentum-transfer method, used previously for finding the composition dependence of the ion mobility.<sup>2</sup> As a side result we make explicit a hidden assumption in the previous work, namely that the dependence of the momentum-transfer cross section on the drift velocity itself could be ignored. This has a distinct effect on the deviations from Blanc's law predicted for high fields.

At low fields, the analogue of Blanc's law for ion mobilities holds to first order in the Chapman-Enskog approximation,

$$1/D = \sum_j (x_j/D_j) , \quad (1)$$

where  $D$  is the diffusion coefficient of the ion in the mixture, the  $x_j$  are mole fractions, and the  $D_j$  are the diffusion coefficients of the ion in the pure component gases at a number density the same as the total number density of the mixture. Higher Chapman-Enskog approximations yield only positive deviations from Blanc's law, but these are usually small.<sup>3</sup> At high fields we

find that appreciable deviations from this linear rule can occur, and that the deviations are sensitive to the form of the ion-neutral interaction. •

## II. MOMENTUM-TRANSFER THEORY

The basic idea in a momentum-transfer calculation is that the momentum given to the ions by external forces must be balanced by the momentum the ions transfer to the neutral molecules by collisions, since the ions are not accelerated on the average. For mobility, the external force is obviously supplied by the electric field. For diffusion, the external force is regarded as coming from the gradient of partial pressure of the diffusing species.<sup>4</sup> Wannier<sup>5,6</sup> has shown that the drift and diffusive motions are separable, in the sense that the diffusion is simply superposed on the net drift velocity. Thus the momentum transferred per unit volume in the field direction (along the z-axis) for ions in a single gas is<sup>2,4</sup>

$$n\xi\mu\bar{v}_z \left\langle Nv_r \int_0^\infty (1 - \cos\theta) 2\pi b db \right\rangle_z = neE - \frac{\partial p_i}{\partial z}, \quad (2)$$

and perpendicular to the field direction is

$$n\xi\mu\bar{v}_x \left\langle Nv_r \int_0^\infty (1 - \cos\theta) 2\pi b db \right\rangle_x = - \frac{\partial p_i}{\partial x}, \quad (3)$$

where  $n$  is the number density of ions,  $\xi$  is a proportionality constant of order unity,  $\mu$  is the reduced mass of an ion-neutral pair,  $\bar{v}_z$  and  $\bar{v}_x$  are the average ion velocity components,  $N$  is the number density of neutrals (it is assumed that  $N \gg n$ ),  $v_r$  is the relative speed of an ion-neutral pair colliding with impact parameter  $b$  and deflection angle  $\theta$ ,  $e$  is the ionic charge,  $E$  the electric field strength, and  $p_i$  the partial pressure of ions. The pointed brackets indicate averages over relative speeds. The average ion velocities are related to the drift velocity  $v_d$  and the diffusive fluxes  $J_x$  and  $J_z$  by the expression

$$\bar{v}_z = v_d + (J_z/n), \quad \bar{v}_x = J_x/n, \quad (4)$$

and the diffusion coefficients parallel and perpendicular to the field direction are defined by the equations

$$J_z = - \frac{D_{||}}{kT} \frac{\partial p_i}{\partial z} , \quad J_x = - \frac{D_{\perp}}{kT} \frac{\partial p_i}{\partial x} . \quad (5)$$

Comparison of Eqs. (2)-(5) identifies  $D_{||}$  and  $D_{\perp}$  in terms of the average momentum-transfer cross sections.

For mixtures, the left-hand sides of Eqs. (2) and (3) become summations over the species in the mixtures. The results for both  $D_{||}$  and  $D_{\perp}$  then become

$$D \sum_j \mu_j N_j \langle v_{rj} Q(v_{rj}) \rangle = kT/\xi , \quad (6)$$

where  $Q(v_{rj})$  is the momentum-transfer or diffusion cross section,

$$Q(v_{rj}) = 2\pi \int_0^{\infty} (1 - \cos \theta_j) b db , \quad (7)$$

in which  $\theta_j$  signifies  $\theta(v_{rj})$ . The diffusion coefficients in pure gas  $j$  of density  $N = \sum_j N_j$  are given by one term in these sums,

$$D_j \mu_j N \langle v_{rj} Q(v_{rj}) \rangle_j = kT/\xi , \quad (8)$$

where now the average  $\langle \rangle_j$  is not the same as the corresponding average in Eq. (6), because the ion velocity distributions are different in the pure gas and in the mixture. From here on the notation  $\langle \rangle_j$  means an average taken in the pure gas  $j$ , whereas  $\langle \rangle$  means an average taken in the mixture. Combining Eqs. (6) and (8), we obtain

$$\frac{1}{D} = \sum_j \frac{x_j}{D_j} \frac{\langle v_{rj} Q(v_{rj}) \rangle}{\langle v_{rj} Q(v_{rj}) \rangle_j} . \quad (9)$$

An analogous treatment of  $v_d$  yields an expression of similar form

$$\frac{1}{v_d} = \sum_j \frac{x_j}{v_{dj}} \frac{\langle v_{rj} Q(v_{rj}) \rangle}{\langle v_{rj} Q(v_{rj}) \rangle_j} \quad (10)$$

However, this does not necessarily mean that the D's have the same composition dependence as does  $v_d$ , because the averages in general depend on  $v_d$  unless the field is very weak.

The problem is now reduced to finding the ratio of the averages,  $\langle v_{rj} Q_j \rangle / \langle v_{rj} Q_j \rangle_j$ . In general this will require further approximations, but two especially simple cases can be distinguished at once. First, if the field is weak the relative velocities are essentially all thermal, the ion distribution functions are the same in the pure gases and in the mixtures, and the ratios of the averages are all unity. Second, if  $Q_j$  is inversely proportional to  $v_{rj}$  (the Maxwell model), the products  $v_{rj} Q_j$  are all constants and the ratios are again unity. Thus Blanc's law holds for both these special cases.

### III. EVALUATION OF AVERAGES

To evaluate the ratios of averages appearing in Eqs. (9) and (10), we make three approximations. We first assume that the averages can be decomposed,

$$\langle v_{rj}^{Q(v_{rj})} \rangle_j \sim \langle v_{rj} \rangle_j^{Q(\langle v_{rj} \rangle_j)} , \quad (11)$$

and second that

$$\langle v_{rj} \rangle_j \propto \langle v_{rj}^2 \rangle_j^{1/2} . \quad (12)$$

It is easy to show that

$$\langle v_{rj}^2 \rangle_j = \langle v^2 \rangle_j + \langle v_j^2 \rangle_j , \quad (13)$$

where  $\langle v^2 \rangle_j$  is the mean square ion velocity and  $\langle v_j^2 \rangle_j$  is the mean square velocity of neutral species  $j$ . The latter is always entirely thermal and equal to  $3kT/M_j$ , where  $M_j$  is the neutral mass, but  $\langle v^2 \rangle_j$  contains both thermal and field components, and the field component is partitioned between a part visible as drift motion and a random part. Thus the first two assumptions reduce the problem to the determination of the partitioning of the ion energy, for which an approximate result is already known.<sup>2,6</sup> For ions in a single gas, Wannier<sup>6</sup> has shown that the energy partitioning is approximately

$$\langle v^2 \rangle_j + \langle v_j^2 \rangle_j = 3kT \left( \frac{1}{m} + \frac{1}{M_j} \right) + v_{dj}^2 \left( \frac{m + M_j}{m} \right) , \quad (14)$$

where  $m$  is the ion mass. This result is exact for the Maxwell model, and is thought to be a reasonable approximation for other ion-neutral interactions. The generalization to mixtures yields<sup>2</sup>

$$\langle v^2 \rangle + \langle v_j^2 \rangle = 3kT \left( \frac{1}{m} + \frac{1}{M_j} \right) + v_d^2 \left( \frac{m + \bar{M}}{m} \right) , \quad (15)$$

where the mean mass of the gas mixture is

$$\bar{M} = \frac{\sum_j \omega_j M_j}{\sum_j \omega_j} , \quad (16)$$

$$\omega_j = x_j M_j Q(\langle v_{rj} \rangle) / (m + M_j)^2 . \quad (17)$$

Equations (15) and (16) can always be considered correct in a formal sense, so the third approximation is embodied in Eq. (17).

Combining the foregoing approximations, we obtain

$$\frac{\langle v_{rj} Q(v_{rj}) \rangle}{\langle v_{rj} Q(v_{rj}) \rangle_j} \sim \frac{v_d Q(\langle v_{rj} \rangle)}{v_{dj} Q(\langle v_{rj} \rangle_j)} \left[ \frac{m + \bar{M}}{m + M_j} + \frac{3kT}{M_j v_d^2} \right]^{1/2} \left[ 1 + \frac{3kT}{M_j v_{dj}^2} \right]^{-1/2} . \quad (18)$$

Substitution of this expression back into Eqs. (9) and (10) yields the final formulas for the composition dependences of  $D_{||}$ ,  $D_{\perp}$ , and  $v_d$ . The previous result<sup>2</sup> for  $v_d$  implicitly assumed that the ratios of the Q's in Eq. (18) were all unity, an assumption valid only for rigid spheres. The velocity dependences of the Q's must be known before numerical results can be calculated; although the same Q's appear in both numerator and denominator in Eq. (18), they are to be evaluated at different mean velocities corresponding to averages in the pure components and in the mixtures.



#### IV. EXAMPLES

Here we present a few illustrations of the foregoing results. If the ion-neutral interaction potentials  $V_j$  vary inversely as some power of the separation  $r_j$ ,

$$V_j = C_j / r_j^{n_j}, \quad (19)$$

where  $C_j$  and  $n_j$  are constants, then the cross sections vary as  $v_{rj}^{-4/n_j}$ ,

$$Q(v_{rj}) = 2\pi A^{(1)}(n_j) \left( \frac{2n_j C_j}{v_{rj}^2} \right)^{2/n_j} \left( \frac{1}{m} + \frac{1}{M_j} \right)^{2/n_j}, \quad (20)$$

where the  $A^{(1)}(n_j)$  are pure numbers that have been evaluated by numerical integration.<sup>7,8</sup> The averages in Eqs. (9) and (10) can then be readily found.

Thus Eq. (9) becomes

$$\frac{1}{D} = \sum_j \frac{x_j}{D_j} \left( \frac{v_d}{v_{dj}} \right)^{1-4/n_j} \left[ \frac{m+\bar{M}}{m+M_j} + \frac{3kT}{M_j v_d^2} \right]^{\frac{1}{2} - \frac{2}{n_j}} \left[ 1 + \frac{3kT}{M_j v_{dj}^2} \right]^{-\frac{1}{2} + \frac{2}{n_j}}, \quad (21)$$

and a similar expression for  $v_d$  follows from Eq. (10). The weight factors  $\omega_j$  of Eq. (17) for finding  $\bar{M}$  can now be written as

$$\omega_j = \frac{x_j M_j A^{(1)}(n_j)}{(m+M_j)^2} \left( \frac{2n_j C_j}{M_j v_d^2} \right)^{2/n_j} \left[ \frac{m+\bar{M}}{m+M_j} + \frac{3kT}{M_j v_d^2} \right]^{-2/n_j}. \quad (22)$$

Since Eq. (22) itself contains  $\bar{M}$ , the actual computation of  $\bar{M}$  may require iteration.

For a numerical example we consider rigid spheres at high fields, for which Eq. (21) can be put in the form

$$\frac{1}{(D/v_d)} = \sum_j \frac{x_j}{(D_j/v_{dj})} \left( \frac{v_d}{v_{dj}} \right)^2 \left( \frac{m + \bar{M}}{m + M_j} \right)^{1/2}. \quad (23)$$

For a binary gas mixture the fractional deviation of  $D$  from Blanc's law can be written in terms of three dimensionless ratios,

$$\begin{aligned} \frac{D(\text{Blanc})}{D} &= \frac{D^{-1} - D^{-1}(\text{Blanc})}{D^{-1}(\text{Blanc})} + 1 = \\ &= \frac{x_1 + x_2 D^* (v^*)^2 (m^*)^{1/2}}{(x_1 + x_2 D^* v^*) [x_1 + x_2 (v^*)^2 (m^*)^{1/2}]^{1/2}} \left[ \frac{x_1 + x_2 (v^*)^2 (m^*)^{1/2}}{x_1 + x_2 (v^*)^2 (m^*)^{3/2}} \right]^{1/4}, \quad (24) \end{aligned}$$

where

$$D^* = (D_1/v_{d1}) / (D_2/v_{d2}), \quad (25)$$

$$v^* = v_{d1}/v_{d2}, \quad (26)$$

$$m^* = (m + M_1) / (m + M_2). \quad (27)$$

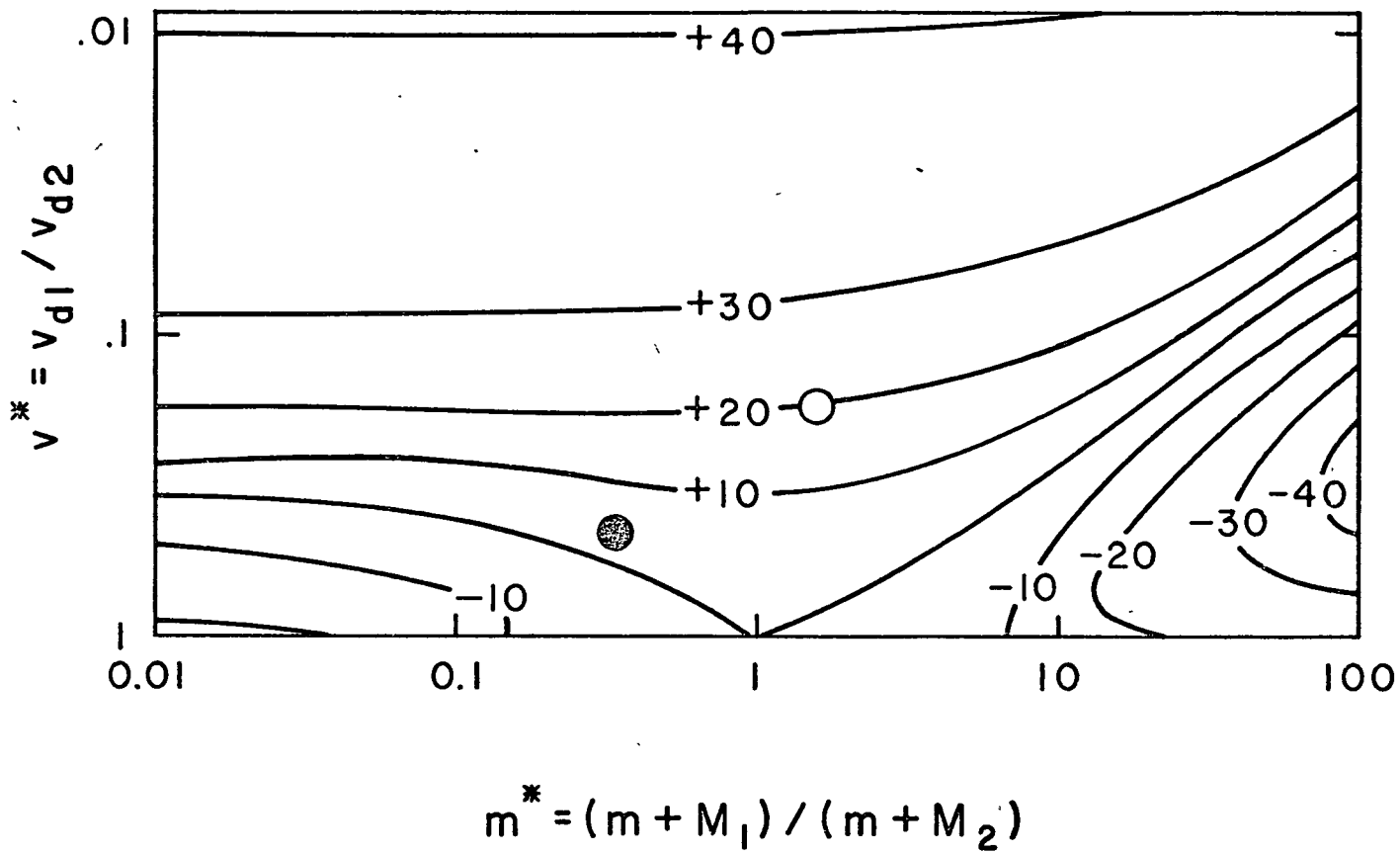
If we choose  $x_1$  and  $D^*$ , then deviations from Blanc's law can be shown as contours in a  $v^* - m^*$  plane. Such a diagram is shown in Fig. 1, with  $x_1 = 1/2 = x_2$  and  $D^* = 1$ . The latter choice means that the deviations from the Einstein relation are the same in the two pure components; as a consequence, the diagram represents the deviations from Blanc's law for  $v_d$  as well as  $D$ . Two points on the diagram refer to  $K^+$  in  $H_2 + N_2$  and to  $He^+$  in  $He + Ne$ . The deviations for the latter case are predicted to be small, in agreement with experiment.<sup>9</sup> Unfortunately there are no measurements for the case with large predicted deviations. Both positive and negative deviations are seen to be possible.

## V. CONCLUSIONS

The present results predict that deviations from Blanc's law for both mobility and diffusion coefficients can be of either sign at high fields, and can be of a magnitude that should be detectable experimentally. The deviations depend explicitly on the nature of the ion-neutral interaction through the averages  $\langle v_{rj} Q_j \rangle$  and  $\langle v_{rj} Q_j \rangle_j$ , and are zero for the Maxwell model in which  $Q_j$  is inversely proportional to  $v_{rj}$  and the mean free time between collisions is the same at all velocities. This behavior is reminiscent of the behavior of thermal diffusion in neutral gases, which has been used as a sensitive probe of intermolecular forces,<sup>10</sup> and indeed the same sort of averages occur in the momentum-transfer theory of gaseous thermal diffusion.<sup>11</sup> This suggests that deviations from Blanc's law at high fields might be a useful probe of ion-neutral forces. Physically, the present theory ascribes the deviations from Blanc's law to the difference in the ion energy partitioning in the gas mixture and in the pure components. Only the particular velocity dependence of the cross section given by the Maxwell model is just right to compensate for this effect.

FIGURE CAPTION

Fig. 1. Percentage deviations from Blanc's law,  $100[D^{-1}-D^{-1}(\text{Blanc})]/D^{-1}(\text{Blanc})$ , at high fields for an equimolar rigid-sphere binary gas mixture, as contours in a  $v^*-m^*$  plane with  $D^* = 1$ . The dimensionless variables are defined in Eqs. (25)-(27). The same contours hold for  $v_d$  as for  $D$  when  $D^* = 1$ . The open circle refers to  $K^+$  in  $H_2 + N_2$ , and the filled circle to  $He^+$  in  $He + Ne$ .



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