THE THEORETICAL TOOLS OF EXPERIMENTAL GRAVITATION[*]

Clifford M. Will

California Institute of Technology, Pasadena, California

Lectures to be presented at Course 56 of the

International School of Physics "Enrico Fermi",

Varenna, Italy, July 17 to July 29, 1972,

and to be published in the Proceedings

of the School.

TABLE OF CONTENTS

i

1. THEORETICAL FRAMEWORKS FOR TESTING
RELATIVISTIC GRAVITY

An astronomer detects mysterious pulses in the light from a star at the center of the Crab Nebula, and theorists speculate that their source is a rotating neutron star (a "pulsar"). Massive aluminum cylinders in Illinois and Maryland are suddenly and simultaneously set into vibration, and theorists suggest that a gravitational wave has just passed through the solar system. Radio astronomers discover that space is filled with black-body radiation with a temperature of about 3° Kelvin, and theorists say that it is a by-product of the initial "big bang" of the Universe. X-ray astronomers discover aperiodic fluctuations in the X-ray emission from Cygnus-X1; optical astronomers discover that Cygnus-X1 is associated with a single-line spectroscopic binary star; and from this evidence theorists speculate that the X-rays come from a Black Hole in orbit around a normal star.

But when the theorists sit down and begin to construct detailed models for these phenomena, they suddenly pull up short. All these phenomena, they notice, involve "relativistic" gravitation in a very crucial way. Newton's theory of gravitation is certainly inadequate to describe these phenomena quantitatively; and two of them (black holes and gravitational waves) it cannot describe even qualitatively. Experimental tests in the solar system up to 1960 seem to confirm Einstein's relativistic theory of gravity, so perhaps that is the theory to use in model-building. But those experiments were of such low accuracy (only 20 per cent precision in most cases), that they also seem to confirm several alternatives: Whitehead's Theory, the Belinfante-Swihart Theory,

Dicke-Brans-Jordan Theory, Yilmaz's Theory, Papapetrou's Theory, ... Theorists are hamstrung. Unless they have some strong reason for believing one of these theories over the others, they can have little confidence in the models they build to explain astrophysical phenomena.

Fortunately, the same advances in laboratory and space technology that made possible the discovery of these astrophysical phenomena, will also, in the coming decade, give the theoretical astrophysicist stronger experimental reasons for believing only one theory of gravitation. The technology of the 60's has handed us a set of high-precision tools for testing gravitational theories in the 70's; radar ranging to planets and satellites, with accuracies better than 15 meters; laser ranging to the Moon, accurate to better than 30 centimeters; long baseline interferometry, capable of measuring angles down to 3×10^{-4} seconds of arc; atomic and molecular clocks, stable to one part in 10^{14} over periods as long as a year; gravimeters, able to measure changes in acceleration on the Earth as small as 10^{-10} g; and many others.

These developments -- discoveries in astronomy and astrophysics, and advancing technology -- have made the systematic, high-precision testing of gravitation theories an important and exciting task for the 70's.

Although there are many new experimental possibilities, the cost of carrying out most in terms of manpower and money is very high. (The megabuck is a useful unit of measure for some of the tests.) For this reason, it is crucial that we have as good a theoretical framework as possible for comparing the relative values of the various experiments and for proposing new ones which might have been overlooked.

The most simple-minded theoretical framework would be a direct comparison of general relativity with Newtonian theory. Indeed, it was just such a comparison which motivated Einstein's original three tests -- the gravitational redshift, the deflection of light, and the perihelion shift of Mercury. One might think that we should merely continue to measure these and other non-Newtonian, general-relativistic effects to higher and higher accuracy; and only if a discrepancy between experiment and theory is found, should we begin to consider other theories.

But because of the lack of high precision data favoring general relativity over any other theory, and because of the large (and growing) number of competing theories, there is a great need for a theoretical framework which is powerful enough that it can be used to design and assess experimental tests in detail, yet is general enough that it is not biased in favor of general relativity. It should also provide a machinery for analysing all the theories of gravity which have been invented as alternatives to Einstein in the past 70 years, for classifying them, for elucidating their similarities and differences, and finally for comparing their predictions with the results of solar-system experiments. We would like to see experiment force us, with very very few a priori assumptions about the nature of gravity, toward general relativity or some other theory.

A leading exponent of this viewpoint is Robert H. Dicke. It has led him and others to perform several high-precision null experiments (Eötvös-Dicke-Braginsky experiment; Hughes-Drever experiment; ether-drift experiments) which greatly strengthen our faith in the foundations of general relativity. (See Dicke 1964; also Sect. 2 and 3 below.) Without

3

this viewpoint, some of the null experiments might not have been performed, and we would certainly not understand so well their significance.

Dicke himself has suggested one type of theoretical framework for comparing various theories of gravity and analysing the significance of various experiments. His framework (see Sect. 2 below) is particularly powerful for discussing the null experiments, for delineating the qualitative nature of gravity, and for devising new covariant theories of gravity.

The Dicke framework assumes almost nothing about the nature of gravity. It helps one to design and discuss experiments which test, at a very fundamental level, the nature of spacetime and gravity. Within it one asks such questions as: Do all bodies respond to gravity with the same acceleration? Is space locally isotropic in its intrinsic properties? What types of fields, if any, are associated with gravity -- scalar fields, vector fields, tensor fields, affine fields, ...?

Crucial among these experiments is the Eötvös experiment (as improved by Dicke and Braginsky and their collaborators) which verifies to high accuracy the composition-independence of acceleration of laboratory-sized bodies ("Universality of Free Fall"). By analysing this experiment within the Dicke framework, one arrives at a number of "fair-confidence" conclusions about the nature of gravity. These are (i) that gravity is associated, at least in part, with a "metric"; (ii) that freely falling test bodies move along geodesics of this metric; and (iii) that in local freely falling frames of this metric, all the non-gravitational laws of physics take on their special-relativistic forms. Current research is groping toward a more convincing proof of this "fair-confidence" conclusion (cf. Sect. 3). Theories of gravity which satisfy conditions (i), (ii)

4

and (iii) are called "metric" theories.

A second theoretical framework starts where the Dicke framework leaves off, by taking the postulates (i), (ii) and (iii) of metric theories as its foundation. This framework is the Parametrized Post-Newtonian (PPN) formalism of Eddington (1922), Robertson (1962), and Schiff (1967), as improved and extended by Schiff (1960a), Baierlein (1967), Nordtvedt (1968b), Will (1971a) and Will and Nordtvedt (1972).

The PPN framework takes the slow-motion, post-Newtonian limit of all conceivable metric theories and characterizes that limit by a set of ~ 9 real-valued parameters (see Sect. 4 for details). Each metric theory of gravity is characterized by a set of particular values for these PPN parameters. The task of solar-system gravity experiments in the coming decade can be regarded as one of measuring the values of these PPN parameters and thereby delineating, hopefully, which theory of gravity is correct.

In the following two sections, we will discuss the Dicke framework, and the foundation which it and the Eötvös-Dicke-Braginsky experiment lay for the PPN formalism. The remaining four sections deal with the PPN formalism in detail. Sect. 4 sets up the formalism and analyses the structure of the PPN metric; Sect. 5 reviews metric theories of gravity and their post-Newtonian limits; Sect. 6 derives PPN equations of motion; and Sect. 7 analyses specific solar-system experiments in detail, and uses the current status of experiment to put limits on the values of the PPN parameters, thereby ruling out several theories of gravity.

2. THE DICKE FRAMEWORK

2·1. Statement of the Framework

The Dicke framework for analyzing experimental tests of gravity was expounded in Appendix 4 of Dicke's (1964) Les Houches lectures. Here we shall present a slightly generalized version of Dicke's framework, and we shall couch it in slightly different language.

Dicke begins with two statements about the type of mathematical formalism to be used in discussing gravity. These statements have little physical content;[1] they serve primarily to delineate the vantage point

---

[1] See, however, Trautman's (1965, p. 101) remarks about the physical significance of assuming spacetime to be a differentiable manifold.

---

from which gravity will be viewed. They say:

Statement (i): Spacetime will be regarded as a 4-dimensional manifold, with each point of the manifold corresponding to a physical event. The manifold need not *a priori* have either a metric or an affine connection.

Statement (ii): The theory of gravity will be expressed in a form that is independent of the particular coordinates used; i.e., the equations of gravity and the mathematical entities in them will be put into covariant form.

Notice that even if there is some physically preferred coordinate system in spacetime, the theory can still be put into covariant form. For example, one can introduce four scalar fields, whose numerical values are equal to the values of the preferred coordinates:

$$\alpha(q) = x(\alpha), \quad \beta(q) = y(q), \quad \gamma(q) = z(q), \quad \delta(q) = t(q),$$

(2.1)

q a point in spacetime; (x,y,z,t) preferred coordinates;

and one can then regard these fields as associated with gravity.

The Newtonian theory of gravity is an example of a theory that is not normally expressed in covariant language; the Newtonian equations -- $\nabla^2 U = -4\pi G\rho$, $\underset{\sim}{F} = m\nabla U$ -- are valid only in a particular class of coordinate frames. However, as Cartan has shown [see Trautman (1965) for a review] , Newtonian theory can be expressed in an alternative co-variant form involving a nonmetrical affine connection.

"Stratified Theories" of gravity (cf. Sect. 5) are also examples of theories which have physically preferred coordinate systems, but which can still be put into covariant form.

Having laid down his mathematical viewpoint (statements i and ii above), Dicke then imposes two constraints, which he requires of all acceptable theories of gravity. They are:

Constraint 1: Gravity must be associated with one or more fields of tensorial character (i.e., scalars, vectors, and tensors of various ranks).

Constraint 2: The dynamical equations which govern gravity must be derivable from an invariant action principle.

These constraints have deep significance; they strongly confine the theory. For this reason, we should be willing to accept them only if they are fundamental to our subsequent arguments. For most applications of the Dicke framework they are not needed at all. Therefore, we shall usually not assume them. If we ever need and use them, we shall

state so explicitly.

There is one final item in the Dicke framework -- an item of great significance:

Guiding principle: Ockham's (1495) razor -- Nature likes things as simple as possible (pluralitas non est ponenda sine necessitate).

This guiding principle is used, of course, to tell us what kinds of theories of gravity are the most likely to be correct -- and, therefore, what kinds of experiments are the most important ones to perform.

Notice that by telling us to apply Ockham's razor within a covariant mathematical framework, Dicke builds a very particular bias into his formalism. Only those theories which look simple when expressed in covariant form are deemed promising. By this criterion, general relativity is very promising -- perhaps the most promising theory of all! However, Newtonian theory is not. In its covariant form (Trautman 1965), by contrast with its conventional form, Newtonian theory is exceedingly complicated. A physicist working in the Dicke framework would never be so pathological as to dream up a theory like that of Newton!

Keeping this bias in mind, we shall proceed to discuss experiments within the Dicke framework.

2·2. The Fields Associated with Gravity

The Dicke framework is particularly useful for designing and interpreting experiments which ask what types of fields are associated with gravity. When Dicke himself uses it for this purpose, he imposes constraint 1 (above) -- i.e., he considers only scalar, vector, and tensor fields. To be on the safe side however, we shall go all the way and admit

8

any field that takes on a covariant form; i.e., we shall abandon constraint 1.

(i) Second-Rank Tensor Field

First let us consider tensor fields of rank $\binom{0}{2}$. There is very strong experimental evidence that at least one such field exists in the Universe: a symmetric field ψ_{ij} which, far from all gravitating bodies, reduces to the Minkowskii metric η_{ij}. Far from gravitating bodies, this η_{ij} has orthonormal tetrads which are related by Lorentz transformations, and determines the ticking rates of atomic and nuclear clocks and the lengths of laboratory rods.

The evidence for such a field comes largely from elementary particle physics. Since these experiments are performed at high energies and velocities, and over very small regions of space and time, the effects of gravity on their outcome are negligible. Thus we may treat such experiments <u>as if they were being performed far from all gravitating matter</u>. The evidence provided by these experiments is of two types: First, experiments which measure space and time intervals directly -- e.g., measurements of the time dilation of the decay rates of unstable particles.[2] Second, experiments which reveal the fundamental role played by the Lorentz group in particle physics,[3] including every-day,

---

[2]For a 2 per cent test of time dilation with muons of $(1 - v^2)^{-1/2} \sim 12$ in a storage ring, see Farley, Bailey, Brown, Giesch, Jöstlein, van der Meer, Picasso, and Tannenbaum (1966). For earlier time dilation experiments see Frisch and Smith (1963); Durbin, Loar, and Havens (1952); Rossi and Hall (1941); Ives and Stilwell (1938, 1941). For an experiment which verifies, to one part in 10^4, that the speed of light (γ rays) is

independent of the velocity of its source (decaying π^o) for source velocities $v > 0.99975c$, see Alväger, Farley, Kjellman, and Wallin (1964).

[3]See Lichtenberg (1965) for a discussion of Lorentz invariance, spin and statistics, the TCP theorem, and relevant experiments.

---

high-precision verifications of four-momentum conservation and of the relativistic laws of kinematics. To cast out the Minkowskii metric η_{ij} entirely would destroy the theoretical backing of such experiments.

Let us notice what particle-physics experiments do and do not tell us about the tensor field, ψ_{ij}: First, they do not guarantee that there exist global Lorentz frames -- i.e., coordinate systems extending throughout all of spacetime, in which[4]

---

[4]Here and throughout most of these lectures we use units in which the speed of light is unity (see Sect. 3 for a discussion of units and notation).

(2.2) $\psi_{ij} =$ Minkowskii metric $\eta_{ij} =$ diag $(1, -1, -1, -1)$.

Nor do they demand that at each event q there exist local frames, related by Lorentz transformations, in which the laws of elementary particle physics take on their special relativity form. They only demand that, in the limit as gravity is "turned off" (either by working far from gravitating bodies or by performing experiments where the effects of gravity can be ignored), the non-gravitational laws of physics reduce to the laws of special relativity.

Second, elementary particle experiments do tell us that the times

measured by atomic clocks in the limit as gravity is turned off depend only on velocity; not upon acceleration. The measured squared interval is $ds^2 = \eta_{ab} dx^a dx^b$, independently of acceleration. Equivalently but more physically, the time interval measured by a clock moving with velocity v^α relative to a coordinate frame in the absence of gravity is

$$(2.3) \qquad ds = (\eta_{ab} dx^a dx^b)^{1/2} = [1 - (v^x)^2 - (v^y)^2 - (v^z)^2]^{1/2} dt,$$

independently of the clock's acceleration $d^2 x^\alpha/dt^2$. If this were not so, then particles moving in circular orbits in strong magnetic fields would exhibit different decay rates than freely moving particles, which they do not (Farley et al. 1966);[5] and ^{57}Fe nuclei would show acceleration

---

[5]The experiment of Farley et al. is a 2 per cent check of acceleration-independence of the muon decay rate for energies $E/m = (1 - v^2)^{-1/2} \sim 12$ and for accelerations, as measured in the muon rest frame, of $a = 5 \times 10^{20}$ cm/sec^2 = 0.6 cm. Note that, at accelerations a factor 10^{13} larger than this ($a \sim 10^{33}$ cm/sec$^2 \sim 10^{12}$/cm), in one light travel time across the muon it accelerates up to near the speed of light, if it was initially at rest. Such large accelerations will probably affect the decay rates -- not because of any breakdown in relativity theory, but because the decay cannot be analysed within a single co-moving Lorentz frame. The muon ceases to be a valid special relativistic clock. See Ageno and Amaldi (1966) and Bailey and Picasso (1970).

---

dependence in the frequency of their Mössbauer transitions, which they do not (Sherwin 1960).

We shall henceforth assume the existence of the symmetric

tensor field ψ_{ij}.

(ii) More than One Second-Rank Tensor

The Hughes-Drever experiments rule out, with very high precision, the existence of more than one second-rank tensor field, both coupling directly to matter. (See pp. 14-22 of Dicke (1964) for discussion.) They do not rule out, however, additional second-rank tensor fields which couple only to gravity or to matter's gravitational self-energy, because the effects of self-gravity in those experiments were negligible. Experiments which may be used to rule out such fields are discussed in Sect. 7.

(iii) Vector Field

Various ether-drift experiments make it unlikely that a vector field coupling directly to matter is present. [See pp. 22-25 of Dicke (1964); also Turner and Hill (1964); Champeney, Isaak, and Khan (1963).] Again, vector fields which couple only to matter's gravitational energy, can only be ruled out by experiments which involve gravity (Sect. 7).

(iv) Scalar Field

No experiment performed thus far has been able to rule out or reveal the presence of a scalar field. However, future studies of the polarization properties of cosmic gravitational waves might reveal the scalar field, if it is present. The deformations produced in a disk placed perpendicular to the incoming waves are area-preserving (quad-rupolar) if the waves are purely tensor in nature; but they can be area-changing (monopolar) if the waves have a scalar component. Other ways of experimentally delineating a scalar field are discussed by Dicke (1964).

(v) Scalar, Vector, and Tensor Densities

When Dicke (1964) writes down his constraint 1 (cf. §II.a above), he explicitly states that he will not consider theories in which boson fields such as gravity transform as tensor densities; he admits only tensorial transformation laws. However, if we conclude that a metric field is present, such a constraint becomes superfluous. Any scalar, vector, or tensor density can be expressed in terms of the determinant of the metric and a corresponding pure scalar, vector, or tensor. Hence, with a metric present we can ignore the densities. Without a metric we must search for experiments to rule out tensor-density fields.

So far we have said nothing about the existence of a metric field in spacetime: none of the experiments discussed above offers any evidence for its existence. For such evidence, we must turn to the Eötvös experiment (and possibly to the gravitational redshift experiment), and to a conjecture which originated with Leonard Schiff.

3. SIGNIFICANCE OF THE EÖTVÖS-DICKE-BRAGINSKY EXPERIMENT

3·1. Introduction

Although Einstein considered the gravitational redshift one of the most important of the predictions of general relativity, it was not until 1965 that a truly accurate confirmation of the redshift could be made. That year, Pound and Snider (1965), using an improved version of the experiment performed five years earlier by Pound and Rebka, confirmed the gravitational redshift of photons climbing up the Harvard tower through the Earth's gravitational field. Their accuracy of one per cent was made possible by the use of the Mössbauer effect (recoilless emission and absorption of photons). However, in the intervening years, the interpretation of the redshift experiment had changed.

The work of Schiff (1960b) and Dicke (1964) suggested that the redshift experiment was not a strong test of general relativity at all. The gravitational redshift, they claimed, could be calculated by appealing to (i) conservation of energy, (ii) elementary quantum theory, and (iii) the Eötvös experiment, i.e. the measurement of the composition-independence of gravitational acceleration for laboratory-sized bodies. This univerality of gravitational acceleration was first verified by Baron Roland v. Eötvös (Eötvös, Pekar and Fekete 1922) to one part in 10^9 precision, and improved by Roll, Krotkov, and Dicke (1964) (one part in 10^{11}) and more recently by Braginsky and Panov (1971) (one part in 10^{12}). Leonard Schiff was working on what he felt would be a more convincing "proof" of this point of view at the time of his tragic death in January 1971.

A second point of view, spelled out by Schild (1962) and others

was that the gravitational redshift, while not a strong test of general relativity itself, does prove that space and time, as measured by rods and atomic clocks, has to be curved by the presence of gravitating masses.

A third point of view has emerged from recent research (Lee, Lightman and Thorne 1972). This interpretation is in some sense an amalgamation of the other two, and is outlined in the following subsections.

3·2. Completeness, Self-Consistency and Agreement with Special Relativistic Physics

Any theory of gravity which is to be taken seriously at all must satisfy the following three constraints.

(i) It must be complete, that is it must be capable of analysing from "first principles" the outcome of every experiment of interest. It is not enough for the theory to postulate that bodies made of different material fall with the same acceleration. The theory must mesh with and incorporate a complete set of electromagnetic and quantum mechanical laws, which can be used to calculate the detailed behavior of atoms in gravitational fields.

(ii) It must be self-consistent. A gravitation theory is consistent if its prediction for the outcome of every experiment is unique, i.e. if, when one calculates the prediction by two different methods, one always gets the same result.

(iii) In the limit as gravity is "turned off", the non-gravitational laws of physics must reduce to the laws of special relativity. We call a theory with this property a "relativistic" theory. Elementary particle

experiments convince us that a theory of gravity must be "relativistic" to be viable.

Table 3.I contains a partial list of theories of gravity which violate some of these constraints, and are thus non-viable. (See Thorne, Will and Ni [1971] for further discussion.)

3·3. Schiff's Conjecture

The Eötvös-Dicke-Braginsky (EDB) experiments verify, to high precision, the composition-independence of the gravitational acceleration of laboratory bodies. This can be restated as the Principle of the Universality of Free Fall (UFF) which states that "if an uncharged test body is placed at an initial event in spacetime and is given an initial velocity there, then its subsequent world line will be independent of its internal structure and composition" (see Misner, Thorne and Wheeler [1972] for detailed definitions and discussion). The EDB experiments are direct tests of UFF.

Dicke's (1964) "Weak Equivalence Principle" and Bondi's (1957) "equality of passive and inertial mass" are equivalent to UFF.

Dicke (1960) and Schiff (1959, 1960b) have discussed the theoretical significance of the EDB experiments. Lee, Lightman, and Thorne (1972) have used their ideas as the foundation of a viewpoint for analysing the EDB experiment, a viewpoint summarized by "Schiff's Conjecture": Any complete and self-consistent gravitation theory which embodies the Universality of Free Fall must also unavoidably embody the Einstein Equivalence Principle, which states that all the non-gravitational laws of physics are the same in every local, freely falling frame. By "local

freely falling frame" we mean a reference frame which falls along one of the universal test-body world lines, and which is small enough that one can ignore inhomogeneities in the gravitational fields.

A useful theorem which can be used to extend Schiff's conjecture is the following: A (complete and self-consistent) gravitation theory is a metric theory if and only if it is relativistic and embodies the Einstein Equivalence Principle. By metric theory of gravity we mean any theory which (i) endows spacetime with a metric, (ii) chooses geodesics of the metric as its universal test-body trajectories, and (iii) chooses the special-relativity laws of physics as the laws to be satisfied in its freely falling frames (see Misner, Thorne and Wheeler [1972] for discussion)[6].

---

[6]This definition of "metric theory" is more restrictive than the one used by Thorne and Will (1971) and by Will (1971a).

This theorem is a straightforward consequence of the definitions of "relativistic" and "metric" theories and of the Einstein Equivalence Principle.

Thus, if the Eötvös-Dicke-Braginsky experiments have been carried out to sufficiently high precision (verification of UFF), and if Schiff's conjecture is correct, then in order to agree with the EDB experiments and to be relativistic, a theory of gravity must be a metric theory.

The EDB experiments therefore become a powerful tool for distinguishing metric theories from non-metric theories, and for ruling out the latter.

3·4. Proofs of Schiff's Conjecture

Leonard Schiff's interest in a proof of the conjecture that the Universality of Free Fall implies the Einstein Equivalence Principle was rekindled in the fall of 1970 during a conference on experimental tests of gravitation theories held at the California Institute of Technology[7],

---

[7]Kip S. Thorne, private communication.

---

and he set to work on a proof, using the sophisticated quantum mechanical techniques he had used in earlier discussions of UFF (Schiff 1959, 1960b). Unfortunately, his untimely death in January 1971 cut short the analysis.

Lee, Lightman and Thorne (1972) have made preliminary steps toward a proof of the conjecture. Central to their discussion is a special case of Schiff's Conjecture: Every relativistic Lagrangian-based theory of gravity (complete and self-consistent) which embodies UFF is necessarily a metric theory. They are tacking this specialized conjecture from two directions:

(i) a direct proof, using general concepts in the "theory of Lagrangian-based theories";

(ii) an indirect proof, by demonstrating the absence of a counter-example: a Lagrangian-based non-metric theory which embodies UFF, i.e. agrees with the Eötvos-Dicke-Braginsky experiment to infinite precision. Several Lagrangian-based, seemingly non-metric theories are known; one of the strongest contenders is a theory due to Belinfante and Swihart (1957a,b,c). But calculations by Lee and Lightman (1972) suggest that the Belinfante-Swihart theory violates the Eötvos-Dicke-Braginsky experiment, although probably at a level higher than the

current experimental limit. Their results also show that, at least to the order (in a power series expansion in the Belinfante-Swihart gravitational field) to which the theory <u>agrees with</u> the EDB experiment, it can be rewritten as a metric theory consistently to that same order.

While these results are still a long way from ruling out all non-metric theories of gravity, they strengthen our conviction that the Eötvös-Dicke-Braginsky experiments and the Universality of Free Fall form a powerful tool for sifting through metric and non-metric theories.

3·5. Significance of the Gravitational Redshift Experiment

The gravitational redshift is a direct consequence of the Einstein Equivalence Principle and of the principle of "relativistic" gravity (which together produce a metric theory of gravity [see Subsect. 3·3]), i.e. it can be derived using the standard "elevator argument" first used by Einstein (see, for example, Schiff 1960b), or directly from the postulates of metric gravitation theories. If the Schiff conjecture is correct, then the redshift becomes a direct consequence of the Universality of Free Fall or the Weak Equivalence Principle. For this reason, many authors have viewed the redshift experiment as a weak test of gravitation theories (Dicke 1964, Schiff 1960b).

However, one should be more generous toward the redshift experiment, because in the absence of a rigorous proof of the Schiff conjecture, it still provides a useful test of gravitation theories (albeit under restricted circumstances). This latter point of view has been spelled out by Schild (1962) and by Thorne and Will (1971). Their viewpoint can be summarized as follows:

19

We first make a restrictive assumption about the nature of gravity
(an assumption less restrictive however, than assuming "relativistic"
gravity plus the Einstein Equivalence Principle). We assume that
gravitation is relativistic, and that there exist local frames in which
atomic clocks measure proper time in the special relativistic manner
(cf. eq. [2.3]). We assume nothing however, about the motion of these
frames, which we denote "local Lorentz frames".

The redshift experiment of highest precision is that of Pound and
Rebka (1960), as improved by Pound and Snider (1965). It reveals a
redshift of $z = \Delta\lambda/\lambda = (gh/c^2)(1 \pm 0.01)$ for photons climbing up through
a height h in the Earth's locally homogeneous gravitational field -- if
the emitter and receiver are at rest relative to the Earth's surface.
Here g is the acceleration of test bodies at the surface of the Earth.
This tells us that the local Lorentz frames accelerate downward with
the same acceleration, g, as acts on a free particle (to within 1 per cent
precision). To arrive at this conclusion from the experiment, we argue
as follows:

We wish our argument to be as independent of the special
relativistic laws of physics as possible. The only aspects of special
relativity that we shall use are (i) the relationship between the Minkowskii
metric of the local Lorentz frames and the ticking rates of atomic clocks;
and (ii) the conservation of wave fronts in electromagnetic waves. Let
us assume (falsely) that the local Lorentz frames were unaccelerated
relative to the walls of the tower used in the Pound-Rebka experiment.
We can then perform a calculation in that particular Lorentz frame which
was attached to the walls of the tower and was large enough to cover the

entire tower. The static nature of the emitter, receiver, gravitational field, and Lorentz coordinate system guaranteed that, although the spacetime trajectories of the wave crests might have been bent by gravity, they were certainly the same from one crest to another, except for a translation Δt_L in the Lorentz time coordinate. Hence, the coordinate rates $1/\Delta t_L$ of emission and reception of wave crests were the same. But by assumption these Lorentz coordinate rates were also the proper rates measured by the atomic clocks (^{57}Fe nuclei) of the experiment. Hence, theory predicts zero redshift, in contradiction with experiment. Our assumption that the local Lorentz frames were unaccelerated must be wrong!

We must assume, then, that the local Lorentz frames were accelerated relative to the tower. Since gravity pointed vertically and all horizontal directions were equivalent in all respects, the acceleration of the Lorentz frames must have been vertical. Denote by $\underline{a}$ its value in the downward direction. As in our previous argument, in a static coordinate system (i.e., in coordinates at rest relative to emitter, receiver, and Earth's static gravitational field) the wave-crest trajectories must have been identical, except for a time translation Δt_s from one crest to the next. But in this case the static coordinates were not Lorentz coordinates. Rather, they were accelerated upward (in the $+z$ direction) relative to the Lorentz frames (here c is the speed of light):

$$ct_L = (z_s + c^2/a) \sinh (at_s/c) \quad ,$$

(3.1)
$$z_L = (z_s + c^2/a) \cosh (at_s/c) \quad ,$$

$$x_L = x_s \quad , \quad y_L = y_s \; \cdot$$

[For an elementary derivation and discussion of this transformation law between Lorentz frames and accelerated frames, see, e.g., Misner, Thorne, and Wheeler (1972).] Hence, proper time as measured by atomic clocks was given by

$$c^2 \, d\tau^2 = c^2 \, dt_L{}^2 - dx_L{}^2 - dy_L{}^2 - dz_L{}^2$$

(3.2)

$$= (1 + az_s/c^2)^2 \, c^2 \, dt_s{}^2 - dx_s{}^2 - dy_s{}^2 - dz_s{}^2.$$

Since, as before, the wave-crest emission and reception rates were the same $(1/\Delta t_s)$ when measured in static coordinate time, they were related by

$$\frac{\Delta\lambda}{\lambda} = \frac{\nu_{em}}{\nu_{rec}} - 1 = \frac{(1 + az_{s\,rec}/c^2)\Delta t_s}{(1 + az_{s\,em}/c^2)\Delta t_s} - 1 = a(z_{s\,rec} - z_{s\,em})/c^2$$

(3.3)

$$= ah/c^2 \, ,$$

when measured in the proper time of the atomic clocks. But the experimentally measured redshift was gh/c^2 to a precision of one per cent. Hence, the downward acceleration of the Lorentz frames was the same as that of a free particle, $g = 980 \, cm/sec^2$, to a precision of one per cent.

In summary, the redshift experiments reveal that, to a precision of $\sim 0.01 \, GM/R^2$ where M and R are the mass and radius of the Earth, the local Lorentz frames at the Earth's surface are unaccelerated relative to freely falling test bodies. Equivalently, test bodies move along straight lines in the local Lorentz frames. If we identify as the metric the unique second rank tensor field $\mathbf{g}$ which takes the Minkowskii form in every freely falling frame, then elementary differential geometry

22

tells us that the test-body trajectories are geodesics of the metric **g** .

Thus, in the absence of a proof of the Schiff Conjecture, gravitational redshift experiments may still be a valuable tool in testing the validity of metric theories of gravity. For this reason, it is very important that the precision of the redshift experiments be improved as much as possible -- both on Earth (homogeneous field) and elsewhere in the solar system (inhomogeneous fields). Of particular interest will be experiments in which atomic clocks are flown in spacecraft and rockets (Kleppner, Vessot and Ramsey 1970, Vessot and Levine 1971, Vessot 1971).

3·6. Gravitation as a Geometric Phenomenon

These analyses of Schiff's Conjecture and of the gravitational redshift have instilled in some theorists a very strong conviction that only metric theories of gravity have a hope of being completely viable. Since the remaining sections deal solely with metric theories of gravity, we will briefly review here some of the key formulas of differential geometry, and will set down once and for all the conventions and notation to be used. We discuss one by one the three "postulates" of metric theories of gravity:

1. Spacetime is a four-dimensional manifold endowed with a metric **g** . The metric is a $\binom{0}{2}$ tensor, with components g_{ij} in a particular x^i coordinate system. This metric endows spacetime with a "Riemannian affinity" Γ^i_{jk} (not a tensor) defined, in a given coordinate system by

$$\Gamma^i_{\ jk} = \frac{1}{2} g^{im} (g_{mj,k} + g_{mk,j} - g_{jk,m}) \ ,$$

(3.4)

$$\Gamma^k_{\ jk} = (\ell n \ \sqrt{-g})_{,j} \ .$$

[Notation: g is the determinant of g_{ij}, comma denotes partial derivative.]

The Riemannian affinity is used to define the covariant derivative

[Notation: semicolon denotes covariant derivative]:

$$A^i_{\ ;j} \equiv A^i_{\ ,j} + \Gamma^i_{\ jk} A^k \ ,$$

(3.5)

$$A_{i;j} \equiv A_{i,j} - \Gamma^k_{\ ij} A_k \ ,$$

where

(3.6)
$$A_i \equiv g_{ij} A^j \ .$$

Under a transformation of coordinates of the form

(3.7)
$$(x^i)^\dagger = f^i(x^j) \ ,$$

the metric transforms according to the standard tensorial transformation law:

(3.8)
$$(g_{ij})^\dagger = \frac{\partial x^k}{(\partial x^i)^\dagger} \frac{\partial x^\ell}{(\partial x^j)^\dagger} g_{k\ell} \ .$$

If the transformation is infinitesimal, i.e.

(3.9)
$$(x^i)^\dagger = x^i + \xi^i(x^j) \ ,$$

then g_{ij} transforms to first order in ξ^i according to

(3.10)
$$g_{ij}^{\ \dagger}[(x^i)^\dagger] = g_{ij}(x^i) - \xi_{i;j} - \xi_{j;i} \ .$$

Covariant derivatives of tensors transform as tensors, while partial

derivatives do not. Thus far there is no physical content, just mathematics.

2. _Test-body trajectories are geodesics of the metric._ A curve is a geodesic of the metric if there exists a parameter λ which parametrizes the curve $x^i(\lambda)$ such that the curve satisfies

$$(3.11) \qquad \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0 \ .$$

Here λ is called an affine parameter. Equation (3.11) can also be derived by minimizing the invariant quantity

$$(3.12) \qquad d\sigma = (g_{ij} \, dx^i \, dx^j)^{1/2} \ ,$$

along the trajectory, i.e. by using the standard variational calculation:

$$(3.13) \qquad 0 = \delta \int d\sigma = \delta \int_{\lambda_1}^{\lambda_2} (g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda})^{1/2} d\lambda \ .$$

Equation (3.13) yields equation (3.14) only if $d\sigma = d\lambda \neq 0$.

In local freely falling frames, coordinates can always be chosen so that at a given event q,

$$(3.14) \qquad g_{ij}(q) = \eta_{ij} \ , \qquad \Gamma^i_{jk}(q) = 0 \ .$$

We still lack a connection between physical objects and the metric. This connection is given by Postulate 3.

3. _In local freely falling frames, the non-gravitational laws of physics take on their special relativistic forms._ These special relativity laws include the following:

     a. Physical rods and atomic clocks measure spacetime intervals

given by

$$(3.15) \qquad ds^2 = \eta_{ij} \, dx^i \, dx^j \ .$$

b. Maxwell's equations have the form

$$(3.16) \qquad F^{ij}{}_{,j} = 4\pi J^i \ , \qquad F_{ij,k} + F_{ki,j} + F_{jk,i} = 0 \ ,$$

where F^{ij} is the Maxwell field tensor, related to the four-vector potential A^i by

$$(3.17) \qquad F_{ij} = A_{i,j} - A_{j,i} \ ,$$

and where J^i is the current four-vector.

c. Equations of Motion for stressed matter have the form

$$(3.18) \qquad T^{ij}{}_{,j} = 0 \ ,$$

where T^{ij} is the stress-energy tensor for matter and non-gravitational fields.

In non-freely-falling frames, the non-gravitational laws of physics take on curved-spacetime forms, which are obtained from the special relativistic forms by invoking the rules:

$$\eta_{ij} \rightarrow g_{ij}$$

"comma" $\rightarrow$ "semicolon"

These rules are sometimes referred to as the "Strong Equivalence Principle". The result:

$$(3.19) \qquad a. \ \ ds^2 = g_{ij} \, dx^i \, dx^j \ .$$

(3.20)
$$\text{b. } F^{ij}{}_{;j} = 4\pi J^i , \quad F_{ij;k} + F_{ki;j} + F_{jk;i} = 0 ,$$

$$F_{ij} = A_{i;j} - A_{j;i} ,$$

(3.21)
$$\text{c. } T^{ij}{}_{;j} = 0 ,$$

and so on. For a discussion of possible "Strong Equivalence Principles" which use non-Riemann affine connections, see Thorne and Will (1971).

A consequence of the geometrical optics limit of the curved-space-time Maxwell equations (3.20) is that photons travel along null geodesics of the metric, i.e. geodesics which satisfy

(3.22)
$$ds^2 = g_{ij} \, dx^i \, dx^j = 0.$$

Because of the arbitrariness of coordinates in spacetime, the results of any experiment must always be expressed in terms of invariant, physically-measurable quantities: times measured by atomic clocks (proper time), distances measured by physical rods or by light signals (proper distance), and so on.

We complete this review of gravity as a metric phenomenon with a summary of notation, conventions and units to be used throughout these lectures:

a. Roman indices run over the values 0, 1, 2 and 3; Greek indices run over the values 1, 2 and 3.

b. Tensors and four-vectors are written abstractly using thick type, e.g. **g** , **K** ; and in component notation using italic type, e.g. g_{ij}, K^i; three-dimensional Euclidean vectors will be written in bold-face type, e.g. $\underset{\sim}{a}$, $\underset{\sim}{v}$.

c. The Minkowskii metric has signature - 2, that is

(3.23) $\eta_{ij} = \text{diag} [1,-1,-1,-1]$.

d. Round brackets surrounding indices denote symmetrization; square brackes denote anti-symmetrization, for example,

(3.24) $A_{(ij)} = \frac{1}{2} (A_{ij} + A_{ji})$,

(3.25) $A_{[ij]} = \frac{1}{2} (A_{ij} - A_{ji})$.

e. We use "geometrized" units: units in which the speed of light is unity and the Newtonian gravitational constant as measured far from the solar system and galaxy, in the mean rest-frame of the Universe, is unity. In these units, the mass of the Sun is 1.473 km.

For a much more thorough discussion of graviation as geometry, the reader is referred to Misner, Thorne and Wheeler (1972).

4. THE PARAMETRIZED POST-NEWTONIAN FORMALISM

4·1. Introduction

In this section, and for the remainder of these lectures, we will assume that whatever the correct theory of gravity is, it must be a metric theory. This assumption rests on our conviction that the high-precision Eötvös-Dicke-Braginsky experiments and the gravitational redshift experiments together may be used to rule out all non-metric theories of gravity (Sect. 3). Of course, until this conviction has been completely justified, we must keep a small portion of our minds open to non-metric theories. But for the remaining sections, we will focus on metric gravitational theories.

When we examine the fundamental postulates of metric theories of gravity (Sect. 3), we notice a crucial feature: no matter how complex the theory, no matter what additional gravitational or cosmological fields it deals with, <u>freely falling matter responds only to the metric</u> g_{ij}. This is embodied in the equations of motion

$$(4.1) \qquad \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{\ jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0, \qquad [\text{test bodies}]$$

$$(4.2) \qquad T^{ij}_{\ ;j} = 0, \qquad [\text{stressed matter}]$$

where the covariant derivative is computed solely from the metric using standard formulas for the Christoffel symbols (see Sect. 3). T^{ij} is the stress-energy tensor for matter and all non-gravitational fields, and λ is an affine parameter along the test-body's world line.

Thus the only gravitational field which enters the equations of motion (4.1) and (4.2) is the metric. The role of the other fields which a given theory may contain can only be that of helping to generate the spacetime curvature associated with the metric. Matter may create these fields, and they plus the matter may generate the metric, but they cannot act back directly on the matter. The matter responds only to the metric.

From this point of view, the metric g_{ij} and the equations of motion become the primary theoretical entities, and all that distinguishes one metric theory from another is the particular way in which matter (and possibly other gravitational fields) generates the metric.

The comparison of metric theories of gravity with each other and with experiment becomes particularly simple when one takes the slow-motion, post-Newtonian limit. Fortunately, the post-Newtonian limit is sufficiently accurate to encompass all solar-system tests that can be performed in the foreseeable future, with the exception of gravity-wave experiments. (Gravity waves do not exist in the post-Newtonian limit.)

4·2. The Post-Newtonian Limit

In the solar system, gravitation is weak enough that Newton's theory of gravity is adequate to explain all but the most minute effects. To an accuracy of about a part in 10^5, light rays travel on straight lines at constant speed, and test bodies move according to

$$(4.3) \qquad \underset{\sim}{a} = \underset{\sim}{\nabla} U \; ,$$

where $\underset{\sim}{a}$ is the body's acceleration, and U is the Newtonian gravitational potential. From the standpoint of a metric theory of gravity, Newtonian physics may be viewed as a first-order approximation, or a weak-field limit. Consider a test body momentarily at rest in a static external gravitational field. From the geodesic equation (4.1), the body's acceleration in a $(t, \underset{\sim}{x})$ coordinate system is given by

$$a^\alpha = - \Gamma^\alpha{}_{oo}$$

$$(4.4) \qquad = \frac{1}{2} g^{\alpha\beta} g_{00,\beta} \,.$$

In the complete absence of gravity, we know the metric must reduce to the special relativity Minkowskii metric

$$(4.5) \qquad g_{ij} \rightarrow \eta_{ij} = \text{diag} \, (1, -1, -1, -1).$$

In the presence of a very weak gravitational field, then, equation (4.4) can yield Newtonian gravitation, equation (4.3) only if

$$(4.6) \qquad g^{\alpha\beta} \simeq - \delta^{\alpha\beta} \,,$$

$$(4.7) \qquad g_{oo} \simeq 1 - 2U \,.$$

Equations (4.6) and (4.7) along with the equations of motion represent the Newtonian limit of any metric theory of gravity. But the Newtonian limit no longer suffices when we begin to demand accuracies greater than a part in 10^5. For example, it cannot account for Mercury's additional perihelion shift of $\sim 5 \times 10^{-7}$ radians per orbit. Thus we need a more accurate approximation to the spacetime metric, which will correctly account for solar-system effects which

go beyond, or "post" Newtonian theory. This higher order approximation to the metric is known as the Post-Newtonian Limit (Chandrasekhar 1965).

The key features of the post-Newtonian limit can be better understood if we first develop a "bookkeeping" system for keeping track of "small quantities". In the solar system, the Newtonian gravitational potential U is nowhere larger than 10^{-5} (in geometrized units, U is dimensionless). Planetary velocities are related to U by virial relations which yield

$$(4.8) \qquad v^2 \lesssim U .$$

The matter making up the Sun and planets is under pressure p, but this pressure is generally smaller than the matter's gravitational energy density ρU, where ρ is the rest-mass density of matter; in other words

$$(4.9) \qquad p/\rho \lesssim U ,$$

(p/ρ is $\sim 10^{-5}$ in the Sun, $\sim 10^{-10}$ in the Earth). Other forms of energy in the solar system (compressional energy, radiation, thermal energy, etc.) are small: the specific energy density Π (ratio of energy density to rest-mass density) is related to U by

$$(4.10) \qquad \Pi \lesssim U ,$$

(Π is $\sim 10^{-5}$ in the Sun, $\sim 10^{-9}$ in the Earth). These four small quantities are assigned a bookkeeping label which denotes their "order of smallness":

(4.11) $U \sim v^2 \sim p/\rho \sim \Pi \sim O(2)$.

Then single powers of velocity v are $O(1)$, U^2 is $O(4)$, Uv is $O(3)$, $U\Pi$ is $O(4)$ and so on. Also, since the time-evolution of the solar system is governed by the motions of its constituents, we have

$$\partial/\partial t \sim \underset{\sim}{v} \cdot \underset{\sim}{\nabla} ,$$

and thus

(4.12) $\dfrac{|\partial/\partial t|}{|\partial/\partial x|} \sim O(1)$.

We can now analyse the "Post-Newtonian" metric using this bookkeeping system. The variational principle, equation (3.13) from which one can derive the geodesic equation (4.1) for a single particle, may be rewritten

$$0 = \delta \int_A^B \left[g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right]^{1/2} dt$$

(4.13) $$= \delta \int_A^B \left[g_{oo} + 2g_{o\alpha} v^\alpha + g_{\alpha\beta} v^\alpha v^\beta \right]^{1/2} dt .$$

The integrand in equation (4.13) may thus be viewed as a Lagrangian L for a single particle in a metric gravitational field. From equations (4.6) and (4.7), we see that the Newtonian limit corresponds to

(4.14) $L = (1 - 2U - v^2)^{1/2}$,

as can verified using the Euler-Lagrange equations. In other words, Newtonian physics is given by an approximation for L correct to $O(2)$. Post-Newtonian physics must therefore involve those terms in L of

next highest order, $O(4)$.

But what happened to odd-order terms, $O(1)$ or $O(3)$? Odd-order terms must contain odd powers of velocity v or time-derivatives $\partial/\partial t$. Since these terms change sign under time reversal, odd-order terms therefore represent energy inflow or loss by the system. But, conservation of baryons, a fundamental low of physics, prohibits terms of $O(1)$ from appearing in L, and conservation of energy in the Newtonian limit prohibits terms of $O(3)$. These conservation laws are discussed further in Subsect. 4·7. In general relativity, the first odd-order terms which can appear in L are $O(7)$ terms: these terms represent energy lost from the system by gravitational radiation (see Chandrasekhar and Esposito [1970], and Burke [1971]).

In order to express L [eq. (4.13)] to $O(4)$, we must know the various metric components to an appropriate order:

$$(4.15) \qquad L = \left\{ 1 - 2U - v^2 + g_{oo}[O(4)] + 2g_{o\alpha}[O(3)]v^\alpha \right.$$
$$\left. + g_{\alpha\beta}[O(2)]v^\alpha v^\beta \right\}^{1/2}.$$

Thus the post-Newtonian limit of any metric theory of gravity requires a knowledge of

$$g_{oo} \quad \text{to} \quad O(4),$$

$$g_{o\alpha} \quad \text{to} \quad O(3),$$

$$g_{\alpha\beta} \quad \text{to} \quad O(2).$$

The post-Newtonian propagation of light rays may also be obtained using the above approximations to the metric. Since light

34

moves along null trajectories $(ds = 0)$, the Lagrangian L must be

formally identical to zero. In the first order, Newtonian limit, this

implies that light must travel along straight lines, at speed 1, i.e.

$$(4.16) \qquad O = L = (1 - v^2)^{1/2}; \qquad v^2 = 1.$$

In the next, post-Newtonian order, we must have

$$(4.17) \qquad O = L = \left\{ 1 - 2U - v^2 + g_{\alpha\beta}[O(2)] v^{\alpha} v^{\beta} \right\}^{1/2}.$$

Thus to obtain post-Newtonian corrections to the propagation of light

rays, we need to know

$$g_{oo} \quad \text{to} \quad O(2),$$

$$g_{\alpha\beta} \quad \text{to} \quad O(2).$$

4·3. The Most General Post-Newtonian Metric

We now proceed to devise the most general post-Newtonian

metric which any reasonable metric theory of gravity might predict

for a system of "perfect" fluid. Recall: a perfect nonviscous fluid is

one which, in the Newtonian limit, obeys the usual Eulerian equations

of hydrodynamics:

$$\partial \rho / \partial t + \partial(\rho v^{\alpha}) / \partial x^{\alpha} = 0,$$

$$(4.18) \qquad \rho \frac{dv^{\alpha}}{dt} = \rho \frac{\partial U}{\partial x^{\alpha}} - \frac{\partial p}{\partial x^{\alpha}},$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v^{\alpha} \frac{\partial}{\partial x^{\alpha}},$$

where v^{α} is the velocity of an element of fluid, ρ is the rest-mass

density of matter in the element, p is the total pressure (matter plus radiation) on the element, d/dt is the time derivative "following the fluid", and U is the Newtonian gravitational potential, defined by

$$(4.19) \qquad U = \int \frac{\rho(\underset{\sim}{x}',t)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \, d\underset{\sim}{x}' \, .$$

The same fluid in a metric theory of gravity is described by an energy-momentum tensor of the form

$$(4.20) \qquad T^{ij} = (\rho + \rho\Pi + p)u^i u^j - pg^{ij} \, ,$$

where u^i is the fluid's four-velocity. We assume throughout that the matter composing the solar system can be idealized as perfect fluid; for the purposes of most solar-system experiments in the coming decades, this has been shown to be an adequate assumption (Will 1971a).

This general post-Newtonian metric should satisfy the following conditions:

(a) The deviations of the metric from flat space are all of Newtonian or post-Newtonian order; no post-post-Newtonian or higher-order deviations are included (see eq. [4.15]).

(b) The metric becomes Minkowskian (flat space) as the distance $|\underset{\sim}{x} - \underset{\sim}{x}'|$ between the field point and the matter becomes large.

(c) The metric is generated only by the rest mass, energy, pressure, and velocity; not by their gradients. This is a reasonable condition to put on physically acceptable metric theories, and is a condition which can be relaxed quite easily if there is ever any reason

to do so.

(d) The coordinates are chosen such that the metric coefficients are dimensionless.

(e) The form of the metric should be independent of our choice of the origin of the coordinate system, and of the orientation of the spatial axes. The metric should depend on time only implicitly, via the evolving distribution of the fluid.

These conditions limit the possible gravitational terms which may appear in the post-Newtonian metric:

i) $g_{\alpha\beta}$ to $O(2)$: From condition (e), $g_{\alpha\beta}$ must behave as a three-dimensional tensor under rotations, thus the only $O(2)$ terms which can appear are

$$(4.21) \qquad g_{\alpha\beta}[\,O(2)\,] : \quad U\delta_{\alpha\beta}, \quad U_{\alpha\beta},$$

where $U_{\alpha\beta}$ is given by

$$(4.22) \qquad U_{\alpha\beta}(\underset{\sim}{x},t) = \int \frac{\rho(\underset{\sim}{x}',t)(x-x')_\alpha(x-x')_\beta}{\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|^3} \, d\underset{\sim}{x}' .$$

ii) $g_{o\alpha}$ to $O(3)$: These metric components must behave as vectors under rotations, and thus must contain at most the terms

$$(4.23) \quad g_{o\alpha}[\,O(3)\,]: \quad V_\alpha, \quad W_\alpha, \quad w_\alpha U, \quad w_\beta U_{\alpha\beta} ,$$

where

$$(4.24) \qquad V_\alpha = \int \frac{\rho' v'_\alpha \, d\underset{\sim}{x}'}{\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|} , \qquad W_\alpha = \int \frac{\rho' \underset{\sim}{v}' \cdot (\underset{\sim}{x}-\underset{\sim}{x}')(x-x')_\alpha}{\left|\underset{\sim}{x} - \underset{\sim}{x}'\right|^3} \, d\underset{\sim}{x}' ;$$

and where the vector $\underset{\sim}{w}$, assumed to be $O(1)$, represents the velocity of our chosen coordinate system relative to the mean rest-frame of the Universe (recall: $\underset{\sim}{v}$ is the velocity of each element of matter relative to our coordinate system).

iii) g_{oo} to $O(4)$: This should be a scalar under rotations, and thus may contain at most the $O(4)$ terms:

(4.25)
$$g_{oo}[\,O(4)\,]:\quad U^2,\quad \Phi_1,\quad \Phi_2,\quad \Phi_3,\quad \Phi_4,\quad \mathsf{G},\quad \mathsf{B},$$
$$w^2 U,\quad w^\alpha w^\beta U_{\alpha\beta},\quad w^\alpha V_\alpha,\quad w^\alpha W_\alpha,$$

where

$$\Phi_1 = \int \frac{\rho' v'^2}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}',\qquad\qquad \Phi_2 = \int \frac{\rho' U'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}',$$

(4.26)
$$\Phi_3 = \int \frac{\rho' \Pi'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}',\qquad\qquad \Phi_4 = \int \frac{p'\, d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|},$$

$$\mathsf{G} = \int \frac{\rho'[\,\underset{\sim}{v}' \cdot (\underset{\sim}{x} - \underset{\sim}{x}')\,]^2}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3}\, d\underset{\sim}{x}',\qquad\qquad \mathsf{B} = \int \frac{\rho'(\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \cdot \frac{d\underset{\sim}{v}'}{dt}\, d\underset{\sim}{x}'.$$

At first glance, the reader might be disturbed by the presence of metric terms which depend on our coordinate system's velocity $\underset{\sim}{w}$ relative to the mean rest-frame of the Universe. However we can find no a priori reason for ignoring such terms. These terms do not violate the principles of special relativity, since they are purely gravitational terms, while special relativity is valid only when the effects of gravitation can be ignored; but they do suggest that the response

of matter to gravitation may be affected by motion relative to the Universe. Nonetheless there is good reason for including them: several metric theories of gravity say they should be included (see Sect. 5).

4·4. Coordinate-System Freedom and the Standard Post-Newtonian Gauge

We can restrict the form of the post-Newtonian metric somewhat by making use of the arbitrariness of coordinates embodied in the postulates of a metric theory. An infinitesimal coordinate or "gauge" transformation

$$(4.27) \qquad (x^i)^\dagger = x^i + \xi^i(x^k)$$

changes the metric to

$$(4.28) \qquad g_{ij}^\dagger = g_{ij} - \xi_{i;j} - \xi_{j;i} .$$

By choosing

$$(4.29) \qquad \xi_\alpha = \epsilon \chi_{,\alpha} ,$$

where χ is the so-called "superpotential" (Chandrasekhar and Lebovitz 1962b) given by

$$(4.30) \qquad \chi = -\int \rho' |\underset{\sim}{x} - \underset{\sim}{x}'| \, d\underset{\sim}{x}' ,$$

$$\chi_{,\alpha\beta} = - \delta_{\alpha\beta} U + U_{\alpha\beta} , \quad \chi_{,o\alpha} = V_\alpha - W_\alpha ,$$

we obtain, to post-Newtonian order,

(4.31) $\qquad g^{\dagger}_{\alpha\beta} = g_{\alpha\beta} + 2\epsilon U \delta_{\alpha\beta} - 2\epsilon U_{\alpha\beta} \cdot$

So by an appropriate choice of ϵ, we can eliminate one of the terms (eq. [4.21]) from $g_{\alpha\beta}$; we will eliminate $U_{\alpha\beta}$. Similarly by choosing

(4.32) $\qquad \xi_0 = \lambda_1 \chi_{,0} + \lambda_2 w^{\alpha} \chi_{,\alpha}$

we obtain, to post-Newtonian order

(4.33) $\qquad g^{\dagger}_{00} = g_{00} + 2\lambda_1 \Phi_1 - 2\lambda_1 \mathbb{G} - 2\lambda_1 \mathbb{B} - 2\lambda_2 w^{\alpha} V_{\alpha} + 2\lambda_2 w^{\alpha} W_{\alpha} \cdot$

By an appropriate choice of λ_1 and λ_2, we can always eliminate the terms $\mathbb{B}$ and $w^{\alpha} W_{\alpha}$ from g_{00}. Note that the transformation equations (4.29) and (4.32) do not introduce any new terms of $O(3)$ into $g_{0\alpha}$.

We will thus adopt a <u>standard Post-Newtonian gauge</u> -- that gauge in which the spatial part of the metric is diagonal and isotropic (i.e. no $U_{\alpha\beta}$), and in which g_{00} contains no terms $\mathbb{B}$ or $w^{\alpha} W_{\alpha}$. There is no physical significance in this gauge choice; it is purely a matter of convenience.

4·5. The Parametrized Post-Newtonian Metric

We now know the most general form for the post-Newtonian perfect-fluid metric in any metric theory of gravity. Now the only way any one theory's metric can differ from any other theory's is in the numerical coefficients which multiply each term in the metric. By replacing each numerical coefficient by an arbitrary parameter, we obtain a "super metric theory of gravity", whose special cases (particular numerical values of the parameters) are the post-Newtonian

metrics of particular theories of gravity. This "super metric" is called the Parametrized Post-Newtonian (PPN) metric, and the parameters are called PPN Parameters.

How many PPN parameters do we need? In the standard PPN gauge, the metric contains a total of fourteen terms, but only nine PPN parameters are needed. We will show in Subsect. 4.6 that the five w-dependent metric terms are not completely independent: their multipliers (parameters) are related to the multipliers of the nine other metric terms.

We will also show in Subsect. 4.6 and 4.7 that we can give a physical significance to our nine PPN parameters if we use certain linear combinations of parameters as multipliers of the individual metric terms. With these remarks, and with detailed proofs left to Subsect. 4.6 and 4.7, we give the PPN metric, in Table 4.I. Table 4.I also includes detailed definitions of the PPN coordinate system and the matter variables, formulas for the perfect-fluid stress-energy tensor to post-Newtonian order, and the equations of motion.

This use of parameters to describe the post-Newtonian limit of metric theories of gravity is called the Parametrized Post-Newtonian (PPN) Formalism. A primitive version of such a formalism was devised and studied by Eddington (1922), Robertson (1962) and Schiff (1967). This Eddington-Robertson-Shiff formalism treated the solar-system metric as that of a spherical non-rotating Sun, and idealized the planets as test bodies moving on geodesics of this metric. The metric in this version of the formalism reads

$$g_{oo} = 1 - 2M/r + 2\beta(M/r)^2 \ ,$$

(4.34) $$g_{o\alpha} = 0 \ ,$$

$$g_{\alpha\beta} = - (1 + 2\gamma M/r)\delta_{\alpha\beta} \ ,$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, M is the mass of the Sun, and β and γ are PPN parameters.

These two parameters may be given a physical interpretation in this formalism. The parameter β measures the amount of non-linearity $[(M/r)^2]$ which a given theory puts into the g_{oo} component of the metric, and the parameter γ measures the amount of curvature of space produced by a given gravitating body in a given metric theory (for a detailed discussion of the physical meaning of space-curvature, see Misner, Thorne, and Wheeler [1972]).

More general versions of the formalism were examined by Shiff (1960a), Baierlein (1967), Nordtvedt (1968b), Will (1971a), and Will and Nordtvedt (1972). In these lectures, we will use the Will-Nordtvedt version of the PPN formalism.

4·6. Lorentz Invariance and the PPN Metric

The PPN metric has a considerable amount of symmetry built into it.

(i) By using metric terms constructed from relative distances $|\underset{\sim}{x} - \underset{\sim}{x}'|$ between field points and the matter, we have made the metric independent of our choice of coordinate system origin. Put differently, we have guaranteed that the metric be invariant under a three-dimensional linear translation:

42

(4.35) $\qquad x^{\alpha\dagger} = x^\alpha + a^\alpha$.

(ii) We have made the metric independent of our choice of the origin of time by making all time dependence in g_{ij} implicit, via the evolving distributions of density, pressure, etc. Thus we have built into the metric, invariance under a linear time translation:

(4.36) $\qquad t^\dagger = t + b$.

(iii) By constructing the metric terms out of three-dimensional scalars, vectors and tensors where appropriate, we have made the metric independent of our choice of axes, i.e. invariant under a three-dimensional rotation:

(4.37) $\qquad x^{\alpha\dagger} = R^\alpha{}_\beta x^\beta$; $\quad R^\alpha{}_\beta \tilde{R}^\beta{}_\gamma = \delta^\alpha{}_\gamma$.

(v) A further symmetry has been built into the metric in Table 4.I. Although the PPN metric contains terms which depend on the (arbitrary) velocity of our chosen coordinate system relative to the Universe, the results of physical measurements clearly must not (this is the fundamental postulate of covariance -- the results of physical measurements cannot depend on arbitrary coordinate systems). For a system such as the Sun and planets, the only unique physically measurable velocities are the velocities of elements of matter relative to each other and to the solar-system center of mass, and the velocity $\underset{\sim}{w}_B$ of the solar-system center of mass (Barycenter) relative to the Universe's comoving frame (as measured, for example, by studying doppler shifts in the cosmic microwave radiation). Thus the PPN prediction for any physical effect

can depend only on these relative velocities and on w_B, never on the arbitrary coordinate-system velocity w. The only way to guarantee this is to demand that the PPN metric have the same functional form (when written in the standard gauge), independent of the velocity of the coordinate system relative to the Universe. Put differently, the PPN metric must be invariant under a Lorentz transformation (of low velocity, of course, to preserve the post-Newtonian approximation):

$$(4.38) \qquad (x^i)^\dagger = L^i_{\ j} x^j \ , \qquad \eta^{ij} = \tilde{L}^i_{\ k} L^j_{\ \ell} \eta^{k\ell} .$$

In this subsection we will prove that by writing the w-dependent metric terms in the manner shown in Table 4.I, we have built this constraint into the PPN metric.

Since the PPN metric (in the standard gauge) contains all possible metric functions consistent with our conditions (a) to (d) in Subsect. 4·3, invariance in the form of the metric under the above transformations is equivalent to invariance in the values of the PPN parameters. The set of linear transformations we have discussed above forms the ten-parameter Inhomogeneous Lorentz Group (four translations, three rotations, three Lorentz transformations -- these are the only linear tranformations which leave invariant the Minkowskii metric, the form taken by g_{ij} far from the matter). We can thus restate our symmetry conditions (i) to (iv) on the PPN metric in the following way:

The PPN parameters must behave as scalars under (post-Newtonian-preserving) transformations in the Inhomogeneous Lorentz Group.

It remains only to prove that the PPN metric in Table 4.I has indeed had Lorentz invariance built in.

Consider the PPN metric (Table 4.I) written in a coordinate system which moves at velocity $\underset{\sim}{w}$ relative to the mean rest-frame of the Universe. We make a Lorentz transformation to a new frame which moves at velocity $\underset{\sim}{u}$ relative to the old frame, assuming $|\underset{\sim}{u}|$ is small, i.e. of $O(1)$. This transformation from old coordinates $(t, \underset{\sim}{x})$ to new coordinates $(\tau, \underset{\sim}{\xi})$ can be expanded in powers of $\underset{\sim}{u}$ to the required order: this approximate form of the Lorentz Transformation has been called a Post-Galilean Transformation (Chandrasekhar and Contopoulos 1967), and has the form

$$(4.39)\qquad \underset{\sim}{x} = \underset{\sim}{\xi} + (1 + \tfrac{1}{2} u^2)\underset{\sim}{u}\tau + \tfrac{1}{2}(\underset{\sim}{\xi} \cdot \underset{\sim}{u})\underset{\sim}{u} + [O(4)] \times \underset{\sim}{\xi} \ ,$$

$$t = \tau(1 + \tfrac{1}{2} u^2 + \tfrac{3}{8} u^4) + (1 + \tfrac{1}{2} u^2)\underset{\sim}{\xi} \cdot \underset{\sim}{u} + [O(5)] \times \tau \ ,$$

where $\underset{\sim}{u}\tau$ is assumed to be $O(0)$.

We now apply this transformation to the PPN metric and show that in the new coordinates $(\tau, \underset{\sim}{\xi})$ it has exactly the same form as it had in the old coordinates $(t, \underset{\sim}{x})$. We use the standard transformation law $(x^0 = t, \ \xi^0 = \tau)$:

$$(4.40)\qquad g_{ij}(\underset{\sim}{\xi}, \tau) = \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^\ell}{\partial \xi^j} g_{k\ell}(\underset{\sim}{x}, t) \ .$$

We must also express the functions (fields) which appear in $g_{k\ell}(\underset{\sim}{x}, t)$ in terms of the new coordinates. Since ρ, Π, and p are all measured in comoving local inertial frames, they are unchanged by the transformation: for any given element of fluid,

$$\rho(\underset{\sim}{x},t) = \rho(\underset{\sim}{\xi},\tau) \ ,$$

$$(4.41) \qquad \Pi(\underset{\sim}{x},t) = \Pi(\underset{\sim}{\xi},\tau) \ ,$$

$$p(\underset{\sim}{x},t) = p(\underset{\sim}{\xi},\tau) \ .$$

If $\underset{\sim}{v}(\underset{\sim}{x},t)$ and $\underset{\sim}{\nu}(\underset{\sim}{\xi},\tau)$ are the matter velocities in the two coordinate systems, they are related by

$$(4.42) \qquad \underset{\sim}{v} = \underset{\sim}{\nu} + \underset{\sim}{u} + O(3) \ .$$

If $\underset{\sim}{\omega}$ is the velocity of the new frame relative to the Universe, it is related to $\underset{\sim}{w}$ by

$$(4.43) \qquad \underset{\sim}{w} = \underset{\sim}{\omega} - \underset{\sim}{u} + O(3) \ .$$

The elements of volume $d\underset{\sim}{x}'$ and $d\underset{\sim}{\xi}'$ in the two frames are related by the usual Jacobian transformation law, which gives

$$(4.44) \qquad d\underset{\sim}{x}' = (1 - \underset{\sim}{\nu}' \cdot \underset{\sim}{u} - \frac{1}{2} u^2 + O(4)) \ d\underset{\sim}{\xi}' \ .$$

We make use of a formula given by Chandrasekhar and Contopoulos (1967), namely,

$$(4.45) \qquad \frac{1}{|\underset{\sim}{x} - \underset{\sim}{x}'|} = \frac{1}{|\underset{\sim}{\xi} - \underset{\sim}{\xi}'|} \left\{ 1 + \frac{1}{2}(\underset{\sim}{u} \cdot \underset{\sim}{n}')^2 + (\underset{\sim}{u} \cdot \underset{\sim}{n}')(\underset{\sim}{\nu}' \cdot \underset{\sim}{n}') + O(4) \right\} \ ,$$

where

$$(4.46) \qquad \underset{\sim}{n}' = (\underset{\sim}{\xi} - \underset{\sim}{\xi}')/|\underset{\sim}{\xi} - \underset{\sim}{\xi}'| \ .$$

We then find, using equations (4.41), (4.42), (4.44), and (4.45), along with the definitions of the metric functions, equations (4.19), (4.22)

(4.24) and (4.26), that

$$(4.47) \quad U(\underset{\sim}{x},t) = (1 - \tfrac{1}{2}u^2)U(\underset{\sim}{\xi},\tau) - u^\alpha V_\alpha(\underset{\sim}{\xi},\tau)$$

$$+ u^\alpha W_\alpha(\underset{\sim}{\xi},\tau) + \tfrac{1}{2} u^\alpha u^\beta U_{\alpha\beta}(\underset{\sim}{\xi},\tau) + O(6),$$

$$(4.48) \quad U_{\alpha\beta}(\underset{\sim}{x},t) = U_{\alpha\beta}(\underset{\sim}{\xi},\tau) + O(4),$$

$$(4.49) \quad V_\alpha(\underset{\sim}{x},t) = V_\alpha(\underset{\sim}{\xi},\tau) + u^\alpha U(\underset{\sim}{\xi},\tau) + O(5),$$

$$(4.50) \quad W_\alpha(\underset{\sim}{x},t) = W_\alpha(\underset{\sim}{\xi},\tau) + u^\beta U_{\alpha\beta}(\underset{\sim}{\xi},\tau) + O(5),$$

$$(4.51\text{a}) \quad \Phi_1(\underset{\sim}{x},t) = \Phi_1(\underset{\sim}{\xi},\tau) + 2u^\alpha V_\alpha(\underset{\sim}{\xi},\tau) + u^2 U(\underset{\sim}{\xi},\tau) + O(6),$$

$$(4.51\text{b}) \quad \Phi_{2(3,4)}(\underset{\sim}{x},t) = \Phi_{2(3,4)}(\underset{\sim}{\xi},\tau) + O(6),$$

$$(4.52) \quad G(\underset{\sim}{x},t) = G(\underset{\sim}{\xi},\tau) + 2u^\beta W_\beta(\underset{\sim}{\xi},\tau) + u^\alpha u^\beta U_{\alpha\beta}(\underset{\sim}{\xi},\tau) + O(6).$$

Applying transformation equations (4.39) to the PPN metric (Table 4.I) and making use of equations (4.40), (4.43), and (4.47) to (4.52), we obtain, for the metric in the $(\tau,\underset{\sim}{\xi})$ system, to post-Newtonian order

$$g_{oo}(\underset{\sim}{\xi},\tau) = 1 - 2U(\underset{\sim}{\xi},\tau) + 2\beta U(\underset{\sim}{\xi},\tau)^2$$

$$- (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1(\underset{\sim}{\xi},\tau) + \zeta_1 G(\underset{\sim}{\xi},\tau)$$

$$- 2[(3\gamma - 2\beta + 1 + \zeta_2)\Phi_2(\underset{\sim}{\xi},\tau) + (1 + \zeta_3)\Phi_3(\underset{\sim}{\xi},\tau)$$

$$(4.53) \qquad\qquad + 3(\gamma + \zeta_4)\Phi_4(\underset{\sim}{\xi},\tau)]$$

$$+ (\alpha_1 - \alpha_2 - \alpha_3)\omega^2 U(\underset{\sim}{\xi},\tau) + \alpha_2 \omega^\alpha \omega^\beta U_{\alpha\beta}(\underset{\sim}{\xi},\tau)$$

$$- (2\alpha_3 - \alpha_1)\omega^\alpha V_\alpha(\underset{\sim}{\xi},\tau) + (1 - \zeta_1 - \alpha_2)u^\alpha \chi(\underset{\sim}{\xi},\tau)_{,o\alpha},$$

$$g_{o\alpha} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_\alpha(\underset{\sim}{\xi},\tau) + \frac{1}{2}(1 + \alpha_2 - \zeta_1)W_\alpha(\underset{\sim}{\xi},\tau)$$

(4.54)
$$+ \frac{1}{2}(\alpha_1 - 2\alpha_2)\omega^\alpha U(\underset{\sim}{\xi},\tau) + \alpha_2\omega^\beta U_{\alpha\beta}(\underset{\sim}{\xi},\tau)$$

$$+ \frac{1}{2}(1 - \zeta_1 - \alpha_2)u^\beta \chi(\underset{\sim}{\xi},\tau),_{\alpha\beta} \ ,$$

(4.55)
$$g_{\alpha\beta} = -\left[1 + 2\gamma U(\xi,\tau)\right]\delta_{\alpha\beta} .$$

Note that because of the presence of gravitating matter, the Lorentz transformation has introduced additional terms into the metric:

$$(1 - \zeta_1 - \alpha_2)u^\alpha \chi,_{o\alpha} \quad \text{in} \quad g_{oo} ,$$

$$\frac{1}{2}(1 - \zeta_1 - \alpha_2)u^\beta \chi,_{\alpha\beta} \quad \text{in} \quad g_{0\alpha} .$$

But these terms can always be removed by an infinitesimal gauge transformation

(4.56)
$$t^\dagger = t + \frac{1}{2}(1 - \zeta_1 - \alpha_2)u^\beta \chi,_\beta .$$

In other words, a pure Lorentz transformation of the PPN metric takes it out of the standard PPN gauge. Thus the Lorentz-transformed PPN metric eqs. (4.53), (4.54) and (4.55), written back in the standard PPN gauge, has exactly the same functional form as the original PPN metric. Q.E.D.

We can now see the physical significance of the PPN parameters α_1, α_2 and α_3, which appear as multipliers of the $\underset{\sim}{w}$-dependent terms in the metric. These parameters measure the extent and manner in which motion relative to the mean rest-frame of the Universe affects

the post-Newtonian metric, and produces observable effects. The

parameters α_1, α_2 and α_3 have been named "preferred frame"

parameters (Will and Nordtvedt 1972) since they measure the sizes

of post-Newtonian effects produced by motion relative to the "preferred"

rest-frame of the Universe. If all three are zero, no such effects are

present. Further discussion of preferred-frame effects appears in

Sect. 6 and 7.

4·7. Conservation Laws in the PPN Formalism

Conservation laws in Newtonian gravitation theory are familiar:

for isolated gravitating systems, mass is conserved, energy is con-

served, linear and angular momenta are conserved, and the center of

mass of the system moves uniformly. Not so in every metric theory

of gravity. Some theories violate some of these conservation laws

at the post-Newtonian level, and it is the purpose of this section to

explore such violations using the PPN formalism.

We begin by making a number of assumptions about the properties

of matter which should be valid in any theory of gravity.

Assumption 1: The total number of baryons δA in any sample of

matter is conserved. Conservation of baryon number is one of the

most fundamental laws of physics, and should certainly be valid in

the presence of gravity. This law can be expressed as a continuity

equation for the baryon number density n: in a local inertial frame

momentarily comoving with the matter, the equation

$$(4.57) \qquad 0 = d(\delta A)/dt = d(n\delta V)/dt \ ,$$

is equivalent to

(4.58) $\partial n / \partial t + (\partial / \partial x^{\alpha})(n v^{\alpha}) = 0$,

where v^{α} is the matter velocity in the comoving frame ($v^{\alpha} = 0$ but $\nabla \cdot \underset{\sim}{v} \neq 0$). The Lorentz-invariant version of this continuity equation, valid in any local inertial frame is

(4.59) $0 = \dfrac{\partial}{\partial t}(n u^{o}) + \dfrac{\partial}{\partial x^{\alpha}}(n u^{\alpha}) = (n u^{i})_{,i}$,

where u^{i} is the baryon four-velocity, given by

(4.60) $u^{i} = dx^{i}/ds$.

Equation (4.59) can then be generalized to curved spacetime using the standard "comma-goes-to-semicolon" rule (Sect. 3)

(4.61) $0 = (n u^{i})_{;i}$.

Equation (4.61) is the <u>law of Baryon Conservation</u> in covariant form.

<u>Assumption 2</u>: <u>Matter is composed of a chemically homogeneous, electrically neutral distribution of atoms composed of neutrons, protons, and electrons. The chemical composition of matter is static.</u>

For the purpose of solar-system experimental tests (although not for the theory of solar and planetary structure) this assumption is a reasonable model for solar-system matter.

The rest-mass density of matter ρ is a physically measured quantity, obtained as follows: in a local inertial frame which comoves momentarily with an element of matter, add together the rest masses

of all the atoms in the element, and divide by the volume δV of the element. Then

$$\rho \equiv \sum_i^{\text{element}} \mu_i / \delta V$$

(4.62)

where μ_i is the rest-mass of the i'th atom.

Lemma: For matter which conforms with Assumption 2,

(4.63)
$$\rho = \mu n \, ,$$

where n is the number density of baryons, and μ is a constant.

Proof: The mass of each atom consists of baryon rest-mass, and binding energy, and may be written

(4.64)
$$\mu_i = \delta A_i \mu_o + \Delta \mu_i \, ,$$

where δA_i is the number of baryons in the atom, μ_o is an atomic mass unit, and $\Delta \mu_i$ is the "mass-excess", related to the nuclear and atomic binding energy. Then from equation (4.62)

$$\rho = (\delta A \mu_o + \sum_i^{\text{element}} \Delta \mu_i) / \delta V$$

(4.65)
$$= (\delta A / \delta V)(\mu_o + \sum_i^{\text{element}} \Delta \mu_i / \delta A) \, .$$

Because of our assumption of chemical homogeneity and time-independence, the rest-mass per baryon

(4.66)
$$\mu_o + \sum_i^{\text{element}} \Delta \mu_i / \delta A \equiv \mu \, ,$$

is a constant. Thus

$$(4.67) \qquad \rho = \mu n \ . \qquad Q.E.D.$$

Lemma: If baryons are conserved, and matter conforms with Assumption 2, then

$$(4.68) \qquad (\rho u^i)_{;i} = 0 \ .$$

Proof: It is sufficient to work in a local comoving inertial frame. From equation (4.57) and the constancy of μ, we get

$$(4.69) \qquad 0 = d(\mu n \delta V)/dt = d(\rho \delta V)/dt \ ,$$

which is equivalent to

$$(4.70) \qquad 0 = \partial \rho / \partial t + \nabla \cdot \rho \underset{\sim}{v} \ .$$

The generalization of this equation to arbitrary inertial frames and to curved spacetime proceeds as before; thus

$$(4.71) \qquad (\rho u^i)_{;i} = 0 \ . \qquad Q.E.D.$$

Equation (4.71) is the <u>Law of Conservation of Rest Mass</u>, in covariant form. For matter which obeys Assumption 2 it is valid in any metric theory of gravity, both at the post-Newtonian level and at the exact, strong-field level.

By combining this law with the equations of motion for stressed matter,

$$(4.72) \qquad T^{ij}_{\ ;j} = 0,$$

we can obtain a further important law, which too is exact and theory-

independent: the <u>Law of Local Energy Conservation</u> or the <u>Law of</u>

<u>Isentropic Flow</u>.

We follow Chandrasekhar (1969) and evaluate the equation

(4.73) $\qquad u_i T^{ij}_{\;;j} = 0$,

using the perfect-fluid stress-energy tensor, equation (4.20). We

work in a local inertial frame, momentarily comoving with the

element δV of fluid. Then from equation (4.73),

(4.74) $\qquad (\partial/\partial t)(\rho + \rho\Pi) + \nabla \cdot (\rho + \rho\Pi + p)\underset{\sim}{v} = 0.$

This can be rewritten

(4.75) $\qquad (d/dt)(\rho + \rho\Pi) + (\rho + \rho\Pi)\nabla \cdot \underset{\sim}{v} + p\nabla \cdot \underset{\sim}{v} = 0$.

But

(4.76) $\qquad \nabla \cdot \underset{\sim}{v} = (1/\delta V)d(\delta V)\,dt$.

Thus equation (4.75) becomes

(4.77) $\qquad (d/dt)\,[(\rho + \rho\Pi)\delta V] + pd(\delta V)/dt = 0$.

So, in a local comoving inertial frame, the change in the <u>total</u> energy

(rest-mass plus internal) of an element of fluid is balanced by the

work done $[\,pd(\delta V)]$; this simply expresses <u>Local Conservation of Energy</u>,

or <u>Isentropic Flow</u>, since from the First Law of Thermodynamics, and

from equation (4.77)

(4.78) $\qquad T\,d\,(\text{Entropy}) = d\,(\text{Energy}) + pdV = 0$.

Because of the Conservation of Rest Mass, $\rho \delta V$ is constant, and equation (4.77) can be written in the form

$$(4.79) \qquad \rho d\Pi/dt - (p/\rho)\, d\rho/dt = 0 \ .$$

In frame-invariant curved-spacetime language, equation (4.79) has the form (Chandrasekhar 1969):

$$(4.80) \qquad u^i[\, \Pi_{,i} + p(1/\rho)_{,i}\,] = 0 \ .$$

These exact, covariant, theory-independent local conservation laws are summarized in Table 4.II.

We can obtain a useful form of the law of Conservation of Rest Mass by noticing that

$$(4.81) \qquad (\rho u^i)_{;i} = (1/\sqrt{-g})(\rho\sqrt{-g}\ u^i)_{,i} \ ,$$

where

$$(4.82) \qquad g = \det \|g_{ij}\| \ .$$

In a coordinate system $(t, \underset{\sim}{x})$, equation (4.71) can thus be written

$$(4.83) \qquad 0 = (\rho\sqrt{-g}\ u^i)_{,i} = (\rho\sqrt{-g}\ u^o)_{,o} + (\rho\sqrt{-g}\ u^o v^\alpha)_{,\alpha} \ ,$$

since

$$(4.84) \qquad u^\alpha = u^o v^\alpha \ .$$

By defining the "conserved density" ρ^*:

$$(4.85) \qquad \rho^* \equiv \rho\sqrt{-g}\ u^o \ ,$$

we can cast equation (4.83) in the form of an "Eulerian" continuity
equation, valid in our $(t, \underset{\sim}{x})$ coordinate system:

(4.86) $\partial \rho^* / \partial t + \underset{\sim}{\nabla} \cdot \rho^* \underset{\sim}{v} = 0.$

This "conserved" density is useful because for any function f
defined in a volume V whose boundary is outside the matter

(4.87) $(d/dt) \int_V \rho^* f \, d\underset{\sim}{x} = \int_V \rho^* (df/dt) \, d\underset{\sim}{x} .$

Notice that equation (4.87) implies

(4.88) $dM/dt = 0, \quad M = \int_V \rho^* \, d\underset{\sim}{x} ,$

where M is the total rest mass of the particles in the volume V;
from equation (4.85), we get,

$$M = \int [\rho u^0 \sqrt{-g}] \, d\underset{\sim}{x}$$

(4.89) $= \int \rho d \text{ (proper volume)}$

$= \text{total rest mass of particles.}$

The conservation laws summarized in Table 4.II are purely
local conservation laws; they depend only on properties of matter as
measured in local, comoving inertial frames, where relativistic and
gravitational effects are negligible (hence they are theory-independent).
Equation (4.89) represents our first "global" or "integral" conservation
law; however, it is really nothing more than conservation of baryons
coupled with our specific model for matter.

However, when we attempt to devise more general integral conservation laws, such as for total energy (as opposed to just rest-mass), total momentum, or total angular momentum we run into difficulties.

It is well known [see, for example, Landau and Lifschitz (1962)] that integral conservation laws cannot be obtained directly from the equation of motion for stressed matter,

$$(4.90) \qquad T^{ij}{}_{;j} = 0 ,$$

because of the presence of the Christoffel symbols in the covariant derivative. Rather, one searches for a quantity Θ^{ij} which reduces to T^{ij} in flat spacetime, and whose ordinary divergence is zero, i.e.,

$$(4.91) \qquad \Theta^{ij}{}_{,j} = 0 .$$

Then, providing Θ^{ij} is symmetric, one finds that the quantities

$$(4.92) \qquad P^i = \int_\Sigma \Theta^{ij} \, d\Sigma_j \ , \quad J^{ij} = 2\int_\Sigma x^{[i}\Theta^{j]k} \, d\Sigma_k ,$$

are conserved, i.e., the integrals in equation (4.92) vanish when taken over a closed 3-dimensional hypersurface Σ. If one chooses a coordinate system $(t,\underset{\sim}{x})$ in which Σ is a constant-time hypersurface and extends infinitely far in all directions, then P^i and J^{ij} are independent of time, and are given by

$$(4.93) \qquad P^i = \int \Theta^{io} \, d\underset{\sim}{x} \ , \quad J^{ij} = 2\int x^{[i}\Theta^{j]o} \, d\underset{\sim}{x} \ ,$$

where $d\underset{\sim}{x}$ is a volume element of ordinary three-dimensional space.

An appropriate choice of $\circledast^{ij}$ allows one to interpret the components of P^i and J^{ij} in the usual way: as measured in the asymptotically flat spacetime far from the matter, P^0 is the total energy, P^α is the total momentum, $J^{\alpha\beta}$ is the total angular momentum and $J^{0\alpha}$ determines the motion of the center of mass of the matter.

The quantity $\circledast^{ij}$, normally called the <u>stress-energy complex</u>, has been found for the exact versions of general relativity (Landau and Lifshitz 1962), Brans-Dicke theory (Nutku 1969b, Dykla 1972) and others (Ni 1972c). It has also been explicitly calculated in the post-Newtonian and post-post-Newtonian approximations of general relativity (Chandrasekhar 1969; Chandrasekhar and Nutku 1969). (A wide variety of non-symmetric stress-energy complexes have been found for general relativity, but only the symmetric version guarantees conservation of angular momentum.

Here we will focus on the post-Newtonian limit, and will use the PPN formalism to attempt to construct a $\circledast^{ij}$.

The most general possible form for $\circledast^{ij}$ which reduces to T^{ij} in flat spacetime (negligible gravitational fields), and which is accurate to post-Newtonian order, is

$$(4.94) \qquad \circledast^{ij} = (1 - aU)(T^{ij} + t^{ij}),$$

where a is a constant (to be determined), and t^{ij} is a quantity (which may be interpreted as gravitational stress-energy) which vanishes in flat space-time, and which is a function of the fields U, $U_{\alpha\beta}$, Φ_1, Φ_2, Φ_3, Φ_4, $\mathscr{A}$, V_α, and W_α, their derivatives, and $\underset{\sim}{w}$ (and may also contain the matter variables ρ, Π, p, and $\underset{\sim}{v}$). We reject terms in $\circledast^{ij}$ of the

form

$$v^2 T^{ij} \ , \quad \Pi T^{ij} \ , \quad (p/\rho) T^{ij} \ , \quad w^2 T^{ij} \ ,$$

since such terms do not vanish for arbitrary distributions of stressed matter in regions of negligible gravitational field.

By combining equations (4.90), (4.91) and (4.94) we find that t^{ij} must satisfy (to post-Newtonian order)

(4.95) $\qquad t^{ij}{}_{,j} - a U_{,j} t^{ij} = \Gamma^i{}_{jk} T^{jk} + \Gamma^j{}_{kj} T^{ik} + a U_{,j} T^{ij} \ .$

In order to solve equation (4.95) for t^{ij} we will use the following equations, which are equivalent to the definitions for the metric functions, equations (4.19), (4.22), (4.24), (4.26) and (4.30), to express matter variables in terms of field quantities:

$$\nabla^2 U = -4\pi\rho \ , \qquad \nabla^2 V_\alpha = -4\pi\rho v_\alpha \ ,$$

$$\nabla^2 \Phi_1 = -4\pi\rho v^2 \ , \qquad \nabla^2 \Phi_2 = -4\pi\rho U \ ,$$

(4.96)

$$\nabla^2 \Phi_3 = -4\pi\rho \Pi \ , \qquad \nabla^2 \Phi_4 = -4\pi p \ ,$$

$$\nabla^2 \chi = -2U \ , \qquad V_{\alpha,\alpha} = -U_{,o} \ ;$$

and we will use the following identity, which is valid for any function f:

(4.97) $\qquad 4\pi\rho f_{,\alpha} = -2(\partial/\partial x_\beta)(U_{,(\alpha} f_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta} U_{,\gamma} f_{,\gamma}) + U_{,\alpha} \nabla^2 f \ .$

We substitute into equation (4.95) the formulas for T^{ij} and for the PPN Christoffel symbols calculated from the metric (Table 4.I), and

use equations (4.96) and (4.97) to obtain (to post-Newtonian order) for $i = 0$,

$$4\pi t^{oj}{}_{,j} = (\partial/\partial t)[\tfrac{1}{2}(6\gamma + 2a - 5)|\nabla U|^2]$$

(4.98)
$$- (\partial \ \partial x^\beta)[(3\gamma + a - 2)U_{,\beta}U_{,o}$$

$$+ (3\gamma + a - 3)U_{,\gamma}(V_{\gamma,\beta} - V_{\beta,\gamma})] \ .$$

Equation (4.98) can be integrated directly (making use of the condition that t^{ij} vanish in flat spacetime) to yield

(4.99)
$$t^{oo} = (8\pi)^{-1}(6\gamma + 2a - 5)|\nabla U|^2 \ ,$$

(4.100)
$$t^{o\alpha} = - (4\pi)^{-1}[(3\gamma + a - 2)U_{,\alpha}U_{,o} + (3\gamma + a - 3)U_{,\gamma}(V_{\gamma,\alpha} - V_{\alpha,\gamma})] \ .$$

An expression for the conserved total energy can be obtained using equations (4.93), (4.94), and (4.99). The result is (after an integration by parts):

(4.101)
$$P^o = \int \rho^*(1 + \tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi) \, d\underset{\sim}{x} \ ,$$

where we have used the PPN version of the "conserved" density (cf. eq. [4.85])

(4.102)
$$\rho^2 = \rho(1 + \tfrac{1}{2}v^2 + 3\gamma U + O(4)) \ .$$

The first term in equation (4.101) is the total conserved rest-mass of particles in the fluid (eq. [4.89]). The other terms in equation (4.102) are the total kinetic, gravitational, and internal energies in the fluid,

whose sum is conserved according to Newtonian theory (which can be used in any post-Newtonian terms). Thus P^0 is simply the total mass-energy of the fluid. So far we have found nothing new. Equation (4.102) for P^0 can be found directly using the conservation of rest-mass (eq. [4.88]) and Newtonian theory.

For $i = \alpha$, we must first compute $t^{\alpha\beta}$ to Newtonian order. Equation (4.95) yields

$$(4.103) \qquad 4\pi t^{\alpha\beta}{}_{,\beta} = (\partial/\partial x^{\beta})(U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}) \,,$$

from which we obtain the standard Newtonian result (Chandrasekhar 1969)

$$(4.104) \qquad t^{\alpha\beta}{}_{\text{Newtonian}} = (4\pi)^{-1}(U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}) \,.$$

This Newtonian approximation for $t^{\alpha\beta}$ can now be used to simplify all post-Newtonian terms in equation (4.95). We obtain after a lengthy calculation, the post-Newtonian equation for $t^{\alpha j}$:

$$
\begin{aligned}
4\pi t^{\alpha j}{}_{,j} = {}& (\partial/\partial t)\Big[\tfrac{1}{2}(4\gamma + 2 + \alpha_1 - 2\alpha_2 + 2\zeta_1)U_{,\alpha}U_{,o} \\
& + \tfrac{1}{2}(4\gamma + 4 + \alpha_1)U_{,\gamma}(V_{\gamma,\alpha} - V_{\alpha,\gamma}) + (5\gamma + a - 1)U_{,\gamma}V_{\alpha,\gamma} \\
& - (5\gamma + a - 1)(\partial/\partial x^{\gamma})(UV_{\alpha,\gamma}) + \alpha_2 w^{\beta}U_{,\beta}U_{,\alpha} - \tfrac{1}{2}\alpha_1 w^{\alpha}U_{,\beta}U_{,\beta}\Big] \\
& + (\partial/\partial x^{\beta})\Big\{[1 - \tfrac{1}{2}(5\gamma + 2\zeta_2 - a - 1)U](U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}) \\
& + [U_{,(\alpha}(4\Phi - \zeta_1 a)_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}(4\Phi - \zeta_1 a)_{,\gamma}] \\
& + (4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)[U_{,(\alpha}V_{\beta),o} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}V_{\gamma,o}] \\
& + (1 + \alpha_2 - \zeta_1)[U_{,(\alpha}W_{\beta),o} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}W_{\gamma,o}]
\end{aligned}
$$

60

$$- 2(4\gamma + 4 + \alpha_1)[V_{[\alpha,\gamma]}V_{[\beta,\gamma]} - \tfrac{1}{4}\delta_{\alpha\beta}V_{[\gamma,\delta]}V_{[\gamma,\delta]}]$$

$$- \tfrac{1}{4}(4\gamma + 2 + \alpha_1 - 2\alpha_2 + 2\zeta_1)\delta_{\alpha\beta}(U_{,o})^2$$

(4.105)

$$+ \tfrac{1}{2}(\alpha_3 - \alpha_1)w^2(U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma})$$

$$- \alpha_2 w^\gamma w^\delta (U_{,(\alpha}U_{\beta)\gamma,\delta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\eta}U_{\eta\gamma,\delta})$$

$$- (\alpha_1 - 2\alpha_3)w^\gamma(V_{\gamma,(\alpha}U_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta}V_{\gamma,\delta}U_{,\delta})$$

$$+ 2\alpha_2 w^\gamma(\chi_{,o\gamma(\alpha}U_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta}\chi_{,o\gamma\delta}U_{,\delta})$$

$$+ \tfrac{1}{2}(\alpha_1 + 2\alpha_2)w_{(\alpha}U_{,\beta)}U_{,o} - \alpha_2\delta_{\alpha\beta}w^\gamma U_{,\gamma}U_{,o}$$

$$+ \tfrac{1}{2}\alpha_1 U_{,\gamma}[V_{\gamma,(\alpha}w_{\beta)} - w_{(\alpha}V_{\beta),\gamma}] + \alpha_2 w^\gamma w_{[\alpha}U_{,\beta]}U_{,\gamma}$$

$$+ \tfrac{1}{2}(\alpha_1 - 2\alpha_2)U_{,o}w_{[\alpha}U_{,\beta]} - \tfrac{1}{2}\alpha_1 U_{,\gamma}[V_{\gamma,[\alpha}w_{\beta]} - w_{[\beta}V_{\alpha],\gamma}]$$

$$+ (5\gamma + a + 1)U(\rho v^\alpha v^\beta + \delta^{\alpha\beta}p)\Big\}$$

$$+ 4\pi Q^\alpha ,$$

where

(4.106)
$$4\Phi = (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1$$

$$+ 2[(3\gamma - 2\beta + 1 + \zeta_2)\Phi_2 + (1 + \zeta_3)\Phi_3 + 3(\gamma + \zeta_4)\Phi_4] ,$$

and where

(4.107)
$$Q^\alpha = (\partial U/\partial x^\alpha)[\tfrac{1}{2}(\alpha_3 + \zeta_1)\rho v^2 + \alpha_3 \rho\underset{\sim}{v}\cdot\underset{\sim}{w} + (8\pi)^{-1}\zeta_2|\underset{\sim}{\nabla}U|^2$$

$$+ \zeta_3\rho\Pi + 3\zeta_4 p + (8\pi)^{-1}\zeta_1\nabla^2 U] .$$

The term Q^α can <u>not</u> in general be written as a combination of

gradients and time derivatives of fluid quantities and gravitational fields -- or so we believe. (We have been unable to develop a completely rigorous proof; but strong arguments that this is so are given in Will [1971c].) Therefore, in order for $t^{\alpha j}$ (and hence $\circledast^{\alpha j}$) to exist and to have a form which involves only matter and gravitational field variables and their derivatives, each PPN parameter combination Q^α (eq. [4.107]) must vanish separately, i.e., the parameters must satisfy

$$(4.108) \qquad \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0 \ .$$

These conditions can also be obtained using Chandrasekhar's (1965) technique which consists of integrating the hydrodynamic equations of motion over all space and calculating a conserved momentum P^α. Using the PPN formalism, the corresponding result is

$$
\begin{aligned}
(4.109) \qquad (d/dt) \int \rho^* &\Big\{ v^\alpha [\, 1 + \tfrac{1}{2} v^2 + (2\gamma + 1)U + \Pi + p/\rho] \\
&- \tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_\alpha - \tfrac{1}{2}(1 + \alpha_2 - \zeta_1)W_\alpha \\
&- \tfrac{1}{2}(\alpha_1 - \alpha_2)w^\alpha U - \tfrac{1}{2}\alpha_2 w^\beta U_{\alpha\beta} \Big\} d\underset{\sim}{x} \\
&+ \int Q^\alpha \, d\underset{\sim}{x} = 0
\end{aligned}
$$

The second term in equation (4.109) can be written as a total time derivative of an integral over all space, only if Q^α can be written as a combination of time derivatives, and spatial divergences (which lead to surface integrals at infinity that vanish). But according to the reasoning given in Will (1971c), this can only be true if the five

parameter constraints of equation (4.108) are satisfied. Then $Q^{\alpha} \equiv 0$, and we have a conserved momentum.

We now see the physical significance of the parameters ζ_1, ζ_2, ζ_3, ζ_4 and α_3: <u>They measure the extent and manner in which a given theory of gravity predicts violations of conservation of total momentum.</u> If all five are zero for any given theory, then momentum is conserved; if some are non-zero, then total momentum may not be conserved. Notice that the parameter α_3 plays a dual role in the PPN formalism, both as a conservation-law parameter, and as a preferred-frame parameter.

In order to guarantee conservation of angular momentum, $t^{\alpha\beta}$ must be symmetric. The $t^{\alpha\beta}$ part of equation (4.105) contains some antisymmetric terms; in order that they vanish for arbitrary systems, we must have

(4.110) $\qquad \alpha_1 = \alpha_2 = 0$.

It is the symmetry of $t^{o\alpha}$, i.e. uniform motion of the center of mass which then fixes the value of a. Comparing $t^{o\alpha}$ of equation (4.98) with $t^{\alpha o}$ of equation (4.105), we find using equations (4.108) and (4.110):

(4.111) $\qquad a = 1 - 5\gamma$.

We apply the name <u>Fully Conservative Theory</u> to any theory of gravity which possesses a full complement of post-Newtonian conservation laws: momentum, angular momentum and center-of-mass motion, i.e. whose PPN parameters satisfy

63

$$(4.112) \qquad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \ .$$

A Fully Conservative Theory cannot be a preferred-frame theory ($\alpha_1 = \alpha_2 = \alpha_3 = 0$). For such theories, t^{ij} and $\circledR^{ij}$ have the form:

$$(4.113) \qquad t^{oo} = - (8\pi)^{-1}(4\gamma + 3)|\underset{\sim}{\nabla} U|^2 \ ,$$

$$(4.114) \qquad t^{o\alpha} = t^{\alpha o} = (4\pi)^{-1}[\, (2\gamma + 1)U_{,\alpha}U_{,o} + (2\gamma + 2)U_{,\beta}(V_{\beta,\alpha} - V_{\alpha,\beta})\,] \ ,$$

$$
\begin{aligned}
(4.115) \qquad t^{\alpha\beta} = (4\pi)^{-1}\ \Big[\ & [1 - (5\gamma - 1)U](U_{,\alpha}U_{,\beta} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}U_{,\gamma}) \\
& + 4[\, U_{,(\alpha}\Phi_{,\beta)} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}\Phi_{,\gamma}] \\
& + (4\gamma + 3)[\, U_{,(\alpha}V_{\beta),o} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}V_{\gamma,o}] \\
& + [\, U_{,\alpha}W_{\beta),o} - \tfrac{1}{2}\delta_{\alpha\beta}U_{,\gamma}W_{\gamma,o}] \\
& - 8(\gamma + 1)[\, V_{[\alpha,\gamma]}V_{[\beta,\gamma]} - \tfrac{1}{4}\delta_{\alpha\beta}V_{[\gamma,\delta]}V_{[\gamma,\delta]}] \\
& - \tfrac{1}{2}(2\gamma + 1)\delta_{\alpha\beta}(U_{,o})^2 \ \Big] \ ,
\end{aligned}
$$

$$(4.116) \qquad \circledR^{ij} = [\, 1 + (5\gamma - 1)U](T^{ij} + t^{ij}), $$

and the conserved quantities are

$$(4.117) \qquad P^o = \int \rho^*(1 + \tfrac{1}{2}v^2 - \tfrac{1}{2}U + \Pi)\, d\underset{\sim}{x} \ ,$$

$$
\begin{aligned}
(4.118) \qquad P^\alpha = \int \rho^* \Big\{ & v^\alpha[\, 1 + \tfrac{1}{2}v^2 + (2\gamma + 1)U + \Pi + p/\rho] \\
& - \tfrac{1}{2}(4\gamma + 3)V^\alpha - \tfrac{1}{2}W^\alpha \Big\}\, d\underset{\sim}{x} \ ,
\end{aligned}
$$

64

$$(4.119) \qquad J^{\alpha\beta} = 2\int \rho^* x^{[\alpha} \left\{ v^{\beta]} \left[1 + \frac{1}{2} v^2 + (2\gamma + 1)U + \Pi + p/\rho \right] \right.$$

$$\left. - \frac{1}{2}(4\gamma + 3)V^{\beta]} - \frac{1}{2} W^{\beta]} \right\} d\underset{\sim}{x} \;,$$

$$(4.120) \qquad J^{\alpha o} = \int \rho^* x^\alpha (1 + \frac{1}{2} v^2 - \frac{1}{2} U + \Pi) \, d\underset{\sim}{x} - P^\alpha t \;.$$

By defining a center of mass X^α, given by

$$(4.121) \qquad X^\alpha = \frac{\int \rho^* x^\alpha (1 + \frac{1}{2} v^2 - \frac{1}{2} U + \Pi) \, d\underset{\sim}{x}}{\int \rho^* (1 + \frac{1}{2} v^2 - \frac{1}{2} U + \Pi) \, d\underset{\sim}{x}} \;,$$

we find from equations (4.117) and (4.120) and the constancy of $J^{\alpha o}$, that

$$(4.122) \qquad dX^\alpha/dt = P^\alpha/P^o \;,$$

i.e., the center of mass moves uniformly with velocity P^α/P^o. The conserved quantities P^i and J^{ij} transform as a four-vector and an antisymmetric tensor under low-velocity Lorentz Transformations (for discussion see Will [1971c]).

Some theories of gravity may possess only energy and momentum conservation laws, i.e., their parameters may satisfy

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0,$$

$$(4.123)$$

$$\text{one of } \{\alpha_1, \alpha_2\} \neq 0.$$

We call such theories Semi-Conservative Theories. Their conserved energy P^o, and momentum P^α, are given by equations (4.101) and (4.109), and P^i transforms under Lorentz Transformations as a

four-vector. There is no conserved J^{ij} in such theories. Semi-conservative theories may be "preferred-frame" theories ($\alpha_1 \neq 0$, $\alpha_2 \neq 0$).

Non-Conservative Theories possess no conservation laws other than for energy; their parameters satisfy

(4.124)   one of $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \alpha_3\} \neq 0$.

There is a close connection between conservation laws and Lagrangian formulations of metric theories. Any metric theory whose field equations are derived from an invariant Lagrangian action principle can be shown to possess integral conservation laws for energy and momentum (Trautman [1962], and references cited therein). Thus any Lagrangian based metric theory of gravity is at least Semi-Conservative, i.e., has PPN parameter values

(4.125) $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0$.

Table 4.III summarizes these conservation law properties of metric theories of gravity.

5. Metric Theories of Gravity and their Post-Newtonian Limit

5·1. Introduction

We now breathe some life into the PPN parameters by examining a wide class of twentieth-century metric theories of gravity and calculating their PPN parameter values. This section will illustrate an important application of the PPN formalism, that of comparing and classifying metric theories of gravity. Much of the discussion in this section is based on work by Wei-Tou Ni (1972a); other pertinent references are Thorne, Will and Ni (1971), and Will and Nordtvedt (1972).

5·2. General Relativity

a. <u>Principal references</u>: Einstein (1916); Standard textbooks and references cited therein, e.g., Synge (1960), and Misner, Thorne, and Wheeler (1972).

b. <u>Gravitational Fields Present</u>: **g**

c. <u>Arbitrary Parameters and Functions</u>: None (we will ignore the "cosmological constant" λ, which is known to be too small to be measurable in the solar system).

d. <u>Field Equations</u>: The field equations are derived from an invariant Lagrangian action principle:

$$(5.1) \qquad 0 = \delta \int (\sqrt{-g}\, R + L_I)\, d^4x,$$

where R is the scalar curvature formed from the metric (see Misner, Thorne and Wheeler [1972] for formulas), and where L_I is the inter-

67

action Lagrangian, which includes the mutual coupling of the gravitational field with all matter and non-gravitational fields. L_I has the form

(5.2) $\qquad L_I = L_I[\, g_{ij},$ matter and non-gravitational field variables$\,]$,

and in local inertial frames of g_{ij}, has the standard special relativistic form. Thus the theory satisfies the postulates of a metric theory (see Sect. 3), and hence,

(5.3) $\qquad T^{ij}{}_{;j} = 0 \; ; \qquad T^{ij} = \dfrac{1}{8\pi\sqrt{-g}} \dfrac{\partial L_I}{\partial g_{ij}} \; .$

By varying the Lagrangian in the usual way we obtain the field equations

(5.4) $\qquad R_{ij} - \dfrac{1}{2} g_{ij} R = 8\pi T_{ij}$,

where R_{ij} is the Ricci Tensor. Equation (5.3) also follows directly from the field equations.

e. <u>The Post-Newtonian Limit</u>: In this section all our discussions of the post-Newtonian limits of metric theories of gravity are based on the standard techniques developed by Chandrasekhar (1965). We will use these techniques here to derive the post-Newtonian metric for general relativity, and thereby obtain its PPN parameter values. Our calculation can then be used as a prototype for obtaining the post-Newtonian limit of nearly every metric theory of gravity (for an exception, see Subsect. 5'8). Using the "bookkeeping" scheme developed in Subsect. 4'2, we solve the field equations (5.4) for g_{oo} to O(4), $g_{o\alpha}$ to O(3), and $g_{\alpha\beta}$ to O(2) for a stress-energy tensor given by equation (4.20). We first rewrite equation (5.4) in the form

(5.5)
$$R_{ij} = 8\pi(T_{ij} - \frac{1}{2} g_{ij} T) \ ,$$

where

(5.6)
$$T = g^{mn} T_{mn} \ .$$

We choose an asymptotically flat coordinate system which is at rest relative to the mean rest-frame of the Universe ($\underset{\sim}{w} = 0$; this assumption is $\underline{not}$ necessary in general relativity, but $\underline{is}$ in many other theories [see Subsect. 5·7]), and write

(5.8)
$$g_{ij} = \eta_{ij} + h_{ij} \ .$$

The calculation then proceeds in stages:

(i) $\underline{h_{oo} \ to \ O(2)}$. To the required order

(5.8)
$$R_{oo} = \frac{1}{2} \nabla^2 h_{oo}, \quad T_{oo} = \rho, \quad T = \rho,$$

then equation (5.5) becomes

(5.9)
$$\nabla^2 h_{oo} = 8\pi\rho \ ,$$

whose solution is

(5.10)
$$h_{oo} = - 2 U.$$

(ii) $\underline{h_{\alpha\beta} \ to \ O(2)}$. When we impose the gauge condition

(5.11)
$$h^i_{\ i,\alpha\beta} - h^\gamma_{\ \beta,\alpha\gamma} - h^\gamma_{\ \alpha,\beta\gamma} = 0 \ ,$$

equation (5.5) becomes, to the required order

(5.12)
$$\nabla^2 h_{\alpha\beta} = 8\pi\rho \, \delta_{\alpha\beta} \ ,$$

whose solution is

(5.13) $\qquad h_{\alpha\beta} = -2U\delta_{\alpha\beta}$.

(iii) $\underline{h_{o\alpha}}$ to $O(3)$. When we impose the gauge condition

(5.14) $\qquad \frac{1}{2} h^{\beta}_{\;\beta,o} - h^{\beta}_{\;o,\beta} = 0$,

we get to the required order,

(5.15) $\qquad R_{o\alpha} = \frac{1}{2}\nabla^2 h_{o\alpha} - \frac{1}{2} U_{,o\alpha}$, $\qquad T_{o\alpha} - \frac{1}{2} g_{o\alpha} T = -\rho v_{\alpha}$.

Then, equation (5.5) becomes

(5.16) $\qquad \nabla^2 h_{o\alpha} = -16\pi\rho v_{\alpha} + U_{,o\alpha}$.

By using the fact that

(5.17) $\qquad U = -\frac{1}{2}\nabla^2 \chi$, $\qquad \chi_{,o\alpha} = V_{\alpha} - W_{\alpha}$,

we obtain

(5.18) $\qquad h_{o\alpha} = \frac{7}{2} V_{\alpha} + \frac{1}{2} W_{\alpha}$.

Notice that our resulting expressions for h_{oo}, $h_{o\alpha}$ and $h_{\alpha\beta}$ do satisfy the gauge conditions [eqs. (5.11) and (5.14)] to the necessary order.

(iv) $\underline{h_{oo}}$ to $O(4)$. In our chosen gauge, R_{oo} evaluated correctly to $O(4)$ is given by

(5.19) $\qquad R_{oo} = \nabla^2(\frac{1}{2} h_{oo} - U^2) + 4U\nabla^2 U$.

To the necessary order, we also have

70

(5.20)
$$T_{oo} - \frac{1}{2} g_{oo} T = 4\pi\rho + 8\pi\rho(v^2 - U + \frac{1}{2}\pi + \frac{3}{2}p/\rho).$$

Then the solution to equation (5.5) is

(5.21)
$$h_{oo} = -2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4.$$

(v) $\underline{g_{ij}}$ and the PPN parameters. The final form for the metric is

$$g_{oo} = 1 - 2U + 2U^2 - 4\Phi_1 - 2[2\Phi_2 + \Phi_3 + 3\Phi_4],$$

(5.22)
$$g_{o\alpha} = \frac{7}{2} V_\alpha + \frac{1}{2} W_\alpha,$$

$$g_{\alpha\beta} = -(1 + 2U)\delta_{\alpha\beta}.$$

Since the metric is already in the standard PPN gauge, we can read off the PPN parameter values immediately (see also Table 5.I):

$$\gamma = \beta = 1,$$

(5.23)
$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.$$

f. Discussion. We note from equations (5.23) that general relativity is a Fully Conservative Theory ($\alpha_i = 0$, $\zeta_i = 0$) with no preferred-frame effects ($\alpha_i = 0$).

5.3. Scalar-Tensor Theories

A variety of metric theories of gravity have been devised which postulate in addition to the metric, a scalar gravitational field φ. The most general such theory was examined by Bergmann (1968) and Wagoner (1970), and special cases were studied earlier by Jordan

(1948, 1955), Thirry (1948), Brans and Dicke (1961) and by Nordtvedt (1970b). We will examine the Bergmann-Wagoner Theory in detail, then will discuss the various special cases.

a. Principal References: Bergmann (1968), Wagoner (1970).

b. Gravitational Fields Present: $\varphi, \mathbf{g}$.

c. Arbitrary Parameters and Functions: Two arbitrary functions of φ, the "coupling" function $\omega(\varphi)$ and the cosmological function $\lambda(\varphi)$. $\lambda(\varphi)$ is known to be too small to be measurable in the solar system, so it will be dropped.

d. Field Equations: The field equations are derived from an invariant action principle

$$0 = \delta \int \left[\sqrt{-g} \left\{ \varphi R - [\omega(\varphi)/\varphi] g^{ij} \varphi_{,i} \varphi_{,j} + 2\varphi \lambda \right\} + L_I(g_{ij}, \ldots) \right] d^4x .$$

(5.24)

In this form, the theory satisfies the postulates of a metric theory (for a discussion of other "non-metric" representations of the theory, see Ni [1972a]). The resulting field equations are

$$R_{ij} - \frac{1}{2} g_{ij} R - \lambda g_{ij} = \frac{8\pi}{\varphi} T_{ij} + \frac{\omega(\varphi)}{\varphi} (\varphi_{,i} \varphi_{,j} - \frac{1}{2} g_{ij} \varphi_{,k} \varphi^{,k})$$

(5.25)

$$+ \varphi^{-1} (\varphi_{,i;j} - g_{ij} \Box \varphi)$$

(5.26) $$\Box \varphi + \frac{2\varphi^2 \, d\lambda/d\varphi - 2\varphi\lambda}{3 + 2\omega(\varphi)} = \frac{1}{3 + 2\omega(\varphi)} \left(8\pi T - \frac{d\omega}{d\varphi} \varphi_{,i} \varphi^{,i} \right) .$$

The stressed-matter equations of motion,

(5.27) $$T^{ij}_{;j} = 0 ,$$

follow from the form of the interaction Lagrangian, or directly from the field equations.

e. The Post-Newtonian Limit: Far from the solar-system, the scalar field φ is assumed to have a present value φ_o, determined by the rest of the Universe. In a coordinate system at rest relative to the Universe, we write

$$g_{ij} = \eta_{ij} + h_{ij} \ ,$$

(5.28)
$$\varphi = \varphi_o + \varphi_1 \ ,$$

$$\omega(\varphi) = \omega + \omega'\varphi_1 + \dots \ ,$$

where

(5.29)
$$\omega = \omega(\varphi_o), \qquad \omega' = d\omega/d\varphi\big|_{\varphi_o} \ .$$

(We assume the cosmological function λ is zero.) Following the technique of Chandrasekhar (1965), the field equations (5.25) and (5.26) may be solved, and the post-Newtonian metric obtained (for details, see Nordtvedt [1970b], Ni [1972a]). The result is

(5.30)
$$g_{oo} = 1 - 2U + 2(1 + \Lambda)U^2 - 4\left(\frac{3 + 2\omega}{4 + 2\omega}\right)\Phi_1$$
$$- 4\left(\frac{1 + 2\omega}{4 + 2\omega} - \Lambda\right)\Phi_2 - 2\Phi_3 - 6\left(\frac{1 + \omega}{2 + \omega}\right)\Phi_4 \ ,$$

$$g_{o\alpha} = \frac{1}{2}\left(\frac{10 + 7\omega}{2 + \omega}\right)V_\alpha + \frac{1}{2}W_\alpha \ ,$$

$$g_{\alpha\beta} = -\left[1 + 2\left(\frac{1 + \omega}{2 + \omega}\right)U\right]\delta_{\alpha\beta} \ ,$$

where

(5.31)
$$\Lambda = \omega'/(3 + 2\omega)^2(4 + 2\omega) \ .$$

By comparing equations (5.30) with the PPN metric in Table 4.I, we can
read off the PPN parameter values for this theory (see also Table 5.1):

$$\gamma = \left(\frac{1 + \omega}{2 + \omega}\right), \qquad \beta = 1 + \Lambda ,$$

(5.32)

$$\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$$

The Newtonian gravitational constant G (as measured by studying
Keplerian orbits far from the gravitating matter) is related to the
scalar field in the theory by

(5.33)
$$G_{Today} = \frac{1}{\varphi_0}\left(\frac{4 + 2\omega}{3 + 2\omega}\right) \equiv 1 ,$$

($G_{Today} = 1$ by our use of geometrized units.) If the asymptotic
scalar field evolves in time as a result of the changing structure of the
Universe, G may change secularly with time from its present value
of unity according to

(5.34)
$$\left(\frac{1}{G}\frac{dG}{dt}\right)_{Today} = \left[1 + \frac{2\omega'}{(3 + 2\omega)^2}\right]\frac{1}{\varphi_0}\left(\frac{d\varphi}{dt}\right)_{Today} .$$

f. Other Theories and Special Cases:

(i) Nordtvedt's (1970b) Scalar-Tensor Theory is equivalent
to the Bergmann-Wagoner Theory in the special case of zero cosmolo-
gical function, i.e., $\lambda = 0$ (Ni 1972a). Its PPN parameters are the
same as in the Bergmann-Wagoner Theory (see Table 5.I).

(ii) Dicke-Brans-Jordan Theory is the special case ω = const.,
$\lambda = 0$ of the Bergmann-Wagoner-Nordtvedt theories (Jordan 1948, 1955;
Thirry 1948; Brans and Dicke 1961). Its post-Newtonian limit was

calculated by Nutku (1969a), and its PPN parameters may be obtained

from the Bergmann-Wagoner-Nordtvedt PPN parameters by setting

$\omega' = 0$ (see Table 5.I). In the limit $\omega \rightarrow + \infty$, Dicke-Brans-Jordan

Theory reduces to general relativity.

g. <u>Discussion</u>: We note that the Scalar-Tensor Theories are

all Fully Conservative Theories ($\alpha_i = 0$, $\zeta_i = 0$), with <u>no</u> preferred

frame effects ($\alpha_i = 0$).

Since gravitation appears to be a long-range, purely attractive

force, one might expect the Universe's global matter distribution to

affect local gravitational physics, and to play a dominant role in

establishing any "preferred" reference frame. In fact, from this

"Machian" point of view, the mystery seems to be: <u>How can a metric</u>

<u>theory of gravity avoid at all having a preferred frame related to the</u>

<u>Universe rest-frame?</u> General relativity and the Scalar-Tensor theories

obviously avoid preferred-frame effects (cf. their parameter values

$\alpha_1 = \alpha_2 = \alpha_3 = 0$ in Table 5.I) even though the Scalar-Tensor Theories

embody several "Machian" concepts [cf. eqs. (5.33) and (5.34)]. Before

we discuss theories which <u>do</u> have preferred frames, we must first

shed some light on how these exceptional theories avoid them. We

confine our attention to Lagrangian-based metric theories (so that

$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$) which assume no preferred frame <u>a priori</u>

(later in this section we will examine theories which do assume "prior

geometry").

Consider a local gravitational system, such as the solar system,

which is embedded in the Universe. We separate the computation of

the metric into two parts: a Universal or cosmological solution, and

a "local" solution. From this viewpoint, the Universe affects the local gravitational physics of the system by establishing the boundary conditions (at a boundary "far" from the matter) for the various fields generated by the local system. The local system "feels" its relationship to the Universe via the asymptotic field values of the fields present; metric g, scalar field φ, vector field K, second-rank tensor field C, and so on. Several conclusions follow:

1) <u>A theory which contains solely a metric field yields local gravitational physics which is identical in all asymptotic Lorentz frames, and which does not change with Universe evolution. In addition, the Newtonian gravitational constant (cf. Sect. 6) is unaffected by the proximity of matter.</u> All this follows from the invariance properties of η_{ij} (the asymptotic form of g_{ij}), the only field coupling the local system asymptotically to the Universe, and from general covariance, which allows us to always find a coordinate system in which the metric field takes this Minkowskii form at the boundary between the Universe and the local system.

2) <u>A theory which contains a metric field and a scalar field φ yields physics which is identical in all asymptotic Lorentz frames, but which may vary with Universe evolution; G may be affected by proximity of matter.</u> These conclusions follow from invariance of both η_{ij} and φ under Lorentz transformations, but now φ may vary with Universe evolution and may depend on the proximity of matter.

3) <u>A theory which contains a vector field K_i and/or an additional second-rank tensor field C_{ij} yields local physics which may depend on</u>

motion relative to a preferred Universe rest-frame, and which may vary with evolution of the Universe; G may be affected by proximity of matter. This follows because the asymptotic values of K_i and C_{ij} are not invariant under Lorentz transformations (an exception would be $C_{ij[asymptotic]}$ proportional to η_{ij}).

In summary, it is the Lorentz invariance of the asymptotic fields φ and η_{ij} which makes it impossible to have preferred-frame effects in theories containing solely those fields. Thus we must appeal to vector- or tensor-metric theories (or to theories with "prior geometry") for preferred-frame effects.

It is commonly believed that cosmological vector fields and additional second-rank tensor fields are absent from physics (cf. Sect. 2). The Hughes-Drever experiment (measurement of the isotropy of inertial mass) and a variety of laboratory "ether-drift" experiments have been used by Dicke [(1964); see also Peebles and Dicke (1962); Peebles (1962)] to rule out these fields. Closer examination of these arguments shows that the experimental evidence rules out only those vector and "second tensor" cosmological fields which couple directly to matter. Since these experiments were performed under conditions where the effects of gravity were negligible, they do not rule out vector and second-tensor fields which couple only to gravity.

We now proceed to discuss such theories.

5·4. Vector-Metric Theories

We assume these theories are derivable from a coordinate-invariant Lagrangian, with no a priori assumption of a special coordinate

frame or of any prescribed form for the tensors in the theory. For simplicity, we restrict ourselves to theories which have linear field equations for the (timelike) vector field; nevertheless, a wide variety of theories is possible. The general Lagrangian has the form

$$L_{V-M} = L_I (g_{ij}; \text{ matter and non-gravitational fields})$$

(5.35)
$$+ k_1 R + k_2 K_i K^i R + k_3 K_i K_j R^{ij} + k_4 K_{i;j} K^{i;j}$$

$$+ k_5 K_{i;j} K^{j;i} + k_6 K_i^{\;;i} K_j^{\;;j} ,$$

where R_{ij} and R are the Ricci tensor and scalar formed from the metric, k_1, ..., k_6 are dimensionless coupling constants. The PPN parameters for a special case of these vector-metric theories have been computed:

    a. <u>Principal References</u>: Will and Nordtvedt (1972).

    b. <u>Gravitational Fields Present</u>: **g**, **K**.

    c. <u>Arbitrary Parameters and Functions</u>: K, the asymptotic "strength" of the vector field.

    d. <u>Field Equations</u>: The field equations are derived from an invariant action principle:

(5.36)
$$0 = \delta \int \left[\sqrt{-g}\, G_o^{-1} \left\{ R + K_{i;j} K_{k;\ell} g^{ik} g^{j\ell} \right\} + L_I \right] d^4 x,$$

where G_o plays the role of an <u>unrenormalized</u> gravitational constant. This theory corresponds to the case [eq. (5.35)] $k_1 = k_4 = G_o^{-1}$, $k_2 = k_3 = k_5 = k_6 = 0$. The field equations are

(5.37)
$$K_{i;j}^{\;;j} = 0,$$

78

$$(5.38) \qquad R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G_o T_{ij} + \theta_{ij} \; ,$$

where θ_{ij} is the vector-field stress-energy tensor given by

$$(5.39) \qquad \theta_{ij} = K_{i;m} K_j{}^{;m} + K_{m;i} K^m{}_{;j} - \frac{1}{2} K^m{}_{;n} K_m{}^{;n} g_{ij}$$
$$+ \frac{1}{2} (K^m S_{ij} - K_i S^m{}_j - K_j S^m{}_i)_{;m} \; ,$$

where

$$(5.40) \qquad S_{ij} = K_{i;j} + K_{j;i} \; .$$

The stressed-matter equations of motion

$$(5.41) \qquad T^{ij}{}_{;j} = 0 \; ,$$

follow from the form of L_I or directly from the field equations.

e. **The Post-Newtonian Limit:** The metric is expanded about the flat-space Minkowskii metric according to

$$(5.42) \qquad g_{ij}(\underset{\sim}{x}, t) = \eta_{ij} + h_{ij}(\underset{\sim}{x}, t)$$

where, far from the matter, h_{ij} tends asymptotically toward zero. We also expand the vector field K_i about its asymptotic value. However, in order to simplify the calculation, we work in a coordinate system in which the asymptotic vector field has only a time component; i.e. asymptotically

$$(5.43) \qquad K_\alpha = 0 \; .$$

Then, in this coordinate system, we have

$$(5.44) \qquad |K|^2 \equiv \eta^{ij} K_i K_j = K_o^2 \; ,$$

far from the configuration of fluid. The frame in which $K_o = K$ and $K_\alpha = 0$ is presumably the rest-frame of the Universe's smoothed-out distribution of matter. The expansion of the vector field can thus be written

$$K_o = K + \varphi(\underset{\sim}{x},t) ,$$

(5.45)

$$K_\alpha = k_\alpha(\underset{\sim}{x},t) ,$$

where φ is $O(2)$ and k_α is $O(3)$, and both go to zero far from the matter. The resulting PPN metric is (see Will and Nordtvedt [1972] for details):

$$g_{oo} = 1 - 2U + 2U^2 - 4\Phi_1 - 4\Phi_2 - 2\Phi_3 - 6\Phi_4 ,$$

(5.46)

$$g_{o\alpha} = \frac{1}{2}\left(7 - \frac{K^2}{1 + \frac{1}{2}K^2}\right)V_\alpha + \frac{1}{2}\left(1 + \frac{K^2}{1 + \frac{1}{2}K^2}\right)W_\alpha ,$$

$$g_{\alpha\beta} = - (1 + 2U)\delta_{\alpha\beta} ,$$

and the PPN parameters have the values (see also Table 5.I)

$$\gamma = \beta = 1 , \quad \alpha_1 = 0 , \quad \alpha_2 = K^2/(1 + \frac{1}{2}K^2) ,$$

(5.47)

$$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 .$$

The gravitational constant G is related to G_o and the vector field by

$$G_{Today} = \frac{G_o}{1 + \frac{1}{2}K^2} \equiv 1 .$$

If the asymptotic vector field evolves in time, G may change from its present value according to

80

$$(5.49) \qquad \frac{1}{G}\left(\frac{dG}{dt}\right)_{Today} = -\left[\frac{K}{1 + \frac{1}{2}K^2}\left(\frac{dK}{dt}\right)\right]_{Today} \quad .$$

f. <u>Discussion</u>: This Vector-Metric Theory is a Semi-Conservative theory $(\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0)$ and does predict preferred-frame effects $(\alpha_2 \neq 0$, see Sect. 7).

5·5. Tensor-Metric Theories

A wide class of Tensor-Metric Theories can be constructed following the same prescription as for the Vector-Metric Theories. For a symmetric tensor field C_{ij}, we can devise a Lagrangian of the form

$$L_{T-M} = L_I(g_{ij}; \text{ matter and non-gravitational fields})$$

$$+ k_1 {}'R + k_2 {}'C_{ij}C^{ij}R + k_3 {}'(C_{ij}g^{ij})^2 R$$

$$(5.50) \qquad + k_4 {}'C_{ij}g^{ij}C_{k\ell}R^{k\ell} + k_5 {}'C_{ik}C^k{}_j R^{ij}$$

$$+ k_6 {}'C_{ij;k}C^{ij;k} + k_7 {}'C_{ij;k}C^{ik;j} + k_8 {}'C_{ik;}{}^k C^{ij}{}_{;j}$$

$$+ k_9 {}'C_{ij;}{}^j g^{k\ell}C_{k\ell}{}^{;i} + k_{10} {}'C_{ij;k}C_{\ell m;}{}^k g^{ij}g^{\ell m} \quad .$$

We point out here that the above equation (5.50) alone does not reveal the full richness of possible Lagrangian terms for tensor-metric theories. If C_{ij} is a non-singular field, one can define Christoffel symbols in terms of C_{ij}. Also, third-rank tensors $S^i{}_{jk}$ can be produced by taking the difference of the Cristoffel symbols

$$(5.51) \qquad S^i{}_{jk} = \Gamma^i{}_{jk}(\mathbf{g}) - \Gamma^i{}_{jk}(\mathbf{C}) \quad .$$

Covariant derivatives can be defined with respect to each tensor, g_{ij} or C_{ij}, and tensor densities can be formed using either

$$\sqrt{-g}$$

or

$$\sqrt{|\det C|} \quad .$$

All these quantities can be used to construct Lagrangian terms. Nordtvedt and Hellings (1972) have studied these Tensor-Metric Theories in detail.

5·6. Conformally Flat Theories

Conformally flat theories possess a global Lorentz metric η_{ij} and a scalar field ψ, which generate the physical metric via the algebraic equation

$$(5.52) \qquad g_{ij} = \psi \eta_{ij} \quad .$$

The scalar field is generated by the matter via a wave-type equation. One of the most famous of the conformally flat theories is Nordstrøm's Second Theory.

a. <u>Principal References</u>: Nordstrøm (1913, 1914), Einstein and Fokker (1914), Whitrow and Morduch (1960, 1965).

b. <u>Gravitational Fields Present</u>: η, φ, g.

c. <u>Arbitrary Parameters and Functions</u>: None.

d. <u>Field Equations</u>: In the form derived by Einstein and Fokker (1914), they are

(5.53) $\qquad C_{ijk\ell} = 0 ,$

(5.54) $\qquad R = 24\pi T ,$

where $C_{ijk\ell}$ is the Weyl conformal tensor constructed from **g**. But the vanishing of the Weyl tensor guarantees the existence of a flat spacetime metric and a scalar field φ which generate g_{ij} by

(5.55) $\qquad g_{ij} = \varphi^2 \eta_{ij} .$

Equations (5.53) and (5.54) also allow φ to be calculated from the variational principle

(5.56) $\qquad 0 = \delta \int (L_I - \tfrac{1}{3} R\sqrt{-g}) \, d^4x .$

The field equation (5.54) then becomes

(5.57) $\qquad \eta^{ij} \varphi_{,i,j} = - 4\pi T \varphi^3 .$

Equation (5.57) is Nordstrøm's original field equation.

      e. <u>The Post-Newtonian Limit</u>: By solving the field equation (5.57) for φ to the appropriate order (see Ni [1972a] for details) we obtain

$$g_{oo} = 1 - 2U + U^2 + 6\Phi_2 - 2\Phi_3 + 6\Phi_4 ,$$

(5.58) $\qquad g_{o\alpha} = - \tfrac{1}{2} V_\alpha + \tfrac{1}{2} W_\alpha ,$

$$g_{\alpha\beta} = - (1 - 2U)\delta_{\alpha\beta} ,$$

from which we get

$$\gamma = -1 , \qquad \beta = \frac{1}{2}$$

(5.59) $\qquad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 .$

Notice that Nordstrøm's Second Theory is a Fully Conservative Theory
($\alpha_i = \zeta_i = 0$) with no preferred frames ($\alpha_i = 0$).

   f.   Other Theories or Special Cases: A variety of conformally
flat theories have been devised.   We list them here, and quote their
PPN parameter values in Table 5.I.   The reader is referred to Ni
(1972a) for details and references.

   (i)   General Conformally Flat Theory (Ni 1972a),

   (ii)   General Conformally Flat Theory, Lagrangian based (Ni 1972a),

   (iii)   Whitrow-Morduch Conformally Flat Theory (Whitrow and

           Morduch 1960, 1965),

   (iv)   Littlewood-Bergmann Theory (Littlewood 1953, Bergmann

           1956),

   (v)   Nordstrøm's First Theory (Nordstrøm 1912).

   g.   Discussion: We will see in Sect. 7 that Conformally-Flat
Theories all predict zero bending of light rays by the Sun and zero
relativistic time-delay of light signals passing by the Sun.   This can
also be deduced from the conformal invariance of Maxwell's equations
(i.e., invariance under a transformation $g_{ij} \rightarrow \psi^2 g_{ij}$): propagation of
light rays in the metric $\varphi^2 \eta_{ij}$ is identical to propagation in the flat
space metric η_{ij}, namely straight-line propagation at constant speed.

5.7. Stratified Theories with Time-Orthogonal Conformally Flat Space Slices

Some metric theories of gravity endow spacetime with a "prior" geometric structure, which exists independently of field equations or the distribution of matter in the Universe. This "prior geometry" can take many different forms. Conformally flat theories of gravity exhibit "prior geometry" by insisting that the metric of spacetime be conformally flat (vanishing Weyl tensor) regardless of the motion and evolution of the matter. An important class of theories of gravity which postulate a "prior geometry" are the "Stratified Theories with Time-Orthogonal Conformally Flat Space Slices" (Ni 1972a).

These theories are devised using the following prescription: The Universe's large-scale distribution of matter determines a preferred reference frame whose space slices ("strata") are conformally flat, although the full spacetime is not. In this preferred frame, the metric has the form

$$(5.60) \qquad ds^2 = e^{2\varphi} dt^2 - e^{2\psi} (dx^2 + dy^2 + dz^2) \,,$$

where φ and ψ are scalar fields. In geometric, coordinate-free language, such theories have (i) a background, flat metric $\boldsymbol{\eta}$; (ii) a Universal time coordinate t (a scalar field) which is covariantly constant and has timelike gradients with respect to $\boldsymbol{\eta}$; (iii) scalar gravitational fields φ and ψ; and (iv) a metric $\boldsymbol{g}$ construction from $\boldsymbol{\eta}$, t, φ, and ψ by

$$(5.61) \qquad \boldsymbol{g} = e^{2\psi} \boldsymbol{\eta} + [e^{2\varphi} - e^{2\psi}] \, \boldsymbol{dt} \otimes \boldsymbol{dt} \,.$$

These theories differ from one another by their field equations for φ and ψ. Some Stratified Theories contain a single gravitational field φ and construct the metric using functions of φ:

$$(5.62) \qquad \mathbf{g} = e^{2g(\varphi)}\boldsymbol{\eta} + [\, e^{2f(\varphi)} - e^{2g(\varphi)}\,]\, \mathbf{dt} \otimes \mathbf{dt} \; .$$

We will describe a recent theory of this type, published by Nathan Rosen (1971a,b)

    a. <u>Principal References</u>: Rosen (1971a,b), Ni (1972a).

    b. <u>Gravitational Fields Present</u>: $\boldsymbol{\eta}$, φ, ψ, t, $\mathbf{g}$.

    c. <u>Arbitrary Parameters and Functions</u>: Three parameters a, b, c. In order to have the correct Newtonian limit, b must satisfy

$$(5.63) \qquad b = 2 + c^2/4a \; .$$

The post-Newtonian limit is independent of a, and so contains only one arbitrary parameter λ, given by

$$(5.64) \qquad \lambda = c/2a \; .$$

    d. <u>Field Equations</u>: The field equations are derived from an invariant action principle

$$(5.65) \qquad 0 = \delta \int \left[\sqrt{-g}\; g^{ij}(a\psi_{,i}\psi_{,j} + b\varphi_{,i}\varphi_{,j} + c\varphi_{,i}\psi_{,j}) + L_I \right] d^4x,$$

and are given by (in the preferred reference frame)

$$a\left\{ \tfrac{1}{2} e^{-2\varphi}(\psi_{,o})^2 + \tfrac{1}{2} e^{-2\psi}|\nabla\psi|^2 \right\}$$

$$(5.66) + b\left\{ e^{-2\varphi}[\,\varphi_{,oo} - \tfrac{1}{2}(\varphi_{,o})^2 + 3\varphi_{,o}\psi_{,o}\,] - e^{-2\psi}[\nabla^2\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + \nabla\varphi \cdot \nabla\psi] \right\}$$

$$+ c\left\{ \tfrac{1}{2} e^{-2\varphi}[\,\psi_{,oo} + 3(\psi_{,o})^2\,] - \tfrac{1}{2} e^{-2\psi}[\nabla^2\psi + |\nabla\psi|^2] \right\} = -\,8\pi T^o{}_o \; ,$$

$$a\left\{-e^{-2\varphi}[\psi_{,oo}+(\psi_{,o})^2+\psi_{,o}\varphi_{,o}]+e^{-2\psi}[\nabla^2\psi+|\underset{\sim}{\nabla}\psi|^2+\underset{\sim}{\nabla}\psi\cdot\underset{\sim}{\nabla}\varphi]\right\}$$

$$+b\left\{e^{-2\varphi}[\varphi_{,oo}+(\psi_{,o})^2+3\psi_{,o}\varphi_{,o}]-e^{-2\psi}[\nabla^2\varphi+|\underset{\sim}{\nabla}\varphi|^2+\underset{\sim}{\nabla}\psi\cdot\underset{\sim}{\nabla}\varphi]\right\}$$

(5.67)

$$+c\left\{\frac{1}{2}e^{-2\varphi}[\psi_{,oo}+3(\psi_{,o})^2-\varphi_{,oo}+(\varphi_{,o})^2]\right.$$

$$\left.-\frac{1}{2}e^{-2\psi}[\nabla^2\psi+|\underset{\sim}{\nabla}\psi|^2-\nabla^2\varphi+|\underset{\sim}{\nabla}\varphi|^2]\right\}=-8\pi T,$$

where

(5.68)
$$T^o{}_o=g^{oo}T_{oo}, \qquad T=g^{ij}T_{ij}.$$

The stressed matter equations of motion follow from the form of L_I, namely

(5.69)
$$T^{ij}{}_{;j}=0.$$

    e. <u>The Post-Newtonian Limit</u>: Working in the preferred reference frame (at rest with respect to the Universe), we may compute the post-Newtonian metric from equations (5.66), (5.67) and (5.68), making use of the constraint (5.63). Then, in our Standard PPN Gauge, the resulting metric takes the form

$$g_{oo}=1-2U+\frac{1}{2}(3+\lambda)U^2-2(1+\lambda)\Phi_1-(5\lambda-1)\Phi_2-2\Phi_3-6\lambda\Phi_4,$$

(5.70)
$$g_{o\alpha}=-\frac{1}{2}V_\alpha+\frac{1}{2}W_\alpha,$$

$$g_{\alpha\beta}=-(1+2\lambda U)\delta_{\alpha\beta},$$

with PPN parameters

$$\gamma = \lambda \, , \qquad \beta = \frac{1}{4}(3 + \lambda) \, , \qquad \alpha_1 = -4(1 + \lambda) \, ,$$

(5.71)

$$\alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \, .$$

Note that Rosen's theory is a Semi-Conservative Theory with preferred-frame effects.

f. <u>Other Theories or Special Cases</u>: There are several other Stratified Theories, many of which are discussed in detail by Ni (1972a). Two additional stratified theories were devised by Ni himself. We list them here and quote their PPN parameter values in Table 5.I.

(i) Page-Tupper Theory (Page and Tupper 1968),

(ii) Papapetrou's Theory (Lagrangian-based) (Papapetrou 1954a,b,c),

(iii) Yilmaz's Theory (Yilmaz 1958, 1962) (as completed by Ni [1972a]),

(iv) Ni's Lagrangian Stratified Theory (Ni 1972a),

(v) Ni's General Stratified Theory (Ni 1972a),

(vi) Coleman's Theory (Coleman 1971),

(vii) Einstein's Theory with "Variable Velocity of Light" (Einstein 1912),

(viii) Whitrow-Morduch Theory with "Variable Velocity of Light" (Whitrow and Morduch 1960, 1965).

g. <u>Discussion</u>: We conclude this subsection with a theorem concerning the PPN parameter α_1 for stratified theories:

<u>In every stratified theory of gravity with time-orthogonal, conformally flat space slices, the PPN parameter α_1 has the value</u>

$$(5.72) \qquad \alpha_1 = -4(1 + \gamma) \ .$$

Note that all the stratified theories in Table 5.I obey equation (5.72). In Sect. 7, we will make use of this result along with the experimentally measured value for γ ($\sim 1.0 \pm 0.1$) to show that <u>all the stratified theories in Table 5.I disagree violently with experiment.</u>

The proof of this theorem goes as follows: Pick any stratified theory with time-orthogonal conformally flat space slices. Using a coordinate system at rest with respect to the Universe, compute the post-Newtonian metric due to an arbitrary configuration of matter. To put this metric into the "standard" PPN gauge, it may be necessary to apply an infinitesimal gauge transformation [eq. (4.27)]

$$(5.73) \qquad t^\dagger = t + \epsilon \chi_{,o} \ , \qquad \underset{\sim}{x}^\dagger = \underset{\sim}{x} \ ,$$

for some value of ϵ. Since $g_{o\alpha}$ was initially identically zero [by assumption, cf. eq. (5.60)], in the new gauge it becomes

$$(5.74) \qquad \begin{aligned} (g_{o\alpha})^\dagger &= \epsilon \chi_{,o\alpha} \\ &= -\epsilon V_\alpha + \epsilon W_\alpha \ . \end{aligned}$$

By comparing this with the PPN metric in Table 4.I we obtain

$$(5.75) \qquad \begin{aligned} \tfrac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1) &= -\epsilon \ , \\ \tfrac{1}{2}(1 - \alpha_2 - \zeta_1) &= \epsilon \ . \end{aligned}$$

Adding equations (5.75) we obtain, finally,

$$(5.76) \qquad \alpha_1 = -(4\gamma + 4) \ ,$$

independent of gauge, ϵ. Q.E.D.

5·8. Whitehead's Theory

    a. <u>Principal References</u>: Whitehead (1922), Synge (1952).

    b. <u>Gravitational Field Present</u>: $\boldsymbol{\eta}$, $\boldsymbol{g}$.

    c. <u>Arbitrary Parameters and Functions</u>: None.

    d. <u>Field Equations</u>: Whitehead's (1922) theory of gravity is a Lorentz-invariant action-at-a-distance metric theory. The metric $\boldsymbol{g}$ determines the geodesics along which freely-falling test bodies move, and in the local inertial frames of $\boldsymbol{g}$, the non-gravitational laws of physics take on their standard special relativistic forms. The theory also contains a global Lorentz metric $\boldsymbol{\eta}$ which is physically un-observable, but which appears in the equations used to calculate $\boldsymbol{g}$: for a field point with four-vector position $\boldsymbol{X}$, g_{ab} is given by the following equations (cf. Fig. 5.1)

$$g_{ab}(\boldsymbol{X}) = \eta_{ab} - 2 \sum_k m_k (y_k)_a (y_k)_b / w_k{}^3 ,$$

$$\boldsymbol{Y}_k = \boldsymbol{X} - \boldsymbol{X}_k , \quad \boldsymbol{Y}_k \cdot \boldsymbol{Y}_k = 0 ,$$

(5.77)

$$w_k = \boldsymbol{Y}_k \cdot (d\boldsymbol{X}_k / d\sigma) ,$$

$$d\sigma^2 = \eta_{ab} \, dx^a \, dx^b ,$$

where " $\cdot$ " means contraction with respect to η_{ab}, and where m_k is the rest mass of the k'th particle. Thus the metric g_{ab} is determined at a point in spacetime by the effect of all other masses along the past $\boldsymbol{\eta}$-"light cone" of the point. In order to generalize equations (5.77) to continuous, fluid systems, we notice that the total rest masses of

particles in a small element of volume along the past η-light cone is equal to the flux of matter across the corresponding element $d\Sigma_c^-$ of the three-dimensional hypersurface which comprises the η-light-cone:

$$(5.78) \qquad \sum_{\substack{\text{element} \\ \text{on } \eta\text{-cone}}} m_k = (\sqrt{-g}\, \rho u^c\, d\Sigma_c)^- \, ,$$

where ρ is the rest mass density of matter, as measured in a local co-moving inertial frame of $\mathbf{g}$, and u^c is the four-velocity of the element of matter. The superscript (-) indicates quantities to be evaluated along the past η-light-cone. Then

$$g_{ab}(\mathbf{X}) = \eta_{ab} - 2 \int_{\Sigma^-} \frac{(y^-)_a (y^-)_b}{(w^-)^3} (\sqrt{-g}\, \rho u^c\, d\Sigma_c)^- \, ,$$

$$(5.79) \qquad \mathbf{Y}^- = \mathbf{X} - \mathbf{X}^- , \quad \mathbf{Y}^- \cdot \mathbf{Y}^- = 0 \, ,$$

$$w^- = \mathbf{Y}^- \cdot (d\mathbf{X}^-/d\sigma) \, .$$

Note that when written in terms of ρ, equation (5.79) for the metric is now an implicit integral equation for g_{ab}, since g_{ab} itself appears inside the integral.

e. The Post-Newtonian Limit: In the post-Newtonian metric, all field quantities are evaluated on a constant-time hypersurface Σ, rather than along the past η-"light cone" Σ^-, at each point. By following each element of fluid from the past η-light cone to the present (Σ), one can show that if $d\Sigma$ is the intersection of the world tube of a given element of matter with the constant-time hypersurface Σ, then, because of conservation of rest-mass, equations (4.71) and (4.81),

$$(5.80) \qquad (\sqrt{-g}\ \rho u^c\ d\Sigma_c)^- = \sqrt{-g}\ \rho u^c\ d\Sigma_c = \sqrt{-g'}\ \rho' u^{0'}\ d\underset{\sim}{x'} \ ,$$

as long as no net flow of matter has occurred across the walls of the element's world tube. We must also express $(y^-)_i$ and w^- in terms of quantities evaluated on Σ. The crucial formulas are

$$(y^-)^\alpha = (x-x')^\alpha + |\underset{\sim}{x} - \underset{\sim}{x}'|v'^\alpha + \underset{\sim}{v}' \cdot (\underset{\sim}{x}-\underset{\sim}{x}')v'^\alpha$$

$$-\frac{1}{2}|\underset{\sim}{x}-\underset{\sim}{x}'|^2\ dv'^\alpha/dt \ ,$$

$$(5.81) \qquad (y^-)^0 = |\underset{\sim}{x}-\underset{\sim}{x}'|\left\{1 + \underset{\sim}{v}' \cdot (\underset{\sim}{x}-\underset{\sim}{x}')/|\underset{\sim}{x}-\underset{\sim}{x}'| + \frac{1}{2}v'^2\right.$$

$$\left. +\frac{1}{2}[\underset{\sim}{v}' \cdot (\underset{\sim}{x}-\underset{\sim}{x}')/|\underset{\sim}{x}-\underset{\sim}{x}'|]^2 - \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{x}') \cdot d\underset{\sim}{v}'/dt\right\} \ ,$$

$$w^- = |\underset{\sim}{x}-\underset{\sim}{x}'|\left\{1 + \frac{1}{2}[\underset{\sim}{v}' \cdot (\underset{\sim}{x}-\underset{\sim}{x}')/|\underset{\sim}{x}-\underset{\sim}{x}'|]^2 + \frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{x}') \cdot d\underset{\sim}{v}'/dt\right\} \ .$$

We make a gauge transformation to put the metric in the standard PPN gauge:

$$x^\alpha \rightarrow x^\alpha + \int \frac{\rho'(x-x')^\alpha}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\ d\underset{\sim}{x}' \ ,$$

$$(5.82)$$

$$x^0 \rightarrow x^0 - 2\int \rho' \ln|\underset{\sim}{x}-\underset{\sim}{x}'|\ d\underset{\sim}{x}' - \frac{5}{2}\chi_{,0} \ ,$$

and obtain

$$g_{00} = 1 - 2U + 2U^2 + 2\Phi_1 - 4\Phi_2 - 6\mathcal{C}$$

$$+ 2\int \frac{\rho'\rho''(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \cdot \left(\frac{(\underset{\sim}{x}'-\underset{\sim}{x}'')}{|\underset{\sim}{x}-\underset{\sim}{x}''|} - \frac{(\underset{\sim}{x}-\underset{\sim}{x}'')}{|\underset{\sim}{x}'-\underset{\sim}{x}''|}\right) \ ,$$

$$(5.83) \qquad g_{0\alpha} = \frac{1}{2}V_\alpha + \frac{7}{2}W_\alpha \ ,$$

$$g_{\alpha\beta} = -(1 + 2U)\delta_{\alpha\beta} \ .$$

Except for the final term in g_{oo}, this is the same as a PPN metric with parameter values

$$\gamma = 1, \quad \beta = 1, \quad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = 0,$$
(5.84)
$$\zeta_1 = -6, \quad \zeta_3 = -1, \quad \zeta_4 = -1.$$

The extra term in g_{oo}, which prevents us from analysing Whitehead's theory completely within the PPN framework, will be shown (Sect. 7) to have striking observable consequences.

    f. <u>Other Theories or Special Cases</u>: Generalizations of Whitehead's theory have been examined by Schild (1956, 1962).

    g. <u>Discussion</u>: In its original form, Whitehead's theory could not describe measurements made by rods and atomic clocks, and said nothing about the trajectories of photons. The interpretation we hse here was first introduced by Synge (1952) to make Whitehead's theory complete. For further discussion of Whitehead's theory, see Temple (1924), Rayner (1954, 1955), Clark (1954), Whitrow and Morduch (1965), and Will (1971d).

6. EQUATIONS OF MOTION IN THE PPN FORMALISM

6·1. Introduction

One of the fundamental postulates of any metric theory of gravity is that "test" bodies, i.e. bodies with negligible size and structure, follow geodesics of the spacetime metric. The early Eddington-Robertson-Schiff versions of the PPN formalism treated the sun and planets as test bodies and used the geodesic equation of motion

$$(6.1) \qquad \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0 \, ,$$

where λ is an affine parameter along the body's world line, to calculate their trajectories. However, the test-body model for the planets is unrealistic and inadequate for studying all the experimental tests made possible by current technology. Rather, the planets must be viewed as finite, self-gravitating "clumps" of matter. One model for the planets builds them up out of swarms of test-bodies which interact with each other only through their mutual gravitational attraction, i.e. an "ideal" gas (Nordtvedt 1968b). Each test body follows a geodesic of the metric produced by the other test bodies in the planet, as well as by other planets, and the motion of each planet is obtained by averaging the motions of all the constituent test bodies. Another model treats the planets as self-gravitating "clumps" of perfect nonviscous fluid (Will 1971a). Each element of fluid obeys the "stressed-matter" equation of motion

$$(6.2) \qquad T^{ij}_{;j} = 0 \, ,$$

94

and the motion of each planet is obtained by averaging over all the constituent fluid elements. Detailed justification of the perfect-fluid model for the planets is given in Will (1971a). In Subsection 6·3, we will derive PPN equations of motion for the planets using the perfect-fluid model.

The PPN formalism treats photons as test-bodies with zero mass; their trajectories are therefore <u>null</u> geodesics of the metric, i.e. geodesics constrained by

(6.3) $$0 = (g_{ij} \, dx^i \, dx^j)^{1/2}$$

This result follows directly from the curved-space vacuum Maxwell Equations, valid in any metric theory:

(6.4)
$$F^{ij}{}_{;j} = 0 \ ,$$
$$F_{[ij;k]} = 0 \ ,$$

when we take the geometrical optics (photon) limit. In Subsection 6·2 we will derive PPN equations of motion for photons.

The remainder of Sect. 6 is devoted to deriving PPN equations for the precession of the spin of a rotating body (gyroscope or planet) (Subsect. 6·4), and to obtaining an expression for the Newtonian gravitational constant as measured by a Cavendish experiment performed in the solar system ("locally measured G") (Subsect. 6·5).

6·2. Equations of Motion for Photons in the PPN Formalism

We begin with the photon geodesic equation

(6.5)
$$\frac{d^2x^i}{d\lambda^2} + \Gamma^i_{\ jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0 \ ,$$

where λ is an "affine" parameter measured along the photon's trajectory ($d\lambda$ is <u>not</u> the same as ds, which is zero for a null trajectory). We can rewrite equation (6.5) using PPN coordinate time t rather than λ as affine parameter by noticing from equation (6.5) that ($x^o \equiv t$):

(6.6)
$$\frac{d^2t}{d\lambda^2} + \Gamma^o_{\ jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0 \ .$$

Then equation (6.5) can be rewritten

(6.7)
$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha_{\ jk} \frac{dx^j}{dt} \frac{dx^k}{dt} - \Gamma^o_{\ jk} \frac{dx^j}{dt} \frac{dx^k}{dt} \frac{dx^\alpha}{dt} = 0 \ .$$

The solutions to equation (6.7) must be null geodesics, and so must satisfy

(6.8)
$$g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0 \ .$$

To post-Newtonian accuracy, equations (6.7) and (6.8) may be written (see Table 6.I for expressions for the Christoffel symbols $\Gamma^i_{\ jk}$):

(6.9)
$$\frac{d^2x^\alpha}{dt^2} = \frac{\partial U}{\partial x^\alpha}\left(1 + \gamma \left|\frac{d\underset{\sim}{x}}{dt}\right|^2\right) - 2\frac{dx^\alpha}{dt}\left(\frac{d\underset{\sim}{x}}{dt} \cdot \underset{\sim}{\nabla} U\right)(1 + \gamma) \ ,$$

(6.10)
$$0 = 1 - 2U - |d\underset{\sim}{x}/dt|^2(1 + 2\gamma U) \ .$$

The Newtonian, or first-order solution of these equations is

(6.11)
$$x^\alpha_{NEWTONIAN} = n^\alpha(t - t_o), \quad |\underset{\sim}{n}| = 1 \ ,$$

in other words, straight-line propagation at constant speed, $|d\underset{\sim}{x}/dt| = 1$. By writing

(6.12)
$$x^\alpha = n^\alpha(t - t_0) + x_p^{\,\alpha} \; ,$$

and substituting into equations (6.9) and (6.10) we obtain post-Newtonian equations for the deviation $x_p^{\,\alpha}$ of the photon's path from uniform, straight-line motion:

(6.13)
$$\frac{d^2 x_p^{\,\alpha}}{dt^2} = (1 + \gamma) \left[\frac{\partial U}{\partial x^\alpha} - 2n^\alpha \underset{\sim}{n} \cdot \underset{\sim}{\nabla} U \right] \; ,$$

(6.14)
$$\underset{\sim}{n} \cdot d\underset{\sim}{x}_p/dt = - (1 + \gamma) U \; .$$

We will use these equations in Sect. 7 to derive expressions for the deflection of light by the Sun and for the time-delay of photons passing close to the Sun.

6˙3. Equations of Motion for Massive Bodies in the PPN Formalism

In Newtonian gravitational theory, massive, self-gravitating bodies, i.e. planets, obey very simple equations of motion, provided we neglect tidal couplings between them (torques, tidal dissipation effects, etc.). By defining an inertial mass and a center of mass for each body according to

(6.15)
$$m_i = \int_{\text{ith body}} \rho \; d\underset{\sim}{x} \; ,$$

(6.16)
$$X_i = (1/m_i) \int_i \rho \underset{\sim}{x} \; d\underset{\sim}{x} \; ,$$

one can show, using the Newtonian equation of continuity (eq. [4.18]) that

(6.17)
$$dm_i/dt = 0 \; ,$$

(6.18) $\qquad \underset{\sim}{v_i} = d\underset{\sim}{X_i}/dt = (1/m_i) \int_i \rho \underset{\sim}{v} \, d\underset{\sim}{x}$,

(6.19) $\qquad \underset{\sim}{a_i} = dv_i/dt = (1/m_i) \int_i \rho (d\underset{\sim}{v}/dt) \, d\underset{\sim}{x}$.

By using the Newtonian perfect-fluid equation of motion (eq. [4.18]) we obtain the following expression for $\underset{\sim}{a_i}$,

(6.20) $\qquad \underset{\sim}{a_i} = \underset{\sim}{\nabla} \mathfrak{U}$,

(6.21) $\qquad \mathfrak{U} = \sum_{j \neq i} \left(\frac{m_j}{r_{ij}} + \frac{1}{2} Q_j{}^{\alpha\beta} \frac{r_{ij}{}^{\alpha} r_{ij}{}^{\beta}}{r_{ij}{}^5} + \ldots \right)$,

where m_j is the inertial mass of the jth massive body, $Q_j{}^{\alpha\beta}$ is its quadrupole moment defined by

(6.22) $\qquad Q_j{}^{\alpha\beta} = \int_j \rho (3x^{\alpha} x^{\beta} - x^2 \delta^{\alpha\beta}) \, d\underset{\sim}{x}$,

and $\underset{\sim}{r_{ij}}$ is given by

(6.23) $\qquad \underset{\sim}{r_{ij}} = \underset{\sim}{X_i} - \underset{\sim}{X_j}$.

We now wish to generalize these equations to the post-Newtonian approximation, using the PPN formalism. Because there are many different "mass densities" in the post-Newtonian limit -- rest-mass of baryons ρ, mass-energy density $\rho(1 + \Pi)$, "conserved" density ρ^*, and so on -- there are a variety of possible definitions for inertial mass. We will use the following definition: Construct a coordinate frame completely surrounding the ith massive body and momentarily at rest with respect to its center of rest mass, i.e. a frame in which

$$(6.24) \qquad \underset{\sim}{v}_{\text{REST MASS}} = \frac{\int_i \rho^* \underset{\sim}{v} \, d\underset{\sim}{x}}{\int_i \rho^* \, d\underset{\sim}{x}} = 0 \quad .$$

Let this frame fall freely along a test-body trajectory in the field of all the other bodies except the ith. In the "interbody" region of this frame (far from the ith body, yet between the ith body and the other bodies), let the coordinate system be asymptotically Minkowskiian to some desired degree of accuracy (we ignore tidal gravitational forces produced by the external bodies). In this frame, the Newtonian gravitational potential is $\overline{U}$, produced (to our desired degree of accuracy) only by the ith body:

$$(6.25) \qquad \overline{U} = \int_i \frac{\rho' \, d\underset{\sim}{x}'}{|x - \underset{\sim}{x}|} \quad .$$

The velocity of each element of matter in the ith body relative to this frame is $\overline{\underset{\sim}{v}}(\underset{\sim}{x})$. Using these quantities we now define the inertial mass of the ith body:

$$(6.26) \qquad m_i \equiv \int_i \rho^* (1 + \tfrac{1}{2} \overline{v}^2 - \tfrac{1}{2} \overline{U} + \Pi) \, d\underset{\sim}{x} \quad .$$

Note that m_i is the total mass-energy of the body -- rest-mass of particles, plus kinetic, gravitational and internal energies -- as measured in a local, comoving "inertial" frame surrounding the body. As long as we ignore tidal forces on the ith body, then according to our discussion of conservation laws in the PPN formalism (cf. equation [4.101]), m_i is conserved to post-Newtonian accuracy, i.e.,

$$(6.27) \qquad dm_i/dt = 0 \quad .$$

This can also be shown by explicit calculation using equations (4.18),

(4.87), and (6.26).

We also define a <u>center of inertial mass</u>, valid in the PPN coordinate system:

(6.28)
$$\underset{\sim}{X}_i = (1/m_i) \int_i \rho^*(1 + \tfrac{1}{2}\bar{v}^2 - \tfrac{1}{2}\bar{U} + \Pi)\underset{\sim}{x}\; d\underset{\sim}{x} \; .$$

By making use of the equation of continuity for ρ^* (eq. [4.86] or [4.87]), and by using Newtonian equations of motion in any post-Newtonian terms, we obtain for the ith body's velocity $\underset{\sim}{v}_i$ relative to the PPN coordinate system:

(6.29)
$$v_i^\alpha \equiv dX_i^\alpha/dt$$
$$= (1/m_i) \int_i \left[\rho^*(1 + \tfrac{1}{2}\bar{v}^2 - \tfrac{1}{2}\bar{U} + \Pi)v^\alpha + p\bar{v}^\alpha - \tfrac{1}{2}\rho^*\bar{W}^\alpha\right] d\underset{\sim}{x} \; ,$$

where

(6.30)
$$\bar{W}^\alpha = \int_i \frac{\rho'\bar{\underset{\sim}{v}}' \cdot (\underset{\sim}{x} - \underset{\sim}{x}')(\underset{\sim}{x} - \underset{\sim}{x}')^\alpha}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} \; d\underset{\sim}{x}' \; .$$

The acceleration a_i^α is thus given by

$$a_i^\alpha \equiv dv_i^\alpha/dt$$

(6.31)
$$= \frac{1}{m_i}\left[\int_i \rho^*(1 + \tfrac{1}{2}\bar{v}^2 - \tfrac{1}{2}\bar{U} + \Pi)\frac{dv^\alpha}{dt}\; d\underset{\sim}{x}\right.$$
$$+ v_i^\beta \int_i \frac{\partial p}{\partial x^\beta}\bar{v}^\alpha \; d\underset{\sim}{x} + \int_i \left(\frac{\partial p}{\partial t}\bar{v}^\alpha - \frac{p}{\rho^*}\frac{\partial p}{\partial x^\alpha}\right) d\underset{\sim}{x}$$
$$\left. - \frac{1}{2}\frac{d}{dt}\int_i \rho^*\bar{W}^\alpha \; d\underset{\sim}{x} + \frac{1}{2}\mathcal{J}_i^\alpha - \frac{1}{2}\mathcal{J}_i^{*\alpha} + \mathcal{P}_i^\alpha\right],$$

where $\mathcal{J}_i^\alpha$, $\mathcal{J}_i^{*\alpha}$ and $\mathcal{P}_i^\alpha$ are determined purely by the internal

structure of the ith body. Formulas for these and other "internal" terms are given in Table 6.II. The acceleration of the center of inertial mass is thus more than just the weighted average of the accelerations of individual fluid elements; it includes accelerations which depend only on the internal structure of the body or on the body's velocity (v_i) relative to the PPN coordinate system. The meaning of many of these additional terms will become clear later.

We now evaluate the first term in equation (6.31) using the PPN perfect-fluid equations of motion. We substitute the post-Newtonian expressions for T^{ij} (Table 4.I) and for the Christoffel symbols Γ^i_{jk} (Table 6.I) into the stressed matter equation of motion

$$(6.32) \qquad T^{ij}_{\ ;j} = 0$$

and rewrite the equation in terms of the conserved density ρ^*. The resulting equation of motion for each element of matter is given by

$$
\rho^* \frac{dv^\alpha}{dt} = \rho^* \frac{\partial U}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha} [p(1 + 3\gamma U)] + \frac{\partial p}{\partial x^\alpha}\left(\frac{1}{2} v^2 + \Pi + \frac{p}{\rho^*}\right)
$$

$$
- \rho^* \frac{d}{dt} [(2\gamma + 2)Uv^\alpha - \frac{1}{2}(4\gamma + 4 + \alpha_1)V^\alpha - \frac{1}{2}\alpha_1 Uw^\alpha]
$$

$$
+ v^\alpha \left(\rho^* \frac{\partial U}{\partial t} - \frac{\partial p}{\partial t}\right) - \frac{1}{2}(1 + \alpha_2 - \zeta_1)\rho^* \frac{\partial}{\partial t}(V^\alpha - W^\alpha)
$$

$$(6.33)$$

$$
- \frac{1}{2}\rho^* [(4\gamma + 4 + \alpha_1)v^\beta + (\alpha_1 - 2\alpha_3)w^\beta] V_{\beta,\alpha}
$$

$$
+ \rho^* \frac{\partial}{\partial x^\alpha} [2\Phi - \frac{1}{2}\zeta_1 G - \frac{1}{2}\alpha_2 w^\beta w^\gamma U_{\beta\gamma} + \alpha_2 w^\beta(V_\beta - W_\beta)]
$$

$$
+ \rho^* \frac{\partial U}{\partial x^\alpha} [\gamma v^2 - \frac{1}{2}\alpha_1 \underset{\sim}{w} \cdot \underset{\sim}{v} + \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)w^2 - (2\beta - 2)U + 3\gamma \frac{p}{\rho^*}] \ ,
$$

where Φ is given in Table 6.I.

We now substitute this expression for $\rho^{*} d\underset{\sim}{v}/dt$ into equation (6.31), and perform the integration, using Newtonian equations where necessary to simplify any post-Newtonian terms. Considerable simplification of the equations results if we assume that the internal structure of each massive body is stationary. This is a reasonable approximation for the solar system, since any <u>secular</u> changes in the structure of the Sun or planets occur over time scales much longer than orbital periods. Moreover, any periodic changes in internal structure, such as tidal deformations of the Earth, can be shown to have negligible post-Newtonian orbital effects when averaged over several periods of oscillation. This assumption of stationary structure allows us to use several Newtonian virial relations to simplify post-Newtonian expressions. These relations, easily derived using equations (4.18) and the formulas given in Table 6.II, have the form for each massive body:

$$(6.34) \qquad \frac{d^2}{dt^2} I^{\alpha\beta} = 0 = 2\overline{\overline{U}}^{\alpha\beta} + \Omega^{\alpha\beta} + \delta^{\alpha\beta}P \ ,$$

$$(6.35) \qquad \frac{d^2}{dt^2} I = 0 = 2\overline{\overline{U}} + \Omega + 3P \ ,$$

$$(6.36) \qquad 2\frac{d}{dt} \overline{\overline{U}}^{\alpha\beta} = 0 = H^{\alpha\beta} + H^{\beta\alpha} - \int \overline{v}^{\alpha} \frac{\partial p}{\partial x^{\beta}} d\underset{\sim}{x} - \int \overline{v}^{\beta} \frac{\partial p}{\partial x^{\alpha}} d\underset{\sim}{x} \ ,$$

$$(6.37) \qquad \frac{d}{dt}\Omega^{\alpha\beta} = 0 = H^{\alpha\beta} + H^{\beta\alpha} - 3K^{\alpha\beta} \ ,$$

$$(6.38) \qquad \frac{d}{dt}\Omega = 0 = - H^{\beta}{}_{\beta} \ ,$$

$$(6.39) \qquad \frac{d}{dt} P = 0 = \int \frac{\partial p}{\partial t} d\underset{\sim}{x} \ ,$$

$$\text{(6.40)} \qquad \frac{d}{dt}\int \rho^* \overline{W}^\alpha \, d\underset{\sim}{x} = 0 = -\mathbb{1}^\alpha - \mathcal{J}^\alpha + \mathcal{J}^{*\alpha} + 3\mathcal{J}^{**\alpha} - \mathbb{III}^{*\alpha} - \mathcal{P}^\alpha \ ,$$

$$\text{(6.41)} \qquad \frac{d}{dt}\int \rho^* \overline{V}^\alpha \, d\underset{\sim}{x} = 0 = \mathcal{J}^\alpha + \mathcal{J}^{*\alpha} + \mathbb{III}^\alpha + \mathcal{P}^\alpha .$$

The final form of the equation of motion is

$$a_i^\alpha = -\left(1/m_i\right)\!\left[\tfrac{1}{2}(\alpha_3 + \zeta_1)\mathbb{1}_i^\alpha + \zeta_1\!\left(\mathcal{J}_i^\alpha - \tfrac{3}{2}\mathcal{J}_i^{**\alpha}\right)\right.$$

$$\left. + \zeta_2 \mathbb{III}_i^\alpha + \zeta_3 \mathbb{E}_i^\alpha + 3\zeta_4 \mathcal{P}_i^\alpha \right]$$

$$- \alpha_3 (w + v_i)_\beta H_i^{\beta\alpha}/m_i$$

$$+ \frac{(m_p)_i^{\alpha\beta}}{m_i} \frac{\partial \mathbb{U}}{\partial x_i^\beta}$$

$$+ \sum_{j\neq i} \frac{m_j r_{ij}^\alpha}{r_{ij}^3} \left\{ (2\gamma+2\beta)\sum_{k\neq i} \frac{m_k}{r_{ik}} + (2\beta-1-\zeta_2)\sum_{k\neq ij} \frac{m_k}{r_{jk}} \right.$$

$$+ (2\gamma+2\beta+1+\tfrac{1}{2}\alpha_1-\zeta_2)\frac{m_i}{r_{ij}} - \tfrac{1}{2}(1+\alpha_2-\zeta_1)\sum_{k\neq ij} \frac{m_k \underset{\sim}{r}_{ij}\cdot \underset{\sim}{r}_{jk}}{r_{jk}^3}$$

$$- \gamma v_i^2 + \tfrac{1}{2}(4\gamma+4+\alpha_1)\underset{\sim}{v}_i \cdot \underset{\sim}{v}_j - \tfrac{1}{2}(2\gamma+2+\alpha_2+\alpha_3)v_j^2$$

$$\text{(6.42)} \qquad - \tfrac{1}{2}(\alpha_2+\alpha_3-\alpha_1)w^2 + \tfrac{1}{2}\alpha_1 \underset{\sim}{w}\cdot \underset{\sim}{v}_i + \tfrac{1}{2}(\alpha_1-2\alpha_2-2\alpha_3)\underset{\sim}{w}\cdot \underset{\sim}{v}_j$$

$$\left. + \tfrac{3}{2}(1+\alpha_2)(\underset{\sim}{v}_j\cdot \hat{\underset{\sim}{r}}_{ij})^2 + \tfrac{3}{2}\alpha_2(\underset{\sim}{w}\cdot \hat{\underset{\sim}{r}}_{ij})^2 + 3\alpha_2(\underset{\sim}{w}\cdot \hat{\underset{\sim}{r}}_{ij})(\underset{\sim}{v}_j\cdot \hat{\underset{\sim}{r}}_{ij}) \right\}$$

$$- \tfrac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1)\sum_{j\neq i} \frac{m_j}{r_{ij}} \sum_{k\neq ij} \frac{m_k r_{jk}^\alpha}{r_{jk}^3}$$

$$+ \sum_{j\neq i} \frac{m_j}{r_{ij}^3} \underset{\sim}{r}_{ij}\cdot \left[(2\gamma+2)\underset{\sim}{v}_i - (2\gamma+1)\underset{\sim}{v}_j\right] v_i^\alpha$$

$$- \tfrac{1}{2}\sum_{j\neq i} \frac{m_j}{r_{ij}^3} \underset{\sim}{r}_{ij}\cdot \left[(4\gamma+4+\alpha_1)\underset{\sim}{v}_i - (4\gamma+2-\alpha_1-2\alpha_2)\underset{\sim}{v}_j + 2\alpha_2\underset{\sim}{w}\right] v_j^\alpha$$

$$- \tfrac{1}{2}\sum_{j\neq i} \frac{m_j}{r_{ij}^3} \underset{\sim}{r}_{ij}\cdot \left[\alpha_1\underset{\sim}{v}_i - (\alpha_1-2\alpha_2)\underset{\sim}{v}_j + 2\alpha_2\underset{\sim}{w}\right] w^\alpha \ ,$$

where

(6.43)
$$\tilde{r}_{ij} = r_{ij}/r_{ij} .$$

The first six terms in equation (6.42) $\mathfrak{I}_i^\alpha$, $\mathcal{J}_i^\alpha$, $\mathcal{J}_i^{**\alpha}$, $\mathbb{M}_i^\alpha$, $\mathfrak{E}_i^\alpha$ and $\mathcal{P}_i^\alpha$ depend only on the internal structure of the ith massive body, and thus represent <u>self-accelerations</u> of the body's center of inertial mass. Such self-accelerations are associated with breakdowns in conservation of total momentum, since they depend on the PPN "conservation law" parameters α_3, ζ_1, ζ_2, ζ_3 and ζ_4. In any Semi-Conservative theory of gravity, <u>i.e.</u> one which has

(6.44)
$$\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 ,$$

these self-accelerations are absent. Note too, that spherically symmetric massive bodies feel no self-accelerations regardless of the theory of gravity, since for them the terms $\mathfrak{I}_i^\alpha$, $\mathcal{J}_i^\alpha$, $\mathcal{J}_i^{**\alpha}$, $\mathbb{M}_i^\alpha$, $\mathfrak{E}_i^\alpha$, and $\mathcal{P}_i^\alpha$ are identically zero.

The next term in equation (6.42):

(6.45)
$$-\alpha_3 (w + v_i)_\beta H_i^{\beta\alpha}/m_i ,$$

is a "self acceleration" which involves the massive body's motion relative to the mean Universal rest-frame. It depends on the "conservation-law-preferred-frame" parameter α_3, which is zero in any Semi-Conservative theory of gravity. For any static massive body, $\bar{v} = 0$ and thus $H_i^{\alpha\beta}$ is zero, but for a body which rotates uniformly with angular velocity λ,

(6.46)
$$\bar{v} = \lambda \times (x - X_i) ,$$

and

$$H_i^{\beta\alpha} = \epsilon^{\beta\gamma\delta}\lambda_\gamma \int_i \rho^* \rho^{*\prime} \frac{(x'-X_i)_\delta (x-x')^\alpha}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

<div align="center">(6.47)</div>

$$= \epsilon^{\beta\gamma\delta}\lambda_\gamma (\Omega_i)^\alpha{}_\delta \ .$$

For a nearly spherical body, the isotropic part of $\Omega^\alpha{}_\delta$ makes the dominant contribution to equation (6.47), i.e.

$$(\Omega_i)^\alpha{}_\delta \approx \frac{1}{3} \delta^\alpha{}_\delta \Omega_i \ ,$$

<div align="center">(6.48)</div>

$$H_i^{\beta\alpha} \approx \frac{1}{3} \epsilon^{\alpha\beta\gamma}\lambda_\gamma \Omega_i \ .$$

Then the acceleration term in equation (6.45) becomes

<div align="center">(6.49)</div>

$$- \alpha_3 (\Omega_i/m_i)(\underset{\sim}{w} + \underset{\sim}{v}_i) \times \underset{\sim}{\lambda}_i \ .$$

In Section 7 we will see that this term may produce strikingly large observable effects in the solar system, if α_3 is different from zero.

The next term in equation (6.42),

<div align="center">(6.50)</div>

$$\frac{(m_p)_i^{\alpha\beta}}{m_i} \frac{\partial \mathfrak{H}}{\partial x_i^\beta} \ ,$$

is the "quasi-Newtonian" acceleration of the massive body. Here $(m_p)_i^{\alpha\beta}$ is the "passive gravitational mass tensor" (Nordtvedt 1968b, Will 1971a), given by

<div align="center">(6.51)</div>

$$(m_p)_i^{\alpha\beta} = m_i \left\{ \delta^{\alpha\beta} [1 + (4\beta - \gamma - 3 - \alpha_1 + \alpha_2 - \zeta_1)\Omega_i/m_i] \right.$$
$$\left. - (\alpha_2 + \zeta_2 - \zeta_1)\Omega_i^{\alpha\beta}/m_i \right\} \ ,$$

and $\mathfrak{H}$ is the "quasi-Newtonian potential", given by

<div align="center">105</div>

$$(6.52) \qquad \mathfrak{U}(\underset{\sim}{X}_i) = \sum_{j \neq i} \left\{ \frac{[m_a(\underset{\sim}{r}_{ij})]_j}{r_{ij}} + \frac{1}{2} Q_j^{\alpha\beta} \frac{r_{ij}^{\alpha} r_{ij}^{\beta}}{r_{ij}^5} \right\},$$

where m_a is the "active gravitational mass" of the jth body (which may be a function of position):

$$
\begin{aligned}
(6.53) \qquad [m_a(r_{ij})]_j = m_j[\, 1 & + (4\beta - \gamma - 3 - \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\zeta_1 - 2\zeta_2)\Omega_j/m_j \\
& + \zeta_3 E_j/m_j - (\tfrac{3}{2}\alpha_3 + 3\zeta_4 + \zeta_1)P_j/m_j \\
& + \tfrac{1}{2}\zeta_j(\Omega_j^{\beta\gamma}/m_j)(\hat{r}_{ij}^{\beta}\hat{r}_{ij}^{\gamma})\,].
\end{aligned}
$$

In equation (6.52), $Q_j^{\alpha\beta}$ is again the quadrupole moment of the jth body, defined by

$$(6.54) \qquad Q_j^{\alpha\beta} = \int_j \rho^*(3x^{\alpha}x^{\beta} - x^2\delta^{\alpha\beta})\, d\underset{\sim}{x} + \begin{pmatrix} \text{post-Newtonian} \\ \text{corrections} \end{pmatrix}.$$

In Newtonian theory, the active gravitational mass, the passive gravitational mass and the inertial mass are the same, hence each massive body's acceleration is independent of its mass or structure ("Equivalence principle" -- see Sect. 3). However, according to equations (6.50) and (6.51), passive gravitational mass need not be equal to inertial mass in a given metric theory of gravity (and in fact may be anisotropic); their difference depends on several PPN parameters, and on the gravitational self-energy (Ω and $\Omega_{\alpha\beta}$) of the body. This "breakdown in the Equivalence Principle" for massive bodies has been called the "Nordtvedt Effect" after its discoverer (Nordtvedt 1968b; see also Dicke 1970, and Will 1971a). The observable consequences of the "Nordtvedt effect" will be discussed in Sect. 7. The existence of this "Nordtvedt effect" does not violate the Eötvös-Dicke-Braginsky experiment (Sect. 3) since

the laboratory-sized bodies used in those experiments had negligible self-gravity, i.e.

$$|\Omega/m|_{\text{LABORATORY BODIES}} < 10^{-39}.$$

According to equation (6.53), "active" gravitational mass for massive bodies may also differ from inertial mass and passive gravitational mass. In Newtonian gravitational theory, uniform center-of-mass motion of an isolated system is a result of the law "action equals reaction", i.e. of the law "active gravitational mass equals passive gravitational mass". In the PPN formalism one can still use such Newtonian language to describe the "quasi-Newtonian" acceleration, equation (6.50). From Subsect. 4·7, we know that uniform center-of-mass motion is a property of Fully-Conservative theories of gravity, i.e. theories which have

(6.55) $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.$

By substituting these parameter values into equations (6.51) and (6.53), we find that for Fully-Conservative theories, the active and passive masses are indeed equal and isotropic, and are given by

(6.56) $m_a = m_p = m[1 + (4\beta - \gamma - 3)(\Omega/m)]$.

In general relativity ($\gamma = \beta = 1$), the active, passive and inertial masses are identical -- there is no Nordtvedt effect in general relativity.

The remaining terms in equation (6.42) we will call N-Body accelerations. These are the post-Newtonian corrections to the Newtonian equations of motion which would result from treating each body as a test

107

body moving along a geodesic of the PPN metric produced by all the other bodies, assumed to be point masses. It is these terms which produce the "classical" perihelion shift of the planets, as well as a host of other effects, to be examined in Sect. 7. For the case of general relativity, the N-Body terms in equation (6.42) are in agreement with the equations obtained by de Sitter (1916) [once a crucial error in de Sitter's work has been corrected], Einstein, Infeld and Hoffmann (1938), Levi-Civita (1964), and Fock (1964). In Dicke-Brans-Jordan theory, the N-Body terms reduce to those obtained by Estabrook (1969).

6·4. Equations of Motion for Spinning Bodies in the PPN Formalism

The motion of spinning bodies (gyroscopes, planets, elementary particles) in curved spacetime has been a subject of considerable research for many years. This research has been aimed at discovering (i) how a body's intrinsic angular momentum (spin) alters its trajectory (deviations from geodesic motion), and (ii) how a body's motion in curved spacetime alters its spin.

No really satisfactory solution is available for the first problem, because of the difficulties in defining a center of mass of a spinning body in curved spacetime. The most successful attempts at a solution have been made by Mathisson (1937), Papapetrou (1951), Corinaldesi and Papapetrou (1951), Tulczyjew and Tulczyjew (1962) and Dixon (1964, 1970). The central conclusion of these calculations has been that the intrinsic spin S^{ij} (i.e. J^{ij} evaluated in the body's "center-of-mass" frame) of a body should produce deviations from geodesic motion of the form

(6.57) $m \delta a^i \sim S^{j\ell} u^k R^i_{\ jk\ell}$

where u^k is the body's four-velocity, and $R^i_{\ jk\ell}$ is the Riemann curvature tensor. However, these calculations differ greatly in details and interpretation. For a spinning body moving with velocity v in a Newtonian gravitational potential $U \sim M/r$, these deviations are, in order of magnitude:

$$\delta a \sim (|S^{\alpha\beta}|/m)|\underset{\sim}{v}|(M/r^3)$$

(6.58)

$$\sim (b^2 \lambda/r)(M/r)^{1/2} a_{NEWTONIAN} \ ,$$

where b is the radius of the body, and λ its rotational angular velocity. For a planet rotating near break-up velocity ($\lambda^2 \sim m/b^3$), we have

$$\delta a \lesssim (m/b)^{1/2}(M/r)^{1/2}(b/r)a_{NEWTONIAN}$$

(6.59)

$$\lesssim 10^{-12} a_{NEWTONIAN} \ ,$$

and for a 4-cm-radius gyroscope orbiting the Earth (frequency 200 rps),

(6.60) $\delta a \lesssim 10^{-20} a_{NEWTONIAN} \ .$

Thus, for the most part, spin-induced deviations from geodesic motion can be ignored in the solar system. In our derivation of massive-body equations of motion (Subsect. 6·3), we ignored the effects of tidal gravitational forces (Riemann curvature tensor); thus our equation of motion (eq. [6.42]) does not include the effects of spin.

It is problem (ii), the effects of a body's motion on its spin which is well understood. All calculations to date have shown that, as long as the direct effects of tidal gravitational forces (Riemann curvature tensor)

on the spinning body can be neglected, <u>the spin **S** is Fermi-Walker</u>

<u>Transported along the body's world line.</u> Here, the four-vector **S**

is given by

(6.61) $$S^i = \frac{1}{2} \epsilon^{ijk\ell} S_{k\ell} u_j \ ,$$

(6.62) $$u^i S_i = 0 \ .$$

The equation of Fermi-Walker transport is then

(6.63) $$u^j S^i_{\ ;j} = u^i (a^j S_j) \ ,$$

where a^i is the body's four-acceleration, given by

(6.64) $$a^i = u^j u^i_{\ ;j} \ .$$

The reader is referred to Misner, Thorne and Wheeler (1972) for more

detailed discussion of Fermi-Walker transport.

It is convenient to analyse equation (6.63) in a local inertial

frame which is comoving with the body. The basis vectors of this

frame are related to the basic vectors of our PPN coordinate system

by a Lorentz transformation plus a normalization, and are given by

(see Misner, Thorne and Wheeler [1972])

$$e_{\hat{0}}^{\ i} = u^i \ ,$$

(6.65) $$e_{\hat{\alpha}}^{\ 0} = v_\alpha + O(3) \ ,$$

$$e_{\hat{\alpha}}^{\ \beta} = (1 - \gamma U) \delta_\alpha^{\ \beta} + \frac{1}{2} v_\alpha v_\beta + O(4) \ .$$

Then, because of equation (6.62), the spin is a purely spatial vector in

the comoving orthonormal frame, i.e.

(6.66) $S_{\hat{0}} = e_{\hat{0}}{}^{i} S_{i} = u^{i} S_{i} = 0$.

We now calculate the precession of the spatial components of the spin $S_{\hat{\alpha}}$. Since $e_{\hat{\alpha}}{}^{i} u_{i} = 0$, we have, from equation (6.63),

$$0 = e_{\hat{\alpha}}{}^{i} u^{j} S_{i;j}$$

(6.67)

$$= u^{j} S_{\hat{\alpha};j} - S_{i} u^{j} e_{\hat{\alpha}}{}^{i}{}_{;j} .$$

Since $S_{\hat{\alpha}}$ is a scalar (scalar product of two vectors), we have:

(6.68) $u^{j} S_{\hat{\alpha};j} = u^{j} S_{\hat{\alpha},j} = dS_{\hat{\alpha}}/ds$.

The second term in equation (6.67) is most easily evaluated in the PPN coordinate frame. Using equations (6.65), we first obtain relations between S^{i} and $S^{\hat{i}}$:

(6.69) $S_{0} = - v_{\alpha} S_{\hat{\alpha}} + O(3) S_{\hat{\alpha}}$,

(6.70) $S_{\alpha} = (1 + \gamma U) S_{\hat{\alpha}} + \frac{1}{2} v_{\alpha} (v_{\beta} S_{\hat{\beta}}) + O(4) S_{\hat{\beta}}$.

Then after some simplification, we get, to post-Newtonian order,

(6.71) $$\frac{dS_{\hat{\alpha}}}{ds} = S_{\hat{\beta}} [v_{[\alpha} a_{\beta]} + g_{0[\beta,\alpha]} + (2\gamma + 1) v_{[\beta} U_{,\alpha]}] ,$$

where a_{i} is the body's four-acceleration. This can be rewritten in three-dimensional vector notation:

(6.72) $d\underset{\sim}{S}/ds = \underset{\sim}{\Omega} \times \underset{\sim}{S}$,

(6.73) $\underset{\sim}{\Omega} = \frac{1}{2} [\underset{\sim}{v} \times \underset{\sim}{a} + \nabla \times \underset{\sim}{h} + (2\gamma + 1) \underset{\sim}{v} \times \nabla U]$,

(6.74) $\underset{\sim}{h} = g_{0\alpha} \underset{\sim}{e_{\hat{\alpha}}}$.

111

In equation (6.71) it does not matter whether the vectors entering into $\underset{\sim}{\Omega}$ are evaluated in the PPN coordinate frame or in the comoving frame since their spatial basis vectors differ only by terms of $O(2)$. We have calculated the precession of the spin relative to a comoving frame which is rotationally tied to the PPN coordinate frame, whose axes are fixed relative to the distant galaxies. Thus we have calculated the spin's precession angular velocity $\underset{\sim}{\Omega}$ relative to a frame fixed with respect to the distant galaxies. We will discuss the observable consequences of this precession in Sect. 7.

6·5. Cavendish Experiments and the Locally-Measured Newtonian Gravitational Constant

Here we derive an equation which is not really an equation of motion, but is nevertheless a fundamental result in the PPN formalism. Since the formulation of the Dicke-Brans-Jordan scalar-tensor theory, considerable interest has focused on the constancy of the Newtonian gravitational constant G. One line of investigation examines the effect of the evolution of the Universe on the value of G as measured in the mean rest-frame of the Universe far from any local distribution of matter. Several theories predict a secular rate of change of this "constant" (see Sect. 5), whose "present" value has been set equal to unity by our choice of units. A second line of investigation examines the effect of nearby matter (planets and stars) and of motion through the Universe on the value of G measured in laboratory Cavendish experiments (Brans 1962a, Nordtvedt 1970b, Will 1971d). This value is normally called the "locally measured" gravitational constant. In an

idealized version of such a Cavendish experiment one measures the relative acceleration of two bodies as a function of their masses and of the distance between them. Distances and times are measured by means of physical rods and atomic clocks at rest in the laboratory. The gravitational constant G is then identified as that number with dimensions $cm^3 g^{-1} sec^{-2}$ which appears in Newton's law of gravitation for the two bodies.

Since gravimeters are extremely sensitive devices for measuring accelerations, we shall use as our Cavendish experiment a gravimeter at rest on the surface of the Earth. A calculation for Cavendish experiments with both bodies of laboratory sizes would proceed similarly and would produce the same final answer.

We idealize our Earth-gravimeter Cavendish experiment as follows: a body of mas m_1 (Earth) is freely falling through spacetime. A test body with negligible mass (gravimeter) is moving through space-time, maintained at a constant proper distance r_p from the Earth by a four-acceleration $\mathbf{F}$. An invariant "radial" unit four-vector $\mathbf{E}_r$, carried by the gravimeter points directly toward the center of mass of the Earth. Then, according to Newton's law of gravitation, the radial component of the acceleration as measured by the gravimeter is given by

(6.75) $$\mathbf{F} \cdot \mathbf{E}_r = Gm_1/r_p^{\,2} + r_p(D\mathbf{E}_r/D\tau) \cdot (D\mathbf{E}_r/D\tau) \ ,$$

where $D/D\tau$ is the covariant derivative with respect to the gravimeter's proper time τ along the gravimeter's world line. The last term in equation (6.75) is simply the centrifugal acceleration, defined in an invariant way (except for corrections of order 10^{-9} $m_1/r_p^{\,2}$ which

113

we ignore; see below). Since $\mathbf{F} \cdot \mathbf{E}_r$ is an invariant quantity, we can calculate it in the PPN coordinate system and then use equation (6.75) to identify the locally-measured gravitational constant G.

Throughout this calculation, we will neglect any terms which produce accelerations of 10^{-9} g or smaller, as measured by the gravimeter. This amounts to neglecting Earth-generated post-Newtonian accelerations of the gravimeter, post-Newtonian corrections to the centrifugal acceleration and to tidal accelerations, and other, more complicated accelerations. We do this because 10^{-9} g seems to be the current limit of reliable Earth-bound gravimeter data. For further discussion of relativistic affects which produce accelerations smaller than 10^{-9} g see Nordtvedt (1971a) and Will (1971d). Of course all these neglected accelerations would be even more negligible ($<< 10^{-9}$ g) in a laboratory-type Cavendish experiment.

We do the calculation in a PPN coordinate system which is momentarily at rest with respect to the Earth. The PPN metric is given by the expression in Table 4.I, where now the velocity $\underset{\sim}{w}$ is the Earth's velocity relative to the mean rest-frame of the Universe (denoted $\underset{\sim}{w}_1$). At any given moment of PPN coordinate time t, each body, denoted by a subscript j, has a three-vector position denoted $\underset{\sim}{x}_j(t)$ and a velocity denoted $\underset{\sim}{v}_j(t)$. We treat the gravimeter (j = 0) and the sun, planets and stars (j = 2,3,...) as point masses, and the Earth (j = 1) as a spherically symmetric distribution of perfect fluid, but with negligible self-gravity. [See Nordtvedt (1971a) for the result of treating the Earth as a massive, self-gravitating body, with resultant accelerations of magnitude $\lesssim 10^{-9}$ g due to the equivalence-principle breakdown, and accelerations dependent

114

on the structure of the Earth.] We will separate the Newtonian gravitational potential U due to the Earth from that due to the other planets and the sun

$$(6.76) \qquad U(\underset{\sim}{x}) = U_1(\underset{\sim}{x}) + \sum_{k \neq 1} m_k / r_k \; ,$$

where

$$(6.77) \qquad \underset{\sim}{r_k} = \underset{\sim}{x} - \underset{\sim}{x_k} \; , \qquad r_k = |\underset{\sim}{x} - \underset{\sim}{x_k}| = \left[\sum_\alpha (x - x_k)^\alpha (x - x_k)^\alpha \right]^{1/2} \; .$$

We first calculate the proper distance r_p from the gravimeter to the center of the Earth. We use a physically reasonable definition for r_p -- namely one half the proper time (as measured by the gravimeter) required for a photon to travel from the gravimeter to the center of the Earth and back:

$$(6.78) \qquad r_p = \frac{1}{2} \int_{t_e}^{t_r} \left\{ 1 - U[\underset{\sim}{x_o}(t)] - \frac{1}{2} v_o^2(t) \right\} dt \; ,$$

where t_e and t_r are the PPN coordinate times corresponding to emission and reception of the light signal, and the integral is taken along the gravimeter's world line. The round trip time $(t_r - t_e)$ is obtained by integrating the geodesic equations (6.13) and (6.14) for the light signal along its path from its emission at $\underset{\sim}{x_o}(t_e)$ to its deflection at the center of the Earth $\underset{\sim}{x_1}(t_d)$ and back to its reception at $\underset{\sim}{x_o}(t_r)$, and is given by (see also Subsect. 7·2)

$$(6.79) \qquad t_r - t_e = |\underset{\sim}{x_o}(t_r) - \underset{\sim}{x_1}(t_d)| + |\underset{\sim}{x_1}(t_d) - \underset{\sim}{x_o}(t_e)|$$
$$+ (1 + \gamma) \int_{\sigma_e}^{\sigma_r} U[\underset{\sim}{x}(\sigma)] \, d\sigma + O(3) \; ,$$

where σ is PPN coordinate time t along the path of the light signal. We take into account the motion of the gravimeter and the Earth during the time of transit of the signal according to

$$x_0^{\alpha}(t_r) \approx x_0^{\alpha}(t_e) + (t_r - t_e)v_0^{\alpha} + \frac{1}{2}(t_r - t_e)^2 \, dv_0^{\alpha}/dt,$$

(6.80)

$$x_1^{\alpha}(t_d) \approx x_1^{\alpha}(t_e) + \frac{1}{2}(t_d - t_e)^2 \, dv_1^{\alpha}/dt \; .$$

The velocity v_0^{α} and accelerations dv_0^{α}/dt and dv_1^{α}/dt are all to be evaluated at $t = t_e$. Equations (6.78), (6.79), and (6.80) lead to the final result

(6.81)
$$r_p = r_{10}\left[1 - \frac{\underset{\sim}{v_0} \cdot \underset{\sim}{r_{10}}}{r_{10}} + \gamma \sum_{k \neq 1,0} \frac{m_k}{r_{1k}}\right] + \delta r_p \; ,$$

where

(6.82)
$$\delta r_p = r_{10}\left[-(d/dt)\underset{\sim}{v_0} \cdot \underset{\sim}{r_{10}} - \frac{1}{2}v_0^2 - U_1(\underset{\sim}{x_0}) \right.$$
$$\left. + \frac{1}{2}\gamma \sum_{k \neq 1,0} \frac{m_k}{r_{1k}^3}(\underset{\sim}{r_{10}} \cdot \underset{\sim}{r_{1k}}) + \frac{1}{2}\frac{(1+\gamma)}{r_{10}}\int_{\sigma_e}^{\sigma_r} U_1[x(\sigma)]\,d\sigma + O(3)\right].$$

The proper distance r_p is to be kept constant (by the force which holds the gravimeter at rest on the surface of the Earth). Thus

(6.83)
$$dr_p/d\tau = dr_p/dt = 0 \; ,$$
$$d^2r_p/d\tau^2 = d^2r_p/dt^2 = 0 \; .$$

Equations (6.81) and (6.82) along with equations (6.83) then give the following results:

(6.84)
$$\underset{\sim}{v_0} \cdot \underset{\sim}{r_{10}}/r_{10} = O(3),$$

116

$$(6.85) \qquad \left(\frac{d\underset{\sim}{v}_1}{dt} - \frac{d\underset{\sim}{v}_0}{dt}\right) \cdot \frac{\underset{\sim}{r}_{10}}{r_{10}} + \frac{v_0^2}{r_{10}} + O(3) = 0 \ .$$

We also find that the term δr_p in equation (6.81) leads only to gravimeter-measured accelerations of less than 10^{-9} g, and can thus be ignored. It is equation (6.85) which we will use to determine the acceleration measured by the gravimeter.

Assume that the Earth follows a geodesic of spacetime (neglect self-accelerations and equivalence-principle violations), but that the four-acceleration of the gravimeter is F :

$$(6.86) \qquad u^j_{Earth} u^i_{Earth;j} = 0 \ ,$$

$$(6.87) \qquad u^j_{gravim} u^i_{gravim;j} = F^i \ ,$$

$$(6.88) \qquad F_i u^i_{gravim} = 0 \ .$$

In PPN coordinates, equations (6.86), (6.87), and (6.88) may be written

$$(6.89) \qquad \frac{dv^\alpha_1}{dt} + \Gamma^\alpha_{oo}(\underset{\sim}{x}_1) = 0 \ ,$$

$$(6.90) \qquad \frac{dv_o^\alpha}{dt} + \Gamma^\alpha_{bc}(\underset{\sim}{x}_o) v_o^b v_o^c - \Gamma^o_{bc}(\underset{\sim}{x}_o) v_o^b v_o^c v_o^\alpha$$
$$= (d\tau_{gravim}/dt)^2 (F^\alpha - v_o^\alpha F^o),$$

$$(6.91) \qquad F^o = \underset{\sim}{F} \cdot \underset{\sim}{v}_o + O(4) \ ,$$

where $v_1^o = v_o^o = dt/dt = 1$, and

$$(6.92) \qquad (d\tau_{gravim}/dt)^2 = 1 - 2 \sum_{k \neq 1,0} (m_k/r_{1k}) - v_o^2 + (10^{-9} \text{ g terms}) \ .$$

By making use of the PPN Christoffel symbols (Table 6.I) along with

117

equations (6.89), (6.90), and (6.91), and using the Newtonian equations

of motion to simplify any post-Newtonian terms, and as usual ignoring

small force terms, we get from equation (6.85):

$$\frac{\underset{\sim}{F} \cdot \underset{\sim}{r}_{10}}{r_{10}} = \sum_{k\neq 1,0} \frac{m_k r_{10}{}^{\alpha} r_{10}{}^{\beta} (3 r_{1k}{}^{\alpha} r_{1k}{}^{\beta} - r_{1k}{}^{2} \delta^{\alpha\beta})}{r_{1k}{}^{5} r_{10}} + \frac{v_o{}^2}{r_{10}}$$

(6.93)
$$+ \frac{\underset{\sim}{r}_{10}}{r_{10}} \cdot \underset{\sim}{\nabla} U_1(\underset{\sim}{x}_o) \left[1 - (4\beta - \gamma - 3 - \zeta_2) \sum_{k\neq 10} \frac{m_k}{r_{1k}} \right.$$

$$\left. - \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) w_1{}^2 \right] - \frac{1}{2} \alpha_2 w_1{}^{\beta} w_1{}^{\gamma} \frac{\underset{\sim}{r}_{10}}{r_{10}} \cdot \underset{\sim}{\nabla} U_{1\beta\gamma}(\underset{\sim}{x}_o) \ .$$

For a spherically symmetric Earth, it is straightforward to show that

(6.94)
$$\underset{\sim}{\nabla} U_1(\underset{\sim}{x}_o) = - \frac{M \underset{\sim}{r}_{10}}{r_{10}{}^3} \left(1 - 3\gamma \sum_{k\neq 10} \frac{m_k}{r_{1k}} \right) + O(6) \ ,$$

$$U_{1\beta\gamma}(\underset{\sim}{x}_o) = - \frac{M}{r_{10}{}^5} (3 r_{10}{}^{\beta} r_{10}{}^{\gamma} r_{10}{}^{\alpha} - r_{10}{}^{2} r_{10}{}^{\gamma} \delta^{\alpha\beta} - r_{10}{}^{2} r_{10}{}^{\beta} \delta^{\alpha\gamma})$$

(6.95)
$$+ \frac{I}{r_{10}{}^7} (5 r_{10}{}^{\beta} r_{10}{}^{\gamma} r_{10}{}^{\alpha} - r_{10}{}^{2} r_{10}{}^{\gamma} \delta^{\alpha\beta} + r_{10}{}^{2} r_{10}{}^{\beta} \delta^{\alpha\beta}$$

$$- r_{10}{}^{2} r_{10}{}^{\alpha} \delta^{\alpha\beta}) + O(4) \ ,$$

(neglecting terms leading to forces smaller than 10^{-9} g), where M and

I are the rest mass of particles and the spherical moment of inertia of

the Earth.

We must now compute the invariant radial unit vector $\mathbf{E}_r$.

The tangent four-vector to the photon path λ^{α} at the moment of

emission by the gravimeter is given according to the photon's geodesic

equation by (ignoring terms leading to small forces)

118

$$\lambda^o = 1 \ ,$$

(6.96)

$$\lambda^\alpha = (r_{10}{}^\alpha/r_{10}) \left[1 - (1+\gamma) \sum_{k \neq 1, 0} m_k/r_{1k} \right] + O(3) \ .$$

The radial unit four-vector $\mathbf{E}_r$ is the direction of the emitted photon, as measured by the gravimeter. This is simply the projection of λ^i onto the hypersurface orthogonal to the gravimeter's four-velocity u^i, suitably normalized (see also Subsect. 7·2):

(6.97)

$$(E_r)^a = (\delta^a{}_b - u^a u_b)\lambda^b / |(\delta^c{}_d - u^c u_d)\lambda^d|$$

$$= [\lambda^a/(\lambda^b u_b)] - u^a \ .$$

Then the invariant radial component of the gravimeter's four-acceleration is

(6.98)

$$F_r = (F^a \lambda_a)/(\lambda^b u_b) - F^a u_a \ .$$

From equations (6.81), (6.84), (6.91), (6.93), (6.94), (6.95), (6.96), and (6.98), we get for the radial acceleration measured by the gravimeter:

(6.99)

$$F_r = - \sum_{k \neq 1, 0} \frac{m_k r_{10}{}^\alpha r_{10}{}^\beta (3 r_{1k}{}^\alpha r_{1k}{}^\beta - r_{1k}{}^2 \delta^{\alpha\beta})}{r_{1k}{}^5 r_{10}} - \frac{v_o{}^2}{r_p}$$

$$+ (M/r_p{}^2) \left[1 - (4\beta - \gamma - 3 - \zeta_2) \sum_{k \neq 10} m_k/r_{1k} \right.$$

$$- \frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) w_1{}^2 - \frac{1}{2}\alpha_2 (\underset{\sim}{w}_1 \cdot \underset{\sim}{e}_r)^2$$

$$\left. + \frac{1}{2} \frac{I}{M r_p{}^2} \left(3(\underset{\sim}{w}_1 \cdot \underset{\sim}{e}_r)^2 - w_1{}^2 \right) \right] \ ,$$

where

(6.100) $\qquad \underset{\sim}{e}_r = \underset{\sim}{r}_{10}/r_{10}.$

The first term in equation (6.99) is simply the Newtonian tidal accelera-tion, which is of the order of 10^{-7} g. The second term is the Newtonian centrifugal acceleration ($\sim 10^{-3}$ g), which is equivalent (to the necessary accuracy) to the invariant expression $r_p(D\underset{\sim}{E}_r/D\tau) \cdot (D\underset{\sim}{E}_r/D\tau)$ in equation (6.75). From the third term we get the locally-measured gravitational constant G:

$$G = 1 - (4\beta - \gamma - 3 - \zeta_2)U_{external}$$

(6.101)
$$+ \frac{1}{2}\left[\alpha_3 - \alpha_1 + \alpha_2\left(1 - \frac{I}{Mr_p^2}\right)\right]w_1^2$$

$$- \frac{1}{2}\alpha_2\left(1 - \frac{3I}{Mr_p^2}\right)(\underset{\sim}{w}_1 \cdot \underset{\sim}{e}_r)^2.$$

We will discuss observable consequences of equation (6.101) in Sect. 7. We note here that general relativity predicts (see Table 5.I)

(6.102) $\qquad G = 1 ,$

and Scalar-Tensor theories predict

(6.103) $\qquad G = 1 - \left[\frac{1}{2 + \omega} + \Lambda\right]U_{external} ,$

in agreement with results obtained by Brans (1962a,b) and Nordtvedt (1970b).

7. OBSERVABLE EFFECTS AND EXPERIMENTAL TESTS OF METRIC THEORIES OF GRAVITY

7·1. Introduction

In this Section we make use of the PPN equations of motion derived in Sect. 6 to analyse specific effects and experimental tests in the solar system.

7·2. Light Bending and Time Delay: Measuring Curvature in the Solar System

The bending of light rays by the Sun, and the added delay in the round-trip travel time of a radar signal which passes the Sun both measure the parameter γ. A light ray (or photon) which grazes the Sun is deflected by an angle

$$(7.1) \qquad \delta\theta \approx \frac{1}{2} (1 + \gamma) 1.''75 \quad ,$$

independent of the frequency of light. A radar signal sent across the solar system past the Sun to a planet or satellite and returned to the Earth suffers an additional non-Newtonian delay in its round-trip travel time, given by, for a ray which passes close to the Sun,

$$(7.2) \qquad \delta t_{round-trip} \approx \frac{1}{2}(1 + \gamma)[\, 250\,\mu sec - 20\,\mu sec\; \ell n(d^2/r)] \quad ,$$

where d is the distance of closest approach of the ray in solar radii, and r is the distance of the planet or satellite from the Sun, in astronomical units.

Measurements of these two effects have given us our most precise measurements of the parameter γ to date.

The prediction of the bending of light by the sun was one of the great successes of Einstein's general relativity. Eddington's confirmation of the bending in the first days following World War I helped make Einstein famous. However, the experiments of Eddington and his co-workers had only 30 per cent accuracy, and succeeding experiments weren't much better: the results were scattered between one half and one and a half times the Einstein value, and the accuracies were low. However, the development of long-baseline radio interferometry has altered the situation. Long-baseline and very-long-baseline (VLBI) interferometric techniques have the capability in principle of measuring angular separations and changes in angles as small as 3×10^{-4} seconds of arc. Coupled with this technological advance is a heavenly coincidence: Each October 8, two strong Quasi Stellar Radio Sources 3C273 and 3C279 pass very close to the Sun (as seen from the Earth), in fact 3C279 actually goes behind the Sun. By measuring the relative bending of the two signals from these quasars, radio astronomers over the past few years have been able to measure the coefficient $1/2(1 + \gamma)$ in equation (7.1), which has the value unity in general relativity. Their results:

October, 1969

Muhleman, Ekers, and Fomalont (1970)	$1.04 \begin{smallmatrix} +0.15 \\ -0.1 \end{smallmatrix}$
Seielstad, Sramek, and Weiler (1970)	1.01 ± 0.12

October, 1970

Hill (1971)	1.07 ± 0.17
Sramek (1971)	0.90 ± 0.05
Shapiro et al. (1971)	1.03 ± 0.2

One of the major sources of error in these experiments is the solar corona, which bends radio waves much more strongly than it bent the visible light rays which Eddington observed. Improvements in dual frequency techniques may improve accuracies by allowing the coronal bending, which depends on the frequency of the wave, to be measured separately from the gravitational bending, which does not.

The "time-delay" effect was not predicted by Einstein; it was 1964 when this effect was discovered by Shapiro (1964) as a theoretical consequence of general relativity and of other theories of gravity (see also Muhleman and Reichley [1964]). In the following years, attempts were made to measure this effect using radar ranging to targets passing through "superior conjunction" (target on the far side of the Sun; radar signals passing close to the Sun). Two types of targets were employed: planets such as Mercury and Venus, used as passive reflectors of the radar signals; and the Mariner VI and VII spacecraft, used as active retransmitters of the radar signals. Detailed analyses of the measured round-trip travel times yielded the following results for the coefficient $1/2(1 + \gamma)$:

Passive radar to Mercury and Venus 1.02 ± 0.05

(Shapiro, Ash, Ingalls, Smith, Campbell, Dyce, Jurgens and Pettengill [1971])

Active radar to Mariners VI and VII 1.00 ± 0.04

(Anderson, Esposito, Martin and Muhleman [1971])

Here, as in the light deflection measurements, the solar corona causes uncertainties in the measurements because of its slowing down of the radar signal; again dual frequency ranging may help to reduce these

errors. For detailed discussion of experimental problems and prospects, the reader is referred to other lectures in this volume, and to Davies (1971).

Expressions for the light bending and time delay can be obtained in a straightforward way using the PPN photon equations of motion (6.13) and (6.14). Consider a light signal emitted at PPN coordinate time t_e at a point x_e in an initial direction described by the unit vector $\underset{\sim}{n}$, where

$$(7.3) \qquad \underset{\sim}{n} \cdot \underset{\sim}{n} = 1 .$$

Including the post-Newtonian correction $\underset{\sim}{x}_p$, the resulting trajectory of the photon then has the form

$$(7.4) \qquad \begin{aligned} x^o(t) &= t , \\ \underset{\sim}{x}(t) &= \underset{\sim}{x}_e + \underset{\sim}{n}(t - t_e) + \underset{\sim}{x}_p(t) . \end{aligned}$$

We compute the components of $\underset{\sim}{x}_p$ parallel and perpendicular to the unperturbed trajectory, given by

$$(7.5) \qquad x_p(t)_{\parallel} = \underset{\sim}{n} \cdot \underset{\sim}{x}_p(t) ,$$

$$(7.6) \qquad \underset{\sim}{x}_p(t)_{\perp} = \underset{\sim}{x}_p(t) - \underset{\sim}{n}(\underset{\sim}{n} \cdot \underset{\sim}{x}_p(t)) .$$

Equations (6.13), (6.14), (7.5) and (7.6) then yield

$$(7.7) \qquad \frac{d^2 x_p{}^{\alpha}{}_{\perp}}{dt^2} = (1 + \gamma)\left[\frac{\partial U}{\partial x^{\alpha}} - n^{\alpha}\underset{\sim}{n} \cdot \underset{\sim}{\nabla} U\right] ,$$

$$(7.8) \qquad \frac{dx_{p\parallel}}{dt} = - (1 + \gamma)U .$$

124

For simplicity we assume the Newtonian gravitational potential U is produced by a static spherical body (Sun) of mass m, i.e.

$$(7.9) \qquad U = m/r \; .$$

Along the unperturbed path of the photon, U then has the form

$$(7.10) \qquad U[\underset{\sim}{x}(t)] = \frac{m}{r(t)} = \frac{m}{|\underset{\sim e}{x} + \underset{\sim}{n}(t - t_e)|} \; .$$

Equations (7.7) and (7.8) can be integrated along the unperturbed photon path using equation (7.10), with the result

$$(7.11) \qquad \frac{d}{dt} \underset{\sim p}{x}(t)_\perp = -(1 + \gamma) \frac{m \underset{\sim}{d}}{d^2} \left\{ \frac{\underset{\sim}{x}(t) \cdot \underset{\sim}{n}}{r(t)} - \frac{\underset{\sim e}{x} \cdot \underset{\sim}{n}}{r_e} \right\} \; ,$$

$$(7.12) \qquad x_p(t)_\| = -(1 + \gamma) m \, \ell n \left[\frac{r(t) + \underset{\sim}{x}(t) \cdot \underset{\sim}{n}}{r_e + \underset{\sim e}{x} \cdot \underset{\sim}{n}} \right] \; ,$$

where

$$(7.13) \qquad \underset{\sim}{d} = \underset{\sim}{n} \times (\underset{\sim e}{x} \times \underset{\sim}{n}) \; .$$

Note that $\underset{\sim}{d}$ is the vector joining the center of the Sun and the point of closest approach of the unperturbed ray (see Fig. 7.1).

Equation (7.11) represents a change in the direction of the photon's trajectory, a deflection toward the Sun (in the direction $-\underset{\sim}{d}$).

Consider an observer at rest on the Earth ($\oplus$), who receives the photon. The angle θ (see Fig. 7.1) between the direction of the incoming photon and the direction of a photon emitted by the Sun and received by the observer is a physically measurable quantity, and can be given an invariant mathematical expression. The tangent four-vectors,

125

dx^i/dt and $dx_\odot^{\ i}/dt$, of the two incoming photons are projected onto the hypersurface orthogonal to the observer's four-velocity, u^i using the "projection operator" (see Subsect. 6·5):

$$(7.14) \qquad P_i^{\ j} = \delta_i^{\ j} - u_i u^j .$$

The inner product between the resulting vectors is related to the cosine of θ ("dot" means d/dt):

$$(7.15) \qquad \cos\theta = - \frac{\dot{x}_\odot^{\ i}(\delta_i^{\ j} - u_i u^j)\, \dot{x}_k(\delta_j^{\ k} - u_j u^k)}{|\dot{x}_\odot^{\ i}(\delta_i^{\ j} - u_i u^j)|\, |\dot{x}_k(\delta_j^{\ k} - u_j u^k)|} .$$

If we ignore the velocity of the Earth, which only produces aberration, then equation (7.15) simplifies to

$$(7.16) \qquad \cos\theta = 1 - [(g_{ij}\dot{x}_\odot^{\ i}\dot{x}^j)/(g_{oo}\dot{x}_\odot^{\ o}\dot{x}^o)] .$$

By substituting equations (7.4), (7.8), and (7.11) into equation (7.16), we get to post-Newtonian accuracy,

$$(7.17) \qquad \cos\theta = \underset{\sim}{n}_\odot \cdot \underset{\sim}{n} - (1+\gamma)\frac{m}{d}\left(\frac{\underset{\sim}{n}_\odot \cdot \underset{\sim}{d}}{d}\right)\left(\frac{\underset{\sim}{x}_\oplus \cdot \underset{\sim}{n}}{r_\oplus} - \frac{\underset{\sim}{x}_e \cdot \underset{\sim}{n}}{r_e}\right) .$$

We define the angle θ_o, the angle between the unperturbed photon path and the photon emitted toward the Earth by:

$$(7.18) \qquad \cos\theta_o = \underset{\sim}{n}_\odot \cdot \underset{\sim}{n} .$$

The difference $\delta\theta = \theta - \theta_o$ between these two angles measures the deflection of the photon's trajectory, and is given to first order from equation (7.17) by

$$(7.19) \qquad \delta\theta = \left(\frac{1+\gamma}{2}\right)\frac{2m}{d}\left(\frac{\underset{\sim}{x}_{\oplus}\cdot\underset{\sim}{n}}{r_{\oplus}} - \frac{\underset{\sim}{x}_{e}\cdot\underset{\sim}{n}}{r_{e}}\right)$$

For a photon emitted from a distant star or galaxy,

$$(7.20) \qquad r_{e} \gg r_{\oplus}, \qquad (\underset{\sim}{x}_{e}\cdot\underset{\sim}{n})/r_{e} \sim -1 .$$

Also, to sufficient accuracy

$$(7.21) \qquad (\underset{\sim}{x}_{\oplus}\cdot\underset{\sim}{n})/r_{\oplus} = \cos\theta_{o} .$$

Thus

$$(7.22) \qquad \delta\theta = \left(\frac{1+\gamma}{2}\right)\frac{4m}{d}\left(\frac{1+\cos\theta_{o}}{2}\right) .$$

For general relativity ($\gamma = 1$), equation (7.22) is in agreement with results obtained by Shapiro (1967) and Ward (1970).

The deflection is a maximum for a ray which just grazes the Sun, i.e. for $\theta_{o} \approx 0$, $d \approx$ solar radius. In this case

$$(7.23) \qquad \delta\theta_{MAX} = \frac{1}{2}(1+\gamma) 1\rlap{.}''75 .$$

The time delay is obtained from equation (7.12). The time taken for a signal to propagate from $\underset{\sim}{x}_{e}$ to $\underset{\sim}{x}$ is given from equations (7.4) and (7.12) by

$$(7.24) \qquad (t - t_{e}) \approx \left|\underset{\sim}{x} - \underset{\sim}{x}_{e}\right| + (1+\gamma)\, m\, \ell n\left[\frac{r(t) + \underset{\sim}{x}(t)\cdot\underset{\sim}{n}}{r_{e} + \underset{\sim}{x}_{e}(t)\cdot\underset{\sim}{n}}\right] .$$

For a signal emitted from the Earth, reflected off a planet or spacecraft at $\underset{\sim}{x}_{P}$ and received back at Earth, the round-trip travel time Δt is given by (Fig. 7.2)

(7.25) $\qquad \Delta t \approx 2\left|x_{\oplus} - x_P\right| + 2(1 + \gamma)\, m\,\ell n\, \dfrac{(r_{\oplus} + \underset{\sim}{x}_{\oplus} \cdot \underset{\sim}{n})(r_P - \underset{\sim}{x}_P \cdot \underset{\sim}{n})}{d^2}$

where $\underset{\sim}{n}$ is the direction of the photon on its return flight. Here we have ignored the motion of the Earth and planets during the round trip of the signal. To be completely correct, the round-trip travel time should be expressed in terms of the proper time elapsed during the round trip, as measured by an atomic clock on Earth; but this introduces no new effects, so we will not do so here. The additional "time delay" δt produced by the second term in equation (7.25) is a maximum when the planet is on the far side of the Sun from the Earth (superior conjunction), i.e. when

(7.26) $\qquad \underset{\sim}{x}_{\oplus} \cdot \underset{\sim}{n} \approx r_{\oplus}\,, \quad \underset{\sim}{x}_P \cdot \underset{\sim}{n} \approx -r_P\,, \quad d \approx$ solar radius;

then

(7.27)
$$\delta t = 2(1 + \gamma)\, m\,\ell n\,(4 r_{\oplus} r_P / d^2)$$
$$= \frac{1}{2}(1 + \gamma)\left[250\ \mu\text{sec} - 20\ \mu\text{sec}\ \ell n \left(\frac{d}{R_{\odot}}\right)^2 \left(\frac{a}{r_P}\right)\right],$$

where $R_{\odot}$ is the radius of the Sun, and a is an astronomical unit. For further discussion of the time delay see Shapiro (1964, 1966a,b) and Ross and Schiff (1966). The time delay can also be measured by analysing its effects on the arrival times of the pulses of radiation from pulsars (Reichley [1971], Kovács, Will and Thorne [1972]).

By comparing the experimentally measured values of the parameter combination $\frac{1}{2}(1 + \gamma)$ with the predictions of various metric theories of gravity (Table 5.I), we can see immediately that all the Conformally Flat theories of gravity ($\gamma = -1$; no bending or time delay) and the

Einstein and the Whitrow-Morduch Stratified theories ($\gamma = 0$; half the observed effect) are ruled out.

Some theories are made uncomfortable but are not quite ruled out by the light deflection and time delay experiments. These are the Scalar-Tensor theories. In order for these theories to agree with the time delay measurements within two standard deviations, their coupling constant ω must be larger than 6. These theories reduce to general relativity in the limit $\omega \rightarrow \infty$.

7·3. Perihelion Shifts

In the past several years, the theoretical interpretation of the perihelion shifts of the planets has become more and more complex. The measured perihelion shifts are accurately known: after the effects of the other planets and of the "general precession" of the Earth's rotation axis have been subtracted out, Mercury has a residual perihelion shift of 43 seconds of arc per century; and this shift is known to a precision of about 1 per cent from radar-ranging data for the planets (Shapiro 1971). For Earth the residual shift is 4 arcseconds per century, known to about 10 per cent accuracy.

The PPN prediction for these effects can be easily obtained from the PPN equation of motion (6.42). We consider a two-body system: one of the bodies is a test body, the other has mass m, self gravitational energy Ω, a small quadrupole moment $Q_{\alpha\beta}$; and rotates uniformly with angular velocity $\underset{\sim}{\lambda}$. The orbit of the test body is in a plane normal to $\underset{\sim}{\lambda}$, and the entire system is moving relative to the mean rest-frame of the Universe with velocity $\underset{\sim}{w}$. We work in a PPN coordinate system

which is at rest with respect to the massive body.

Then, from equation (6.42), the acceleration $\underset{\sim}{a}_T$ of the test body is given by

(7.28)
$$
\begin{aligned}
\underset{\sim}{a}_T = \underset{\sim}{\nabla} \mathcal{H} \\
+ \frac{m\underset{\sim}{r}}{r^3} \left\{ (2\gamma + 2\beta)\frac{m}{r} - \gamma v^2 - \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)w^2 \right. \\
\left. + \frac{1}{2}\alpha_1 (\underset{\sim}{w} \cdot \underset{\sim}{v}) + \frac{3}{2}\alpha_2 (\underset{\sim}{w} \cdot \underset{\sim}{\hat{r}})^2 \right\} \\
+ (2\gamma + 2)\underset{\sim}{v} \frac{m\underset{\sim}{r} \cdot \underset{\sim}{v}}{r^3} - \underset{\sim}{w} \frac{m\underset{\sim}{r}}{r^3} \cdot (\frac{1}{2}\alpha_1 \underset{\sim}{v} + \alpha_2 \underset{\sim}{w}) \; .
\end{aligned}
$$

The massive body is not affected by the test body (negligible mass) but does feel its own "self accelerations" (see Subsect. 6·3, eq. [6.42]). Because we have assumed the massive body is nearly spherical, the only significant "self-acceleration" is the term (eq. [6.49])

(7.29)
$$
\underset{\sim}{a}_m = - \alpha_3 (\Omega/m)\underset{\sim}{w} \times \underset{\sim}{\lambda} \; .
$$

Making use of equations (6.52) and (6.53), along with our assumption of a nearly spherical massive body $(\Omega^{\alpha\beta} \sim \frac{1}{3}\Omega\delta^{\alpha\beta})$, we obtain for $\underset{\sim}{\nabla}\mathcal{H}$:

(7.30)
$$
\frac{\partial \mathcal{H}}{\partial x^\alpha} = - \frac{m_a r^\alpha}{r^3} - \frac{1}{2} Q^{\beta\gamma} \left(\frac{5r^\beta r^\gamma r^\alpha}{r^7} - \frac{2\delta^{\alpha\beta} r^\gamma}{r^5} \right) .
$$

For a body which is axially symmetric about its rotation axis $\underset{\sim}{\lambda}$, $Q^{\beta\gamma}$ can be shown to have the form

(7.31)
$$
Q^{\beta\gamma} = mR^2 J_2 (\delta^{\beta\gamma} - 3\hat{\lambda}^\beta\hat{\lambda}^\gamma) \; ,
$$

where J_2 is a dimensionless measure of the quadrupole moment, given by

(7.32)
$$
J_2 = (C - A)/mR^2 \; ;
$$

130

where

$$C = [\text{moment of inertia about rotation axis}] \, ,$$

(7.33)
$$A = [\text{moment of inertia about equatorial axis}] \, ,$$

$$R = \text{radius of massive body}] \, .$$

The relative acceleration between the test body and the massive body
is thus

(7.34)
$$\underset{\sim}{a} \equiv \underset{\sim}{a}_T - \underset{\sim}{a}_m \, ,$$

and is given from equations (7.28), (7.29), (7.30) and (7.31) by

$$
\begin{aligned}
\underset{\sim}{a} = & -m_a \underset{\sim}{r}/r^3 + \tfrac{1}{2}(mR^2 J_2/r^4)[15(\hat{\underset{\sim}{\lambda}} \cdot \hat{\underset{\sim}{r}})\hat{\underset{\sim}{r}} - 6(\hat{\underset{\sim}{\lambda}} \cdot \hat{\underset{\sim}{r}})\hat{\underset{\sim}{\lambda}} - 3\hat{\underset{\sim}{r}}] \\
& + \frac{m\underset{\sim}{r}}{r^3}\Big\{ (2\gamma + 2\beta)\frac{m}{r} - \gamma v^2 - \tfrac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1) w^2 \\
& \qquad\qquad + \tfrac{1}{2}\alpha_1 (\underset{\sim}{w} \cdot \underset{\sim}{v}) + \tfrac{3}{2}\alpha_2 (\underset{\sim}{w} \cdot \hat{\underset{\sim}{r}})^2 \Big\} \\
& + (2\gamma + 2)\underset{\sim}{v}\frac{m\underset{\sim}{r} \cdot \underset{\sim}{v}}{r^3} - \underset{\sim}{w}\frac{m\underset{\sim}{r}}{r^3} \cdot (\tfrac{1}{2}\alpha_1 \underset{\sim}{v} + \alpha_2 \underset{\sim}{w}) \\
& + \alpha_3 (\Omega/m)\,\underset{\sim}{w} \times \underset{\sim}{\lambda} \, .
\end{aligned}
$$

(7.35)

We consider a planetary orbit with the following instantaneous
orbital elements: eccentricity e, semi-major axis a, and angle of
perihelion relative to the equinox $\tilde{\omega}$.

Following the standard procedure for computing perturbations of
orbital elements (Smart [1953], Robertson and Noonan [1968]), we
resolve the acceleration $\underset{\sim}{a}$ (eq. [7.35]) into a radial component $\mathcal{R}$, a
component $\mathcal{W}$, normal to the orbital plane, and a component $\mathcal{S}$ normal

to $\widetilde{\omega}$ and $\Re$, and calculate the rates of change of the orbital elements using the formulae (in the notation of Robertson and Noonan [1968]):

$$(7.36) \qquad \frac{d\widetilde{\omega}}{dt} = -\frac{p\Re}{he} \cos \varphi + \frac{\mathcal{S}(p+r)}{he} \sin \varphi \ ,$$

$$(7.37) \qquad \frac{de}{dt} = \frac{1-e^2}{h} \left[a\Re \sin \varphi + \frac{\mathcal{S}}{e} \left(\frac{ap}{r} - r \right) \right] \ ,$$

$$(7.38) \qquad \frac{da}{dt} = \frac{2a^2}{h} \left(\frac{\mathcal{S}p}{r} + \Re e \sin \varphi \right) \ ,$$

where h is the angular momentum per unit mass of the orbit, φ is the angle of the planet measured from perihelion, and p is the semi-latus rectum given by

$$(7.39) \qquad p = a(1 - e^2) \ .$$

We calculate the perturbations (eqs. [7.36], [7.37], and [7.38]), to first order, retaining only secular terms, and using a Keplerian ellipse as unperturbed orbit, given by

$$(7.40) \qquad r = p(1 + e \cos \varphi)^{-1} \ ,$$

$$(7.41) \qquad r^2 (d\varphi/dt) = h = \text{constant} \ .$$

For the secular changes over one orbit, we obtain, to zero'th order in the eccentricity $(e \neq 0)$:

$$(7.42) \qquad \Delta\widetilde{\omega} = \frac{6\pi m}{p} \left\{ \frac{1}{3}(2 + 2\gamma - \beta) + J_2 \left(\frac{R^2/m}{2p} \right) \right\}$$

$$-2\pi \left[\frac{1}{4}\alpha_1 \left(\frac{m}{p} \right)^{1/2} \frac{w_Q}{e} + \frac{1}{8}\alpha_2 (w_P^2 - w_Q^2) - \frac{1}{2}\alpha_3 \left(\frac{|\Omega|}{m} \right) \left(\frac{\lambda p^2}{me} \right) w_Q \right] \ ,$$

$$(7.43) \qquad \frac{\Delta e}{e} = -2\pi \left[\frac{1}{4}\alpha_1 \left(\frac{m}{p} \right)^{1/2} \frac{w_P}{e} - \frac{1}{4}\alpha_2 w_P w_Q - \frac{1}{2}\alpha_3 \left(\frac{|\Omega|}{m} \right) \left(\frac{\lambda p^2}{me} \right) w_P \right] \ ,$$

(7.44) $\Delta a = O(e^2)$,

where w_P and w_Q are the components of $\underset{\sim}{w}$ in the direction of the planet's perihelion (w_P) and in the direction at right angles to this (w_Q), in the plane of the orbit. The perturbations in equation (7.35) can also be shown to produce secular changes in the inclination and angle of nodes of orbits, proportional to J_2 and to the component of $\underset{\sim}{w}$ normal to the orbital plane.

We focus on the perihelion shift, equation (7.42). The first term is the "classical" perihelion shift, which depends on the PPN parameters γ and β and on the quadrupole moments J_2 and which would be present even if the solar system were at rest in the Universe $(\underset{\sim}{w} = 0)$. The other terms in equation (7.42) are "preferred-frame" perihelion shifts. We now evaluate the perihelion shift for Mercury and Earth, using standard values for the orbital elements (Allen 1963), numerical values for the Sun's gravitational energy and rotational angular velocity:

$$(|\Omega|/m)_{\odot} \approx 4 \times 10^{-6} , \qquad \lambda_{\odot} \approx 3 \times 10^{-6} \ \text{sec}^{-1} ,$$

and an adopted value for $\underset{\sim}{w}$. Throughout this section, we assume that the solar system's motion through the Universe is due to its (nearly circular) orbital motion around the Galaxy, i.e. we assume $\underset{\sim}{w}$ is 200 km/sec in the direction given by galactic coordinates $\ell^{II} = 90°$, $b^{II} = 0°$. In terms of the Geocentric Ecliptic coordinate system (z-axis normal to the Earth's orbit, x-axis directed toward the Sun at vernal equinox) the direction of $\underset{\sim}{w}$ is given by $\lambda = 346°$, $\beta = 60°$, and in the Geocentric Equatorial coordinate system (z-axis normal to the Earth's equator, x-axis same as

before), it is given by $\alpha = 318^\circ$, $\delta = 48^\circ$ (see Smart [1960] for definitions of these astronomical coordinate systems, and for equations to transform from one to the other). For Mercury ($\mathbf{\breve{\varphi}}$) and the Earth ($\oplus$), equation (7.42) yields, in seconds of arc per century:

$$(7.45) \qquad (\Delta\tilde{\omega}_{\breve{\varphi}})_{PPN} = 43[\tfrac{1}{3}(2\gamma+2-\beta)] + 4[J_2/3\times10^{-5}] + 35\alpha_1 + 8\alpha_2 - 4\times10^4\alpha_3,$$

$$(7.46) \qquad (\Delta\tilde{\omega}_{\oplus})_{PPN} = 4[\tfrac{1}{3}(2\gamma+2-\beta)] + 57\alpha_1 + \alpha_2 - 7\times10^5\alpha_3 .$$

Note that J_2 for the Sun is positive (oblateness), but is probably smaller than 5×10^{-5}, so we have normalized J_2 by writing equation (7.45) in terms of $(J_2/3\times10^{-5})$. The effect of J_2 on the Earth's perihelion is below the experimental uncertainty. The measured perihelion shifts are

$$(\Delta\omega_{\breve{\varphi}})_{OBSERVED} = 43 \pm 0.4 ,$$
$$(7.47)$$
$$(\Delta\omega_{\oplus})_{OBSERVED} = 4 \pm 0.4 .$$

Because of the complicated PPN-parameter dependence of equations (7.45) and (7.46), reliable separate measurements of all the PPN parameters and of J_2 cannot be made. However, we can obtain a useful limit on the parameters α_1, α_2 and α_3, by combining equations (7.45) and (7.46), eliminating the term involving γ and β, and treating J_2 as an experimental uncertainty: then in order that the PPN perihelion shifts agree with the measured shifts within the experimental error, the parameters α_1, α_2 and α_3 must satisfy

$$(7.48) \qquad |13\,\alpha_1 + 0.07\,\alpha_2 - 170,000\,\alpha_3| < 0.2 .$$

Note that general relativity is in excellent agreement with perihelion-shift measurements $(\gamma = \beta = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 0)$, as long as the effect of J_2 is smaller than the experimental error, i.e. as long as

(7.49) $J_2 \lesssim 3 \times 10^{-6}$.

Since light-deflection and time-delay experiments limit the coupling constant ω of Scalar-Tensor Theories to values larger than ~ 6 (two standard deviations), these theories can agree with perihelion shift measurements for $\omega \sim 6$ only if

$$J_2 \sim 3 \times 10^{-5} \quad [\text{Dicke-Brans-Jrodan}] ,$$
(7.50)
$$J_2 - 10^{-4}\Lambda \sim 3 \times 10^{-5} \quad [\text{Bergmann-Wagoner-Nordtvedt}] .$$

7·4. Geophysical "Preferred-Frame" Effects

We focus attention on the "locally-measured" Newtonian gravitational constant, which was calculated in Sect. 6 by considering the Earth's gravitational force on a gravimeter at rest on the surface of the Earth:

$$G = 1 - (4\beta - \gamma - 3 - \zeta_2)U_{\text{external}}$$
(7.51)
$$+ \frac{1}{2}\left[(\alpha_3 - \alpha_1) + \alpha_2\left(1 - \frac{I}{MR^2}\right)\right]w_\oplus{}^2$$
$$- \frac{1}{2}\alpha_2\left(1 - \frac{3I}{MR^2}\right)(\underset{\sim}{w}_\oplus \cdot \underset{\sim}{e}_r)^2 ,$$

where M is the mass of the Earth, R its radius and I its spherical moment of inertia; $\underset{\sim}{w}_\oplus$ is the Earth's velocity through the preferred Universal rest-frame, and $\underset{\sim}{e}_r$ is a unit vector joining the gravimeter

and the center of the Earth.

Measurements of the absolute value of G are accurate to at most a part in 10^5 (Rose et al. 1969), and so cannot discern the post-Newtonian corrections to G in equation (7.51). However, measurements of the effects of variations in G are much more accurate; hence we will concentrate on such variations.

Because of the Earth's eccentric orbital motion, the external potential produced by the Sun varies on Earth by only a part in 10^{10}, too small to be detected with confidence by Earth-bound gravimeters or Cavendish experiments [see, however Wilk (1971) for preliminary studies of an orbiting Cavendish experiment to be performed in a highly eccentric orbit, with variations in U as large as a part in 10^7]. We will thus consider only the "preferred-frame" effects in equation (7.51), which depend on the Earth's velocity relative to the rest-frame of the Universe.

The Earth's velocity $\underset{\sim}{w}_{\oplus}$ is made up of two parts, a uniform velocity $\underset{\sim}{w}$ of the solar system relative to the preferred frame, and the Earth's orbital velocity $\underset{\sim}{v}$ around the Sun, thus

$$w_{\oplus}{}^2 = w^2 + 2\underset{\sim}{w} \cdot \underset{\sim}{v} + v^2 \ ,$$

(7.52)

$$(\underset{\sim}{w}_{\oplus} \cdot \underset{\sim}{e}_r)^2 = (\underset{\sim}{w} \cdot \underset{\sim}{e}_r)^2 + 2(\underset{\sim}{w} \cdot \underset{\sim}{e}_r)(\underset{\sim}{v} \cdot \underset{\sim}{e}_r) + (\underset{\sim}{v} \cdot \underset{\sim}{e}_r)^2 \ .$$

So because of the Earth's rotation (changing $\underset{\sim}{e}_r$) and orbital motion (changing $\underset{\sim}{v}$), there will be variations in the gravimeter measurements of G, given by (we retain only terms which vary with amplitude larger than 10^{-9} G)

$$\Delta G/G = (\tfrac{1}{2}\alpha_2 + \alpha_3 - \alpha_1)\underset{\sim}{w}\cdot\underset{\sim}{v}$$

(7.53)

$$+\tfrac{1}{4}\alpha_2[\,(\underset{\sim}{w}\cdot\underset{\sim}{e}_r)^2 + 2(\underset{\sim}{w}\cdot\underset{\sim}{e}_r)(\underset{\sim}{v}\cdot\underset{\sim}{e}_r) + (\underset{\sim}{v}\cdot\underset{\sim}{e}_r)^2\,]\;.$$

where we have used the fact that, for the Earth,

(7.54) $I \simeq \tfrac{1}{2}MR^2\,.$

In order to compare this variation in G with gravimeter data, we must perform a harmonic analysis of the terms in equation (7.53). The frequencies involved will be the sidereal rotation of the Earth Ω, due to the changing $\underset{\sim}{e}_r$ relative to the fixed direction of $\underset{\sim}{w}$, and its orbital sidereal frequency ω due to the changing direction of $\underset{\sim}{v}$ relative to $\underset{\sim}{w}$, along with harmonics and linear combinations of these frequencies. We work in Geocentric Ecliptic Coordinates (see Subsect. 7·3), and assume a circular Earth orbit, with the Earth at vernal equinox at $t = 0$.   Then

(7.55) $\underset{\sim}{w} = w[\,\cos\beta\,(\cos\lambda\,\underset{\sim}{e}_x + \sin\lambda\,\underset{\sim}{e}_y) + \sin\beta\,\underset{\sim}{e}_z\,]\,,$

(7.56) $\underset{\sim}{v} = v(\sin\omega t\,\underset{\sim}{e}_x - \cos\omega t\,\underset{\sim}{e}_y)\,.$

For a gravimeter stationed at Earth latitude L,

$$\underset{\sim}{e}_r = \cos L \cos(\Omega t - \epsilon)\underset{\sim}{e}_x$$

(7.57) $+ [\,\cos L \sin(\Omega t - \epsilon)\cos\theta + \sin L \sin\theta\,]\,\underset{\sim}{e}_y$

$- [\,\cos L \sin(\Omega t - \epsilon)\sin\theta - \sin L \cos\theta\,]\,\underset{\sim}{e}_z\,,$

where ϵ is related to the longitude of the gravimeter on the Earth, and θ is the "tilt" ($23\text{-}1/2^{\circ}$) of the Earth relative to the Earth's orbit

(ecliptic). Equations (7.55), (7.56), and (7.57) give

(7.58)
$$\underset{\sim}{w} \cdot \underset{\sim}{v} = wv \cos \beta \sin (\omega t - \lambda),$$

$$(\underset{\sim}{w} \cdot \underset{\sim}{e}_r)^2 = w^2 \left[\frac{1}{3} + \frac{3}{2} (\frac{1}{3} - \sin^2 \delta)(\frac{1}{3} - \sin^2 L) \right.$$

(7.59)
$$+ \frac{1}{2} \sin 2\delta \sin 2L \cos (\Omega t - \epsilon - \alpha)$$

$$\left. + \frac{1}{2} \cos^2 \delta \cos^2 L \cos 2(\Omega t - \epsilon - \alpha) \right] ,$$

$$(\underset{\sim}{w} \cdot \underset{\sim}{e}_r)(\underset{\sim}{v} \cdot \underset{\sim}{e}_r) = wv \left\{ \frac{1}{3} \cos \beta \sin (\omega t - \lambda) \right.$$

$$+ (\frac{1}{3} - \sin^2 L)[\frac{1}{2} \cos \beta \sin (\omega t - \lambda) + \frac{3}{2} \sin \delta \sin\theta \cos \omega t]$$

$$+ \frac{1}{4} \sin \delta (1 - \cos \theta) \sin 2L \sin [(\Omega + \omega)t - \epsilon]$$

$$- \frac{1}{4} \cos \delta \sin \theta \sin 2L \cos [(\Omega + \omega)t - \epsilon - \alpha]$$

(7.60)
$$- \frac{1}{4} \sin \delta (1 + \cos \theta) \sin 2L \sin [(\Omega - \omega)t - \epsilon]$$

$$- \frac{1}{4} \cos \delta \sin \theta \sin 2L \cos [(\Omega - \omega)t - \epsilon - \alpha]$$

$$+ \frac{1}{4} \cos \delta (1 - \cos \theta) \cos^2 L \sin [(2\Omega + \omega)t - 2\epsilon - \alpha]$$

$$\left. - \frac{1}{4} \cos \delta (1 + \cos \theta) \cos^2 L \sin [(2\Omega + \omega)t - 2\epsilon - \alpha] \right\}$$

$$(\underset{\sim}{v} \cdot \underset{\sim}{e}_r)^2 = v^2 \left\{ \frac{1}{3} + \frac{3}{2} (\frac{1}{3} - \sin^2 L)(\frac{1}{3} - \frac{1}{2} \sin^2 \theta) \right.$$

$$- \frac{3}{4} (\frac{1}{3} - \sin^2 L) \sin^2 \theta \cos 2\omega t$$

$$+ \frac{1}{4} \sin 2\theta \sin 2L \sin (\Omega t - \epsilon)$$

$$- \frac{1}{4} \sin \theta (1 - \cos \theta) \sin 2L \sin [(\Omega + 2\omega)t - \epsilon]$$

(7.61)
$$+ \frac{1}{4} \sin \theta (1 + \cos \theta) \sin 2L \sin [(\Omega - 2\omega)t - \epsilon]$$

$$+ \frac{1}{4} \sin^2\theta \cos^2 L \cos 2(\Omega t - \epsilon)$$

$$- \frac{1}{8}(1 - \cos\theta)^2 \cos^2 L \cos [2(\Omega + \omega)t - 2\epsilon]$$

$$- \frac{1}{8}(1 + \cos\theta)^2 \cos^2 L \cos [2(\Omega - \omega)t - 2\epsilon] \Big\}$$

where we have used both the ecliptic coordinates (λ, β) and the equatorial coordinates (α, δ) corresponding to the direction of $\underset{\sim}{w}$, in order to simplify the various expressions.

Equations (7.58), (7.59), (7.60, and (7.61) reveal four different types of variations in G.

i) <u>Semi-Diurnal Variations</u>: These are the terms which vary with frequency around 2Ω: 2Ω, $2\Omega + \omega$, $2\Omega - \omega$, $2(\Omega + \omega)$, $2(\Omega - \omega)$; i.e. have periods around twelve hours $(\omega \ll \Omega)$ and vary with latitude according to $\cos^2 L$. These variations are completely analogous to the twelve-hour solid-Earth tides produced by the Sun and Moon, called "Semi-Diurnal sectorial Waves" by Melchior (1966). The true gravimeter measurements for these tides are affected not only by the variation in G, but also by the displacement of the Earth's surface relative to the center of the Earth, and by the deformation of the Earth. This variation in gravimeter readings is related to the variation in G by

$$(7.62) \qquad (\Delta g/g)_{\text{SEMI-DIURNAL}} = 1.18(\Delta G/G)_{\text{SEMI-DIURNAL}} ,$$

where the factor 1.18 is a combination of so-called "Love Numbers", which depend on the detailed structure of the Earth (Melchior 1966).

ii) <u>Diurnal Variations</u>: These are the terms which vary with a frequency around Ω: Ω, $\Omega + \omega$, $\Omega - \omega$, $\Omega + 2\omega$, $\Omega - 2\omega$; i.e. have periods

around 24 hours, and vary with latitude according to sin 2L. These variations are completely analogous to the 24-hour "Diurnal Tesseral Waves" of the solid Earth (Melchior 1966), and give gravimeter readings related to the variation in G by the same factor:

(7.63) $(\Delta g/g)_{DIURNAL} = 1.18 (\Delta G/G)_{DIURNAL}$.

iii) <u>Long-Period Zonal Variations</u>: These are the variations with frequencies ω and 2ω, and with latitude dependence $(1/3 - \sin^2 L)$, which are completely analogous to the long-period tides produced by the Sun and Moon, called "Long-Period Zonal Waves" by Melchior (1966). These long-period zonal waves produces variations in the Earth's moment of inertia, which in turn cause variations in the rotation rate of the Earth. These rotation-rate variations are related to the amplitude of the zonal variations by (Mintz and Munk 1953; Melchior 1966)

(7.64) $(\Delta \Omega/\Omega)_{ZONAL} = 0.41 \, A_{ZONAL}$,

where A_{ZONAL} is related to the zonal variations in G in equations (7.60) and (7.61) by

(7.65) $(\Delta G/G)_{ZONAL} = A_{ZONAL} (\frac{1}{3} - \sin^2 L)$.

iv) <u>Long-Period Spherical Variations</u>: These are the variations (eqs. [7.58] and [7.60]) which have frequency ω, but no latitude dependence; they represent a yearly variation in the <u>strength</u> of G, and have no counterpart in Newtonian tidal theory. These variations produce a purely spherical deformation of the Earth, as opposed to the Sectorial, Tesseral, and Zonal waves which produce purely quadrupole deformations.

140

This yearly spherical "breathing" of the Earth as G varies causes a variation in the Earth's moment of inertia, which in turn causes a variation in the rotation frequency, given by

$$(7.66) \qquad (\Delta\Omega/\Omega)_{\text{SPHERICAL}} = -(\Delta I/I) \simeq (1/10)(\Delta G/G)_{\text{SPHERICAL}} .$$

Detailed calculations of this change in the Earth's moment of inertia due to the spherical variation in G are given in Nordtvedt and Will (1972).

By combining equations (7.58), (7.59), (7.60), and (7.61) with the expression for $\Delta G/G$, equation (7.53), substituting numerical values

$$v \simeq 30 \text{ km/sec}, \quad w \simeq 200 \text{ km/sec}, \quad \theta \simeq 23\frac{1}{2}^{\circ},$$

$$(7.67) \qquad \lambda \simeq 346^{\circ}, \quad \alpha \simeq 318^{\circ},$$

$$\beta \simeq 60^{\circ}, \quad \delta \simeq 48^{\circ},$$

and using equations (7.62), (7.63), (7.64), and (7.66), we may compute the amplitudes of all the various components of the Earth tides $(\Delta g/g)$ and of the variations in the Earth's rotation rate $(\Delta\Omega/\Omega)$. These amplitudes are listed in Table 7.I.

The largest predicted Earth-tide components are the sidereal diurnal and semi-diurnal (Ω and 2Ω) tides. Other, smaller components include diurnal and semi-diurnal solar-time tides [$(\Omega - \omega)$ and $2(\Omega - \omega)$], and tides with frequencies $2\Omega - \omega$ and $\Omega - 2\omega$. The most important of these is the 12-hour sidereal time (2Ω) tide, with amplitude (see Table 7.I)

$$(7.68) \qquad (\Delta g/g)_{\text{PPN}} \approx 3 \times 10^{-8} \, \alpha_2 \cos^2 L .$$

The semi-diurnal tides predicted by Newtonian theory have three principal frequency components, a 12-hour lunar-time component (denoted M_2), a 12-hour solar-time component (S_2), and a 12-hour sidereal-time component (K_2). (The sidereal component depends on the declination (tilt) of the lunar and solar orbits relative to the Earth's equatorial plane.) These components of the tides have amplitudes of

$$(\Delta g/g)_{M_2} \sim 9 \times 10^{-8} \cos^2 L \, ,$$

(7.69) $$(\Delta g/g)_{S_2} \sim 4 \times 10^{-8} \cos^2 L \, ,$$

$$(\Delta g/g)_{K_2} \sim 1 \times 10^{-8} \cos^2 L \, .$$

The M_2 tide is easily separated from the other two semi-diurnal tides by means of Fourier analysis of one month's gravimeter data. However, separation of the S_2 and K_2 tides requires at least one year of continuous gravimeter data. Because of gravimeter drift and long-period tides, such a separation is not easy to obtain [see Barsenkov (1967) for a partial separation of S_2 and K_2 using 19 months of data taken at Talgar, U.S.S.R.].

Experimental measurements of the combined S_2 and K_2 tides (amplitude $\sim 5 \times 10^{-8}$ g) are found to agree with the predictions of Newtonian gravitation (coupled with reasonable models for the structure of the Earth) to a precision of 2 per cent (Harrison, Ness, Longman, Forbes, Kraut, and Slichter 1963; Pariiskii, Barsenkov, Volkov, Gridnev, and Kramer 1967). Thus any discrepancy between Newtonian theory and experiment for this component of the tides must be less than one part in

10^9 at any latitude for which reliable gravimeter data is available. By comparing equation (7.68) with this experimental limit, we find that α_2 must satisfy

(7.70) $\qquad |\alpha_2| < 3 \times 10^{-2}.$

We have used the semi-diurnal sidereal tide (2Ω) to put an experimental upper limit on the value of α_2 (eq. [7.70]) rather than the larger diurnal sidereal tide (Ω), because agreement between Newtonian theory and observation is not as good for the diurnal as for the semi-diurnal tidal components, possibly because of diurnal effects due to heating by the Sun (Harrison et al. 1963). It may be possible to improve this upper limit by as much as an order of magnitude, by analysing long (one year) Earth-tide time series, coupled with improved models for the effects of ocean tides on the gravimeter measurements (Berger et al. 1971, Harrison et al. 1972).

We can also put an experimental limit on the size of the predicted yearly variation in the Earth's rotation frequency Ω with amplitude, according to the PPN formalism (cf. Table 7.I)

(7.71) $\qquad (\Delta\Omega/\Omega)_{PPN} \approx (\tfrac{2}{3}\alpha_2 + \alpha_3 - \alpha_1)(3 \times 10^{-9}).$

The observed yearly variation in the Earth's rotation rate (measured by comparing astronomical time with atomic time standards) has an amplitude of

(7.72) $\qquad (\Delta\Omega/\Omega)_{OBSERVED} \approx 4 \times 10^{-9}$

(Smith and Tucker 1953). But this variation can be readily understood

using Newtonian geophysics: it is produced by an annual variation in the angular momentum of the atmosphere due to seasonal changes in wind patterns and by a long-period (one year) Earth-tide produced by the Sun (Mintz and Munk 1953; see also Melchior 1966). These calculations yield agreement with the observed variation in Ω with uncertainties around 15 per cent, hence the PPN variation in the Earth's rotation rate must satisfy

$$(7.73) \qquad (\Delta\Omega/\Omega)_{PPN} < 6 \times 10^{-10} .$$

Equations (7.71) and (7.73) thus show that the PPN parameter combination $(2/3\,\alpha_2 + \alpha_3 - \alpha_1)$ must satisfy

$$(7.74) \qquad \left| \tfrac{2}{3}\,\alpha_2 + \alpha_3 - \alpha_1 \right| < 0.2 .$$

Although the zonal yearly variation in the Earth's rotation rate (Table 7.I) is apparently larger than the spherical variation, we have ignored it, because the limit set on α_2 by the Earth tides (3×10^{-2}), makes this effect too small to be discernible. Thus we have focused on the spherical variation in Ω in order to set a limit (eq. [7.74]) on the combination $(2/3\,\alpha_2 + \alpha_3 - \alpha_1)$.

We now combine the three experimentally determined limits on the PPN preferred-frame parameters, equations (7.48), (7.70), and (7.74) and obtain individual upper limits on the values of α_1, α_2, and α_3, as listed in Table 7.II. For comparison, Table 7.II also lists the predicted values for α_1, α_2, and α_3 for all the metric theories which until now, were condisered viable, i.e. agreed with the light-deflection and time-delay tests.

From Table 7.II, we can see immediately that the stratified theories due to Page and Tupper, Yilmaz, Papapetrou, Ni, Coleman, and Rosen cannot be correct theories of gravitation -- they disagree violently with experiment. In fact, we can be more general. We have shown (Sect. 5) that any Stratified Theory (with time-orthogonal, conformally flat space slices) which agrees with light-deflection and time-delay experiments must have $\alpha_1 \sim -8$, which is forty times larger than our experimental upper limit. Hence no "Stratified Theory" of gravity, past, present, or future, can be the correct theory of gravity.

A second conclusion emerges from Table 7.II: the squared magnitude of the cosmological vector field (K) in the Vector-Metric Theory must satisfy

$$(7.75) \qquad K^2 < 3 \times 10^{-2},$$

in order to agree (within the experimental uncertainty) with Earth-tide data. From this point of view, these results are complementary to the Hughes-Drever (isotropy of inertial mass) and "ether-drift" experiments discussed in Sect. 2. Those experiments put limits on the strengths of cosmological vector or tensor fields which couple to matter's nuclear or electromagnetic energy (Peebles 1962, Peebles and Dicke 1962, Dicke 1964), while our results put limits on vector or tensor fields which couple to matter's gravitational energy. There is a wide class of such vector-metric and tensor-metric theories (Sect. 5) and the limits on α_1, α_2, and α_3 should be pushed as low as possible in order to put more stringent limits on these cosmological vector and tensor fields (see Subsect. 7·9).

A third conclusion obtained from Table 7.II is that the limits on α_1, α_2, and α_3 do not distinguish between general relativity and scalar-tensor theories; these theories are $\underline{\text{not}}$ preferred-frame theories and predict no preferred-frame effects (for a discussion see Sect. 5).

Throughout this section we have assumed that the solar system moves through the Universe with a velocity $\underset{\sim}{w}$ equal to its nearly circular orbital velocity around the Galaxy ($\sim$ 200 km/sec in the direction $\ell^{II} = 90^\circ$, $b^{II} = 0^\circ$). A more realistic value for $\underset{\sim}{w}$ would be the solar system's velocity through the cosmic microwave radiation, when it is ultimately measured with confidence. Current measurements of this velocity, obtained by studying the anisotropy in the measured temperature of the microwave radiation (caused by the Doppler shift) are not yet completely reliable (Conklin [1969]; Boughn, Fram, and Partridge [1971]). Those results which have been obtained, however, are in rough agreement with measurements of the solar system's velocity relative to clusters of galaxies, obtained by studies of galactic red-shifts (de Vaucouleurs and Peters 1968), and suggest a net velocity of $\sim$ 200 km/sec in the direction $\ell^{II} \sim 290^\circ$, $b^{II} \sim 24^\circ$ (see Sciama [1971] for a discussion). But because of the experimental uncertainties, this value for our velocity relative to the Universe should not be given much weight at this time.

For the sake of illustration, however, one can repeat the calculations of Subsect. 7·3 and 7·4 using this new value for $\underset{\sim}{w}$. The resulting limits on the PPN parameters α_1, α_2, and α_3 are

(7.76) $|\alpha_1| < 0.1$, $|\alpha_2| < 2 \times 10^{-2}$, $|\alpha_3| < 8 \times 10^{-6}$,

which are not significantly different than those given in Table 7.II.

7·5. Experimental Disproof of Whitehead's Theory

Ever since its inception in 1922, Whitehead's theory of gravitation has been a thorn in Einstein's side, because it agrees with general relativity in its predictions for all the "classical tests" -- light bending, time delay, perihelion shifts. However, it has recently been shown (Will 1971d) to predict Earth-tides caused by the Galaxy which are 200 times larger than observations will permit.

To see this, we repeat the calculation of the locally measured gravitational constant (Subsect. 6·5), using the metric of Whitehead's theory. According to equation (5.83), the Whitehead metric is the same as a PPN metric with parameter values

$$\gamma = 1, \quad \beta = 1, \quad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = 0,$$

(7.77)

$$\zeta_1 = -6, \quad \zeta_3 = -1, \quad \zeta_4 = -1,$$

except for the following additional term in g_{oo}:

$$(7.78) \qquad \delta g_{oo} = 2 \int \frac{\rho' \rho''(\underset{\sim}{x} - \underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} \cdot \left[\frac{(\underset{\sim}{x}' - \underset{\sim}{x}'')}{|\underset{\sim}{x} - \underset{\sim}{x}''|} - \frac{(\underset{\sim}{x} - \underset{\sim}{x}'')}{|\underset{\sim}{x}' - \underset{\sim}{x}''|} \right] d\underset{\sim}{x}' \, d\underset{\sim}{x}'' \ .$$

This extra metric term changes only the equation of motion for the gravimeter via the Christoffel symbol Γ^α_{oo} -- other effects of δg_{oo} are either of post-post-Newtonian order or produce forces smaller than 10^{-9} g. The resulting change in the force $\underset{\sim}{F}$ is (cf. Subsect. 6·5):

$$(7.79) \quad \delta \underset{\sim}{F} \cdot \underset{\sim}{r}_{10}/r_{10} = -(M/r_{10}^2) \left[2 \sum_{k \neq 1,0} m_k/r_{1k} + \sum_{k \neq 1,0} m_k (\underset{\sim}{e}_r \cdot \underset{\sim}{r}_{1k})^2/r_{1k}^3 \right] .$$

Combining equation (7.79) with equations (6.98) and (6.99), and substituting the parameter values of equation (7.77), we obtain

$$F_r = - \sum_{k \neq 1, 0} \frac{m_k r_{10}^{\alpha} r_{10}^{\beta} (3 r_{1k}^{\alpha} r_{1k}^{\beta} - r_{1k}^2 \delta^{\alpha\beta})}{r_{1k}^5 r_{10}} - \frac{v_o^2}{r_p}$$

(7.80)

$$+ (M/r_p^2) \left[1 + 2 \sum_{k \neq 1, 0} m_k / r_{1k} + \sum_{k \neq 1, 0} m_k (\underset{\sim}{e}_r \cdot \underset{\sim}{r}_{1k})^2 / r_{1k}^3 \right].$$

The value of G identified with equation (7.80) is

(7.81)
$$G_{\text{WHITEHEAD}} = 1 + 2U + \sum_k U_k (\underset{\sim}{e}_k \cdot \underset{\sim}{e}_r)^2 .$$

Here U_k is the Newtonian gravitational potential due to the kth external body (including Sun, Moon, planets and stars) and $\underset{\sim}{e}_k$ is a unit vector from the Earth to the center of mass of the kth body. There are no "preferred frame" effects in equation (7.81), since $\alpha_1 = \alpha_2 = \alpha_3 = 0$ for Whitehead's theory.

Since the Whitehead metric must be calculated in a global Lorentz coordinate system of $\eta_{\alpha\beta}$ [or at best in a spacetime of constant curvature (Temple 1924)], the field due to the galaxy cannot be removed by transformation to a local inertial frame surrounding the solar system (as one would do in general relativity). Thus the anisotropic term in equation (7.81) will be dominated by the central regions of the galaxy:

(7.82)
$$G_{\text{WHITEHEAD}} = 1 + 2U + (M_{\text{gal}} / R_{\text{gal}}) (\underset{\sim}{e}_{\text{gal}} \cdot \underset{\sim}{e}_r)^2 .$$

The dominant effect is a twelve-hour sidereal-time Earth tide, analogous to the $(\underset{\sim}{w} \cdot \underset{\sim}{e}_r)^2$ tides of equation (6.101), with amplitude

148

(7.83) $\qquad (\Delta g/g)_{\text{WHITEHEAD}} \approx (2 \times 10^{-7}) \cos^2(L)$.

This is 200 times larger than the experimental limit of a part in 10^9 (Subsect. 7·4), and so, after 50 years, Whitehead's theory is ruled out.

7·6. Precession of an Orbiting Gyroscope

Since 1960, much effort has been directed toward testing theories of gravitation using an orbiting supercounducting gyroscope (Schiff [1960a], articles in Davies [1971]). The object of the experiment is to measure the precession of the gyroscope's spin axis $\underset{\sim}{S}$ relative to the distant stars as the gyroscope orbits the Earth. According to the PPN formalism, this precession is given by (Subsect. 6·4):

(7.84) $\qquad dS/ds = \underset{\sim}{\Omega} \times \underset{\sim}{S}$,

(7.85) $\qquad \underset{\sim}{\Omega} = \frac{1}{2} [\underset{\sim}{v} \times \underset{\sim}{a} + \underset{\sim}{\nabla} \times \underset{\sim}{h} + (2\gamma + 1)\underset{\sim}{v} \times \underset{\sim}{\nabla} U]$,

(7.86) $\qquad \underset{\sim}{h} = g_{oj} \underset{\sim}{e}_j$.

Here $\underset{\sim}{a}$ is the gyroscope's four-acceleration, which is zero for an orbiting body (free fall). In a PPN coordinate system at rest with respect to the Earth, equation (7.85) along with the expression for $g_{o\alpha}$ in Table 4.I, yields

(7.87) $\qquad \underset{\sim}{\Omega} = \frac{1}{4}(4\gamma + 4 + \alpha_1)\underset{\sim}{\nabla} \times \underset{\sim}{V} - \frac{1}{4}\alpha_1 \underset{\sim}{w}_{\oplus} \times \underset{\sim}{\nabla} U + \frac{1}{2}(2\gamma + 1)\underset{\sim}{v} \times \underset{\sim}{\nabla} U$,

where

(7.88) $\qquad \underset{\sim}{V} = V_\alpha \underset{\sim}{e}_\alpha$.

Since the Earth is momentarily at rest in the PPN coordinate system,

V_α can be shown to have the form

$$(7.89) \qquad V_\sim = \int \frac{\rho' v'_\sim}{|x_\sim - x'_\sim|} \, dx' \approx -\frac{1}{2} \frac{r_\sim \times J_\sim}{r^3} + O\left(\frac{1}{r^3}\right) ,$$

where $J_\sim$ is the Earth's angular momentum vector, and $r_\sim$ is the vector from the Earth to the gyroscope. Using

$$(7.90) \qquad U \sim m/r ,$$

we get

$$\Omega_\sim = -\frac{1}{8}(4\gamma + 4 + \alpha_1) \frac{1}{r^3}[J_\sim - 3\hat{r}_\sim(\hat{r}_\sim \cdot J_\sim)] + \frac{1}{4}\alpha_1 \frac{m}{r^3}(w_{\oplus\sim} \times r_\sim)$$

$$(7.91)$$

$$-\frac{1}{2}(2\gamma + 1)\frac{m}{r^3}(v_\sim \times r_\sim) .$$

The first term in equation (7.91) is called the Lens-Thirring precession or the "dragging of inertial frames" (for a detailed discussion of this effect see Thorne [1971]). For a gyroscope in a polar orbit with its spin axis directed normal to the orbital plane, this dragging of inertial frames produces a secular precession of the spin axis with amplitude (see Fig. 7.3)

$$(7.92) \qquad \begin{bmatrix} \text{Lens-Thirring} \\ \text{Precession} \end{bmatrix} \sim \frac{1}{8}(4\gamma + 4 + \alpha_1)[0.''05 \text{ of arc in one year}] .$$

The second term in equation (7.91) is a preferred-frame effect which produces only periodic precessions of amplitude

$$(7.93) \qquad \begin{bmatrix} \text{Preferred Frame} \\ \text{Precessions} \end{bmatrix} \sim \alpha_1[10^{-3} \text{ '' of arc}] ,$$

which are probably too small to be measurable. The third term is called the Geodetic precession, caused by the curvature of space around the

150

Earth. For a polar orbit with the gyroscope axis in the plane of the orbit, the Geodetic precession produces a secular rotation of the spin axis of amplitude (see Fig. 7.3)

(7.94)   [Geodetic Precession] $\sim \frac{1}{3}(1+2\gamma)[\,7''$ of arc in one year$]$.

For a review of experimental and theoretical aspects of this experiment, to be attempted by Fairbank and collaborators at Stanford University, see papers by Everitt, Fairbank and O'Connell in Davies (1971), and lectures by Fairbank in this volume.

7·7. The Nordtvedt Effect

The breakdown in the Equivalence Principle for massive, self-gravitating bodies, which many theories predict, has a variety of observable consequences. In quasi-Newtonian language, this Equivalence-Principle violation may be expressed by attributing to each massive body a "passive gravitational mass tensor" $m_p^{\alpha\beta}$ which may differ from its inertial mass m. According to equation (6.42), the quasi-Newtonian part of the body's acceleration may be written

(7.95) $a^\alpha = (m_p^{\alpha\beta}/m)\,\partial\mathfrak{U}/\partial x^\beta$,

where $\mathfrak{U}$ is the quasi-Newtonian gravitational potential and $m_p^{\alpha\beta}$ is given according to the PPN formalism by

(7.96) $m_p^{\alpha\beta} = m\left\{(1 - \eta\,|\Omega\,|/m)\delta^{\alpha\beta} + \xi\,\Omega^{\alpha\beta}/m\right\}$,

where

(7.97) $\eta = 4\beta - \gamma - 3 - \alpha_1 + \alpha_2 - \zeta_1$,

151

(7.98) $\qquad \xi = \zeta_1 - \alpha_2 - \zeta_2$,

and where Ω and $\Omega^{\alpha\beta}$ are the body's internal gravitational energy and gravitational energy tensor (see Table 6.II).

The most important consequence of the Nordtvedt effect is a polarization of the Moon's orbit about the Earth (Nordtvedt [1968c]). Because the Moon's self-gravitational energy is much smaller than the Earth's, the Nordtvedt effect causes the Earth and Moon to fall toward the Sun with slightly different accelerations. Including their mutual attraction, we have (from equations (6.50) and (6.52), neglecting quadrupole moments, and post-Newtonian effects on <u>active</u> gravitational masses):

$$(a_\oplus)^\alpha = - (m^{\alpha\beta}/m)(MR^\beta/R^3 - m_o r^\beta/r^3) \ ,$$

(7.99)

$$(a_{\mathbb{C}})^\alpha = - (MR_o{}^\alpha/R_o{}^3 + mr^\alpha/r^3) \ ,$$

where m, M, and m_o are the masses of the Earth, Sun, and Moon respectively; $\underset{\sim}{R}$ is the vector from the Sun to the Earth, $\underset{\sim}{r}$ is the vector from the Earth to the Moon, and $\underset{\sim}{R}_o$ is the vector from the Sun to the Moon; $m^{\alpha\beta}$ is the Earth's passive gravitational mass tensor. We have neglected the contribution of the Moon's Ω and $\Omega^{\alpha\beta}$. Since $r \ll R$, we can make the approximation

(7.100) $\qquad R_o{}^\alpha/R_o{}^3 \simeq R^\alpha/R^3 - (r_\beta/R^5)(3R^\alpha R^\beta - R^2 \delta^{\alpha\beta})$.

Then the <u>relative</u> Earth-Moon acceleration a^α, defined by

(7.101) $\qquad a^\alpha = (a_{\mathbb{C}})^\alpha - (a_\oplus)^\alpha$,

can be calculated using equations (7.96), (7.99), and (7.100), giving

$$a^\alpha = -\mu r^\alpha/r^3 - (\eta \delta^{\alpha\beta}|\Omega|/m - \xi \Omega^{\alpha\beta}/m)(MR^\beta/R^3)$$

(7.102)

$$-\xi(\Omega^{\alpha\beta}/m - \frac{1}{3}\delta^{\alpha\beta}\Omega/m)m_o r^\beta/r^3 + (3R^\alpha R^\beta - R^2\delta^{\alpha\beta})Mr^\beta/R^5,$$

where

(7.103) $\mu = m_o + m - (\eta + \frac{1}{3}\xi)|\Omega|$.

The first term in equation (7.102) is the Newtonian acceleration between the Earth and Moon and the second term is the difference between the Earth's and Moon's acceleration toward the Sun (Nordtvedt effect). The third term results from the fact that the acceleration of the Earth toward the Moon has a component normal to the Earth-Moon direction (caused by the anisotropy in $\Omega^{\alpha\beta}$); it is two orders of magnitude smaller than the second term, and will be neglected. The fourth term is the classical tidal perturbation on the Moon's orbit; since it is a purely non-relativistic perturbation, we will not consider it further. Hence the equation of motion of the Moon relative to the Earth, including the perturbation arising from the Nordtvedt effect is

(7.104) $$a^\alpha = -\mu r^\alpha/r^3 - (\eta \delta^{\alpha\beta}|\Omega|/m - \xi \Omega^{\alpha\beta}/m)(MR^\beta/R^3) .$$

For an axially symmetric Earth which rotates with angular velocity λ with moment of inertia I, and which has negligible stresses, one can show that $\Omega^{\alpha\beta}$ has the form (in a coordinate system whose x - y plane is the plane of the Earth's orbit around the Sun, and whose x-axis points in the direction of the vernal equinox):

153

$$(7.105) \qquad \Omega^{\alpha\beta} = \frac{1}{3}\,\delta^{\alpha\beta}\Omega + \frac{1}{3}\Psi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - 3\sin^2\theta & 3\sin\theta\cos\theta \\ 0 & 3\sin\theta\cos\theta & 1 - 3\cos^2\theta \end{pmatrix} \;,$$

where $\theta = 23\text{-}1/2^{\circ}$ (the "tilt" of the Earth), and (Chandrasekhar and Lebovitz 1962a)

$$(7.106) \qquad \Psi \simeq -\frac{1}{2}\lambda^2 I \;.$$

For the Earth we get [see MacDonald (1966) for values for Ω]:

$$(7.107) \qquad \Omega/m = -4.85 \times 10^{-10}, \qquad \Psi/m = -3.95 \times 10^{-13} \;.$$

We assume that the Moon's unperturbed orbit is circular with angular velocity ω_o and in the x - y plane, and also that the orbit of the Earth around the sun is circular with angular velocity ω_s.

We work in an inertial coordinate system centered at the Sun. Then the acceleration a^α and the angular momentum per unit mass h^α are given by

$$(7.108) \qquad a^\alpha = d^2 r^\alpha/dt^2 \qquad \text{and} \qquad h^\alpha = [\,\underset{\sim}{r} \times (d\underset{\sim}{r}/dt)]^{\,\alpha} \;,$$

and the following relations hold

$$d^2 r/dt^2 = \underset{\sim}{r} \cdot \underset{\sim}{a}/r + h^2/r^3 \;,$$
$$(7.109)$$
$$dh^\alpha/dt = (\underset{\sim}{r} \times \underset{\sim}{a})^\alpha \;.$$

Thus, by making use of equations (7.104) and (7.105), and by defining

$$\delta a^o = [\,(\eta + \frac{1}{3}\xi\,)\,|\Omega|\,/m - \frac{1}{3}\xi\Psi/m]\ M/R^2 \;,$$
$$(7.110)$$
$$\delta a' = \xi(\Psi/m)\sin^2\theta\ (M/R^2) \;,$$

we obtain $(\Lambda = \omega_0 - \omega_s)$

$$\frac{d^2 r}{dt^2} = \frac{h^2}{r^3} - \frac{\mu}{r^2} + (\delta a^0 + \frac{1}{2} \delta a') \cos \Lambda t$$

(7.111)
$$- \frac{1}{2} \delta a' \cos [(\omega_0 + \omega_s)t + 2\varphi_0] \, ,$$

$$\frac{dh}{dt} = - r \left\{ (\delta a^0 + \frac{1}{2} \delta a') \sin \Lambda t - \frac{1}{2} \delta a' \sin [(\omega_0 + \omega_s)t + 2\varphi_0] \right\} \, .$$

where the phase φ_0 is related to the angular position of the Sun and Moon at $t = 0$. We next linearize about a circular orbit:

$$h = h_0 + \delta h \, ,$$

(7.112)

$$r = r_0 + \delta r \, ,$$

and use $\mu/r_0^3 = h_0^2/r_0^4 = \omega_0^2$. Integration of the resulting equations gives

(7.113)
$$\delta h = \frac{r_0}{\Lambda} (\delta a^0 + \frac{1}{2} \delta a') \cos \Lambda t - \frac{r_0 \delta a'}{2(\omega_0 + \omega_s)} \cos [(\omega_0 + \omega_s)t + 2\varphi_0] \, ,$$

$$\delta r = (\delta a^0 + \frac{1}{2} \delta a') \left[\frac{1 + 2\omega_0/\Lambda}{\omega_0^2 - \Lambda^2} \right] \cos \Lambda t$$

(7.114)

$$- \frac{1}{2} \delta a' \left[\frac{1 + 2\omega_0/(\omega_0 + \omega_s)}{\omega_0^2 - (\omega_0 + \omega_s)^2} \right] \cos [(\omega_0 + \omega_s)t + 2\varphi_0] \, .$$

The first term in equation (7.114) is the perturbation due to the isotropic part of the Nordtvedt effect (Nordtvedt 1968c). It represents a polarization of the Earth-Moon system by the external field of the Sun. This "polarization of the orbit" is always directed toward the Sun as it rotates around the Earth (cf. Fig. 7.4). The second term in equation (7.114)

155

is a direct result of the anisotropy in $m^{\alpha\beta}$. It is a polarization which rotates in a sense opposite to that of the rotating Sun.

Using equations (7.107), (7.110) and (7.114) and the values $M/R^2 = 5.9 \times 10^{-6}$ km/sec^2 and $\theta = 23\text{-}1/2°$, and making use of the fact that $\omega_s \simeq \omega_o/13 = 2 \times 10^{-7}$ sec^{-1}, we get for the isotropic Nordtvedt effect

(7.115) $\delta r_{(\text{isotropic})} \simeq 840 \; (\eta + \frac{1}{3}\xi) \; \cos \; (\omega_o - \omega_s)t \quad$ cm,

and for the anisotropic effect

(7.116) $\delta r_{(\text{anisotropic})} \simeq - \; 5.3 \times 10^{-2} \xi \; \cos \left[(\omega_o + \omega_s)t + 2\varphi_o \right] \quad$ cm.

Hoped-for accuracy of laser ranging to the corner reflectors on the Moon is $\sim$ 10 cm, so analysis of the ranging data should yield an estimate or an upper limit for the parameter $\eta + \frac{1}{3}\xi$. In terms of PPN parameters,

(7.117) $\eta + \frac{1}{3}\xi = 4\beta - \gamma - 3 - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2$.

The amplitude of the anisotropic effect (eq. [7.116]) is four orders of magnitude smaller than that of the isotropic effect, putting it well below the expected 10-cm accuracy of lunar laser ranging, and hence unmeasurable (Will 1971b).

The predictions of various theories of gravity for the parameters $\eta + \frac{1}{3}\xi$ and ξ are shown in Table 7.III.

Other potentially observable consequences of the Nordtvedt effect are discussed by Nordtvedt (1968a, 1970a, 1971a,b). These include shifts

in the stable Lagrange points of Jupiter (measurable by ranging to the Trojan asteroids), modifications of Kepler's third law, and polarizations of orbits produced by Jupiter.

7·8. Other Perturbations on the Earth-Moon System

There are a variety of other perturbations of the Moon's orbit which are potentially measurable by lunar laser ranging. These include

i) periodic perturbations in the Earth-Moon range produced by the non-linear superposition of the gravitational fields of the Sun and the Earth (Baierlein [1967], Krogh and Baierlein [1968], Nordtvedt [1972]). Further research is needed before the complete PPN-parameter-dependences of these effects are known with confidence.

ii) "preferred-frame" perturbations caused by the solar system's motion through the Universe. These periodic perturbations depend on PPN parameters α_1, α_2 and α_3, and some may have amplitudes as large as 70 km. For a partial catalogue of these effects, see Nordtvedt and Will (1972).

7·9. White-Dwarf Pulsations and Preferred-Frame Parameters

Recent research by Ni (1972b) has shown that it may be possible to push the limits on α_1, α_2 and α_3 as low as 10^{-6} by means of studies of the pulsations of white dwarf stars. According to the PPN formalism, motion of pulsating white dwarfs relative to the mean rest-frame of the Universe should produce instabilities, i.e. exponential growth or decay of the amplitudes of their pulsation. Observational studies of white-dwarf pulsations put stringent limits on pulsation instabilities. These limits,

coupled with studies of the proper motions of white dwarfs put stringent

limits -- as low as 10^{-6} -- on α_1, α_2 and α_3, provided one assumes

there are no other sources of pulsation damping which could stabilize

these preferred-frame instabilities. Unfortunately, there remain crucial

uncertainties in the theory of the damping of white dwarf pulsations,

which must be understood before white dwarfs may become a high-

precision tool for measuring PPN parameters.

7·10. Secular Variation in the Newtonian Gravitational Constant

Several metric theories of gravity predict that the Newtonian

gravitational constant as measured far from the solar system in the

rest-frame of the Universe should vary with time as the Universe evolves.

For Scalar-Tensor theories (eq. [5.34])

(7.118)
$$\left(\frac{\dot{G}}{G}\right)_{Today} = \left[1 + \frac{2\omega'}{(3+2\omega)^2}\right] \frac{1}{\varphi}\left(\frac{d\varphi}{dt}\right)_{Today} \quad ,$$

and for the Vector-Metric theory discussed in Sect. 5,

(7.119)
$$\left(\frac{\dot{G}}{G}\right)_{Today} = -\left[\frac{K}{1 + \frac{1}{2}K^2}\right]\left(\frac{dK}{dt}\right)_{Today} \quad ,$$

where φ and K are the magnitudes of the scalar and vector fields of

those theories. Since these fields vary as a result of the evolution of the

Universe, their logarithmic derivatives should be of the order of the

inverse Hubble time, i.e.

(7.120)
$$\frac{1}{\varphi}\left(\frac{d\varphi}{dt}\right) \sim \frac{1}{K}\left(\frac{dK}{dt}\right) \sim \frac{1}{H_o} \sim \frac{1}{10^{10} \text{ years}} \quad .$$

Thus for Scalar-Tensor theories,

(7.121) $(\dot{G}/G)_{\text{Scalar-Tensor}} \sim 10^{-10} \text{ (years)}^{-1}$,

and for the Vector-Metric Theory, taking into account the experimental upper limit on the value of K^2 (eq. [7.75]),

(7.122) $(\dot{G}/G)_{\text{Vector-Metric}} \sim 10^{-12} \text{ (years)}^{-1}$.

Dicke has suggested that a variation in G of the order suggested by equation (7.121) would not be inconsistent with geophysical, planetary and astrophysical data (see Dicke and Peebles [1965] for discussion and references).

Such a variation in G should cause the orbital periods of planets to vary when compared with an atomic-time scale, and recent analysis of planetary radar-ranging data has put a limit on such variations of

(7.123) $(\dot{G}/G) < 4 \times 10^{-10} \text{ (years)}^{-1}$

(Shapiro, Smith, Ash, Ingalls, and Pettengill 1971). A more useful limit on $\dot{G}$ must therefore await further improvements (or more data!). Lunar laser ranging may also be able to put a limit on $\dot{G}$, as long as the secular effects of the Earth-Moon tidal interactions do not obscure the effects of a changing G.

7·11. Currently Viable Metric Theories (as of June, 1972)

We have assumed that the correct theory of gravity must be a metric theory (see Sect. 3 for a partial justification), and have used the PPN formalism to analyse experimental tests of these theories. Experiments to date have proclaimed many theories to be non-viable:

159

(i) Conformally Flat Theories -- disagree violently with light-deflection and time-delay experiments.

(ii) Stratified Theories (with time-orthogonal conformally flat space slices) -- predict "preferred-frame" effects in violent disagreement with observations.

(iii) Whitehead's Theory -- predicts Galaxy-induced Earth tides, in violent disagreement with observations.

Other theories are still viable, or may be made so by an appropriate choice of a "coupling constant"

(i) General relativity -- agrees with all experiments to date.

(ii) Scalar-Tensor theories with $\omega > 6$ -- agree with all experiments. Improvements in the light-deflection and time delay experiments may push ω higher (toward the general relativity limit).

(iii) Vector-Metric Theory with $K^2 < 3 \times 10^{-2}$ -- agrees with all experiments. Improved Earth-tide measurements may push K^2 lower (toward the general relativity limit).

(iv) Theories yet to be invented or found in the literature -- Lurking in the literature may be other metric theories which pass all experimental tests; and theories currently being invented (in particular by Ni and by Nordtvedt and Hellings) may also be viable.

Thus there is still an important role for the theorist in the field of Experimental Gravitation. He must continue to examine theories of gravity, search for new experiments, evaluate the significance of current experiments, until the results of the experimenters can allow him to say he has confidence in only one theory of gravitation.

REFERENCES

Ageno, M. and Amaldi, E. 1966, *Academia Nazionale dei Lincei*, 8, 1 (Series VIII).

Allen, C. W. 1963, *Astrophysical Quantities* (2nd ed.; London The Athlone Press).

Alväger, T., Farley, F. J. M., Kjellman, J., and Wallin, I. 1964, *Phys. Letters*, 12, 260.

Anderson, J. D., Esposito, P. B., Martin, W. L., and Muhleman, D. O. 1971, in *Proceedings of the Conference on Experimental Tests of Gravitational Theories*, ed. R. W. Davies (NASA-JPL Technical Memorandum 33-499).

Baierlein, R. 1967, *Phys. Rev.*, 162, 1275.

Bailey, J. and Picasso, E. 1970, *Prog. in Nucl. Phys.*, 12, 43.

Barsenkov, S. N. 1967, *Bull. (Izv.) Acad. Sci. U.S.S.R., Earth Phys.*, 3, 170.

Belinfante, F. J. and Swihart, J. C. 1957a, *Ann. Phys. (N.Y.)*, 1, 168.
_____ 1957b, *ibid.*, 1, 196.
_____ 1957c, *ibid.*, 2, 81.

Berger, J., Haubrich, R. A., Block, B., and Gilbert, F. 1971, private communication.

Bergmann, O. 1956, *Am. J. Phys.*, 24, 39.

Bergmann, P. G. 1968, *Int. J. Theor. Phys.*, 1, 25.

Birkhoff, G. D. 1943, *Proc. Nat. Acad. Sci.*, 29, 231.

Bondi, H. 1957, *Rev. Mod. Phys.*, 29, 423.

Boughn, S. P., Fram, D. M., and Partridge, R. B. 1971, *Ap. J.*, 165, 439.

Braginsky, V. B. and Panov, V. I. 1971, Zh. Eksp. Teor. Fiz., 61, 875.

Brans, C. H. 1962a, Phys. Rev., 125, 388.

_____ 1962b, Ibid., 125, 2194.

Brans, C. H. and Dicke, R. H. 1961, Phys. Rev., 124, 925.

Burke, W. L. 1971, J. Math. Phys., 12, 401.

Champeney, D. C., Isaak, G. R., and Khan, A. M. 1963, Phys. Letters, 7, 241.

Chandrasekhar, S. 1965, Ap. J., 142, 1488.

_____ 1969, Ibid., 158, 45.

Chandrasekhar, S., and Contopoulos, G. 1967, Proc. Roy. Soc., 298A, 123.

Chandrasekhar, S. and Esposito, P. 1970, Ap. J., 160, 153.

Chandrasekhar, S. and Lebovitz, N. R. 1962a, Ap. J., 135, 238.

_____ 1962b, Ibid., 136, 1032.

Chandrasekhar, S., and Nutku, Y. 1969, Ap. J., 158, 55.

Clark, G. L. 1954, Proc. Roy. Soc. Edinburgh, 64A, 49.

Coleman, C. J. 1971, J. Phys. A., 4, 611.

Conklin, E. K. 1969, Nature, 222, 971.

Corinaldesi, E. and Papapetrou, A. 1951, Proc. Roy. Soc., 209A, 259.

Davies, R. W., ed. 1971, Proceedings of the Conference on Experimental Tests of Gravitational Theories (NASA-JPL Technical Memorandum 33-499).

De Sitter, W. 1916, M.N.R.A.S., 67, 155.

Dicke, R. H. 1960, Am. J. Phys., 28, 344.

_____ 1964, The Theoretical Significance of Experimental Relativity (New York: Gordon and Breach).

Dicke, R. H. 1970, _Gravitation and the Universe_, Jayne Lectures for

 1969 (Philadelphia: American Philosophical Society).

Dicke, R. H. and Peebles, P. J. E. 1965, _Space Science Reviews_,

 4, 419.

Dixon, W. G. 1964, _Nuovo Cimento_, 34, 317.

_____ 1970, _Proc. Roy. Soc._, 314A, 499.

Durbin, R. P., Loar, H. H., and Havens, W. W., Jr. 1952, _Phys. Rev._

 88, 179.

Dykla, J. J. 1972, paper in preparation.

Eddington, A. S. 1922, _The Mathematical Theory of Relativity_ (London:

 Cambridge University Press).

Einstein, A. 1912, _Ann. Physik_, 38, 443.

_____ 1916, _ibid._, 49, 769.

Einstein, A., and Fokker, A. D. 1914, _Ann. Physik_, 44, 321.

Einstein, A., Infeld, L., and Hoffman, B. 1938, _Ann. Math._, 39, 65.

Eötvös, R. V., Pekar, D., and Fekete, E. 1922, _Ann. Physik_, 68, 11.

Estabrook, F. E. 1969, _Ap. J._, 158, 81.

Farley, F. J. M., Bailey, J., Brown, R. C. A., Giesch, M.,

 Jöstlein, H., van der Meer, S., Picasso, E., and Tannenbaum, M.

 1966, _Nuovo Cimento_, 45, 281.

Fock, V. 1964, _The Theory of Space, Time and Gravitation_ (New York:

 The Macmillan Co.).

Frisch, D. H. and Smith, J. H. 1963, _Am. J. Phys._, 31, 342.

Harrison, J. C., Farrell, W. E., Johnson, L. E., and Levine, J.

 1972, private communication.

Harrison, J. C., Ness, N. F., Longman, I. M., Forbes, R. F. S.,
Kraut, E. A., and Slichter, L. B. 1963, J. Geophys. Res.,
68, 1497.

Hill, J. M., 1971, M.N.R.A.S., 153, 7P.

Hoyle, F. 1960, M.N.R.A.S., 120, 266.

Hoyle, F. and Narlikar, J. V. 1963, Proc. Roy. Soc., 273A, 1.

_____ 1964, ibid., 278A, 465.

Ives, H. E. and Stilwell, G. R. 1938, J. Opt. Soc. Am., 28, 215.

_____ 1941, ibid., 31, 369.

Jordan, P. 1948, Astr. Nachr., 276, 193.

_____ 1955, Schwerkraft and Weltall, (Braunschweig:
Friedrich Vieweg and Son).

Kleppner, D., Vessot, R. F., and Ramsey, N. F. 1970, Ap. and Space
Sci., 6, 13.

Kovács, S. J., Will, C. M., and Thorne, K. S. 1972, B.A.P.S.,
17, 141.

Krogh, C. and Baierlein, R. 1968, Phys. Rev., 175, 1576.

Kustaanheimo, P. 1957, Ann. Akad. Sci. Fenn., 228, 3.

_____ 1966, Phys. Letters, 23, 75.

Kustaanheimo, P. and Nuotio, V. S. 1967, Relativistic Theories of
Gravitation. I. One-Body Problem (Dept. of Applied Mathe-
matics, University of Helsinki).

Landau, L. D., and Lifschitz, E. M. 1962, The Classical Theory of
Fields (2nd ed.; Reading, Massachusetts: Addison-Wesley Press).

Lee, D., and Lightman, A. 1972, papers in preparation.

Lee, D., Lightman, A., and Thorne, K. S. 1972, papers in
preparation.

Levi-Civita, T. 1964, The n-Body Problem in General Relativity
(Dordrecht: D. Reidel Publishing Co.).

Lichtenberg, D. B. 1965, in Meson and Baryon Spectroscopy
(New York: Springer-Verlag), p. 18.

Littlewood, D. E. 1953, Proc. Camb. Phil. Soc., 49, 90.

MacDonald, G. F. 1966, in Advances in Earth Science, ed.
P. M. Hurley (Cambridge, Mass: MIT Press).

Mathisson, M. 1937, Acta Physica Polonica, 6, 163.

Melchior, P. 1966, The Earth Tides (Oxford: Pergamon Press).

Milne, E. A. 1937, Proc. Roy. Soc., 160A, 7.

_____ 1948, Kinematical Relativity (London: Oxford
University Press).

Mintz, Y., and Munk, W. 1953, M.N.R.A.S. Geophys. Suppl., 6,
566.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972,
Gravitation (San Francisco: W. H. Freeman and Company,
publishers).

Muhleman, D. O., Ekers, R. D., and Fomalont, E. B. 1970,
Phys. Rev. Letters, 24, 1377.

Muhleman, D. O. and Reichley, P. 1964, Jet Propulsion Laboratory
Space Program Summary No. 37-29, Vol. IV, 342.

Ni, W.-T. 1972a, Ap. J., in press.

_____ 1972b, paper in preparation.

_____ 1972c, paper in preparation.

Nordstrøm, G. 1912, Phys. Zeit., 13, 1126.

_____ 1913, Ann. Physik, 42, 533.

_____ 1914, ibid., 43, 1101.

Nordtvedt, K., Jr. 1968a, Phys. Rev., 169, 1014.

_____ 1968b, ibid., 169, 1017.

_____ 1968c, ibid., 170, 1186.

_____ 1970a, Icarus, 12, 1.

_____ 1970b, Ap. J., 161, 1059.

_____ 1971a, Phys. Rev. D., 3, 1683.

_____ 1971b, in Proceedings of the Conference on

Experimental Tests of Gravitational Theories, ed. R. W.

Davies (NASA-JPL Technical Memorandum 33-499).

_____ 1972, paper in preparation.

Nordtvedt, K., Jr., and Hellings, R. 1972, paper in preparation.

Nordtvedt, K., Jr., and Will, C. M. 1972, Ap. J., in press.

Nutku, Y. 1969a, Ap. J., 155, 999.

_____ 1969b, ibid., 158, 991.

Ockham, William of, 1495, Super quattuor libros sententiarum

subtilissimae quaestiones (Lyons).

Page, C., and Tupper, B. O. J. 1968, M.N.R.A.S., 138, 67.

Papapetrou, A. 1951, Proc. Roy. Soc., 209A, 248.

_____ 1954a, Math. Nachr., 12, 129.

_____ 1954b, ibid., 12, 143.

_____ 1954c, Z. Phys., 139, 518.

Pariiskii, N. N., Barsenkov, S. N., Volkov, V. A., Gridnev, D. G.,

and Kramer, M. V. 1967, Bull. (Izv.) Acad. Sci. U.S.S.R.,

Earth Phys., 2, 107.

Peebles, P. J. E. 1962, Ann. Phys. (N.Y.), 20, 240.

Peebles, P. J. E., and Dicke, R. H. 1962, Phys. Rev., 127, 629.

Poincaré, H. 1906, Rend. Circ. Mat. di Palermo, 21, 166.

Pound, R. V. and Rebka, G. A. 1960, Phys. Rev. Letters, 4, 337.

Pound, R. V. and Snider, J. L. 1965, Phys. Rev., 140, B788.

Rayner, C. B. 1954, Proc. Roy. Soc., 222A, 509.

_____ 1955, ibid., 232A, 135.

Reichley, P. E. 1971, in Proceedings of the Conference on Experi-
mental Tests of Gravitational Theories, ed. R. W. Davies
(NASA-JPL Technical Memorandum 33-499).

Robertson, H. P. 1962, in Space Age Astronomy, eds. A. J. Deutsch
and W. B. Klemperer (New York: Academic Press), p. 228.

Robertson, H. P., and Noonan, T. W. 1968, Relativity and Cos-
mology (Philadelphia: W. B. Saunders).

Roll, P. G., Krotkov, R., and Dicke, R. H. 1964, Ann. Phys. (N.Y.),
26, 442.

Rose, R. D., Parker, H. M., Lowry, R. A., Kuhlthau, A. R., and
Beams, J. W. 1969, Phys. Rev. Letters, 23, 655.

Rosen, N. 1971a, Phys. Rev. D, 3, 2317.

_____ 1971b, General Relativity and Gravitation, 2, 129.

Ross, D. K. and Schiff, L. I. 1966, Phys. Rev., 141, 1215.

Rossi, B. and Hall, D. B. 1941, Phys. Rev., 59, 223.

Schiff, L. I. 1959, Proc. Nat. Acad. Sci. U.S.A., 45, 69.

_____ 1960a, ibid., 46, 871.

_____ 1960b, Am. J. Phys., 28, 340.

Schiff, L. I. 1967, in Relativity Theory and Astrophysics.

I. Relativity and Cosmology, ed. J. Ehlers (Providence, R.I.:

American Mathematical Society).

Schild, A. 1956, Proc. Roy. Soc., 235A, 202.

_____ 1962, in Evidence for Gravitational Theories,

ed. C. Møller (New York: Academic Press).

Sciama, D. W. 1971, in General Relativity and Cosmology,

ed. R. K. Sachs (New York: Academic Press).

Seielstad, G. A., Sramek, R. A., and Weiler, K. W. 1970,

Phys. Rev. Letters, 24, 1373.

Shapiro, I. I. 1964, Phys. Rev. Letters, 13, 789.

_____ 1966a, Phys. Rev., 141, 1219.

_____ 1966b, ibid., 145, 1005.

_____ 1967, Science, 157, 806.

_____ 1971, in Proceedings of the Conference on Experi-

mental Tests of Gravitational Theories, ed. R. W. Davies

(NASA-JPL Technical Memorandum 33-499).

Shapiro, I. I. et al. 1971, preliminary results reported at the

California Institute of Technology, November 17, 1971.

Shapiro, I. I., Ash, M. E., Ingalls, R. P., Smith, W. B.,

Campbell, D. B., Dyce, R. B., Jurgens, R. F., and

Pettengill, G. H. 1971, Phys. Rev. Letters, 26, 1132.

Shapiro, I. I., Smith, W. B., Ash, M. E., Ingalls, R. P., and

Pettengill, G. H. 1971, Phys. Rev. Letters, 26, 27.

Sherwin, C. W. 1960, Phys. Rev., 120, 17.

168

Smart, W. M. 1953, Celestial Mechanics (London: Longman Green & Co.)

_____ 1960, Spherical Trigonometry (4th ed.; Cambridge:

Cambridge University Press).

Smith, H. M., and Tucker, R. H. 1953, M.N.R.A.S., 113, 251.

Sramek, R. A. 1971, Ap. J. Letters, 167, 55.

Synge, J. L. 1952, Proc. Roy. Soc., 211A, 303.

_____ 1960, Relativity, the General Theory, (Amsterdam:

North-Holland Publishing Company).

Temple, G. 1924, Proc. Phys. Soc., 36, 176.

Thirry, Y. R. 1948, Compt. Rend., 226, 216.

Thorne, K. S. 1971, in General Relativity and Cosmology,

ed. R. K. Sachs (New York: Academic Press).

Thorne, K. S., and Will, C. M. 1971, Ap. J., 163, 595.

Thorne, K. S., Will, C. M., and Ni, W.-T. 1971, in Proceedings

of the Conference on Experimental Tests of Gravitational

Theories, ed. R. W. Davies (NASA-JPL Technical Memoran-

dum 33-499).

Trautman, A. 1962 in Gravitation, an Introduction to Current Research,

ed. L. Witten (New York: J. Wiley and Sons).

_____ 1965, in Lectures on General Relativity, by

A. Trautman, F. A. E. Pirani, H. Bondi (Englewood Cliffs,

N.J.: Prentice-Hall).

Tulczyjew, B. and Tulczyjew, W. 1962, in Recent Developments in

General Relativity (London: Pergamon Press).

Turner, K. C. and Hill, H. A. 1964, Phys. Rev., 134, B252.

Vessot, R. F. C. and Levine, M. W. 1971, in Proceedings of the

Conference on Experimental T_ests of Gravitational Theories,

ed. R. W. Davies (NASA-JPL Technical Memorandum 33-499).

Vessot, R. F. C. 1971, private communication.

Wagoner, R. V. 1970, Phys. Rev. D., 1, 3209.

Ward, W. R. 1970, Ap. J., 162, 345.

Whitehead, A. N. 1922, The Principle of Relativity (Cambridge:

Cambridge University Press).

Whitrow, G. J. and Morduch, G. E. 1960, Nature, 188, 790.

_____ 1965, in Vistas in Astronomy, Vol. 6, ed. A. Beer

(Oxford: Pergamon Press).

Wilk, L. S. ed. 1971, MIT Measurement Systems Laboratory,

Progress Report, PR8.

Will, C. M. 1971a, Ap. J., 163, 611.

_____ 1971b, ibid., 165, 409.

_____ 1971c, ibid., 169, 125.

_____ 1971d, ibid., 169, 141.

Will, C. M., and Nordtvedt, K., Jr. 1972, Ap. J., in press.

Yilmaz, H. 1958, Phys. Rev., 111, 1417.

_____ 1962, in Evidence for Gravitational Theories,

ed. C. Møller (New York: Academic Press).

_____ 1971, Phys. Rev. Letters, 27, 1399.

TABLE 3.I. A Partial List of Non-Viable Gravitation Theories

Theories and References	Comments, including some but not all the reasons why the theory is non-viable[*]
Newtonian Gravitation theory	Is not relativistic.
Hoyle's C-field theory [Hoyle, 1960; Hoyle and Narlikar (1963, 1964)]	Was devised originally as a foundation for the steady-state model of the Universe. Is incomplete -- lacks an equation governing the rate at which the C-field creates particles.
Milne's kinematical relativity [Milne (1937, 1948)]	Was devised originally to handle certain cosmological problems. Is complete -- makes no redshift prediction; predicts zero deflection of light.
Poincaré's (1906) theory as generalized by Whitrow and Morduch (1965)	Action-at-a-distance theory in flat space-time with an adjustable parameter n. For $n < 2$ predicts zero redshift. For $n \geq 2$ gives internally inconsistent treatment of light propagation.
Whitrow-Morduch vector theory [Whitrow and Morduch (1960, 1965)]	Contains a vector gravitational field that resides in flat spacetime; possesses a freely specifiable parameter p. For $p = 0$, predicts no redshift. For $p \neq 0$ gives internally inconsistent treatment of light propagation.

TABLE 3.I. Continued

Theory and References	Comments, including some, but not all the reasons why the theory is non-viable[*]
Kustaanheimo's various vector theories [Kustaanheimo (1957, 1966) Kustaanheimo and Nuotio (1967); Whitrow and Morduch (1960, 1965)]	Contain a vector gravitational field that resides in flat spacetime; possess several freely specifiable functions, which can be adjusted to give agreement with the "four standard tests". Are incomplete -- do not mesh with other laws of physics, except by imposing them in the global Lorentz frame of the flat spacetime metric, which then gives internal inconsistencies: different redshifts for light viewed as photons and light viewed as waves; non-zero redshifts for photons between points at the same gravitational potential. Also several scalar and tensor theories which suffer the same problems.
Birkhoff's (1943) theory	Gives same prediction as general relativity for 4 standard tests; but predicts crazy results for internal behavior of matter -- $-p = \rho c^2$, $v_{sound} = v_{light}$.
Yilmaz's (1971) theory	Contains a tensor gravitational field used to construct a metric. Is inconsistent -- differential equation for the metric in terms of the tensor field is not integrable.

172

TABLE 3.I. footnote

---

*These theories are non-viable in their <u>present</u> form. Future modifi-

cations or specializations might make some of them viable. If we have

misinterpreted any theory here, we apologize to its proponents, and we

urged them to demonstrate explicitly its self-consistency, completeness,

and correct experimental predictions.

TABLE 4.I.   The Parametrized Post-Newtonian Formalism

A.   Coordinate System: the framework uses a particular, nearly globally Lorentz coordinate system in which the coordinates are (t, x^1, x^2, x^3).   Three-dimensional, Euclidean vector notation is used throughout.   All coordinate arbitrariness ("gauge freedom") has been removed by specialization of the coordinates.

B.   Matter Variables:

   1. ρ = density of rest mass as measured in a local Lorentz frame momentarily comoving with the gravitating matter.

   2. $v_\alpha = (dx^\alpha/dt)$ = coordinate velocity of the matter.

   3. w_α = coordinate velocity of PPN coordinate system relative to the mean rest-frame of the Universe.

   4. p = pressure as measured in a local Lorentz frame momentarily comoving with the matter.

   5. Π = internal energy per unit rest mass.   It includes all forms of non-rest-mass, non-gravitational energy -- e.g., energy of compression and thermal energy.

C.   PPN Parameters:

$\gamma, \ \beta, \ \alpha_1, \ \alpha_2, \ \alpha_3, \ \zeta_1, \ \zeta_2, \ \zeta_3, \ \zeta_4 .$

$C.4$

174

TABLE 4.I (Continued)

D. <u>Metric</u>

$$g_{oo} = 1 - 2U + 2\beta U^2 - (2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1 + \zeta_1 \mathbb{G}$$

$$- 2[(3\gamma - 2\beta + 1 + \zeta_2)\Phi_2 + (1 + \zeta_3)\Phi_3 + 3(\gamma + \zeta_4)\Phi_4]$$

$$+ (\alpha_1 - \alpha_2 - \alpha_3)w^2 U + \alpha_2 w^\alpha w^\beta U_{\alpha\beta} - (2\alpha_3 - \alpha_1)w^\alpha V_\alpha$$

$$g_{o\alpha} = \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_\alpha + \frac{1}{2}(1 + \alpha_2 - \zeta_1)W_\alpha$$

$$+ \frac{1}{2}(\alpha_1 - 2\alpha_2)w^\alpha U + \alpha_2 w^\beta U_{\alpha\beta}$$

$$g_{\alpha\beta} = - (1 + 2\gamma U)\delta_{\alpha\beta}$$

E. <u>Stress-Energy Tensor:</u>

$$T^{oo} = \rho(1 + \Pi + v^2 + 2U)$$

$$T^{o\alpha} = \rho(1 + \Pi + v^2 + 2U + p/\rho)v_\alpha$$

$$T^{\alpha\beta} = \rho v_\alpha v_\beta(1 + \Pi + v^2 + 2U + p/\rho) + p\delta_{\alpha\beta}(1 - 2\gamma U)$$

F. <u>Equations of Motion</u>

1. Stressed Matter

$$T^{ij}_{;j} = 0 \quad \text{i.e.,} \quad T^{ij}_{,j} + \Gamma^i_{jk}T^{jk} + \Gamma^j_{kj}T^{ik} = 0$$

2. Test Bodies

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

TABLE 4.II. Local Conservation Laws

Law	Covariant Form of Equations
Conservation of Baryons	$(nu^i)_{;i} = \dfrac{1}{\sqrt{-g}} (\sqrt{-g}\, nu^i)_{,i} = 0$
Conservation of Rest Mass	$(\rho u^i)_{;i} = \dfrac{1}{\sqrt{-g}} (\sqrt{-g}\, \rho u^i)_{,i} = 0$
Isentropic Flow	$u^i[\, \Pi_{,i} + p(1/\rho)_{,i}] = 0$

TABLE 4.III. Integral Conservation Laws

PPN Parameter Values $\{\zeta_1,\zeta_2,\zeta_3,\zeta_4,\alpha_3\}$	$\{\alpha_1,\alpha_2\}$	Type of Theory	Conserved Quantities
all zero	all zero	Fully Conservative	P^i, J^{ij}
all zero	may be non-zero	Semi-Conservative	P^i
may be non-zero	any values	Non-Conservative	P^o

TABLE 5.1. Metric Theories of Gravity and Their PPN Parametric Values

Theory and its adjustable parameters[†]	γ	β	α_1	α_2	α_3	ζ_1	ζ_2	ζ_3	ζ_4
1. General Relativity[L]	1	1	0	0	0	0	0	0	0
2. Scalar-Tensor Theories									
a. Bergmann-Wagoner-Nordtvedt (ω, Λ)[L]	$\frac{1+\omega}{2+\omega}$	$1+\Lambda$	0	0	0	0	0	0	0
b. Dicke-Brans-Jordan (ω)[L]	$\frac{1+\omega}{2+\omega}$	1	0	0	0	0	0	0	0
3. Vector-Metric Theory (K)[L]	1	1	0	$\dfrac{K^2}{1+\frac{1}{2}K^2}$	0	0	0	0	0
4. Conformally Flat Theories									
a. Nordstrøm-Einstein-Fokker[L]	-1	$\frac{1}{2}$	0	0	0	0	0	0	0
b. Ni's Lagrangian Theory (q)[L]	-1	$1-q$	0	0	0	0	0	0	0
c. Ni's General Theory (p,q)	-1	$1-q$	0	0	0	0	$4+p-2q$	0	0
d. Nordstrøm	-1	1	0	0	0	0	0	0	0
e. Littlewood-Bergmann	-1	$\frac{1}{2}$	0	0	0	0	-1	0	0
f. Whitrow-Morduch (q)	-1	$1-q$	0	0	0	0	$-2q$	0	0

TABLE 5.I (cont'd)

Theory and its adjustable parameters[†]	γ	β	α_1	α_2	α_3	ζ_1	ζ_2	ζ_3	ζ_4
5. Stratified Theories (with Time-Orthogonal Conformally Flat Space Slices)									
a. Rosen $(\lambda)^L$	λ	$\frac{1}{4}(3+\lambda)$	$-4(1+\lambda)$	0	0	0	0	0	0
b. Papapetrou^L	1	1	-8	-4	0	0	0	0	0
c. Ni's Lagrangian Stratified Theory^L	1	1	-8	0	0	0	0	0	0
d. Ni's General Stratified Theory (p,q)	1	1-q	-8	0	-4	0	p-2q-2	0	-2
e. Yilmaz	1	1	-8	0	-4	0	-2	0	-2
f. Page-Tupper (a,c)	a	1+c	-4(1+a)	0	-2(1+a)	0	1+a+2c	0	-(1+a)
g. Coleman (p)	1	1	-8	0	-4	0	p-2	0	-2
h. Einstein's 1912 Theory	0	0	-4	0	-2	0	-1	0	-1
i. Whitrow-Morduch Stratified Theory	0	-1	-4	0	-2	0	-3	0	-1
6. Whitehead's Theory	1	1	0	0	0	-6	0	-1	-1
								(plus additional metric terms)	

[†] The superscript ^{L} refers to theories which are Lagrangian-based.

TABLE 6.I.   Christoffel Symbols for the PPN Metric

$\Gamma^o{}_{oo} = -\partial U/\partial t$, $\Gamma^o{}_{o\alpha} = -\partial U/\partial x^\alpha$,

$\Gamma^o{}_{\alpha\beta} = \gamma\delta_{\alpha\beta}\,\partial U/\partial t + \frac{1}{8}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)V_{(\alpha,\beta)}$

$\qquad + \frac{1}{8}(1 + \alpha_2 - \zeta_1)W_{(\alpha,\beta)} + \frac{1}{8}(\alpha_1 - 2\alpha_2)w_{(\alpha}U_{,\beta)}$

$\qquad + \frac{1}{4}\alpha_2 w^\gamma U_{\gamma(\alpha,\beta)}$,

$\Gamma^\alpha{}_{oo} = -\partial U/\partial x^\alpha + \partial/\partial x^\alpha\Big[(\beta + \gamma)U^2 - 2\Phi + \frac{1}{2}\zeta_1 \mathbb{G}$

$\qquad + \frac{1}{2}\alpha_2 w^\beta w^\gamma U_{\beta\gamma} - \frac{1}{2}(\alpha_2 + \alpha_3 - \alpha_1)w^2 U + \frac{1}{2}(\alpha_1 - 2\alpha_3)w^\beta V_\beta\Big]$

$\qquad - \partial/\partial t\Big[\frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)\,V_\alpha + \frac{1}{2}(1 + \alpha_2 - \zeta_1)W_\alpha$

$\qquad + \frac{1}{2}(\alpha_1 - 2\alpha_2)w_\alpha U + \alpha_2 w^\beta U_{\alpha\beta}\Big]$,

$\Gamma^\alpha{}_{o\beta} = \gamma\delta_{\alpha\beta}\,\partial U/\partial t - \frac{1}{8}(4\gamma + 4 + \alpha_1)V_{[\alpha,\beta]}$

$\qquad - \frac{1}{8}\alpha_1 w_{[\alpha}U_{,\beta]}$,

$\Gamma^\alpha{}_{\beta\zeta} = \gamma(\delta_{\alpha\beta}U_{,\zeta} + \delta_{\alpha\zeta}U_{,\beta} - \delta_{\beta\zeta}U_{,\alpha})$,

where

$\Phi = \frac{1}{4}(2\gamma + 2 + \alpha_3 + \zeta_1)\Phi_1 + \frac{1}{2}(3\gamma - 2\beta + 1 + \zeta_2)\Phi_2$

$\qquad + \frac{1}{2}(1 + \zeta_3)\Phi_3 + \frac{3}{2}(\gamma + \zeta_4)\Phi_4$.

TABLE 6.II. Massive-Body Integrals in the PPN Formalism

Vector Integrals

$$\mathbb{W}_i^{\alpha} = \int_i \frac{\rho^* \rho^{*'} \rho^{*''}(x-x')^{\alpha}}{|x'-x''||x-x'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' \, d\underset{\sim}{x}'' , \qquad \mathbb{W}_i^{*\alpha} = \int_i \frac{\rho^* \rho^{*'} \rho^{*''}(\underset{\sim}{x}'-\underset{\sim}{x}'')\cdot(\underset{\sim}{x}-\underset{\sim}{x}')(x-x')^{\alpha}}{|\underset{\sim}{x}'-\underset{\sim}{x}''|^3 |\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' \, d\underset{\sim}{x}''$$

$$\mathfrak{l}_i^{\alpha} = \int_i \frac{\rho^* \rho^{*'} \bar{v}'^2 (x-x')^{\alpha}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' , \qquad \mathscr{T}_i^{\alpha} = \int_i \frac{\rho^* \rho^{*'} \bar{v}'^{\alpha} \underset{\sim}{\bar{v}}'\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

$$\mathscr{T}_i^{*\alpha} = \int_i \frac{\rho^* \rho^{*'} \bar{v}^{\alpha} \underset{\sim}{\bar{v}}'\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' , \qquad \mathscr{T}_i^{**\alpha} = \int_i \frac{\rho^* \rho^{*'}[\underset{\sim}{\bar{v}}'\cdot(\underset{\sim}{x}-\underset{\sim}{x}')]^2 (x-x')^{\alpha}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^5} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

$$P_i^{\alpha} = \int_i \frac{\rho^* p'(x-x')^{\alpha}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' , \qquad \mathbb{E}_i^{\alpha} = \int_i \frac{\rho^* \rho^{*'} \Pi'(x-x')^{\alpha}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

Tensor and Scalar Integrals

$$\mathbb{T}_i^{\alpha\beta} = \tfrac{1}{2} \int \rho^* \bar{v}^{\alpha} \bar{v}^{\beta} \, d\underset{\sim}{x} \qquad\qquad \mathbb{T}_i = \tfrac{1}{2} \int_i \rho^* \bar{v}^2 \, d\underset{\sim}{x}$$

$$\Omega_i^{\alpha\beta} = -\tfrac{1}{2}\int_i \frac{\rho^* \rho^{*'}(x-x')^{\alpha}(x-x')^{\beta}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' , \qquad \Omega_i = -\tfrac{1}{2}\int_i \frac{\rho^* \rho^{*'}}{|\underset{\sim}{x}-\underset{\sim}{x}'|} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

$$I_i^{\alpha\beta} = \int_i \rho^*(x-X_i)^{\alpha}(x-X_i)^{\beta} \, d\underset{\sim}{x} , \qquad I_i = \int_i \rho^* |\underset{\sim}{x}-\underset{\sim}{X}_i|^2 \, d\underset{\sim}{x}$$

$$P_i = \int_i p \, d\underset{\sim}{x} , \qquad\qquad E_i = \int_i \rho^* \Pi \, d\underset{\sim}{x}$$

$$H_i^{\alpha\beta} = \int_i \frac{\rho^* \rho^{*'} \bar{v}'^{\alpha}(x-x')^{\beta}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^3} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' , \qquad K_i^{\alpha\beta} = \int_i \frac{\rho^* \rho^{*'} \underset{\sim}{\bar{v}}'\cdot(\underset{\sim}{x}-\underset{\sim}{x}')(x-x')^{\alpha}(x-x')^{\beta}}{|\underset{\sim}{x}-\underset{\sim}{x}'|^5} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$

TABLE 7.I. Geophysical Preferred-Frame Effects

Effect	Frequency	Amplitude
i) Semi-Diurnal Earth-Tides $(\Delta g/g)$	$2\Omega - 2\omega$	$1 \times 10^{-9}\, \alpha_2 \cos^2 L$
	$2\Omega - \omega$	$1 \times 10^{-8}\, \alpha_2 \cos^2 L$
	$2\Omega^{\dagger}$	$3 \times 10^{-8}\, \alpha_2 \cos^2 L$
	$2\Omega + \omega$	$< 10^{-9}$
	$2\Omega + 2\omega$	$< 10^{-9}$
ii) Diurnal Earth-Tides $(\Delta g/g)$	$\Omega - 2\omega$	$< 10^{-9}$
	$\Omega - \omega$	$1 \times 10^{-8}\, \alpha_2 \sin 2L$
	Ω	$7 \times 10^{-8}\, \alpha_2 \sin 2L$
	$\Omega + \omega$	$3 \times 10^{-9}\, \alpha_2 \sin 2L$
	$\Omega + 2\omega$	$< 10^{-9}$
iii) Zonal Variations in Earth's Rotation $(\Delta\Omega/\Omega)$	ω	$8 \times 10^{-9}\, \alpha_2$
	2ω	$< 10^{-9}\, \alpha_2$
iv) Spherical Variation in Earth Rotation $(\Delta\Omega/\Omega)$	$\omega^{\dagger}$	$3 \times 10^{-9}\, (2/3\, \alpha_2 + \alpha_3 - \alpha_1)$

$\dagger$ Used to put limits on the values of the parameters α_1, α_2, and α_3 (Subsect. 7'4).

TABLE 7.II. Experimental Limits and Theoretical Predictions

for the PPN Parameters α_1, α_2, and α_3

EXPERIMENTAL LIMITS ON α_1, α_2, AND α_3

$$|\alpha_1| < 0.2$$
$$|\alpha_2| < 3 \times 10^{-2}$$
$$|\alpha_3| < 2 \times 10^{-5}$$

THEORETICAL PREDICTIONS FOR α_1, α_2, AND α_3

Theory and its adjustable parameters	α_1	α_2	α_3
General Relativity	0	0	0
Scalar-Tensor Theories (ω, Λ)	0	0	0
Vector-Metric Theory (K)	0	$K^2/(1 + 1/2\ K^2)$	0
Stratified Theories:			
a. Page and Tupper (a,c)	$-4(1+a)$	0	$-2(1+a)$
b. Yilmaz	-8	0	-4
c. Papapetrou	-8	-4	0
d. Lagrangian Stratified Theory (Ni)	-8	0	0
e. General Stratified Theory (Ni) (p,q)	-8	0	-4
f. Coleman (p)	-8	0	-4
g. Rosen (λ)	$-4(1+\lambda)$	0	0

TABLE 7.III. <u>Theoretical Predictions for "Nordtvedt Effect" Parameters</u>

Theory	Nordtvedt Effect Parameter	
	(a) Isotropic: $(\eta + \frac{1}{3}\xi)$	(b) Anisotropic: ξ
General Relativity	0	0
Scalar-Tensor Theories		
a. Bergmann-Wagoner-Nordtvedt	$4\Lambda + 1/(2 + \omega)$	0
b. Dicke-Brans-Jordan	$1/(2 + \omega)$	0
Vector-Metric Theory	$\frac{2}{3} K^2/(1 + \frac{1}{2} K^2)$	$- K^2/(1 + \frac{1}{2} K^2)$

Caption for Fig. 5.1.

Fig. 5.1 - Kinematical quantities used to calculate the metric for
Whitehead's theory. The metric at a point P is deter-
mined by all the masses which cross the past "η-light-cone"
Σ^- of P.

Figure Captions

Fig. 7.1. Schematic diagram of gravitational light bending.

Fig. 7.2. Round-trip trajectory of a photon from the Earth to a planet
or spacecraft.

Fig. 7.3. Two gyroscopes in polar orbit around the Earth. The
gyroscope with its axis pointed normal to the orbital plane suffers
a precession due to the "dragging of inertial frames". The
gyroscope whose axis lies in the orbital plane undergoes a
"geodetic precession".

Fig. 7.4. Isotropic Nordtvedt Effect -- a polarization of the Moon's
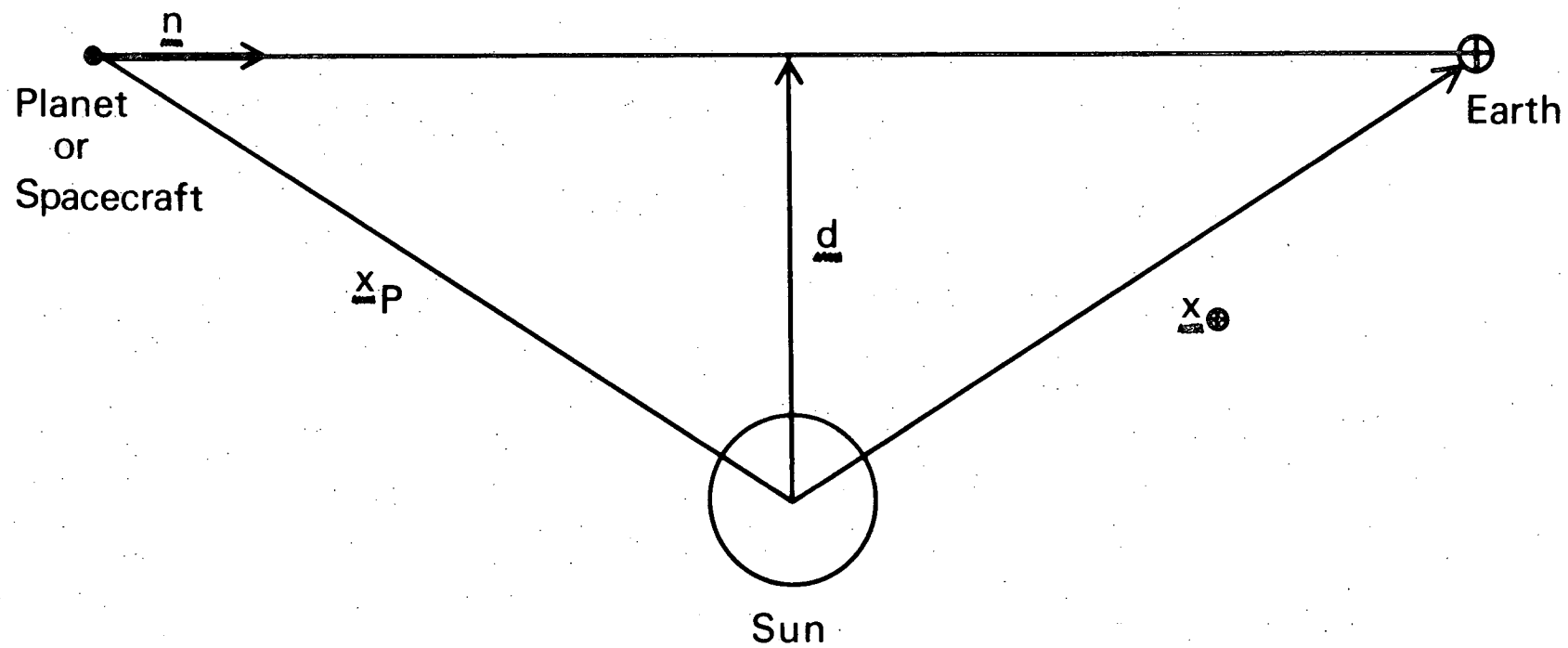orbit with the maximum (apogee) always directed toward the
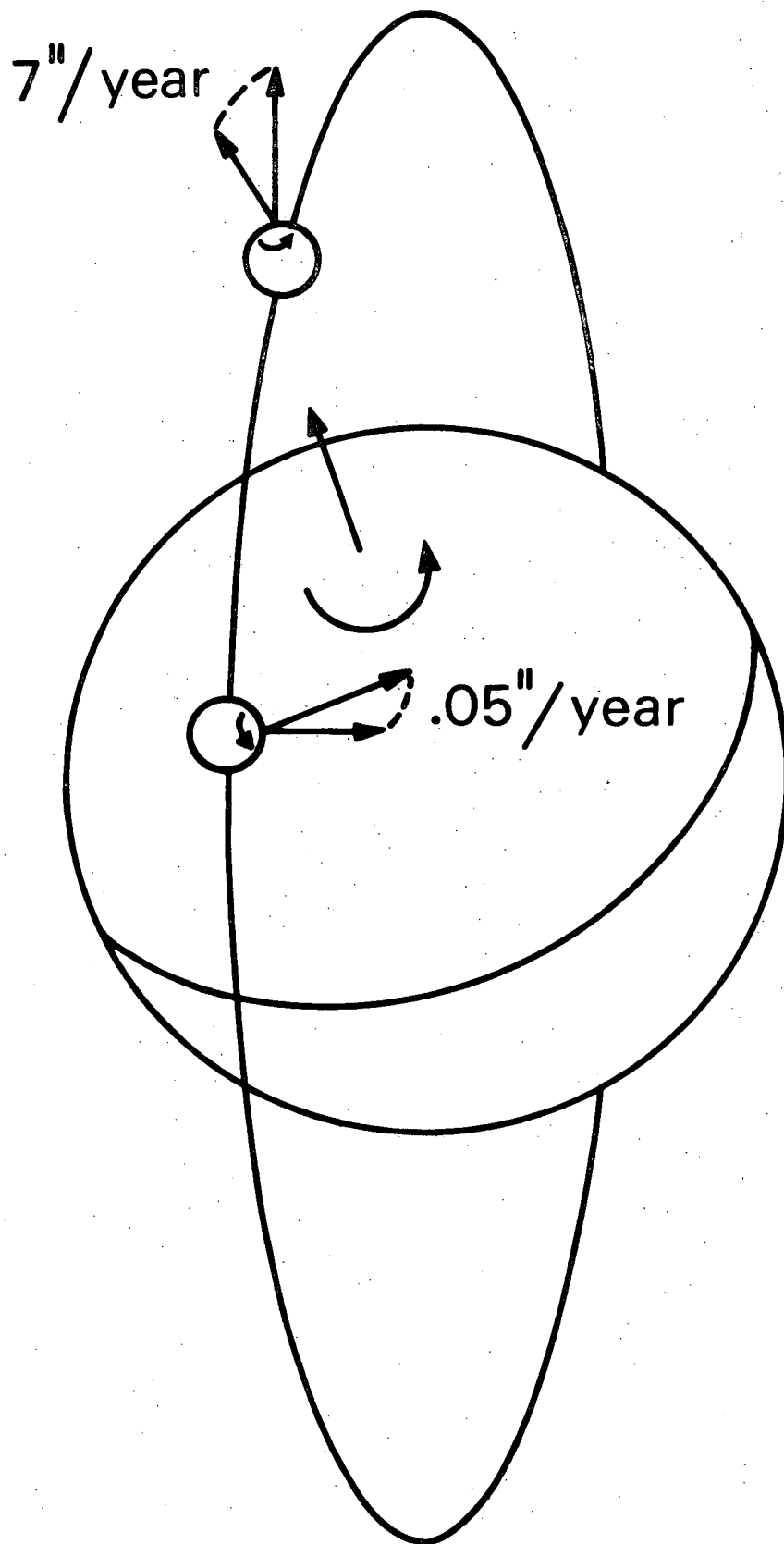Sun.
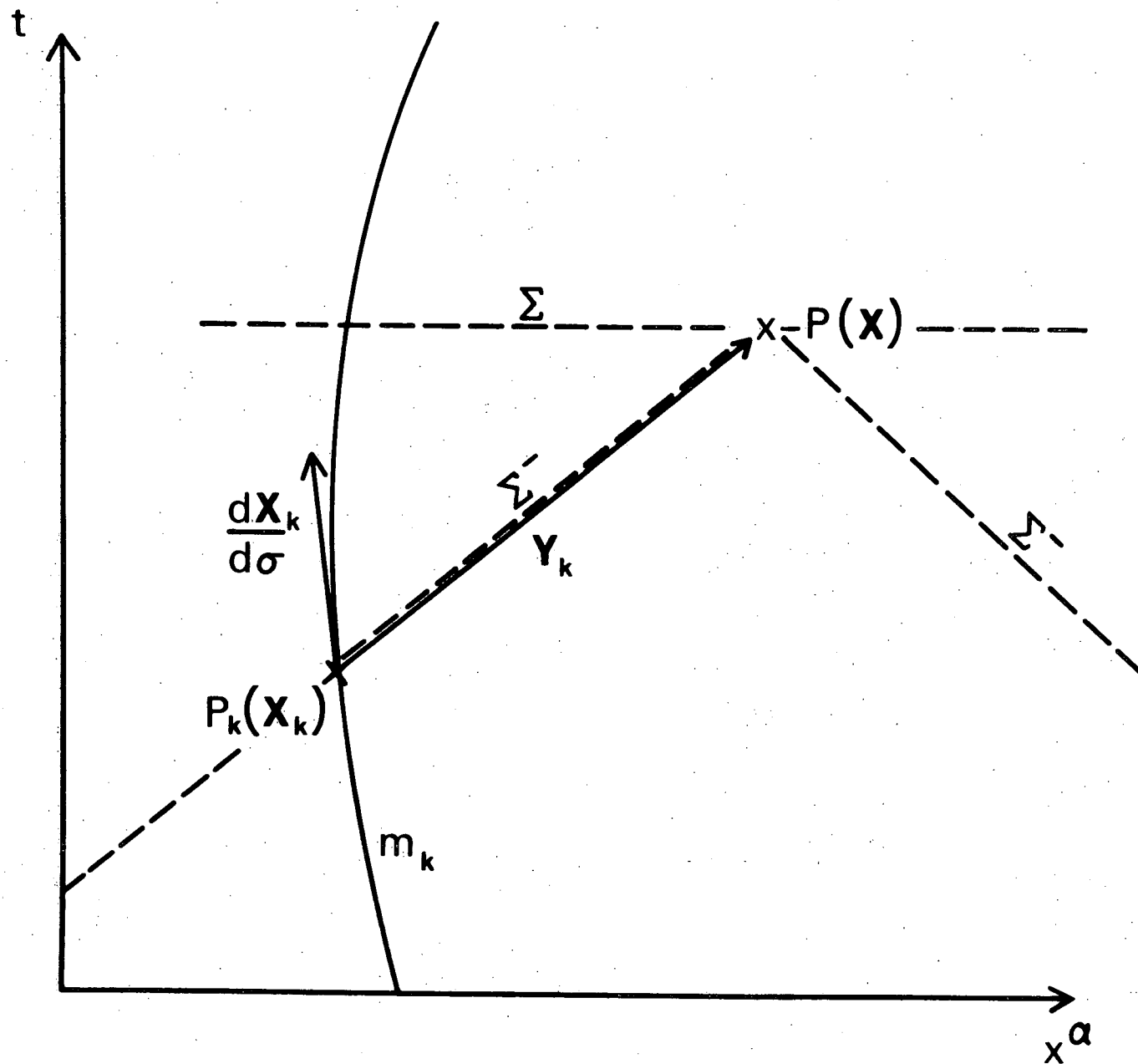
Fig. 5.1

Fig. 7.1

Fig. 7.2

Fig. 7.3

Fig. 7.4