VIBRATION AND DAMPING OF LAMINATED, COMPOSITE-MATERIAL PLATES INCLUDING THICKNESS-SHEAR EFFECTS

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16. Abstract

An analytical investigation of sinusoidally forced vibration of laminated, anisotropic plates including bending-stretching coupling, thickness-shear flexibility, all three types of inertia effects, and material damping was conducted. The analysis begins with the anisotropic stiffness and damping constitutive relations for a single layer and proceeds through the analogous relations for a laminate. Then the various types of energy and work terms are derived and the problem is formulated as an eigenvalue problem by application of the extended Rayleigh-Ritz Method. The effects of thickness-shear deformation are considered by the use of a shear correction factor analogous to that used by Mindlin for homogeneous plates. The general analysis is applicable to plates with any combination of natural boundary conditions at their edges.

Natural frequencies of boron/epoxy plates calculated on the basis of two different resonance criteria (peak-amplitude and modified Kennedy-Pancu techniques) and the associated nodal patterns are in good agreement with previously published experimental data.

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CONTENTS

ABSTRACT

This report is concerned with analytical investigation of sinusoidally forced vibration of laminated, anisotropic plates including bending-stretching coupling, thickness-shear flexibility, all three types of inertia effects, and material damping.

In the analysis the effects of thickness-shear deformation are considered by the use of a shear correction factor K, analogous to that used by Mindlin for homogeneous plates. Two entirely different approaches for calculating the thickness-shear factor for a laminate are presented. Numerical examples indicate that the value of K depends on the layer properties and the stacking sequence of the laminate.

The general analysis is applicable to plates with any combination of natural boundary conditions at their edges. The analysis begins with the anisotropic stiffness and damping constitutive relations for a single layer and proceeds through the analogous relations for a laminate. Then the various types of energy and work terms are derived and the problem is formulated as an eigenvalue problem by application of the extended Rayleigh-Ritz method.

The first five resonant frequencies of boron/epoxy plates with all edges free are calculated on the basis of two different resonance criteria: the peak-amplitude and modified Kennedy-Pancu techniques. The results show that the resonant frequencies obtained by the two techniques differ by only a very small amount, and are in good agreement with the results obtained

both experimentally and analytically by Clary. Furthermore, the nodal patterns obtained agree satisfactorily with those of Clary. Finally, the damping values in this investigation are in good agreement with the experimental ones obtained by Clary.

SYMBOLS

A	cross-sectional area
A_o	area involved in shear-stress integration (fig. 8)
A_i, B_i	constants depending upon the boundary conditions (Appendix E)
A_i	hereditary constants, eq. (B-21)
A_{ij}	stretching stiffness of the plate
a	length of plate
$a(x_1), a(x_2)$	wave amplitudes at positions x_1 and x_2
a_i, a_{i+1}, a_{i+n}	free-vibration amplitudes corresponding to the i-th, (i+1)th, and (i+n)th cycles
a_o	initial amplitude
B_{ij}	bending-stretching coupling stiffness of the plate
$\bar{B}_{ij}$	complex stiffness coefficient
b	width of plate
b	material damping coefficient defined in eq. (B-10)
b_c	critical material damping coefficient
C_d	damping coefficient defined in eq. (B-22)
C'_d	damping coefficient defined in eq. (B-8)
C_i	constants depending upon the initial conditions
C_{ij}	Cauchy elastic coefficient
$\bar{C}_{ij}$	complex version of C_{ij}
$\bar{C}'_{ij}$	effective stiffness coefficients defined following eq. (F-7)
C_m, C_n	characteristic parameters tabulated in ref. 44
c	viscous damping coefficient

$\bar{c}$	Kelvin-Voigt complex damping coefficient
c_c	critical value of c
c_s	shear wave-propagation velocity
D	total dissipative energy
D_A	energy dissipated per unit of plate area
D_{ij}	bending and twisting stiffnesses of the plate
$\bar{D}_{ij}$	complex version of D_{ij}
E	Young's modulus
E_{11}, E_{22}	Young's moduli in the x,y directions
Ei(u)	the exponential integral defined in eq. (B-18)
e	base of the natural logarithms, e ≈ 2.7183
F_d	damping force
F_s	spring force
$\tilde{F}$	exciting force amplitude
ΔF	horizontal shear force per unit width
F_{ij}	thickness-shear stiffnesses of the plate
$\bar{F}_{ij}$	complex stiffness coefficients associated with F_{ij}
$\{f\}$	generalized-force column matrix
g	loss tangent
g_1	parameter of the Biot model
g_k	dimensionless geometric parameter defined in ref. 35
$g_{A_{11}}$, etc.	subscripted loss tangents where the subscripts (i.e. A_{11}) refer to the associated stiffness; for example, $g_{A_{11}}$ signifies the loss tangent associated with the longitudinal stretching stiffness
H	factor $\equiv (h/m)^{\frac{1}{2}}$
HPMF	half-power magnification factor
$\bar{H}$	factor $\equiv H/\omega_n$

$H^{(a)}, H^{(b)}$	thickness fractions of layers a and b, respectively
h	total thickness of the plate; parameter in eq. (H-2)
I	integral form defined in Appendix G
i	unit imaginary number $\equiv \sqrt{-1}$
K	thickness shear factor
$\bar{K}$	factor defined in eq. (H-10)
$\bar{K}_c$	complex-flexibility factor defined in eq. (F-15)
K_{ij}	composite shear coefficient
k	spring rate
$\bar{k}$	complex spring constant
k'	stiffness coefficient associated with the total in-plane force
k''	$C_1 e^{im}$ (C_1 and m are constants)
$\bar{k}$	Kelvin-Voigt complex stiffness defined in eq. (B-3)
L	effective length of spring
$\tilde{L}$	amplitude of Lagrangian energy difference
$\bar{L}_c$	complex-stiffness factor defined in eq. (F-16)
ln	natural logarithm
log	logarithm with base 10
[M]	mass matrix
M_{ij}	stress couples, moment per unit width
MF, MF'	magnification factors defined in eqs. (B-38) and (B-16)
m	damping exponent defined in eq. (B-22)
m	mass
m, n	cos θ, sin θ
m_o, m_1, m_2	mass per unit of plate area, first moment of mass per unit area, second moment of mass per unit area

N_{ij}	membrane stress resultants, force per unit width
n	degree of heredity
Q	quality factor = reciprocal of the dimensionless bandwidth
Q_i	thickness-shear stress resultant
$Q_{k\ell}$	reduced stiffness coefficients transformed to plate coordinates (x,y); see eq. (A-16)
$Q^*_{k\ell}$	reduced (plane-stress) stiffness coefficients with respect to material symmetry axes (X,Y); defined in eq. (A-6)
q	normal pressure acting on plate
R	density ratio ($\equiv \rho^{(1)}/\rho^{(2)}$)
RMF	resonant magnification factor
$\bar{S}_{ij}$	complex flexibility defined in eq. (F-22)
$[S]$	complex stiffness matrix
T	kinetic energy; period of damped oscillation (Appendix B only)
T_A	kinetic energy per unit of plate area
$[T_r]$	transformation matrix defined in eq. (A-12)
t	time
t_1, t_2	two different specific values of time
U	strain energy
U_A	strain energy per unit of plate area
U_d	energy dissipated per cycle
U_{dv}	damping energy per cycle and per unit of volume
U_{mn}	undetermined longitudinal-displacement parameters
U_s	shear strain energy
U'_s	shear strain energy associated with equivalent uniform, longitudinal thickness-shear strain
U_v	strain energy per unit of plate volume

u,v,w	displacements in the x,y,z directions
u_o,v_o,w_o	displacements of middle surface in x,y,z directions
$\tilde{u}$	displacement amplitude of vibration
u'	trasient solution in eq. (B-70)
u_{st}	static displacement; see Appendix B
V	volume
V_{mn}	undetermined transverse-displacement parameter
v_s	spatial attenuation constant defined in eq. (B-33)
v_t	temporal decay constant defined in eq. (B-30)
W	total work done
W_{mn}	undetermined normal-displacement parameter
X,Y,Z	coordinates of the material-symmetry axes
x,y,z	rectangular coordinates in the longitudinal, transverse, and thickness directions
x_1,x_2	two different specific values of position
Z_m,Z_n	characteristic parameters tabulated in ref. 44
z_k,z_{k-1}	thickness-direction coordinates of outer and inner faces of the k-th layer
$\{\ \}$	column matrix
$[\ ]$	square matrix
α,β	normalized arguments in the x,y directions
$\bar{\alpha}$	α_1/α_n
α_i	heredity constants in eq. (B-21)
β	$C_{55}^{(1)}/C_{55}^{(2)}$ in Appendix E only
γ	loss angle defined in eq. (B-27)
γ_s	spatial attenuation rate defined in eq. (B-34)
γ_t	decay rate defined in eq. (B-31)

$\bar{\Delta}$	comples factor defined in eq. (F-22)
δ	logarithmic decrement defined in eq. (B-28)
δ_s	logarithmic attenuation defined in eq. (B-32)
ϵ	Biot parameter in eq. (B-17)
ϵ_{ij}	strain components with respect to x,y axes
ϵ_{IJ}	strain components with respect to X,Y axes
ϵ_{xz}	strain corresponding to longitudinal thickness-shear action (Section III and Appendix E)
ϵ'_{xz}	longitudinal thickness-shear strain weighted according to eq. (55)
ζ	damping ratio
ζ_2	z_2/z_1 (Appendix E)
θ	angle of orientation of an individual layer
κ_{ij}	curvature changes
μ	parameter ($\equiv \epsilon/\nu_n$)
ν	Poisson's ratio
ν_{12}, ν_{21}	major and minor in-plane Poisson's ratios
ξ	generalized coordinate
$\{F_{mn}\}$	column matrix representing generalized displacements
π	pi $\approx$ 3.1415962
ρ	density
$\sum$	summation symbol
$\bar{\sigma}$	stress amplitude
σ_{ij}	stress components with respect to x,y axes
σ_{IJ}	stress components with respect to X,Y axes
τ	dummy time variable in eq. (B-19)
Φ	assumed modal function
$\phi(t)$	hereditary kernel defined in eq. (B-21)

φ	phase angle between response and exciting force
Ψ_{xmn}, Ψ_{ymn}	undetermined rotation parameters
Ψ_x, Ψ_y	angles of rotation in the xz and yz planes
Ω	dimensionless frequency ($\equiv \omega/\omega_n$)
Ω_{hp}	Ω at half-power points
Ω_s	ω/c_s
ω	circular frequency of vibration
ω_n	natural frequency
ω_1, ω_2	angular frequencies defining the half-power points of the frequency spectrum

Superscripts:

(k)	denotes a typical kth layer
$\hat{}$	denotes a damping quantity
$\cdot$	denotes differentiation with respect to time
$\sim$	denotes that the quantity is an amplitude
o	refers to the middle surface of the plate
I,R	denote the respective imaginary and real parts of a complex quantity

Subscripts:

,	denotes partial differentiation with respect to the variable following the comma, i.e. $u_{o,x} = \partial u_o/\partial x$
k	integers ; k = 1,2,... K
ℓ	integers ; ℓ = 1,2,... L
m	integers ; m = 1,2,... M
n	integers ; n = 1,2,... N
R	denotes resonance

SECTION I

INTRODUCTION

1.1 Introductory Remarks

The continuing demand for increased structural efficiency in many advanced aerospace vehicles has resulted in development of laminated structures made of advanced fiber-reinforced composite materials, such as boron-epoxy, graphite-epoxy and boron-aluminum (reference 1). One of the most common structural elements for such vehicles is the rectangular plate or panel, for which numerous stable static, buckling, and free-vibrational analyses have been performed (reference 2). These advanced vehicles must maintain their structural integrity not only during high staticly applied mechanical and thermal loadings, but also in a variety of dynamic environments. In order to predict the dynamic stresses to which a structure will be subjected, it is necessary to perform a dynamic response analysis including the effects of damping properties as well as the various stiffnesses. An example of this type of requirement is found in the Space Shuttle now under NASA preliminary development (reference 3).

A composite material is defined as two or more materials joined together to form a nonhomogeneous material, which is used in constructing structures. Although such a material is nonhomogeneous on a micro scale, it behaves macroscopically as if it were a homogeneous, anisotropic material. An anisotropic material is one which exhibits different properties when

tested at different directional orientations within the body. For example, a single layer of composite material containing unidirectionally oriented fibers (hereafter referred to as a unidirectional composite) has considerably greater stiffness and strength in the direction of the fibers than it does in a direction transverse to the fibers (see figure 1). This anisotropic aspect is one of several which makes structural analyses for composite-material structures more complicated (reference 4) than those for ordinary isotropic materials, such as aluminum alloys or many unfilled plastics.

A single layer of unidirectional composite has certain orthogonal axes of material symmetry and thus is said to be orthotropic, and behaves in considerably simpler fashion than a general anisotropic material. If the material is thin it is called plane orthotropic. However, for purposes of obtaining a more efficient design, it is often advantageous to place different layers or plies at different orientations. For the case of a thin plate, this means that we must consider it as a plane anisotropic material. This treatment is intermediate between the complicated general anisotropic case and the much simpler plane orthotropic case.

A laminate consists of two or more layers integrally joined together. It is said to be laminated symmetrically if all of the layers above and below the midplane of the laminate have the same dimensions, properties, and orientation (if the layers are orthotropic); see figure 2. A laminate is said to be balanced if all of the layers oriented at $+\theta$ are balanced by an equal number of identical layers at an orientation of $-\theta$; see figure 3. If a laminate is not symmetrically laminated, coupling occurs between in-plane (either normal or shear) stress on one hand and either bending or twisting deformation on the other hand. In such a case as a laminate

consisting of a metallic substrate and a few layers of overlaying filamentary composite material, considerable bending-stretching coupling can occur and thus, it is necessary to consider not only the in-plane and bending stiffnesses but also the coupling stiffnesses which couple together the other two effects (reference 4). Obviously, this is considerably more complicated to analyze than a simple single layer.

Unfortunately, it appears that no one has yet succeeded in devising a lamination scheme that is both symmetrical and balanced. However, in the special, yet practical, case of multiple identical layers oriented alternately at $+\theta$ and $-\theta$, as the number of layers is increased, the bending-coupling effect diminishes (reference 5). For more than ten such layers, the bending-stretching coupling may be neglected for most engineering purposes and, thus, the laminate may be treated macroscopically as if it were a single-layer plate.

It has long been recognized that, due to the relatively low shear stiffnesses of composite materials in planes normal to the laminating plane, the thickness-shear* deformations must be included in the analysis, even when the plate is relatively thin (references 6-8), in order to achieve reasonably good predictions of laminate flexural behavior. Although several existing laminated plate theories include thickness-shear flexibility, all of them require an ad hoc assumption of the required shear correction factor. In this report, two different, relatively simple procedures are

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*Sometimes referred to as "transverse shear", especially in the case of beams. The terminology used here follows that established by Yu for sandwich plates (reference 9). Here the term, transverse, is reserved for the in-plane direction which is normal to the longitudinal direction.

presented to enable rational prediction of the appropriate composite shear correction factor (K). The frequencies calculated using these values for K in shear-flexible laminated plate vibrational analysis give good agreement with the exact three-dimensional elasticity solution, which is computationally much more complicated.

The use of high-damping polymeric materials in the form of a thin layer or tape has come into widespread use as a structural damper to reduce the vibrational response of aircraft panels, especially in high-noise regions such as in the vicinity of jet engines. When the polymeric material is added on either one side or both sides, it is known as an unconstrained damping layer; when the polymeric material is placed between two or more layers, it is called a constrained damping layer (or layers). Appropriate analyses and design procedures have been developed for including either type of damping layer, assuming all layers (metal and polymer) are isotropic (references 10-12). However, so far as known, no detailed solutions have been published on vibration of laminated composite-material plates which included material damping. Such an analysis is presented in this report and the results are compared with experimental data reported recently by Clary (reference 13).

1.2 A Brief Survey of Selected Vibrational Analyses of Laminated Plates

Apparently the first vibrational analysis was carried out by Pister (reference 14) for a thin plate arbitrarily laminated of isotropic layers. In this case, the net effect of the bending-stretching coupling resulted in a reduced flexural stiffness.

Stavsky (reference 15) formulated a coupled bending-stretching dynamic theory for thin plates laminated of composite-material layers, but he did not present any numerical results. Apparently the first published results of the vibrational analysis of such plates is due to Ashton and Waddoups (reference 16), who used the Rayleigh-Ritz method to analyze rectangular plates. These results compared reasonably well with experimental results for the completely free and cantilever cases. A similar analytical and experimental study, but involving simply supported and free boundary conditions, was carried out by Hikami (reference 17). Additional analysis of the free-edge case was carried out by Ashton (reference 18).

For simply supported plates, Whitney and Leissa (references 19,20) presented closed-form solutions for the natural frequencies in the case of cross-ply and angle-ply lamination schemes. As would be expected, their results showed a strong effect of bending-stretching coupling in lowering the natural frequencies.

The case of clamped boundary conditions is more complicated analytically, but more representative of practical aerospace structures. Rayleigh-Ritz and experimental investigations of such structures were carried out independently by Ashton and Anderson (reference 21) and by Bert and Mayberry (reference 22).

The first more accurate vibrational analysis of laminated plates including thickness-shear flexibility was made by Ambartsumyan; see reference 23. He assumed an arbitrary distribution of thickness-shear stresses through the thickness. However, in carrying out his actual calculations, Ambartsumyan assumed a simple parabolic distribution. It can be shown by a simple mechanics-of-materials analysis (Jourawski shear theory; see Section

III) that the thickness-shear stress distribution must be discontinuous from layer to layer; thus, the simple parabolic distribution does not hold. Ambartsumyan did not give any numerical results for vibration of laminated plates; however, Whitney (reference 24) did so, using Ambartsumyan's basic theory.

Using another approach, Yang et al (reference 25) entended the Mindlin homogeneous, isotropic, dynamic plate analysis (reference 26) to the laminated anisotropic case. They assumed a thickness-shear angle which is independent of the thickness coordinate (z) and then integrated the stress equations of motion to obtain the governing partial differential equations. After integration, they introduced a thickness-shear coefficient in an ad hoc fashion to correlate the predicted frequencies with known results.

Also Yang et al introduced the coupling inertial effect (present only in the case of plates laminated unsymmetrically with respect to mass distribution), as well as the familiar translational and rotatory inertia effects present in homogeneous (or mass-symmetrically laminated) plates. The inclusion of these higher-order inertial effects is consistent with the inclusion of thickness-shear flexibility.

The analysis presented in this report is an improvement over both the Ambartsumyan and Yang et al analyses, in that it presents simple rational means for calculating the shear coefficient, rather than assuming it á priori.

An alternative to considering shear deformation per se is to make a microlaminar analysis, such as considered by Biot (reference 27) and Bolotin (reference 28). However, this approach has not been used very extensively so far.

Another approximate approach is to use the Voigt and Reuss models to

determine the properties of an equivalent homogeneous, anisotropic, shear-flexible plate. This method was originated independently by Postma (reference 29), White and Angona (reference 30), and Rytov (reference 31) to investigate wave propagation in a continuum consisting of alternating layers of stiff and flexible isotropic materials. This concept was applied recently to plates with experimental verification for the special case of a beam, by Achenbach and Zerbe (reference 32). A new analysis, extending the Postma approach to the more general case of a laminate consisting of alternating orthotropic layers, is presented in **Appendix F.**

All of the analyses mentioned above are approximate formulations in this sense: interlaminar compatibility is impossible unless both of the in-plane Poisson's ratios are identical in all of the layers. The reason for this is that, in a plate, one can have discontinuities in the strain component in a given direction in the plane, due to the Poisson contraction caused by a stress resultant or a stress couple acting in the perpendicular direction in the plane. This deficiency can be removed by using the approach introduced recently by Hsu and Wang (reference 33) and Wang (reference 34) for laminated shells. Another technique is to apply the nonhomogeneous three-dimensional elasticity approach, such as used recently by Srinivas et al (references 35-37) for a special case of simply supported edges. Still another method is to use finite elements in the thickness direction, as introduced recently by Tso et al (reference 38). Unfortunately, all of the more accurate analyses mentioned in this paragraph are quite complicated computationally and thus are not amendable to engineering analyses of practical structural elements. Furthermore, the work of reference 37 showed that for structurally reasonable values of the following modal parameter, the Mindlin-type theory (such as extended here) is

sufficiently accurate for determination of natural frequencies, but not for determination of the associated stress distribution.

There are numerous analyses in the literature on vibration of damped plates with isotropic layers; also there are a few vibrational analyses of single-layer, anisotropic plates and a few quasi-static (creep) analyses. of laminated, anisotropic plates. However, vibrational analyses of laminated, anisotropic plates are quite limited. Dong (reference 39) indicated the solution for the dynamic response of a simply-supported rectangular plate arbitrarily laminated of orthotropic, viscoelastic plies modeled as a standard linear solid.

SECTION II

FORMULATION OF THE THEORY OF LAMINATED, SHEAR-FLEXIBLE PLATES

In this section is presented a theoretical analysis of sinusoidally forced vibration of laminated, anisotropic plates including bending-stretching coupling, thickness-shear, all three types of inertia effects, and material damping. First the assumptions, on which the analysis is based, are stated explicitly. The general analysis is applicable to plates with any combination of natural boundary conditions at their edges. The analysis begins with the anisotropic stress-strain relations for a single layer and proceeds through the stiffness and damping constitutive relations, formulation of the various types of energy and work terms, and culminates in the formulation of an eigenvalue problem by application of the extended Rayleigh-Ritz method.

2.1 Hypotheses

The following assumptions are made in the analysis presented here:

H1. All displacements are assumed to be sufficiently small so that the linear strain-displacement relations are sufficiently accurate.

H2. The layers which make up the plate are linearly elastic and may be isotropic or orthotropic with any arbitrary orientation in the plane of the plate.

H3. The layers are sufficiently thin that thickness-normal-stress

effects may be neglected, i.e. the layers are assumed to have finite stiffnesses which resist membrane, bending, and thickness shearing stresses.

H4. The layers are assumed to be bonded together perfectly.

H5. The plate is assumed to have all components of translational, translational-rotatory coupling, and rotatory inertia.

H6. All material damping effects are assumed to be small. They are incorporated by using stiffnesses which are complex rather than real (see Appendix B); the complex-stiffness array is assumed to be symmetric (see Appendix I). All other thermal effects are neglected.

H7. All initial-stress effects are neglected.

H8. All interactions with a surrounding fluid can be either neglected or considered to be included in the values used for the material-damping coefficients.

H9. The excitation consists of a uniformly distributed normal pressure loading, sinusoidal with respect to time.

2.2 Kinematics

The displacement field is assumed to be (hypothesis H3):

$$\begin{aligned} u\,(x,y,z,t) &= u_o\,(x,y,t) + z\psi_x(x,y,t) \\ v\,(x,y,z,t) &= v_o\,(x,y,t) + z\psi_y(x,y,t) \\ w\,(x,y,z,t) &= w_o\,(x,y,t) \end{aligned} \qquad (1)$$

where u,v,w are the displacement components in the x,y,z directions (see figure 4); u_o,v_o,w_o are the displacement components at the middle surface

of the multi-layer plate; ψ_x and ψ_y are the weighted middle-surface rotations in the respective xz and yz planes; and t is time.

Within the framework of hypothesis H1, the total engineering strains are given by the following in-plane strain-displacement relations:

$$\epsilon_{ij} = \epsilon^{o}_{ij} + z\kappa_{ij} \qquad \text{(no sum)} \quad (ij = xx, yy, xy) \qquad (2)$$

where

$$\epsilon^{o}_{xx} = u_{o,x} \quad ; \quad \epsilon^{o}_{yy} = v_{o,y} \quad ; \quad \epsilon^{o}_{xy} = v_{o,x} + u_{o,y} \qquad (3)$$

$$\kappa_{xx} = \psi_{x,x} \quad ; \quad \kappa_{yy} = \psi_{y,y} \quad ; \quad \kappa_{xy} = \psi_{y,x} + \psi_{x,y} \qquad (4)$$

Also the following thickness-shear strain-displacement relations hold:

$$\epsilon_{xz} = w_{o,x} + \psi_x \quad ; \quad \epsilon_{yz} = w_{o,y} + \psi_y \qquad (5)$$

where a subscript comma denotes partial differentiation with respect to the variable following the comma, i.e. $u_{o,x} = \partial u_o / \partial x$.

2.3 Stiffness Constitutive Relations

In line with hypothesis H2, the following stress-strain relations are assumed to hold for each individual layer of which the plate is comprised:

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \sigma_{yz}^{(k)} \\ \sigma_{xz}^{(k)} \\ \sigma_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & Q_{16}^{(k)} \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & Q_{26}^{(k)} \\ 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} & 0 \\ 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} & 0 \\ Q_{16}^{(k)} & Q_{26}^{(k)} & 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx}^{(k)} \\ \epsilon_{yy}^{(k)} \\ \epsilon_{yz}^{(k)} \\ \epsilon_{xz}^{(k)} \\ \epsilon_{xy}^{(k)} \end{Bmatrix} \quad (6)$$

where $\sigma_{xx}^{(k)}$, $\sigma_{yy}^{(k)}$ are the in-plane normal stresses; $\sigma_{xz}^{(k)}$, $\sigma_{yz}^{(k)}$ are the thickness-shear stresses; $\sigma_{xy}^{(k)}$ is the in-plane shear stress; the Q_{ij} are the reduced stiffness coefficients which are applicable to a thin layer (see Appendix A); superscript (k) denotes a typical layer k, where k = 1,2,3...,n ; and n is the total number of layers of which the plate is comprised.

It is noted that it is more consistent mathematically to write the stress-strain relations in either matrix form as

$$\left\{ \sigma_i^{(k)} \right\} = \left[Q_{ij}^{(k)} \right] \left\{ \epsilon_j^{(k)} \right\} \quad (i,j=1,2,4,5,6). \quad (7)$$

or in tensor notation as

$$\sigma_{ij}^{(k)} = Q_{ijk\ell}^{(k)} \; \epsilon_{k\ell}^{(k)} \quad (i,j=x,y,z). \quad (8)$$

However, here a mixed notation is used for engineering convenience: double lettered subscripts (xx,yy,yz,xz,xy) following classical elasticity theory notation for stresses and strains, and double numbered subscripts (i,j=1,2,4,5,6) to reduce the number of subscripts used on the stiffness coefficients.

As is conventional in plate and shell theory, it is convenient to introduce generalized forces which are applicable to the whole laminate, rather than only a specific distance from the plate middle surface. The stress components, $\sigma_{ij}^{(k)}$, are functions of x,y,z; therefore, integrating them through the thickness yields the following generalized forces, which are functions of x,y only:

$$\left\{N_{xx},N_{yy},N_{xy},Q_x,Q_y\right\} \equiv \int_{-h/2}^{h/2} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)},\sigma_{xz}^{(k)},\sigma_{yz}^{(k)}\right\} dz$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)},\sigma_{xz}^{(k)},\sigma_{yz}^{(k)}\right\} dz \qquad (9)$$

$$\left\{M_{xx},M_{yy},M_{xy}\right\} \equiv \int_{-h/2}^{h/2} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)}\right\} z\, dz$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{\sigma_{xx}^{(k)},\sigma_{yy}^{(k)},\sigma_{xy}^{(k)}\right\} z\, dz \qquad (10)$$

where h = total plate thickness, the quantity $(z_k - z_{k-1})$ is the thickness of an individual layer (see figure 5), N_{ij} = membrane stress resultants (force per unit length), Q_i = thickness-shear stress resultants (force per unit length), and M_{ij} are the stress couples (moment per unit length).

Inserting the stress-strain relations, given by equations (6) into equations (9) and (10), one obtains the following constitutive relations for the composite:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ \hline M_{xx} \\ M_{yy} \\ M_{xy} \\ \hline Q_y \\ Q_x \end{Bmatrix} = \left[\begin{array}{ccc:ccc:cc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & F_{44} & F_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & F_{55} \end{array}\right] \begin{Bmatrix} \epsilon^o_{xx} \\ \epsilon^o_{yy} \\ \epsilon^o_{xy} \\ \hline \kappa_{xx} \\ \kappa_{yy} \\ \kappa_{xy} \\ \hline \epsilon_{yz} \\ \epsilon_{xz} \end{Bmatrix} \quad (11)$$

where the respective stretching, bending-stretching coupling, and bending stiffnesses of the plate are defined as follows:

$$\{A_{ij}, B_{ij}, D_{ij}\} = \int_{-h/2}^{h/2} Q_{ij}^{(k)} \{1, z, z^2\} \, dz \quad (i,j=1,2,6) \qquad (12)$$

and the thickness-shear stiffnesses of the plate are as follows:

$$F_{ij} = K_{ij} \int_{-h/2}^{h/2} Q_{ij}^{(k)} \, dz \quad (i,j=4,5) \qquad (13)$$

where K_{ij} ≡ composite shear coefficient, introduced to account for the thickness-shear strain variation through the thickness. Section III presents several approximate analyses to predict K_{ij} analytically.

The A_{ij}, B_{ij}, and D_{ij} submatrices appearing in equation (11) are present in the classical Kirchhoff-type theory of laminated plates (see refs. 4,5,15-22). In the case of an orthotropic plate all of the terms with subscripts 16 and 26, called the cross-elasticity or shear-coupling terms, vanish. Furthermore, when the layers are all isotropic materials,

only two of each set of the terms with subscripts 11,12,22, and 66 are independent.

If the plate is either homogeneous (i.e. single layer) or laminated symmetrically about the plate middle surface, the bending-stretching coupling stiffness submatrix B_{ij} vanishes. Then the only remaining terms are the A_{ij} and D_{ij} submatrices, which are present in classical, homogeneous thin-plate theory.

The thickness-shear stiffness coefficients F_{ij} account for the presence of thickness-shear strains, in a manner which can be considered to be a generalization of that used by Mindlin (ref. 26) for homogeneous, isotropic plates. To reduce the present theory to the classical Kirchhoff-type plate theory, the thickness-shear strains would be omitted; thus, from equations (5), one would obtain:

$$\psi_x = - w_{o,x} \; ; \quad \psi_y = - w_{o,y} \tag{14}$$

Furthermore, one would delete the thickness-shear strains, ϵ_{xz} and ϵ_{yz}, from equations (11) and the thickness-shear stress resultants, Q_x and Q_y, would be computed from equilibrium considerations only.

2.4 Strain Energy

The differential of the strain-energy density (strain energy per unit volume) is given by:

$$dU_v^{(k)} = \sigma_{xx}^{(k)} \, d\epsilon_{xx}^{(k)} + \sigma_{yy}^{(k)} \, d\epsilon_{yy}^{(k)} + \sigma_{xy}^{(k)} \, d\epsilon_{xy}^{(k)}$$

$$+\sigma_{xz}^{(k)}\, d\epsilon_{xz}^{(k)} + \sigma_{yz}^{(k)} d\epsilon_{yz}^{(k)} \qquad (15)$$

Integration of equation (15) yields:

$$U_v^{(k)} = (1/2)\,[\sigma_{xx}^{(k)}\,\epsilon_{xx}^{(k)} + \sigma_{yy}^{(k)}\,\epsilon_{yy}^{(k)} + \sigma_{xy}^{(k)}\,\epsilon_{xy}^{(k)} + \sigma_{yz}^{(k)}\,\epsilon_{yz}^{(k)} + \sigma_{xz}^{(k)}\,\epsilon_{xz}^{(k)}\,] \qquad (16)$$

The strain energy per unit of plate area (U_A) is the integral of $U_v^{(k)}$ over the total thickness of the plate:

$$U_A = \int_{-h/2}^{h/2} U_v^{(k)} dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} U_v^{(k)} dz \qquad (17)$$

Substituting equations (16), (6), and (2) into equation (17), performing the integration, and using equations (12) and (13), one obtains the following expression for the strain energy per unit area:

$$\begin{aligned} U_A = (1/2)\,[&A_{11}(\epsilon_{xx}^o)^2 + 2A_{12}\,\epsilon_{xx}^o\,\epsilon_{yy}^o + A_{22}(\epsilon_{yy}^o)^2 + 2A_{16}\,\epsilon_{xx}^o\,\epsilon_{xy}^o \\ &+ 2A_{26}\,\epsilon_{yy}^o\,\epsilon_{xy}^o + A_{66}\,(\epsilon_{xy}^o)^2 + 2B_{11}\,\epsilon_{xx}^o\,\kappa_{xx} \\ &+ 2\,B_{12}(\epsilon_{xx}^o\kappa_{yy} + \epsilon_{yy}^o\,\kappa_{xx}) + 2\,B_{22}\,\epsilon_{yy}^o\,\kappa_{yy} \\ &+ 2\,B_{16}(\epsilon_{xx}^o\,\kappa_{xy} + \epsilon_{xy}^o\,\kappa_{xx}) + 2B_{26}(\epsilon_{yy}^o\,\kappa_{xy} + \epsilon_{xy}^o\,\kappa_{yy}) \\ &+ 2\,B_{66}\,\epsilon_{xy}^o\,\kappa_{xy} + D_{11}\,\kappa_{xx}^2 + 2\,D_{12}\,\kappa_{xx}\kappa_{yy} + D_{22}\,\kappa_{yy}^2 \\ &+ 2\,D_{16}\,\kappa_{xx}\,\kappa_{xy} + 2\,D_{26}\,\kappa_{yy}\,\kappa_{xy} + D_{66}\,\kappa_{yy}^2 + F_{44}\,\epsilon_{yz}^2 \\ &+ 2\,F_{45}\,\epsilon_{yz}\,\epsilon_{xz} + F_{55}\,\epsilon_{xz}^2\,] \end{aligned} \qquad (18)$$

2.5 Damping Coefficients and Dissipative Energy

For a material with Kimball-Lovell structural damping (see Appendix B), the stress-strain rate relation for sinusoidal motion at a circular frequency ω can be expressed as follows:

$$\hat{\sigma} = (b/\omega)\ \dot{\epsilon} \tag{19}$$

where b is a material constant and $\dot{\epsilon}$ is the strain rate.

The strain is given by

$$\epsilon = |\epsilon|\ e^{i\omega t} \tag{20}$$

where $i \equiv \sqrt{-1}$.

Thus,

$$\dot{\epsilon} = i\omega\epsilon \tag{21}$$

Substituting equation (21) into equation (19), one obtains:

$$\hat{\sigma} = \hat{Q}\epsilon \tag{22}$$

where

$$\hat{Q} = ib \tag{23}$$

Generalizing equation (23) to the entire array of stresses and strains in layer "k", in contracted notation analogous to equation (7), we obtain the following expression, where $\hat{Q}_{ij}^{(k)}$ is symmetric (hypothesis H6):

$$\left\{ \hat{\sigma}_i^{(k)} \right\} = [\hat{Q}_{ij}^{(k)}] \left\{ \epsilon_j^{(k)} \right\} \tag{24}$$

Equation (24) is the same as equation (7), except that here the presence of the hat symbols ($\hat{}$) denotes damping quantities rather than elastic quantities.

The energy dissipated per unit of plate area, due to damping, is denoted by the symbol D_A. The expression for it in terms of the midplane strains (ϵ^o_{ij}) and the curvatures ($\varkappa_{ij}$) is the same as equation (18), except for the presence of the hat symbols over all of the A_{ij}, B_{ij}, D_{ij}, and F_{ij} quantities, where

$$\hat{A}_{ij} = ig^{(A)}_{ij} A_{ij} \quad , \quad \hat{B}_{ij} = ig^{(B)}_{ij} B_{ij} \quad , \quad \hat{D}_{ij} = ig^{(D)}_{ij} D_{ij} \quad ,$$

$$\hat{F}_{ij} = ig^{(F)}_{ij} F_{ij} \tag{25}$$

where $g^{(\ )}_{ij}$ are loss tangents (see Appendix B).

2.6 Kinetic Energy

The differential kinetic energy of an elemental volume (dx by dy by dz) is given by:

$$(\rho^{(k)}/2)\ (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\ dx\ dy\ dz$$

where $\rho^{(k)}$ is the density at the point (x,y,z) and $\dot{u},\dot{v},\dot{w}$ are the velocity components in the x,y,z directions respectively.

Now the kinetic energy per unit of plate area is obtained from the differential quantity given above by dividing by dx dy and integrating over the entire thickness of the plate, as follows:

$$T_A = \int_{-h/2}^{h/2} (\rho^{(k)}/2)(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)\, dz \qquad (26)$$

Substituting the kinematic relations, equations (1), into equation (26) and performing the integration, assuming that the density is uniform through the thickness of each individual layer, one arrives at the following expression:

$$T_A = (m_o/2)(\dot{u}_o^2 + \dot{v}_o^2 + \dot{w}_o^2) + m_1 (\dot{u}_o \dot{\psi}_x + \dot{v}_o \dot{\psi}_y)$$

$$+ (m_2/2) (\dot{\psi}_x^2 + \dot{\psi}_y^2) \qquad (27)$$

where m_o, m_1, and m_2 are respectively the mass per unit of plate area, first moment of mass per unit area (coupling inertia), and second moment of mass per unit area (mass moment of inertia), defined as follows:

$$\{m_o, m_1, m_2\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho^{(k)} \{1, z, z^2\}\, dz \qquad (28)$$

Only the coefficient m_o appears in classical Kirchhoff theory for the dynamics of thin, homogeneous plates. The Mindlin dynamic plate theory of homogeneous plates (ref. 26) contains both m_o and m_2, while the dynamic theory originated by Yang et al (ref. 25) for laminated plates contains m_o, m_1, and m_2.

2.7 Work Done by External Forces

The only external force considered here is a uniformly distributed normal pressure which has a sinusoidal wave form in time. Thus, it can be written in complex variable notation as follows:

$$q(x,y,t) = \tilde{q}\, e^{i\omega t} \tag{29}$$

where $\tilde{q}$ = amplitude of the normal pressure.

The following expression for the total work done may be derived easily from the principle of virtual work:

$$W = \int_o^a \int_o^b q\,(x,y,t)\; w_o\,(x,y)\; dx\, dy \tag{30}$$

2.8 Application of the Extended Rayleigh-Ritz Method

Since the present problem is one of steady-state harmonic excitation denoted in complex-variable exponential forms, the displacements and rotations are proportional to $e^{i\omega t}$ also. It is noted that the phase angle between the response and the excitation is taken care of by the imaginary components of these quantities. Thus, the time dependence cancels out in all of the energy terms which appear in Lagrange's equation. Therefore, it is necessary to consider only the amplitudes of the respective energy and work terms. Then, as shown in detail in Appendix C, the amplitude of the Lagrangian energy difference can be expressed as follows:

$$\tilde{L} = (\tilde{T} + \tilde{W}) - (\tilde{U} + \tilde{D}) \tag{31}$$

where $\tilde{D},\tilde{T},\tilde{U},\tilde{W}$ are the amplitudes of the dissipative, kinetic, and strain energies and of the work done by the external forces, respectively.

The strain and damping energy terms can be combined by introducing complex stiffnessses, as discussed in Appendix B. Making this substitution and integrating over the plate area, we obtain the following:

$$\begin{aligned}\tilde{U}+\tilde{D} = \tfrac{1}{2}\int_0^a\int_0^b \{&\bar{A}_{11}\tilde{u}_{o,x}^2 + 2\bar{A}_{12}\tilde{u}_{o,x}\tilde{v}_{o,y} + \bar{A}_{22}\tilde{v}_{o,y}^2 + 2(\bar{A}_{16}\tilde{u}_{o,x}+\bar{A}_{26}\tilde{v}_{o,y})(\tilde{v}_{o,x}+\tilde{u}_{o,y})\\ &+\bar{A}_{66}(\tilde{v}_{o,x}+\tilde{u}_{o,y})^2 + 2\bar{B}_{11}\tilde{u}_{o,x}\tilde{\psi}_{x,x} + 2\bar{B}_{12}(\tilde{u}_{o,x}\tilde{\psi}_{y,y}+\tilde{v}_{o,y}\tilde{\psi}_{x,x})+2\bar{B}_{22}\tilde{v}_{o,y}\tilde{\psi}_{y,y}\\ &+2\bar{B}_{16}[\tilde{u}_{o,x}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})+(\tilde{v}_{o,x}+\tilde{u}_{o,y})\tilde{\psi}_{x,x}] + 2\bar{B}_{26}[\tilde{v}_{o,y}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})\\ &+ (\tilde{v}_{o,x}+\tilde{u}_{o,y})\tilde{\psi}_{y,y}] + 2\bar{B}_{66}(\tilde{v}_{o,x}+\tilde{u}_{o,y})(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) + \bar{D}_{11}\tilde{\psi}_{x,x}^2\\ &+2\bar{D}_{12}\tilde{\psi}_{x,x}\tilde{\psi}_{y,y}+\bar{D}_{22}\tilde{\psi}_{y,y}^2+ 2\bar{D}_{16}\tilde{\psi}_{x,x}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y}) +2\bar{D}_{26}\tilde{\psi}_{y,y}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})\\ &+\bar{D}_{66}(\tilde{\psi}_{y,x}+\tilde{\psi}_{x,y})^2 + \bar{F}_{44}(\tilde{w}_{o,y}+\tilde{\psi}_y)^2 + 2\bar{F}_{45}(\tilde{w}_{,y}+\tilde{\psi}_y)(\tilde{w}_{o,x}+\tilde{\psi}_x)\\ &+\bar{F}_{55}(\tilde{w}_{o,x}+\tilde{\psi}_x)^2\} \, dx\, dy \end{aligned} \tag{32}$$

where the superscript $(\tilde{\ })$ denotes that the quantity is an amplitude, and

$$\bar{A}_{ij},\ \bar{B}_{ij},\ \bar{D}_{ij} = (A_{ij}+\hat{A}_{ij}),\ (B_{ij}+\hat{B}_{ij}),\ (D_{ij}+\hat{D}_{ij});\ i,j=1,2,6$$

$$\bar{F}_{ij} = F_{ij}+\hat{F}_{ij}\ ;\ i,j=4,5 \tag{33}$$

The sum of the amplitudes of the kinetic energy and the work done by external forces can be expressed as follows:

$$\tilde{T}+\tilde{W} = (\omega^2/2) \int_o^a \int_o^b [m_o (\tilde{u}_o^2 + \tilde{v}_o^2 + \tilde{w}_o^2) + 2m_1(\tilde{u}_o\tilde{\psi}_x + \tilde{v}_o\tilde{\psi}_y)$$

$$+ m_2 (\tilde{\psi}_x^2 + \tilde{\psi}_y^2)] \, dx \, dy + \int_o^a \int_o^b \tilde{q} \, \tilde{w}_o \, dx \, dy \qquad (34)$$

To apply the extended Rayleigh-Ritz method, the assumed functions for the amplitudes of the displacements and rotations are given the following general form:

$$\begin{aligned}
\tilde{u}_o &= \sum_{m=1}^{M} \sum_{n=1}^{N} U_{mn} \Phi_{um}(\alpha) \Phi_{un}(\beta) \\
\tilde{v}_o &= \sum_{m=1}^{M} \sum_{n=1}^{N} V_{mn} \Phi_{vm}(\alpha) \Phi_{vn}(\beta) \\
\tilde{w}_o &= \sum_{m=1}^{M} \sum_{n=1}^{N} W_{mn} \Phi_{wm}(\alpha) \Phi_{wn}(\beta) \qquad (35) \\
\tilde{\psi}_y &= \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{ymn} \Phi_{\psi ym}(\alpha) \Phi_{\psi yn}(\beta) \\
\tilde{\psi}_x &= \sum_{m=1}^{M} \sum_{n=1}^{N} \Psi_{xmn} \Phi_{\psi xm}(\alpha) \Phi_{\psi xn}(\beta)
\end{aligned}$$

The modal coefficients $U_{mn}, V_{mn}, W_{mn}, \Psi_{ymn}, \Psi_{xmn}$ are the undetermined parameters, the Φ's are assumed modal functions, $\alpha = x/a$, $\beta = y/b$.

Substituting equations (32), (34), and (35) into equation (31) for the Lagrangian energy-difference amplitude $(\tilde{L})$, one obtains the result presented in Appendix D1.

Hamilton's principle can be stated mathematically as follows:

$$\delta \int_{t_1}^{t_2} \tilde{L}\, dt = 0 \tag{36}$$

To achieve as close of an approximation as possible to equation (36), the Lagrangian function $\tilde{L}$ is minimized by setting its partial derivatives with respect to the modal coefficients respectively equal to zero, i.e.

$$\partial\tilde{L}/\partial U_{k\ell} = 0 \; ; \quad \partial\tilde{L}/\partial V_{k\ell} = 0 \; ; \quad \partial\tilde{L}/\partial W_{k\ell} = 0 \; ; \quad \partial\tilde{L}/\partial \Psi_{xk\ell} = 0$$

$$\partial\tilde{L}/\partial \Psi_{yk\ell} = 0 \quad ; \quad k = 1,2,\ldots, M \quad ; \quad \ell = 1,2,\ldots, N \tag{37}$$

Equations (37) represent a set of M x N nonhomogeneous, linear algebraic equations.

2.9 Reduction to Matrix Form

For computational convenience, it is desirable to express the set of equations (37) in matrix form as follows:

$$[S] \left\{\xi_{mn}\right\} - \omega^2 [M] \left\{\xi_{mn}\right\} = \left\{f\right\} \tag{38}$$

where [S] = complex stiffness matrix, [M] = mass matrix, and $\left\{\xi_{mn}\right\}$ and $\left\{f\right\}$ are column matrices representing the generalized displacements and generalized forces, respectively, i.e.

$$\left\{\xi_{mn}\right\} \equiv \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \Psi_{ymn} \\ \Psi_{xmn} \end{Bmatrix} \quad ; \quad \left\{f\right\} \equiv \begin{Bmatrix} 0 \\ 0 \\ \tilde{q} \\ 0 \\ 0 \end{Bmatrix} \tag{39}$$

Equation (38) can be written in explicit complex form as follows:

$$\left[S^{(R)} + iS^{(I)}\right] \left\{\xi_{mn}^{(R)} + i\xi_{mn}^{(I)}\right\} - \omega^2 \left[M^{(R)}\right] \left\{\xi_{mn}^{(R)} + i\xi_{mn}^{(I)}\right\} = \left\{f^{(R)}\right\} \tag{40}$$

where superscripts (R) and (I) denote the real and imaginary parts of the complex quantities appearing in equation (38).

Equation (40) can be solved for the response matrix, partitioned into real and imaginary parts, as follows:

$$\begin{Bmatrix} \xi_{mn}^{(R)} \\ \xi_{mn}^{(I)} \end{Bmatrix} = \begin{bmatrix} S^{(R)} - \omega^2 M^{(R)} & -S^{(I)} \\ S^{(I)} & S^{(R)} - \omega^2 M^{(R)} \end{bmatrix}^{-1} \begin{Bmatrix} f^{(R)} \\ 0 \end{Bmatrix} \tag{41}$$

The modal coefficients are placed in the inverse matrix in equation (41). For the specific numerical case treated M=N=2, i.e. m,n=1,2. Thus, the complete matrix is of order 40 x 40 with each submatrix being of order 2 x 2. Thus, the problem has been reduced to a standard algebraic eigenvalue problem which can be solved by an available computer suboutline for the IBM 360 Series 50 digital computer. Complete computer program documentation is presented in Appendix J.

SECTION III

DERIVATION OF THICKNESS-SHEAR FACTORS FOR LAMINATES

In this section are presented two entirely different approaches for calculating the thickness-shear factors for a laminate. The first approach is to extend the Jourawski static theory of shear-flexible beams (reference 40), as presented in elementary mechanics-of-materials textbooks, to a laminated beam. The other approach is to extend, to the laminate case, Mindlin's method (ref. 26) of matching the pure thickness-shear-mode frequencies predicted by two-dimensional dynamic elasticity and by Timoshenko one-dimensional, shear-flexible beam theory (references 41, 42).

In the case of isotropic plates, there is only one independent thickness-shear factor (K), since $K_{45}=0$ and $K_{55}=K_{44}=K$. The static approach mentioned above yields a value of K = 0.833 (reference 43), while Mindlin's dynamic approach results in a value of 0.822 for K. Srinivas et al. (ref. 35) showed that use of either of these values for K in Mindlin's plate theory results in a very close approximation to the lower natural frequencies computed by exact, dynamic, three-dimensional theory of elasticity for rather thick, homogeneous isotropic rectangular plates simply supported on all edges.

In the case of orthotropic plates, there are two independent thickness-shear factors (K_{44} and K_{55}), since $K_{45}=0$ and $K_{55}\neq K_{44}$. In the case of a plane anisotropic plate, such as one consisting of a unidirectional

composite material with its major material-symmetry axis oriented at an acute angle (θ) with its edges, there are three independent, non-zero K_{ij}. However, for this case, Appendix A presents transformations from which the three F_{ij}, proportional to K_{ij} and defined in equation (13), can be calculated from the following data: F_{44} and F_{55} for the orthotropic case ($\theta=0^{o}$) and the value of θ in the anisotropic case. Thus, the general problem of determining K_{ij} is reduced to one of calculating K_{44} and K_{55} for the orthotropic case.

To calculate K_{55} for an orthotropic laminate by either of the two approaches mentioned above, one considers a beam oriented in the x direction and laminated of layers having properties $E_{11}^{(k)*}$ and $C_{55}^{(k)}$. To calculate K_{44}, $x \rightarrow y$, $E_{11}^{(k)} \rightarrow E_{22}^{(k)}$, and $C_{55}^{(k)} \rightarrow C_{44}^{(k)}$.

To provide a check for the results, the fundamental eigenvalues for free vibration calculated by laminated, shear-flexible plate theory (Section II), using these values of K_{ij}, are compared with the exact laminated elasticity theory values given by Srinivas et al. (ref. 35).

3.1 Static Approach

To calculate the horizontal shear force per unit width, ΔF, acting on a cross section, figures 6 and 7 are used. For consistency plate notation is used, even though the theory is only one dimensional.

The bending stress in typical layer "k", at a distance z from the midplane of the laminate, is calculated as follows:

$$\sigma_{xx}^{(k)} = E_{11}^{(k)} \kappa_{xx} z \qquad (42)$$

*For thin plates as treated in this report, $E_{11}^{(k)} \rightarrow Q_{11}^{(k)}$, where $Q_{11}^{(k)} \equiv E_{11}^{(k)} / (1-\nu_{12}^{(k)} \nu_{21}^{(k)})$.

where κ_{xx} = longitudinal bending curvature and $E_{11}^{(k)}$ = longitudinal Young's modulus of layer "k".

The longitudinal stress couple (bending moment per unit width) is defined as follows:

$$M_{xx} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} z \, dz \tag{43}$$

The longitudinal curvature can be expressed in terms of M_{xx} as follows:

$$\kappa_{xx} = M_{xx}/D_{11} \tag{44}$$

where D_{11} = longitudinal flexural rigidity.

From equations (42) and (44), the bending stresses acting on the left ($x=x_1$) and right ($x = x_1+\Delta x$) sides of the elements are:

$$\begin{aligned} \sigma_{xx}^{(k)}(x_1) &= (E_{11}^{(k)} z/D_{11}) M_{xx} \quad ; \\ \sigma_{xx}^{(k)}(x_1+\Delta x) &= (E_{11}^{(k)} z/D_{11})(M_{xx}+\Delta M_{xx}) \end{aligned} \tag{45}$$

Since both of these bending stress distributions act on identical cross-sectional areas, the horizontal shear force per unit width, which acts on the element, is given by:

$$\Delta F = \int_{A_o} [\sigma_{xx}^{(k)}(x_1+\Delta x) - \sigma_{xx}^{(k)}(x_1)] \, dA \tag{46}$$

where A_o = area shown in figure 8.

Combining equations (45) and (46), one obtains:

$$\Delta F = (\Delta M_{xx}/D_{11})\,Y(z) \tag{47}$$

where

$$Y(z) \equiv \int_{A_o} E_{11}^{(k)} z \, dA \ .$$

The force per unit width, ΔF, must equilibrate the horizontal shear stress, $\sigma_{xz}^{(k)}$, acting on the bottom face of the element shown in figure 7. Thus, we have:

$$\Delta F = \sigma_{xz}^{(k)} \, \Delta x \tag{48}$$

Hooke's law in shear for layer "k" is

$$\sigma_{xz}^{(k)} = C_{55}^{(k)} \epsilon_{xz}^{(k)} \tag{49}$$

where $C_{55}^{(k)}$ and $\epsilon_{xz}^{(k)}$ are the modulus and strain corresponding to longitudinal thickness-shear action in layer "k".

Substituting equation (49) into equation (48), one obtains the following result:

$$\Delta F = C_{55}^{(k)} \epsilon_{xz}^{(k)} \Delta x \tag{50}$$

The following expression for the longitudinal shear strain at a distance z is obtained by equating the right-hand sides of equation (47) and (50):

$$\epsilon_{xz}^{(k)}(x,z) = [Y(z)/(C_{55}^{(k)} D_{11})]\, Q_x(x) \tag{51}$$

where Q_x is the thickness-shear stress resultant (force per unit width), which is related to M_{xx} by static equilibrium as follows:

$$Q_x = \Delta M_{xx}/\Delta x \tag{52}$$

The shear strain energy for a laminated, rectangular-cross-section beam, shown in figure 9, is:

$$U_s = \tfrac{1}{2} \int_A C_{55}^{(k)} [\epsilon_{xz}^{(k)}]^2 \, dA \tag{53}$$

The shear strain energy associated with an equivalent uniform, longitudinal thickness-shear strain in a laminated, rectangular-cross-section beam of cross-sectional area A is:

$$U_s' = (K_{55}/2)(\epsilon_{xz}')^2 \int_A C_{55}^{(k)} \, dA \tag{54}$$

where K_{55} is the longitudinal thickness-shear factor and ϵ_{xz}' is weighted according to:

$$\epsilon_{xz}' = [\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)} \epsilon_{xz}^{(k)} \, dz][\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)} \, dz]^{-1} \tag{55}$$

Equating the right-hand sides of equations (53) and (54) yields an equation; then substituting equations (49) and (51) into that equation, one arrives at the following explicit expression for the longitudinal

thickness-shear factor

$$K_{55} = \frac{\left[\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Y(z)\, dz\right]^2}{\left[\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} C_{55}^{(k)}\, dz\right] \left\{\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} (C_{55}^{(k)})^{-1} [Y(z)]^2\, dz\right\}} \qquad (56)$$

Equation (56) is a general expression for the static, longitudinal thickness-shear factor for an arbitrarily laminated beam. It is a relatively simple algebraic expression which depends upon only the longitudinal Young's and thickness-shear moduli of the individual layers ($E_{11}^{(k)}$ and $C_{55}^{(k)}$) and the lamination geometry. Numerical results are presented in later examples.

3.2 Dynamic Approach

Due to the inherent mathematical form of the equations of dynamic elasticity theory for a multi-material (multi-layer) medium, it is not feasible to present a general solution for the thickness-shear factor obtained by using the dynamic approach. Even in the case of a specific class of laminate, such as the three-layer, symmetrically laminated one, it is not possible to obtain an explicit expression for the thickness-shear factor. In fact, the complexity of the dynamic case, which involves the dynamic elasticity-theory analysis of a shear-flexible laminated beam, approaches that of the shear-flexible laminated plate and thus negates much of the advantage of using the thickness-shear factor

approach to laminated, shear-flexible plate dynamics. However, in order to illustrate the dynamic approach and to provide a numerical comparison with the static approach and with the exact results of Srinivas et al (ref. 35), an analysis for the symmetrical three-layer case is presented in Appendix E.

SECTION IV

MODAL FUNCTIONS USED FOR VARIOUS PLATE BOUNDARY CONDITIONS

In applying the extended Rayleigh-Ritz method, the approximate modal shapes assumed must satisfy the kinematic boundary conditions. However, to improve convergence, it is desirable to satisfy the force-type boundary conditions also. For a rectangular plate, the three cases of boundary conditions most commonly encountered in practice are discussed in Sections 4.1-4.3.

It should be noted tha C_m, C_n, Z_m, Z_n are characteristic parameters tabulated in reference 44. Also, five boundary conditions per edge need to be prescribed rather than four as in classical theory.

4.1 Simply Supported on All Edges

The boundary conditions considered here are simply-supported edges in the sense that

$$\begin{aligned} N_{xx} = M_{xx} = \tilde{v}_o = \tilde{w}_o = \tilde{\psi}_y = 0 \qquad & \text{at } x = 0,a \ (\alpha=0,1) \\ N_{yy} = M_{yy} = \tilde{u}_o = \tilde{w}_o = \tilde{\psi}_x = 0 \qquad & \text{at } y = 0,b \ (\beta=0,1) \end{aligned} \tag{57}$$

As was pointed out by Wang (reference 45), it is impossible for a separable-form deflection function to satisfy the above boundary conditions for the case of a plate of anisotropic material (Q_{16} and $Q_{26} \neq 0$). However, the following modal functions permit eqs. (35) to satisfy eq. (57) exactly

in the orthotropic case, but only approximately in the anisotropic case:

$$\Phi_{um} = \Phi_{\psi xm} = \cos m\pi\alpha \; ; \quad \Phi_{vm} = \Phi_{wm} = \Phi_{\psi ym} = \sin m\pi\alpha$$
$$\Phi_{vn} = \Phi_{\psi yn} = \cos n\pi\beta \; ; \quad \Phi_{un} = \Phi_{wn} = \Phi_{\psi xn} = \sin n\pi\beta \tag{58}$$

For the simply supported case, we substitute equations (58) into equations (D-1 through D-6). The evaluations of the integral forms are presented in Appendix G.

4.2 Fully Clamped on All Edges

The boundary conditions for a fully clamped laminated plate are as follows:

$$\begin{aligned} &\tilde{u}_o = \tilde{v}_o = 0 \quad \text{at } x = 0,a \text{ and } y = 0,b \\ &\tilde{w}_o = \tilde{w}_{o,x} = \tilde{\psi}_y = 0 \quad \text{at } x = 0,a \\ &\tilde{w}_o = \tilde{w}_{o,y} = \tilde{\psi}_x = 0 \quad \text{at } y = 0,b \end{aligned} \tag{59}$$

The assumed modal functions selected to permit equations (35) to satisfy equations (59) are as follows:

$$\Phi_{um} = \Phi_{vm} = \Phi_{\psi xm} = \Phi_{\psi ym} = \sin 2m\pi\alpha$$
$$\Phi_{un} = \Phi_{vn} = \Phi_{\psi xn} = \Phi_{\psi yn} = \sin 2n\pi\beta$$
$$\Phi_{wm} = \cosh Z_m\alpha - \cos Z_m\alpha - C_m (\sinh Z_m\alpha - \sin Z_m\alpha) \tag{60}$$
$$\Phi_{wn} = \cosh Z_n\beta - \cos Z_n\beta - C_n (\sinh Z_n\beta - \sin Z_n\beta)$$

For the fully clamped case, we substitute equations (60) into equations (D1 through D6). The evaluations of the integral forms are presented in Appendix G.

4.3 Free on All Edges

The boundary conditions appropriate for a laminated plate free on all edges can be expressed mathematically as follows:

$$\begin{aligned} N_{xy} &= M_{xy} = 0 && \text{at } x = 0,a \text{ and } y = 0,b \\ N_{xx} &= Q_x = M_{xx} = 0 && \text{at } x = 0,a \\ N_{yy} &= Q_y = M_{yy} = 0 && \text{at } y = 0,b \end{aligned} \tag{61}$$

The assumed modal functions selected to permit equations (35) to satisfy equations (61) are as follows:

$$\begin{aligned} \Phi_{um} &= \Phi_{vm} = \Phi_{\psi xm} = \Phi_{\psi ym} = 1 - \cos 2m\pi\alpha \\ \Phi_{un} &= \Phi_{vn} = \Phi_{\psi xn} = \Phi_{\psi yn} = 1 - \cos 2n\pi\alpha \\ \Phi_{wm} &= \cos m\pi\alpha \quad ; \qquad \Phi_{wn} = \cos n\pi\beta \end{aligned} \tag{62}$$

For the free edge case, we substitute equations (62) into equations (D1 through D6). The evaluation of the resulting integral forms are presented in Appendix G.

SECTION V

NUMERICAL RESULTS AND COMPARISON WITH RESULTS OF OTHER INVESTIGATORS

In this section a comparison is made between the numerical results obtained in this investigation and those obtained by other investigators.

5.1 Thickness-Shear Factors for Laminates

Two example problems are analyzed and, where possible, results of Example 1 are compared with solutions obtained from other sources. The results of Example 2 are presented for use in design.

Example 1: Three-layer, symmetric, isotropic laminate. - The coordinate system is shown in figure 10. All geometrical and material property ratios correspond to the data of ref. 35 given as follows:

$$z_2/z_1 = h^{(2)}/[h^{(2)}+2h^{(1)}] = 0.8 \ , \quad E_{11}^{(1)}/E_{11}^{(2)} = 15,$$

$$C_{55}^{(1)}/C_{55}^{(2)} = 15, \quad \rho^{(1)}/\rho^{(2)} = 1 \ , \text{ and } \nu^{(1)} = \nu^{(2)} = 0.3$$

Assuming only the x-direction is considered in the static and dynamic approaches which were developed in Section III, then the following results are obtained: (I) static case - the result obtained from equation (56) can be expressed as $K_{55}=0.651$, (II) dynamic case - the result obtained by equating eqs. (E-13) and (E-27) is $K_{55}=0.350$.

When the three-layer, laminate thickness parameter is chosen to be $z_2/z_1 = 1$, the analysis given in Section III can be reduced to a single-layer homogeneous, isotropic material. Thus, using the data $E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = \rho^{(1)}/\rho^{(2)} = 1$ and $\nu^{(1)} = \nu^{(2)} = 0.3$, the static case gives $K_{55} = 0.833$ and dynamic case results in $K_{55} = 0.822$. These results are identical with the values of the static and the dynamic shear factors obtained for single-layer plates by previous investigators (refs. 43 and 16).

Example 2: Multiple alternating layers of two materials. - The geometrical and material property ratios are given as

$$h^{(2)}/h^{(1)} = 1, \quad E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)} = \text{from 0.01 to 100}$$

The laminates vary from two layers to nine layers. Since the static case is a relatively simple algebraic expression which considers only longitudinal Young's and thickness-shear moduli of the individual layers ($E_{11}^{(k)}$ and $C_{55}^{(k)}$) and the laminate geometry, for design purposes, eq. (56) will be used. The results are shown in table 1 and figure 11.

5.2 Plate Simply Supported on All Edges

Here we consider only free vibration of plates made of composite material having material-symmetry axes coinciding with the geometric coordinates (plate edges) as shown in figure 12. Neglecting energy dissipation, the time integral of the difference between the potential and kinetic energies attains a stationary value. The displacement and rotations

are proportional to $e^{i\omega t}$. The time dependence in the strain and kinetic energies cancel each other, i.e. only the amplitudes of the strain energy and kinetic energy need be considered. Thus, we use only the real part of homogeneous eqs. (D1-D6) with proper boundary conditions, see eq. (57), to apply Example 3.

Example 3: Three-layer, symmetric, isotropic plate. - The elastic coefficients are those of isotropic material:

$$Q_{11}^{(k)} = Q_{22}^{(k)} = E^{(k)} / \left\{ 1 - [\nu^{(k)}]^2 \right\}, Q_{12}^{(k)} = \nu^{(k)} E^{(k)} / \left\{ 1-[\nu^{(k)}]^2 \right\}$$

$$Q_{44}^{(k)} = Q_{55}^{(k)} = Q_{66}^{(k)} = E^{(k)} / \left\{ 2[1+\nu^{(k)}] \right\} , \quad Q_{45}^{(k)}=0 \quad (k=1,2)$$

Geometrical and material property ratios correspond to the data of ref. 35, given as follows

$$z_2/z_1 = h^{(2)}/[h^{(2)}+2h^{(1)}]= 0.8, \; E_{11}^{(1)}/E_{11}^{(2)} = C_{55}^{(1)}/C_{55}^{(2)}=15,$$

$$\rho^{(1)}/\rho^{(2)} = 1, \quad \nu^{(1)} = \nu^{(2)} = 0.3$$

and the following dimensionless geometric parameter defined in ref. 35:

$$g_k^2 \equiv \left\{ [m\pi h^{(k)}/a]^2 + [n\pi h^{(k)}/b]^2 \right\}^{\frac{1}{2}} = \begin{cases} 0.0002\pi^2 & \text{for } k=1 \\ 0.0128\pi^2 & \text{for } k=2 \end{cases}$$

When the thickness of three-ply laminates is chosen to be $z_2/z_1 = 1$, it can be reduced to a single-layer material as a special case. Then the material properties are as follows:

$$Q_{ij}^{(1)} = Q_{ij}^{(2)} \qquad i,j = 1, 2, 4, 5, 6$$

$$E_{11}^{(1)} / E_{11}^{(2)} = C_{55}^{(1)} / C_{55}^{(2)} = \rho^{(1)} / \rho^{(2)} = 1 \; ; \; \nu^{(1)} = \nu^{(2)} = 0.3$$

$$g_1 = 0, \quad g_2 = 12.$$

A FORTRAN IV program is employed to compute the lowest eigenvalues of plates with simply supported edges. The two cases considered are: (1) single-layer homogeneous, isotropic, and (2) three-ply symmetrical construction. The results are shown as a solid line in figures 13 and 14. The static and dynamic shear factors calculated in examples 1 and 2 are also shown for comparison.

The static Jourawski shear theory and dynamic Timoskenko type theory appear to give good results for K as evaluated by inserting in laminated, shear flexible plate theory and comparing the lowest eigenvalues with those given in refs. 35 and 14.

5.3 Plate Free on All Edges

The problem considered here is forced vibration of a rectangular plate made of an arbitrary number of composite-material layers each having its major material-symmetry axis oriented at an arbitrary angle with geometrical coordinates as shown in figure A-2.

Since the present problem is one of steady-state harmonic excitation including material damping effects, the complete set of eqs. (D1-D6) with proper boundary conditions, eqs. (61), must be used.

Example 4: Boron/epoxy plate with twenty-four parallel plies oriented at an arbitrary angle. - The input-data material properties are Young's and shear moduli (E_{11}, E_{22}, C_{44}, C_{55}, C_{66}), major Poisson's ratio (ν_{12}), bending and twisting stiffnesses (D_{11}, D_{22}, D_{12}, D_{66}), and their corresponding loss tangents, as generated in ref. 46 from constituent-material experimental properties. It is noticed that all of the above input data are functions of frequency. Geometrical and material properties correspond to the data of ref. 13, as follows:

length, $a = 18.19$ in ; width, $b = 2.75$ in.

thickness, $h = 0.034$ in. ; density, $\rho = 0.000194$ lb-sec^2/in^4

angle of orientation, $\theta = 0^\circ, 10^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$

A FORTRAN IV program is employed to compute the eigenvectors of a plate with all edges free. To obtain the resonant frequencies of the first five modes, two different resonance criteria are applied. In the peak-amplitude method, the system is excited harmonically and the amplitude at a particular point ($\alpha = 0$, $\beta = 1$) is measured over a range of frequency. The amplitude of the response depends not only on the dynamic characteristics of the system, but also on the amplitude of the force applied to it. For a linear system the peak amplitude is taken as maximum displacement per unit amplitude of force. Throughout this report, this ratio is referred to as the response. A typical result of response with varying frequency is shown in figure 15 for an angle of orientation, $\theta = 10^\circ$. The complete results are as tabulated in Table II.

In the modified Kennedy-Pancu method, the modal shapes, phase relations between motions at various points, and coupling between the

various degrees of freedom are assumed to be unaffected by the presence of a small amount of damping. Vectors are used for the analysis of the results of forced vibration by making a polar plot (called the Argand plane) of the displacement vector (amplitude and phase angle). In the vicinity of resonance, the resulting curve would be an approximately circular arc. Each point on the circle locus corresponds to a value of frequency.

In the modified Kennedy-Pancu method, the resonant frequency is defined to correspond to the point of the Argand-plot curve having maximum change in arc length (Δs) per fixed change in frequency ($\Delta\omega$). The change in arc length Δs can be determined approximately in analytical fashion as the response amplitude $(W/\tilde{q})$ multiplied by the difference in the phase angles corresponding to the two distinct frequencies (separated by $\Delta\omega$), i.e.

$$\Delta s = (W/\tilde{q})\ \Delta\varphi$$

The parameter $\Delta s/\Delta\omega$ is plotted versus excitation frequency in figure 16 for a specific angle of orientation, $\theta = 10^{o}$.

As can be seen in the figure, the curves exhibits sharp spikes and thus the Kennedy-Pancu resonant frequencies, i.e. the frequencies corresponding to the maximum points, can readily be found. The results obtained in this fashion for all of the plates used by Clary are tabulated in Table II.

A discussion of experimental methods used to determine material damping in composite materials is presented in Appendix H. Here the damping ratios are calculated by using the modified Kennedy-Pancu method as expressed in equation (B-69), Appendix B. It is noted that the damping ratio is the

ratio of material damping coefficient to critical material damping coefficient. The results associated with each mode are tabulated completely in Table II.

The nodal patterns of the first five modes are calculated from the response at selected points on the surface of the plate. Twenty-five points in the x direction and five points in the y direction are used. The nodal patterns are shown in figures 17-22. The results appear to be in good agreement with the data of ref. 13.

VI. CONCLUSIONS

The analyses in Sections II and IV were developed for the vibrational problem of composite-material plates with thickness-shear flexibility and material damping. The theory used in the analyses is that of Yang, Norris, and Stavsky (ref. 25), which can be considered to be the laminated, anisotropic version of Mindlin's dynamic plate theory.

To account for the effects of thickness-shear deformation, a shear correction factor K was introduced. A variety of methods to arrive at an appropriate shear factor have been proposed for homogeneous plates; the most popular ones resulted in K = 0.833 for a static distribution (ref. 44) and 0.822 for the dynamic case (ref. 26). Srinivas et al. (ref. 35) showed that use of either of these values for K in Mindlin's plate theory gives a very close approximation to the lower natural frequencies computed by exact, dynamic, three-dimensional elasticity theory for rather thick, homogeneous, isotropic rectangular plates simply supported on all edges.

Unfortunately to date no one has proposed a means of rational calculation of the shear factor for a laminated plate. Srinivas et al. made an exact, dynamic three-dimensional elasticity analysis of simply supported laminated plates. However, their analysis is quite tedious computationally and thus they have presented numerical results for only a very limited number of cases.

In the present investigation, both static (Jourauski shear theory) and dynamic (Mindlin type analysis for pure thickness-shear motion) approaches were used to derive a shear factor K for laminates.

An assessment of the accuracy of these theories was made by comparing the lowest eigenvalues calculated by the laminated, shear-flexible plate theory with the laminated, three-dimensional exact values given by Srivinvas et al. (ref. 35).

It has been found that the shear factor K for a laminate is not the same value as that for a homogeneous, isotropic member. The numberical examples indicated that the value of K depends on the layer properties and the stacking sequence of the laminate.

Using peak-amplitude and modified Kennedy-Pancu methods, the first five resonant frequencies were calculated for various laminated boron/epoxy plates with all edges free. The resonant frequencies obtained by the two techniques differed by only a very small amount, and were in good agreement with the results obtained both experimentally and analytically by Clary (ref. 13). Furthermore, the nodal patterns have been found to be satisfactory or as good as the data given by Clary. Finally, the damping-ratio results were in good agreement with the experimental ones obtained by Clary. No comparison with analytical results for lmainated-plate damping could be made, since they have not been available previously.

APPENDIX A

NOTATION AND TRANSFORMATION FOR ELASTIC COEFFICIENTS

The generalized Hooke's law (Cauchy equations) for a general anisotropic material in terms of rectangular coordinates x,y,z are as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{21} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad \text{(A-1)}$$

where σ_{ij} and ϵ_{ij} are stress and strain components, and $C_{k\ell}$ are the Cauchy stiffness coefficients for a three-dimensional body. It can be shown that the matrix of Cauchy coefficients is symmetric, provided that the material is conservative (elastic), so that

$$C_{\ell k} = C_{k\ell} \quad \text{(A-2)}$$

Usually a single layer of fiber-reinforced composite material has the fibers oriented more or less parallel to the top and bottom surfaces

of the layer, i.e. material-symmetry plane XY coincides with surface plane xy. Thus, it is orthotropic in the yz and xz planes, and the shear-normal or cross-elasticity coefficients with subscripts i4 and i5 (i=1,2,3,6) vanish:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{Bmatrix} \quad \text{(A-3)}$$

The only exception to the above, i.e. the only case when the general form (A-1), rather than (A-3), must be used is the case of a shingle-laminated composite, as shown in figure A-1 (reference 47).

If, in addition to being parallel to the top and bottom faces of the layer, the fibers are also oriented in a direction parallel to one of the coordinate directions in the plane of the layer, the material-symmetry plates XY,YZ,ZX all coincide with the coordinate planes xy, yz,zx and the material is said to be completely <u>orthotropic</u>. In this case the Cauchy relations are:

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{ZZ} \end{Bmatrix} = \begin{bmatrix} C^*_{11} & C^*_{12} & C^*_{13} \\ C^*_{12} & C^*_{22} & C^*_{23} \\ C^*_{13} & C^*_{23} & C^*_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \epsilon_{ZZ} \end{Bmatrix} ; \begin{Bmatrix} \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{Bmatrix} = \begin{Bmatrix} C^*_{44}\ \epsilon_{YZ} \\ C^*_{55}\ \epsilon_{XZ} \\ C^*_{66}\ \epsilon_{XY} \end{Bmatrix} \quad \text{(A-4)}$$

where the asterisks denote the orthotropic case.

In the case of a thin layer of fiber-reinforced composite material in which the thickness-normal stress (σ_{ZZ}) can be neglected (see Hypothesis H3, Section 2.1), the third equation in set (A-4) can be solved for ϵ_{ZZ} in terms of ϵ_{XX} and ϵ_{YY}. Then equations (A-4) can be simplified as follows:

$$\begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \end{Bmatrix} = \begin{bmatrix} Q^*_{11} & Q^*_{12} \\ Q^*_{12} & Q^*_{22} \end{bmatrix} \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \end{Bmatrix} ; \begin{Bmatrix} \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{Bmatrix} = \begin{Bmatrix} Q^*_{44} & \epsilon_{YZ} \\ Q^*_{55} & \epsilon_{XZ} \\ Q^*_{66} & \epsilon_{XY} \end{Bmatrix} \tag{A-5}$$

where the reduced orthotropic stiffness coefficients ($Q^*_{k\ell}$) are related to the Cauchy three-dimensional orthotropic stiffness coefficients ($C^*_{k\ell}$) as follows:

$$\begin{aligned} Q^*_{11} &\equiv C^*_{11} - (C^*_{13})^2/C^*_{33} \quad ; \quad Q^*_{12} \equiv C^*_{12} - (C^*_{13}\, C^*_{23}/C^*_{33}) \; ; \\ Q^*_{22} &\equiv C^*_{22} - (C^*_{23})^2/C^*_{33} \quad ; \quad Q^*_{ij} \equiv C^*_{ij} \; (ij = 44,55,66) \end{aligned} \tag{A-6}$$

Equation (A-5) can be written in the form of a single matrix equation as follows:

$$\{\sigma_{IJ}\} = [Q^*_{k\ell}] \{\epsilon_{IJ}\} \tag{A-7}$$

where

$$\{\sigma_{IJ}\} \equiv \begin{Bmatrix} \sigma_{XX} \\ \sigma_{YY} \\ \sigma_{YZ} \\ \sigma_{XZ} \\ \sigma_{XY} \end{Bmatrix}, \quad \{\epsilon_{IJ}\} \equiv \begin{Bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \tfrac{1}{2}\epsilon_{YZ} \\ \tfrac{1}{2}\epsilon_{XZ} \\ \tfrac{1}{2}\epsilon_{XY} \end{Bmatrix} \tag{A-8}$$

$$[Q^*_{k\ell}] \equiv \begin{bmatrix} Q^*_{11} & Q^*_{12} & 0 & 0 & 0 \\ Q^*_{12} & Q^*_{22} & 0 & 0 & 0 \\ 0 & 0 & 2Q^*_{44} & 0 & 0 \\ 0 & 0 & 0 & 2Q^*_{55} & 0 \\ 0 & 0 & 0 & 0 & 2Q^*_{66} \end{bmatrix} \tag{A-9}$$

where the presence of the factor 1/2 in the ϵ_{IJ} matrix is used to make it a second-rank tensor and this necessitates the presence of the factor 2 appearing in equation (A-9).

In structural-panel applications of composite materials, usually there are design requirements for multiple orientations of fiber-reinforced composite-material layers. Therefore, it is essential that a set of transformation relations be used to calculate the stiffness coefficients for any desired orientation from that associated with the major material-symmetry direction (fiber direction) as shown in figure A-2. Such relations are developed in the ensuing paragraphs.

Arbitrary orthogonal axes in the plane of the laminate and making an angle θ with the material-symmetry axes (X,Y) are designated as the x,y axes. Then the components of stress and strain can be transformed from the x,y axes to the X,Y axes as follows:

$$\{\sigma_{IJ}\} = [T_r]\{\sigma_{ij}\} \tag{A-10}$$

$$\{\epsilon_{IJ}\} = [T_r]\{\epsilon_{ij}\} \tag{A-11}$$

where $\{\sigma_{IJ}\}$ and $\{\epsilon_{IJ}\}$ are as defined in equations (A-8); $\{\sigma_{ij}\}$ and $\{\epsilon_{ij}\}$ are similar except that the subscripts X,Y,Z are replaced by the subscripts x,y,z; and $[T_r]$ is the transformation, defined as follows:

$$[T_r] \equiv \begin{bmatrix} m^2 & n^2 & 0 & 0 & -2mn \\ n^2 & m^2 & 0 & 0 & 2mn \\ 0 & 0 & m & -n & 0 \\ 0 & 0 & n & m & 0 \\ mn & -mn & 0 & 0 & m^2-n^2 \end{bmatrix} \tag{A-12}$$

where

$$m = \cos\theta \quad , \quad n = \sin\theta. \tag{A-13}$$

For the case when $\theta \neq 0$, we have

$$\{\sigma_{ij}\} = [Q_{k\ell}]\{\epsilon_{ij}\} \tag{A-14}$$

In view of equations (A-7,A-10,A-11,A-14),

$$[Q_{k\ell}] = [T_r]^{-1}[Q^*_{k\ell}][T_r] \qquad \text{(A-15)}$$

By substituting equations (A-9) and (A-12) and then performing the matrix operations indicated in equation (A-15) the elements of the $[Q_{k\ell}]$ matrix are related to those of the $[Q^*_{k\ell}]$ matrix as follows:

$$\begin{aligned}
Q_{11} &= Q^*_{11}m^4 + 2\,(Q^*_{12} + 2Q^*_{66})\,m^2n^2 + Q^*_{22}n^4 \\
Q_{12} &= Q^*_{12}m^4 + (Q^*_{11} + Q^*_{22} - 4Q^*_{66})m^2n^2 + Q^*_{12}n^4 \\
Q_{22} &= Q^*_{22}m^4 + 2(Q^*_{12} + 2Q^*_{66})m^2n^2 + Q^*_{11}n^4 \\
Q_{66} &= Q^*_{66}m^4 + (Q^*_{11} + Q^*_{22} - 2Q^*_{12} - 2Q^*_{66})m^2n^2 + Q^*_{66}n^4 \\
Q_{16} &= (2Q^*_{66} + Q^*_{12} - Q^*_{11})m^3n - (2Q^*_{66} + Q^*_{12} - Q^*_{22})mn^3 \qquad \text{(A-16)} \\
Q_{26} &= -(2Q^*_{66} + Q^*_{12} - Q^*_{22})m^3n + (2Q^*_{66} + Q^*_{12} - Q^*_{11})mn^3 \\
Q_{44} &= Q^*_{44}m^2 + Q^*_{55}n^2 \\
Q_{45} &= Q^*_{55}mn - Q^*_{44}mn \\
Q_{55} &= Q^*_{55}m^2 + Q^*_{44}n^2
\end{aligned}$$

APPENDIX B

MODELS AND MEASURES OF MATERIAL DAMPING

B1. Mathematical Models for Material Damping

Internal friction or damping in materials can be caused by a variety of combinations of fundamental physical mechanisms, depending upon the specific material. For metals, these mechanisms include thermoelasticity on both the micro and macro scales, grain boundary viscosity, point-defect relaxations, eddy-current effects, stress-induced ordering, interstitial inpurities, and electronic effects (reference 48). According to Lazan (ref. 48), little is known about the physical micromechanisms operative in most nonmetallic materials. However, for one important class of these, namely polymers and elastomers, considerable phenomenological data have been obtained. Due to the long-range molecular order associated with their giant molecules, polymers exhibit rheological behavior intermediate between that of a crystalline solid and a simple liquid. Of particular importance is the marked dependence of both stiffness and damping on frequency and temperature.

The purpose of developing a mathematical model for the rheological behavior of a solid is to permit realistic results to be obtained from mathematical analyses of complicated structures under various conditions, such as sinusoidal, random, and transient loading. According to reference 48, as early as 1784 Coulomb recognized that the mechanisms of damping

operative at low stresses may be different than those at high stresses. Even today, after more than 2500 publications on damping have appeared, major emphasis is placed on linear models of damping for several reasons: they have sufficient accuracy for the low-stress regime and linear analyses are computationally more economical than nonlinear ones.

The simplest mathematical models of rheological systems are single-parameter models: (1) an idealized spring, which exhibits a restoring force linearly proportional to displacement and thus displays no damping whatsoever, and (2) an idealized dashpot, which produces a force linearly proportional to velocity. Obviously, neither of these models are very appropriate to represent the behavior of most real materials.

The next most complicated models of rheological systems are the two-parameter models: (1) the Maxwell model, which consists of a spring and dashpot in mechanical series, as shown schematically in figure B-1(a); and (2) the Kelvin-Voigt model, which is comprised of a spring in parallel with a dashpot, as shown in figure B-1(b). The Maxwell model is a fair approximation to the behavior of a viscoelastic liquid. However, as a model for a viscoelastic solid, it has several very serious drawbacks (reference 48); there are no means to provide for internal stress and for afterworking. The Kelvin-Voigt model overcomes these deficiencies and is a first approximation to the behavior of a viscoelastic solid. However, it has the following disadvantages (reference 48): there is no elastic response during application or release of loading, the creep rate approaches zero for long durations of loading, and there is no permanent set irrespective of the loading history.

It should be mentioned that the Kelvin-Voigt model is the simplest one which permits representation as a complex quantity when subjected to sinusoidal motion. To show this, one can write the following differential equation governing the motion of a single-degree-of-freedom system consisting of a sinusoidally excited mass attached to a Kelvin-Voigt element:

$$m\ddot{u} + c\dot{u} + k\,u = \tilde{F}\,e^{i\omega t} \qquad \text{(B-1)}$$

where m ≡ mass, c ≡ viscous damping coefficient, k ≡ spring rate, $\tilde{F}$ ≡ exciting force amplitude, ω ≡ circular frequency of exciting force, u ≡ displacement, i ≡ $\sqrt{-1}$, t ≡ time, and a dot denotes a derivative with respect to time. Assuming that sufficient time has passed for the transients to die out,* we can write the steady state solution of equation (B-1) as follows:

$$u = \tilde{u}\,e^{i(\omega t-\varphi)} = \tilde{F}\,\left[(k-m\omega^2)+ i\omega c\right]^{-1}\,e^{i\omega t} \qquad \text{(B-2)}$$

where φ ≡ phase angle between the response (u) and the exciting force, and $\tilde{u}$ ≡ displacement amplitude.

It is noted that the terms containing Kelvin-Voigt coefficients k and iωc can be grouped into a single Kelvin-Voigt <u>complex stiffness</u> ($\tilde{k}$) as follows:

*It can be shown that they do die out for a mechanical system, for which c, m, and k are all positive quantities.

$$\bar{k} = k + i\omega c \tag{B-3}$$

Alternatively, the Kelvin-Voigt element coefficient could have been grouped into a single complex damping coefficient ($\bar{c}$) as follows:

$$\bar{c} = c - (i\ k/\omega) \tag{B-4}$$

The force (F_d) in a dashpot is given by

$$F_d = c\ \dot{u} = i\omega c\tilde{u}\, e^{i(\omega t - \varphi)} \tag{B-5}$$

The energy dissipated per cycle (U_d) is a dashpot undergoing sinusoidal motion can be calculated as follows:

$$U_d = \oint F_d du = \int_0^{2\pi/\omega} F_d\ \dot{u}\ dt = \pi c\omega \tilde{u}^2 \tag{B-6}$$

In 1927, Kimball and Lovell (reference 49) observed that many engineering materials exhibited energy losses which were contradictory to equation (B-6). Specifically, they found that U_d was proportional to $\tilde{u}^2$ but independent of ω, i.e.

$$U_d = C_d' \tilde{u}^2 \tag{B-7}$$

Later work by Wegel and Walther (reference 50) also showed that $U_d \propto \tilde{u}^2$ but C_d' was a weak function of ω. Still later, for stresses below the fatigue limit of the material, Lazan (references 48,51) showed that $U_d \propto \tilde{u}^2$ and C_d' was practically independent of ω.

Equating the energy dissipated per cycle in the Kimball-Lovell material, equation (B-7), to that in the dashpot, equation (B-6), one obtains

$$C_d' = \pi c \omega \qquad \text{(B-8)}$$

To utilize the Kimball-Lovell observation in practical structural dynamic analyses, it was desirable to have an expression for the associated damping force, i.e. an equation analogous to equation (B-5), which governs a dashpot. According to Bishop (reference 52) this was first done by Collar. Combining equations (B-5 , B-8), one obtains the following "frequency-dependent damping" relation:

$$F_d = (b/\omega)\ \dot{u} \qquad \text{(B-9)}$$

where b is a constant given by

$$b = C_d'/\pi \qquad \text{(B-10)}$$

The kind of damping represented by equation (B-9) has been called frequency-dependent damping, since the usual dashpot coefficient c is now replaced by the quantity b/ω.

For the damper represented by Eq. (B-9), it is convenient to replace k in the undamped system by the following Kimball-Lovell <u>complex stiffness</u> (reference 53):

$$\bar{k}' = k + ib \qquad \text{(B-11)}$$

where it is now noted that both k and b are assumed to be independent of frequency. This representation has been used extensively in aircraft

structural dynamic and flutter analyses (references 54,55).

An alternative to equation (B-11), is to use the concept of a complex damping coefficient, originated by Myklestad (reference 56), in which the spring constant is replaced by

$$\bar{k}'' = C_1 e^{im} \qquad \text{(B-12)}$$

where C_1 and m are constants.

Although Kimball and Lovell's original work was based on data obtained during steady-state sinusoidal motion, the models represented by equations (B-9, B-11, B-12) have been applied to free vibrations as well (refs. 52,53,57). To alleviate the difficulty of the interpreting of ω in equation (B-9) for the case of free vibration or for multiple-frequency forced vibration, Reid (reference 58) suggested the following form:

$$F_d = b \left| u/\dot{u} \right| \dot{u} \qquad \text{(B-13)}$$

where the bars denote that the absolute value of the quantity between them is to be used.

In summary, four different versions of material damping have been discussed. In terms of the differential equation for free vibration, they may be written as follows:

$$m\ddot{u} + (b/\omega)\,\dot{u} + ku = 0 \qquad \text{(B-14a)}$$

$$m\ddot{u} + (k + ib)\,u = 0 \qquad \text{(B-14b)}$$

$$m\ddot{u} + C_1 e^{im} u = 0 \qquad \text{(B-14c)}$$

$$m\ddot{u} + b\left|u/\dot{u}\right|\dot{u} + ku = 0 \qquad \text{(B-14d)}$$

The models represented by equations (B-14) have been criticized by many investigators for various reasons:

1. In equation (B-14a), the interpretation of α is dubious (references 58,60). Milne (reference 59) suggested that it is associated with the imaginary part of the pair of complex roots, but went on to state that there is no clear justification for this interpretation.

2. Equations (B-14b) and (B-14c) are equations with complex coefficients and the meaning of this is not clear (references 57,59). Furthermore, although they have complex exponential solutions, neither the real nor imaginary parts alone are solutions (references 59,60).

3. Equation (B-14d) is a nonlinear differential equation, due to the behavior of $|u/\dot{u}|$ (reference 59).

4. None of the models represented by equations (B-14) can be simulated, even on a conceptual basis, on an analog computer (references 58, 60).

5. The model represented by equation (B-14a) does not meet the Wiener-Paley condition (references 61, 62) of causality for physically realizable systems, as was shown in references 60, 63-65.

6. Equation (B-14b) fails to give the proper relations of a damping force that is proportional to displacement and in phase with velocity. This can be alleviated by use of a more unwieldy expression proposed in references 66,67.

Incidentally reference 68 claimed that none of the equation (B-14) models result in energy dissipation that is independent of ω (the reason for abandoning the Kelvin-Voigt model). However, an error in this reasoning, which makes it invalid, was pointed out in reference 69.

It can be shown that the sinusoidally forced vibration version of free vibration equations (B-14a,b,c) are exactly equivalent. Probably the most popular form for writing this is as follows:

$$m\ddot{u} + (k + ib)u = \tilde{F} e^{i\omega t} \qquad \text{(B-15)}$$

The major criticism of the use of equation (B-15) is that it results in a conventionally defined magnification factor (see Section B2), corresponding to zero frequency, which is not equal to unity for $g \neq 0$ (reference 52). This difficulty can be eliminated by redefining the magnification factor as follows:

$$MF' \equiv \tilde{u}/u'_{st} \qquad \text{(B-16)}$$

where

$$u'_{st} \equiv \tilde{F}/k'$$

and k' is the stiffness coefficient associated with the total in-plane force amplitude, i.e.

$$k' \equiv (k^2 + b^2)^{\frac{1}{2}}$$

Thus,

$$MF' = (k^2+b^2)^{\frac{1}{2}}/[(k-m\omega^2)^2 + b^2]^{\frac{1}{2}} = (1 + g^2)^{\frac{1}{2}}/[(1-\omega^2/\omega_n^2)^2+g^2]^{\frac{1}{2}}$$

The major advantages of using the equation (B-15) model for steady-state vibrational and flutter analyses are as follows:

1. The analysis is considerably simplified in comparison with that for viscous damping, as pointed out in reference 70.

2. The analysis is linear.

3. The complex-modulus concept has long been in common use, especially

in connection with measurement of the damping properties of elastomers, polymers, etc. (reference 69).

4. The complex-stiffness approach has been used very successfully in flutter analysis for approximately three decades refs. 54, 55, 69, 71).

In view of these considerations, the complex-stiffness approach is used in the main portion of the present report. However, for completeness and comparison, a number of other more complicated approaches are also discussed in this section.

To overcome some of the deficiencies of the Kelvin-Voigt and Maxwell models, they were combined in various ways to obtain the three-parameter models shown in figure B-1(c) and (d). It can be shown (reference 72) that by properly selecting the values of the coefficients, the two models can be used interchangably. In other words, the model representation is not unique. In fact, both representations have been called the "standard linear solid", "standard linear material," "simple anelastic model", or "standard model of a viscoelastic body" (references 73-76). Unfortunately, few materials have been characterized by the use of the standard linear solid. Perhaps the most extensive characterization has been carried out for flexural vibration of 2024-T4 aluminum alloy: by Granick and Stern (reference 77,78) in air and by Gustafson et al (reference 79) in vacuo.

Obviously one can go on from a three-parameter model to a four-parameter one, etc. Continuing this process, one obtains two well-known models (reference 74): the Kelvin chain, shown in figure B-2(a), and the generalized Maxwell model, shown in figure B-2(b). The behavior of such systems can be written in either differential or integral form, as discussed in detail in reference 74. However, the complexity of

such a model has limited its use both experimentally (in characterizing engineering materials) and analytically (in performing dynamic analyses). Thus, for engineering applications, there is a continuing search for models which are simple, yet represent the behavior of real materials with sufficient accuracy.

A relatively simple model, which is somewhat similar to the generalized Maxwell one, is due to Biot (reference 80); see figure B-2(c). The number of simple Voigt elements in this model is allowed to increase without bound, so that the damping force is represented by the following expression:

$$F_d = g_1 \int_{t_1}^{t} Ei\,[-\epsilon(t-\tau)](du/d\tau)\,d\tau \qquad \text{(B-17)}$$

where τ is a dummy variable; $\epsilon, g_1 \equiv$ parameters, and Ei(u) is the exponential integral

$$Ei\,(u) \equiv \int_{\infty}^{-u} (e^{-\xi}/\xi)\,d\xi \qquad \text{(B-18)}$$

Caughey (ref. 60) made a detailed study of Biot's model for both free and forced vibration. He showed that for $\omega/\epsilon > 10$, the energy dissipated in Biot's model is within 4% of being independent of frequency (ω) and thus essentially depends only upon the square of the displacement amplitude. Milne (ref. 59) studied the Biot model by means of the convolution integral.

Neubert (reference 81) introduced a variant of the Biot model, in which a different distribution function for the local stiffness-damping ratio is used. Milne (ref. 59) also studied this model via the convolution integral. Milne introduced a new synthesized model of his own. However,

the main conclusion of his investigation was that, for practical engineering purposes, the choice of a rheological model is not critical, provided that the range of constant damping is kept the same and the damping is small.

The linear hereditary theory of material damping was originated in 1876 by Boltzmann (reference 82). In this theory, the energy loss is attributed to the elastic delay by which the deformation lags behind the applied force. It is called a hereditary theory because the instantaneous deformation depends upon all of the stresses applied to the body previously as well as upon the stress at that instant. The damping force is given by

$$F_d = \int_{-\infty}^{t} \Phi(t,\tau)\ u(\tau)\ d\tau \qquad \text{(B-19)}$$

where t ≡ actual time, τ ≡ an instant of time (dummy variable), u ≡ displacement, and Φ ≡ hereditary kernel or memory function. Often the hereditary kernel depends upon the difference (t-τ) only; then equation B-19) becomes:

$$F_d = \int_{-\infty}^{t} \Phi(t - \tau)\ u\ (\tau)\ d\tau \qquad \text{(B-20)}$$

It is most advantageous to determine the hereditary kernel from experimental data. It has been found to be a monotonically decreasing function which can be represented mathematically as follows:

$$\Phi\ (t) = \sum_{i=1}^{n} A_i e^{-\alpha_i t} \qquad \text{(B-21)}$$

Such a system is said to have a heredity of degree n and A_i and α_i

are called heredity constants. Hobbs (ref. 75) has shown that a first-degree hereditary model is equivalent to a simple anelastic model (standard linear solid).

Volterra (reference 83) has presented a method of determining the hereditary kernel from material response to either suddenly applied loading or to sinusoidal excitation. He also presented plots of magnification factor and phase shift verus frequency for a wide range of parameters of a single-degree-of-treedom system with first-degree hereditary damping. For certain combinations of the parameters, the magnification and phase characteristics are quite different from that of a viscously damped system; yet for certain other combinations, the characteristics are quite similar.

Numerous nonlinear mathematical models have been proposed to permit better correspondence between theory and experiment. However, all of them, by their very nonlinear nature, are more complicated to apply in structural dynamic analyses of practical engineering systems. Perhaps the simplest one is damping energy per unit volume and per cycle proportional to stress to a power (ref. 48):

$$U_{dv} = C_d \tilde{\sigma}^m \qquad \text{(B-22)}$$

where C_d and m are material constants and $\tilde{\sigma}$ is the cyclic stress amplitude. It is noted that m=2 corresponds to the Kimball-Lovell relation (linear theory) discussed previously. It has been found that m varies from 1.8 to 6.0, generally being near 2.0 at low stress amplitudes.

The hysteresis loop associated with the complex-modulus model, equation (B-11), is elliptic in shape; see figure B-3(a). However,

the hysteresis loops determined experimentally have sharp corners at both ends and are nearly linear in the low-stress, low-strain region, as shown in figure B-3(b). According to ref. 48 (page 97), in 1938 Davidenkov proposed a nonlinear mathematical model which results in a similar shape of hysteresis loop. Further work on this class of model has been carried out by Pisarenko (reference 84).

Rosenblueth and Herrera (reference 85) proposed a nonlinear model which eliminates the causality difficulties of the complex-modulus model. Chang and Bieber (reference 86) introduced a nonlinear hysteresis model in which there is no hysteresis damping unless the displacement amplitude exceeds a certain threshold value.

B2. Measures of Material Damping

In this section, various definitions of damping are discussed in the context of a complex-modulus material.

Energy Dissipation Under Steady-State Sinusoidal Vibration. - Energy dissipated per cycle (U_d) in the form of internal frictional heating is one measure of damping. However, this quantity depends upon the size, shape, and dynamic stress distribution (which in turn, depends upon the particular mode of vibration). In view of the above difficulties, the specific damping energy (U_{dv}) is usually considered to be a more basic property of the material, rather than the structure. The U_{dv} is defined as the damping energy per cycle and per unit volume, assuming a uniform dynamic stress distribution throughout the volume considered. Thus, the total damping energy is

$$U_d = \oint_V U_{dv}\, dV \qquad \text{(B-23)}$$

where V = volume. The usual units of U_d are in-lb/cycle and of U_{dv}, in-lb/in^3-cycle.

Resonant Magnification Factor Under Steady-State Sinusoidal Excitation. - The resonant magnification factor (RMF) is defined as the dimensionless ratio of the response at resonance to the excitation, where both the response and the excitation must be specified in the same units (see figure B-4). These quantities may be in units of displacement, velocity, acceleration, or strain. Unfortunately, the resonant magnification factor is dependent upon the structural system configuration as well as upon the damping property of the material, so that it is considered to be a system characteristic, rather than a basic material property.

Incidentally RMF is not applicable to nonlinear systems, since then it depends upon the excitation level. (In a linear system, RMF is independent of the level of excitation.)

Bandwidth of Half-Power Points Under Steady-State Sinusoidal Excitation.- The separation $(\omega_2-\omega_1)$ between the frequencies associated with the half-power points increases with an increase in damping (see figure B-5) and thus can be used as a measure of damping. A more meaningful definition is to normalize by the associated resonant frequency (ω_n), i.e. use the dimensionless bandwidth given by:

$$(\omega_2 - \omega_1)/\omega_n \quad .$$

This measure is the basis for determination of damping by the original Kennedy-Pancu method (reference 87).

The quality factor Q is defined as the reciprocal of the dimensionless

bandwidth.

Derivative of Phase Angle with Respect to Frequency, at Resonance. - This measure is the basis for determination of damping by an improvement (reference 88) of the Kennedy-Pancu method.

Loss Tangent Under Steady-State Sinusoidal Excitation. - Applying the complex-stiffness (see Section B1) to the material property representing stiffness, namely the Young's modulus (E), we obtain

$$E \rightarrow E^R(1 + ig) \tag{B-24}$$

Thus, the loss tangent* is defined as follows:

$$g = E^I/E^R \tag{B-25}$$

where $E^I \equiv$ loss modulus** and E^R is called the storage modulus.***

Referring to figure B-6, we obtain the following relation:

$$g = \tan \gamma = E^I/E^R \tag{B-26}$$

or

$$\gamma = \arctan E^I/E^R \tag{B-27}$$

*Also known as the "loss coefficient", "loss factor", or "damping factor".

**Also sometimes called the "dissipation modulus."

***Also sometimes called "elastic modulus" or "real modulus."

The quantity γ is called the loss angle and equation (B-26) explains the origin of the term loss tangent for g.

Cyclic Decay of Free Vibrations. - In a linear system, free vibrations decay exponentially (see figure B-7). The larger the damping, the faster is the decay. Thus, the logarithmic decrement δ is defined as follows:

$$\delta = \ell n \, (a_i / a_{i+1}) \tag{B-28}$$

where ℓn = natural logarithm.

For a power-law material, eq. (B-22), to have δ independent of amplitude, m must be 2 (see Section B3 for proof). When m=2, the following means of calculating δ is more practical, especially when damping is small:

$$\delta = (1/n) \, \ell n \, (a_i / a_{i+n}) \tag{B-29}$$

where n is any arbitrary integer.

Temporal Decay of Free Vibrations. - Another measure associated with the decay of free vibrations is the temporal decay constant v_t, defined as follows (reference 48, foldout opposite p. 35):

$$v_t = (t_2 - t_1)^{-1} \, \ell n \, [u(t_1)/u(t_2)] \tag{B-30}$$

where t_1 and t_2 are two different values of time, arbitrary except that $t_2 > t_1$ and $(t_2 - t_1)$ must be an integer number of periods of damped vibration. The quantities $u(t_1)$ and $u(t_2)$ are the respective displacements at times t_1 and t_2. It is seen that the usual units for v_t are sec^{-1}.

An alternate way of specifying the temporal decay is the decay rate γ_t, which has units of decibals per second (db/sec) and is defined as follows:

$$\gamma_t \equiv 20\ (t_2-t_1)^{-1}\ \log\ [u(t_1)/u(t_2)] \qquad \text{(B-31)}$$

or

$$\gamma_t = 20(t_2-t_1)^{-1}\log\ e^{v_t(t_2-t_1)} = (20\ \log\ e)v_t \approx 8.68\ v_t$$

where log ≡ logarithm to the base 10.

Spatial Attenuation of a Plane Wave in a Slender Bar. - If one propagates a plane, harmonic wave in a slender bar made of a homogeneous, linear material, the wave exhibits exponential decay with axial position (x), analogous to the temporal decay of free vibration (which may be consiered as a standing wave). Analogous to the logarithmic decrement (see above), we have the logarithmic attenuation δ_s, defined as follows in terms of amplitudes $a(x_1)$ and $a(x_2)$ at stations x_1 and x_2:

$$\delta_s = \ell n\ [a(x_1)/a(x_2)] \qquad \text{(B-32)}$$

Analogous to the temporal decay constant, we define the spatial attenuation constant v_s as follows:

$$v_s \equiv (x_2-x_1)^{-1}\ \ell n\ [a(x_1)/a(x_2)] \qquad \text{(B-33)}$$

The unit of v_s is in^{-1}.

Finally, analogous to the decay rate, we define the spatial attenuation rate γ_s as follows:

$$\gamma_s = 20(x_2-x_1)^{-1} \log [a(x_1)/a(x_2)] \approx 8.68\ v_s. \qquad \text{(B-34)}$$

The unit of γ_s is db per unit length.

It should be cautioned that the determination of damping by wave attenuation requires that the cross section of the bar be compact (preferably round or square) and that the largest cross-sectional dimension be very small compared to the wave length of the traveling wave. Furthermore, the two stations along the bar, at which the wave measurements are made, must be: (1) sufficiently far from the point of impact that initial transients have died out and (2) sufficiently far from each other so that a measurable change in amplitude occurs. Kolsky discussed these points in detail in reference 89.

B3. Inter-Relationships Among Various Measures of Damping for Homogeneous Materials

Here we consider a single degree-of-freedom system consisting of a mass supported on a massless spring made of a homogeneous Kimball-Lovell or complex-modulus material. Then the governing differential equation for simple harmonic excitation is equation (B-15), which has the following steady-state solution:

$$u = \tilde{u}\ e^{i(\omega t-\varphi)} \qquad \text{(B-35)}$$

where

$$\tilde{u} = \tilde{F}\ [(k-m\omega^2)^2 + b^2]^{-1/2} \qquad \text{(B-36)}$$

$$\varphi \equiv \arctan [b/(k-m\omega^2)] \qquad \text{(B-37)}$$

The magnification factor (MF) is defined as follows:

$$MF \equiv \tilde{u}/u_{st} \tag{B-38}$$

where u_{st} is the so-called static displacement, defined as follows:

$$u_{st} \equiv \tilde{F}/k \tag{B-39}$$

Combining equations (B-36, B-38, and B-39), one obtains the following relation:

$$MF = \left\{ [1 - (\omega/\omega_n)^2]^2 + (b/m\omega_n^2)^2 \right\}^{-1/2} \tag{B-40}$$

where ω_n is the natural frequency (i.e. the resonant frequency of the undamped system), given by:

$$\omega_n = (k/m)^{1/2} \tag{B-41}$$

The <u>critical</u> material damping coefficient (b_c) is the minimum value of b for which the <u>free</u> vibratory motion is non-oscillatory. It is obtained from the following relationship:

$$(b_c/2m\omega_n)^2 = k/m = \omega_n^2$$

or

$$b_c = 2m\omega_n^2 \tag{B-42}$$

The damping ratio ζ is defined as follows:

$$\zeta = b/b_c \tag{B-43}$$

Combining equations (B-40, B-42, and B-43), one arrives at the following general, dimensionless relationship:

$$MF = \left\{\left[1-(\omega/\omega_n)^2\right]^2 + 4\zeta^2\right\}^{-1/2} \qquad \text{(B-44)}$$

It is seen easily that MF is a function of only two parameters: the frequency ratio (ω/ω_n) and the damping ratio (ζ). It should be mentioned that equation (B-44) is somewhat <u>different</u> than the known relationship for a single-degree-of-freedom Kelvin-Voigt viscously damped system, even though the damping ratio ζ is defined analogously ($\equiv c/c_c$, where $c_c = 2m\omega_n$).

The <u>resonant magnification factor</u> (RMF) is defined as the value of MF at resonance, which is taken here to occur when $\varphi = 90^o$. In view of equation (B-37), this implies that the resonant frequency is ω_n. Thus, at resonance, equation (B-44) reduces to the following simple expression:

$$\boxed{RMF = (2\zeta)^{-1}} \qquad \text{(B-45)}$$

Thus, it is seen that the same simple relationship between the damping ratio and the <u>resonant</u> magnification factor holds as does for a Kelvin-Voigt system. Although the resonant amplitude can be measured accurately, there are problems in measuring the static deflection in complicated structures (due to signal-to-noise ratio limitations of instrumentation). For this reason, in structural dynamics, RMF is seldom used as a measure of damping.

The damping energy, i.e. the energy dissipated per cycle, can be computed as follows:

$$U_d = \oint F_d \, du = \int_o^{2\pi/\omega} F_d \dot{u} \, dt \qquad \text{(B-46)}$$

where F_d is the damping force, given by

$$F_d = (b/\omega)\dot{u} \tag{B-47}$$

Substituting equations (B-35) and (B-47) into equation (B-46) and integrating, we obtain the following result:

$$U_d = \pi b \tilde{u}^2 \tag{B-48}$$

To convert equation (B-48) to the form of equation (B-22), with m=2, it is necessary to divide by the volume and to relate $\tilde{u}$ to $\tilde{\sigma}$. Now the force amplitude is

$$\tilde{F} = A\tilde{\sigma} = k\tilde{u} \tag{B-49}$$

where A = cross-sectional area.

Assuming the spring is a massless and uniform bar, we have

$$k = AE/L \tag{B-50}$$

where L = effective length of spring and E = Young's modulus of the bar. Combining equations (B-48, B-49, B-50), one obtains the following result:

$$U_{ds} = U_d/V = (\pi bL/A)(\tilde{\sigma}/E)^2 \tag{B-51}$$

Comparing the form of equations (B-22) and (B-51), we see that for a Kimball-Lovell material, m = 2 and

$$C_d = \pi bL/AE^2 \tag{B-52}$$

It should be noted that Lazan (reference 48, chap. II) has shown that these results are independent of the stress distribution and the specimen geometry, provided that m = 2.

In view of equations (B-42, B-43, and B-44), equation (B-52) can be expressed in the following more useful form:

$$\boxed{C_d = 2\pi\, \zeta/E} \qquad \text{(B-53)}$$

According to equation (B-48), the damping energy (and thus, excitation power) is proportional to the square of the displacement amplitude u_o (and thus, the MF). Then the power is one half of what it is at resonance when MF = RMF/$\sqrt{2}$. In view of equation (B-45), the half-power magnification factor (HPMF) is related to the damping ratio as follows:

$$\text{HPMF} = \text{RMF}/\sqrt{2} = (2\sqrt{2}\ \zeta)^{-1} \qquad \text{(B-54)}$$

To determine the two sideband frequencies ($\omega = \omega_1$ and ω_2) corresponding to half power, we combine equations (B-44) and (B-54) to obtain the following quadratic expression in ω^2:

$$(2\sqrt{2}\ \zeta)^2 = [1 - (\omega/\omega_n)^2]^2 + 4\zeta^2$$

which has the following positive roots:

$$\omega_{1,2} = \omega_n(1 \pm 2\ \zeta)^{\frac{1}{2}} \qquad \text{(B-55)}$$

It is interesting to note that equation (B-55) is identical to the analogous expression for the lightly damped Kelvin-Voigt system, i.e. $(c/c_c)^2 \ll 1$.

For small damping ($\zeta \ll \frac{1}{2}$),

$$\omega_{1,2} \approx (1 \pm \zeta)\ \omega_n$$

Thus, we get the following relation between the dimensionless bandwidth and the damping ratio ζ:

$$\boxed{(\omega_2-\omega_1)/\omega_n \approx 2\zeta} \qquad \text{(B-56)}$$

In the original Kennedy-Pancu method (reference 87), this relationship is used to determine the damping ratio from the geometry of an experimentally determined Argand plot (a polar-coordinate plot of response amplitude versus phase angle).

In terms of the quality factor Q, equation (B-56) may be rewritten as follows:

$$\boxed{Q \approx (2\zeta)^{-1}} \qquad \text{(B-57)}$$

It is interesting to note that in order for the half-power frequencies to be real, the damping ratio ζ must be no greater than $\sqrt{2}/2 \approx 0.707$. However, this is not a severe limitation for actual structural materials.

The potential energy (strain energy) stored in our single-degree-of-freedom mathematical model is given by:

$$U = \oint F_s \, du \qquad \text{(B-58)}$$

where F_s is the spring force, given by

$$F_s = ku \qquad \text{(B-59)}$$

Thus,

$$U = k \int_o^{\tilde{u}} u \, du = k\tilde{u}^2/2 \qquad \text{(B-60)}$$

Combining equations (B-48) and (B-60), we obtain

$$U_d/U = 2\pi b/k \qquad \text{(B-61)}$$

Using equations (B-41, B-42, B-43), we can rewrite equation (B-61) in the following more useful form:

$$U_d/U = 4\pi\zeta \qquad \text{(B-62)}$$

or

$$\boxed{\zeta = (1/4\pi)(U_d/U)} \qquad \text{(B-63)}$$

The loss tangent is defined as follows for the system considered here:

$$g \equiv b/k \qquad \text{(B-64)}$$

Inserting equations (B-41, B-42, B-43) into equation (B-64) we obtain the following useful relationship:

$$\boxed{g = 2\zeta} \qquad \text{(B-65)}$$

The following useful relationship, which is sometimes taken as a fundamental definition of damping (reference 66), is obtained here by combining equations (B-63) and (B-65):

$$\boxed{g = (1/2\pi)(U_d/U)} \qquad \text{(B-66)}$$

Taking the derivative of phase angle φ with respect to frequency ω, one obtains the following result from equation (B-37):

$$d\varphi/d\omega = 2bm\omega/[(k-m\omega^2)^2+b^2] \qquad \text{(B-67)}$$

At resonance (subscript R), defined by $\omega=\omega_n = \sqrt{k/m}$, equation (B-67) becomes:

$$(d\varphi/d\omega)_R = 2m\omega_n/b \qquad \text{(B-68)}$$

Combining equations (B-42, B-43, and B-68), we obtain the following

very useful result:

$$\boxed{\zeta = \left[\omega_n \left(d\varphi/d\omega\right)_R\right]^{-1}} \qquad \text{(B-69)}$$

It is interesting to note that although the MF vs. ω relation, equation (B-44), is different than the corresponding relation for a Kelvin-Voigt viscously damped system, equation (B-69) is identical to its corresponding relationship for a Kelvin-Voigt system. Equation (B-69) is used to determine the damping ratio from an experimentally determined Argand diagram in the improved Kennedy-Pancu method (reference 88).

The transient solution of the system represented by equation (B-14a) can be written as follows:

$$u' = \tilde{u}' e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\ \omega_n t - \varphi\right) \qquad \text{(B-70)}$$

From its definition in equation (B-28), the logarithmic decrement δ can be determined as follows:

$$\delta = \ln \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1+T)}} = \ln e^{\zeta\omega_n T} = \zeta\omega_n T \qquad \text{(B-71)}$$

where T is the period of damped oscillation, given by

$$T = (2\pi/\omega_n)(1-\zeta^2)^{-1/2} \qquad \text{(B-72)}$$

Thus, we now have

$$\boxed{\delta = 2\pi\zeta\ (1-\zeta^2)^{-1/2}} \qquad \text{(B-73)}$$

For small damping ($\zeta^2 \ll 1$), equation (B-73) can be replaced by the following approximate expression:

$$\boxed{\delta \approx 2\pi\zeta} \qquad \text{(B-74)}$$

Combining equations (B-65) and (B-74), one obtains the following:

$$\boxed{\delta = \pi g} \tag{B-75}$$

In view of equation (B-63) and since the specific energies are given by:

$$U_{ds} \equiv U_d/V, \; U_s \equiv U/V, \tag{B-76}$$

we have

$$\zeta = U_{ds}/4\pi U_s \tag{B-77}$$

However, the specific strain energy is

$$U_s = \tilde{\sigma}^2/2E \tag{B-78}$$

A power-law-damping material is defined by equation (B-22). Combining this relation with equation (B-78), one obtains:

$$\zeta = (E\, C_d/8\pi)\tilde{\sigma}^{m-2} \tag{B-79}$$

Thus, it is clear that ζ is independent of stress amplitude only for a KL (Kimball-Lovell) material (m=2). Then

$$\boxed{\zeta = E\, C_d/8\pi} \tag{B-80}$$

or, in view of equation (B-65),

$$\boxed{g = E\, C_d/4\pi} \tag{B-81}$$

Furthermore, since the logarithmic decrement of a KL material with small damping is approximately proportional to the damping ratio as shown by equation (B-74), the following approximate relation holds:

$$\boxed{\delta \approx E\, C_d/4} \tag{B-82}$$

APPENDIX C

DERIVATION OF ENERGY DIFFERENCE

It is assumed that the generalized coordinate ξ and generalized force f have the following forms:

$$\xi = \tilde{\xi}\, e^{i\omega t}$$
$$f = \tilde{q}\, e^{i\omega t} \qquad \text{(C-1)}$$

where $\tilde{\xi}$ is a complex form representing $\tilde{u}_o, \tilde{v}_o, \tilde{w}_o, \tilde{\psi}_x$ and $\tilde{\psi}_y$.

The strain energy can be expressed as

$$U = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, \xi_i \xi_j = \tilde{U}\, e^{2i\omega t} \qquad \text{(C-2)}$$

where Q_{ij} are stiffness coefficients and $\tilde{U}$ is the amplitude of the strain energy given by:

$$\tilde{U} = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}\, \tilde{\xi}_i\, \tilde{\xi}_j \qquad \text{(C-3)}$$

For the damping energy, a dissipation function suitable for material damping (reference 90) is developed. Hence, we introduce a dissipation energy function for material damping as follows:

$$D = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (b_{ij}/\omega)\ \dot{\xi}_i\, \dot{\xi}_j = (i\omega/2) \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{Q}_{ij}\, \tilde{\xi}_i \tilde{\xi}_j e^{2i\omega t} = (i\omega)\tilde{D}\, e^{2i\omega t} \qquad \text{(C-4)}$$

where b_{ij} are structural damping coefficients, $\hat{Q}_{ij} = ib_{ij}$ and $\tilde{D}$ is the amplitude of dissipation energy given by:

$$\tilde{D} = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{Q}_{ij} \tilde{\xi}_i \tilde{\xi}_j \tag{C-5}$$

The kinetic energy of the system is

$$T = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{\xi}_i \dot{\xi}_j = -(\omega^2/2) \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \tilde{\xi}_i \tilde{\xi}_j \, e^{2i\omega t} = -\tilde{T} \, e^{2i\omega t} \tag{C-6}$$

where m_{ij} are inertia coefficients and $\tilde{T}$ is the amplitude of kinetic energy given by

$$\tilde{T} = (\omega^2/2) \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \tilde{\xi}_i \tilde{\xi}_j \tag{C-7}$$

The work done by the uniformly distributed normal force is

$$W = f \sum_{i=1}^{n} \xi_i = \tilde{q} \, e^{i\omega t} \sum_{i=1}^{n} \xi_i = \tilde{W} \, e^{2i\omega t} \tag{C-8}$$

where $\tilde{q}$ is the amplitude of the normal pressure and $\tilde{W}$ is the amplitude of the work done

$$\tilde{W} = \tilde{q} \sum_{i=1}^{n} \tilde{\xi}_i \tag{C-9}$$

The Lagrangian equation takes the following form (reference 91)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) + \frac{\partial D}{\partial \dot{\xi}} + \frac{\partial U}{\partial \xi} = f \qquad \text{(C-10)}$$

Each term of equation (C-10) can be expressed in terms of its amplitude as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = -\frac{\partial \tilde{T}}{\partial \tilde{\xi}} e^{i\omega t}, \quad \frac{\partial D}{\partial \dot{\xi}} = \frac{\partial \tilde{D}}{\partial \tilde{\xi}} e^{i\omega t}$$

$$\frac{\partial U}{\partial \xi} = \frac{\partial \tilde{U}}{\partial \tilde{\xi}} e^{i\omega t}, \quad f = \frac{\partial W}{\partial \xi} = \frac{\partial \tilde{W}}{\partial \tilde{\xi}} e^{i\omega t} \qquad \text{(C-11)}$$

With equations (C-11) introduced, one can rewrite equation (C-10) as follows:

$$\frac{\partial}{\partial \tilde{\xi}} (\tilde{T} + \tilde{W} - \tilde{U} - \tilde{D}) e^{i\omega t} = 0 \qquad \text{(C-12)}$$

or

$$\frac{\partial}{\partial \tilde{\xi}} (\tilde{T} + \tilde{W} - \tilde{U} - \tilde{D}) = 0 \qquad \text{(C-13)}$$

Equation (C-13) is analogous to one presented by Volterra (reference 83).

Thus, the amplitude of the Lagrangian energy difference can be expressed as follows

$$\tilde{L} = (\tilde{T} + \tilde{W}) - (\tilde{U} + \tilde{D}) \qquad \text{(C-14)}$$

when used in conjunction with Hamilton's principle, expressed in equation (36).

APPENDIX D

COMPLETE ENERGY EXPRESSIONS

D1. The Energy Difference

$$
\begin{aligned}
\tilde{L} = \tfrac{1}{2}\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\int_0^1 \Big\{ & (\bar{A}_{11}/a^2)(U_{mn}\Phi'_{um}\Phi_{un})^2+(2\bar{A}_{12}/ab)(U_{mn}\Phi'_{um}\Phi_{un})(V_{mn}\Phi_{vm}\Phi'_{vn}) \\
& +2\bar{A}_{16}(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un}) \\
& +2\bar{A}_{26}(b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})+\bar{A}_{66}(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})^2 \\
& +2\bar{B}_{11}(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})+2\bar{B}_{12}[(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn} \\
& +b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})]+2\bar{B}_{22}(b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn}) \\
& +2\bar{B}_{16}[(a^{-1}U_{mn}\Phi'_{um}\Phi_{un})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn} \\
& +b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})]+2\bar{B}_{26}[b^{-1}V_{mn}\Phi_{vm}\Phi'_{vn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn} \\
& +b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})] \\
& +2\bar{B}_{66}(a^{-1}V_{mn}\Phi'_{vm}\Phi_{vn}+b^{-1}U_{mn}\Phi_{um}\Phi'_{un})(a^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn}) \\
& +(\bar{D}_{11}/a^2)(\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})^2+(2\bar{D}_{12}/ab)(\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})(\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn}) \\
& +(\bar{D}_{22}/b^2)(\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})^2+2\bar{D}_{16}(a^{-1}\Psi_{xmn}\Phi'_{\psi xm}\Phi_{\psi xn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn} \\
& +b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})+2\bar{D}_{26}(b^{-1}\Psi_{ymn}\Phi_{\psi ym}\Phi'_{\psi yn})(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn}) \\
& +\bar{D}_{66}(a^{-1}\Psi_{ymn}\Phi'_{\psi ym}\Phi_{\psi yn}+b^{-1}\Psi_{xmn}\Phi_{\psi xm}\Phi'_{\psi xn})^2+K_{55}\bar{A}_{55}(a^{-1}W_{mn}\Phi'_{wm}\Phi_{wn}+\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})^2 \\
& +2K_{45}\bar{A}_{45}(a^{-1}W_{mn}\Phi'_{wm}\Phi_{wn}+\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})(b^{-1}W_{mn}\Phi_{wm}\Phi'_{wn}+\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn}) \\
& +K_{44}\bar{A}_{44}(b^{-1}W_{mn}\Phi_{wn}\Phi'_{wn}+\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})^2\Big\}\ ab\ d\alpha\ d\beta \\
-(\omega^2/2)\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\int_0^1 \Big\{ & m_o[(U_{mn}\Phi_{um}\Phi_{un})^2+(V_{mn}\Phi_{vm}\Phi_{vn})^2+(W_{mn}\Phi_{wm}\Phi_{wn})^2] \\
& +2m_1[(U_{mn}\Phi_{um}\Phi_{un})(\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn})+(V_{mn}\Phi_{vm}\Phi_{vn})(\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn})]
\end{aligned}
$$

$$+m_2\left[\left(\Psi_{xmn}\Phi_{\psi xm}\Phi_{\psi xn}\right)^2+\left(\Psi_{ymn}\Phi_{\psi ym}\Phi_{\psi yn}\right)^2\right]\Big\}\ ab\ d\alpha\ d\beta$$

$$-\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\int_0^1 \tilde{q}\ W_{mn}\Phi_{wm}\Phi_{wn}\ ab\ d\alpha\ d\beta \qquad \text{(D-1)}$$

where

$$\Phi'_m = \partial\Phi_m/\partial\alpha\ ,\ \Phi'_n = \partial\Phi_n/\partial\beta$$

D2. Equations for Minimizing the Energy Difference

$$\partial\tilde{L}/\partial U_{k\ell} = (b/a)\bar{A}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_0^1 \Phi'_{um}\Phi'_{uk}\,d\alpha\int_0^1 \Phi_{un}\Phi_{u\ell}\,d\beta$$

$$+\bar{A}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_0^1 \Phi_{vm}\Phi'_{uk}\,d\alpha\int_0^1 \Phi'_{vn}\Phi_{u\ell}\,d\beta$$

$$+(b/a)\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_0^1 \Phi'_{vm}\Phi'_{uk}\,d\alpha\int_0^1 \Phi_{vn}\Phi_{u\ell}\,d\beta$$

$$+\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_0^1 \Phi_{um}\Phi'_{uk}\,d\alpha\int_0^1 \Phi'_{un}\Phi_{u\ell}\,d\beta$$

$$+\bar{A}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_0^1 \Phi'_{um}\Phi_{uk}\,d\alpha\int_0^1 \Phi_{un}\Phi'_{u\ell}\,d\beta$$

$$+(a/b)\bar{A}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_0^1 \Phi_{vm}\Phi_{uk}\,d\alpha\int_0^1 \Phi'_{vn}\Phi'_{u\ell}\,d\beta$$

$$+\bar{A}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N} V_{mn}\int_0^1 \Phi'_{vm}\Phi_{uk}\,d\alpha\int_0^1 \Phi_{vn}\Phi'_{u\ell}\,d\beta$$

$$+(a/b)\bar{A}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N} U_{mn}\int_0^1 \Phi_{um}\Phi_{uk}\,d\alpha\int_0^1 \Phi'_{un}\Phi'_{u\ell}\,d\beta$$

$$+(b/a)\bar{B}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N} \Psi_{xmn}\int_0^1 \Phi'_{\psi xm}\Phi'_{uk}\,d\alpha\int_0^1 \Phi_{\psi xn}\Phi_{u\ell}\,d\beta$$

$$+\bar{B}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi'_{uk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi_{u\ell}d\beta$$

$$+(b/a)\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Psi'_{uk}d\alpha\int_0^1\Phi_{\psi yn}\Phi_{u\ell}d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi'_{uk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi_{u\ell}d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi'_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi xn}\Phi'_{u\ell}d\beta$$

$$+(a/b)\bar{B}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{uk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi'_{u\ell}d\beta$$

$$+\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi yn}\Phi'_{u\ell}d\beta$$

$$+(a/b)\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi'_{u\ell}d\beta$$

$$-ab\,\omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_o U_{mn}\int_0^1\Phi_{um}\Phi_{uk}d\alpha\int_0^1\Phi_{un}\Phi_{u\ell}d\beta$$

$$-ab\,\omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_1\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{uk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{u\ell}d\beta=0 \qquad (D\text{-}2)$$

There is another equation, for $\partial\tilde{L}/\partial V_{k\ell}=0$, which is analogous to eq. (D-2) with the following substitutions:

$$u\leftrightarrow v\ ,\quad U\leftrightarrow V,\ x\leftrightarrow y,\quad 1\leftrightarrow 2,\quad \Phi\leftrightarrow\Phi',\quad a\leftrightarrow b. \qquad (D\text{-}3)$$

$$\partial\tilde{L}/\partial W_{k\ell} = (b/a)K_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi'_{wm}\Phi'_{wk}d\alpha\int_0^1\Phi_{wn}\Phi_{w\ell}d\beta$$

$$+bK_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi'_{wk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{wxn}\Phi_{w\ell}d\beta$$

$$+K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi'_{wm}\Phi_{wk}d\alpha\int_0^1\Phi_{wn}\Phi'_{w\ell}d\beta$$

$$+K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi_{wm}\Phi'_{wk}d\alpha\int_0^1\Phi'_{wn}\Phi_{w\ell}d\beta$$

$$+aK_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{wk}d\alpha\int_0^1\Phi_{\psi xn}\Phi'_{w\ell}d\beta$$

$$+bK_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi'_{wk}d\alpha\int_0^1\Phi_{\psi yn}\Phi_{w\ell}d\beta$$

$$+(a/b)K_{44}\bar{A}_{44}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi_{wm}\Phi_{wk}d\alpha\int_0^1\Phi'_{wn}\Phi'_{w\ell}d\beta$$

$$+aK_{44}\bar{A}_{44}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{wk}d\alpha\int_0^1\Phi_{\psi yn}\Phi'_{w\ell}d\beta$$

$$-ab\omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_o W_{mn}\int_0^1\Phi_{wm}\Phi_{wk}d\alpha\int_0^1\Phi_{wn}\Phi_{w\ell}d\beta$$

$$-ab\tilde{q}\sum_{m=1}^{M}\sum_{n=1}^{N}\int_0^1\Phi_{wk}d\alpha\int_0^1\Phi_{w\ell}d\beta = 0 \qquad \text{(D-4)}$$

$$\partial\tilde{L}/\partial\Psi_{xk\ell} = (b/a)\bar{B}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N}U_{mn}\int_0^1\Phi'_{um}\Phi'_{\psi xk}d\alpha\int_0^1\Phi_{un}\Phi_{\psi x\ell}d\beta$$

$$+\bar{B}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N}V_{mn}\int_0^1\Phi_{vm}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi'_{vn}\Phi_{\psi x\ell}\,d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}U_{mn}\int_0^1\Phi'_{um}\Phi_{\psi xk}\,d\alpha\int_0^1\Phi_{un}\Phi'_{\psi x\ell}\,d\beta$$

$$+(b/a)\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}V_{mn}\int_0^1\Phi'_{vm}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi_{vn}\Phi_{\psi x\ell}\,d\beta$$

$$+\bar{B}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}U_{mn}\int_0^1\Phi_{um}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi'_{un}\Phi_{\psi x\ell}\,d\beta$$

$$+(a/b)\bar{B}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N}V_{mn}\int_0^1\Phi_{vm}\Phi_{\psi xk}\,d\alpha\int_0^1\Phi'_{vn}\Phi'_{\psi x\ell}\,d\beta$$

$$+\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}V_{mn}\int_0^1\Phi'_{vm}\Phi_{\psi xk}\,d\alpha\int_0^1\Phi_{vn}\Phi'_{\psi x\ell}\,d\beta$$

$$+(a/b)\bar{B}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}U_{mn}\int_0^1\Phi_{um}\Phi_{\psi xk}\,d\alpha\int_0^1\Phi'_{un}\Phi'_{\psi x\ell}\,d\beta$$

$$+(b/a)\bar{D}_{11}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi'_{\psi xm}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi_{\psi xn}\Phi_{\psi x\ell}\,d\beta$$

$$+\bar{D}_{12}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi'_{\psi yn}\Phi_{\psi x\ell}\,d\beta$$

$$+(b/a)\bar{D}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi_{\psi yn}\Phi_{\psi x\ell}\,d\beta$$

$$+\bar{D}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi'_{\psi xm}\Phi_{\psi xk}\,d\alpha\int_0^1\Phi_{\psi xn}\Phi'_{\psi x\ell}\,d\beta$$

$$+\bar{D}_{16}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi'_{\psi xk}\,d\alpha\int_0^1\Phi'_{\psi xn}\Phi_{\psi x\ell}\,d\beta$$

$$+(a/b)\bar{D}_{26}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{\psi yn}\Phi'_{\psi x\ell}d\beta$$

$$+\bar{D}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi'_{\psi ym}\Phi_{\psi x\ell}d\alpha\int_0^1\Phi_{\psi yn}\Phi'_{\psi x\ell}d\beta$$

$$+(a/b)\bar{D}_{66}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{\psi xn}\Phi'_{\psi x\ell}d\beta$$

$$+bK_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi'_{wm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{wn}\Phi_{\psi x\ell}d\beta$$

$$+ab\ K_{55}\bar{A}_{55}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{\psi x\ell}d\beta$$

$$+a\ K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}W_{mn}\int_0^1\Phi_{wm}\Phi_{\psi xk}d\alpha\int_0^1\Phi'_{wn}\Phi_{\psi x\ell}d\beta$$

$$+ab\ K_{45}\bar{A}_{45}\sum_{m=1}^{M}\sum_{n=1}^{N}\Psi_{ymn}\int_0^1\Phi_{\psi ym}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi yn}\Phi_{\psi x\ell}d\beta$$

$$-ab\ \omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_1U_{mn}\int_0^1\Phi_{um}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{un}\Phi_{\psi x\ell}d\beta$$

$$-ab\ \omega^2\sum_{m=1}^{M}\sum_{n=1}^{N}m_2\Psi_{xmn}\int_0^1\Phi_{\psi xm}\Phi_{\psi xk}d\alpha\int_0^1\Phi_{\psi xn}\Phi_{\psi x\ell}d\beta=0 \qquad \text{(D-5)}$$

There is an analogous expression for $\partial\tilde{L}/\partial\Psi_{yk\ell}=0$, using the transformations (D-3), plus the following additional one:

$$45 \leftrightarrow 55 \qquad \text{(D-6)}$$

APPENDIX E

DERIVATION OF THICKNESS-SHEAR FACTOR FOR THE THREE-LAYER, SYMMETRICALLY LAMINATED CASE USING THE DYNAMIC APPROACH

E1. Dynamic Elasticity Analysis of an Individual Layer Undergoing Pure Thickness-Shear Motion

For an individual layer undergoing pure thickness-shear motion in the xz plane, the only non-zero strain component is ϵ_{xz}. Now it is assumed that the layer is orthotropic, so that the only non-zero stress component is σ_{xz}, given by equation (49). Then the general stress equations of motion for three-dimensional, dynamic elastic theory reduce to the following two equations:

$$\sigma_{xz,z} = \rho u_{,tt} \quad ; \quad \sigma_{xz,x} = \rho w_{,tt} \tag{E-1}$$

The *longitudinal thickness-shear strain* can be calculated from the displacements (u,w) in the x,z directions, respectively, as follows:

$$\epsilon_{xz} = u_{,z} + w_{,x} \tag{E-2}$$

The following displacement equations of motion are obtained by substituting equations (49) and (E-2) into equations (E-1):

$$c_s^2(u_{,zz} + w_{,xz}) = u_{,tt} \; ; \; c_s^2(u_{,xz} + w_{,xx}) = w_{,tt} \tag{E-3}$$

where c_s is the shear wave-propagation velocity defined by:

$$c_s^2 = C_{55}/\rho \tag{E-4}$$

However, for pure longitudinal thickness-shear motion, the displacements are independent of axial position x. Thus, all derivatives of displacements with respect to x vanish and equations (E-3) reduce to the following single expression:

$$c_s^2 \, u_{,zz} = u_{,tt} \tag{E-5}$$

Equation (E-5) is the familiar, one-dimensional wave equation, which is solved easily by the separation-of-variables method, with the following solution for simple harmonic motion:

$$u(z,t) = (A \cos \Omega_s z + B \sin \Omega_s z)(C_1 \cos \omega t + C_2 \sin \omega t) \tag{E-6}$$

where $\Omega_s \equiv \omega/c_s$, A and B are constants depending upon the boundary conditions, and C_1 and C_2 are constants which depend upon the initial conditions.

E2. Dynamic Elasticity Analysis for Three-Layer, Symmetrically Laminated Case

Here we consider the special case of a three-layer, symmetrically laminated member, in which the two identical outer layers are designated by superscript 1 and the middle layer is denoted by superscript 2, as

shown in figure E-1. The proper boundary conditions for the anti-symmetric modes in pure thickness-shear motion are

$$u^{(1)}(z_2,t) = u^{(2)}(z_2,t) \; ; \; u,_z^{(1)}(z_1,t) = 0 \; ;$$
$$u,_z^{(1)}(z_2,t) = u,_z^{(2)}(z_2,t) \; ; \; u^{(2)}(0,t) = 0 \qquad \text{(E-7)}$$

where z_1 and z_2 are dimensions shown in figure E-1.

Substituting equation (E-6) into equations (E-7) yields the following set of expressions:

$$A_1 \cos \Omega_s^{(1)} z_2 + B_1 \sin \Omega_s^{(1)} z_2 = A_1 \cos \Omega_s^{(2)} z_2$$
$$+ B_2 \sin \Omega_s^{(2)} z_2 \; ; \; - \Omega_s^{(1)} A_1 \sin \Omega_s^{(1)} z_2 + \Omega_s^{(1)} B_1 \cos \Omega_s^{(1)} z_2 =$$
$$- \Omega_s^{(2)} A_2 \sin \Omega_s^{(2)} z_2 + \Omega_s^{(2)} B_2 \cos \Omega_s^{(2)} z_2 \; ; \; -\Omega_s^{(1)} A_1 \sin \Omega_s^{(1)} z_1$$
$$+ \Omega_s^{(1)} B_1 \cos \Omega_s^{(1)} z_1 = 0 \; ; \; A_2 = 0 \qquad \text{(E-8)}$$

or

$$\begin{bmatrix} \cos \Omega_s^{(1)} z_2 & \sin \Omega_s^{(1)} z_2 & - \sin \Omega_s^{(2)} z_2 \\ -\Omega_s^{(1)} \sin \Omega_s^{(1)} z_2 & \Omega_s^{(1)} \cos \Omega_s^{(1)} z_2 & -\Omega_s^{(2)} \cos \Omega_s^{(2)} z_2 \\ -\Omega_s^{(1)} \sin \Omega_s^{(1)} z_1 & \Omega_s^{(1)} \cos \Omega_s^{(1)} z_1 & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{(E-9)}$$

The homogeneous system of linear algebraic equations (E-9) has a nontrivial solution if, and only if, the determinant of its coefficient

matrix is equal to zero. The resulting determinantal equation has as its solutions the roots of the following transcendental equation:

$$\Omega_s^{(1)} \tan \Omega_s^{(2)} z_2 = - \Omega_s^{(2)} \cot (\Omega_s^{(1)} z_2 - \Omega_s^{(1)} z_1) \qquad \text{(E-10)}$$

From the definitions of $\Omega^{(k)}$ and $c_s^{(k)}$, we have

$$\Omega_s^{(1)}/\Omega_s^{(2)} = c_s^{(2)}/c_s^{(1)} = (C_{55}^{(2)}/C_{55}^{(1)})^{\frac{1}{2}}(\rho^{(1)}/\rho^{(2)})^{\frac{1}{2}} = (R/\beta)^{\frac{1}{2}} \qquad \text{(E-11)}$$

where

$$\beta \equiv C_{55}^{(1)}/C_{55}^{(2)} \quad ; \quad R \equiv \rho^{(1)}/\rho^{(2)} \qquad \text{(E-12)}$$

Then equation (E-10) can be expressed as follows:

$$(R/\beta)^{\frac{1}{2}} \tan [(\zeta_2 \Omega_s^{(1)} z_1 (\beta/R)^{\frac{1}{2}}] = \cot [(1-\zeta_2)\Omega_s^{(1)} z_1] \qquad \text{(E-13)}$$

where

$$\zeta_2 \equiv z_2/z_1 \qquad \text{(E-14)}$$

E3. Dynamic Analysis of a Symmetrically Laminated Timoshenko Beam Undergoing Pure Thickness-Shear Motion

Here we consider a symmetrically laminated Timoshenko beam*. The

*A beam exhibiting both thickness-shear flexibility and rotatory inertia is generally referred to as a Timoshenko beam (refs. 41,42).

axial and thickness (or depth) directions are designated as the x and z axes, respectively. Such a beam could be analyzed as a special case of the laminated plate theory presented in Section II by merely deleting all derivatives with respect to y. However, the beam case is so much simpler and pure thickness-shear motion is such a simple type of motion; therefore, it was decided to make an exact analysis for the present case.

The following kinematic relations hold throughout the entire thickness of the laminate:

$$\kappa_{xx} = \psi_{x,x} \quad ; \quad \epsilon_{xz} = w_{o,x} + \psi_x \tag{E-15}$$

The following stress-strain relations are applicable to a typical layer "k":

$$\sigma_{xx}^{(k)} = E_{11}^{(k)} \epsilon_{xx} = E_{11}^{(k)} \kappa_{xx} z \; ; \; \sigma_{xz}^{(k)} = C_{55}^{(k)} \epsilon_{xz} \tag{E-16}$$

The bending moment and shear force, expressed on the basis of a unit width as in plate theory (Section 2.3) are:

$$M_{xx} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xx}^{(k)} z \, dz \quad ; \quad Q_x = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \sigma_{xz}^{(k)} \, dz \tag{E-17}$$

Substituting equations (E-15) and (E-16) into equation (E-17) and introducing the shear factor K_{55} as a correction factor to be determined later, one obtains:

$$M_{xx} = D_{11} \psi_{x,x} \quad ; \quad Q_x = K_{55} A_{55} (w_{o,x} + \psi_x) \tag{E-18}$$

where

$$\left\{D_{11} \ , \ A_{55}\right\} \equiv \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \left\{E_{11}^{(k)} z^2 \ , \ C_{55}^{(k)}\right\} dz \qquad \text{(E-19)}$$

The equations of motion for a symmetrically laminated Timoshenko beam are identical in form to those governing a homogeneous Timoshenko beam (refs. 41,42), namely:

$$Q_{x,x} = m_o \, w_{o,tt} \quad ; \quad M_{xx,x} - Q_x = m_2 \psi_{x,tt} \qquad \text{(E-20)}$$

where m_o and m_2 are defined in equations (28).

Inserting equations (E-18) into equation (E-20), one obtains the following set of two coupled equations of motion in terms of the generalized displacements w_o and ψ_x:

$$K_{55} A_{55} (w_{o,xx} + \psi_{x,x}) = m_o \, w_{o,tt} \qquad \text{(E-21)}$$

$$D_{11} \psi_{x,xx} - K_{55} A_{55} (w_{o,x} + \psi_x) = m_2 \, \psi_{x,tt} \qquad \text{(E-22)}$$

For pure thickness-shear motion, w_o and ψ_x are independent of axial position x, so that equations (E-21) and (E-22) uncouple and become:

$$w_{o,tt} = 0 \qquad \text{(E-23)}$$

$$m_2 \, \psi_{x,tt} + K_{55} A_{55} \psi_x = 0 \qquad \text{(E-24)}$$

Since equation (E-23) does not contain K_{55} and since w_o is not

in equation (E-24), we have no further need for equation (E-23).

For steady-state harmonic motion, the solution of equation (E-24) can be expressed as follows:

$$\psi_x = \tilde{\psi}_x e^{i\omega t} \tag{E-25}$$

where $\tilde{\psi}_x$ is a constant.

Substituting equation (E-25) into equation (E-24), we are led to the following relationship:

$$\omega^2 = K_{55} A_{55}/m_2 \tag{E-26}$$

This equation is applicable to any symmetrical laminate. For the special case of a three-layer one, using the notation depicted in figure 10 and the definitions of A_{55} and m_2 from equations (E-19) and (28), one obtains:

$$\omega^2 = 3K_{55} z_1^{-2} (C_{55}^{(1)}/\rho^{(1)})[(\zeta_2/\beta) + 1-\zeta_2] \cdot [(\zeta_2^3/R) + 1 - \zeta_2^3]^{-1} \tag{E-27}$$

where β,R, and ζ_2 are as defined previously.

E4. Determinations of the Thickness-Shear Factor

The longitudinal thickness-shear factor, K_{55}, is determined implicitly by equating ω^2 associated with the lowest non-trivial solution of equation (E-13) to that given by equation (E-27).

APPENDIX F

COMPLEX STIFFNESS COEFFICIENTS FOR A LAMINATE HAVING ALTERNATING PLIES OF TWO DIFFERENT COMPOSITE MATERIALS*

F1. Introduction

The theory presented in Section II can be used to calculate the stiffness and damping coefficients for an arbitrary laminate, i.e. one consisting of any number of plies of any thickness, material, and orientation. However, in many cases of practical importance, laminates are designed to have many plies of two alternating materials. In such cases, the approximate approach used in refs. 29-31 is sufficiently accurate. Although this approach was originated for application to wave propagation in an infinite medium, it is applicable also to plates. In the latter configuration, it has been verified experimentally in several instances. By comparison with static experimental results, Rose and Tshirschnitz (reference 92) found that it gave good predictions of the in-plane elastic modulus and in-plane shear modulus. Also Achenbach and Zerbe (ref. 32) found that it gave a frequency versus wavelength relationship for longitudinal vibration of a laminated beam which was in excellent agreement with experimental results.

In all of the work mentioned above, all of the individual layers were isotropic. However, in many cases of increasing importance, at least one of the sets of layers may be made of composite materials. Some examples

*After completion of this derivation, the work of Chou et al. (Reference 149) came to the attention of the authors. In their work, Chou et al. derived purely elastic equations which are analogous to the complex ones derived here.

are as follows:

1. Alternating layers of an isotropic material and an orthotropic material. Examples: armor plate consisting of alternating layers of a hard ceramic material (isotropic) and a lossy glass fiber-epoxy matrix composite (orthotropic); a laminate consisting of alternate layers of high-modulus orthotropic composite material (such as boron-epoxy or graphite-epoxy) and low-modulus, high-damping polymer (to increase the damping capacity of the laminate).

2. Alternating layers of two different orthotropic composite materials. Example: Boron-epoxy (for high stiffness) and glass-epoxy (for low cost). The lamination scheme may be either parallel ply (unidirectional) or cross ply.

3. Alternating layers of the same composite material and thickness, but oriented alternately at $+\theta$ and $-\theta$, where $0 < \theta < 90^{\circ}$. This is the so-called angle-ply lamination arrangement, which is very popular in a variety of aerospace structures.

An original derivation is presented here which is applicable to determination of both the stiffness and damping of the above three classes of laminates. It may be considered to be a generalization of the work of refs. 29-31; necessarily the resulting equations reduce to theirs in the case when both materials are isotropic.

F2. Analysis

The bases for the present analysis, as well as that of refs. 29-31, are the following hypotheses:

1. Strains in the plane of the laminations are equal.

2. Stresses in the direction normal to lamination planes are equal.

Hypothesis 1 leads to the Voigt upper-bound estimate (reference 93) of the equivalent macroscopic properties of a two-phase material having arbitrary geometrical configuration of individual constituents. Except for the presence of an additional Poisson's ratio effects term, this is known also as the "rule of mixtures". Hypothesis 2 leads to the correspond-Reuss lower-bound estimate (reference 94), which is almost identical with the so-called "inverse rule of mixtures".

Here we consider a medium consisting of repeating alternating layers denoted by "a" and "b", as shown in figure F-1. The plane of the laminations is designated as the xy plane and z is the normal to this plane. Then the two hypotheses mentioned above can be stated mathematically as follows in contracted notation:

$$\epsilon_j^{(a)} = \epsilon_j^{(b)} = \epsilon_j \qquad (j=1,2,6) \qquad \text{(F-1)}$$

$$\sigma_i^{(a)} = \sigma_i^{(b)} = \sigma_i \qquad (i=3,4,5) \qquad \text{(F-2)}$$

where the ϵ_j are strain components, the σ_i are stress components, and superscripts (a) and (b) refer to layers "a" and "b". Subscripts 1 and 2 refer to normal strain (or normal stress) in the plane of the layers, 3 refers to thickness-normal effects, 4 and 5 refer to thickness shear, and 6 refers to in-plane shear. This notation is consistent with that most widely used in the field of composite-material mechanics (ref. 4) but differs from the mixed notation used in the body of this report.

As a consequence of the complex, orthotropic version of Hooke's law

and static equilibrium, equations (F-1) and (F-2) imply the following relations:

$$\sigma_j = H^{(a)}\sigma_j^{(a)} + H^{(b)}\sigma_j^{(b)} \qquad (j=1,2,6) \qquad \text{(F-3)}$$

$$\epsilon_i = H^{(a)}\epsilon_i^{(a)} + H^{(b)}\epsilon_i^{(b)} \qquad (i=3,4,5) \qquad \text{(F-4)}$$

where $H^{(a)}$ and $H^{(b)}$ are the thickness fractions of layers "a" and "b", respectively. Thus,

$$H^{(a)} + H^{(b)} = 1 \qquad \text{(F-5)}$$

The complex, orthotropic version of Hooke's law, alluded to above, holds for each type of layer as follows:

$$\begin{Bmatrix} \sigma_1^{(k)} \\ \sigma_2^{(k)} \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6^{(k)} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11}^{(k)} & \bar{C}_{12}^{(k)} & \bar{C}_{13}^{(k)} & 0 & 0 & \bar{C}_{16}^{(k)} \\ \bar{C}_{12}^{(k)} & \bar{C}_{22}^{(k)} & \bar{C}_{23}^{(k)} & 0 & 0 & \bar{C}_{26}^{(k)} \\ \bar{C}_{13}^{(k)} & \bar{C}_{23}^{(k)} & \bar{C}_{33}^{(k)} & 0 & 0 & \bar{C}_{36}^{(k)} \\ 0 & 0 & 0 & \bar{C}_{44}^{(k)} & \bar{C}_{45}^{(k)} & 0 \\ 0 & 0 & 0 & \bar{C}_{45}^{(k)} & \bar{C}_{55}^{(k)} & 0 \\ \bar{C}_{16}^{(k)} & \bar{C}_{26}^{(k)} & \bar{C}_{36}^{(k)} & 0 & 0 & \bar{C}_{66}^{(k)} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3^{(k)} \\ \epsilon_4^{(k)} \\ \epsilon_5^{(k)} \\ \epsilon_6 \end{Bmatrix} \qquad (k=a,b) \qquad \text{(F-6)}$$

where the $\bar{C}_{ij}^{(k)}$ are the complex stiffness coefficients.

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_3 \end{Bmatrix} = \begin{bmatrix} \bar{C}'_{11} & \bar{C}'_{12} & \bar{C}'_{13} & \bar{C}''_{13} & \bar{C}'_{16} \\ \bar{C}'_{12} & \bar{C}'_{22} & \bar{C}'_{23} & \bar{C}''_{23} & \bar{C}'_{26} \\ \bar{C}^{(a)}_{13} & \bar{C}^{(a)}_{23} & \bar{C}^{(a)}_{33} & 0 & \bar{C}^{(a)}_{36} \\ \bar{C}^{(b)}_{13} & \bar{C}^{(b)}_{23} & 0 & \bar{C}^{(b)}_{33} & \bar{C}^{(b)}_{36} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3^{(a)} \\ \epsilon_3^{(b)} \\ \epsilon_6 \end{Bmatrix} \qquad \text{(F-7)}$$

where

$$\bar{C}'_{ij} = H^{(a)}\,\bar{C}^{(a)}_{ij} + H^{(b)}\bar{C}^{(b)}_{ij} \quad ; \quad i,j = 11,\ 12,\ 16,\ 22,\ 26$$

$$\bar{C}'_{ij} = H^{(a)}\,\bar{C}^{(a)}_{ij} \quad ; \quad i,j = 13,\ 23$$

$$\bar{C}''_{ij} = H^{(b)}\bar{C}_{ij}^{(b)} \quad ; \quad i,j = 13,\ 23$$

Combining equations (F-4) and the last two of equations (F-7), we get the following relations:

$$\epsilon_3^{(a)} = [C^{(b)}_{33}\,\epsilon_3 - H^{(b)}(\bar{C}^{(a)}_{13} - \bar{C}^{(b)}_{13})\epsilon_1 - H^{(b)}(\bar{C}^{(a)}_{23} - \bar{C}^{(b)}_{23})\epsilon_2$$

$$- H^{(b)}(\bar{C}^{(b)}_{36} - \bar{C}^{(a)}_{36})\epsilon_6]\,[H^{(a)}\bar{C}^{(b)}_{33} + H^{(b)}\,\bar{C}^{(a)}_{33}]^{-1} \qquad \text{(F-8)}$$

$$\epsilon_3^{(b)} = [\bar{C}^{(a)}_{33}\epsilon_3 - H^{(a)}(C^{(b)}_{13} - \bar{C}^{(a)}_{13})\epsilon_1 - H^{(a)}(\bar{C}^{(b)}_{23} - \bar{C}^{(a)}_{23})\epsilon_2$$

$$-H^{(a)}(\bar{C}^{(b)}_{36} - \bar{C}^{(a)}_{36})\epsilon_6]\,[H^{(a)}\bar{C}^{(b)}_{33} + H^{(b)}\,\bar{C}^{(a)}_{33}]^{-1} \qquad \text{(F-9)}$$

Substituting equations (F-8) and (F-9) into the first of equations (F-7), one is led to the following result:

$$\sigma_1 = \bar{C}_{11}\epsilon_1 + \bar{C}_{12}\epsilon_2 + \bar{C}_{13}\epsilon_3 + \bar{C}_{16}\epsilon_6 \qquad \text{(F-10)}$$

where

$$\bar{C}_{11} \equiv H^{(a)}\bar{C}_{11}^{(a)} + H^{(b)}\bar{C}_{11}^{(b)} - \bar{K}_C\,(\bar{C}_{13}^{(a)} - \bar{C}_{13}^{(b)})^2 \qquad \text{(F-11)}$$

$$\bar{C}_{12} \equiv H^{(a)}\bar{C}_{12}^{(a)} + H^{(b)}\bar{C}_{12}^{(b)} - \bar{K}_C\,(\bar{C}_{13}^{(a)} - \bar{C}_{13}^{(b)})(\bar{C}_{23}^{(a)} - \bar{C}_{23}^{(b)}) \qquad \text{(F-12)}$$

$$\bar{C}_{13} \equiv [\bar{C}_{13}^{(a)}(H^{(a)}/\bar{C}_{33}^{(a)}) + \bar{C}_{13}^{(b)}(H^{(b)}/\bar{C}_{33}^{(b)})]\,\bar{L}_c \qquad \text{(F-13)}$$

$$\bar{C}_{16} \equiv H^{(a)}\,\bar{C}_{16}^{(a)} + H^{(b)}\bar{C}_{16}^{(b)} - \bar{K}_c(\bar{C}_{13}^{(a)} - \bar{C}_{13}^{(b)})(\bar{C}_{36}^{(a)} - \bar{C}_{36}^{(b)}) \qquad \text{(F-14)}$$

$$\bar{K}_c \equiv [(\bar{C}_{33}^{(a)}/H^{(a)}) + (\bar{C}_{33}^{(b)}/H^{(b)})]^{-1}]^{-1} \qquad \text{(F-15)}$$

$$\bar{L}_c \equiv \lfloor (H^{(a)}/\bar{C}_{33}^{(a)}) + (H^{(b)}/\bar{C}_{33}^{(b)})\,]^{-1} \qquad \text{(F-16)}$$

Analogous equations to equations (F-10) - (F-14) can be obtained readily by interchanging the roles of subscripts 1 and 2.

From the last two of equations (F-7), one gets:

$$2\sigma_3 = (\bar{C}_{13}^{(a)} + \bar{C}_{13}^{(b)})\epsilon_1 + (\bar{C}_{23}^{(a)} + \bar{C}_{23}^{(b)})\epsilon_2 + \bar{C}_{33}^{(a)}\,\epsilon_3^{(a)} + \bar{C}_{33}^{(b)}\,\epsilon_3^{(b)}$$

$$+ (\bar{C}_{36}^{(a)} + \bar{C}_{36}^{(b)})\,\epsilon_6 \qquad \text{(F-17)}$$

Then substituting equations (F-8) and (F-9) into equation (F-17), one finds the following result:

$$\sigma_3 = \bar{C}_{13}\,\epsilon_1 + \bar{C}_{23}\,\epsilon_2 + \bar{C}_{33}\,\epsilon_3 + \bar{C}_{36}\epsilon_6 \qquad \text{(F-18)}$$

where $\bar{C}_{13}$ and $\bar{C}_{23}$ are given by equation (F-13) and an analogous equation, and

$$\bar{C}_{33} = \bar{L}_c \qquad \text{(F-19)}$$

$$\bar{C}_{36} = [\bar{C}_{36}^{(a)} \ (H^{(a)}/\bar{C}_{33}^{(a)}) + \bar{C}_{36}^{(b)} (H^{(b)}/\bar{C}_{33}^{(b)}) \ ] \ \bar{L}_c \qquad \text{(F-20)}$$

Still another set of equations analogous to equations (F-10)-(F-14) can be obtained by interchanging the roles of subscripts 1 and 6.

As the first step toward determining expressions involving the composite coefficients with subscripts 44, 45, and 55, it is expedient to invert the pertinent equations in set (F-6), thus arriving at the following expressions:

$$\begin{Bmatrix} \epsilon_4^{(k)} \\ \epsilon_5^{(k)} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{44}^{(k)} & \bar{S}_{45}^{(k)} \\ \bar{S}_{45}^{(k)} & \bar{S}_{55}^{(k)} \end{bmatrix} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} \qquad \text{(F-21)}$$

where

$$\begin{aligned} \bar{S}_{44}^{(k)} &\equiv \bar{C}_{55}^{(k)}/\bar{\Delta}^{(k)}, \quad \bar{S}_{45}^{(k)} \equiv -\bar{C}_{45}^{(k)}/\bar{\Delta}^{(k)}, \\ \bar{S}_{55}^{(k)} &= \bar{C}_{44}/\bar{\Delta}^{(k)}, \quad \bar{\Delta}^{(k)} \equiv \bar{C}_{44}^{(k)} \bar{C}_{55}^{(k)} - (\bar{C}_{45}^{(k)})^2 \end{aligned} \qquad \text{(F-22)}$$

Inserting equations (F-21) into equations (F-4), we arrive at the following expressions:

$$\begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix} = \begin{bmatrix} \bar{S}_{44} & \bar{S}_{45} \\ \bar{S}_{45} & \bar{S}_{55} \end{bmatrix} \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} \qquad \text{(F-23)}$$

where

$$\bar{S}_{44} = H^{(a)} \bar{S}_{44}^{(a)} + H^{(b)} \bar{S}_{44}^{(b)} \qquad \text{(F-24)}$$

There are three expressions, analogous to equation (F-24), which are obtained by merely substituting subscripts 45 and 55, respectively, for subscript 44 in equation (F-24).

We now have derived expressions for all twenty of the laminate coefficients which appear in equation (F-23), above, and in the following expression:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & \bar{C}_{36} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_6 \end{Bmatrix} \qquad \text{(F-25)}$$

The effective density of the laminate is given by the following expression:

$$\rho = H^{(a)}\rho^{(a)} + H^{(b)}\rho^{(b)} \qquad \text{(F-26)}$$

APPENDIX G

IDENTIFICATION OF INTEGRAL FORMS

G1. Trigonometric Integrals

The integral forms for three boundary conditions are tabulated in the following form, where $m,k,n,\ell = 1$ to 2:

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{1mk} \equiv \int_o^1 \Phi_{um}\Phi_{uk}\,d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	3/2 ; $m=k\neq 0$ 1 ; $m\neq k$
$I_{1n\ell} \equiv \int_o^1 \Phi_{un}\Phi_{u\ell}\,d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	3/2 ; $n=\ell\neq 0$ 1 ; $n\neq \ell$
$I_{2mk} \equiv \int_o^1 \Phi_{vm}\Phi_{vk}\,d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{2n\ell} \equiv \int_o^1 \Phi_{vn}\Phi_{v\ell}\,d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{3mk} \equiv \int_o^1 \Phi_{um}\Phi_{vk}\,d\alpha$	0	I_{1mk}	I_{1mk}
$I_{3n\ell} \equiv \int_o^1 \Phi_{un}\Phi_{v\ell}\,d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{4mk} \equiv \int_o^1 \Phi_{um}\Phi'_{uk}\,d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	0 ; $m=k\neq 0$ 0 ; $m\neq k$	0 ; $m=k\neq 0$ 0 ; $m\neq k$
$I_{4n\ell} \equiv \int_o^1 \Phi_{un}\Phi'_{v\ell}\,d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	0 ; $n=\ell\neq 0$ 0 ; $n\neq \ell$	0 ; $n=\ell\neq 0$ 0 ; $n\neq \ell$

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{5mk} \equiv \int_o^1 \Phi_{vm} \Phi'_{uk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{5n\ell} \equiv \int_o^1 \Phi_{vn} \Phi'_{u\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{6mk} \equiv \int_o^1 \Phi_{um} \Phi'_{uk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{6n\ell} \equiv \int_o^1 \Phi_{un} \Phi'_{u\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{7mk} \equiv \int_o^1 \Phi_{vm} \Phi'_{vk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{7n\ell} \equiv \int_o^1 \Phi_{vn} \Phi'_{v\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{8mk} \equiv \int_o^1 \Phi'_{um} \Phi'_{uk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	$2mk\pi^2$; $m=k\neq 0$ 0 ; $m\neq k$	$2mk\pi^2$; $m=k\neq 0$ 0 ; $m\neq k$
$I_{8n\ell} \equiv \int_o^1 \Phi'_{un} \Phi'_{u\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$2n\ell\pi^2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$2n\ell\pi^2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$
$I_{9mk} \equiv \int_o^1 \Phi'_{vm} \Phi'_{vk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{9n\ell} \equiv \int_o^1 \Phi'_{vn} \Phi'_{v\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{10mk} \equiv \int_o^1 \Phi'_{um} \Phi'_{vk} d\alpha$	0	I_{8mk}	I_{8mk}
$I_{10n\ell} \equiv \int_o^1 \Phi'_{un} \Phi'_{v\ell} d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{11mk} \equiv \int_o^1 \Phi'_{\psi xm} \Phi_{wk} d\alpha$	0	See G2 for clamped case	0
$I_{11n\ell} \equiv \int_o^1 \Phi_{\psi xn} \Phi_{w\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	See G2	0

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{12mk} \equiv \int_0^1 \Phi_{\psi ym} \Phi_{wk} d\alpha$	$\frac{1}{2}$; m=k≠0 0 ; m≠k	I_{11mk}	I_{11mk}
$I_{12n\ell} \equiv \int_0^1 \Phi_{\psi yn} \Phi_{w\ell} d\beta$	0	$I_{11n\ell}$	$I_{11n\ell}$
$I_{13mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi'_{wk} d\alpha$	kπ/2 ; m=k≠0 0 ; m≠k	See G2	0
$I_{13n\ell} \equiv \int_0^1 \Phi_{\psi xn} \Phi'_{w\ell} d\beta$	0	See G2	0
$I_{14mk} \equiv \int_0^1 \Phi'_{\psi xm} \Phi_{wk} d\alpha$	kπ/2 ; m=k≠0 0 ; m≠k	I_{13mk}	I_{13mk}
$I_{14n\ell} \equiv \int_0^1 \Phi'_{\psi xn} \Phi_{w\ell} d\beta$	0	$I_{13n\ell}$	$I_{13n\ell}$
$I_{15mk} \equiv \int_0^1 \Phi_{\psi ym} \Phi'_{wk} d\alpha$	0	I_{13mk}	I_{13mk}
$I_{15n\ell} \equiv \int_0^1 \Phi_{\psi yn} \Phi'_{w\ell} d\beta$	nπ/2 ; n=ℓ≠0 0 ; n≠ℓ	$I_{13n\ell}$	$I_{13n\ell}$
$I_{16mk} \equiv \int_0^1 \Phi'_{\psi ym} \Phi_{wk} d\alpha$	0	I_{13mk}	I_{13mk}
$I_{16n\ell} \equiv \int_0^1 \Phi'_{\psi yn} \Phi_{w\ell} d\beta$	nπ/2 ; n=ℓ≠0 0 ; n≠ℓ	$I_{13n\ell}$	$I_{13n\ell}$
$I_{17mk} \equiv \int_0^1 \Phi_{wm} \Phi_{wk} d\alpha$	$\frac{1}{2}$; m=k≠0 0 ; m≠k	1	$\frac{1}{2}$; m=k≠0 0 ; m≠k
$I_{17n\ell} \equiv \int_0^1 \Phi_{wn} \Phi_{w\ell} d\beta$	$\frac{1}{2}$; n=ℓ≠0 0 ; n≠ℓ	1	$\frac{1}{2}$; n=ℓ≠0 0 ; n≠ℓ
$I_{18mk} \equiv \int_0^1 \Phi'_{wm} \Phi_{wk} d\alpha$	0	0	0
$I_{18n\ell} \equiv \int_0^1 \Phi'_{wn} \Phi_{w\ell} d\beta$	0	0	0

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{19mk} \equiv \int_o^1 \Phi'_{wm} \Phi_{wk} d\alpha$	0	I_{18mk}	I_{18mk}
$I_{19n\ell} \equiv \int_o^1 \Phi'_{wn} \Phi_{w\ell} d\beta$	0	$I_{18n\ell}$	$I_{18n\ell}$
$I_{20mk} \equiv \int_o^1 \Phi'_{wm} \Phi'_{wk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	$C_m Z_m (C_m Z_m - 2)$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$
$I_{20n\ell} \equiv \int_o^1 \Phi'_{wn} \Phi'_{w\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$C_n Z_n (C_n Z_n - 2)$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$
$I_{21mk} \equiv \int_o^1 \Phi_{\psi xm} \Phi_{\psi xk} d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{21n\ell} \equiv \int_o^1 \Phi_{\psi xn} \Phi_{\psi x\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{22mk} \equiv \int_o^1 \Phi_{\psi ym} \Phi_{\psi yk} d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{22n\ell} \equiv \int_o^1 \Phi_{\psi yn} \Phi_{\psi y\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{23mk} \equiv \int_o^1 \Phi_{\psi xm} \Phi_{\psi yk} d\alpha$	0	I_{1mk}	I_{1mk}
$I_{23n\ell} \equiv \int_o^1 \Phi_{\psi xn} \Phi_{\psi y\ell} d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{24mk} \equiv \int_o^1 \Phi_{\psi xm} \Phi'_{\psi yk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{24n\ell} \equiv \int_o^1 \Phi_{\psi xn} \Phi'_{\psi y\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{25mk} \equiv \int_o^1 \Phi_{\psi ym} \Phi'_{\psi xk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{25n\ell} \equiv \int_o^1 \Phi_{\psi yn} \Phi'_{\psi x\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{4n\ell}$	$I_{4n\ell}$

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{26mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi'_{\psi xk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{26n\ell} \equiv \int_0^1 \Phi_{\psi xn} \Phi'_{\psi x\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{27mk} \equiv \int_0^1 \Phi_{\psi ym} \Phi'_{\psi yk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{27n\ell} \equiv \int_0^1 \Phi_{\psi yn} \Phi'_{\psi y\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{28mk} \equiv \int_0^1 \Phi'_{\psi xm} \Phi'_{\psi xk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{28n\ell} \equiv \int_0^1 \Phi'_{\psi xn} \Phi'_{\psi x\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{29mk} \equiv \int_0^1 \Phi'_{\psi ym} \Phi'_{\psi yk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{29n\ell} \equiv \int_0^1 \Phi'_{\psi yn} \Phi'_{\psi y\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{30mk} \equiv \int_0^1 \Phi'_{\psi xm} \Phi'_{\psi yk} d\alpha$	0	I_{8mk}	I_{8mk}
$I_{30n\ell} \equiv \int_0^1 \Phi'_{\psi xn} \Phi'_{\psi y\ell} d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{31mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi_{uk} d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{31n\ell} \equiv \int_0^1 \Phi_{\psi xn} \Phi_{u\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{32mk} \equiv \int_0^1 \Phi_{um} \Phi_{\psi xk} d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{32n\ell} \equiv \int_0^1 \Phi_{un} \Phi_{\psi x\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{33mk} \equiv \int_0^1 \Phi'_{\psi xm} \Phi_{uk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{33n\ell} \equiv \int_0^1 \Phi'_{\psi xn} \Phi_{u\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{34mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi'_{uk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{34n\ell} \equiv \int_0^1 \Phi_{\psi xn} \Phi'_{u\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{35mk} \equiv \int_0^1 \Phi_{\psi ym} \Phi_{uk} d\alpha$	0	I_{1mk}	I_{1mk}
$I_{35n\ell} \equiv \int_0^1 \Phi_{\psi yn} \Phi_{u\ell} d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{36mk} \equiv \int_0^1 \Phi'_{\psi ym} \Phi_{uk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{36n\ell} \equiv \int_0^1 \Phi'_{\psi yn} \Phi_{u\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{37mk} \equiv \int_0^1 \Phi_{\psi ym} \Phi'_{uk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{37n\ell} \equiv \int_0^1 \Phi_{\psi yn} \Phi'_{u\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq \ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{38mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi_{vk} d\alpha$	0	I_{1mk}	I_{1mk}
$I_{38n\ell} \equiv \int_0^1 \Phi_{\psi xn} \Phi_{v\ell} d\beta$	0	$I_{1n\ell}$	$I_{1n\ell}$
$I_{39mk} \equiv \int_0^1 \Phi'_{\psi xm} \Phi_{vk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{39n\ell} \equiv \int_o^1 \Phi'_{\psi xn} \Phi_{v\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{40mk} \equiv \int_o^1 \Phi_{\psi xm} \Phi'_{vk} d\alpha$	$k\pi/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{4mk}	I_{4mk}
$I_{40n\ell} \equiv \int_o^1 \Phi_{\psi xn} \Phi'_{v\ell} d\beta$	$n\pi/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{4n\ell}$	$I_{4n\ell}$
$I_{41mk} \equiv \int_o^1 \Phi_{\psi ym} \Phi_{vk} d\alpha$	$\frac{1}{2}$; $m=k\neq 0$ 0 ; $m\neq k$	I_{1mk}	I_{1mk}
$I_{41n\ell} \equiv \int_o^1 \Phi_{\psi yn} \Phi_{v\ell} d\beta$	$\frac{1}{2}$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{1n\ell}$	$I_{1n\ell}$
$I_{42mk} \equiv \int_o^1 \Phi'_{\psi ym} \Phi_{vk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{42n\ell} \equiv \int_o^1 \Phi'_{\psi yn} \Phi_{v\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{43mk} \equiv \int_o^1 \Phi_{\psi ym} \Phi'_{vk} d\alpha$	0	I_{4mk}	I_{4mk}
$I_{43n\ell} \equiv \int_o^1 \Phi_{\psi yn} \Phi'_{v\ell} d\beta$	0	$I_{4n\ell}$	$I_{4n\ell}$
$I_{44mk} \equiv \int_o^1 \Phi'_{\psi xm} \Phi'_{uk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{44n\ell} \equiv \int_o^1 \Phi'_{\psi xn} \Phi'_{u\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$
$I_{55mk} \equiv \int_o^1 \Phi'_{\psi ym} \Phi'_{uk} d\alpha$	0	I_{8mk}	I_{8mk}
$I_{55n\ell} \equiv \int_o^1 \Phi'_{\psi yn} \Phi'_{u\ell} d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{56mk} \equiv \int_o^1 \Phi'_{\psi xm} \Phi'_{vk} d\alpha$	0	I_{8mk}	I_{8mk}

Integral Form	Evaluation for Boundary Condition Listed		
	Simply Supported	Clamped Edges	Free Edges
$I_{56n\ell} \equiv \int_0^1 \Phi'_{\psi xn} \Phi'_{v\ell} d\beta$	0	$I_{8n\ell}$	$I_{8n\ell}$
$I_{57mk} \equiv \int_0^1 \Phi'_{\psi ym} \Phi'_{vk} d\alpha$	$mk\pi^2/2$; $m=k\neq 0$ 0 ; $m\neq k$	I_{8mk}	I_{8mk}
$I_{57n\ell} \equiv \int_0^1 \Phi'_{\psi yn} \Phi'_{v\ell} d\beta$	$n\ell\pi^2/2$; $n=\ell\neq 0$ 0 ; $n\neq\ell$	$I_{8n\ell}$	$I_{8n\ell}$

G2. Combination Trigonometric-Beam Type Integrals

These integrals are related to I_{11mk}, $I_{11n\ell}$, I_{13mk}, and $I_{13n\ell}$ for clamped and free edges of the plate mentioned in Appendix G1. Therefore, these integrals were evaluated and are listed below:

All Edges Clamped.

$$I_{11mk} \equiv \int_0^1 \Phi_{\psi xm} \Phi_{wk} d\alpha$$

$$= \frac{1}{(Z_k)^2 + (2m\pi)^2} \left\{ Z_k \sinh Z_k \sin 2m\pi - 2m\pi \cosh Z_k \cos 2m\pi + 2m\pi \right\}$$

$$+ \left\{ \frac{\cos[2m\pi + Z_k] - 1}{2[2m\pi + Z_k]} + \frac{\cos[2m\pi - Z_k] - 1}{2[2m\pi - Z_k]} \right.$$

$$- \frac{C_k}{(Z_k)^2 + (2m\pi)^2} \left\{ Z_k \cosh Z_k \sin 2m\pi - 2m\pi \sinh Z_k \cos 2m\pi \right\}$$

$$\left. + C_k \left\{ \frac{\sin[2m\pi - Z_k]}{2[2m\pi - Z_k]} - \frac{\sin[2m\pi + Z_k]}{2[2m\pi + Z_k]} \right\} \right.$$

$I_{11n\ell}$ is analogous to I_{11mk} with the following substitutions:

$$m \leftrightarrow n \ , \qquad k \leftrightarrow \ell \ , \qquad \alpha \leftrightarrow \beta$$

$$I_{13mk} \equiv \int_o^1 \Phi_{\psi xm} \Phi'_{wk} d\alpha = \frac{Z_k}{(Z_k)^2 + (2m\pi)^2} \left\{ Z_k \cosh Z_k \sin 2m\pi \right.$$

$$\left. - 2m\pi \sinh Z_k \cos 2m\pi \right\} + Z_k \left\{ \frac{\sin[2m\pi - Z_k]}{2[2m\pi - Z_k]} - \frac{\sin[2m\pi + Z_k]}{2[2m\pi + Z_k]} \right\}$$

$$- \frac{Z_k \ C_k}{(Z_k)^2 + (2m\pi)^2} \left\{ Z_k \sinh Z_k \sin 2m\pi - 2m\pi \cosh Z_k \cos 2m\pi + 2m\pi \right\}$$

$$- Z_k \ C_k \left\{ \frac{\cos[2m\pi + Z_k]-1}{2[2m\pi + Z_k]} + \frac{\cos[2m\pi - Z_k]-1}{2[2m\pi - Z_k]} \right\}$$

$I_{13n\ell}$ is analogous to I_{13mk} with the following substitutions:

$$m \leftrightarrow n \ , \qquad k \leftrightarrow \ell \ , \qquad \alpha \leftrightarrow \beta$$

APPENDIX H

EVALUATION OF EXPERIMENTAL METHODS USED TO DETERMINE MATERIAL DAMPING IN COMPOSITE MATERIALS

H1. Specimens and Experimental Techniques

The following experimental techniques and specimen types have been used to determine material damping by experimental means (references 48, 79):

1. Decay of free vibration:
   - a. Torsional pendulum
   - b. Axially vibrating bar* (reference 95)
   - c. Beam
     - (1) Free-free* (reference 96)
     - (2) Cantilever* (references 97, 98)
2. Resonant response:
   - a. Torsional* (reference 99)
   - b. Axial
   - c. Beam
     - (1) Free-free* (references 99, 100)
     - (2) Cantilever* (references 101-104)

*Apparently only these methods have been applied to measurement of damping in composite materials; each is discussed briefly in succeeding paragraphs.

d. Cubic block

e. Plate* (reference 13)

f. Shell* (reference 105)

3. Rotating-beam deflection* (reference 106)

4. Composite oscillator

5. Ultrasonic methods

6. Thermal methods

Pottinger (reference 95) measured the temporal decay of axial vibrations of bars (Type 1b, above) of glass fiber-epoxy and boron fiber-aluminum composites in the frequency range of 1 kHz to 100 kHz.

Among the earliest uses of free vibration to determine the logarithmic decrement of a composite material were the free-free beam experiments reported by Bert et al. (reference 96). The specimens were sandwich beams with glass-epoxy facings and hexagonal-cell honeycomb cores of either aluminum or glass-phenolic. Good correlation was obtained between the measured values of δ and those predicted by an energy analysis of a Timoshenko beam, which includes thickness-shear flexibility and rotatory inertia. However, the complexity of the analysis prevents explicit determination of the δ's of the facings and core.

Later Schultz and Tsai (references 97, 98) used free flexural vibration of very thin cantilever beams to determine δ for glass-epoxy in two configurations: (1) unidirectional with the fibers at an arbitrary orientation θ with respect to the longitudinal axis of the beam, and (2) symmetrically laminated quasi-isotropic layups (layer orientations of 0°, -60°, 60°, -60°, 0° and of 0°, 90°, 45°, -45°, -45°, 45°, 90°, 0°). In the $\theta \neq 0^\circ$, 90° case, the coupling between bending and twisting significantly invalidated the simple flexural theory used in the data reduction.

Adams et al. (reference 99) used torsional and flexural resonant response to determine the damping of glass fiber-epoxy and carbon fiber-epoxy. The magnetic-coil driver was located at the one end of the test bar to excite torsion, Type 2a, and at the center to excite flexure, Type 2c(1). Similar tests were conducted by Wells et al. (reference 100) on graphite-epoxy.

Cantilever beams, Type 2c(2), were used by Keer and Lazan (reference 101), Gustafson et al. (reference 102), Mazza et al. (reference 103), and Tauchert and Moon (reference 104). It is noted that Keer and Lazan used sandwich-type beams with glass reinforced plastic facings and core, and determined the resonant energy dissipated per cycle as the measure of damping, and found it to be proportional approximately to the square of the stress amplitude, i.e. Kimball-Lovell behavior (see Appendix B).

Clary (ref. 13) conducted resonant response tests on long free-edge plates (Type 2e) made of unidirectional boron fiber-epoxy. He used the modified Kennedy-Pancu method of data reduction, and studied the effect of fiber orientation. The analytical results of the present study are compared with his experimental results in Section 5.3.

Bert and Ray (reference 105) carried out resonant response measurements on a free-edge, circular, truncated conical shell (Type 2f) with an aluminum honeycomb core and glass fiber-epoxy facings. They used the original Kennedy-Pancu data reduction method.

Richter (reference 106) used the rotating-beam deflection technique (Type 3) to determine the damping characteristics of glass fiber-epoxy at low frequency (0.01 to 1.67 Hz).

It is apparent that the most popular techniques that have been used

in connection with measurement of composite-material damping are the free-vibration decay and resonant-response methods. Differences in results obtained by these two methods have been studied analytically by Parke (reference 73), using a standard linear solid, and by Heller and Nederveen (reference 107), using a generalized Maxwell model.

H 2. Excitation and Data-Reduction Techniques for Modal Resonant Response of Complicated Structures

The experimental and data-reduction techniques for determining damping from free-vibration decay are rather straight forward. In contrast, a great variety of techniques are in use for modal response at resonance, especially in regard to data reduction.

One approach is to use multiple shakers located at various locations on the structure and having controlled force amplitudes and phase relationships. This approach was origated by Lewis and Wrisley (reference 108) and has been used and extended by Fraeijs de Veubeke (reference 109), Asher (references 110,111),Traill-Nash et al (references 112,113), Hawkins et al (references 114,115), and de Vries (reference 116). The major disadvantage of this approach is the complexity and expense, in time and cost, of installing multiple shakers.

Another approach is to use a single excitation, even when the structure is complicated. In conjunction with this method, there are a variety of resonance-determination methods; these are discussed in the paragraphs which follow. Comparative critical evaluations of a number of these methods have been presented by Bishop and Gladwell (reference 117), Pendered and Bishop (reference 41), and Pendered (reference 118).

Peak-Amplitude Method. - In this simple, classical method, resonance is defined very simply to occur when the amplitude response reaches a peak. The major disadvantage of this method is that it is limited to lightly damped systems without closely spaced frequencies. The reason for this is that both of two closely-spaced-frequency modes may contribute to the amplitude at an intermediate frequency resulting in an indication of only one mode (a pseudo mode, of course). In an analytical study of a typical system with two natural frequencies, closely spaced, Turner (reference 119) showed that the peak-amplitude method results in a 130% error in determination of the resonant amplitude.

In general, there is no direct means of determining the modal damping, i.e. the magnitude of damping associated with various vibrational modes. However, it can be determined from the response at the anti-resonant frequencies, using the equations derived by Brann et al (reference 120). Also, for the special case of a base-excited system, the ratio of the tip-to-base amplitude to the base amplitude is related to the damping, as was shown in references 121,122.

Peak-Quadrature-Component Method. - This method was originated by Stahle and Forlifer (reference 123) in 1958. In this method, resonance is defined to occur when the quadrature component (90^{o} out of phase with the excitation) of the response peaks in either a positive or negative direction. Damping is determined from the in-phase-component peaks which occur at a frequency below resonance and another frequency above resonance. Applications of the method are discussed in references 124-126.

Quadrature-Response Method. - In this method, resonance is defined to occur when the in-phase component of the response vanishes. The errors associated with this method have been discussed by Pendered (ref. 118).

Kennedy-Pancu Method and Its Modifications. - This method was originated by Kennedy and Pancu (reference 87) in 1947. Its basis is the geometry of the theoretical Argand diagram for a single-degree-of-freedom damped system, which is a circle (see figure H-1). The data-reduction procedure is to fair a circle through an Argand diagram obtained from experimental data corresponding to a range of frequencies. The tacit assumptions inherent in the method are:

1. The system is a lightly damped, linear system.

2. In the vicinity of a particular resonance, the extraneous contributions from the off-resonant modes are either negligible or invariant with respect to frequency.

3. There is no damping between the normal modes, i.e. the inertia, damping and stiffness matrices are all diagonal.

Advantages of the method are:

1. It permits resonant frequencies, modal amplitudes, and modal damping values all to be determined from the Argand plot.

2. It has been demonstrated on a comparative basis that the KP (Kennedy-Pancu) has better capability in separating closely spaced resonant frequencies.

In the KP method, two ways to determine the resonant frequencies are possible. One way is to use the frequency associated with the point (on the faired-in circle) which is most distant from the real axis. The other way is to use the maximum frequency spacing technique, in which resonance is defined to occur when the rate of change of the arc length (of the Argand plot) with respect to frequency attains a local maximum.

In the original Kennedy-Pancu method, the damping ratio was determined from the half-power points, which are two ends of the diameter (of the

Argand circle) which is parallel to the real axis. See equation (B-56). Pendered and Bishop (reference 88) introduced an improved method, based upon the change in phase with respect to frequency, evaluated at resonance. See equation (B-69).

Gladwell (reference 127) proposed a method of determining the true response peaks and their associated damping ratios, using three points on the frequency-response curve. However, Pendered and Bishop (ref. 88) showed that variations of the two side-points caused changes of 135% in the damping coefficient of the system that they investigated.

Keller (reference 128). Smart (reference 129), and Pallett (reference 130), discussed improvements in instrumentation used in conjunction with the KP method. In his analytical investigation of a typical two-degree-of freedom system with closely spaced natural frequencies, Turner (ref. 119) indicated a 26% error in resonant amplitude determination, which is a considerable improvement over the 130% error for the peak-amplitude method. The modal-shape-determination aspects of the KP method were discussed in reference 131.

The KP method has been applied successfully to a great variety of vibrating systems, including:

1. Aircraft ground and flight testing (ref. 87)
2. Beam-type structures (ref. 88)
3. Two-degree-of-freedom system (ref. 120)
4. Liquid-propellant launch-vehicle axial vibration (refs. 132,133)
5. Unstiffened, stringer-stiffened, and ring-stiffened sandwich cylindrical shells(refs. 103,130,134,135).
6. Sandwich conical shell with composite-material facings (refs. 102-104)

7. Composite-material plates (ref. 13).

The last three categories listed above are particularly useful applications of the mode-separating capability of the KP method, since they often have numerous closely spaced natural frequencies. Pendered and Bishop (reference 136) used the KP method in connection with determination of the dynamic characteristics of a sub-system from resonance test results on a complete system consisting of two sub-systems.

Woodcock (reference 137) extended the KP method by providing for any arbitrary amount of damping of the classical viscous type and by making no assumptions about the form of the damping, stiffness, and inertia matrices. However, as pointed out by Nissim (reference 138), it is necessary that the inertia matrix be known a priori in order to determine experimentally the complete dynamic description of the system. (It is noted here that the inertia matrix could be determined experimentally, at the cost of additional tests, by the displaced-frequency method, reference 139). Nissim went on to present a set of data-reduction equations with which the disadvantages of ref. 138 are eliminated. It should be pointed out that apparently no actual applications of the methods of refs. 137 and 138 have been documented so far.

H3. New Equations for Applying Two Different Versions of the Kennedy-Pancu Method to Two More Realistic Classes of Solids

From linear vibration theory, it is well known that the largest value of the damping ratio ζ for which free vibrations can occur is $\zeta=\zeta_{cr}=1$; at higher values of ζ, the motion is nonperiodic. For a standard linear solid, Zener (reference 140) showed that the critical loss tangent is 53°, which corresponds to approximately 1.183 for g_{cr}. (This compares with a

value of 2.00 for g_{cr} of a Kelvin-Voigt or Kimball-Lovell solid, since equation (B-65) is assumed to hold for such materials). Parks (ref. 73 showed the error resulting from the use of equation (B-75) for a standard linear solid.

As mentioned previously in the preceding section, in applying the Kennedy-Pancu method, it is tacitly assumed that the inertia, damping and stiffness matrices are all diagonal. Although this assumption may be satisfied in the case of a simple structure made of one material, it is certainly not justified in the case of a laminated composite-material structure. However, if the damping is sufficiently small that it does not affect the modal shapes (although, of course, it affects the modal amplitudes) and the intermodal damping coupling is negligible, then the KP method should still be sufficiently accurate to justify its use for engineering purposes.

It was mentioned in Section B3 that, in order for two half-power points to exist for a Kimball-Lovell material, the damping ratio ζ must be less than $\sqrt{2}/2$. However, when the damping is greater than this value, one half-power point (denoted as ω_2) still exists. Thus, the method can be modified to accommodate this by using equation (B-55) re-written as follows:

$$\zeta = (1/2)\left[(\omega_2/\omega_n)^2 - 1\right] \qquad \text{(H-1)}$$

Just as the modification of the original KP method as described above, equation (H-1), permits its use regardless of the amount of damping, other appropriate equations can be derived to extend the use of the method to linear systems composed of materials represented by other damping models.

Here we present original equations for two cases: the standard linear solid (or first-degree hereditary mode l) and the Biot model. Since these models contain two parameters rather than one as in the cases of viscous and Kimball-Lovell damping, more experimental data will be necessary, of course.

Standard Linear Solid (or First-Degree Hereditary Model). - Volterra (ref. 83) presented an analysis of the sinusoidally forced vibration of a single-degree-of-freedom system exhibiting first-degree hereditary damping, which is equivalent to a standard linear solid. The equation of motion can be written in the present notation as follows:

$$m\ddot{u} + ku + h\int_0^t e^{-\alpha_1(t-\tau)}[du(\tau)/d\tau]d\tau = F_o \sin \omega t \qquad \text{(H-2)}$$

or

$$m\dddot{u} + \alpha_1 m\ddot{u} + (k+h)\dot{u} + \alpha_1 ku = F_o(\omega \cos \omega t + \alpha_1 \sin \omega t) \qquad \text{(H-3)}$$

The steady-state-response solution of equation (H-3) is of the form:

$$u = u_o \sin (\omega t - \varphi) \qquad \text{(H-4)}$$

where

$$u_o \equiv (F_o/m)\left\{[\alpha^2 \omega_n^2 - (\alpha_1^2 - H^2 - \omega_n^2)\omega^2 - \omega^4]^2 + [\alpha_1 H^2 \omega]^2\right\}^{1/2}$$

$$\cdot \left\{\alpha_1^2 (\omega_n^2 - \omega^2)^2 + \omega^2(\omega^2 - \omega_n^2 - H^2)^2\right\}^{-1} \qquad \text{(H-5)}$$

$$\varphi \equiv \arctan\left\{\alpha_1 H^2 \omega\, [\alpha_1^2 \omega_n^2 - (\alpha_1^2 - H^2 - \omega_n^2)\omega^2 - \omega^4]^{-1}\right\} \qquad \text{(H-6)}$$

$$H^2 \equiv h/m \quad , \quad \omega_n^2 \equiv k/m \quad .$$

Equation (H-5) can be written in dimensionless form as follows:

$$MF = \left\{ [\bar{\alpha}^2 - (\bar{\alpha}^2 - \bar{H}^2 - 1)\Omega^2 - \Omega^4]^2 + (\bar{\alpha}\,\bar{H}^2\Omega)^2 \right\}^{1/2}$$

$$\cdot \left\{ \bar{\alpha}^2 \,(1 - \Omega^2)^2 + \Omega^2(\Omega^2 - 1 - \bar{H}^2)^2 \right\}^{-1} \qquad \text{(H-7)}$$

where

$$\bar{\alpha} \equiv \alpha_1/\omega_n \quad , \quad \bar{H} \equiv H/\omega_n \quad .$$

At resonance defined by $\omega = \omega_n$ $(\Omega=1)$, the resonant magnification factor is:

$$RMF = [1 + (\bar{\alpha}\bar{H})^2]^{1/2}/\bar{H}^3 \qquad \text{(H-8)}$$

At the half-power points

$$HPMF = RMF/\sqrt{2} \qquad \text{(H-9)}$$

Combining equations (H-7, H-8, H-9), we obtain the following result:

$$\bar{K}\,[\bar{\alpha}^2(1-\Omega_{hp}^2)^2 + \Omega_{hp}^2(\Omega_{hp}^2 - 1 - \bar{H}^2)^2]^2 = [\bar{\alpha}^2 - (\bar{\alpha}^2 - \bar{H}^2 - 1)\Omega_{hp}^2$$

$$- \Omega_{hp}^4]^2 + (\bar{\alpha}\bar{H}^2)^2\,\Omega_{hp}^2 \qquad \text{(H-10)}$$

where

$$\bar{K} \equiv [1 + (\bar{\alpha}\,\bar{H})^2]\,(2\,\bar{H}^6)^{-1}$$

Unfortunately, equation (H-10) is quartic in Ω_{hp}^2 and thus is very unwieldy for practical use.

To develop equations for the Pendered improvement of the KP method for a standard linear solid, we diffrentiate equation (H-6) with respect to time, obtaining the following result:

$$d\varphi/d\omega = \alpha^2 h^2 \omega \left\{ [\alpha^2 \omega_n^2 - (\alpha^2 - h^2 - \omega_n^2)\omega^2 - \omega^4]^2 + \alpha^2 b^4 \omega^2 \right\}^{-1} \qquad \text{(H-11)}$$

At resonance defined by $\omega = \omega_n$ and denoted by subscript R, we can simplify equation (H-11) as follows:

$$(d\varphi/d\omega)_R = \left\{ [1 + (\omega_n/\alpha)^2]\omega_n \right\}^{-1}$$

or

$$\boxed{\alpha = \omega_n \left\{ [\omega_n (d\varphi/d\omega)_R]^{-1} - 1 \right\}^{-1/2}} \qquad \text{(H-12)}$$

It is interesting to note that at resonance, the parameter h drops out, so that α can be determined directly from equation (H-12). To determine h, it is necessary to measure $d\varphi/d\omega$ at some other frequency (say ω') and then use equation (H-11), with α known from equation (H-12). The second frequency ω' should not be too close to ω_n to avoid inaccuracy. However, it should not be too far from ω_n that the MF has dropped down to such a low value that signal-to-noise-ratio difficulties develop.

Biot Solid. - Caughey (ref. 60) presented an analysis of the sinusoidally forced vibration of a single-degree-of-freedom system with Biot damping. In the present notation, the equation of motion is:

$$m\ddot{u} + ku - g_1 k \int_0^t Ei\,[-\epsilon(t-\tau)]\,(du/d\tau)\,d\tau = \tilde{F}\sin\omega t \qquad \text{(H-13)}$$

where g_1 and ϵ are parameters of the model.

The steady-state-response solution of equation (H-13) is of the same form as equation (H-4), where here

$$\tilde{u} \equiv (\tilde{F}/k)\left\{[1+g_1 \ell n \sqrt{1+(\Omega/\mu)^2} - \Omega^2]^2+[g_1 \arctan(\Omega/\mu)]^2\right\}^{-\frac{1}{2}} \qquad \text{(H-14)}$$

$$\varphi \equiv \arctan\,(\pi/2) g_1 [1+(g_1/2) \ell n\,(1/\mu) - \Omega^2]^{-1} \qquad \text{(H-15)}$$

where

$$\Omega \equiv \omega/\omega_n \,, \quad \mu \equiv \epsilon/\omega_n \,.$$

Equation (H-14) may be rewritten in terms of the magnification factor as follows:

$$MF = \left\{[1+g_1 \ell n \sqrt{1+(\Omega/\mu)^2} - \Omega^2]^2 + [g_1 \arctan\,(\Omega/\mu)]^2\right\}^{-1/2} \qquad \text{(H-16)}$$

At resonance defined by $\omega = \omega_n$ (i.e. $\Omega = 1$), the resonant magnification factor is:

$$\boxed{RMF = g_1^{-1}\,[(\ell n \sqrt{1+\mu^{-2}})^2 + (\arctan \mu^{-1})^2]^{-1/2}} \qquad \text{(H-17)}$$

It is noted that the RMF defined in this way is not the peak MF, but it is very close to it in practical systems. Also, t is emphasized that the RMF is a function of <u>both</u> parameters, g and μ. Thus, data from some other frequency is required. At the half-power points (subscript hp)

defined by HPMF = RMF/$\sqrt{2}$, the following relation must hold:

$$\boxed{\left[1+g_1 \ell n \sqrt{1+(\Omega_{hp}/\mu)^2} - \Omega_{hp}^2\right]^2 + \left[g_1 \arctan\ (\Omega_{hp}/\mu)\right]^2 = 2/(RMF)^2} \qquad \text{(H-18)}$$

Using the necessary experimental data, equations (H-17) and (H-18) can be used to determine the Biot material parameters g and μ.

To derive Pendered type KP relations, we take the derivative of equation (H-15) with respect to time, with the following result:

$$\boxed{d\varphi/d\omega \approx -(\pi g_1 \Omega/\omega_n) \left\{\left[1-\Omega^2+(g_1/2)\ell n\ \mu^{-1}\right]^2 + (\pi/2)^2 g_1^2\right\}^{-1}} \qquad \text{(H-19)}$$

At resonance defined by $\Omega = 1$ and denoted by subscript R, we can simplify equation (H-19) to the following result:

$$\boxed{(d\varphi/d\omega)_R = -(4/g_1 \omega_n)\left[1+(\pi^{-1}\ \ell n\ \mu^{-1})^2\right]^{-1}} \qquad \text{(H-20)}$$

Since equation (H-20) contains both parameters (g_1 and μ), it is necessary to measure $d\varphi/d\omega$ at some other frequency (say Ω') and then use equations (H-19) and (H-20) to solve for the two parameters.

APPENDIX I

DISCUSSION ON SYMMETRY OF THE ARRAY OF CONSTITUTIVE COEFFICIENTS

In the analyses made in the body of this report, we have assumed that the arrays of both the elastic coefficients and the damping coefficients are symmetric, i.e.

$$Q_{\ell k} = Q_{k\ell}$$

$$\hat{Q}_{\ell k} = \hat{Q}_{k\ell}$$

In the present appendix, we discuss both classical and modern thinking, as well as experimental results which are pertinent to this topic of symmetry of the constitutive-coefficient arrays. In recent years, the validity of the symmetry hypothesis has been questioned for a wide variety of materials such as soil, geological formations, laminated wood, fabrics subjected to membrane loads, and certain composite materials (references 141-144).

I1. Elastic Coefficients

More than a century ago, there was considerable controversy regarding the symmetry of the stiffness matrix (or the compliance matrix) of a perfectly elastic material. About a century ago, the argument was supposedly resolved by the brilliant mathematical analysis due to George Green and

the experimental work of Woldemar Voigt; see reference 145. Although Green's analysis was purely mathematical, its importance to modern energy methods of static and dynamic stress analysis was recognized by Love (reference 146), who stated that symmetry of the elastic coefficient array is a necessary condition for the existence of the strain energy function. Voigt verified that symmetry did hold, within experimental accuracy, for the wide variety of single crystals which he investigated. Thus, this hypothesis became well established in anisotropic elasticity theory and it was tacitly assumed to hold for all materials.

Incidentally symmetry of the compliance matrix array requires that the following so-called reciprocal relationship must hold among four of the engineering elastic properties:

$$\nu_{\ell k}/E_{\ell} = \nu_{k\ell}/E_{k} \quad (k,\ell=1,2,3;\ \text{no sum}) \tag{I-1}$$

where the E's are Young's moduli and the ν's are Poisson's ratios.

Recently Alley and Faison (ref. 143) clearly demonstrated experimentally that equation (I-1) does not hold for fabrics subjected to biaxial loadings. They attributed this result to fiber friction, which is a nonconservative process. Table III lists some filamentary-composite-material test values for which eq. (I-1) does not hold. The viscosity of the epoxy matrix may have been the nonconservative process operative in the case of the glass-epoxy and boron-epoxy composites; see Section I2 of this Appendix. Also, the glass was in fabric form, so fabric friction could have played a part. Furthermore, in the case of the boron-epoxy composite, the minor Poisson's ratio is so very small that it is difficult to measure

with a high degree of confidence and experimental scatter effects become highly magnified. However, none of these explanations account for the case of graphite-graphite composite. It does not creep at the test temperature (room temperature); the fibers were filament wound, not in fabric form; and the minor Poisson's ratio is large enough to be measured accurately (especially since strain-gage transverse sensitivity was provided for in the data reduction). Thus, so far, there has not been a satisfactory explanation for this anomaly.

I2. Viscoelastic Material

For viscoelastic materials which are homogeneous and anisotropic on a macroscopic basis, Rogers and Pipkin (ref. 141) claimed that there are no thermodynamic bases which require symmetry of the array of constitutive coefficients. Thus, they indicated that for such materials, symmetry can be established by experiment only. For composite materials having a viscoelastic (epoxy resin) matrix, some experimental data suggest that symmetry may not hold in the case of reinforcement by glass fibers and boron fibers (see table III). However, other experimental data on a viscoelastic composite (nylon-fiber-reinforced rubber) suggest that symmetry does hold.

In view of the lack of a definite proof regarding symmetry of a viscoelastic material, symmetry is assumed to hold in the present investigation (see hypothesis H6, Section 2.1).

APPENDIX J

COMPUTER PROGRAM DOCUMENTATION

The program described in this appendix was programmed to accomplish the following three computations:

1. Calculate the shear factor K for laminates, using Jourawski static shear theory.
2. Calculate the lowest eigenvalue for a simply supported laminated plate without damping.
3. Calculate the amplitude frequency response and modified Kennedy-Pancu frequency response and damping data for a free-edge anisotropic plate with material damping.

Computation 1 was accomplished by using an explicit algebraic expression. Computation 2 was performed by using IBM System/360 Scientific Subroutine Packages NROOT and EIGEN. Computation 3 was performed by Package SMIQ. A complete description of the variables, operations, etc. may be found in IBM Manual 360A-CM-03X, version III, for the subroutines NROOT, EIGEN and SMIQ.

The program was written in FORTRAN IV language as prescribed in IBM System Reference Library Form C-28-6274-3.

The input-data deck was set up as follows:

Computation 1 -

(a) Thickness of each layer (lower and upper limit)

(b) Elastic and shear moduli

Computation 2 -

(a) Plate geometry

(b) Lamination geometry (specially orthotropic)

(c) Moduli and Poisson's ratio data for each ply

(d) Density for each ply

(e) Shear factor (as calculated in Computation 1)

Computation 3 -

(a) Young's and shear moduli for each layer

(b) Poisson's ratios for each layer

(c) Bending and twisting stiffnesses for each layer

(d) Loss tangents corresponding to moduli, Poisson's ratios, and stiffnesses for each layer

(e) Plate geometry

(f) Lamination geometry, including angle of orientation for each layer

(g) Density of each layer

(h) Shear factor (as calculated in Computation 1)

(i) Mode numbers of the assumed modes.

A complete listing of the computer program is presented at the end of this report.

REFERENCES

1. Engler, E.E.; and Cataldo, C.E.: Design and Test of Advanced Structural Components and Assemblies. NASA Space Shuttle Technology Conference, vol. II, NASA TM X-2273, Apr. 1971, pp. 527-583.

2. Ashton, J.E.; and Whitney, J.M.: Theory of Laminated Plates. Technomic Publishing Co., Inc., 1970.

3. Ojalvo, I.U.; and Arcas, N.: Approximate Analysis and Dynamic Tests for a Thermal Protection System Panel. NASA Space Shuttle Technology Conference, vol. III, NASA TM X-2274, Apr. 1971, pp. 49-93.

4. Ashton, J.E.; Halpin, J.C.; and Petit, P.H.: Primer on Composite Materials: Analysis. Technomic Publishing Co., Inc., 1969.

5. Tsai, S.W.: Structural Behavior of Composite Materials. NASA CR-71, July 1964.

6. Chambers, R.E.; and McGarry, F.J.: Shear Effects in Glass Fiber Reinforced Plastics Laminates. ASTM Bull., no. 238, May 1959, pp. 38-41.

7. Khischenko, Yu. M.: Effect of Shear on the Modulus of Elasticity of Specimens of Glas-Fiber Reinforced Plastics Tested in Transverse Bending. Indust. Lab., vol. 30, no. 6, June 1964, pp. 937-939.

8. Tarnopol'skii, Yu. .; Roze, A.V.; and Polyakov, V.A.: Shear Effects During Bending of Oriented Glass Reinforced Plastics. Polymer Mechanics,

vol. 1, no. 2, Mar.-Apr. 1965, pp. 31-37.

9. Yu., Y.-Y.: Simple Thickness-Shear Modes of Vibration of Infinite Sandwich Plates. J. Appl. Mech., vol. 26, no. 4, Dec. 1959, pp. 679-681.

10. Ross, D.; Ungar, E.E.; and Kerwin, E.M., Jr.: Damping of Plate Flexural Vibrations by Means of Viscoelastic Laminae. Structural Damping, J.E. Ruzicka, ed., Amer. Soc. Mech. Engrs., 1959, pp. 49-87.

11. Ruzicka, J.E.; Derby, T.F.; Schubert, D.W.; and Pepi, J.S.: Damping of Structural Composites with Viscoelastic Shear-Damping Mechanisms. NASA CR-742, Mar. 1967.

12. Derby, T.F.; and Ruzicka, J.E.: Loss Factor and Resonant Frequency of Viscoelastic Shear-Damped Structural Composites. NASA CR-1269, Feb. 1969.

13. Clary, R.R.: Vibration Characteristics of Unidirectional Filamentary Composite Material Panels. Composite Materials: Testing and Design (Second Conference), Amer. Soc. Testing & Matls., Spec. Tech. Publ. 497, 1972, pp. 415-438.

14. Pister, K.S.: Flexural Vibration of Thin Laminated Plates. J. Acoust. Soc. Amer., vol. 31, no. 2, Feb. 1959, pp. 233-234.

15. Stavsky, Y.: Thermoelastic Vibrations of Heterogeneous Membranes and Inextensional Plates. AIAA J., vol. 1, no. 3, Mar. 1963, pp. 722-723.

16. Ashton, J.E.: Analysis of Anisotropic Plates. J. Composite Matls., vol. 3, no. 1, Jan. 1969, pp. 148-165.

17. Hikami, Y.: Transverse Vibrations of Laminated Orthotropic Plates. Rept. No. 32, Div. of Solid Mech., Structures, & Mech. Design, Case Western Reserve Univ., Apr. 1969.

18. Ashton, J.E.: Natural Modes of Free-Free Anisotropic Plates. Shock & Vib. Bull., Bull. 39, Pt. 4, U.S. Dept. Defense, Apr. 1969, pp. 93-97.

19. Whitney, J.M.; and Leissa, A.W.: Analysis of Heterogeneous Anisotropic Plates. J. Appl. Mech., vol. 36, no. 2, June 1969, pp. 261-266.

20. Whitney, J.M.; and Leissa, A.W.: Analysis of a Simply Supported Laminated Anisotropic Rectangular Plate. AIAA J., vol. 8, no. 1, Jan. 1970, pp. 28-33.

21. Ashton, J.E.; and Anderson, J.D.: The Natural Modes of Vibration of Boron-Epoxy Plates. Shock & Vib. Bull., Bull. 39, Pt. 4, U.S. Dept. Defense, Apr. 1969, pp. 81-91.

22. Bert, C.W.; and Mayberry, B.L.: Free Vibrations of Unsymmetrically Laminated Anisotropic Plates with Clamped Edges. J. Composite Matls., vol. 3, no. 2, Apr. 1969, pp. 282-293.

23. Ambartsumyan, S.A.: Theory of Anisotropic Plates. Technomic Publishing Co., Inc., 1970.

24. Whitney, J.M.: The Effect of Transverse Shear Deformation on the Bending of Laminated Plates. J. Composite Matls., vol. 3, no. 3,

July 1969, pp. 534-547.

25. Yang, P.C.; Norris, C.H.; and Stavsky, Y.: Elastic Wave Propagation in Heterogeneous Plates. Int. J. Solids & Struct., vol. 2, no. 4, Oct. 1966, pp. 665-684.

26. Mindlin, R.D.: Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates. J. Appl. Mech., vol. 18, no. 1, Mar. 1951, pp. 31-38.

27. Biot, M.A.: Continuum Dynamics of Elastic Plates and Multilayered Solids Under Initial Stress. J. Math. & Mech., vol. 12, no. 6, Nov. 1963, pp. 793-803.

28. Bolotin, V.V.: Vibration of Layered Elastic Plates. Proc. Vib. Probs. (Pol. Acad. Sci.), vol. 4, no. 4, 1963, pp. 331-345.

29. Postma, G.W.: Wave Propagation in a Stratified Medium. Geophysics, vol. 20, no. 4, Oct. 1955, pp. 780-806.

30. White, J.E.; and Angona, F.A.: Elastic Wave Velocities in Laminated Media. J. Acoust. Soc. Amer., vol. 27, no. 2, Mar. 1955, pp. 310-317.

31. Rytov, S.M.: Acoustical Properties of a Thinly Laminated Medium. Sov. Phys.-Acoust., vol. 2, no. 1, Jan. 1956, pp. 68-80.

32. Achenbach, J.D.; and Zerbe, T.R.: Flexural Vibrations of Laminated Plate. J. Eng. Mech. Div., Proc. ASCE, vol. 97, no. EM3, June 1971, pp. 619-628.

33. Hsu, T.M.; and Wang, J.T.: A Theory of Laminated Cylindrical Shells Consisting of Layers of Orthotropic Laminae. AIAA J., vol. 8, no. 12,

Dec. 1970, pp. 2141-2146.

34. Wang, J.T.-S.: Axisymmetric Deformation Theory of Orthotropic Layered Cylindrical Shells. Developments in Mechanics, vol. 6 (Proc. 12th. Midwestern Mech. Conf., Aug. 1971), Univ. of Notre Dame Press, 1971, pp. 787-800.

35. Srinivas, S.; Joga Rao, C.V.; and Rao, A.K.: An Exact Analysis for Vibration of Simply-Supported Homogeneous and Laminated Thick Rectangular Plates. J. Sound & Vib., vol. 12, no. 2, June 1970, pp. 187-200.

36. Srinivas, S.; Joga Rao, C.V.; and Rao, A.K.: Some Results from an Exact Analysis of Thick Laminates in Vibration and Buckling. J. Appl. Mech., vol. 37, no. 3, Sept. 1970, pp. 868-870.

37. Srinivas, S.; and Rao, A.K.: Bending, Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates. Int. J. Solids & Structures, vol. 6, no. 11, Nov. 1970, pp. 1463-1481.

38. Tso, F.K.W.; Dong, S.B.; and Nelson, R.B.: Natural Vibrations of Rectangular Laminated Orthotropic Plates. Developments in Mechanics, vol. 6 (Proc. 12th. Midwestern Mech. Conf., Aug. 1971), Univ. of Notre Dame Press, 1971, pp. 891-905.

39. Dong, S.B.: Behavior of Laminated Orthotropic Viscoelastic Plates. J. Spacecraft & Rkts., vol. 4, no. 10, Oct. 1967, pp. 1385-1388.

40. Timoshenko, S.P.: History of Strength of Materials. McGraw-Hill

Book Co., Inc., 1953, p. 141.

41. Timoshenko, S.: On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars. Phil. Mag., ser. 6, vol. 41, 1921, pp. 744-746.

42. Timoshenko, S.: Vibration Problems in Engineering, 3rd. Ed. D. Van Nostrand Co., Inc., 1955, p. 329.

43. Reissner, E.: The Effect of Transverse-Shear Deformation on the Bending of Elastic Plates. J. Appl. Mech., vol. 12, no. 1, Mar. 1945, pp. A69-A77.

44. Young, E.; and Felgar, R.P., Jr.: Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam. Bulletin No. 4913, Bureau of Engineering Research, University of Texas, Austin, Tex., July 1, 1949.

45. Wang, J.T.: On the Solution of Plates of Composite Materials. J. Composite Matls., vol. 3, no. 3, July 1969, pp. 590-592.

46. Bert, C.W.; and Chang, S.: In-Plane, Flexural, Twisting and Thickness-Shear Coefficients for Stiffness and Damping of a Monolayer Filamentary Composite. Final Report (Part I) on NASA Research Grant NGR-37-003-055, School of Aerospace, Mechanical and Nuclear Engineering, University of Oklahoma, Norman, Okla., Mar. 1972.

47. Ganov, E.V.: Investigation of Bending Stresses in Shingle-Laminated Glass Reinforced Plastic. Polymer Mech., vol. 1, no. 5, Sept.-Oct.

1965, pp. 101-102.

48. Lazan, B.J.: Damping of Materials and Members of Structural Mechanics. Pergamon Press, 1968.

49. Kimball, A.L.; and Lovell, D.E.: Internal Friction in Solids. Physical Rev., ser. 2, vol. 30, 1927, pp. 948-959.

50. Wegel, R.L.; and Walther, H.: Internal Dissipation in Solids for Small Cyclic Strains. Physics, vol. 6, 1935, pp. 141-157.

51. Lazan, B.J.: A Study with New Equipment of the Effects of Fatigue Stress on the Damping Capacity and Elasticity of Mild Steel. Trans. Am. Soc. Metals, vol. 42, 1950, pp. 499-549.

52. Bishop, R.E.D.: The Theory of Damping Forces in Vibration Theory. J. Roy. Aeron. Soc., vol. 59, no. 539, Nov. 1955, pp. 738-742.

53. Soroka, W.W.: Note on the Relations Between Viscous and Structural Damping Coefficients. J. Aeron. Sci., vol. 16, no. 7, Jul. 1949, pp. 409-410, 448.

54. Theodorsen, T.; and Garrick, I.E.: Mechanism of Flutter - a Theoretical and Experimental Investigation of the Flutter Problem. NACA TR 685, 1940.

55. Scanlan, R.H.; and Rosenbaum, R.: Introduction to the Study of Aircraft Vibration and Flutter. Macmillan, 1951; Dover, 1968.

56. Myklestad, N.O.: The Concept of Complex Damping. J. Appl. Mech., vol. 19, no. 3, Sep. 1952, pp. 284-286.

57. Lancaster, P.: Free Vibration and Hysteretic Damping. J. Roy. Aeron. Soc., vol. 64, no. 592, Apr. 1960, p. 229.

58. Reid, T.J.: Free Vibration and Hysteretic Damping. J. Roy. Aeron. Soc., vol. 60, no. 544, Apr. 1956, p. 283.

59. Milne, R.D.: A Constructive Theory of Linear Damping. Paper C.4, Symposium on Structural Dynamics, Loughborough Univ. of Technology, Longhborough, England, 1970.

60. Caughey, T.K.: Vibration of Dynamic Systems with Linear Hysteretic Damping (Linear Theory). Proc. 4th. U.S. Nat. Congr. Appl. Mech., vol. 1, 1962, pp. 87-97.

61. Paley, R.E.A.C.; and Wiener, N.: Fourier Transforms in the Complex Domain. Am. Math. Soc. Colloquium Publicat., vol. 19, 1934.

62. Garcia-Moliner, F.; and Thomson, R.: Linear Response Functions and the Phenomenological Equations of Internal Friction. J. Appl. Phys., vol. 37, no. 1, Jan. 1966, pp. 83-89.

63. Fraeijs de Veubeke, B.M.: Influence of Internal Damping on Aircraft Resonance. AGARD Manual on Elasticity, vol. 1, chap. 3, 1960.

64. Crandall, S.H.: Dynamic Response of Systems with Structural Damping. Air, Space, and Instruments (C.S. Draper Anniv. Vol.), S. Lees, ed. McGraw-Hill Book Co., 1963, pp. 183-193.

65. Crandall, S.H.: The Role of Damping in Vibration Theory. J. Sound & Vib., vol. 11, no. 1, Jan. 1970, pp. 3-18.

66. Pinsker, W.: Structural Damping. J. Aeron. Sci., vol. 16, no. 11, Nov. 1949, p. 699.

67. Scanlan, R.H.; and Mendelson, A.: Structural Damping. AIAA J., vol. 1, no. 4, Apr. 1963, pp. 938-939.

68. Naylor, V.D.: Some Fallacies in Modern Damping. J. Sound & Vib., vol. 11, no. 2, Feb. 1970, pp. 278-280.

69. Scanlan, R.H.: Linear Damping Models and Causality in Vibrations. J. Sound & Vib., vol. 13, no. 4, Dec. 1970, pp. 499-509.

70. Bishop, R.E.D.: The Behavior of Damped Linear Systems in Steady Oscillation. Aeron. Quart., vol. 7, no. 2, May 1956, pp. 156-168.

71. Cunningham, H.J.: In Defense of Modern Damping Theory in Flutter Analysis. J. Sound & Vib., vol. 14, no. 1, Jan. 1971, pp. 142-144.

72. Bland, D.R.: The Theory of Linear Viscoelasticity. Pergamon Press, 1960, pp. 3-4.

73. Parke, S.: Logarithmic Decrements at High Damping. Brit. J. Appl. Phys., vol. 17, no. 2, Feb. 1966, pp. 271-273.

74. Flügge, W.: Viscoelasticity. Blaisdell Publishing Company, 1967, p. 14.

75. Hobbs, G.K.: Methods of Treating Damping in Structures. AIAA Paper 71-349, presented at the AIAA/ASME 12th Structures, Structural Dynamics and Materials Conference, Anaheim, Calif., Apr. 19-21, 1971.

76. Huang, T.C.; and Huang, C.C.: Free Vibrations of Viscoelastic Timoshenko Beams. J. Appl. Mech., vol. 38, no. 2, June 1971, pp. 515-521.

77. Granick, N. and Stern, J.E.: Material Damping of Aluminum by a Resonance-Dwell Technique. NASA TN D-2893, Aug. 1965.

78. Granick, N.; and Stern, J.E.: Material Damping of Aluminum by Resonance-Dwell Technique. Shock & Vib. Bull., Bull. 34, Pt. 5, U.S. Dept. Defense, Feb. 1965, pp. 177-195.

79. Gustafson, A.J.; Mazza, L.T.; Rodgers, R.L.; and McIlwean, E.A.: Development of Test Methods for Measuring Damping of Fiber-Reinforced Materials. Rept. USAAVLABS TR 70-42, U.S. Army Aviation Materiel Laboratories, Ft. Eustis, Va., June 1970 (AD-873990).

80. Biot, M.A.: Linear Thermodynamics and the Mechanics of Solids. Proc. 3rd. U.S. Nat. Congr. Appl. Mech., 1958, pp. 1-18.

81. Neubert, H.K.P.: A Simple Model Representing Internal Damping in Solid Materials. Aeron. Quart., vol. 14, no. 2, May 1963, pp. 187-210.

82. Boltzmann, L.: Zur Theorie der elastische Nachwirkung. Annalen der Physik und Chemie, suppl. 7, 1876, pp. 624-654.

83. Volterra, E.: Vibrations of Elastic Systems Having Hereditary Characteristics. J. Appl. Mech., vol. 17, no.4, Dec. 1950, pp. 363-371.

84. Pisarenko, G.S.: Vibrations of Elastic Systems Taking Account of Energy Dissipation in the Material. Kiev, 1955; Engl. transl., WADD TR-60-582, Wright Air Development Div., Feb. 1962 (AD-274743).

85. Rosenblueth, E.; and Herrera, I.: On a Kind of Hysteretic Damping. J. Eng. Mech. Div., Proc. ASCE, vol. 90, no. EM4, Aug. 1964, pp. 37-48.

86. Chang, C.S.; and Bieber, R.E.: Synthesis of Structural Damping. AIAA J., vol. 6, no. 4, Apr. 1968, pp. 718-719.

87. Kennedy, C.C.; and Pancu, C.D.P.: Use of Vectors in Vibration Measurement and Analysis. J. Aeron. Sci., vol. 14, no. 11, Nov. 1947, pp. 603-625.

88. Pendered, J.W.; and Bishop, R.E.D.: A Critical Introduction to Some Industrial Resonance Testing Techniques. J. Mech. Eng. Sci., vol. 5, no. 4, Dec. 1963, pp. 345-367.

89. Kolsky, H.: Stress Waves in Solids. Oxford Univ. Press, 1953; Dover, 1963.

90. Bishop, R.E.D.: The General Theory of "Hysteretic Damping". Aeron. Quart., vol. 7, no. 1, Feb. 1956, pp. 60-70.

91. Meirovitch, L.: Analytical Methods in Vibrations. Macmillan, 1967, p. 40.

92. Rose, J.L.; and Tshirschnitz, R.: Static Material Constants for a Layered Composite Material. Paper No. 1845A, Spring Meeting,

Soc. for Exper. Stress Anal., Salt Lake City, Utah, May 18-21, 1971.

93. Voigt, W.: Lehrbuch der Kristallphysik. B.G. Teubner, Leipzig, Germany, 1928, pp. 738-741.

94. Reuss, A.: Berechnung der Fliessgrenze von Mischkristallen auf Grund der Plastizitätsbedingung für Einkristalle. Zeits. f. ang. Math. u. Mech., vol. 9, no. 1, Feb. 1929, pp. 48-58.

95. Pottinger, M.G.: Material Damping of Glass Fiber-Epoxy and Boron Fiber-Aluminum Composites. Rept. ARL 70-0237, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, Oct. 1970 (AD-721191).

96. Bert, C.W.; Wilkins, D.J., Jr.; and Crisman, W.C.: Damping in Sandwich Beams with Shear-Flexible Cores. J. Eng. Industry, Trans. ASME, vol. 89B, no. 4, Nov. 1967, pp. 662-670.

97. Schultz, A.B.; and Tsai, S.W.: Dynamic Moduli and Damping Ratios in Fiber-Reinforced Composites. J. Composite Matls., vol. 2, no. 3, July 1968, pp. 368-379.

98. Schultz, A.B.; and Tsai, S.W.: Measurements of Complex Dynamic Moduli for Laminated Fiber-Reinforced Composites. J. Composite Matls., vol. 3, no. 3, July 1969, pp. 434-443.

99. Adams, R.D.; Fox, M.A.O.; Floord, R.J.L.; Friend, R.J.; and Hewitt, R.L.: The Dynamic Properties of Unidirectional Carbon and Glass Fiber Reinforced Plastics in Torsion and Flexure. J. Composite Matls., vol. 3, no. 4, Oct. 1969, pp. 594-603.

100. Wells, H.; Colclough, W.J.; and Goggin, P.R.: Some Mechanical Properties

of Carbon Fibre Composites. Proc. 24th. Ann. Tech. & Mgt. Conf., Reinf. Plastics/Composites Div., Soc. Plastics Industry, Inc., 1969, sect. 2-C.

102. Gustafson, A.J.; Mazza, L.T.; Rodgers, R.L.; and McIlwean, E.H.: Development of Test Methods for Measuring Damping of Fiber-Reinforced Materials. Rept. USAAVLABS TR 70-42, U.S. Army Aviation Materiel Laboratories, Ft. Eustis, Va., 1970 (AD-873990).

103. Mazza, L.T.; Paxson, E.B.; and Rodgers, R.L.: Measurement of Damping Coefficients and Dynamic Modulus of Fiber Composites. Rept. USAAVLABS-TN-2, U.S. Army Aviation Materiel Laboratories, Ft. Eustis, Va., 1971 (AD-869025).

104. Tauchert, T.R.; and Moon, F.C.: Propagation of Stress Waves in Fiber-Reinforced Composite Rods. AIAA J., vol. 9, no. 8, Aug. 1971, pp. 1492-1498.

105. Bert, C.W.; and Ray, J.D.: Vibrations of Orthotropic Sandwich Conical Shells with Free Edges. Int. J. Mech. Sci., vol. 11, no. 9, Sept. 1969, pp. 767-779.

106. Richter, H.P.H.: Photographic Method for Measuring Material Damping and Dynamic Young's Modulus at Low Frequencies Applied to a Fiberglass Reinforced Resin Structure. Proc. 18th.Ann. Tech. & Mgt. Conf., Reinf. Plastics Div., Soc. Plastics Industry, Inc., 1963, sect. 4-D.

107. Heller, R.A.; and Nederveen, C.J.: Comparison of Complex Moduli Obtained from Forced and Free Damped Oscillations. Trans. Soc. Rheology, vol. 11, no. 1, 1967, pp. 113-123.

108. Lewis, A.C.; and Wrisley, D.L.: A System for the Excitation of Pure Modes of Complex Structures. J. Aeron. Sci., vol. 17, no. 11, Nov. 1950, pp. 705-722.

109. Fraeijs de Veubeke, B.M.: A Variational Approach to Pure Mode Excitation Based on Characteristic Phase Lag Theory. AGARD Rept. 39, Advisory Group for Aeron. Res. & Dev., North Atlantic Treaty Org., Apr. 1956.

110. Asher, G.W.: A Method of Normal Mode Excitation Utilizing Admittance Measurements. Proc. Nat. Specialists Meeting on Dynamics & Aeroelasticity, Ft. Worth, Tex., Nov. 6-7, 1958; Inst. of Aeron. Sci., 1959, pp. 69-76.

111. Asher, G.W.: A Note on the Effective Degrees of Freedom of a Vibrating Structure. AIAA J., vol. 5, no. 4, Apr. 1967, pp. 822-824.

112. Traill-Nash, R.W.: On the Excitation of Pure Natural Modes in Aircraft Resonance Testing. J. Aero/Space Sci., vol. 25, no. 12, Dec. 1958, pp. 775-778.

113. Traill-Nash, R.W.; Long, G.; and Bailey, C.M.: Experimental Determination of the Complete Dynamical Properties of a Two-Degree-of-Freedom Model Having Nearly Coincident Natural Frequencies. J. Mech. Eng. Sci., vol. 9, no. 5, Dec. 1967, pp. 402-413.

114. Hawkins, F.J.: An Automatic Shake Testing Technique for Exciting the Normal

Modes of Vibration of Complex Structures. AIAA Sympos. on Struct. Dynamics & Aeroelasticity, Boston, Mass., Aug. 30-Sept. 1, 1965, pp. 452-464.

115. Hawkins, F.J.; Skingle, C.W.; and Taylor, G.A.: A Multiple-Frequency, Shake-Testing Technique for Structures with Rapidly-Changing Dynamic Characteristics. Shock & Vib. Bull., Bull. 35, Pt. 2, U.S. Dept. Defense, Jan. 1966, pp. 107-116.

116. de Vries, G.: The Principles of Global Tests of a Structure. NASA TT F-10,003, Feb. 1966.

117. Bishop, R.E.D.; and Gladwell, G.M.L.: An Investigation into the Theory of Resonance Testing. Phil. Trans. Roy. Soc., London, ser. A, vol. 255, no. 1055, Jan. 17, 1963, pp. 241-280.

118. Pendered, J.W.: Theoretical Investigation into the Effects of Close Natural Frequencies in Resonance Testing. J. Mech. Eng. Sci., vol. 7, no. 4, Dec. 1965, pp. 372-379.

119. Turner, L.J., Jr.: An Analytical Investigation of a Vector Technique for Determining Normal Mode Amplitudes from Vibration Data. M.S. Thesis, Virginia Polytech. Inst., May 1968.

120. Brann, R.P., Devney, A.J.; and Robinson, C.D.: Harmonic Excitation of a Lightly Damped Multi-Degree System. Bull. Mech. Eng. Education, vol. 5, no. 2, 1966, pp. 95-111.

121. Rodden, W.P.; and Whittier, J.S.: Damping of Shaker-Excited Beams Calculated Solely from Displacement-Amplitude Measurements. J.

Acoust. Soc. Amer., vol. 34, no. 4, Apr. 1962, pp. 469-471.

122. Wambsganss, M.W., Jr.; Boers, B.L.; and Rosenberg, G.S.: Method for Identifying and Evaluating Linear Damping Models in Beam Vibrations. Shock & Vib. Bull., Bull. 36, Pt. 4, U.S. Dept. Defense, Jan. 1967, pp. 65-74.

123. Stahle, C.V.; and Forlifer, W.R.: Ground Vibration Testing of Complex Structures. Proc. AIA-AFOSR Flight Flutter Testing Sympos., May 1958.

124. Stahle, C.V.: A Phase Separation Technique for the Experimental Determination of Normal Vibration Modes of Flight Vehicles. Shock & Vib. Bull., Bull. 29, Pt. 4, U.S. Dept. Defense, 1961, pp. 30-42.

125. Stahle, C.V., Jr.: Phase Separation Technique for Ground Vibration Testing. Aerospace Eng., vol. 21, no. 7, July 1962, pp. 56-57, 91-96.

126. Hutton, F.E.: The Application of the Component Analyzer in Determining Modal Patterns, Modal Frequencies, and Damping Factors of Lightly Damped Structures. Shock & Vib. Bull., Bull. 33, Pt. 2, U.S. Dept. Defense, Feb. 1964, pp. 264-272.

127. Gladwell, G.M.L.: A Refined Estimate for the Damping Coefficient. J. Roy. Aeron. Soc., vol. 66, no. 614, Feb. 1962, pp. 124-125.

128. Keller, A.C.: Vector Component Techniques: A Modern Way to Measure Modes. Sound & Vib., vol. 3, no. 3, Mar. 1969, pp. 18-26.

129. Smart, T.E.: Dynamic Phase Plotting. Shock & Vib. Bull., Bull. 37, Pt. 2, U.S. Dept. Defense, Jan. 1968, pp. 77-85.

130. Pallett, D.S.: System for the Complex Plane Representation of Mechanical Vibration Response Parameters. Rept. TM 107.2811-06, Ord. Res. Lab., Penn. State Univ., State College, Pa., Mar. 13, 1969.

131. Pendered, J.W.; and Bishop, R.E.D.: The Determination of Modal Shapes in Resonance Testing. J. Mech. Eng. Sci., vol. 5, no. 4, Dec. 1965, pp. 379-385.

132. Schoenster, J.A.; and Clary, R.R.: Experimental Investigation of the Longitudinal Vibration of a Representative Launch Vehicle with Simulated Propellants. NASA TN D-4502, May 1968.

133. Clary, R.R.; and Turner, L.J., Jr.: Measured and Predicted Longitudinal Vibrations of a Liquid Propellant Two-Stage Launch Vehicle. NASA TN D-5943, Aug. 1970.

134. Bray, F.M.; and Egle, D.M.: An Experimental Investigation of the Vibration of Thin Cylindrical Shells with Discrete Longitudinal Stiffening. Jour. Sound & Vib., vol. 12, no. 2, June 1970, pp. 152-164.

135. Schoenster, J.A : Measured and Calculated Vibration Properties of Ring-Stiffened Honeycomb Cylinders. NASA TN D-6090, Jan. 1971.

136. Pendered, J.W.; and Bishop, R.E.D.: Extraction of Data for a Sub-System from Resonance Test Results. J. Mech. Eng. Sci., vol. 5, no. 4, Dec. 1965, pp. 368-378.

137. Woodcock, D.L.: On the Interpretation of the Vector Plots of Forced Vibrations of a Linear System with Viscous Damping. Aeron. Quart.,

vol. 14, no. 1, Feb. 1963, pp. 45-62.

138. Nissim, E.: On a Simplified Technique for the Determination of the Dynamic Properties of a Linear System with Damping. J. Roy. Aeron. Soc., vol. 71, no. 674, Feb. 1967, pp. 126-128.

139. de Vries, G.: Sondage des systemes vibrants par masses additionelles. La Recherche Aeronautique, no. 30, Nov.-Dec. 1952, pp. 47-49.

140. Zener, C.: Mechanical Behavior of High Damping Metals. J. Appl. Phys., vol. 18, no. 11, Nov. 1947, pp. 1022-1025.

141. Rogers, T.G.; and Pipkin, A.C.: Asymmetric Relaxation or Compliance Matrices in Linear Viscoelasticity. Zeits. f. angew. Math. u. Physik, vol. 14, 1963, pp. 334-343.

142. Busching, H.W.: Mechanical Behavior of Transversely Anisotropic Materials. Proc. (First) Canadian Congr. Appl. Mech., Laval Univ., Quebec, Canada, May 22-26, 1967, vol. 1.

143. Alley, V.L., Jr.; and Faison, R.W.: Decelerator Fabric Constants Required by the Generalized Form of Hooke's Law. AIAA Paper 70-1179, presented at the AIAA 3rd. Aerodynamic Deceleration Systems Conference, Wright-Patterson Air Force Base, Ohio, Sept. 14-16, 1970.

144. Bert, C.W.; and Guess, T.R.: Mechanical Behavior of Carbon-Carbon Filamentary Composites. Composite Materials: Testing and Design (Second Conference), Amer. Soc. Testing & Matls. Spec. Tech.

Publ. 497, 1972, pp. 89-106.

145. Todhunter, I.; and Pearson, K.: A History of the Theory of Elasticity and of the Strength of Materials. Dover Publications, Inc., New York, 1960.

146. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, 4th. Ed. Dover Publications, Inc., New York, 1944, chap. III.

147. Bert, C.W.; Mayberry, B.L.; and Ray, J.D.: Behavior of Fiber-Reinforced Plastic Laminates Under Biaxial Loading. Composite Materials: Testing and Design, Amer. Soc. Testing & Matls., Spec. Tech. Publ. 460, 1969, pp. 362-380.

148. Hadcock, R.N.: Boron-Epoxy Aircraft Structures. Handbook of Fiberglass and Advanced Plastics Composites, ed. by G. Lubin, Van Nostrand Reinhold, 1969, pp. 628-660.

149. Chou, P.C., Carleone, J., and Hsu, C.M.: Elastic Constants of Layered Media. J. Composite Matls., vol. 6, no. 1, Jan. 1972, pp. 80-93.

TABLE I. SHEAR FACTOR FOR LAMINATES WITH MULTIPLE ALTERNATING LAYERS OF TWO MATERIALS

Modulus ratio, $E_{11}^{(1)} / E_{11}^{(2)} = C_{55}^{(1)} / C_{55}^{(2)}$	Number of Layers, n							
	2	3	4	5	6	7	8	9
1	0.833	0.833	0.833	0.833	0.833	0.833	0.833	0.833
10	0.173	0.190	0.203	0.201	0.195	0.193	0.188	0.186
20	0.091	0.087	0.087	0.082	0.072	0.069	0.063	0.062
30	0.062	0.056	0.050	0.047	0.037	0.036	0.032	0.031
40	0.047	0.041	0.033	0.031	0.023	0.022	0.019	0.018
50	0.038	0.032	0.022	0.016	0.015	0.011	0.013	0.012
60	0.032	0.027	0.017	0.016	0.011	0.011	0.009	0.009
70	0.027	0.023	0.014	0.013	0.009	0.008	0.007	0.006
80	0.024	0.020	0.011	0.010	0.008	0.007	0.005	0.005
90	0.021	0.018	0.009	0.008	0.005	0.005	0.004	0.004
100	0.020	0.016	0.007	0.007	0.004	0.004	0.003	0.003

Table II. FREQUENCIES AND DAMPING FOR A LAMINATED COMPOSITE-MATERIAL PLATES

Angle, deg.	Mode No.	Reference 13			Present Results		
		Exp. freq., Hz.	Anal. freq., Hz.	Exp. damping ratio, %	Peak-ampl. freq., Hz.	Kennedy-Pancu freq., Hz.	Calc. damping ratio, %
0	1	141.8	138.2	0.1	138.9	140.0	0.22
	2	182.2	164.0	1.9	170.0	170.4	1.83
	3	386.2	365.4	0.9	359.4	360.1	0.97
	4	401.1	381.1	0.2	389.7	390.2	0.24
	5	649.5	639.4	1.4	639.0	639.6	0.95
10	1	100.6	93.4	0.6	93.0	92.0	1.19
	2	234.0	226.9	3.4	226.1	225.2	1.28
	3	275.2	257.9	0.9	264.0	265.5	1.06
	4	485.3	497.5	1.6	495.2	495.0	0.09
	5	523.4	512.8	0.9	512.3	511.0	0.66
30	1	57.2	56.1	1.0	57.7	58.7	1.37
	2	163.4	164.2	1.0	166.1	165.0	0.82
	3	314.9	325.9	1.1	320.0	320.0	1.03
	4	378.4	385.9	1.6	388.0	387.2	1.64
	5	534.2	572.1	1.4	564.3	565.0	0.97

TABLE II. CONTINUED

Angle, deg.	Mode No.	Reference 13			Present Results		
		Exp. freq., Hz.	Anal. freq., Hz.	Exp. damping ratio, %	Peak-ampl. freq., Hz.	Kennedy-Pancu freq., Hz.	Calc. damping ratio, %
45	1	46.2	42.4	0.9	45.2	45.1	0.98
	2	131.4	122.0	1.2	124.0	124.0	1.25
	3	257.6	250.0	1.1	249.6	251.0	0.95
	4	278.4	291.5	2.2	293.0	293.4	1.59
	5	433.3	432.0	1.5	429.9	430.0	1.18
60	1	44.1	42.4	0.9	43.0	42.3	1.00
	2	127.0	119.0	1.5	121.5	122.0	1.60
	3	260.8	226.3	3.9	228.0	228.1	2.05
	4	239.9	239.3	1.1	237.0	237.6	1.06
	5	405.1	404.0	1.5	402.3	401.5	1.17
90	1	48.0	49.3	0.8	48.1	48.0	0.87
	2	137.6	136.0	1.2	136.0	136.1	1.23
	3	188.9	161.6	5.3	165.6	164.4	3.31
	4	250.6	267.1	1.0	259.8	259.0	1.10
	5	331.7	331.7	2.1	332.3	331.0	1.98

TABLE III. EXPERIMENTAL EXAMPLES OF COMPOSITE MATERIALS WHICH APPEAR TO HAVE UNSYMMETRIC CONSTITUTIVE-COEFFICIENT ARRAYS

Composite	E_{11} ($psix10^{-6}$)	E_{22} ($psix10^{-6}$)	ν_{12}	ν_{21}	ν_{12}/E_{11}	ν_{21}/E_{22}	$100x\frac{(\nu_{12}/E_{11})-(\nu_{21}K_{22})}{(\nu_{12}/E_{11})+(\nu_{21}/E_{22})}$
Glass cloth-epoxy (ref. 147)	3.30	3.15	0.141	0.188	0.0427	0.0597	16.6
Boron-epoxy (ref. 148)	30.6	3.5	0.36	0.033	0.01177	0.00944	-11.0
Graphite-graphite (ref. 144)							
$\pm 60^{\circ}$ wrap	2.98	2.47	0.21	0.18	0.070	0.073	- 2.1
$\pm 70^{\circ}$ wrap	3.44	2.28	0.16	0.15	0.046	0.066	-17.8
$\pm 85^{\circ}$ wrap	4.10	1.83	0.11	0.13	0.027	0.071	-43.1
90° wrap	4.30	1.35	0.09	0.20	0.021	0.148	-75.0

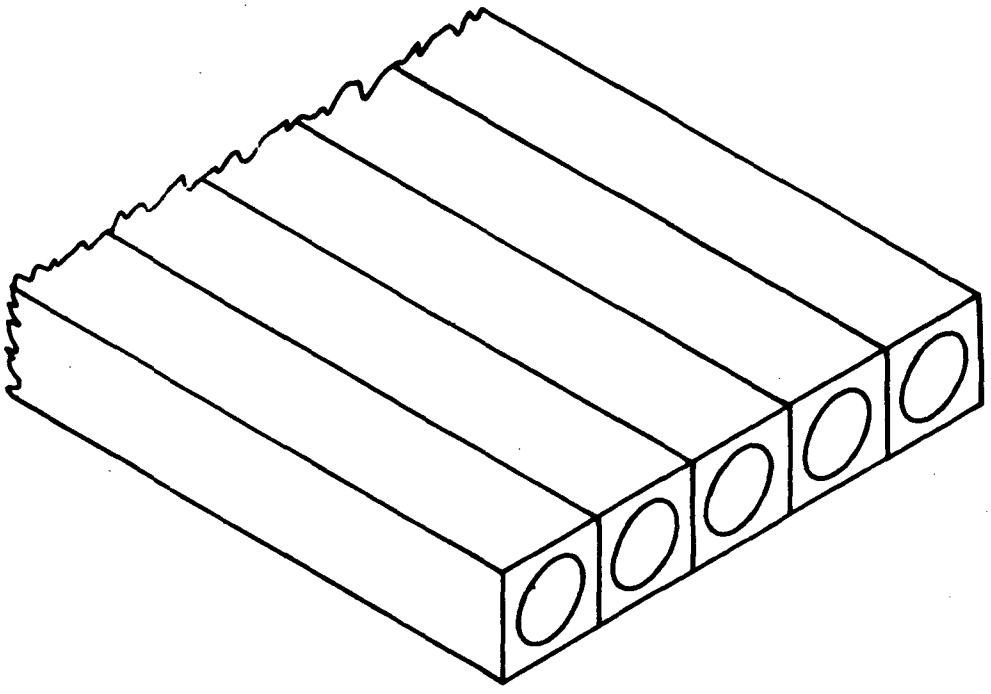


Figure 1. Composite containing undirectional fibers.

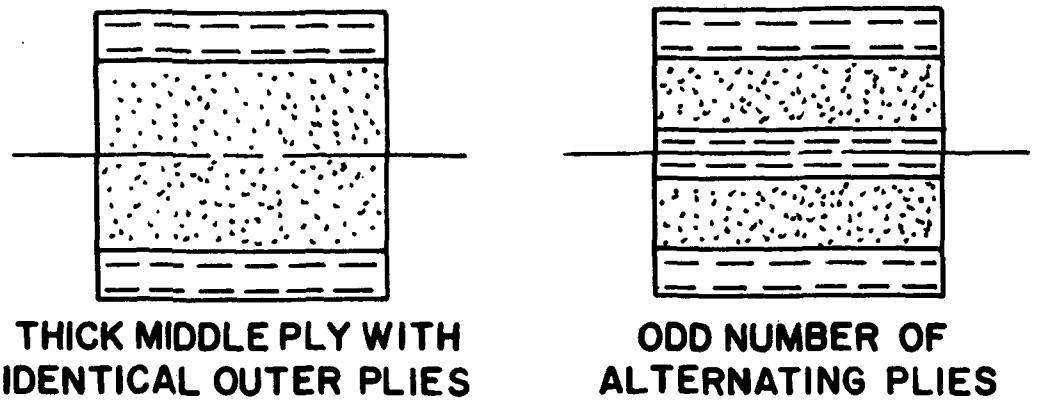


Figure 2. Examples of symmetrical laminates.

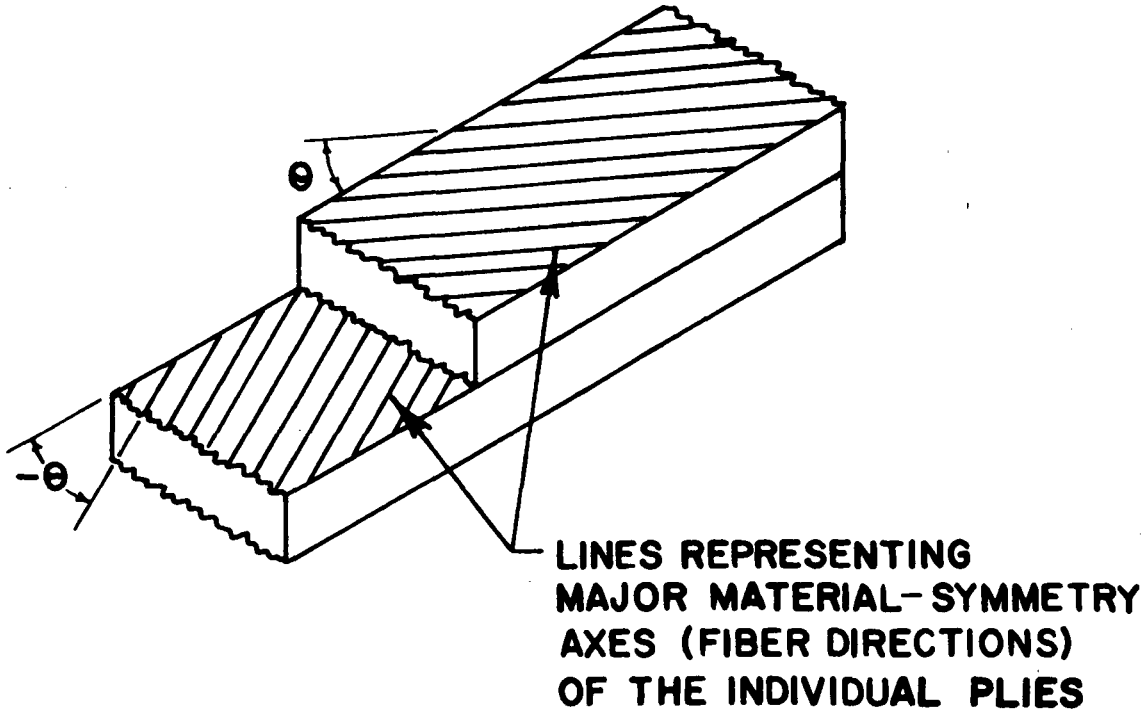


Figure 3. An example of a balanced laminate.

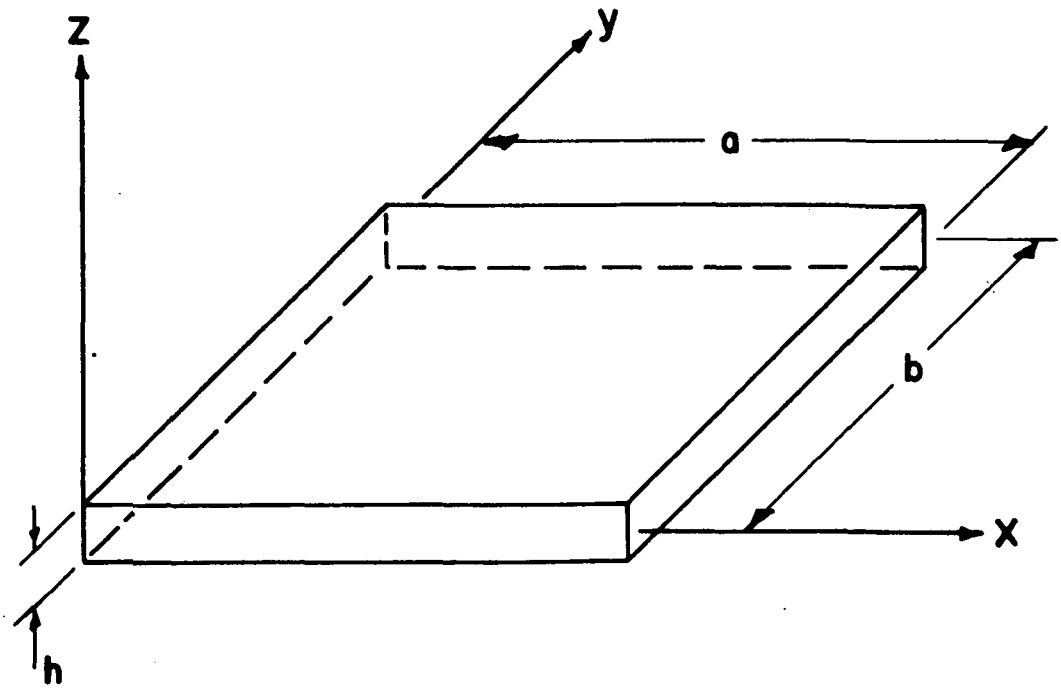


Figure 4. Plate coordinate system.

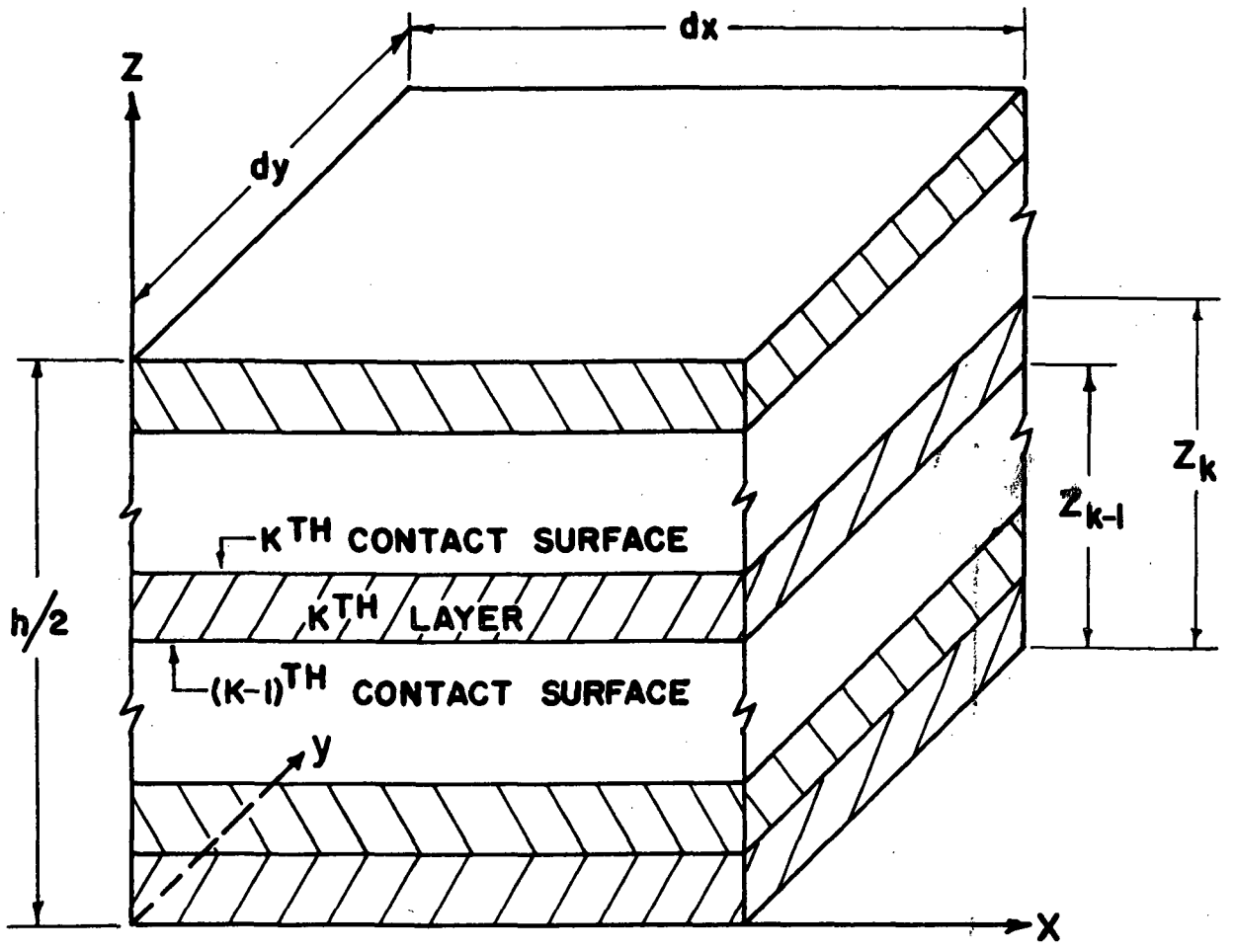


Figure 5. Lamination geometry, showing upper half of plate. (Plane xy is the midplane of the plate).

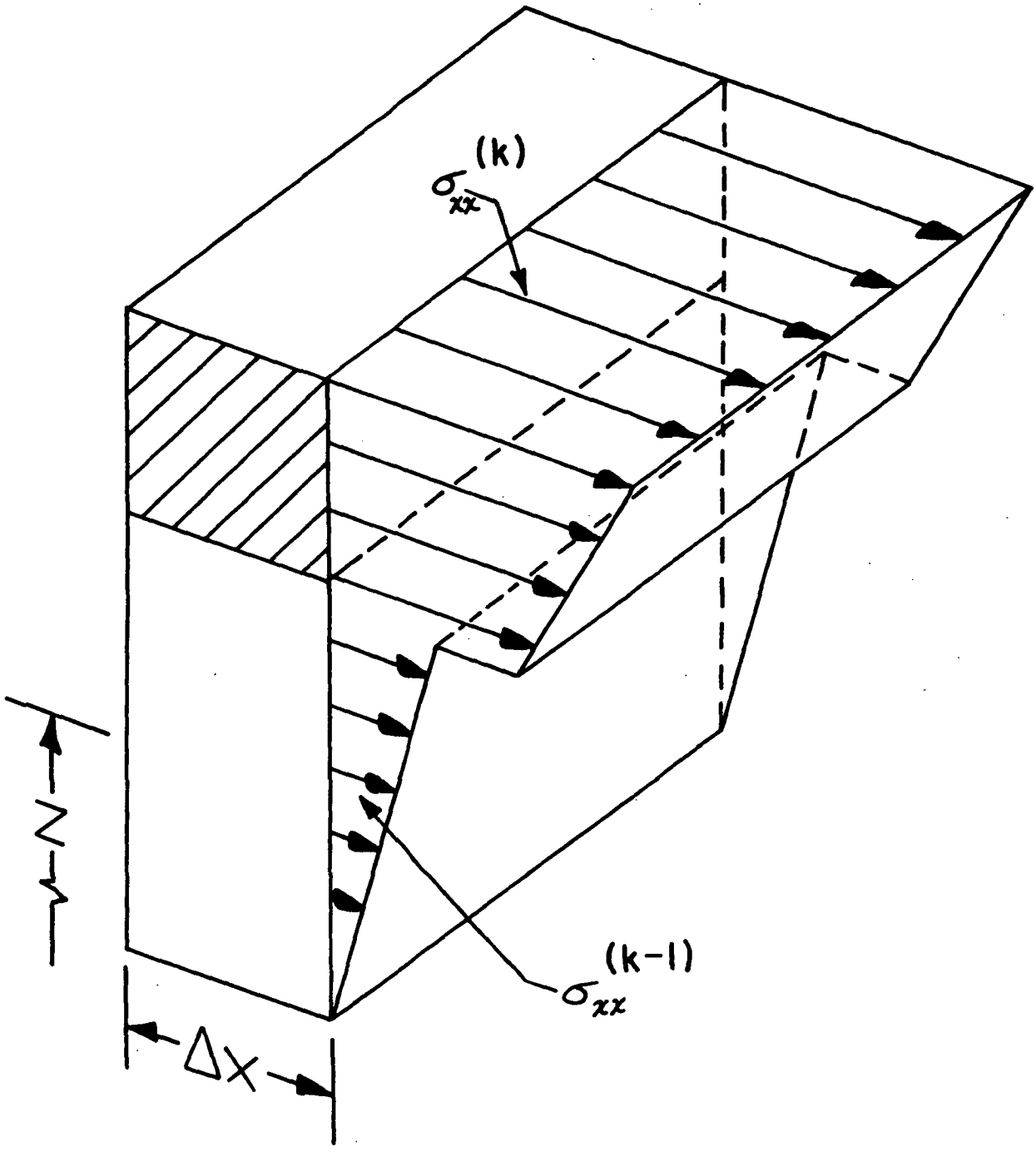


Figure 6. Bending-stress distribution.

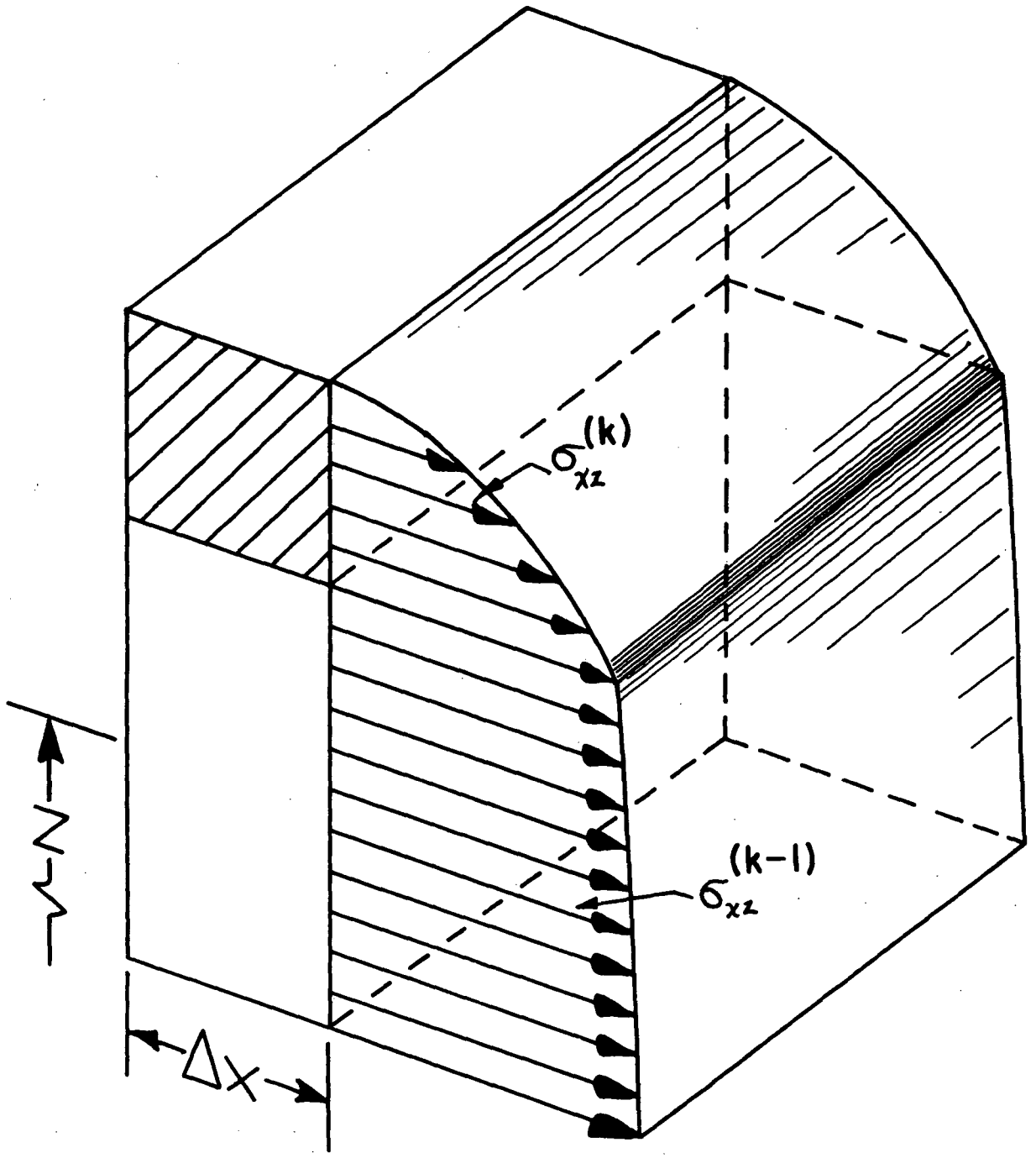


Figure 7. Shear-stress distribution.

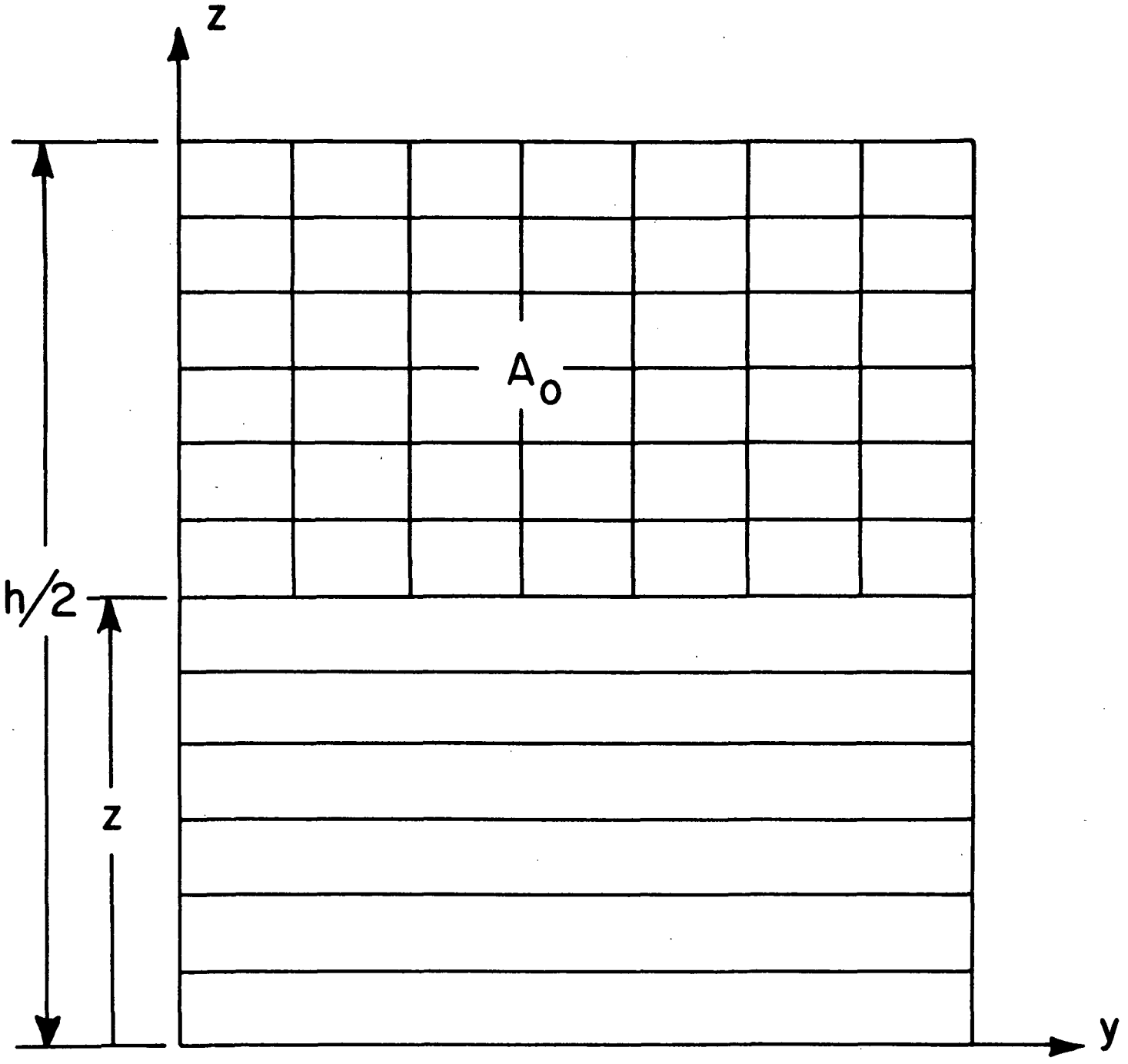


Figure 8. Area of integration A_o.

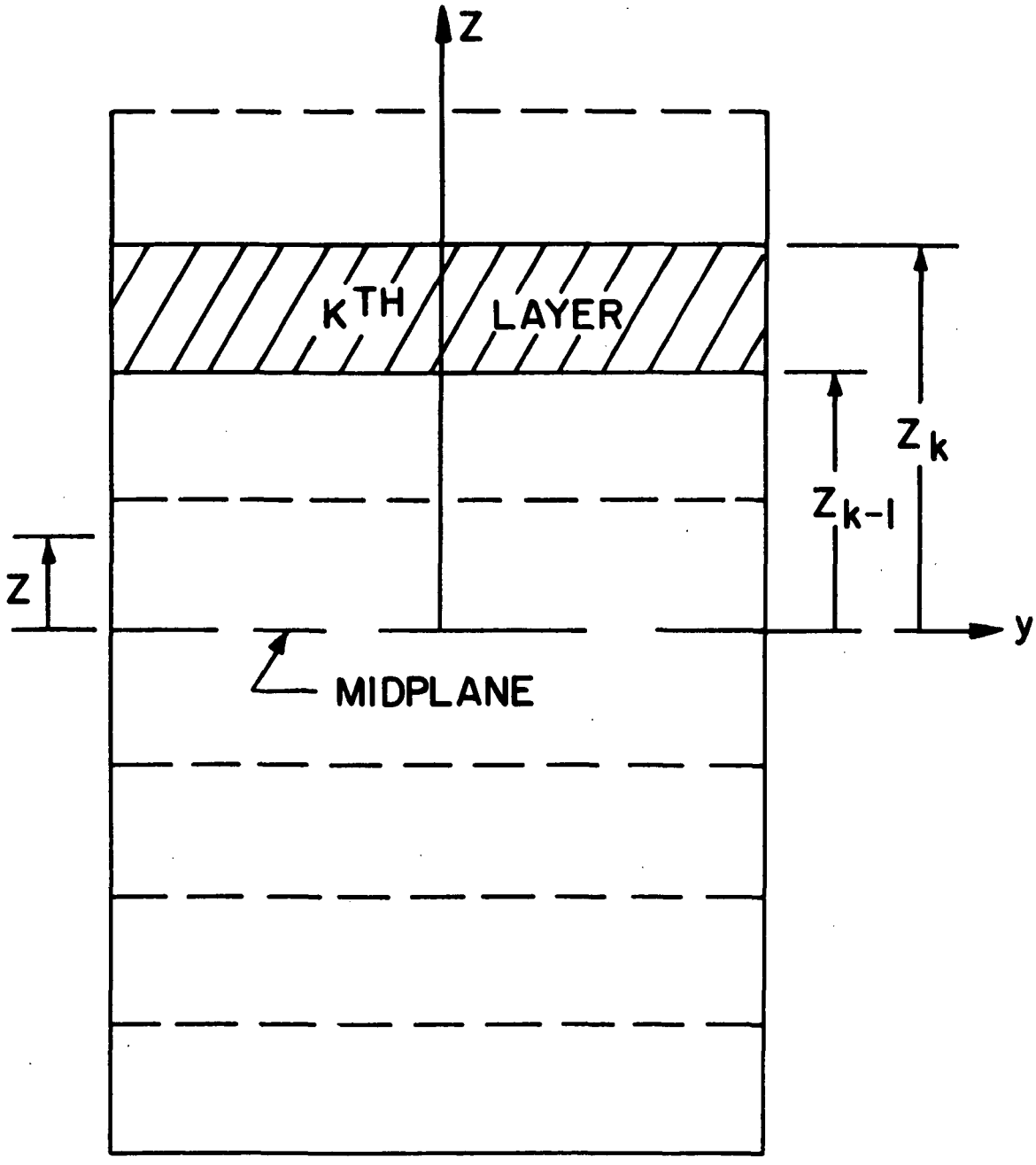


Figure 9. Cross section of laminated beam.

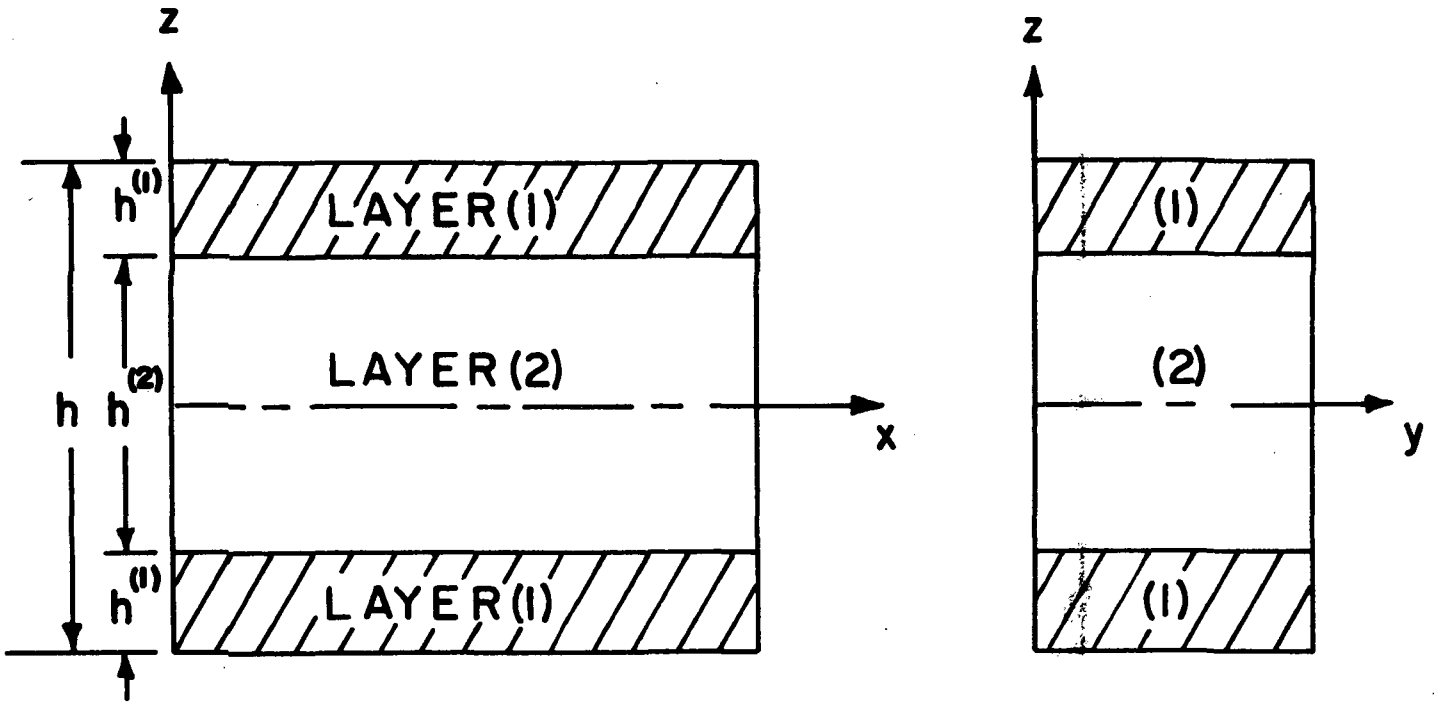


Figure 10. Three-ply symmetrically laminated plate.

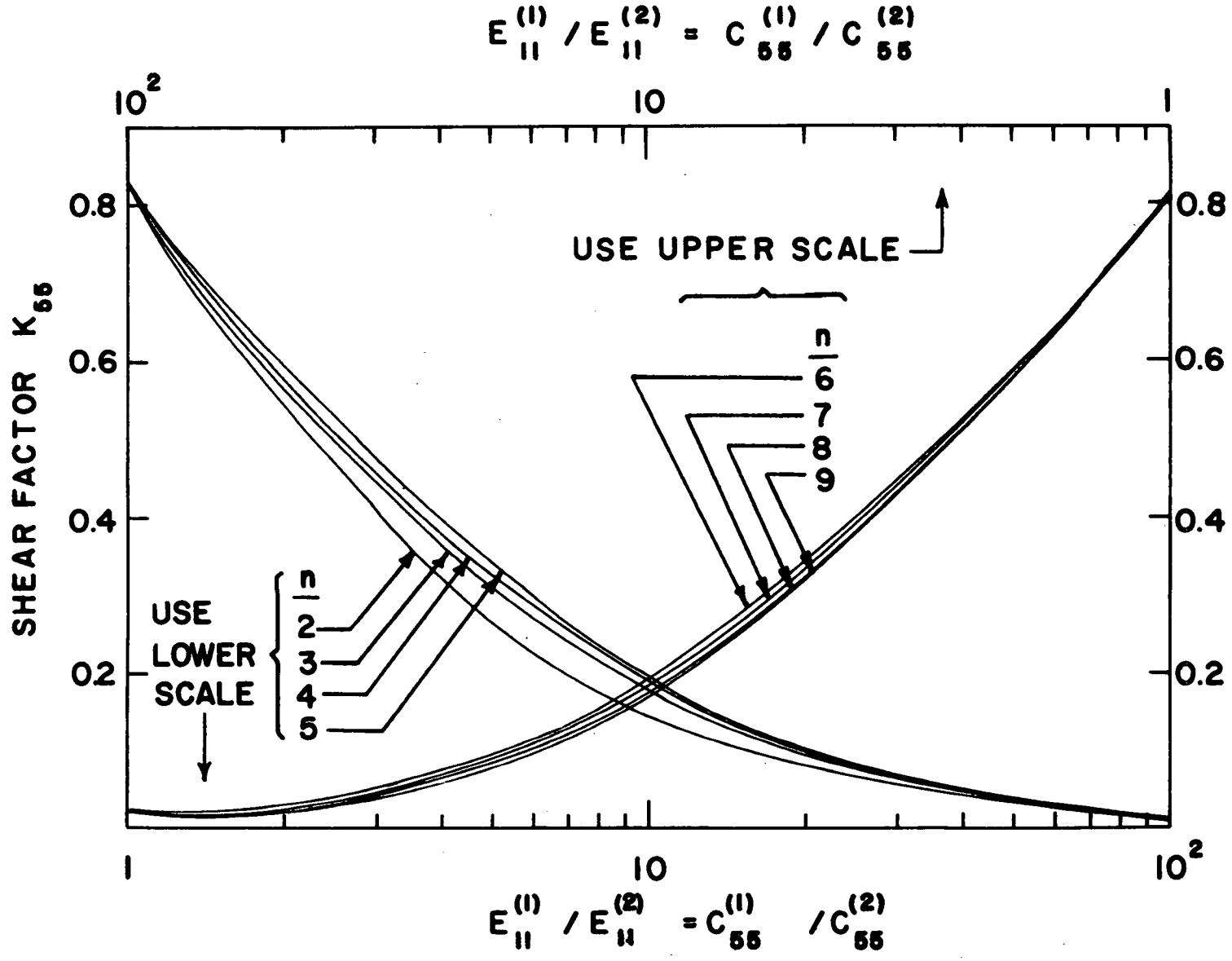


Figure 11. Shear factor for laminates consisting of multiple alternating layers of two materials.

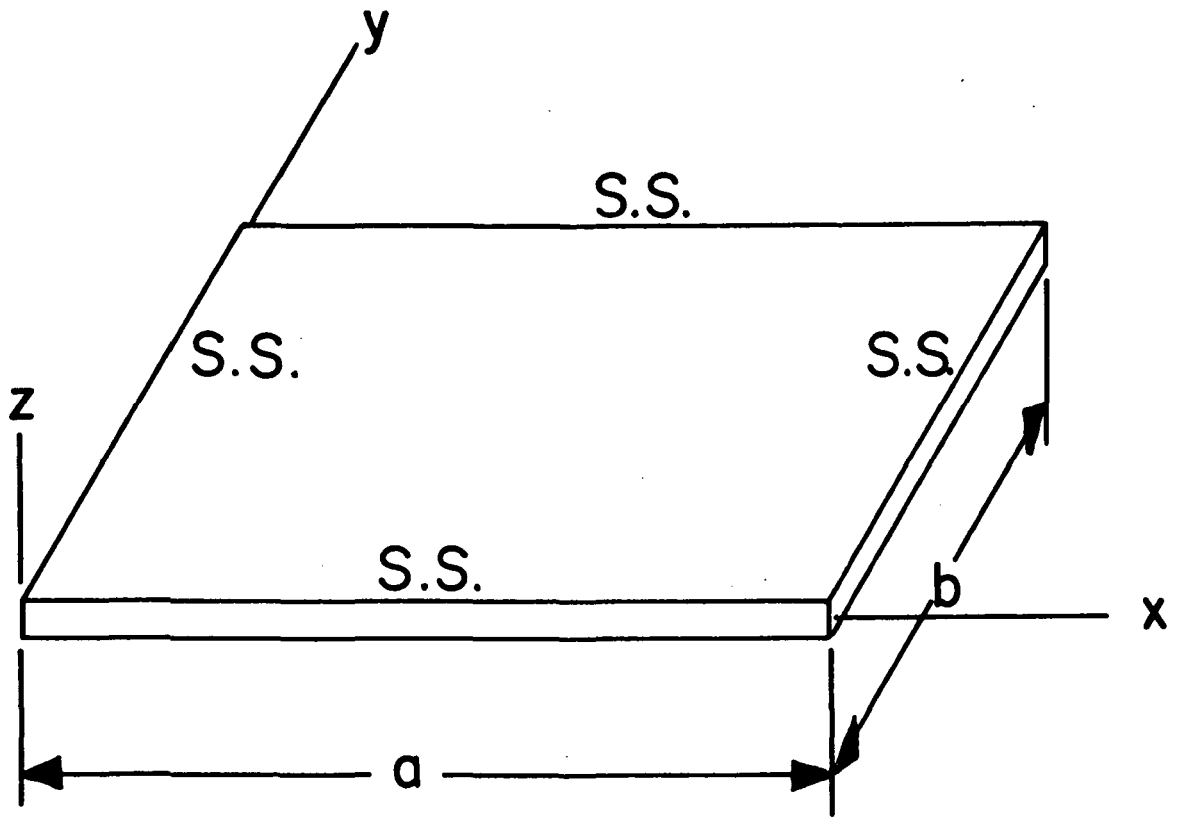


Figure 12. Rectangular plate with simply supported edges.

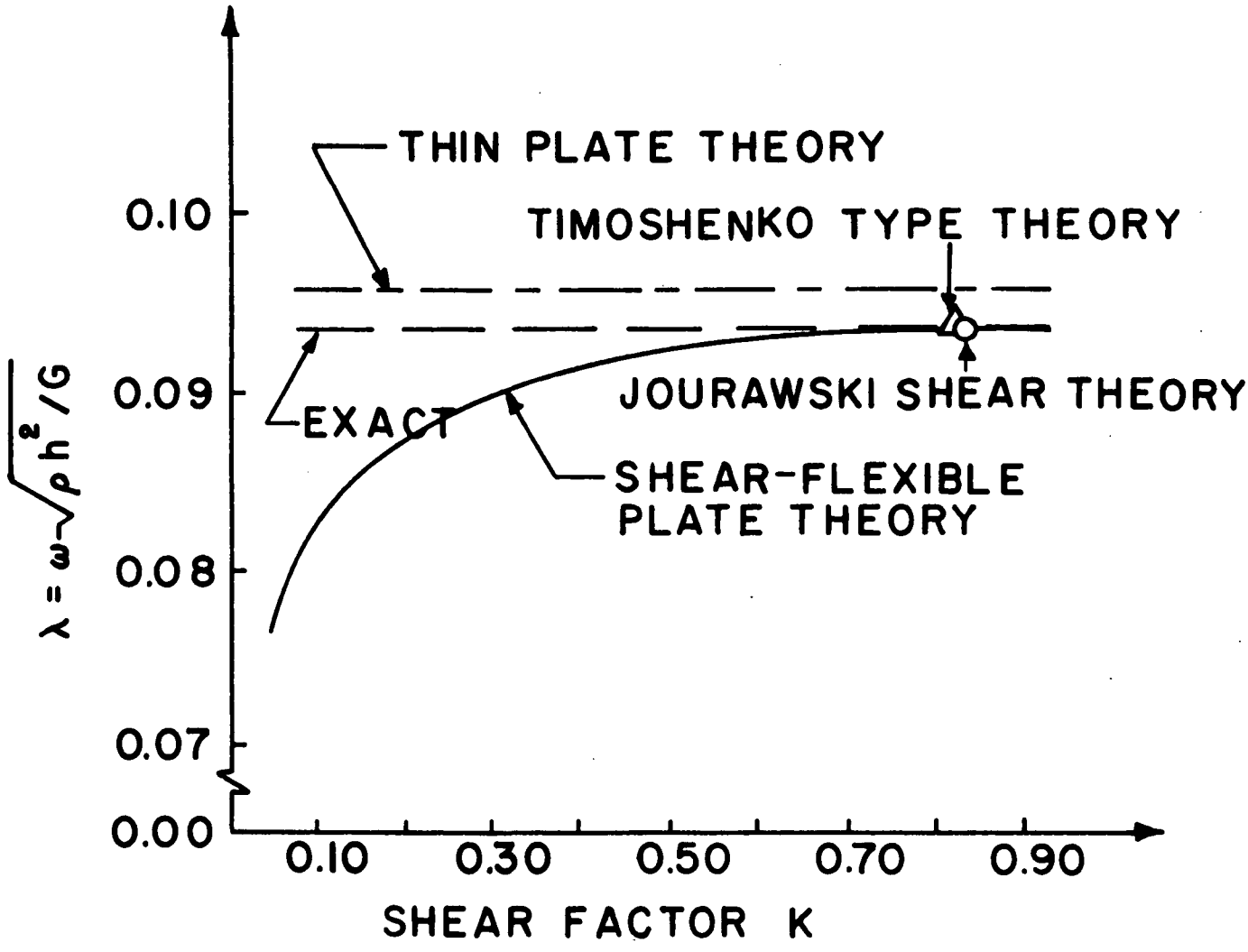


Figure 13. Lowest Eigenvalue (λ) for homogeneous, isotropic plate.

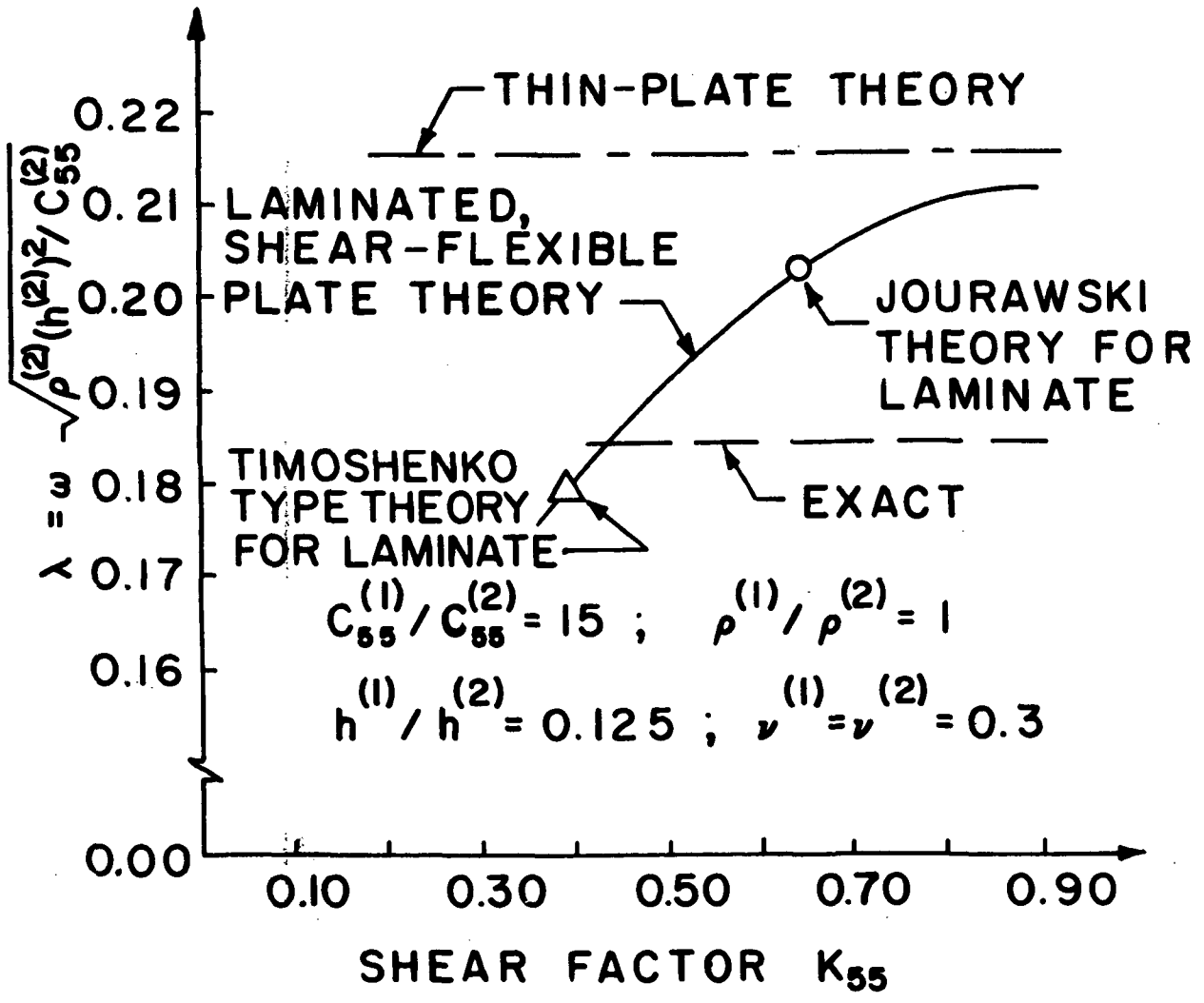


Figure 14. Lowest eigenvalues (λ) for three-ply symmetrically laminated plate.

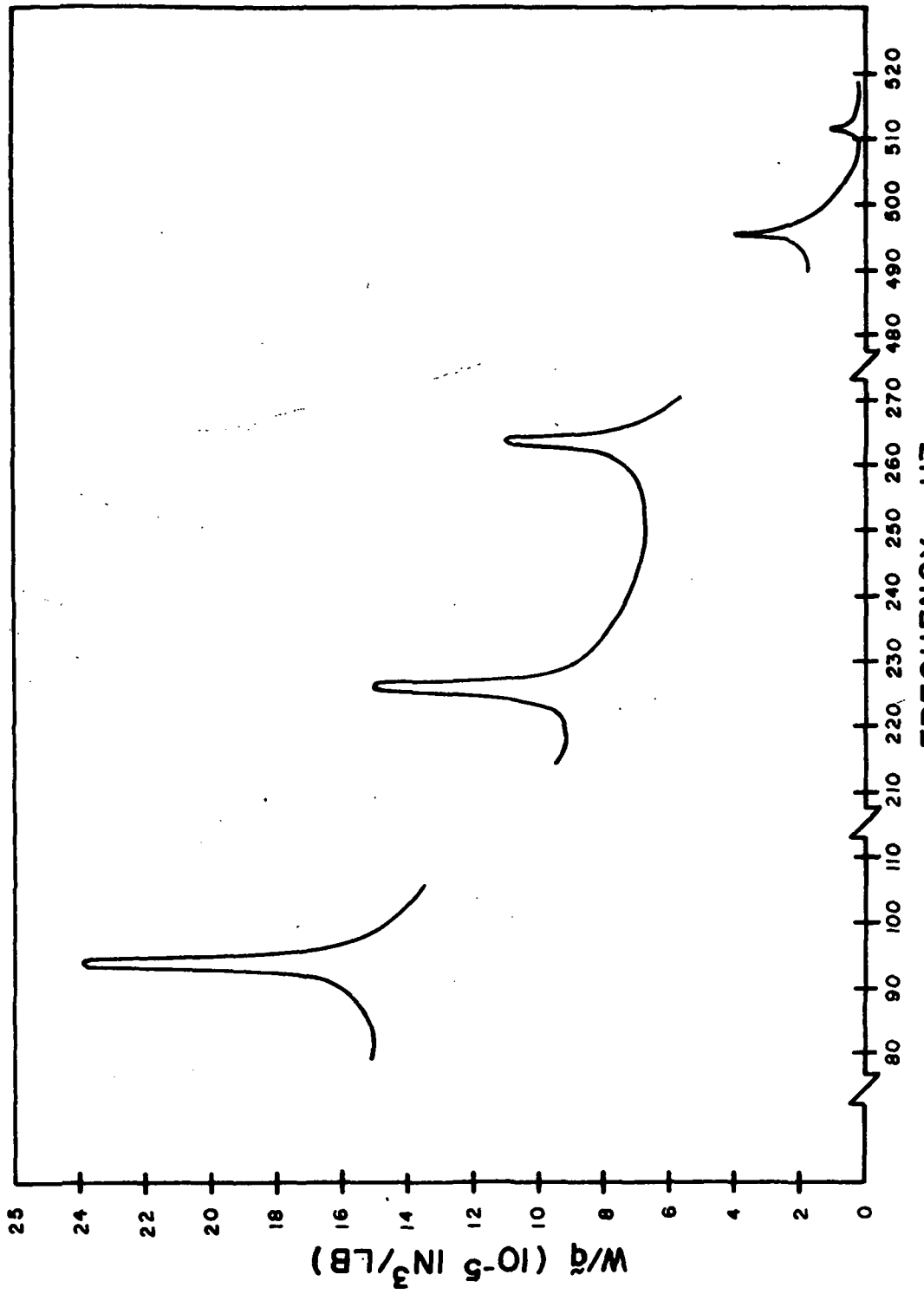
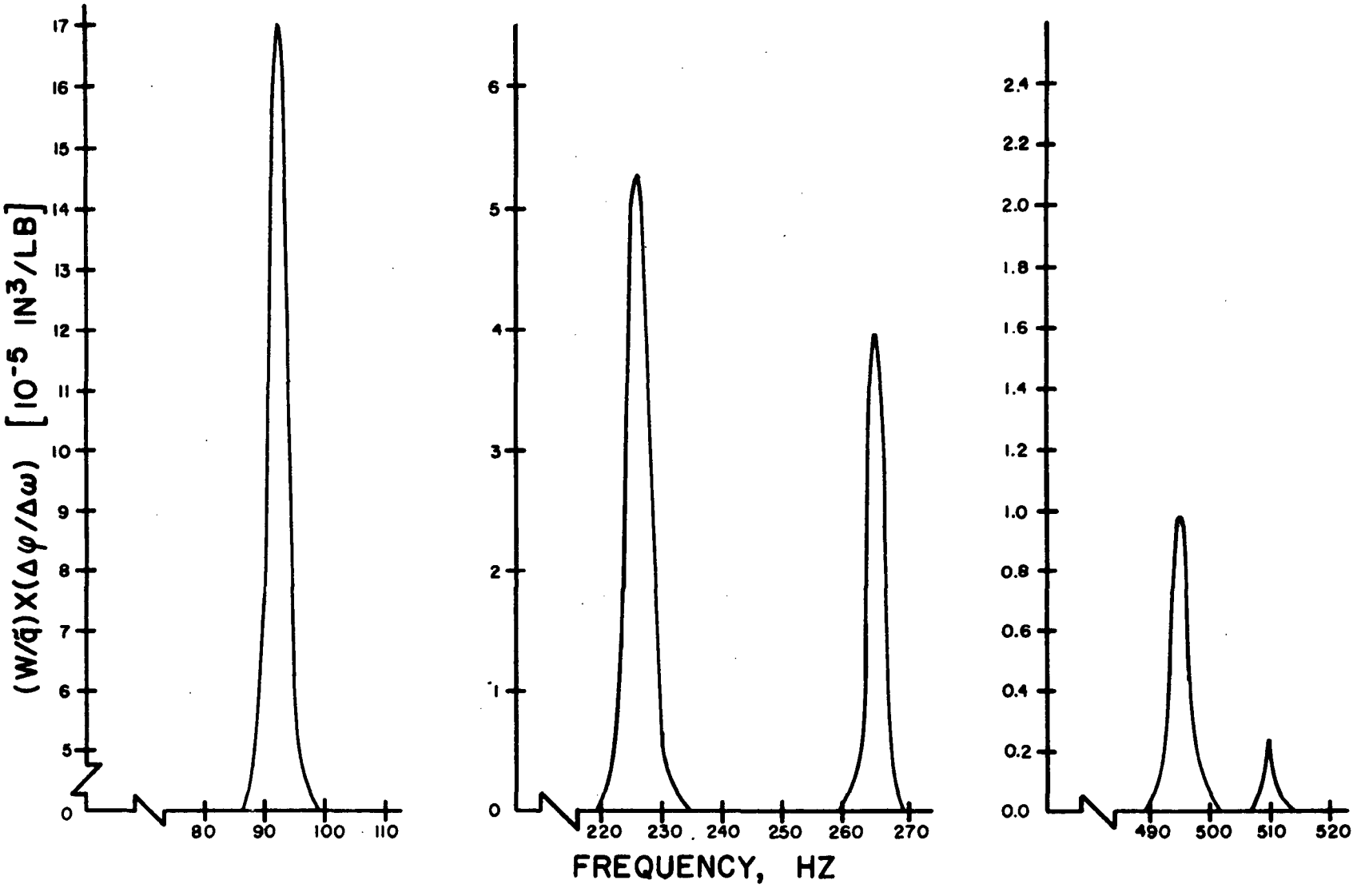


Figure 11. Amplitude frequency response



F . M K -Pa es s

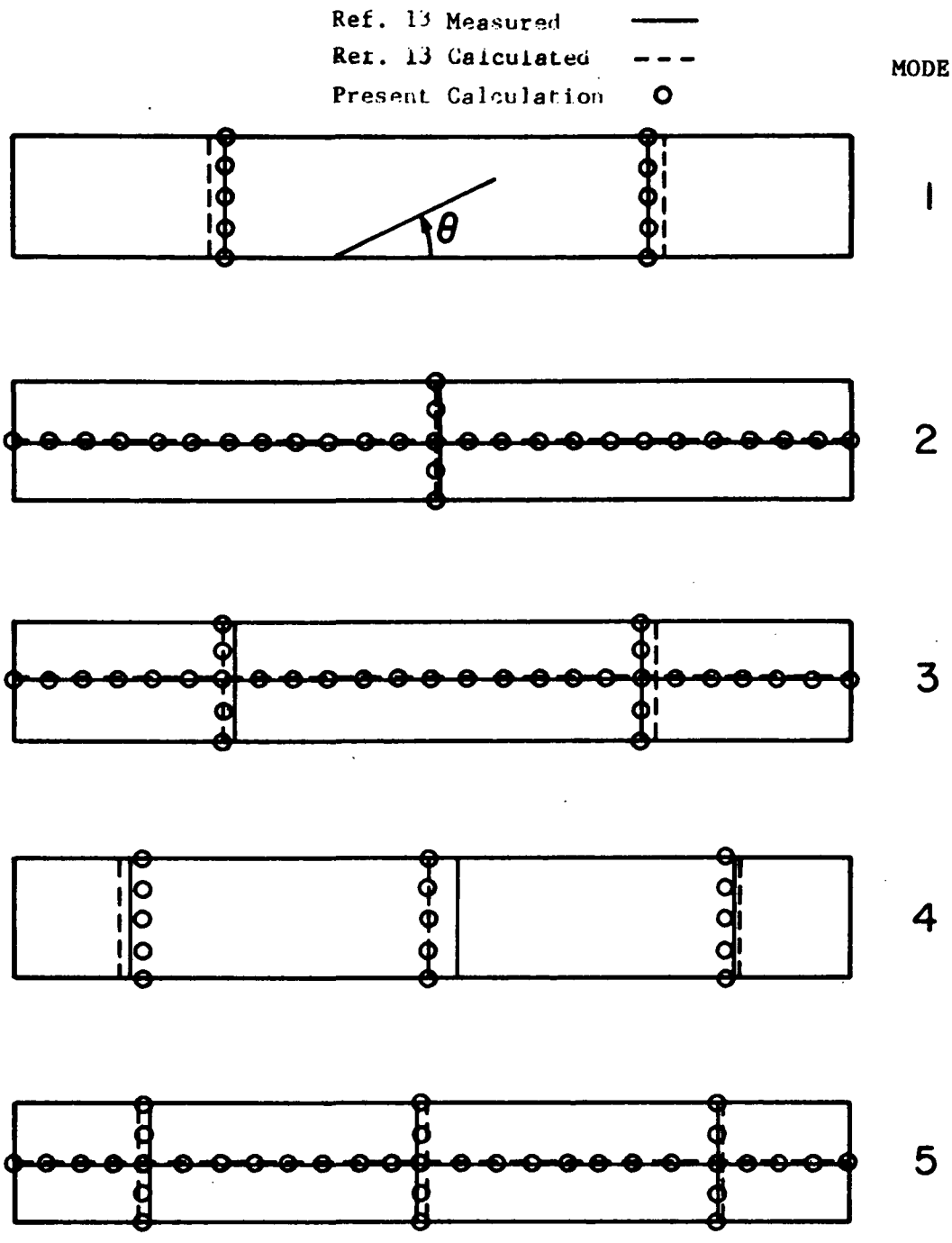


Figure 17. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 0^{o}$

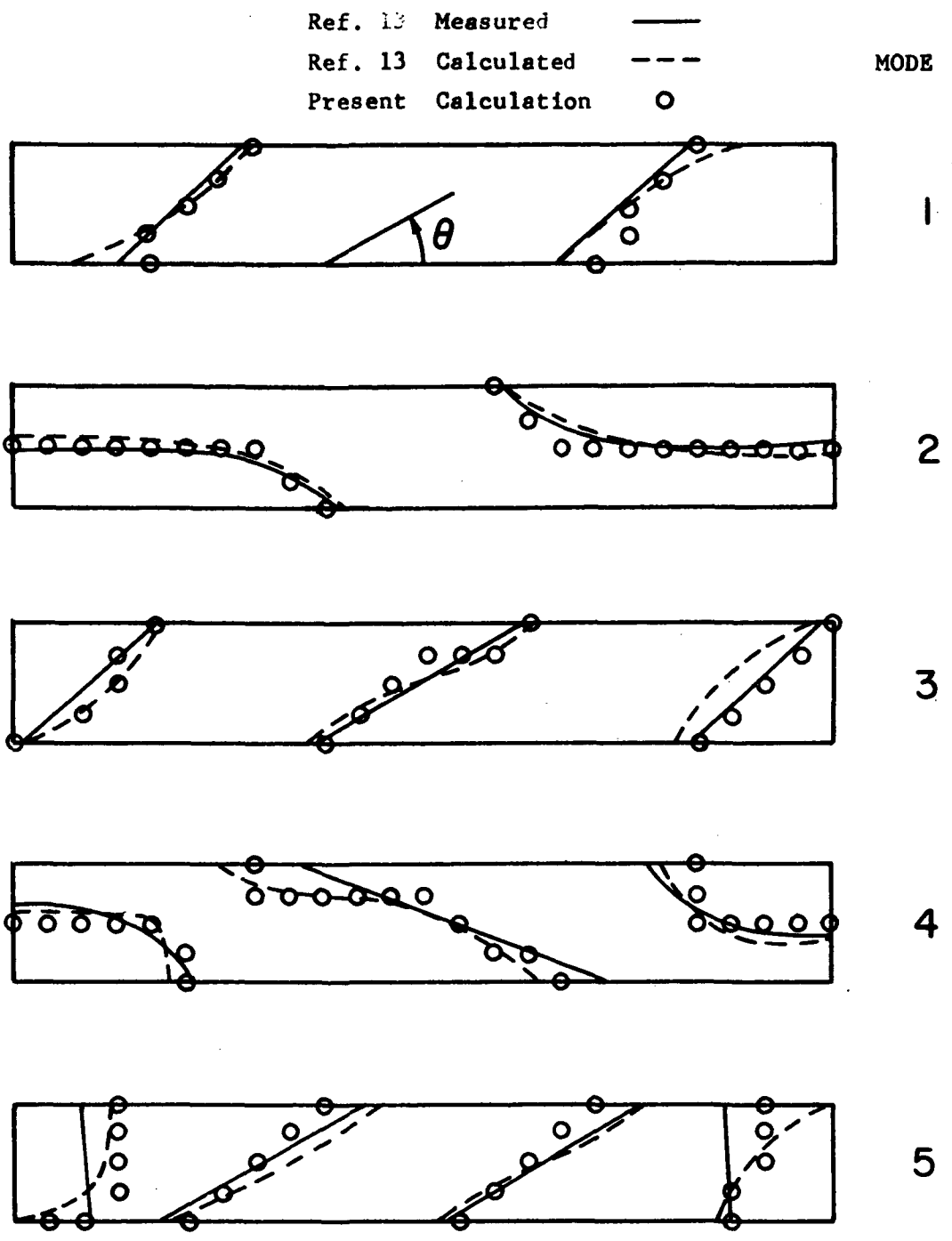


Figure 18. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 10^{\circ}$

Ref. 13 Calculated — — —

Present Calculation O

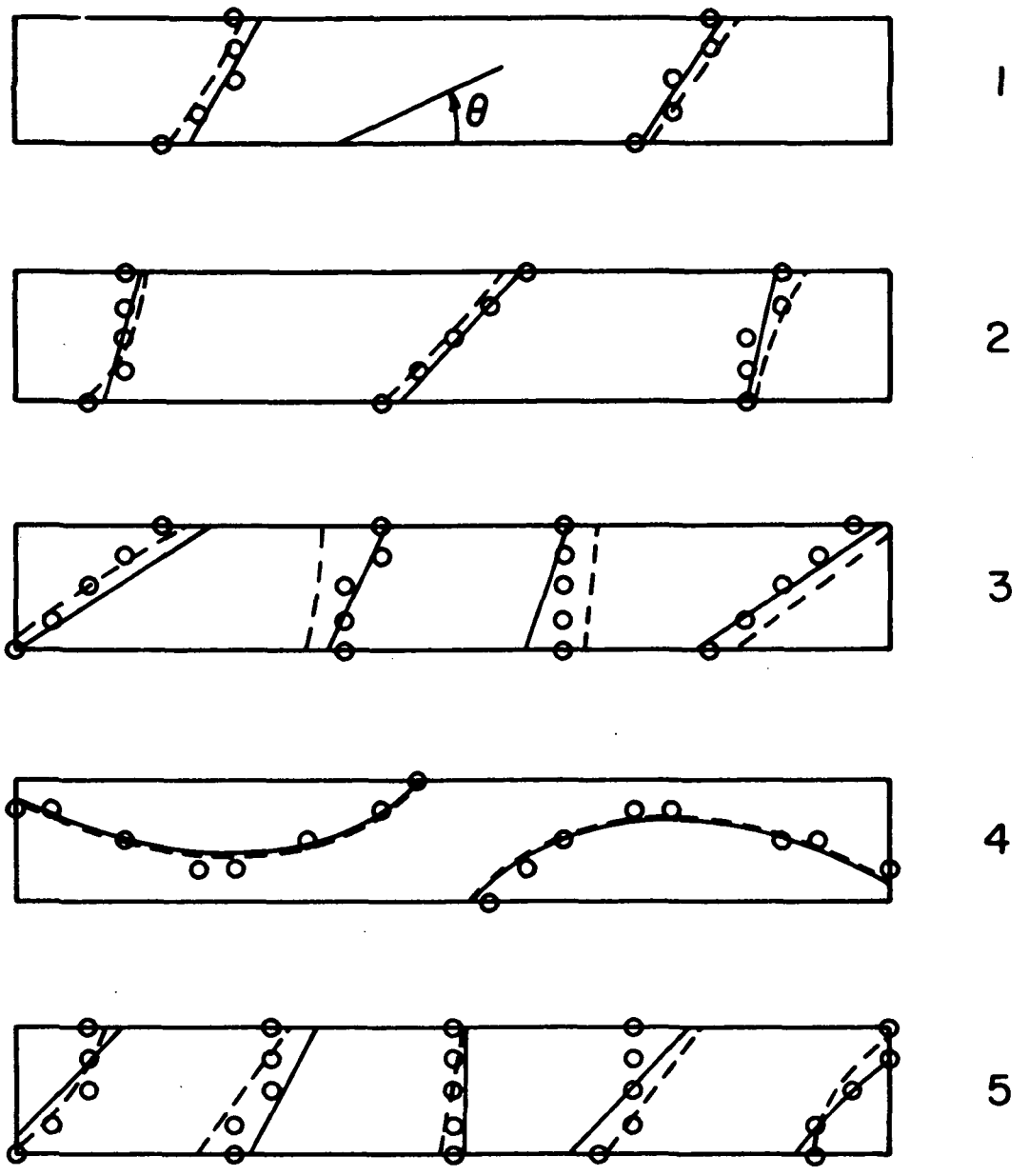


Figure 19. Nodal patterns of the first five modes of composite material plates at angle of orientation θ = 30°

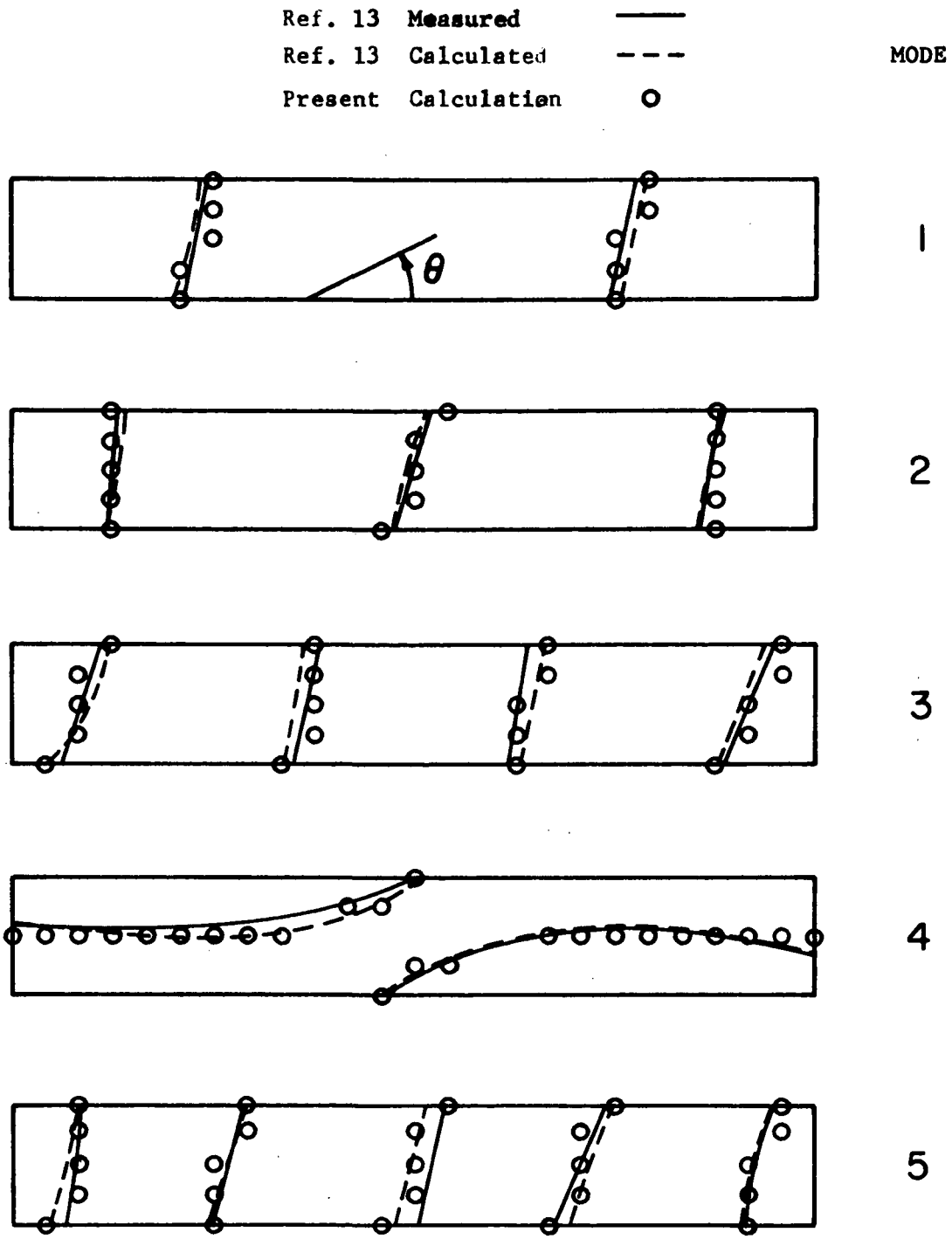


Figure 20. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 45^{c}$

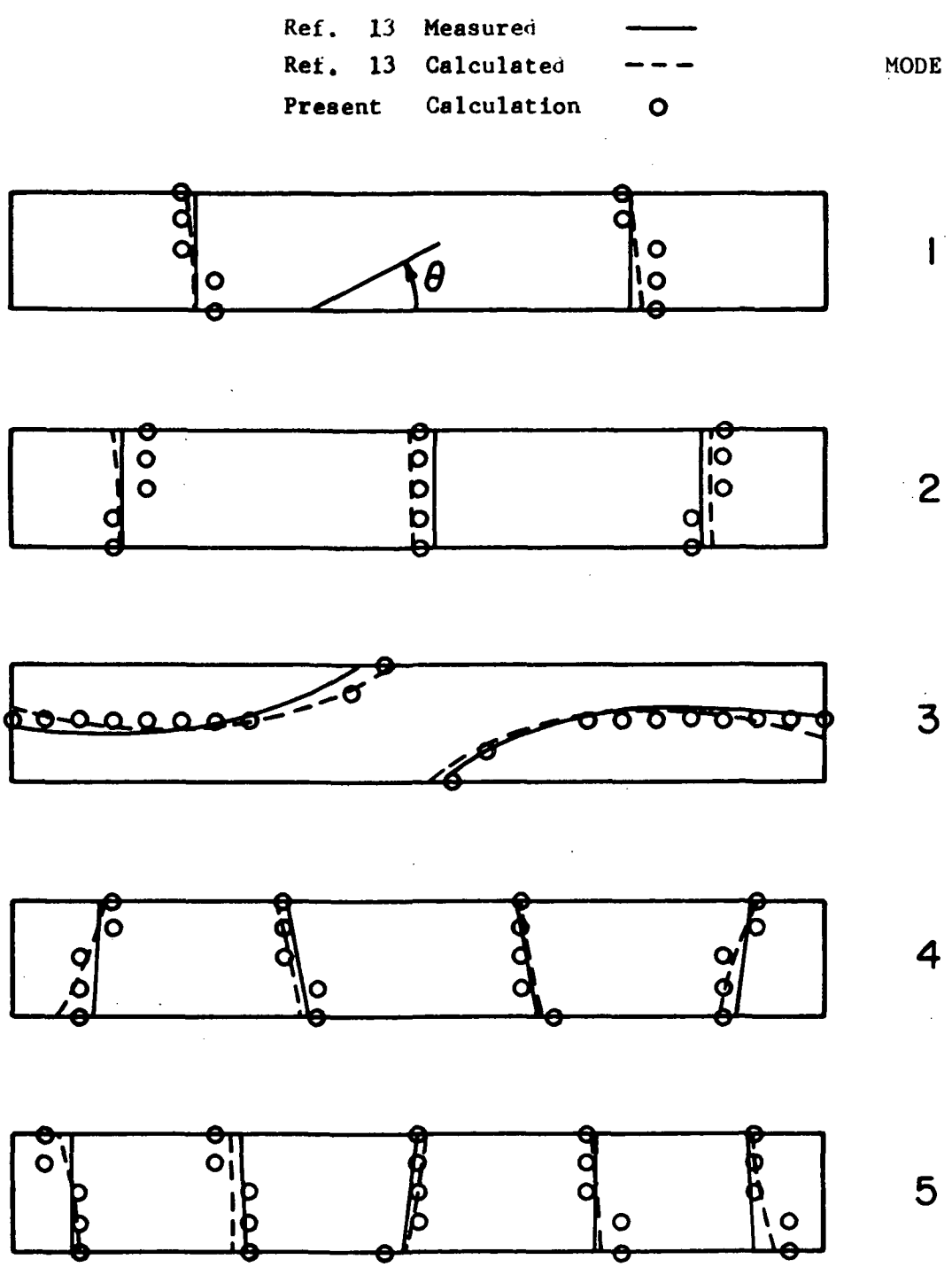


Figure 21. Nodal patterns of the first five modes of composite material plates at angle of orientation $\theta = 60^{\circ}$

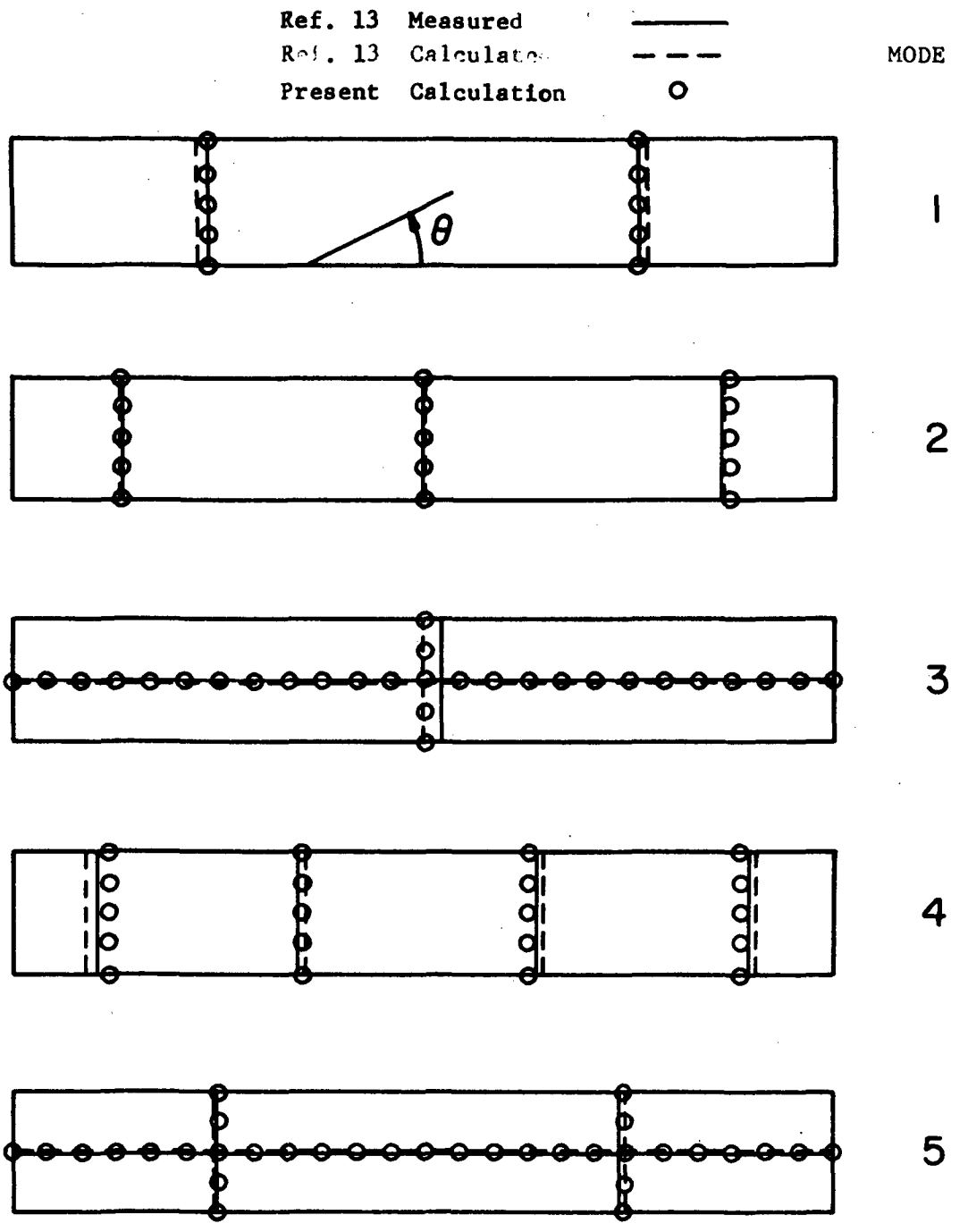


Figure 22. Nodal patterns of the first five modes of composite material plates at angle of orientation $\vartheta = 90^{\circ}$

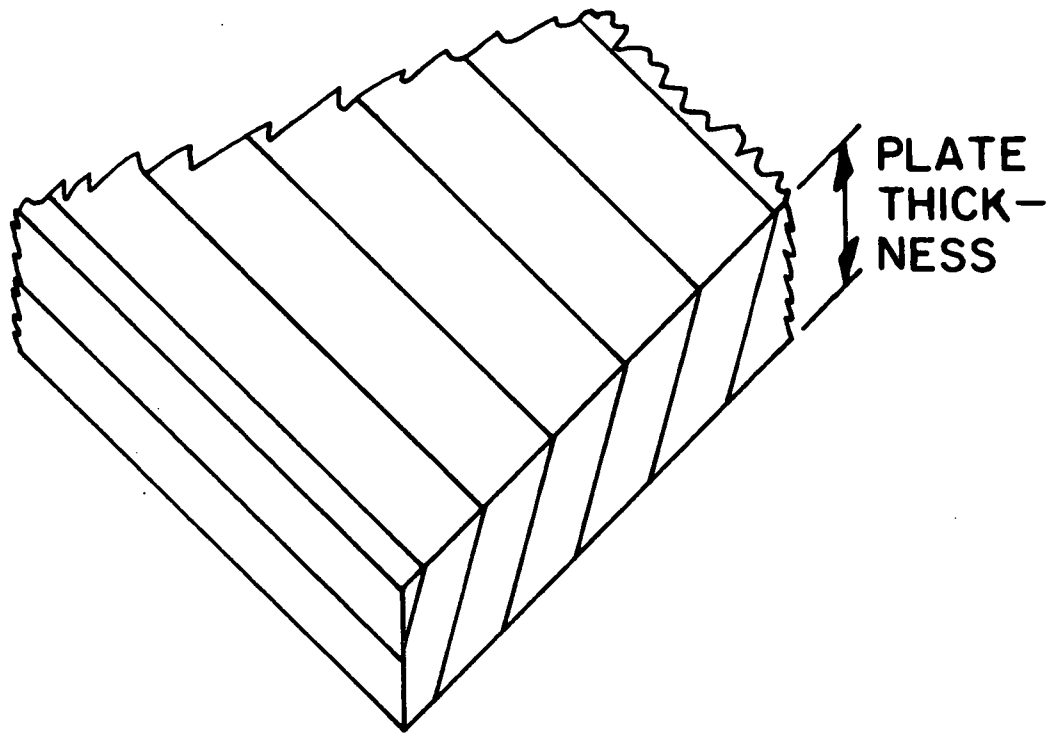


Figure A-1. Shingle-laminated plate.

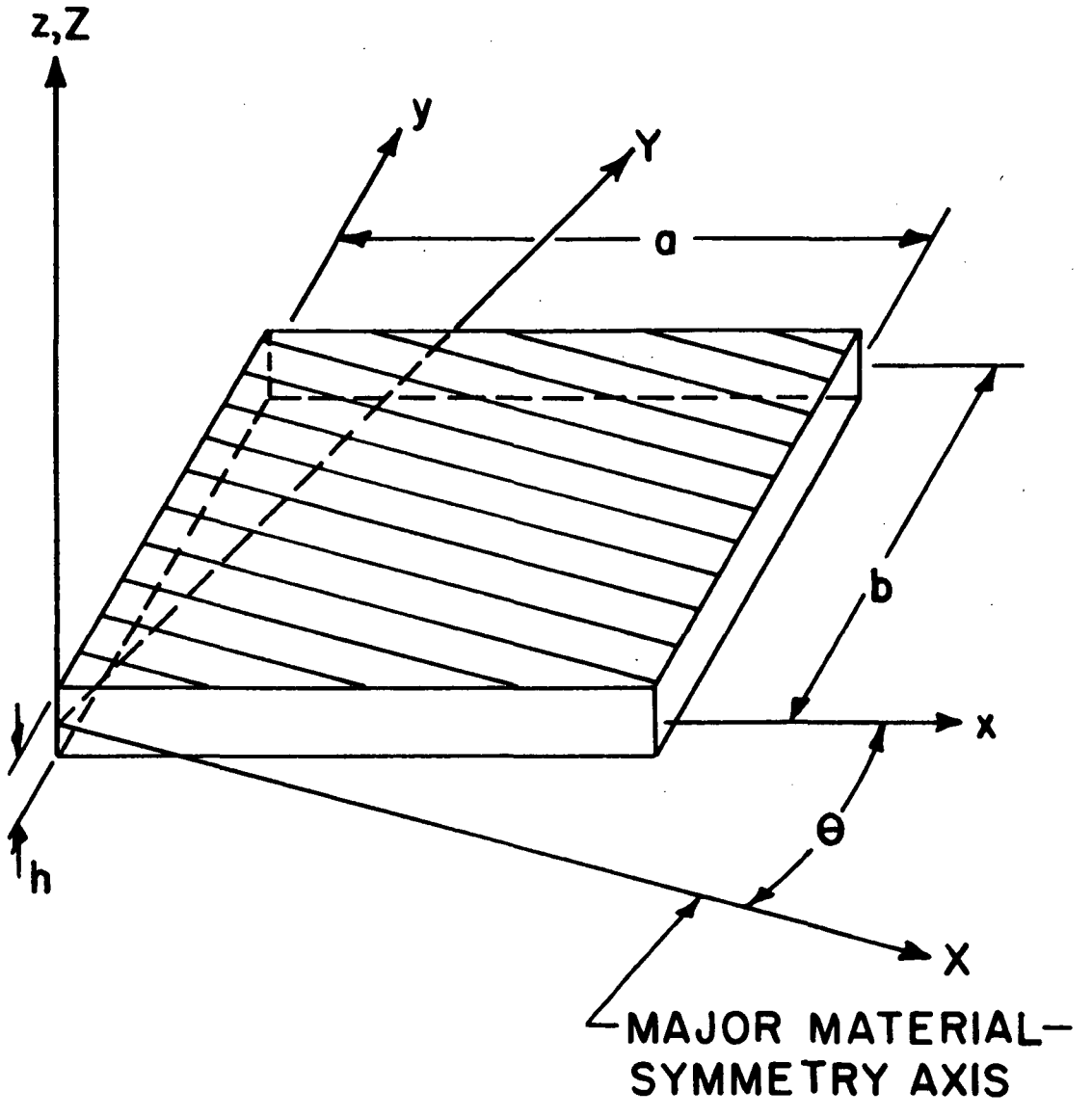


Figure A-2. Major Material-Symmetry Axis Direction.

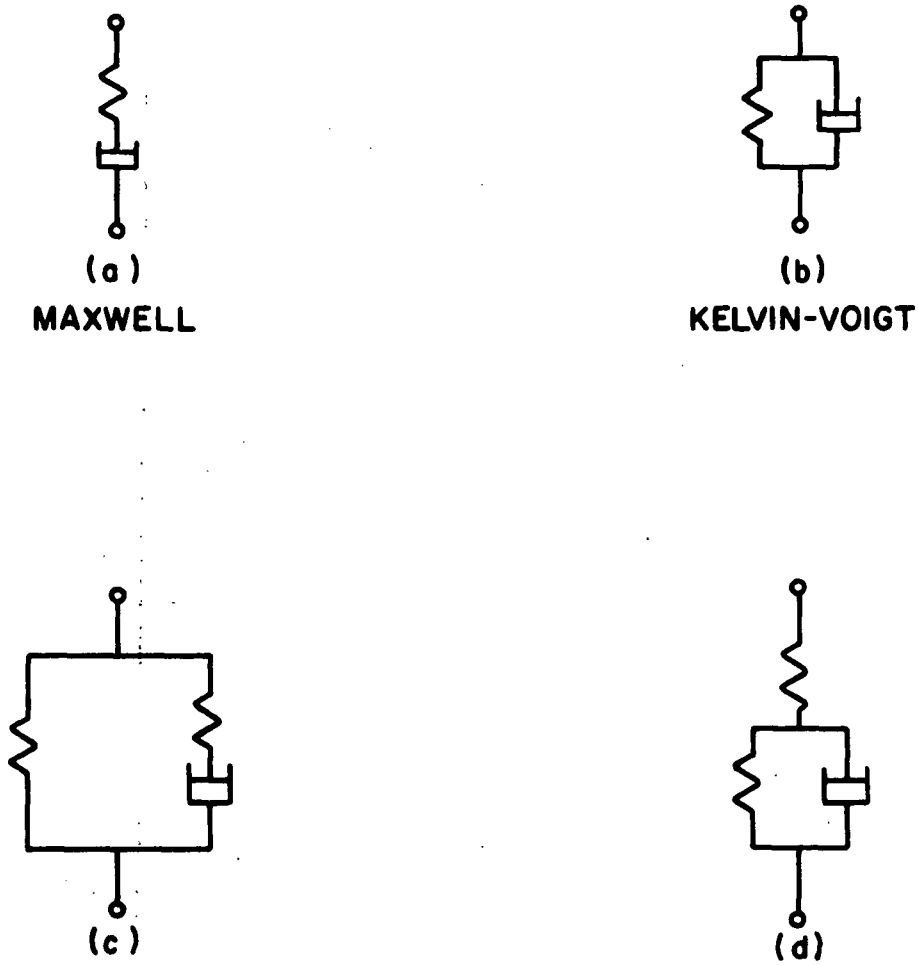


Figure B-1. Two-and three-parameter viscoelastic models.

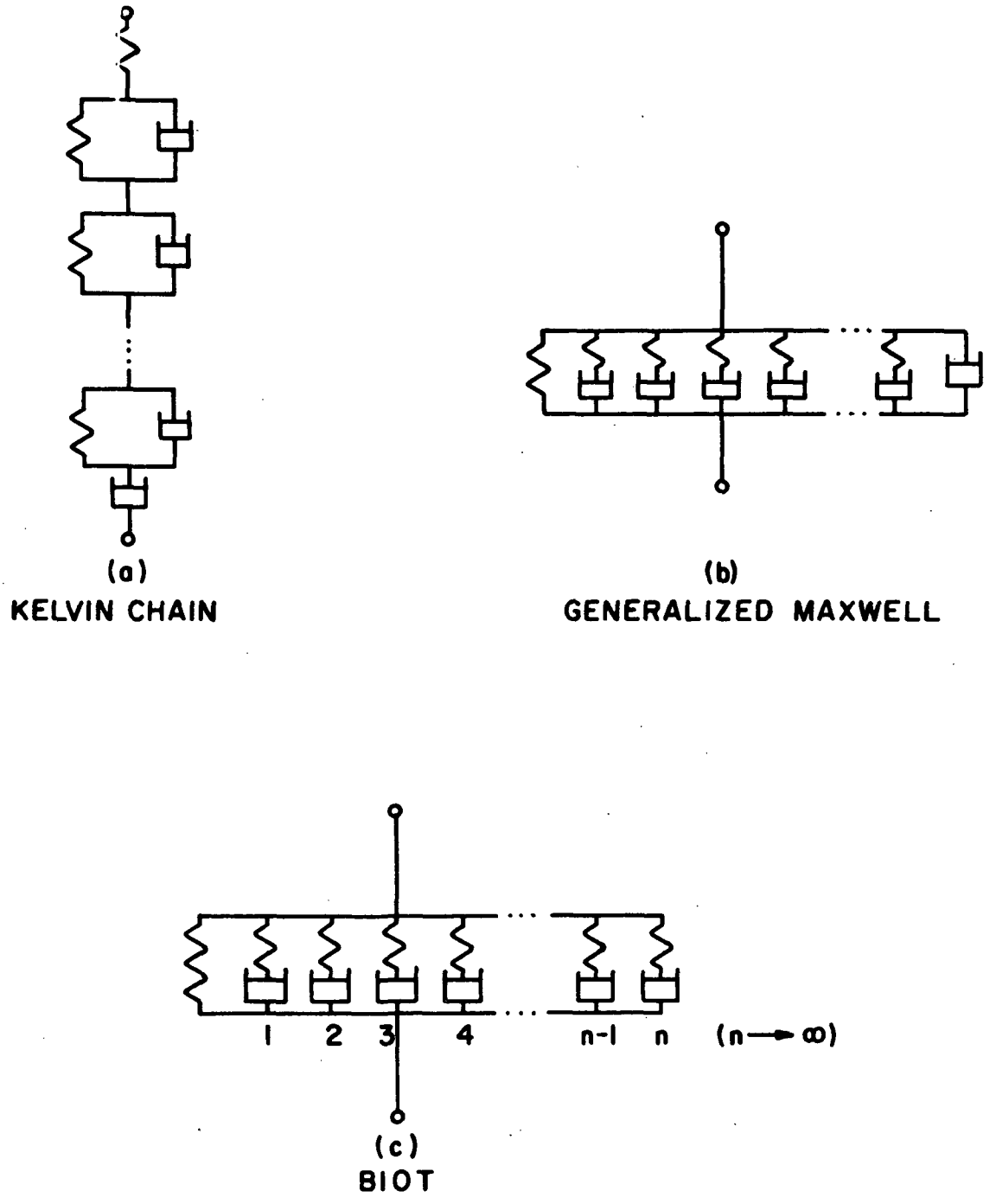
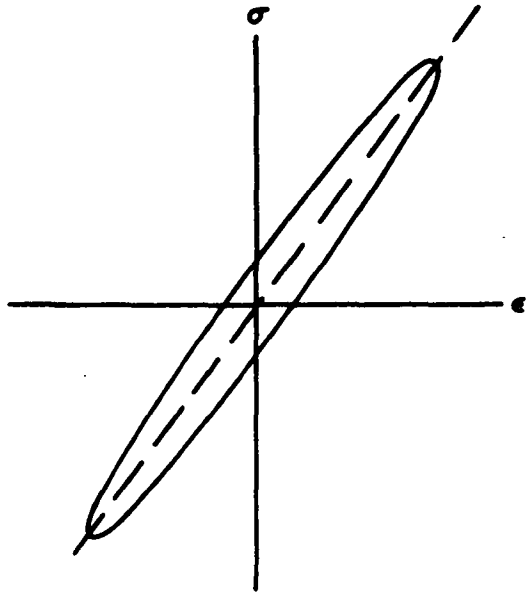
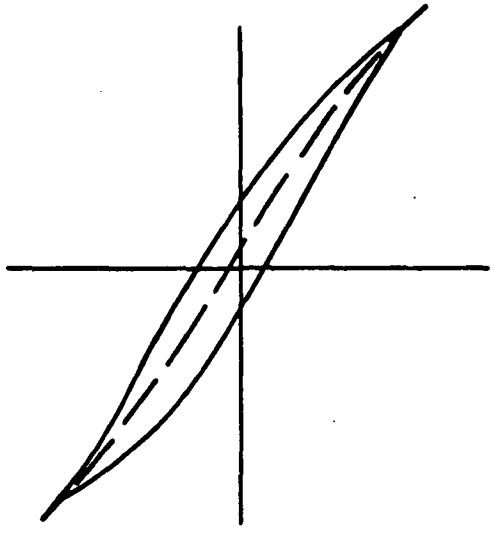


Figure B-2. Many-element viscoelastic models.



(a) ELLIPTICAL HYSTERESIS LOOP



(b) POINTED-END, STRAIGHT-SIDED HYSTERESIS LOOP

Figure B-3. Hysteresis-loop shapes.

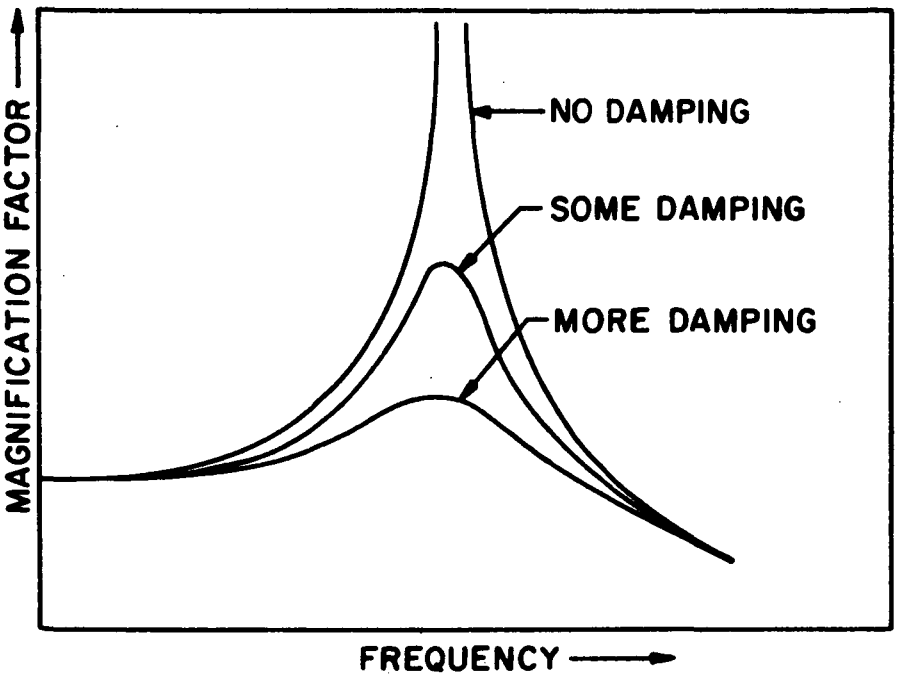


Figure B-4. Effect of damping on magnification factor. (Not drawn to scale).

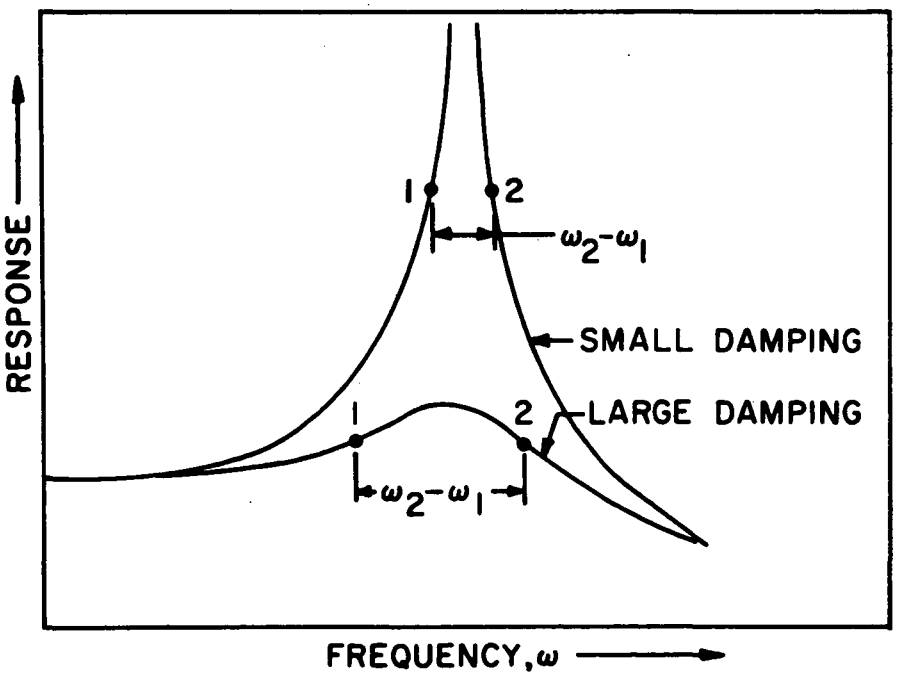


Figure B-5. Effect of damping on half-power-point frequency separation (ω_2-ω_1). (Not drawn to scale).

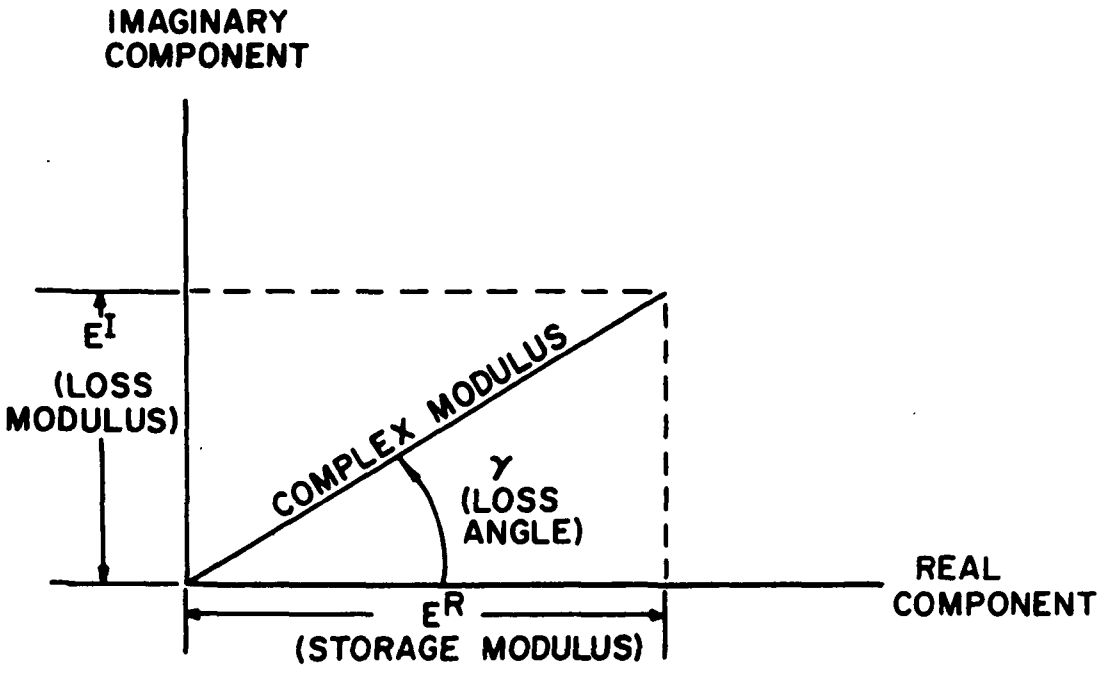


Figure B-6. Concept of complex modulus and loss angle.

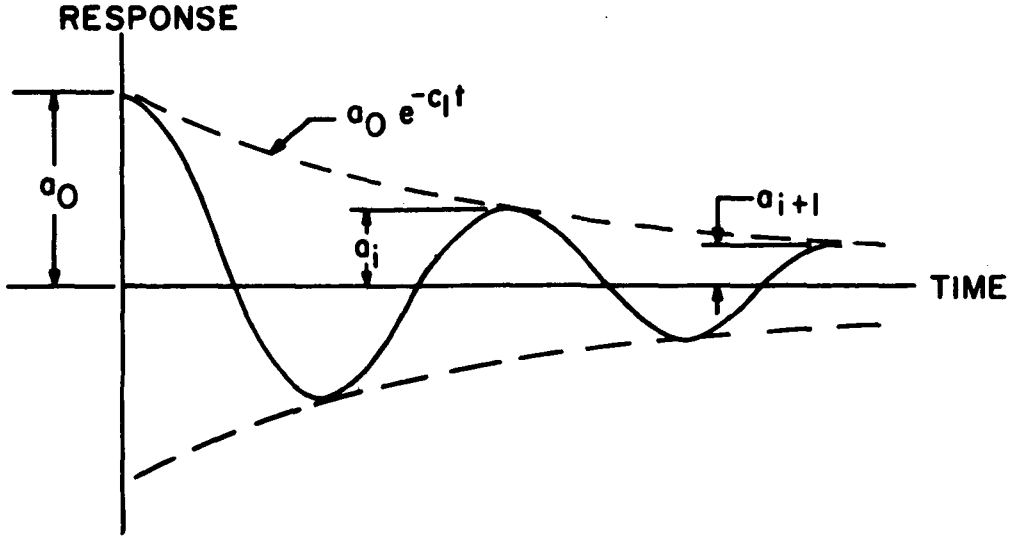


Figure B-7. Exponential decay. (Not drawn to scale).

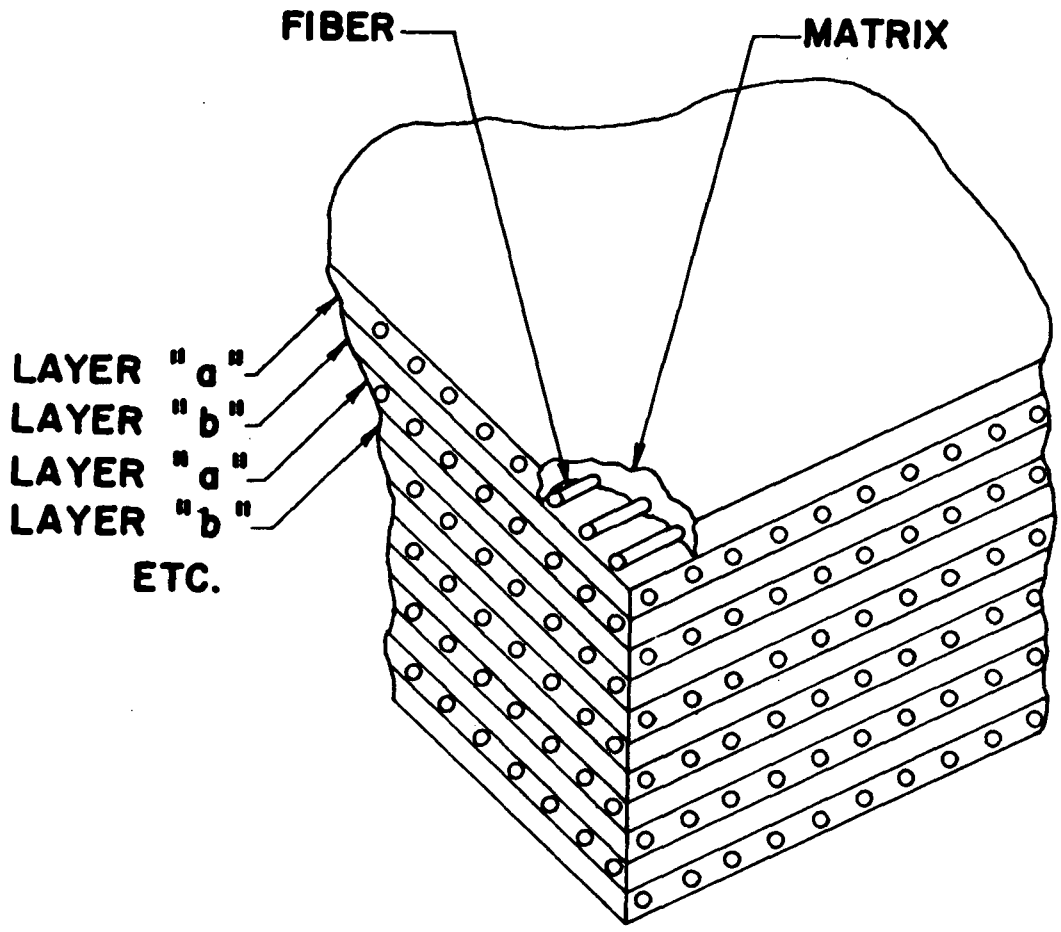


Figure F-1. Medium containing many repeating alternating layers of two different types.

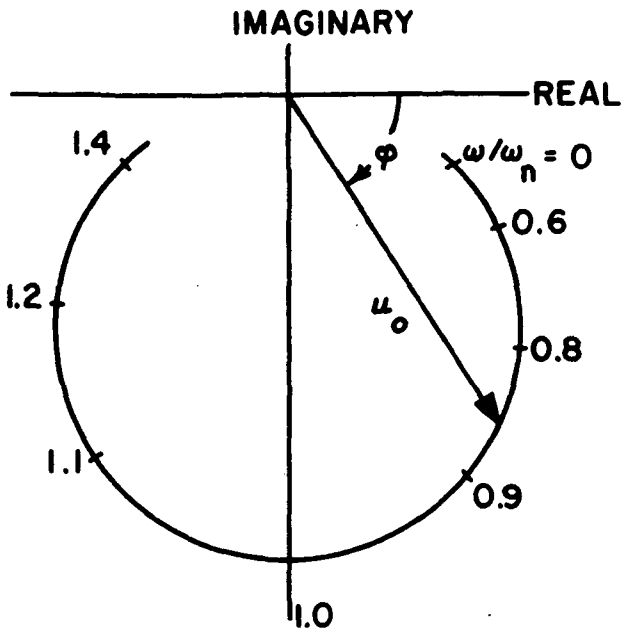


Figure H-1. Theoretical Argand diagram.

```
      DIMENSION E(20),G(20),Z(20),A(20,20)
  120 READ(5,10)N
 10   FORMAT(I3)
      IF(N) 130,130,140
  140 NN=N-1
      READ(5,20)(E(J),G(J),Z(J),J=1,N)
20    FORMAT(3F10.5)
      SUMGZ=G(1)*Z(1)
      DO 30 J=2,N
      SUMGZ=SUMGZ+G(J)*(Z(J)-Z(J-1))
30    CONTINUE
      GE2Z5=E(1)**2/(4.*G(1))*(8./15.*Z(1)**5)
      DO 40 J=2,N
40    GE2Z5=GE2Z5+E(J)**2/(4.*G(J))*(Z(J)**4*(Z(J)-Z(J-1))-
     12./3.*Z(J)**2
     2*(Z(J)**3-Z(J-1)**3)+0.2*(Z(J)**5-Z(J-1)**5))
      E2Z4=E(1)*E(1)*Z(1)**4
      DO 50 J=2,N
50    E2Z4=E2Z4+E(J)*E(J)*(Z(J)**2-Z(J-1)**2)**2
      E2Z4=E2Z4-E(1)*E(1)*Z(1)**4
      GE2Z4=1.0/(4.0*G(1))*E2Z4*Z(1)
      DO 60 J=2,NN
      E2Z4=E2Z4-E(J)*E(J)*(Z(J)**2-Z(J-1)**2)**2
60    GE2Z4=GE2Z4+1.0/(4.0*G(J))*E2Z4*(Z(J)-Z(J-1))
      A(1,1)=E(1)*(2./3.)*Z(1)**3
      DO 70 K=2,NN
70    A(1,K)=E(K)*(Z(K)**2*(Z(K)-Z(K-1))-1./3.*(Z(K)**3-Z(K-1)**3))
      DO 71 J=2,N
71    A(J,1)=E(J)*(Z(J)**2-Z(J-1)**2)
      LL=1
      DO 72 K=2,NN
      DO 73 J=2,N
      A(J,K)=A(J+LL,K-LL)
73    CONTINUE
72    LL=LL+1
      GEZ=0.0
```

```
      II=N
      KK=NN
      DO 74 M=1,NN
      AA=0.0
      DO 75 I=1,KK
      L=I+1
      DO 75 J=L,II
      B=A(I,M)*A(J,M)
      AA=AA+B
75    CONTINUE
      C=0.5*G(M)*AA
      GEZ=GEZ+C
      II=II-1
74    KK=KK-1
      SGEZ=GE2Z4+GE2Z5+GEZ
      E1Z3=(1./3.)*E(1)*Z(1)**3
      DO 80 J=2,N
      E1Z3=E1Z3+0.5*E(J)*(Z(J)**2*(Z(J)-Z(J-1))-1./3.*(Z(J)**3-Z(J-1)
     1**3))
80    CONTINUE
      E1Z2Z=0.5*E(1)*Z(1)**2
      DO 90 J=2,N
90    E1Z2Z=E1Z2Z+0.5*E(J)*(Z(J)**2-Z(J-1)**2)
      E1Z2Z=E1Z2Z-0.5*E(1)*Z(1)**2
      AE1Z3=E1Z2Z*Z(1)
      DO 100 J=2,NN
      E1Z2Z=E1Z2Z-0.5*E(J)*(Z(J)**2-Z(J-1)**2)
      AE1Z3=AE1Z3+E1Z2Z*(Z(J)-Z(J-1))
100   CONTINUE
      SE1Z3=(E1Z3+AE1Z3)**2
      SFK=SE1Z3/(SUMGZ*SGEZ)
      WRITE(6,110)N,SFK
110   FORMAT(' N=',I2,5X,'SHEAR FACTOR K =',F7.4//)
      GO TO 120
  130 CALL EXIT
      END
```

```
C
C      PLATE VIBRATION CAL. BY C.C. SIU
C      COMPOSITE SHAPE FACTOR   K11X2,K22X2,K12X2
C      USING AN EXACT ANALYSIS BY S. SRINIVAS TO COMPARE
C      WITH PLATE THEORY INCLUDING TRANSVERSE SHEAR DEFORMATION
C      STIFFNESS COEFFICENTS FOR  3 LAYER
C      1ST LAYER IS SAME AS 3RD LAYER
C
       DOUBLE PRECISION H,H1,H2,RO1,RO2,G1,G2,K11X2,K22X2,K12X2,ANU,AN
       DOUBLE PRECISION C11I,C111I,C12I,C12II,C22I,C22II,C44I,C44II,
      1C45I,C45II,C55I,C55II,C66I,C66II
       DOUBLE PRECISION AS(5,5),AI(5,5)
       DOUBLE PRECISION EVAL(5),EVEC(5,5)
       INTEGER RA,RB,RC,RD,RE
       INTEGER CA,CB,CC,CD,CE
C
C      READ PLATE PROPERTIES
C
       JJJ=1
    5  READ(5,7) K11X2,K22X2,K12X2,ANU,AN
    7  FORMAT(3F10.4,2F10.2)
C
C      WRITE PLATE PROPERTIES
C
       WRITE(6,8)K11X2,K22X2,K12X2,ANU,AN
    8  FORMAT(3F10.4,2F10.2)
C
C      CAL. COEFFICIENT
C
       C11I=1.D0/(1.D0-ANU**2)
       C111I=C11I
       C12I=ANU/(1.D0-ANU**2)
       C12II=C12I
       C22I=C11I
       C22II=C11I
       C44I=1.D0/(2.D0*(1.D0+ANU))
```

```
      C44II=C44I*(1.D0/AN)
      C45I=0.D0
      C45II=0.D0
      C55I=C44I
      C55II=C44II
      C66I=C44I
      C66II=C44II
      PI=3.1415926536D0
      G1=0.0001**0.5*PI
      G2=0.0064**0.5*PI
      H1=1.D0
      H2=8.D0
      H=10.D0
      RO1=1.D0
      RO2=1.D0
      RA=1
      RB=2
      RC=3
      RD=4
      RE=5
      CA=1
      CB=2
      CC=3
      CD=4
       CE=5
C
C     CALCULATE STIFFNESS MATRIX
C
      AS(RA,CA)=25.D0*(C11I/C66II+C66I/C66II)*G1**2
     1+25.D0/16.D0*(C11II/C66II+1.D0)*G2**2
      AS(RA,CB)=25.D0*(C12I/C66II+C66I/C66II)*G1**2
     1+25.D0/16.D0*(C12II/C66II+1.D0)*G2**2
      AS(RA,CC)=0.D0
      AS(RA,CD)=0.D0
      AS(RA,CE)=0.D0
      AS(RB,CA)=25.D0*(C12I/C66II+C66I/C66II)*G1**2
```

```
     1+25.D0/16.D0*(C12II/C66II+1.D0)*G2**2
      AS(RB,CB)=25.D0*(C22I/C66II+C66I/C66II)*G1**2
     1+25.D0/16.D0*(C22II/C66II+1.D0)*G2**2
      AS(RB,CC)=0.D0
      AS(RB,CD)=0.D0
      AS(RB,CE)=0.D0
      AS(RC,CA)=0.D0
      AS(RC,CB)=0.D0
      AS(RC,CC)=(200.D0*K11X2*(H1/H2)*(C44I/C66II)+200.D0*K22X2*
     1(H1/H2)*(C55I/C66II)+400.D0*K12X2*(H1/H2)*(C45I/C66II))*G1**2
     2+(25.D0/16.D0*K11X2*(C44II/C66II)+25.D0/16.D0*K22X2*(C55II/C66II)
     3+25.D0/8.D0*K12X2*(C45II/C66II))*G2**2
      AS(RC,CD)=20.D0*K11X2*(H1/H2)*(C44I/C66II)*G1
     1+(5.D0/4.D0)*K11X2*(C44II/C66II)*G2
      AS(RC,CE)=20.D0*K22X2*(H1/H2)*(C55I/C66II)*G1
     1+(5.D0/4.D0)*K22X2*(C55II/C66II)*G2
      AS(RD,CA)=0.D0
      AS(RD,CB)=0.D0
      AS(RD,CC)=20.D0*K11X2*(H1/H2)*(C44I/C66II)*G1
     1+(5.D0/4.D0)*K11X2*(C44II/C66II)*G2
      AS(RD,CD)=61.D0/12.D0*(C11I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
     1(C11II/C66II+1.D0)*G2**2+K11X2*(2.D0*(H1/H2)*(C44I/C66II)+(C44II
     2/C66II))
      AS(RD,CE)=61.D0/12.D0*(C12I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
     1(C12II/C66II+1.D0)*G2**2
      AS(RE,CA)=0.D0
      AS(RE,CB)=0.D0
      AS(RE,CC)=20.D0*K22X2*(H1/H2)*(C55I/C66II)*G1
     1+(5.D0/4.D0)*K22X2*(C55II/C66II)*G2
      AS(RE,CD)=61.D0/12.D0*(C12I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
     1(C12II/C66II+1.D0)*G2**2
      AS(RE,CE)=61.D0/12.D0*(C22I/C66II+C66I/C66II)*G1**2+1.D0/12.D0*
     1(C22II/C66II+1.D0)*G2**2+K22X2*(2.D0*(H1/H2)*(C55I/C66II)+(C55II
     2/C66II))
C
C     CALCULATE INERTIA MATRIX
```

```
C
      AI(RA,CA)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RA,CB)=0.D0
      AI(RA,CC)=0.D0
      AI(RA,CD)=0.D0
      AI(RA,CE)=0.D0
      AI(RB,CA)=0.D0
      AI(RB,CB)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RB,CC)=0.D0
      AI(RB,CD)=0.D0
      AI(RB,CE)=0.D0
      AI(RC,CA)=0.D0
      AI(RC,CB)=0.D0
      AI(RC,CC)=(H/H2)**2*(2.D0*(H1/H2)*R01/R02+1.D0)
      AI(RC,CD)=0.D0
      AI(RC,CE)=0.D0
      AI(RD,CA)=0.D0
      AI(RD,CB)=0.D0
      AI(RD,CC)=0.D0
      AI(RD,CD)=1.D0/12.D0+61.D0/768.D0*(R01/R02)
      AI(RD,CE)=0.D0
      AI(RE,CA)=0.D0
      AI(RE,CB)=0.D0
      AI(RE,CC)=0.D0
      AI(RE,CD)=0.D0
      AI(RE,CE)=1.D0/12.D0+61.D0/768.D0*(R01/R02)
C
C     CALL DNRGOT
C
      CALL DNROOT(5,AS,AI,EVAL,EVEC)
C
C     WRITE EIGENVALUES
C
      DO 910 I=1,5
      EVAL(I)=DSQRT(EVAL(I))
  910 WRITE(6,933)I,EVAL(I)
```

```
  933 FORMAT('0',40X,'EIGENVALUE(',I2,')=',D16.8)
      JJJ=JJJ+1
      IF(50-JJJ)951,5,5
  951 STOP
      END
      SUBROUTINE DNROOT (M,A,B,XL,X)
C     SEE WRITE-UP IN IBM SSP, PAGE 31
      DIMENSION A(1),B(1),XL(1),X(1)
      DOUBLE PRECISION A,B,XL,X,SUMV
C     COMPUTE EIGENVALUES AND EIGENVECTORS OF B
      K=1
      DO 100 J=2,M
      L=M*(J-1)
      DO 100 I=1,J
      L=L+1
      K=K+1
  100 B(K)=B(L)
C     THE MATRIX B IS A REAL SYMMETRIC MATRIX
      MV=0
       CALL DEIGEN (B,X,M,MV)
C     FORM RECIPROCALS OF SQUARE ROOT OF EIGENVALUES. THE RESULTS
C     ARE PREMULTIPLIED BY THE ASSOCIATED EIGENVECTORS.
      L=0
      DO 110 J=1,M
      L=L+J
  110 XL(J)=1.0/DSQRT(DABS(B(L)))
      K=0
      DO 115 J=1,M
      DO 115 I=1,M
      K=K+1
  115 B(K)=X(K)*XL(J)
C     FORM (B**(-1/2))PRIME * A * (B**(-1/2))
      DO 120 I=1,M
      N2=0
      DO 120 J=1,M
      N1=M*(I-1)
```

```
      L=M*(J-1)+I
      X(L)=0.0
      DO 120 K=1,M
      N1=N1+1
      N2=N2+1
  120 X(L)=X(L)+B(N1)*A(N2)
      L=0
      DO 130 J=1,M
      DO 130 I=1,J
      N1=I-M
      N2=M*(J-1)
      L=L+1
      A(L)=0.0
      DO 130 K=1,M
      N1=N1+M
      N2=N2+1
  130 A(L)=A(L)+X(N1)*B(N2)
C     COMPUTE EIGENVALUES AND EIGENVECTORS OF A
      CALL DEIGEN (A,X,M,MV)
      L=0
      DO 140 I=1,M
      L=L+I
  140 XL(I)=A(L)
C     COMPUTE THE NORMALIZED EIGENVECTORS
      DO 150 I=1,M
      N2=0
      DO 150 J=1,M
      N1=I-M
      L=M*(J-1)+I
      A(L)=0.0
      DO 150 K=1,M
      N1=N1+M
      N2=N2+1
  150 A(L)=A(L)+B(N1)*X(N2)
      I=0
      K=0
```

```
      DO 180 J=1,M
      SUMV=0.0
      DO 170 I=1,M
      L=L+1
  170 SUMV=SUMV+A(L)*A(L)
  175 SUMV=DSQRT(SUMV)
      DO 180 I=1,M
      K=K+1
  180 X(K)=A(K)/SUMV
      RETURN
      END
C
C     ..................................................................
C
C        SUBROUTINE DEIGEN
C
C        PURPOSE
C           COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC
C           MATRIX
C
C        USAGE
C           CALL DEIGEN(A,R,N,MV)
C
C        DESCRIPTION OF PARAMETERS
C           A - ORIGINAL MATRIX (SYMMETRIC), DESTROYED IN COMPUTATION.
C               RESULTANT EIGENVALUES ARE DEVELOPED IN DIAGONAL OF
C               MATRIX A IN DESCENDING ORDER.
C           R - RESULTANT MATRIX OF EIGENVECTORS (STORED COLUMNWISE,
C               IN SAME SEQUENCE AS EIGENVALUES)
C           N - ORDER OF MATRICES A AND R
C           MV- INPUT CODE
C                     0   COMPUTE EIGENVALUES AND EIGENVECTORS
C                     1   COMPUTE EIGENVALUES ONLY (R NEED NOT BE
C                         DIMENSIONED BUT MUST STILL APPEAR IN CALLING
C                         SEQUENCE)
C
```

```
C        REMARKS                                                        EIGEN026
C           ORIGINAL MATRIX A MUST BE REAL SYMMETRIC (STORAGE MODE=1)   EIGEN027
C           MATRIX A CANNOT BE IN THE SAME LOCATION AS MATRIX R         EIGEN028
C                                                                       EIGEN029
C        SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                  EIGEN030
C           NONE                                                        EIGEN031
C                                                                       EIGEN032
C        METHOD                                                         EIGEN033
C           DIAGONALIZATION METHOD ORIGINATED BY JACOBI AND ADAPTED     EIGEN034
C           BY VON NEUMANN FOR LARGE COMPUTERS AS FOUND IN 'MATHEMATICALEIGEN035
C           METHODS FOR DIGITAL COMPUTERS', EDITED BY A. RALSTON AND    EIGEN036
C           H.S. WILF, JOHN WILEY AND SONS, NEW YORK, 1962, CHAPTER 7   EIGEN037
C                                                                       EIGEN038
C     .....EIGEN039
C                                                                       EIGEN040
      SUBROUTINE DEIGEN(A,R,N,MV)                                       EIGEN041
      DIMENSION A(1),R(1)                                               EIGEN042
C                                                                       EIGEN043
C        .....EIGEN044
C                                                                       EIGEN045
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  EIGEN046
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION      EIGEN047
C        STATEMENT WHICH FOLLOWS.                                       EIGEN048
C                                                                       EIGEN049
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,COSX,         EIGEN050
     1                 COSX2,SINCS                                      EIGEN051
C                                                                       EIGEN052
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS    EIGEN053
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS      EIGEN054
C        ROUTINE.                                                       EIGEN055
C                                                                       EIGEN056
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO      EIGEN057
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTSEIGEN058
C        40, 68, 75, AND 78 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT EIGEN059
C        62 MUST BE CHANGED TO DABS.                                    EIGEN060
C                                                                       EIGEN061
```

```
C        ..................................................................EIGEN062
C                                                                          EIGEN063
C           GENERATE IDENTITY MATRIX                                       EIGEN064
C                                                                          EIGEN065
      IF(MV-1) 10,25,10                                                    EIGEN066
   10 IQ=-N                                                                EIGEN067
      DO 20 J=1,N                                                          EIGEN068
      IQ=IQ+N                                                              EIGEN069
      DO 20 I=1,N                                                          EIGEN070
      IJ=IQ+I                                                              EIGEN071
      R(IJ)=0.D+00                                                         EIGEN072
      IF(I-J) 20,15,20                                                     EIGEN073
   15 R(IJ)=1.D+00                                                         EIGEN074
   20 CONTINUE                                                             EIGEN075
C                                                                          EIGEN076
C           COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)             EIGEN077
C                                                                          EIGEN078
   25 ANORM=0.D+00                                                         EIGEN079
      DO 35 I=1,N                                                          EIGEN080
      DO 35 J=I,N                                                          EIGEN081
      IF(I-J) 30,35,30                                                     EIGEN082
   30 IA=I+(J*J-J)/2                                                       EIGEN083
      ANORM=ANORM+A(IA)*A(IA)                                              EIGEN084
   35 CONTINUE                                                             EIGEN085
      IF(ANORM) 165,165,40                                                 EIGEN086
   40 ANORM=1.414D+00*DSQRT(ANORM)                                         EIGEN087
      ANRMX=ANORM*1.0D-06/FLOAT(N)                                         EIGEN088
C                                                                          EIGEN089
C           INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR               EIGEN090
C                                                                          EIGEN091
      IND=0                                                                EIGEN092
      THR=ANORM                                                            EIGEN093
   45 THR=THR/FLOAT(N)                                                     EIGEN094
   50 L=1                                                                  EIGEN095
   55 M=L+1                                                                EIGEN096
C                                                                          EIGEN097
```

```
C         COMPUTE SIN AND COS                                                  EIGEN098
C                                                                              EIGEN099
   60 MQ=(M*M-M)/2                                                             EIGEN100
      LQ=(L*L-L)/2                                                             EIGEN101
      LM=L+MQ                                                                  EIGEN102
   62 IF(DABS(A(LM))-THR) 130,65,65                                            EIGEN103
   65 IND=1                                                                    EIGEN104
      LL=L+LQ                                                                  EIGEN105
      MM=M+MQ                                                                  EIGEN106
      X=0.5D+00*(A(LL)-A(MM))                                                  EIGEN107
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)                                          EIGEN108
      IF(X) 70,75,75                                                           EIGEN109
   70 Y=-Y                                                                     EIGEN110
   75 SINX=Y/DSQRT(2.D+00*(1.D+00+(DSQRT(1.D+00-Y*Y))))                        EIGEN111
      SINX2=SINX*SINX                                                          EIGEN112
   78 COSX=DSQRT(1.0D+00-SINX2)                                                EIGEN113
      COSX2=COSX*COSX                                                          EIGEN114
      SINCS =SINX*COSX                                                         EIGEN115
C                                                                              EIGEN116
C         ROTATE L AND M COLUMNS                                               EIGEN117
C                                                                              EIGEN118
      ILQ=N*(L-1)                                                              EIGEN119
      IMQ=N*(M-1)                                                              EIGEN120
      DO 125 I=1,N                                                             EIGEN121
      IQ=(I*I-I)/2                                                             EIGEN122
      IF(I-L) 80,115,80                                                        EIGEN123
   80 IF(I-M) 85,115,90                                                        EIGEN124
   85 IM=I+MQ                                                                  EIGEN125
      GO TO 95                                                                 EIGEN126
   90 IM=M+IQ                                                                  EIGEN127
   95 IF(I-L) 100,105,105                                                      EIGEN128
  100 IL=I+LQ                                                                  EIGEN129
      GO TO 110                                                                EIGEN130
  105 IL=L+IQ                                                                  EIGEN131
  110 X=A(IL)*COSX-A(IM)*SINX                                                  EIGEN132
      A(IM)=A(IL)*SINX+A(IM)*COSX                                              EIGEN133
```

```
      A(IL)=X                                                          EIGEN134
  115 IF(MV-1) 120,125,120                                             EIGEN135
  120 ILR=ILQ+I                                                        EIGEN136
      IMR=IMQ+I                                                        EIGEN137
      X=R(ILR)*COSX-R(IMR)*SINX                                        EIGEN138
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX                                   EIGEN139
      R(ILR)=X                                                         EIGEN140
  125 CONTINUE                                                         EIGEN141
      X=2.D+00*A(LM)*SINCS                                             EIGEN142
      Y=A(LL)*COSX2+A(MM)*SINX2-X                                      EIGEN143
      X=A(LL)*SINX2+A(MM)*COSX2+X                                      EIGEN144
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)                    EIGEN145
      A(LL)=Y                                                          EIGEN146
      A(MM)=X                                                          EIGEN147
C                                                                      EIGEN148
C        TESTS FOR COMPLETION                                          EIGEN149
C                                                                      EIGEN150
C        TEST FOR M = LAST COLUMN                                      EIGEN151
C                                                                      EIGEN152
  130 IF(M-N) 135,140,135                                              EIGEN153
  135 M=M+1                                                            EIGEN154
      GO TO 60                                                         EIGEN155
C                                                                      EIGEN156
C        TEST FOR L = SECOND FROM LAST COLUMN                          EIGEN157
C                                                                      EIGEN158
  140 IF(L-(N-1)) 145,150,145                                          EIGEN159
  145 L=L+1                                                            EIGEN160
      GO TO 55                                                         EIGEN161
  150 IF(IND-1) 160,155,160                                            EIGEN162
  155 IND=0                                                            EIGEN163
      GO TO 50                                                         EIGEN164
C                                                                      EIGEN165
  160 IF(THR-ANRMX) 165,165,45                                         EIGEN168
C        COMPARE THRESHOLD WITH FINAL NORM                             EIGEN166
C                                                                      EIGEN167
C                                                                      EIGEN169
```

```
C         SORT EIGENVALUES AND EIGENVECTORS                              EIGEN170
C                                                                        EIGEN171
  165 IQ=-N                                                              EIGEN172
      DO 185 I=1,N                                                       EIGEN173
      IQ=IQ+N                                                            EIGEN174
      LL=I+(I*I-I)/2                                                     EIGEN175
      JQ=N*(I-2)                                                         EIGEN176
      DO 185 J=I,N                                                       EIGEN177
      JQ=JQ+N                                                            EIGEN178
      MM=J+(J*J-J)/2                                                     EIGEN179
      IF(A(LL)-A(MM)) 170,185,185                                        EIGEN180
  170 X=A(LL)                                                            EIGEN181
      A(LL)=A(MM)                                                        EIGEN182
      A(MM)=X                                                            EIGEN183
      IF(MV-1) 175,185,175                                               EIGEN184
  175 DO 180 K=1,N                                                       EIGEN185
      ILR=IQ+K                                                           EIGEN186
      IMR=JQ+K                                                           EIGEN187
      X=R(ILR)                                                           EIGEN188
      R(ILR)=R(IMR)                                                      EIGEN189
  180 R(IMR)=X                                                           EIGEN190
  185 CONTINUE                                                           EIGEN191
      RETURN                                                             EIGEN192
      END                                                                EIGEN193
      SUBROUTINE CHECK (I)
      WRITE (3,1) I
    1 FORMAT (' CHECK ',I3)
      RETURN
      END
```

```
C
C       VIBRATION OF LAMINATED ANISOTROPIC PLATES
C       CONSIDERING THICKNESS SHEAR AND DAMPING EFFECTS
C
        DIMENSION S(40,40),T(40)
        DIMENSION ANGLE(8),H(24),RO(24)
        DIMENSION ER11(24),ER22(24),GE11(24),GE22(24)
        DIMENSION VR12(24),VR21(24)
        DIMENSION GR12(24),GR13(24),GR23(24),GG12(24),GG13(24),GG23(24)
        DIMENSION CRT11(24),CIT11(24),CRT12(24),CIT12(24)
        DIMENSION CRT22(24),CIT22(24),CRT66(24),CIT66(24)
        DIMENSION CRT16(24),CIT16(24),CRT26(24),CIT26(24)
        DIMENSION CRT44(24),CIT44(24),CRT45(24),CIT45(24)
        DIMENSION CRT55(24),CIT55(24)
        REAL K11,K22,K12
        READ(5,15)J,NN,INPU1,INPU2
   15   FORMAT(4I5)
        READ(5,32)A,B,K11,K22,K12
   32   FORMAT(5F10.4)
   10   READ(5,35,END=3333)F
   35   FORMAT(F10.4)
        READ(5,36)(ANGLE(I),H(I),H(I+1),RO(I),I=1,J)
   36   FORMAT(4F10.0)
        READ(5,37)(ER11(I),ER22(I),GE11(I),GE22(I),I=1,J)
   37   FORMAT(4E10.4)
        READ(5,39)(VR12(I),VR21(I),I=1,J)
   39   FORMAT(2F10.6)
        READ(5,40)(GR12(I),GR13(I),GR23(I),GG12(I),GG13(I),GG23(I),I=1,J)
   40   FORMAT(6E10.4)
C
C       CALCULATE STIFFNESS,DAMPING, AND DENSITY COEFFICIENTS
C
        SMAR11=0.
        SMAI11=0.
        SMAR12=0.
        SMAI12=0.
```

```
SMAR16=0.
SMAI16=0.
SMAR22=0.
SMAI22=0.
SMAR26=0.
SMAI26=0.
SMAR66=0.
SMAI66=0.
SMAR44=0.
SMAI44=0.
SMAR45=0.
SMAI45=0.
SMAR55=0.
SMAI55=0.
SMBR11=0.
SMBI11=0.
SMBR12=0.
SMBI12=0.
SMBR16=0.
SMBI16=0.
SMBR22=0.
SMBI22=0.
SMBR26=0.
SMBI26=0.
SMBR66=0.
SMBI66=0.
SMDR11=0.
SMDI11=0.
SMDR12=0.
SMDI12=0.
SMDR16=0.
SMDI16=0.
SMDR22=0.
SMDI22=0.
SMDR26=0.
SMDI26=0.
```

```
      SMDR66=0.
      SMDI66=0.
      SUMP0=0.
      SUMP1=0.
      SUMP2=0.
      PI=3.141592653589793
      DO 150 I=1,J
      CR11=ER11(I)/(1.-VR12(I)*VR21(I))
      CI11=ER11(I)/(1.-VR12(I)*VR21(I))*GE11(I)
      CR22=ER22(I)/(1.-VR12(I)*VR21(I))
      CI22=ER22(I)/(1.-VR12(I)*VR21(I))*GE22(I)
      CR12=VR12(I)*ER22(I)/(1.-VR12(I)*VR21(I))
      CR12=VR12(I)*ER22(I)/(1.-VR12(I)*VR21(I))*GE22(I)
      CR44=GR13(I)
      CI44=GR13(I)*GG13(I)
      CR55=GR23(I)
      CI55=GR23(I)*GG23(I)
      CR66=GR12(I)
      CI66=GR12(I)*GG12(I)
      THETA=ANGLE(I)*PI/180.
      C4=COS(THETA)*COS(THETA)*COS(THETA)*COS(THETA)
      S4=SIN(THETA)*SIN(THETA)*SIN(THETA)*SIN(THETA)
      C3S1=COS(THETA)*COS(THETA)*COS(THETA)*SIN(THETA)
      S3C1=SIN(THETA)*SIN(THETA)*SIN(THETA)*COS(THETA)
      C2S2=COS(THETA)*COS(THETA)*SIN(THETA)*SIN(THETA)
      C2=COS(THETA)*COS(THETA)
      S2=SIN(THETA)*SIN(THETA)
      CS=COS(THETA)*SIN(THETA)
      CRT11(I)=CR11*C4+2.*(CR12+2.*CR66)*C2S2+CR22*S4
      CIT11(I)=CI11*C4+2.*(CI12+2.*CI66)*C2S2+CI22*S4
C
C     COEFFICIENT TRANSFORMATION
C
      CRT12(I)=CR12*C4+(CR11+CR22-4.*CR66)*C2S2+CR12*S4
      CIT12(I)=CI12*C4+(CI11+CI22-4.*CI66)*C2S2+CI12*S4
      CRT22(I)=CR22*C4+2.*(CR12+2.*CR66)*C2S2+CR11*S4
```

```
CIT22(I)=CI22*C4+2.*(CI12+2.*CI66)*C2S2+CI11*S4
CRT66(I)=CR66*C4+(CR11+CR22-2.*CR12-2.*CR66)*C2S2+CR66*S4
CIT66(I)=CI66*C4+(CI11+CI22-2.*CI12-2.*CI66)*C2S2+CI66*S4
CRT16(I)=(2.*CR66+CR12-CR11)*C3S1-(2.*CR66+CR12-CR22)*S3C1
CIT16(I)=(2.*CI66+CI12-CI11)*C3S1-(2.*CI66+CI12-CI22)*S3C1
CRT26(I)=-(2.*CR66+CR12-CR22)*C3S1+(2.*CR66+CR12-CR11)*S3C1
CIT26(I)=-(2.*CI66+CI12-CI22)*C3S1+(2.*CI66+CI12-CI11)*S3C1
CRT44(I)=CR44*C2+CR55*S2
CIT44(I)=CI44*C2+CI55*S2
CRT45(I)=CR55*CS-CR44*CS
CIT45(I)=CI55*CS-CI44*CS
CRT55(I)=CR55*C2+CR44*S2
CIT55(I)=CI55*C2+CI44*S2
AAR11=CRT11(I)*(H(I+1)-H(I))
SMAR11=SMAR11+AAR11
AAI11=CIT11(I)*(H(I+1)-H(I))
SMAI11=SMAI11+AAI11
AAR12=CRT12(I)*(H(I+1)-H(I))
SMAR12=SMAR12+AAR12
AAI12=CIT12(I)*(H(I+1)-H(I))
SMAI12=SMAI12+AAI12
AAR22=CRT22(I)*(H(I+1)-H(I))
SMAR22=SMAR22+AAR22
AAI22=CIT22(I)+(H(I+1)-H(I))
SMAI22=SMAI22+AAI22
AAR66=CRT66(I)*(H(I+1)-H(I))
SMAR66=SMAR66+AAR66
AAI66=CIT66(I)*(H(I+1)-H(I))
SMAI66=SMAI66+AAI66
AAR16=CRT16(I)*(H(I+1)-H(I))
SMAR16=SMAR16+AAR16
AAI16=CIT16(I)*(H(I+1)-H(I))
SMAI16=SMAI16+AAI16
AAR26=CRT26(I)*(H(I+1)-H(I))
SMAR26=SMAR26+AAR26
AAI26=CIT26(I)*(H(I+1)-H(I))
```

```
SMAI26=SMAI26+AAI26
AAR44=CRT44(I)*(H(I+1)-H(I))
SMAR44=SMAR44+AAR44
AAI44=CIT44(I)*(H(I+1)-H(I))
SMAI44=SMAI44+AAI44
AAR45=CRT45(I)*(H(I+1)-H(I))
SMAR45=SMAR45+AAR45
AAI45=CIT45(I)*(H(I+1)-H(I))
SMAI45=SMAI45+AAI45
AAR55=CRT55(I)*(H(I+1)-H(I))
SMAR55=SMAR55+AAR55
AAI55=CIT55(I)*(H(I+1)-H(I))
SMAI55=SMAI55+AAI55
BBR11=CRT11(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR11=SMBR11+BBR11
BBI11=CIT11(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI11=SMBI11+BBI11
BBR12=CRT12(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR12=SMBR12+BBR12
BBI12=CIT12(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI12=SMBI12+BBI12
BBR22=CRT22(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR22=SMBR22+BBR22
BBI22=CIT22(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI22=SMBI22+BBI22
BBR66=CRT66(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR66=SMBR66+BBR66
BBI66=CIT66(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI66=SMBI66+BBI66
BBR16=CRT16(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR16=SMBR16+BBR16
BBI16=CIT16(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBI16=SMBI16+BBI16
BBR26=CRT26(I)*(H(I+1)*H(I+1)-H(I)*H(I))
SMBR26=SMBR26+BBR26
BBI26=CIT26(I)*(H(I+1)*H(I+1)-H(I)*H(I))
```

```
      SMBI26=SMBI26+BBI26
      DDR11=CRT11(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR11=SMDR11+DDR11
      DDI11=CIT11(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI11=SMDI11+DDI11
      DDR12=CRT12(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR12=SMDR12+DDR12
      DDI12=CIT12(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI12=SMDI12+DDI12
      DDR22=CRT22(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR22=SMDR22+DDR22
      DDI22=CIT22(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI22=SMDI22+DDI22
      DDR66=CRT66(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR66=SMDR66+DDR66
      DDI66=CIT66(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI66=SMDI66+DDI66
      DDR16=CRT16(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR16=SMDR16+DDR16
      DDI16=CIT16(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI16=SMDI16+DDI16
      DDR26=CRT26(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDR26=SMDR26+DDR26
      DDI26=CIT26(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
      SMDI26=SMDI26+DDI26
      PP0=R0(I)*(H(I+1)-H(I))
      SUMP0=SUMP0+PP0
      PP1=R0(I)*(H(I+1)*H(I+1)-H(I)*H(I))
      SUMP1=SUMP1+PP1
      PP2=R0(I)*(H(I+1)*H(I+1)*H(I+1)-H(I)*H(I)*H(I))
  150 SUMP2=SUMP2+PP2
      AR11=SMAR11
      AI11=SMAI11
      AR12=SMAR12
      AI12=SMAI12
      AR22=SMAR22
```

```
AI22=SMAI22
AR66=SMAR66
AI66=SMAI66
AR16=SMAR16
AI16=SMAI16
AR26=SMAR26
AI26=SMAI26
AR44=SMAR44
AI44=SMAI44
AR45=SMAR45
AI45=SMAI45
AR55=SMAR55
AI55=SMAI55
BR11=0.5*SMBR11
BI11=0.5*SMBI11
BR12=0.5*SMBR12
BI12=0.5*SMBI12
BR22=0.5*SMBR22
BI22=0.5*SMBI22
BR66=0.5*SMBR66
BI66=0.5*SMBI66
BR16=0.5*SMBR16
BI16=0.5*SMBI16
BR26=0.5*SMBR26
BI26=0.5*SMBI26
DR11=SMDR11/3.
DI11=SMDI11/3.
DR12=SMDR12/3.
DI12=SMDI12/3.
DR22=SMDR22/3.
DI22=SMDI22/3.
DR66=SMDR66/3.
DI66=SMDI66/3.
DR16=SMDR16/3.
DI16=SMDI16/3.
DR26=SMDR26/3.
```

```
      DI26=SMDI26/3.
      P0=SUMP0
      P1=0.5*SUMP1
      P2=SUMP2/3.
C
C     WRITE MATERIAL PROPERTIES AND STIFFNESS COEFFICIENTS
C     CONSIDERING THICKNESS SHEAR AND DAMPING EFFECTS
      WRITE (6,50) AR11,BR11,DR11
   50 FORMAT('0',18X,'AR11= ',E12.5,5X, 'BR11= ',E12.5,5X,
     1'DR11= ',E12.5)
      WRITE(6,55)AI11,BI11,DI11
   55 FORMAT('0',18X,'AI11= ',E12.5,5X, 'BI11= ',E12.5,5X,
     1'DI11= ',E12.5)
      WRITE(6,60)AR12,BR12,DR12
   60 FORMAT('0',18X,'AR12= ',E12.5,5X, 'BR12= ',E12.5,5X,
     1'DR12= ',E12.5)
      WRITE(6,65)AI12,BI12,DI12
   65 FORMAT('0',18X,'AI12= ',E12.5,5X, 'BI12= ',E12.5,5X,
     1'DI12= ',E12.5)
      WRITE(6,70)AR22,BR22,DR22
   70 FORMAT('0',18X,'AR22= ',E12.5,5X, 'BR22= ',E12.5,5X,
     1'DR22= ',E12.5)
      WRITE(6,75)AI22,BI22,DI22
   75 FORMAT('0',18X,'AI22= ',E12.5,5X, 'BI22= ',E12.5,5X,
     1'DI22= ',E12.5)
      WRITE(6,80)AR66,BR66,DR66
   80 FORMAT('0',18X,'AR66= ',E12.5,5X, 'BR66= ',E12.5,5X,
     1'DR66= ',E12.5)
      WRITE(6,85)AI66,BI66,DI66
   85 FORMAT('0',18X,'AI66= ',E12.5,5X, 'BI66= ',E12.5,5X,
     1'DI66= ',E12.5)
      WRITE(6,90)AR16,BR16,DR16
   90 FORMAT('0',18X,'AR16= ',E12.5,5X, 'BR16= ',E12.5,5X,
     1'DR16= ',E12.5)
      WRITE(6,95)AI16,BI16,DI16
   95 FORMAT('0',18X,'AI16= ',E12.5,5X, 'BI16= ',E12.5,5X,
```

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```
      JXI=8*NN**2
      JYI=9*NN**2
      JJUR=JJUR+1
      DO 500 M1=1,NN
      M=M1+INPU1
      DO 500 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI8MK,CI8NL
      IF(M-K)502,504,502
  502 CI8MK=0.
      GO TO 505
  504 CI8MK=2.*M*K*PI**2
  505 CONTINUE
      IF(N-L)506,508,506
  506 CI8NL=0.
      GO TO 509
  508 CI8NL=2.*N*L*PI**2
  509 CONTINUE
C
C     INTEGRAL CI1MK, CI1NL
C
      IF(M-K)512,514,512
  512 CI1MK=1.
      GO TO 515
  514 CI1MK=1.5
  515 CONTINUE
      IF(N-L)516,518,516
  516 CI1NL=1.
      GO TO 519
  518 CI1NL=1.5
  519 CONTINUE
C
C     INTEGRAL CI6MK,CI6NL
C
```

```
      IF(M-K)522,524,522
  522 CI6MK=0.
      GO TO 525
  524 CI6MK=0.
  525 CONTINUE
      IF(N-L)526,528,526
  526 CI6NL=0.
      GO TO 529
  528 CI6NL=0.
  529 CONTINUE
C
C     INTEGRAL CI5MK,CI5NL
C
      CI5MK=CI6MK
      CI5NL=CI6NL
C
C     INTEGRAL CI4MK,CI4NL
C
      CI4MK=CI6MK
      CI4NL=CI6NL
C
C     INTEGRAL CI10MK,CI10NL
C
      CI10MK=CI8MK
      CI10NL=CI8NL
C
C     INTEGRAL CI3MK,CI3NL
C
      CI3MK=CI1MK
      CI3NL=CI1NL
C
C     INTEGRAL CI44MK,CI44NL
C
      CI44MK=CI8MK
      CI44NL=CI8NL
C
```

```
C       INTEGRAL CI31MK,CI31NL
C
        CI31MK=CI1MK
        CI31NL=CI1NL
C
C       INTEGRAL CI34MK,CI34NL
C
        CI34MK=CI4MK
        CI34NL=CI4NL
C
C       INTEGRAL CI33MK, CI33NL
C
        CI33MK=CI4MK
        CI33NL=CI4NL
C
C       INTEGRAL CI37MK, CI37NL
C
        CI37MK=CI4MK
        CI37NL=CI4NL
C
//216 CI17NL=1.
        JYI=JYI+1
  800 S(JJXR,JYI)=-(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
     1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
     2*CI23NL)
C       CALCULATE SUBMATRIX
C       YRUR,YRVR,YRWR,YRXR,YRYR,YRUI,YRVI,YRWI,YRXI,YRYI
        JJYR=4*NN**2
        DO 900 K1=1,NN
        K=K1+INPU1
        DO 900 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
```

```
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJYR=JJYR+1
      DO 900 M1=1,NN
      M=M1+INPU1
      DO 900 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI37MK, CI37NL
C
      IF(M-K)902,904,902
  902 CI37MK=0.
      GO TO 905
  904 CI37MK=0.
  905 CONTINUE
      IF(N-L)906,908,906
  906 CI37NL=0.
      GO TO 909
  908 CI37NL=0.
  909 CONTINUE
C
C     INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI37MK
      CI36NL=CI37NL
C
C     INTEGRAL CI55MK, CI55NL
C
      IF(M-K)912,914,912
  912 CI55MK=0.
      GO TO 915
```

```
  914 CI55MK=2.*M*K*PI**2
  915 CONTINUE
      IF(N-L)916,918,916
  916 CI55NL=0.
      GO TO 919
  918 CI55NL=2.*N*L*PI**2
  919 CONTINUE
C
C     INTEGRAL CI35MK, CI35NL
C
      IF(M-K)922,924,922
  922 CI35MK=1.
      GO TO 925
  924 CI35MK=1.5
  925 CONTINUE
      IF(N-L)926,928,926
  926 CI35NL=1.0
      GO TO 929
  928 CI35NL=1.5
  929 CONTINUE
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI35MK
      CI41NL=CI35NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI55MK
      CI57NL=CI55NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI37MK
      CI42NL=CI37NL
C
```

```
C       INTEGRAL CI43MK, CI43NL
C
        CI43MK=CI37MK
        CI43NL=CI37NL
C
C       INTEGRAL CI15MK, CI15NL
C
        CI15MK=0.
        CI15NL=0.
C
C       INTEGRAL CI12MK, CI12NL
C
        CI12MK=CI35MK
        CI12NL=CI35NL
C
C       INTEGRAL CI24MK, CI24NL
C
        CI24MK=CI37MK
        CI24NL=CI37NL
C
C       INTEGRAL CI25MK, CI25NL
C
        CI25MK=CI37MK
        CI25NL=CI37NL
C
C       INTEGRAL CI23MK, CI23NL
C
        CI23MK=CI35MK
        CI23NL=CI35NL
C
C       INTEGRAL CI30MK, CI30NL
C
        CI30MK=CI55MK
        CI30NL=CI55NL
C
C       INTEGRAL CI22MK, CI22NL
```

```
C
      CI22MK=CI35MK
      CI22NL=CI35NL
C
C     INTEGRAL CI29MK, CI29NL
C
      CI29MK=CI55MK
      CI29NL=CI55NL
C
C     INTEGRAL CI27MK, CI27NL
C
      CI27MK=CI37MK
      CI27NL=CI37NL
      S(JJYR,JUR)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
      JVR=JVR+1
      S(JJYR,JVR)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK
     2*CI41NL
      JWR=JWR+1
      S(JJYR,JWR)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
      JXR=JXR+1
      S(JJYR,JXR)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JYR=JYR+1
      S(JJYR,JYR)=(A/B)*DR22*CI22MK*CI29NL+2.*DR26*CI27MK*CI27NL+(B/A)
     1*DR66*CI29MK*CI22NL+A*B*K22*AR55*CI22MK*CI22NL-W2*A*B*P2*CI22MK
     2*CI22NL
      JUI=JUI+1
      S(JJYR,JUI)=-(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
     1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
      JVI=JVI+1
      S(JJYR,JVI)=-((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
      JWI=JWI+1
```

```
      S(JJYR,JWI)=-(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
      JXI=JXI+1
      S(JJYR,JXI)=-(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
     1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
     2*CI23NL)
      JYI=JYI+1
  900 S(JJYR,JYI)=-((A/B)*DI22*CI22MK*CI29NL+2.*DI26*CI27MK*CI27NL+(B/A)
     1*DI66*CI29MK*CI22NL+A*B*K22*AI55*CI22MK*CI22NL)
C     CALCULATE SUBMATRIX
C     UIUR,UIVR,UIWR,UIXR,UIYR,UIUI,UIVI,UIWI,UIXI,UIYI
      JJUI=5*NN**2
      DO 1000 K1=1,NN
      K=K1+INPU1
      DO 1000 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJUI=JJUI+1
      DO 1000 M1=1,NN
      M=M1+INPU1
      DO 1000 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI8MK,CI8NL
      IF(M-K)1002,1004,1002
 1002 CI8MK=0.
      GO TO 1005
```

```
 1004 CI8MK=2.*M*K*PI**2
 1005 CONTINUE
      IF(N-L)1006,1008,1006
 1006 CI8NL=0.
      GO TO 1009
 1008 CI8NL=2.*N*L*PI**2
 1009 CONTINUE
C
C     INTEGRAL CI1MK, CI1NL
C
      IF(M-K)1012,1014,1012
 1012 CI1MK=1.
      GO TO 1015
 1014 CI1MK=1.5
 1015 CONTINUE
      IF(N-L)1016,1018,1016
 1016 CI1NL=1.
      GO TO 1019
 1018 CI1NL=1.5
 1019 CONTINUE
C
C     INTEGRAL CI6MK,CI6NL
C
      IF(M-K)1022,1024,1022
 1022 CI6MK=0.
      GO TO 1025
 1024 CI6MK=0.
 1025 CONTINUE
      IF(N-L)1026,1028,1026
 1026 CI6NL=0.
      GO TO 1029
 1028 CI6NL=0.
 1029 CONTINUE
C
C     INTEGRAL CI5MK,CI5NL
C
```

```
      CI5MK=CI6MK
      CI5NL=CI6NL
C
C     INTEGRAL CI4MK,CI4NL
C
      CI4MK=CI6MK
      CI4NL=CI6NL
C
C     INTEGRAL CI10MK,CI10NL
C
      CI10MK=CI8MK
      CI10NL=CI8NL
C
C     INTEGRAL CI3MK,CI3NL
C
      CI3MK=CI1MK
      CI3NL=CI1NL
C
C     INTEGRAL CI44MK,CI44NL
C
      CI44MK=CI8MK
      CI44NL=CI8NL
C
C     INTEGRAL CI31MK,CI31NL
C
      CI31MK=CI1MK
      CI31NL=CI1NL
C
C     INTEGRAL CI34MK,CI34NL
C
      CI34MK=CI4MK
      CI34NL=CI4NL
C
C     INTEGRAL CI33MK, CI33NL
C
      CI33MK=CI4MK
```

```
      CI33NL=CI4NL
C
CC    INTEGRAL CI37MK, CI37NL
C
      CI37MK=CI4MK
      CI37NL=CI4NL
C
CC    INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI4MK
      CI36NL=CI4NL
C
CC    INTEGRAL CI55MK, CI55NL
C
      CI55MK=CI8MK
      CI55NL=CI8NL
C
CC    INTEGRAL CI35MK, CI35NL
C
      CI35MK=CI1MK
      CI35NL=CI1NL
      S(JJUI,JUR)=+((B/A)*AI11*CI8MK*CI1NL+2.*AI16*CI6MK*CI6NL+(A/B)
     1*AI66*CI1MK*CI8NL)
      JVR=JVR+1
      S(JJUI,JVR)=+(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JWR=JWR+1
      S(JJUI,JWR)=0.
      JXR=JXR+1
      S(JJUI,JXR)=+((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL+BI16
     1*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
      JYR=JYR+1
      S(JJUI,JYR)=+(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
     1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
      JUI=JUI+1
      S(JJUI,JUI)=(B/A)*AR11*CI8MK*CI1NL+2.*AR16*CI6MK*CI6NL+(A/B)
```

```
     1*AR66*CI1MK*CI8NL-W2*A*B*P0*CI1MK*CI1NL
      JVI=JVI+1
      S(JJUI,JVI)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JWI=JWI+1
      S(JJUI,JWI)=0.
      JXI=JXI+1
      S(JJUI,JXI)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL
     1+BR16*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK
     2*CI31NL
      JYI=JYI+1
 1000 S(JJUI,JYI)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
C     CALCULATE SUBMATRIX
C     VIUR,VIVR,VIWR,VIXR,VIYR,VIUI,VIVI,VIWI,VIXI,VIYI
      JJVI=6*NN**2
      DO 1100 K1=1,NN
      K=K1+INPU1
      DO 1100 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJVI=JJVI+1
      DO 1100 M1=1,NN
      M=M1+INPU1
      DO 1100 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
```

```
C
C       INTEGRAL CI4MK, CI4NL
C
        IF(M-K)1102,1104,1102
 1102 CI4MK=0.
        GO TO 1105
 1104 CI4MK=0.
 1105 CONTINUE
        IF(N-L)1106,1108,1106
 1106 CI4NL=0.
        GO TO 1109
 1108 CI4NL=0.
 1109 CONTINUE
C
C       INTEGRAL CI5MK, CI5NL
C
        CI5MK=CI4MK
        CI5NL=CI4NL
C
C       INTEGRAL CI3MK, CI3NL
C
        IF(M-K)1112,1114,1112
 1112 CI3MK=1.
        GO TO 1115
 1114 CI3MK=1.5
 1115 CONTINUE
        IF(N-L)1116,1118,1116
 1116 CI3NL=1.
        GO TO 1119
 1118 CI3NL=1.5
 1119 CONTINUE
C
C       INTEGRAL CI10MK, CI10NL
C
        IF(M-K)1122,1124,1122
 1122 CI10MK=0.
```

```
      GO TO 1125
 1124 CI10MK=2.*M*K*PI**2
 1125 CONTINUE
      IF(N-L)1126,1128,1126
 1126 CI10NL=0.
      GO TO 1129
 1128 CI10NL=2.*N*L*PI**2
 1129 CONTINUE
C
C     INTEGRAL CI2MK, CI2NL
C
      CI2MK=CI3MK
      CI2NL=CI3NL
C
C     INTEGRAL CI9MK, CI9NL
C
      CI9MK=CI10MK
      CI9NL=CI10NL
C
C     INTEGRAL CI7MK, CI7NL
C
      CI7MK=CI4MK
      CI7NL=CI4NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI4MK
      CI39NL=CI4NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI4MK
      CI40NL=CI4NL
C
C     INTEGRAL CI56MK, CI56NL
C
```

```
      CI56MK=CI10MK
      CI56NL=CI10NL
C
C     INTEGRAL CI38MK,CI38NL
C
      CI38MK=CI3MK
      CI38NL=CI3NL
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI3MK
      CI41NL=CI3NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI10MK
      CI57NL=CI10NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI4MK
      CI42NL=CI4NL
C
C     INTEGRAL CI43MK, CI43NL
C
      CI43MK=CI4MK
      CI43NL=CI4NL
      S(JJVI,JUR)=+(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JVR=JVR+1
      S(JJVI,JVR)=+((A/B)*AI22*CI2MK*CI9NL+2.*AI26*CI7MK*CI7NL+(B/A)
     1*AI66*CI9MK*CI2NL)
      JWR=JWR+1
      S(JJVI,JWR)=0.
      JXR=JXR+1
      S(JJVI,JXR)=+(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
```

```
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JYR=JYR+1
      S(JJVI,JYR)=+((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
      JUI=JUI+1
      S(JJVI,JUI)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JVI=JVI+1
      S(JJVI,JVI)=(A/B)*AR22*CI2MK*CI9NL+2.*AR26*CI7MK*CI7NL+(B/A)
     1*AR66*CI9MK*CI2NL-W2*A*B*P0*CI2MK*CI2NL
      JWI=JWI+1
      S(JJVI,JWI)=0.
      JXI=JXI+1
      S(JJVI,JXI)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JYI=JYI+1
 1100 S(JJVI,JYI)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK*CI41NL
C     CALCULATE SUBMATRIX
C     WIUR,WIVR,WIWR,WIXR,WIYR,WIUI,WIVI,WIWI,WIXI,WIYI
      JJWI=7*NN**2
      DO 1200 K1=1,NN
      K=K1+INPU1
      DO 1200 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJWI=JJWI+1
```

```
      DO 1200 M1=1,NN
      M=M1+INPU1
      DO 1200 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI20MK, CI20NL
C
      IF(M-K)1202,1204,1202
 1202 CI20MK=0.
      GO TO 1205
 1204 CI20MK=2.0*M*K*PI**2
 1205 CONTINUE
      IF(N-L)1206,1208,1206
 1206 CI20NL=0.
      GO TO 1209
 1208 CI20NL=2.0*N*L*PI**2
 1209 CONTINUE
C
C     INTEGRAL CI17MK,CI17NL
C
      IF(M-K)1212,1214,1212
 1212 CI17MK=1.
      GO TO 1215
 1214 CI17MK=1.5
 1215 CONTINUE
      IF(N-L)1216,1218,1216
C     INTEGRAL CI36MK, CI36NL
C
      CI36MK=CI4MK
      CI36NL=CI4NL
C
C     INTEGRAL CI55MK, CI55NL
C
      CI55MK=CI8MK
      CI55NL=CI8NL
```

```
C
C       INTEGRAL CI35MK, CI35NL
C
        CI35MK=CI1MK
        CI35NL=CI1NL
        S(JJUR,JUR)=(B/A)*AR11*CI8MK*CI1NL+2.*AR16*CI6MK*CI6NL+(A/B)
       1*AR66*CI1MK*CI8NL-W2*A*B*P0*CI1MK*CI1NL
        JVR=JVR+1
        S(JJUR,JVR)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
       1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
        JWR=JWR+1
        S(JJUR,JWR)=0.
        JXR=JXR+1
        S(JJUR,JXR)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL
       1+BR16*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK
       2*CI31NL
        JYR=JYR+1
        S(JJUR,JYR)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
       1*BR26*CI35MK*CI55NL+BR66*CI36MK*CI37NL
        JUI=JUI+1
        S(JJUR,JUI)=-((B/A)*AI11*CI8MK*CI1NL+2.*AI16*CI6MK*CI6NL+(A/B)
       1*AI66*CI1MK*CI8NL)
        JVI=JVI+1
        S(JJUR,JVI)=-(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
       1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
        JWI=JWI+1
        S(JJUR,JWI)=0.
        JXI=JXI+1
        S(JJUR,JXI)=-((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL+BI16
       1*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
        JYI=JYI+1
  500   S(JJUR,JYI)=-(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
       1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
C       CALCULATE SUBMATRIX
C       VRUR,VRVR,VRWR,VRXR,VRYR,VRUI,VRVI,VRWI,VRXI,VRYI
        JJVR=1*NN**2
```

```
      DO 600 K1=1,NN
      K=K1+INPU1
      DO 600 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJVR=JJVR+1
      DO 600 M1=1,NN
      M=M1+INPU1
      DO 600 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI4MK, CI4NL
C
      IF(M-K)602,604,602
  602 CI4MK=0.
      GO TO 605
  604 CI4MK=0.
  605 CONTINUE
      IF(N-L)606,608,606
  606 CI4NL=0.
      GO TO 609
  608 CI4NL=0.
  609 CONTINUE
C
C     INTEGRAL CI5MK, CI5NL
C
```

```
      CI5MK=CI4MK
      CI5NL=CI4NL
C
C     INTEGRAL CI3MK, CI3NL
C
      IF(M-K)612,614,612
  612 CI3MK=1.
      GO TO 615
  614 CI3MK=1.5
  615 CONTINUE
      IF(N-L)616,618,616
  616 CI3NL=1.
      GO TO 619
  618 CI3NL=1.5
  619 CONTINUE
C
C     INTEGRAL CI10MK, CI10NL
C
      IF(M-K)622,624,622
  622 CI10MK=0.
      GO TO 625
  624 CI10MK=2.*M*K*PI**2
  625 CONTINUE
      IF(N-L)626,628,626
  626 CI10NL=0.
      GO TO 629
  628 CI10NL=2.*N*L*PI**2
  629 CONTINUE
C
C     INTEGRAL CI2MK, CI2NL
C
      CI2MK=CI3MK
      CI2NL=CI3NL
C
C     INTEGRAL CI9MK, CI9NL
C
```

```
      CI9MK=CI10MK
      CI9NL=CI10NL
C
C     INTEGRAL CI7MK, CI7NL
C
      CI7MK=CI4MK
      CI7NL=CI4NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI4MK
      CI39NL=CI4NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI4MK
      CI40NL=CI4NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI10MK
      CI56NL=CI10NL
C
C     INTEGRAL CI38MK,CI38NL
C
      CI38MK=CI3MK
      CI38NL=CI3NL
C
C     INTEGRAL CI41MK, CI41NL
C
      CI41MK=CI3MK
      CI41NL=CI3NL
C
C     INTEGRAL CI57MK, CI57NL
C
      CI57MK=CI10MK
```

```
      CI57NL=CI10NL
C
C     INTEGRAL CI42MK, CI42NL
C
      CI42MK=CI4MK
      CI42NL=CI4NL
C
C     INTEGRAL CI43MK, CI43NL
C
      CI43MK=CI4MK
      CI43NL=CI4NL
      S(JJVR,JUR)=AR12*CI5MK*CI4NL+(B/A)*AR16*CI10MK*CI3NL+(A/B)
     1*AR26*CI3MK*CI10NL+AR66*CI4MK*CI5NL
      JVR=JVR+1
      S(JJVR,JVR)=(A/B)*AR22*CI2MK*CI9NL+2.*AR26*CI7MK*CI7NL+(B/A)
     1*AR66*CI9MK*CI2NL-W2*A*B*P0*CI2MK*CI2NL
      JWR=JWR+1
      S(JJVR,JWR)=0.
      JXR=JXR+1
      S(JJVR,JXR)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JYR=JYR+1
      S(JJVR,JYR)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK*CI41NL
      JUI=JUI+1
      S(JJVR,JUI)=-(AI12*CI5MK*CI4NL+(B/A)*AI16*CI10MK*CI3NL+(A/B)
     1*AI26*CI3MK*CI10NL+AI66*CI4MK*CI5NL)
      JVI=JVI+1
      S(JJVR,JVI)=-((A/B)*AI22*CI2MK*CI9NL+2.*AI26*CI7MK*CI7NL+(B/A)
     1*AI66*CI9MK*CI2NL)
      JWI=JWI+1
      S(JJVR,JWI)=0.
      JXI=JXI+1
      S(JJVR,JXI)=-(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JYI=JYI+1
```

```
  600 S(JJVR,JYI)=-((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
     1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
C       CALCULATE SUBMATRIX
C       WRUR,WRVR,WRWR,WRXR,WRYR,WRUI,WRVI,WRWI,WRXI,WRYI
        JJWR=2*NN**2
        DO 700 K1=1,NN
        K=K1+INPU1
        DO 700 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
        JYR=4*NN**2
        JUI=5*NN**2
        JVI=6*NN**2
        JWI=7*NN**2
        JXI=8*NN**2
        JYI=9*NN**2
        JJWR=JJWR+1
        DO 700 M1=1,NN
        M=M1+INPU1
        DO 700 N2=1,NN
        N=N2+INPU2
        JUR=JUR+1
C
C       INTEGRAL CI20MK, CI20NL
C
        IF(M-K) 702,704,702
  702   CI20MK=0.
        GO TO 705
  704   CI20MK=2.0*M*K*PI**2
  705   CONTINUE
        IF (N-L) 706,708,706
  706   CI20NL=0.
        GO TO 709
```

```
  708 CI20NL=2.0*N*L*PI**2
  709 CONTINUE
C
C      INTEGRAL CI17MK,CI17NL
C
       IF(M-K)712,714,712
  712 CI17MK=1.
       GO TO 715
  714 CI17MK=1.5
  715 CONTINUE
       IF(N-L)716,718,716
  716 CI17NL=1.
       GO TO 719
  718 CI17NL=1.5
  719 CONTINUE
C
C      INTEGRAL CI18MK ,CI18NL
C
       CI18MK=0.
       CI18NL=0.
C
C      INTEGRAL CI13MK, CI13NL
C
       CI13MK=0.
       CI13NL=0.
C
C      INTEGRAL CI11MK, CI11NL
C
       CI11MK=CI17MK
       CI11NL=CI17NL
C
C      INTEGRAL CI15MK, CI15NL
C
       CI15MK=CI13MK
       CI15NL=CI13NL
C
```

```
C       INTEGRAL CI12MK, CI12NL
C
        CI12MK=CI11MK
        CI12NL=CI11NL
        S(JJWR,JUR)=0.
        JVR=JVR+1
        S(JJWR,JVR)=0.
        JWR=JWR+1
        S(JJWR,JWR)=(B/A)*K11*AR44*CI20MK*CI17NL+2.*K12*AR45*CI18MK
       1*CI18NL+(A/B)*K22*AR55*CI17MK*CI20NL-W2*A*B*P0*CI17MK*CI17NL
        JXR=JXR+1
        S(JJWR,JXR)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
        JYR=JYR+1
        S(JJWR,JYR)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
        JUI=JUI+1
        S(JJWR,JUI)=0.
        JVI=JVI+1
        S(JJWR,JVI)=0.
        JWI=JWI+1
        S(JJWR,JWI)=-((B/A)*K11*AI44*CI20MK*CI17NL+2.*K12*AI45*CI18MK
       1*CI18NL+(A/B)*K22*AI55*CI17MK*CI20NL)
        JXI=JXI+1
        S(JJWR,JXI)=-(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK*CI13NL)
        JYI=JYI+1
  700   S(JJWR,JYI)=-(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
C       CALCULATE SUBMATRIX
C       XRUR,XRVR,XRWR,XRXR,XRYR,XRUI,XRVI,XRWI,XRXI,XRYI
        JJXR=3*NN**2
        DO 800 K1=1,NN
        K=K1+INPU1
        DO 800 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
```

```
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJXR=JJXR+1
      DO 800 M1=1,NN
      M=M1+INPU1
      DO 800 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI44MK, CI44NL
C
      IF(M-K)802,804,802
  802 CI44MK=0.
      GO TO 805
  804 CI44MK=2.*M*K*PI**2
  805 CONTINUE
      IF(N-L)806,808,806
  806 CI44NL=0.
      GO TO 809
  808 CI44NL=2.*N*L*PI**2
  809 CONTINUE
C
C     INTEGRAL CI31MK, CI31NL
C
      IF(M-K)812,814,812
  812 CI31MK=1.
      GO TO 815
  814 CI31MK=1.5
  815 CONTINUE
      IF(N-L)816,818,816
  816 CI31NL=1.
      GO TO 819
```

```
  818 CI31NL=1.5
  819 CONTINUE
C
C     INTEGRAL CI33MK, CI33NL
C
      IF(M-K)822,824,822
  822 CI33MK=0.
      GO TO 825
  824 CI33MK=0.
  825 CONTINUE
      IF(N-L)826,828,826
  826 CI33NL=0.
      GO TO 829
  828 CI33NL=0.
  829 CONTINUE
C
C     INTEGRAL CI34MK, CI34NL
C
      CI34MK=CI33MK
      CI34NL=CI33NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI33MK
      CI39NL=CI33NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI33MK
      CI40NL=CI33NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI44MK
      CI56NL=CI44NL
C
```

```
C       INTEGRAL CI38MK, CI38NL
C
        CI38MK=CI31MK
        CI38NL=CI31NL
C
C       INTEGRAL CI13MK, CI13NL
C
        CI13MK=0.
        CI13NL=0.
C
C       INTEGRAL CI11MK, CI11NL
C
        CI11MK=CI31MK
        CI11NL=CI31NL
C
C       INTEGRAL CI28MK, CI28NL
C
        CI28MK=CI44MK
        CI28NL=CI44NL
C
C       INTEGRAL CI21MK, CI21NL
C
        CI21MK=CI31MK
        CI21NL=CI31NL
C
C       INTEGRAL CI26MK, CI26NL
C
        CI26MK=CI33MK
        CI26NL=CI33NL
C
C       INTEGRAL CI25MK, CI25NL
C
        CI25MK=CI33MK
        CI25NL=CI33NL
C
C       INTEGRAL CI24MK, CI24NL
```

```
C
      CI24MK=CI33MK
      CI24NL=CI33NL
C
C     INTEGRAL CI30MK, CI30NL
C
      CI30MK=CI44MK
      CI30NL=CI44NL
C
C     INTEGRAL CI23MK, CI23NL
C
      CI23MK=CI31MK
      CI23NL=CI31NL
      S(JJXR,JUR)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL+BR16
     1*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK*CI31NL
      JVR=JVR+1
      S(JJXR,JVR)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
     1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
      JWR=JWR+1
      S(JJXR,JWR)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
      JXR=JXR+1
      S(JJXR,JXR)=(B/A)*DR11*CI28MK*CI21NL+2.*DR16*CI26MK*CI26NL+(A/B)
     1*DR66*CI21MK*CI28NL+A*B*K11*AR44*CI21MK*CI21NL-W2*A*B*P2*CI21MK
     2*CI21NL
      JYR=JYR+1
      S(JJXR,JYR)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JUI=JUI+1
      S(JJXR,JUI)=-((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL
     1+BI16*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
      JVI=JVI+1
      S(JJXR,JVI)=-(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
     1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
      JWI=JWI+1
      S(JJXR,JWI)=-(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK
```

```
     1*CI13NL)
      JXI=JXI+1
      S(JJXR,JXI)=-((B/A)*DI11*CI28MK*CI21NL+2.*DI16*CI26MK*CI26NL
     1+(A/B)*DI66*CI21MK*CI28NL+A*B*K11*AI44*CI21MK*CI21NL)
      GO TO 1219
 1218 CI17NL=1.5
 1219 CONTINUE
C
C     INTEGRAL CI18MK ,CI18NL
C
      CI18MK=0.
      CI18NK=0.
C
C     INTEGRAL CI13MK, CI13NL
C
      CI13MK=0.
      CI13NL=0.
C
C     INTEGRAL CI11MK, CI11NL
C
      CI11MK=CI17MK
      CI11NL=CI17NL
C
C     INTEGRAL CI15MK, CI15NL
C
      CI15MK=CI13MK
      CI15NL=CI13NL
C
C     INTEGRAL CI12MK, CI12NL
C
      CI12MK=CI11MK
      CI12NL=CI11NL
      S(JJWI,JUR)=0.
      JVR=JVR+1
      S(JJWI,JVR)=0.
      JWR=JWR+1
```

```
      S(JJWI,JWR)=+((B/A)*K11*AI44*CI20MK*CI17NL+2.*K12*AI45*CI18MK
     1*CI18NL+(A/B)*K22*AI55*CI17MK*CI20NL)
      JXR=JXR+1
      S(JJWI,JXR)=+(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK*CI13NL)
      JYR=JYR+1
      S(JJWI,JYR)=+(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
      JUI=JUI+1
      S(JJWI,JUI)=0.
      JVI=JVI+1
      S(JJWI,JVI)=0.
      JWI=JWI+1
      S(JJWI,JWI)=(B/A)*K11*AR44*CI20MK*CI17NL+2.*K12*AR45*CI18MK
     1*CI18NL+(A/B)*K22*AR55*CI17MK*CI20NL-W2*A*B*P0*CI17MK*CI17NL
      JXI=JXI+1
      JYI=JYI+1
      S(JJWI,JXI)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
 1200 S(JJWI,JYI)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
C     CALCULATE SUBMATRIX
C     XIUR,XIVR,XIWR,XIXR,XIYR,XIUI,XIVI,XIWI,XIXI,XIYI
      JJXI=8*NN**2
      DO 1300 K1=1,NN
      K=K1+INPU1
      DO 1300 L2=1,NN
      L=L2+INPU2
      JUR=0
      JVR=1*NN**2
      JWR=2*NN**2
      JXR=3*NN**2
      JYR=4*NN**2
      JUI=5*NN**2
      JVI=6*NN**2
      JWI=7*NN**2
      JXI=8*NN**2
      JYI=9*NN**2
      JJXI=JJXI+1
      DO 1300 M1=1,NN
```

```
      M=M1+INPU1
      DO 1300 N2=1,NN
      N=N2+INPU2
      JUR=JUR+1
C
C     INTEGRAL CI44MK, CI44NL
C
      IF(M-K)1302,1304,1302
 1302 CI44MK=0.
      GO TO 1305
 1304 CI44MK=2.*M*K*PI**2
 1305 CONTINUE
      IF(N-L)1306,1308,1306
 1306 CI44NL=0.
      GO TO 1309
 1308 CI44NL=2.*N*L*PI**2
 1309 CONTINUE
C
C     INTEGRAL CI31MK, CI31NL
C
      IF(M-K)1312,1314,1312
 1312 CI31MK=1.
      GO TO 1315
 1314 CI31MK=1.5
 1315 CONTINUE
      IF(N-L)1316,1318,1316
 1316 CI31NL=1.
      GO TO 1319
 1318 CI31NL=1.5
 1319 CONTINUE
C
C     INTEGRAL CI33MK, CI33NL
C
      IF(M-K)1322,1324,1322
 1322 CI33MK=0.
      GO TO 1325
```

```
 1324 CI33MK=0.
 1325 CONTINUE
      IF(N-L)1326,1328,1326
 1326 CI33NL=0.
      GO TO 1329
 1328 CI33NL=0.
 1329 CONTINUE
C
C     INTEGRAL CI34MK, CI34NL
C
      CI34MK=CI33MK
      CI34NL=CI33NL
C
C     INTEGRAL CI39MK, CI39NL
C
      CI39MK=CI33MK
      CI39NL=CI33NL
C
C     INTEGRAL CI40MK, CI40NL
C
      CI40MK=CI33MK
      CI40NL=CI33NL
C
C     INTEGRAL CI56MK, CI56NL
C
      CI56MK=CI44MK
      CI56NL=CI44NL
C
C     INTEGRAL CI38MK, CI38NL
C
      CI38MK=CI31MK
      CI38NL=CI31NL
C
C     INTEGRAL CI13MK, CI13NL
C
      CI13MK=0.
```

```
      CI13NL=0.
C
CC    INTEGRAL CI11MK, CI11NL
C
      CI11MK=CI31MK
      CI11NL=CI31NL
C
CC    INTEGRAL CI28MK, CI28NL
C
      CI28MK=CI44MK
      CI28NL=CI44NL
C
CC    INTEGRAL CI21MK, CI21NL
C
      CI21MK=CI31MK
      CI21NL=CI31NL
C
CC    INTEGRAL CI26MK, CI26NL
C
      CI26MK=CI33MK
      CI26NL=CI33NL
C
CC    INTEGRAL CI25MK, CI25NL
C
      CI25MK=CI33MK
      CI25NL=CI33NL
C
CC    INTEGRAL CI24MK, CI24NL
C
      CI24MK=CI33MK
      CI24NL=CI33NL
C
CC    INTEGRAL CI30MK, CI30NL
C
      CI30MK=CI44MK
      CI30NL=CI44NL
```

```
C
C       INTEGRAL CI23MK, CI23NL
C
        CI23MK=CI31MK
        CI23NL=CI31NL
        S(JJXI,JUR)=+((B/A)*BI11*CI44MK*CI31NL+BI16*CI34MK*CI33NL
       1+BI16*CI33MK*CI34NL+(A/B)*BI66*CI31MK*CI44NL)
        JVR=JVR+1
        S(JJXI,JVR)=+(BI12*CI39MK*CI40NL+(B/A)*BI16*CI56MK*CI38NL+(A/B)
       1*BI26*CI38MK*CI56NL+BI66*CI40MK*CI39NL)
        JWR=JWR+1
        S(JJXI,JWR)=+(B*K11*AI44*CI13MK*CI11NL+A*K12*AI45*CI11MK
       1*CI13NL)
        JXR=JXR+1
        S(JJXI,JXR)=+((B/A)*DI11*CI28MK*CI21NL+2.*DI16*CI26MK*CI26NL
       1+(A/B)*DI66*CI21MK*CI28NL+A*B*K11*AI44*CI21MK*CI21NL)
        JYR=JYR+1
        S(JJXI,JYR)=+(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
       1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
       2*CI23NL)
        JUI=JUI+1
        S(JJXI,JUI)=(B/A)*BR11*CI44MK*CI31NL+BR16*CI34MK*CI33NL+BR16
       1*CI33MK*CI34NL+(A/B)*BR66*CI31MK*CI44NL-W2*A*B*P1*CI31MK*CI31NL
        JVI=JVI+1
        S(JJXI,JVI)=BR12*CI39MK*CI40NL+(B/A)*BR16*CI56MK*CI38NL+(A/B)
       1*BR26*CI38MK*CI56NL+BR66*CI40MK*CI39NL
        JWI=JWI+1
        S(JJXI,JWI)=B*K11*AR44*CI13MK*CI11NL+A*K12*AR45*CI11MK*CI13NL
        JXI=JXI+1
        S(JJXI,JXI)=(B/A)*DR11*CI28MK*CI21NL+2.*DR16*CI26MK*CI26NL+(A/B)
       1*DR66*CI21MK*CI28NL+A*B*K11*AR44*CI21MK*CI21NL-W2*A*B*P2*CI21MK
       2*CI21NL
        JYI=JYI+1
 1300   S(JJXI,JYI)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
       1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
       2*CI23NL
```

```
C       CALCULATE SUBMATRIX
C       YIUR,YIVR,YIWR,YIXR,YIYR,YIUI,YIVI,YIWI,YIXI,YIYI
        JJYI=9*NN**2
        DO 1400 K1=1,NN
        K=K1+INPU1
        DO 1400 L2=1,NN
        L=L2+INPU2
        JUR=0
        JVR=1*NN**2
        JWR=2*NN**2
        JXR=3*NN**2
        JYR=4*NN**2
        JUI=5*NN**2
        JVI=6*NN**2
        JWI=7*NN**2
        JXI=8*NN**2
        JYI=9*NN**2
        JJYI=JJYI+1
        DO 1400 M1=1,NN
        M=M1+INPU1
        DO 1400 N2=1,NN
        N=N2+INPU2
        JUR=JUR+1
C
C       INTEGRAL CI37MK, CI37NL
C
        IF(M-K)1402,1404,1402
 1402 CI37MK=0.
        GO TO 1405
 1404 CI37MK=0.
 1405 CONTINUE
        IF(N-L)1406,1408,1406
 1406 CI37NL=0.
        GO TO 1409
 1408 CI37NL=0.
 1409 CONTINUE
```

```
C
C       INTEGRAL CI36MK, CI36NL
C
        CI36MK=CI37MK
        CI36NL=CI37NL
C
C       INTEGRAL CI55MK, CI55NL
C
        IF(M-K)1412,1414,1412
 1412 CI55MK=0.
        GO TO 1415
 1414 CI55MK=2.*M*K*PI**2
 1415 CONTINUE
        IF(N-L)1416,1418,1416
 1416 CI55NL=0.
        GO TO 1419
 1418 CI55NL=2.*N*L*PI**2
 1419 CONTINUE
C
C       INTEGRAL CI35MK, CI35NL
C
        IF(M-K)1422,1424,1422
 1422 CI35MK=1.
        GO TO 1425
 1424 CI35MK=1.5
 1425 CONTINUE
        IF(N-L)1426,1428,1426
 1426 CI35NL=1.0
        GO TO 1429
 1428 CI35NL=1.5
 1429 CONTINUE
C
C       INTEGRAL CI41MK, CI41NL
C
        CI41MK=CI35MK
        CI41NL=CI35NL
```

```
C
C       INTEGRAL CI57MK, CI57NL
C
        CI57MK=CI55MK
        CI57NL=CI55NL
C
C       INTEGRAL CI42MK, CI42NL
C
        CI42MK=CI37MK
        CI42NL=CI37NL
C
C       INTEGRAL CI43MK, CI43NL
C
        CI43MK=CI37MK
        CI43NL=CI37NL
C
C       INTEGRAL CI15MK, CI15NL
C
        CI15MK=0.
        CI15NL=0.
C
C       INTEGRAL CI12MK, CI12NL
C
        CI12MK=CI35MK
        CI12NL=CI35NL
C
C       INTEGRAL CI24MK, CI24NL
C
        CI24MK=CI37MK
        CI24NL=CI37NL
C
C       INTEGRAL CI25MK, CI25NL
C
        CI25MK=CI37MK
        CI25NL=CI37NL
C
```

```
C       INTEGRAL CI23MK, CI23NL
C
        CI23MK=CI35MK
        CI23NL=CI35NL
C
C       INTEGRAL CI30MK, CI30NL
C
        CI30MK=CI55MK
        CI30NL=CI55NL
C
C       INTEGRAL CI22MK, CI22NL
C
        CI22MK=CI35MK
        CI22NL=CI35NL
C
C       INTEGRAL CI29MK, CI29NL
C
        CI29MK=CI55MK
        CI29NL=CI55NL
C
C       INTEGRAL CI27MK, CI27NL
C
        CI27MK=CI37MK
        CI27NL=CI37NL
        S(JJYI,JUR)=+(BI12*CI37MK*CI36NL+(B/A)*BI16*CI55MK*CI35NL+(A/B)
       1*BI26*CI35MK*CI55NL+BI66*CI36MK*CI37NL)
        JVR=JVR+1
        S(JJYI,JVR)=+((A/B)*BI22*CI41MK*CI57NL+BI26*CI42MK*CI43NL+BI26
       1*CI43MK*CI42NL+(B/A)*BI66*CI57MK*CI41NL)
        JWR=JWR+1
        S(JJYI,JWR)=+(B*K12*AI45*CI15MK*CI12NL+A*K22*AI55*CI12MK*CI15NL)
        JXR=JXR+1
        S(JJYI,JXR)=+(DI12*CI25MK*CI24NL+(B/A)*DI16*CI30MK*CI23NL+(A/B)
       1*DI26*CI23MK*CI30NL+DI66*CI24MK*CI25NL+A*B*K12*AI45*CI23MK
       2*CI23NL)
        JYR=JYR+1
```

```
      S(JJYI,JYR)=+((A/B)*DI22*CI22MK*CI29NL+2.*DI26*CI27MK*CI27NL+(B/A)
     1*DI66*CI29MK*CI22NL+A*B*K22*AI55*CI22MK*CI22NL)
      JUI=JUI+1
      S(JJYI,JUI)=BR12*CI37MK*CI36NL+(B/A)*BR16*CI55MK*CI35NL+(A/B)
     1*BR26*CI35MK*CI55NL+3R66*CI36MK*CI37NL
      JVI=JVI+1
      S(JJYI,JVI)=(A/B)*BR22*CI41MK*CI57NL+BR26*CI42MK*CI43NL+BR26
     1*CI43MK*CI42NL+(B/A)*BR66*CI57MK*CI41NL-W2*A*B*P1*CI41MK
     2*CI41NL
      JWI=JWI+1
      S(JJYI,JWI)=B*K12*AR45*CI15MK*CI12NL+A*K22*AR55*CI12MK*CI15NL
      JXI=JXI+1
      S(JJYI,JXI)=DR12*CI25MK*CI24NL+(B/A)*DR16*CI30MK*CI23NL+(A/B)
     1*DR26*CI23MK*CI30NL+DR66*CI24MK*CI25NL+A*B*K12*AR45*CI23MK
     2*CI23NL
      JYI=JYI+1
 1400 S(JJYI,JYI)=(A/B)*DR22*CI22MK*CI29NL+2.*DR26*CI27MK*CI27NL+(B/A)
     1*DR66*CI29MK*CI22NL+A*B*K22*AR55*CI22MK*CI22NL-W2*A*B*P2*CI22MK
     2*CI22NL
C     CALCULATE B MATRIX
      IJUR=0
      IJVR=1*NN**2
      IJWR=2*NN**2
      IJXR=3*NN**2
      IJYR=4*NN**2
      IJUI=5*NN**2
      IJVI=6*NN**2
      IJWI=7*NN**2
      IJXI=8*NN**2
      IJYI=9*NN**2
      DO 1500 M1=1,NN
      M=M1+INPU1
      DO 1500 N2=1,NN
      N=N2+INPU2
      IJUR=IJUR+1
C
```

```
C     INTEGRAL CIMK, CINL
C
      CIMK=1.
      CINL=1.
      T(IJUR)=0.
      IJVR=IJVR+1
      T(IJVR)=0.
      IJWR=IJWR+1
      T(IJWR)=1.*A*B*CIMK*CINL
      IJXR=IJXR+1
      T(IJXR)=0.
      IJYR=IJYR+1
      T(IJYR)=0.
      IJUI=IJUI+1
      T(IJUI)=0.
      IJVI=IJVI+1
      T(IJVI)=0.
      IJWI=IJWI+1
      T(IJWI)=0.
      IJXI=IJXI+1
      T(IJXI)=0.
      IJYI=IJYI+1
 1500 T(IJYI)=0.
C
C     CALL SIMQ
C
      NI=10*NN**2
      CALL SIMQ(S,T,NI,KS)
      WRITE(6,2050)KS
 2050 FORMAT('      KS=',I3//)
      KN1=2*NN**2+1
      KN2=3*NN**2
      KN3=7*NN**2+1
      KN4=8*NN**2
      WRITE(6,2055) (T(J),J=KN1,KN2)
      WRITE(6,2055) (T(I),I=KN3,KN4)
```

```
 2055 FORMAT(E12.5)
      WRITE(6,2067)F
 2067 FORMAT('    FQ=',F10.3)
      GO TO 10
 3333 CALL EXIT
      END
C     PEAK-AMPLITUDE AND MODIFIED KENNEDY-PANCU METHODS
C     WR(J)=T(J) AND WI(I)=T(I)
      REAL M1PIA,N1PIB,M2PIA,N2PIB,M3PIA,N3PIB,M4PIA,N4PIB
   10 READ(5,20)FQ
   20 FORMAT(F10.4)
      IF(FQ)80,80,25
   25 READ(5,30)WR1,WR2,WR3,WR4
   30 FORMAT(4F10.4)
      READ(5,40)WI1,WI2,WI3,WI4
   40 FORMAT(4F10.4)
      READ(5,50)M1,M2,M3,M4,N1,N2,N3,N4
   50 FORMAT(8I5)
      READ(5,55)A,B
   55 FORMAT(2F10.4)
      PI=3.1416
      M1PIA=M1*PI*A
      M2PIA=M2*PI*A
      M3PIA=M3*PI*A
      M4PIA=M4*PI*A
      N1PIB=N1*PI*B
      N2PIB=N2*PI*B
      N3PIB=N3*PI*B
      N4PIB=N4*PI*B
      WRT=WR1*COS(M1PIA)*COS(N1PIB)+WR2*COS(M2PIA)*COS(N2PIB)
     1+WR3*COS(M3PIA)*COS(N3PIB)+WR4*COS(M4PIA)*COS(N4PIB)
      WIT=WI1*COS(M1PIA)*COS(N1PIB)+WI2*COS(M2PIA)*COS(N2PIB)
     1+WI3*COS(M3PIA)*COS(N3PIB)+WI4*COS(M4PIA)*COS(N4PIB)
      WT=(WRT**2+WIT**2)**0.5
      WXOY1=ABS(WT)
      RW=WIT/WRT
```

```
      PHA=ATAN(RW)
      DELS=WT*PHA
      WRITE(6,60)PHA,DELS,FQ
   60 FORMAT('    PHA=',F10.4,'    DELS=',F10.4,'    FQ=',F10.4)
      WRITE(6,65) WX0Y1
   65 FORMAT('    WX0Y1',E12.5)
      GO TO 10
   80 CALL EXIT
      END
C     NODAL PATTERNS FOR ANG=0,10,30,45,60,90
      WRITE(6,35)
   35 FORMAT('      X      ','      Y      ','      W      '//)
   10 READ(5,20,END=220)WM1N1,WM1N2,WM2N1,WM2N2
   20 FORMAT(4F10.4)
      READ(5,30)M1,N1,M2,N2
   30 FORMAT(4I5)
      PI=3.1416
      DO 100 J=1,5
      Y=(J-1)*0.4
      B=Y/1.6
      X=-0.5
   36 X=X+0.5
      A=X/12.
      M1PIA=M1*PI*A
      N1PIB=N1*PI*B
      M2PIA=M2*PI*A
      N2PIB=N2*PI*B
      W=WM1N1*COS(M1PIA)*COS(N1PIB)+WM1N2*COS(M1PIA)*COS(N2PIB)
     1+WM2N1*COS(M2PIA)*COS(N1PIB)+WM2N2*COS(M2PIA)*COS(N2PIB)
      W1=ABS(W)
      IF(W1.LT.0.01)WRITE(6,40)A,B,W1
      IF(A.LT.1.0)GO TO 36
   40 FORMAT(3F10.4)
  100 CONTINUE
      GO TO 10
  220 CALL EXIT
```