EVALUATION OF MINIMUM VARIANCE ESTIMATORS FOR SIGNAL DERIVATIVES IN REAL NOISE ENVIRONMENTS

R. W. Snelsire

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Prepared Under Research Grant NGR-41-001-024

by

Clemson University
Clemson, South Carolina 29631

RECEIVED SEP 1972

(NASA-CR-112146) EVALUATION OF MINIMUM VARIANCE ESTIMATORS FOR SIGNAL DERIVATIVES IN REAL NOISE ENVIRONMENTS R.W. Snelsire (Clemson Univ.) [1972] 37 p CSCL 01B

N72-29999

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I

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1. Introduction:

Economic operation of aircraft depends heavily upon aircraft landing under all weather conditions. Profit or loss is determined by the time required for landing, especially for short haul aircraft. This condition is well known, as is evident by the existence and efforts of organizations such as RTCA (Radio Technical Commission for Aeronautics) and ARINC (Aeronautics Radio, Inc.).

As noted by Mr. Lynn L. Hisk (Hughes Aircraft) "Commuters (short distance passenger aircraft) cannot make the grade financially unless they can maintain an on-time, all-weather operation . . . that does not appear practical within the context of present navigation and ATC (air traffic control) systems." Boeing's D. Clifford pointed out that an AFM (automatic flight management) system could be realized only with "precise position and velocity information."

Precise velocity information, especially rate of change of altitude, is difficult to obtain. In this report a filter is developed which gives the optimum estimate of the rate of change of the state of a system. The actual noise environment in which the filter will operate is then determined and the operation of the filter simulated in this environment.

II. Optimal Filters:

In this section the optimal filters for the estimation of the state (Kalman filter) and the rate of change of state (Martin filter) will be developed.

The development of the Kalman filter which follows will closely follow that of Liebelt and Bergeson.[1] Their derivation is repeated here because it is not easily available in the literature.

Consider, a system described by the following sets of difference equations and observation equations:

$$\bar{x}_{n+1} = \phi\bar{x}_n + \bar{u}_n \tag{1}$$

$$\bar{y}_n = C\bar{x}_n + \bar{v}_n \tag{2}$$

Where $\bar{u}_n$ and $\bar{v}_n$ are random sequences with the properties:

$$E(\bar{u}_n) = E(\bar{v}_n) = 0$$

$$\mathrm{Cov}[\bar{u}_n, \bar{u}_n] = P_1$$

$$\mathrm{Cov}[\bar{v}_n, \bar{v}_n] = Q$$

$$\mathrm{Cov}[\bar{u}_n, \bar{v}_m] = 0$$

$$\mathrm{Cov}[\bar{u}_n, \bar{u}_m] = \mathrm{Cov}[\bar{v}_n, \bar{v}_m] = 0 \qquad m \neq n$$

in which E( ) is the expectation operator and Cov [ ] denotes the covariance matrix of the vector in the brackets.

It is shown in Liebelt and Bergeson that the optimum estimator of the state has the form:

$$\hat{\bar{x}}_{n+1} = \phi\hat{\bar{x}} + K_n(y_n - C\hat{\bar{x}}_n) \tag{3}$$

where

$\hat{\bar{x}}_n$ is the estimate of $\bar{x}_n$, and K_n is a matrix chosen to minimize the mean-squared-error in $\hat{\bar{x}}_n$. Let $\tilde{\tilde{x}}_n = \bar{x}_n - \hat{\bar{x}}_n$ and $R_n = E[\tilde{\tilde{x}}_n \quad \tilde{\tilde{x}}^1]$ where $\tilde{\tilde{x}}_n^1$ is the transpose of $\tilde{\tilde{x}}$. The Kalman filter problem is, given R_n, find K_n so as to minimize the diagonal elements of R_{n+1}. This is equivalent to minimizing the mean-squared error in $\hat{\bar{x}}_n$.

Substituting Equation 2 into Equation 3 and subtracting the result from Equation 1 gives:

$$\bar{x}_{n+1} - \hat{\bar{x}}_{n+1} = \phi(\bar{x}_n - \hat{\bar{x}}_n) - K_n C(\bar{x}_n - \hat{\bar{x}}_n) + u_n - K_n v_n$$

Letting $\bar{x}_n - \hat{\bar{x}}_n = \tilde{x}_n$ and combining terms gives:

$$\tilde{x}_{n+1} = \phi\tilde{x}_n + u_n - K_n v_n \tag{4}$$

The transpose of Equation 4 is:

$$\tilde{x}^1_{n+1} = \tilde{x}^1_n \phi^1 + u^1_n - v^1_n K^1_n \tag{5}$$

Multiplying Equation 4 times Equation 5 gives:

$$\begin{aligned}\tilde{x}_{n+1}\ \tilde{x}^1_{n+1} = {} & (\phi - K_n C)\tilde{x}_n\ \tilde{x}^1_n(\phi - K_n C)^1 + u_n u^1_n + K_n v_n v^1_n K^1_n \\ & (\phi - K_n C)\tilde{x}_n(u^1_n - v^1_n K^1_n) + u_n[\tilde{x}^1_n(\phi - K_n C)^1 - v^1_n K^1_n] \\ & - K_n v_n[x^1_n(\phi - K_n C)^1 + u^1_n]\end{aligned}$$

Taking the expected value of both sides of this equation and noting that $\tilde{x}_n$, u_n, and v_n are independent gives the following expression for R_{n+1}

$$R_{n+1} = E(\tilde{x}_{n+1}\ \tilde{x}^1_{n+1}) = (\phi - K_n C)R_n(\phi - K_n C)^1 + P_1 + K_n Q K^1_n \tag{6}$$

Expanding and collecting terms gives the matrix equivalent of a quatratic in K_n.

$$R_{n+1} = K_n[CRC^1 + Q]K^1_n - K_n C R_n \phi^1 - \phi R_n\ C^1 K^1_n + \phi R_n \phi^1 + P_1 \tag{7}$$

To find the value of K_n which minimizes R_{n+1} it is necessary to complete the square. Let R_{n+1} be expressed in the form:

$$R_{n+1} = (K_n Y - Z)(K_n Y - Z)^1 + U \tag{8}$$

If values of U, Y, and Z can be found which make Equations 7 and 8 identical then Equation 8 can be used to find the optimum value of K_n together with the resulting minimum R_{n+1}.

Expanding Equation 8 gives:

$$R_{n+1} = K_n YY^1 K_n^1 - K_n YZ^1 - ZY^1 K_n^1 + ZZ^1 + U \tag{9}$$

For Equations 7 and 9 to be equivalent the following relations must hold:

$$CRC^1 + Q = YY^1 \qquad (10) \qquad\qquad CR_n \phi^1 = YZ^1 \qquad (12)$$

$$\phi R_n C^1 = ZY^1 \qquad (11) \qquad\qquad \phi R_n \phi^1 + P_1 = ZZ^1 + U \qquad (13)$$

Relation 11 is the transpose of Relation 12 once it is realized that $R_n = R_n^1$.

Since R and Q are symmetric matrices $CRC^1 + Q$ is symmetric. Since any symmetric matrix can be represented as the product of a matrix and its transpose, the matrix Y can be found; once Y is found Z may be found from Relation 11. U can then be found from Relation 13.

For these reasons Equation 7 can be expressed as Equation 8. Since $(K_n Y - Z)(K_n Y - Z)^1$ is positive definite the minimum value of R_{n+1} occurs when:

$$K_n Y = Z \tag{14}$$

multiplying by Y^1 on the right gives

$$K_n YY^1 = ZY^1$$

Substituting Relation 10 into the left hand side and Relation 11 into the right side gives:

$$K_n[CR_nC^1 + Q] = \phi R_n C^1 \tag{15}$$

Multiplying on the right by $[CR_nC^1 + Q]^{-1}$ gives

$$K_n = \phi R_n C^1 [CR_n C^1 + Q]^{-1} \tag{16}$$

The minimum value of $R_{n+1} = U$ can then be easily calculated.

$$R_{n+1} = U = \phi R_n \phi^1 + P_1 - ZZ^1 \tag{17}$$

From Relations 11 and 12 ZZ^1 may be obtained

$$ZZ^1 = \phi R_n C^1 [yy^1]^{-1} \, CR_n \phi^1 \tag{18}$$

or from Equation 16

$ZZ^1 = K_n CR_n \phi^1$, therefore, the minimum value of R_{n+1} is:

$$R_{n+1} = \phi R_n \phi^1 + P_1 - K_n CR_n \phi^1 \tag{19}$$

Equations 3, 16, and 19 then represent the basic operating equations for the Kalman filter.

The only additional information required to run the filter is an initial estimate of the covariance of the error, R_0, and the observed samples, Y_n.

Estimates of the Rate-of-Change of State

The Kalman filter gives the best estimate, $\hat{x}_n$ of the state of a system. There are many situations where the rate-of-change of the state of a system is the quantity of interest.

Assume that Equation 1 is a discrete version of the set of continuous differential equations:

$$\dot{x}(t) = Ax(t) + u(t) \tag{20}$$

If the only information available at the n'th time interval is $\hat{x}_n$, the best estimate of the state of the system, then the best estimate of $\dot{x}(t)$ will be:

$$\hat{\dot{x}}_n = A\hat{x}_n \tag{21}$$

assuming that the noise u(t) has zero mean. This will be called the Kalman estimation since it can be directly obtained from $\hat{x}_n$ the output of the Kalman filter.

Since $\hat{x}_n$ is independent of y_n the estimate in Equation 21 does not use all of the information available at the n'th time period. The Martin filter,[2] has been developed to take advantage of the information in y_n to obtain a better estimate of $\dot{x}_n$.

The Martin filter uses the following equation to estimate $\dot{x}_n$.

$$\hat{\dot{x}}_n = A\hat{x}_n + H_n(y_n - C\hat{x}_n) \tag{22}$$

This is the same as the Kalman estimation except that the term $H_n(y_n - C\hat{x}_n)$ is added to correct the estimate on the basis of the new observed y_n. The matrix H_n will be chosen to minimize the covariance of the error in $\hat{\dot{x}}_n$.

The optimum value of H_n will now be derived. Equation 20 evaluated at the n'th time interval can be written:

$$\dot{\bar{x}}_n = A\bar{x}_n + u(n\tau) \tag{23}$$

$u(n\tau)$ in Equation 23 is a different quantity than $\bar{u}_n$ in Equation 1. $\bar{u}_n$ is the forced response of the system due to the input u(t) over the time interval $(n-1)\tau < t < n\tau$. Let $P_2 = E[u(n\tau)\ u(n\tau)^1]$

Let the estimation error be:

$$e_n = \dot{\bar{x}} - \dot{\hat{\bar{x}}} = A\bar{x}_n + u(n\tau) - A\hat{\bar{x}}_n = H_n(y_n - C\hat{\bar{x}}_n) \tag{24}$$

Substituting y_n from Equation 2 and collecting terms gives:

$$e_n = (A - H_nC)\tilde{x}_n + u(n\tau) - H_nv_n \tag{25}$$

where $\tilde{x}_n = \bar{x}_n - \hat{\bar{x}}_n$

$$e_n^1 = \tilde{x}_n^1(A - H_nC)^1 + u(n\tau)^1 - v_n^1H_n^1 \tag{26}$$

Therefore:

$$\begin{aligned} e_ne_n^1 = {} & (A - H_nC)\tilde{x}_n\tilde{x}_n^1\ (A - H_nC)^1 + u(n\tau)\ u(n\tau)^1 + H_nv_nv_n^1H_n^1 \\ & + (A - H_nC)\tilde{x}_n\ u(n\tau)^1 - (A - H_nC)\tilde{x}_n\ v_n^1H_n^1 + u(n\tau)\tilde{x}_n^1\ (A - H_nC)^1 \\ & - u(n\tau)\ v_n^1H_n^1 - H_nv_n\ \tilde{x}_n^1(A - H_nC)^1 - H_nv_n\ u(n\tau)^1 \end{aligned}$$

Taking the expected value and assuming that

$$E(\tilde{x}_nu(n\tau)^1) = E(\tilde{x}_nv_n^1) = E(u(n\tau)v_n^1) = 0 \qquad \text{gives}$$

$$E(e_ne_n^1) = (A - H_nC)\ R_n(A - H_nC)^1 + P_2 + H_n\ QH_n^1 \tag{27}$$

Expanding Equation 27 and combining the various "powers" of H_n gives:

$$E(e_ne_n^1) = H_n[CR_nC^1 + Q]H_n^1 - AR_nC^1H_n^1 - H_n\ CR_nA^1 + AR_nA^1 + P_2 \tag{28}$$

The equation is in exactly the same format as Equation 7 and exactly the same minimization procedure gives the optimum value of H_n. The value is:

$$H_n = AR_nC^1[CR_nC^1 + Q]^{-1} \tag{29}$$

Let $E(e_ne_n^1)^*$ be the minimum value of $E(e_ne_n^1)$ then:

$$E(e_ne_n^1)^* = AR_nA^1 + P_2 - H_nCR_nA^1 \tag{30}$$

Comparison of Kalman and Martin Filters

Since the Martin filter can be reduced to the Kalman filter, Equation 21, by letting $H_n = 0$. The covariance of the error for the Kalman filter may be obtained from Equation 30 by letting $H_n = 0$. Thus

$$\text{(Kalman filter) } E(e_n e_n^1)^* = AR_n A^1 + P_2 \tag{31}$$

Letting Δ be the difference between Equations 31 and 30 gives:

$$\Delta_n = H_n C R_n A^1 \tag{32}$$

Δ is then the improvement in the covariance of the error caused by using the Martin instead of the Kalman filter. In each application Δ will have to be evaluated to see if the reduction in error is sufficient to justify the additional complexity of the Martin filter.

The effectiveness of the Martin filter in estimating the rate of descent of an aircraft is a function of the noise characteristics of the signals from the pitch gyro and the radar altimeter. The next section of this report is devoted to a study of these noises.

III. Noise Analysis:

In this section statistical characteristics of the measurement noise are determined using real data determined during flight. The noise data was obtained from NASA Guidance and Control Branch, Ames Research Center, Moffet Field, California. A C8A, STOL aircraft was flown over level ground and through several landings. The outputs of the pitch gyroscope and radar altimeter were recorded on an Ampex FR-1300 seven channel FM tape recorder.

In the development of both the Kalman and Martin filters it was assumed that the measurement noises were uncorrelated. That is $E(v_n v_m^1) = 0$ if $n \neq m$. This is equivalent to saying that the autocorrelation function $R(\tau) = 0$ for all $\tau \geq \tau_o$, where τ_o is the time interval for the discrete Kalman filter. Thus, it was necessary to obtain $R(\tau)$ for both the altimeter and pitch gyroscope noise. The power spectrum of the noise was obtained to determine if periodic signals such as power supply hum were present in the noise. Any such periodic noise must be filtered out if $R(\tau)$ is to approach zero as τ tends to infinity.

In both the Kalman and Martin filters quadratic loss functions of the form $E(\tilde{x}_n \tilde{x}_n^1)$ are minimized. It is well known that the optimum filter is not dependent on the shape of the loss function if the noise is Gaussian. For this reason the first order density function was obtained to determine if the noise was Gaussian.

The only measurement noise parameters actually used in either filter are the diagonal elements of the Q matrix. These are the variances of the altimeter noise and the pitch gyroscope noise. These were obtained by evaluating $R(\tau)$ at $\tau = 0$ and as a check were obtained from the density functions.

The signals of the altimeter and pitch gyroscope were recorded by an onboard, FM tape recorder. They contain the real values of the altitude and pitch angle plus the noises to be studied. For both the altitude and pitch angle signals, the slowly varying D-C signals were blocked by a 5 mfd. capacitor. The imput impedance of the Signal Analyzer is of the order of one megohm. Therefore the half power cut off frequency 1/RC $\doteq$ 0.2 Hertz, and thus only noise frequencies less than 0.2 Hertz is lost.

The noises are typified by Figure 1 and Figure 2.

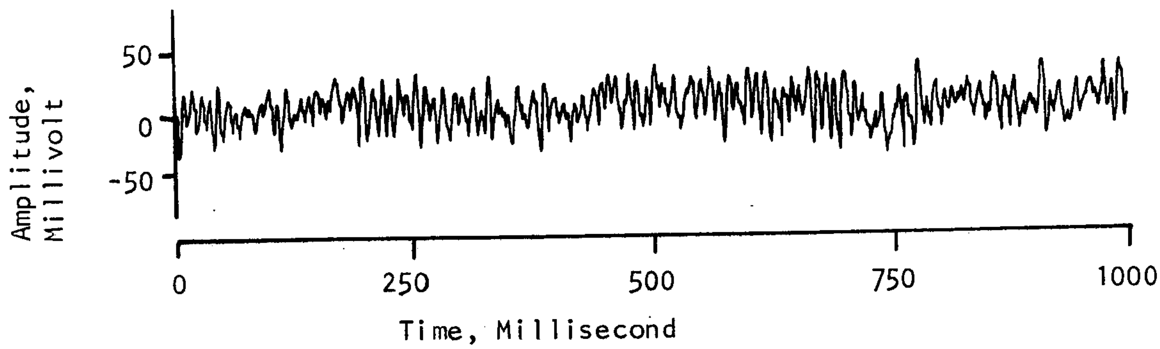

Figure 1. Noise of Radar Altimeter

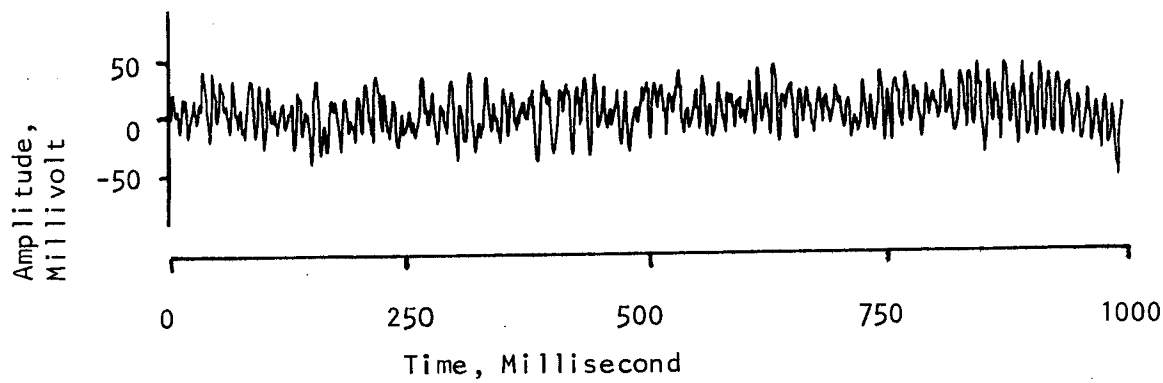

Figure 2. Noise of Pitch Gyroscope

The Power Spectrum Function

With the assumption that the Ergodicity Theorem for Power Spectrums is satisfied for these noises, there are several time averaging techniques available for obtaining the Power Spectrum. The method used employed a Model CAS 8330

Signal Analyzer and a specialized hybrid computing system. The Algorithm and the curves are presented in Figures 3, 4, and 5. In the curves only relative amplitudes are given.

From Figure 4 and Figure 5 it is clear that the power spectra of both noises have significant components at 400 Hz and negligible components at frequencies larger than 400 Hz. Since the FM tape recorder and the aircraft instruments were all driven from the 400 Hz power available in the aircraft it is not possible to determine if this large 400 Hz component was present in the altimeter and pitch gyro outputs, or was introduced by the FM tape recorder. In any case it can easily be removed. Before the autocorrelation function, $R(\tau)$, was calculated the 400 Hz component was removed with an electronic low-pass filter which suppressed it 40 d.b.

These filtered noises, are typified in Figure 6 and Figure 9. Their power spectrums are shown in Figure 7 and Figure 8.

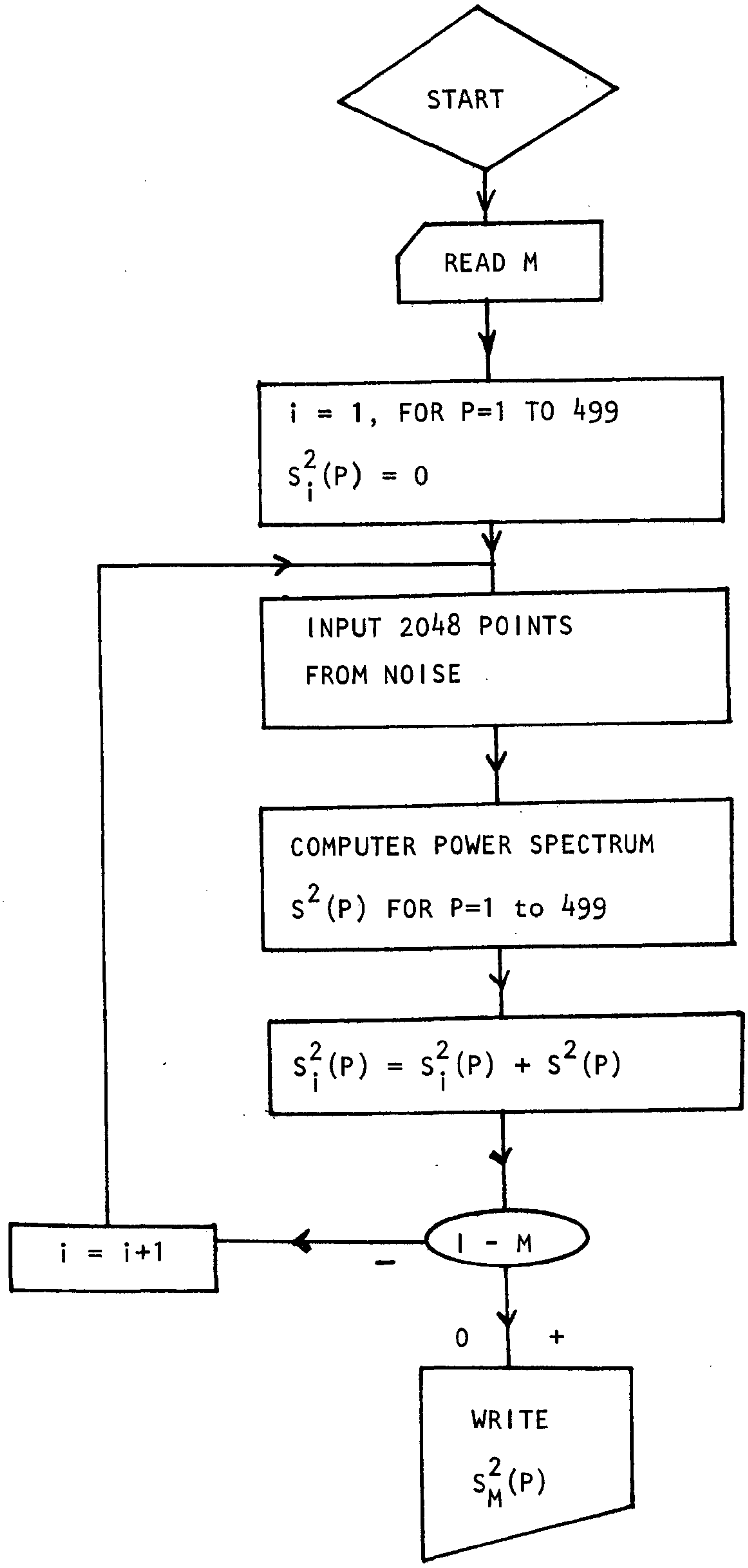

Figure 3. Algorithm for Computing Power Spectrum

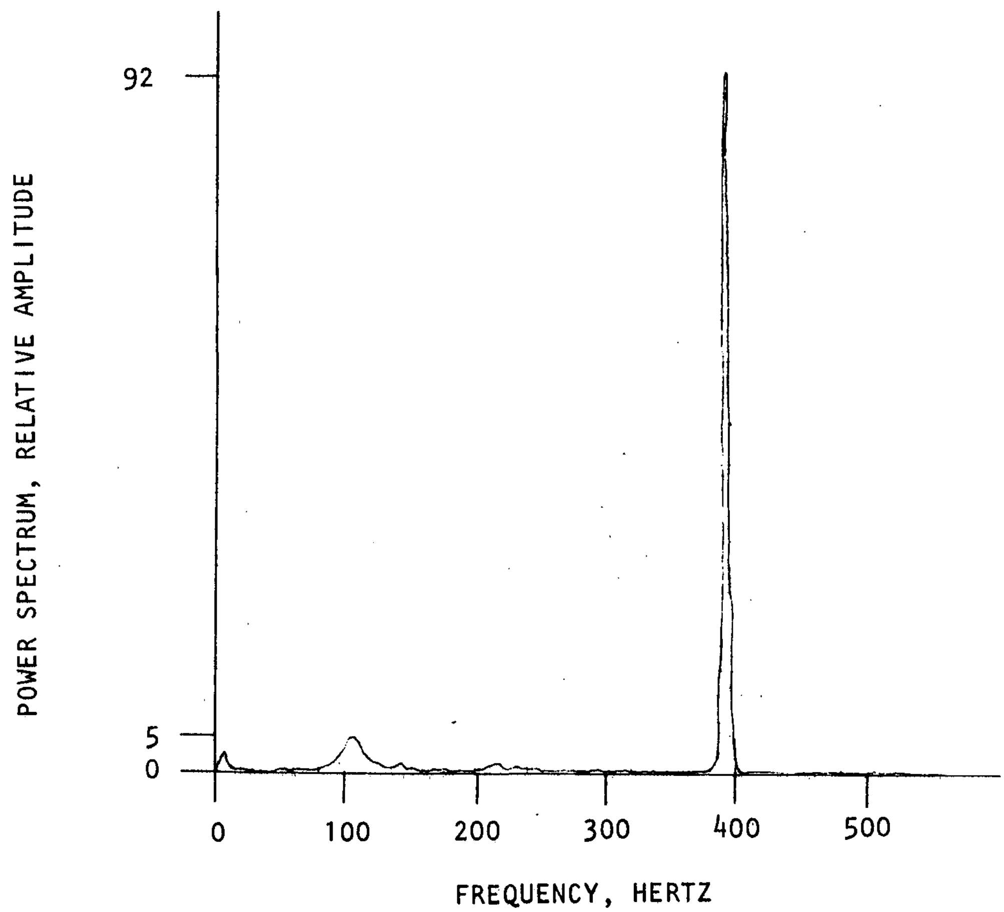

Figure 4. Power Spectrum of Altimeter Noise

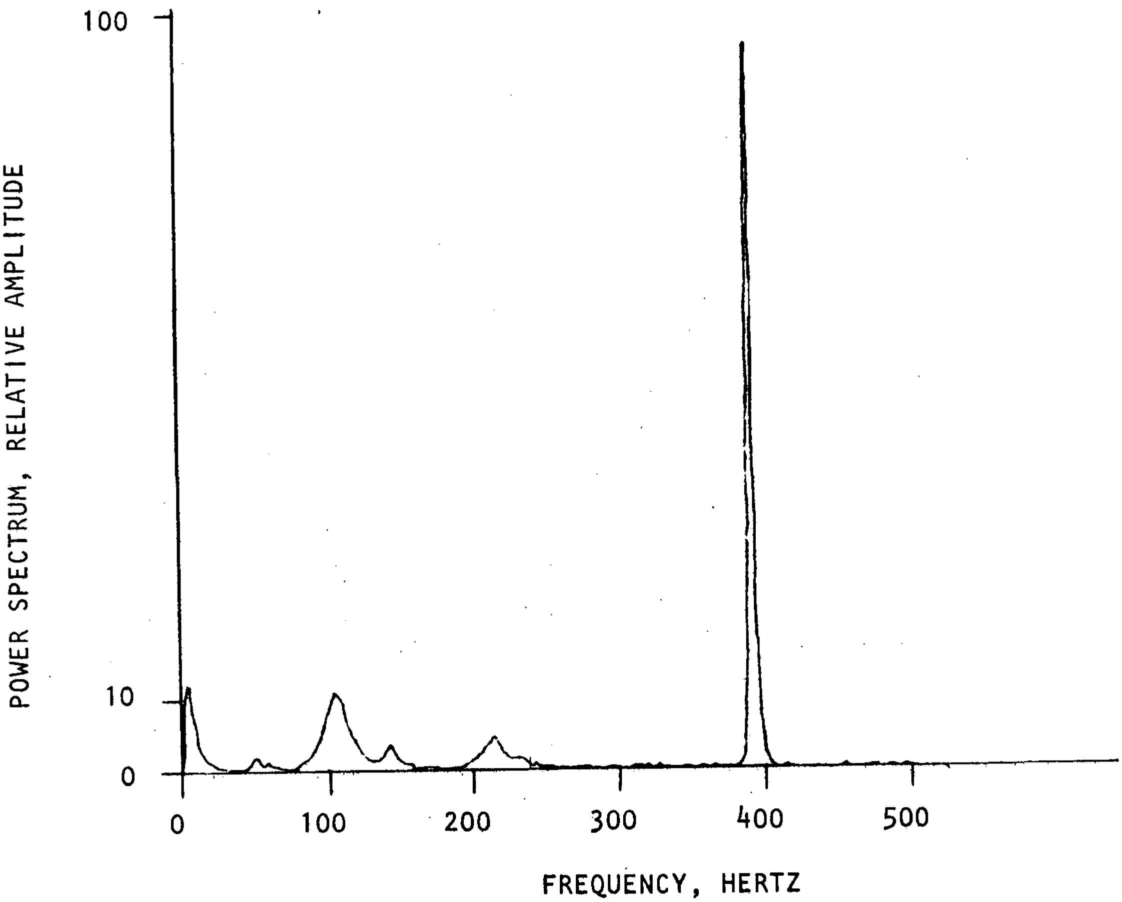

Figure 5. Power Spectrum of Pitch Gyroscope Noise

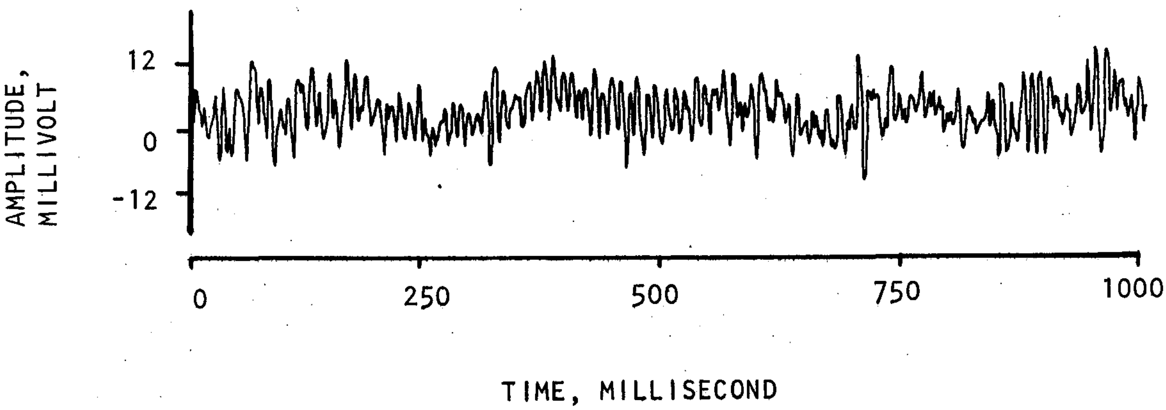

Figure 6. Altimeter Noise with 400 Hz. Removed

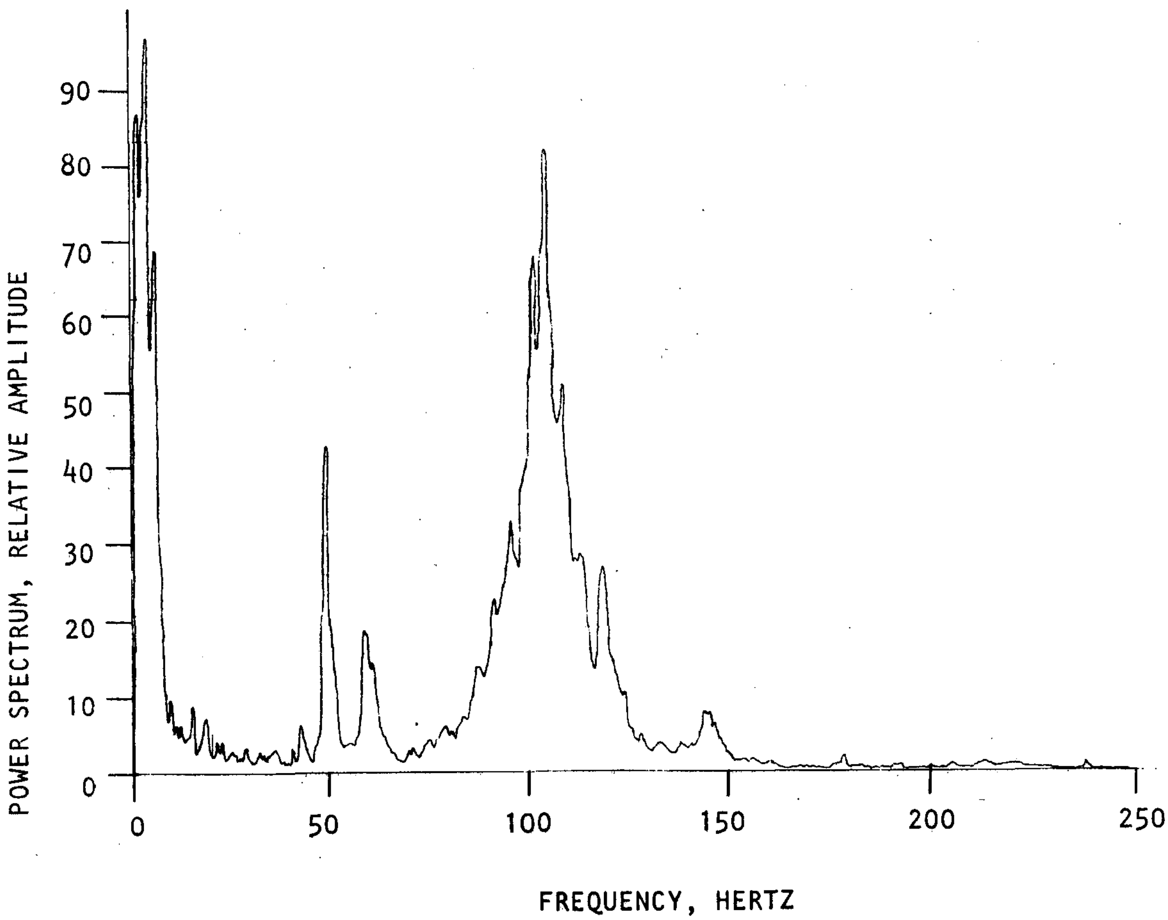

Figure 7. Power Spectrum of Altimeter Noise with 400 Hz. Removed

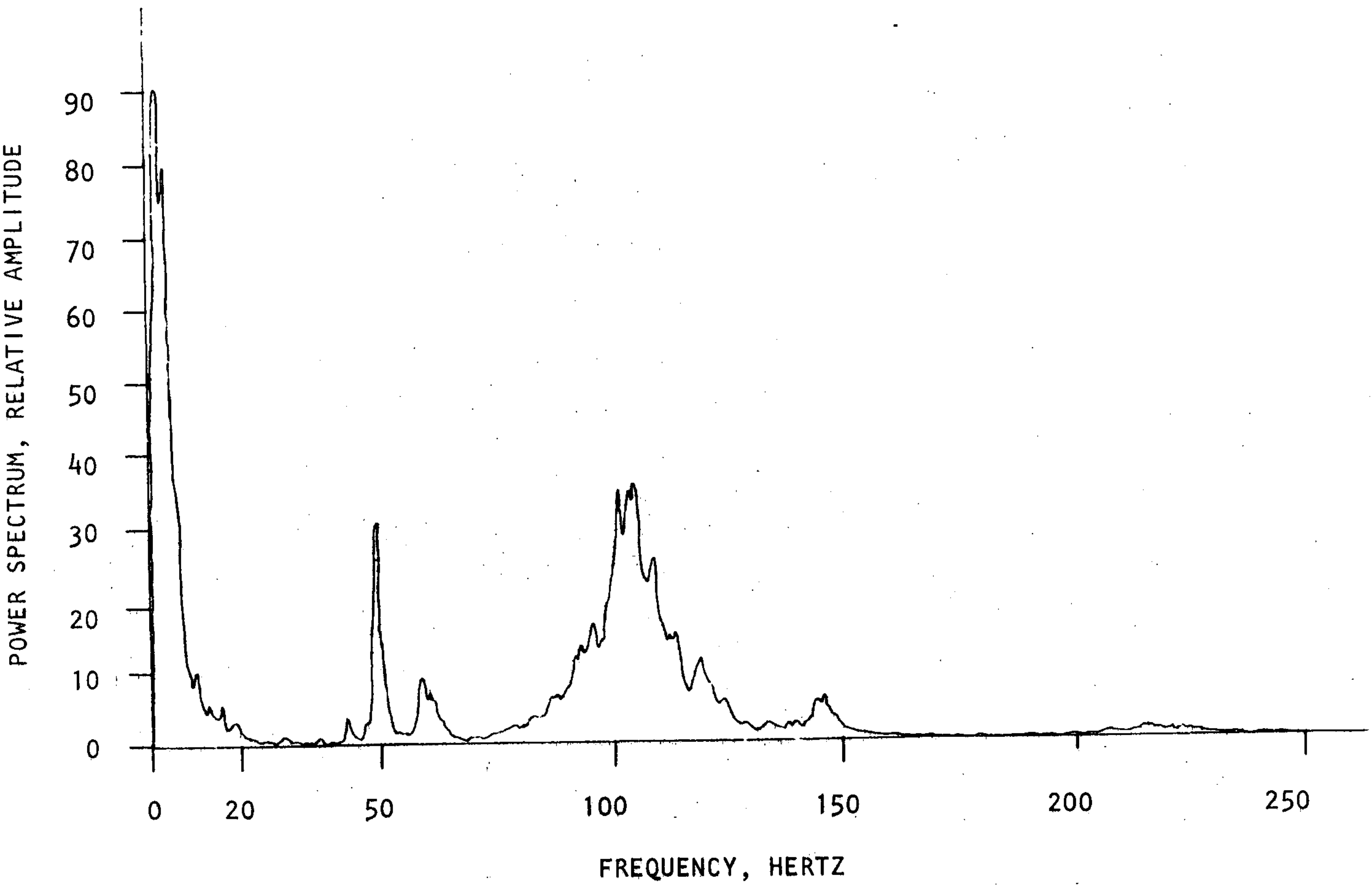

Figure 8. Power Spectrum of Pitch Gyroscope Noise with 400 Hz Removed

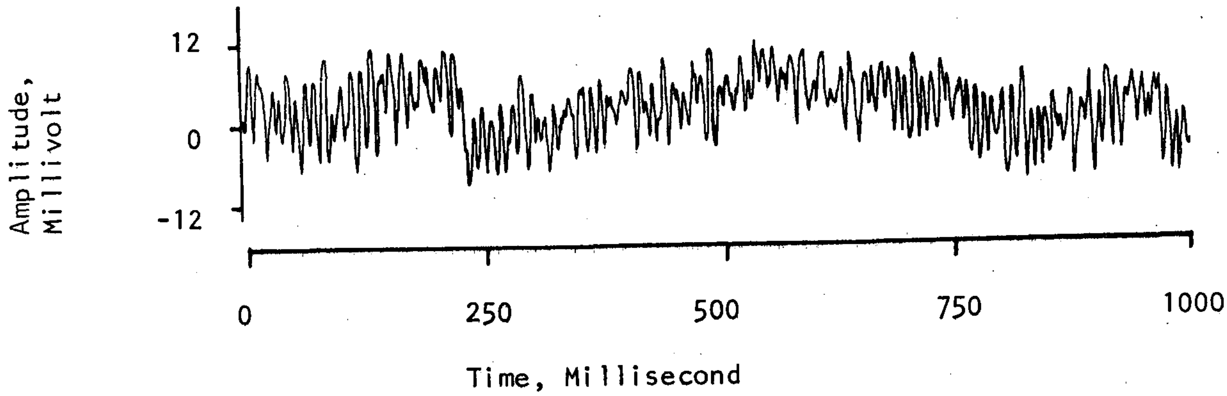

Figure 9. Pitch Gyroscope Noise with 400 Hz. Removed

The Autocorrelation Function

With the assumption that the Ergodicity Theorem for Autocorrelation functions is satisfied for these noises, there are several ways to obtain the Autocorrelation functions. The method used here employed the Model CAS 8330 Signal Analyzer and a specialized hybrid computing system. The Algorithm and curves follow. The gain of the analyzer was determined by using a known sinusoidal input signal. The gain adjustment was left at this setting during the noise analysis. The value of the gain permitted the calibration of the data scale.

The First Order Density Function

The first order density function was obtained from a strip-chart recording of the filtered noises. Samples were taken every 0.1 second. From Figure 11 and Figure 12 it can be seen that these samples are not highly correlated and

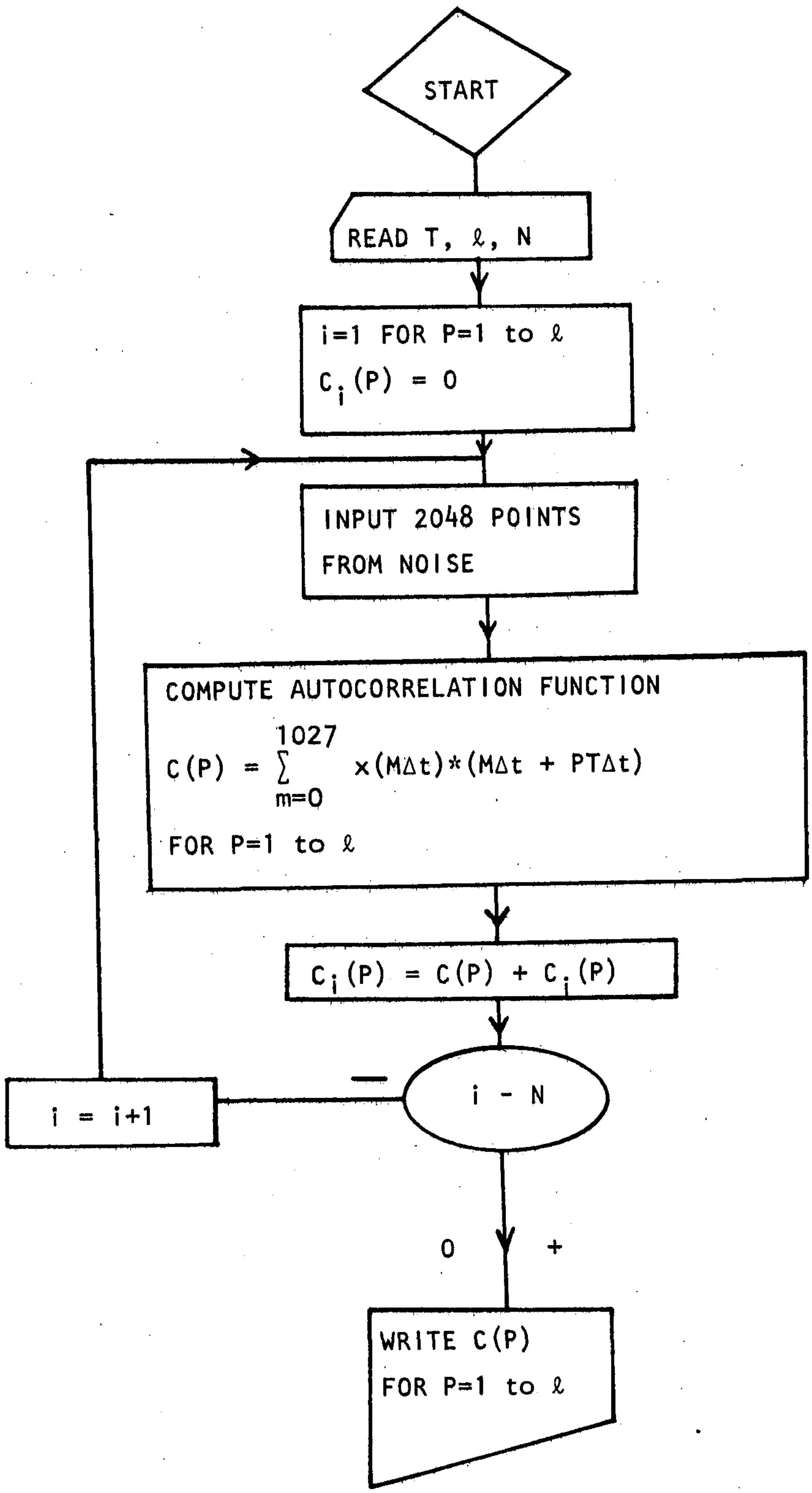

Figure 10. Algorithm for Computing Autocorrelation Function

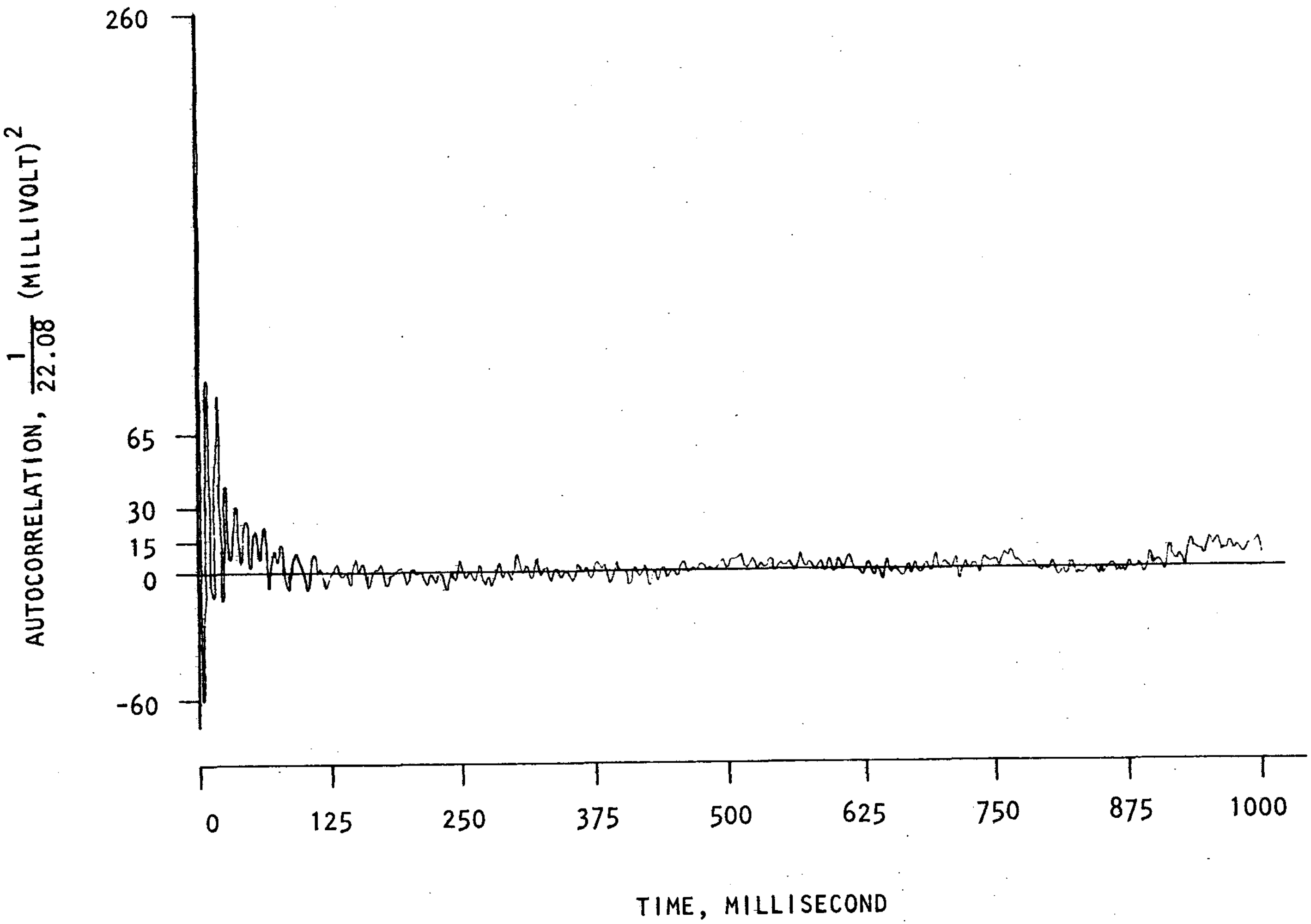

Figure 11. Autocorrelation Function of the Altimeter Noise

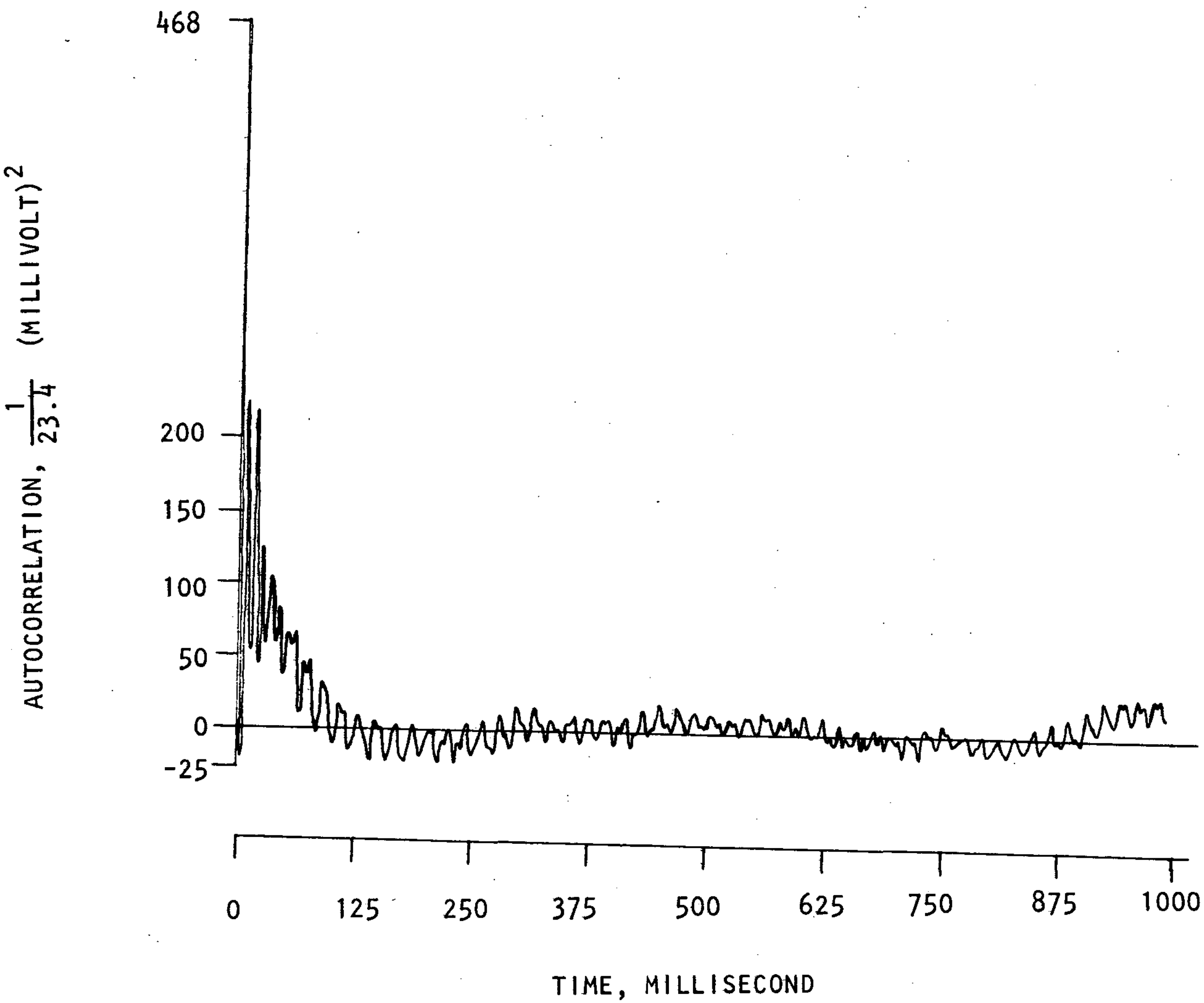

Figure 12. Autocorrelation Function of the Pitch Gyroscope Noise

therefore, it is reasonable to assume that the samples are independent. After sampling 3,487 points for the altimeter noise and 4,000 points for the pitch gyroscope noise, the number of points at the same voltage were counted and their frequencies at 25 discrete values from -12 millivolts to 12 millivolts were determined. The relative frequencies are given in Tables 1 and 2. Plots of relative frequencies versus voltages are shown in Figures 13 and 14.

These relative frequencies come very close to Gaussian curves with variances of 11.8 $(\text{millivolt})^2$ and 19.5 $(\text{millivolt})^2$ respectively.

TABLE 1

FIRST ORDER DENSITY FUNCTION (RELATIVE FREQUENCIES)
OF THE ALTIMETER NOISE

Number of Samples = 3487

Amplitude of Noise (millivolt)	No. of Occurrence	Relative Frequency
12	1	2.87×10^{-4}
11	6	1.72×10^{-3}
10	6	1.72×10^{-3}
9	11	3.16×10^{-3}
8	29	8.36×10^{-3}
7	72	2.07×10^{-2}
6	101	2.09×10^{-2}
5	154	4.42×10^{-2}
4	207	5.93×10^{-2}
3	258	7.4×10^{-2}
2	330	9.46×10^{-2}
1	386	1.11×10^{-1}
0	406	1.17×10^{-1}
-1	375	1.07×10^{-1}
-2	325	9.35×10^{-2}
-3	256	7.34×10^{-2}
-4	183	5.23×10^{-2}
-5	142	4.04×10^{-2}
-6	100	2.87×10^{-2}
-7	59	1.69×10^{-2}
-8	40	1.15×10^{-2}
-9	20	5.73×10^{-3}
-10	13	3.73×10^{-3}
-11	6	1.72×10^{-3}
-12	1	2.87×10^{-4}

TABLE 2

FIRST ORDER DENSITY FUNCTION (RELATIVE FREQUENCIES)
OF THE PITCH GYROSCOPE NOISE

Number of Samples = 4000

Amplitude of Noise (millivolt)	No. of Occurrence	Relative Frequency
12	1	2.5×10^{-4}
11	8	2×10^{-3}
10	10	2.5×10^{-3}
9	24	6×10^{-3}
8	24	6×10^{-3}
7	92	2.3×10^{-2}
6	132	3.3×10^{-2}
5	200	5×10^{-2}
4	255	6.4×10^{-2}
3	344	8.6×10^{-2}
2	353	8.8×10^{-2}
1	362	9.05×10^{-2}
0	366	9.15×10^{-2}
-1	355	8.9×10^{-2}
-2	350	8.75×10^{-2}
-3	350	8.75×10^{-2}
-4	280	7.0×10^{-2}
-5	212	5.3×10^{-2}
-6	140	3.5×10^{-2}
-7	72	1.8×10^{-2}
-8	64	1.6×10^{-2}
-9	42	1.05×10^{-2}
-10	21	5.25×10^{-3}
-11	3	7.5×10^{-4}
-12	0	0

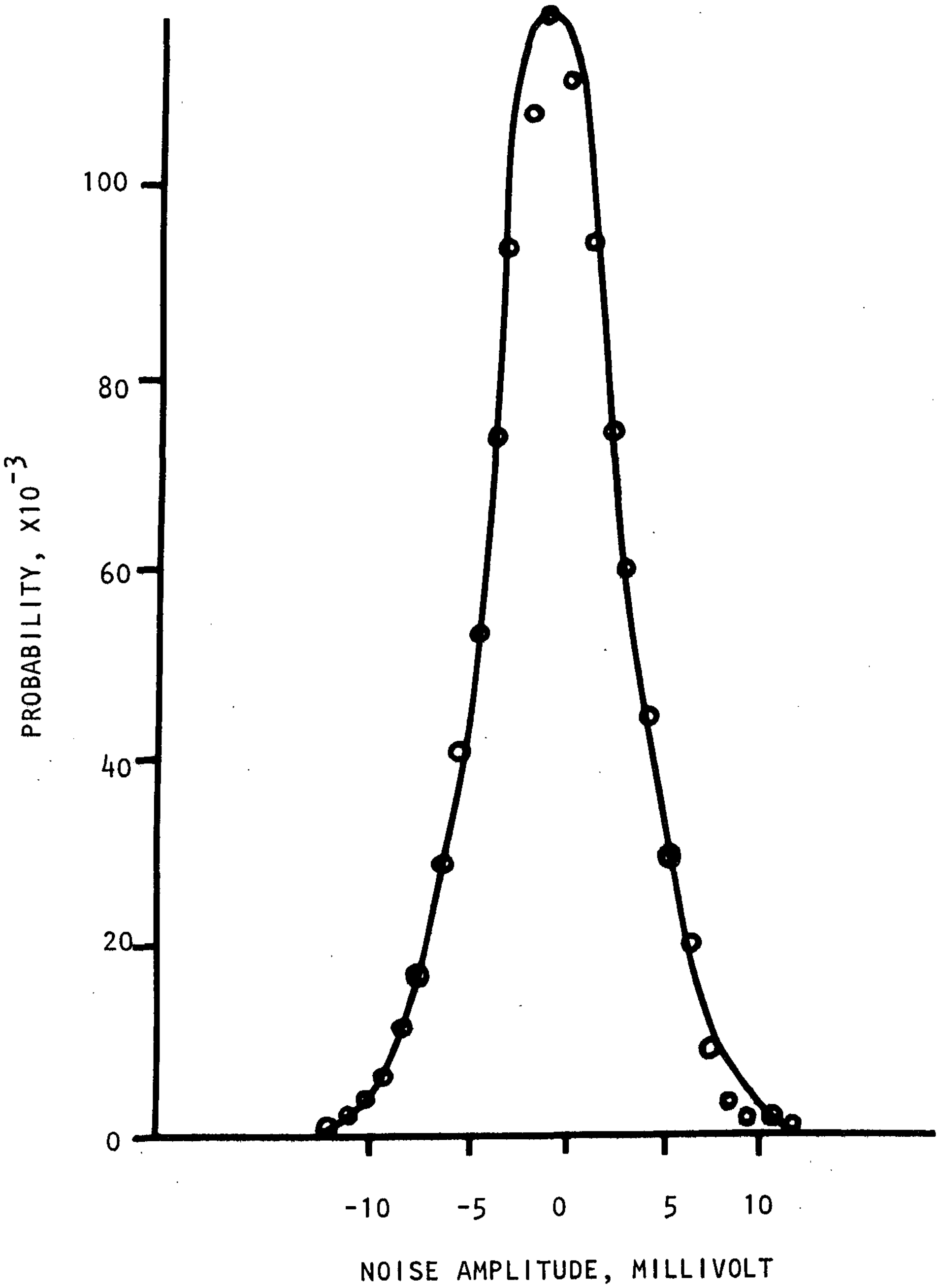

Figure 13. Relative Frequencies of the Noise of Altimeter and Gaussian Noise with Variance Equal to 11.8 (millivolt)^2

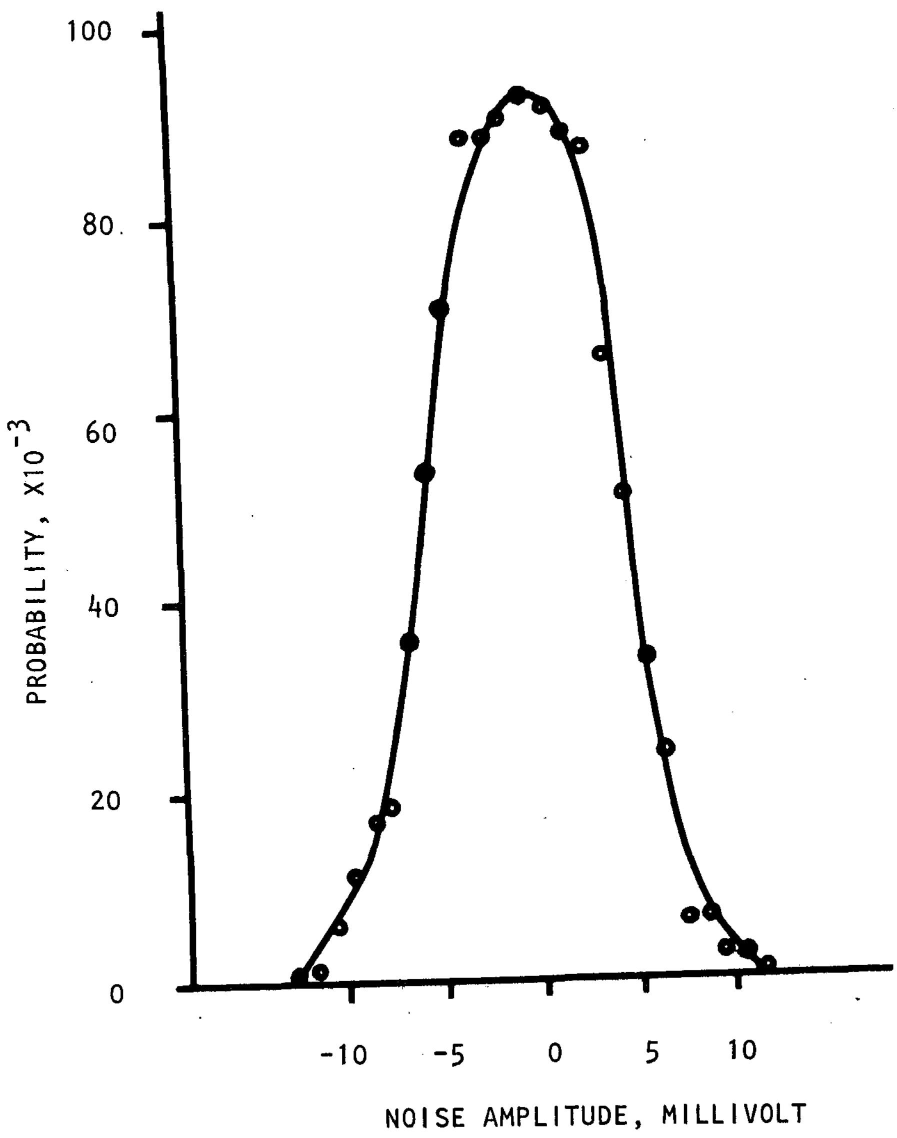

Figure 14. Relative Frequencies of the Noise of Pitch Gyroscopy and the Gaussian Noise with Variance Equal to 19.5 $(\text{millivolt})^2$

Summary of Results

The implications of the noise analysis on the operation of the Kalman and Martin filters can be summarized as follows:

1. The autocorrelation function drops to a small part of its value at $\tau = 0$ for values of $\tau \geq 0.1$ second. This implies that if the filter is updated every 0.1 second the noise samples can be assumed independent.

2. The fact that the noises are Gaussian means that the filters derived in Section II are optimum for any loss function, not just the quadratic loss function used in the derivation. Thus, any discussion of the appropriateness of the chosen loss function is meaningless.

3. The noise variances which are:

   11.8 $(\text{millivolt})^2$ altimeter

   19.5 $(\text{millivolt})^2$ pitch gyroscope

   Converting to feet and milliradians gives:

   0.540 $(\text{feet})^2$ altimeter

   0.147 $(\text{milliradians})^2$ pitch gyroscope

   These are the diagonal elements in the Q matrix.

IV. Evaluation of Filters for Aircraft Use:

It is a simple matter to invent situations in which the Martin filter is significantly better than the Kalman filter or to invent situations in which the improvement is zero or negligible.

In this chapter the relative effectiveness of the Martin filter in a specific situation will be evaluated. The filter will be used to give an optimal estimate of the rate of descent of an aircraft. The aircraft will be modeled as a linear fourth-order system with state variables:

$$\bar{x}_n = \begin{pmatrix} \dot{\theta} \\ \theta \\ \dot{h} \\ h \end{pmatrix} \qquad \begin{array}{l} \theta = \text{pitch angle in radians} \\ \\ h = \text{altitude in feet} \end{array}$$

The general form of the A matrix is:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in which the undertermined a's are functions of the particular aircraft under consideration.

The C matrix is:

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The position of the ones in C indicates that only the pitch and altitude are observed. Therefore, if the error in $\dot{h}$ is uncorrelated with the errors in both θ and h, $\overline{y}_n$ will give no information about h, and the Martin filter will give the same estimate of $\dot{h}$ as the Kalman filter.

In terms of matrices this implies that if R_n is diagonal then the fourth row in H_n will have all zero terms and no correction in the Kalman estimate of $\dot{h}$ will occur. Multiplying the matrices in H_n and keeping track of the zeros in C and R_n shows that this is true if R_n is diagonal. R_0, the initial R_n, can reasonably be assumed diagonal since there is no reason that errors in θ, $\dot{\theta}$, h, and $\dot{h}$ should be correlated. R_n will not be diagonal in general since errors in one state variable will propagate into the other state variables through the action of the ϕ matrix.

Thus it is to be expected that the Martin filter will not be a significant improvement until sufficient time has passed for R_n to develop significant terms.

If P = 0 then R_n will tend to zero as time increases and it is possible that no time interval will exist in which the Martin filter is a significant improvement over the Kalman filter.

To give meaningful insight into the effectiveness of the Martin filter for estimating the variance of $\dot{h}$ typical numbers have been chosen, and $E(e_n e_n^1)^*$ evaluated for both the Martin and Kalman filter. The 4-4 element in $E(e_n e_n^1)^*$ is the variance of $\dot{h}$.

The numbers chosen are:

$$A = \begin{pmatrix} -0.60 & -0.76 & 0.172 & 0 \\ 1.00 & 0 & 0 & 0 \\ 0 & 0.0179 & -0.40 & 0 \\ 0 & 0 & 1.00 & 0 \end{pmatrix}$$

$$R_0 = \begin{pmatrix} 0.00328 & 0 & 0 & 0 \\ 0 & 0.328 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 25.0 \end{pmatrix}$$

$$Q = \begin{pmatrix} .0164 & 0 \\ 0 & 25 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\tau = 0.1$ sec.

This value of Q was chosen from published manufacturer's specifications and seems to reflect the actual signal variances better than the noise variances alone.

The fact that the 3-3 term in R_0 is 0.25 means that initially the rate of descent is well known. Equations 30 and 31 were then implemented in FORTRAN and the variance of $\hat{\dot{h}}$ was calculated for both filters.

The results are shown in Figure 15.

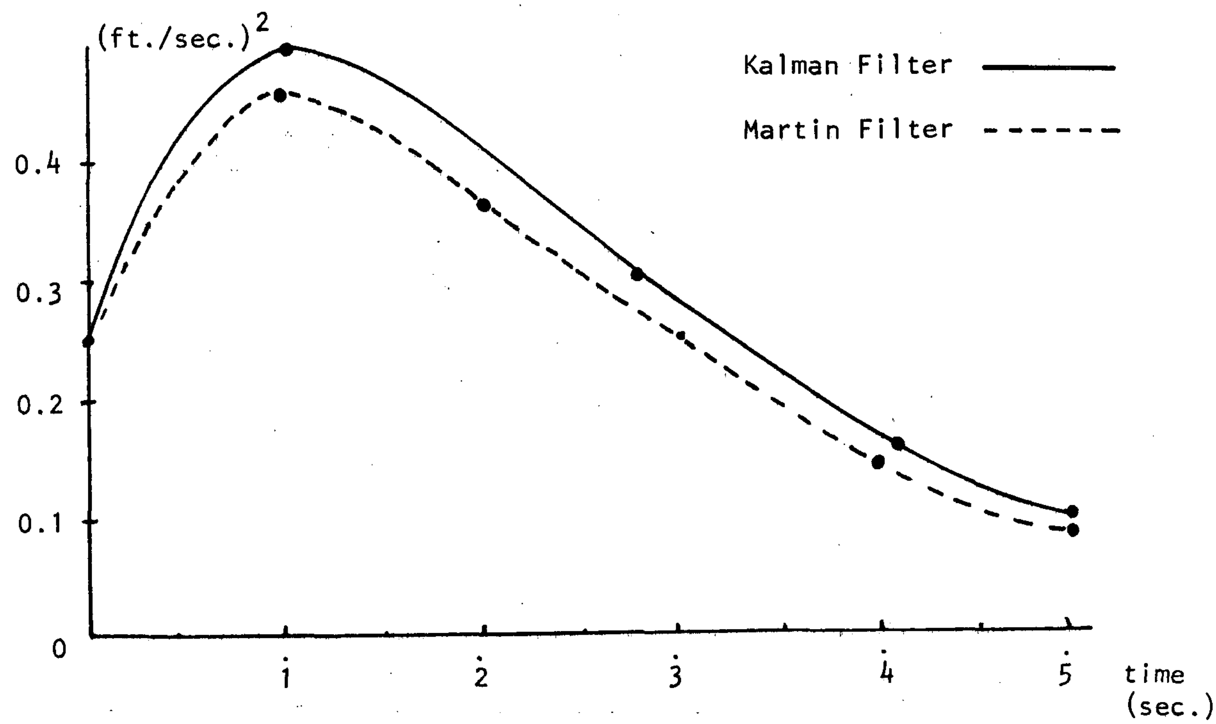

Figure 15. The Variance of $\hat{\dot{h}}$ for the Martin and Kalman Filters

From Figure 15 it can be seen that for $t \geq 2$ seconds the Martin filter is approximately 12 percent better than the Kalman filter. For $t > 5$ both filters are so accurate that there is no reason to implement the more complex Martin filter. If $P_1 \neq 0$ then R_n would not approach 0 and the 12 percent improvement of the Martin filter would continue to be important for all times.

A more dramatic improvement is obtained if the initial estimate of $\dot{h}$ is inaccurate. To illustrate this R was changed to:

$$R = \begin{pmatrix} 3.28\times10^{-3} & 0 & 0 & 0 \\ 0 & 3.28\times10^{-1} & 0 & 0 \\ 0 & 0 & 1.0\times10^{4} & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}$$

The 10^4 in the 3-3 position indicates that the standard deviation of the error in the estimate of $\dot{h}$ is 100 feet/second. The error variances for both filters are plotted in Figure 16.

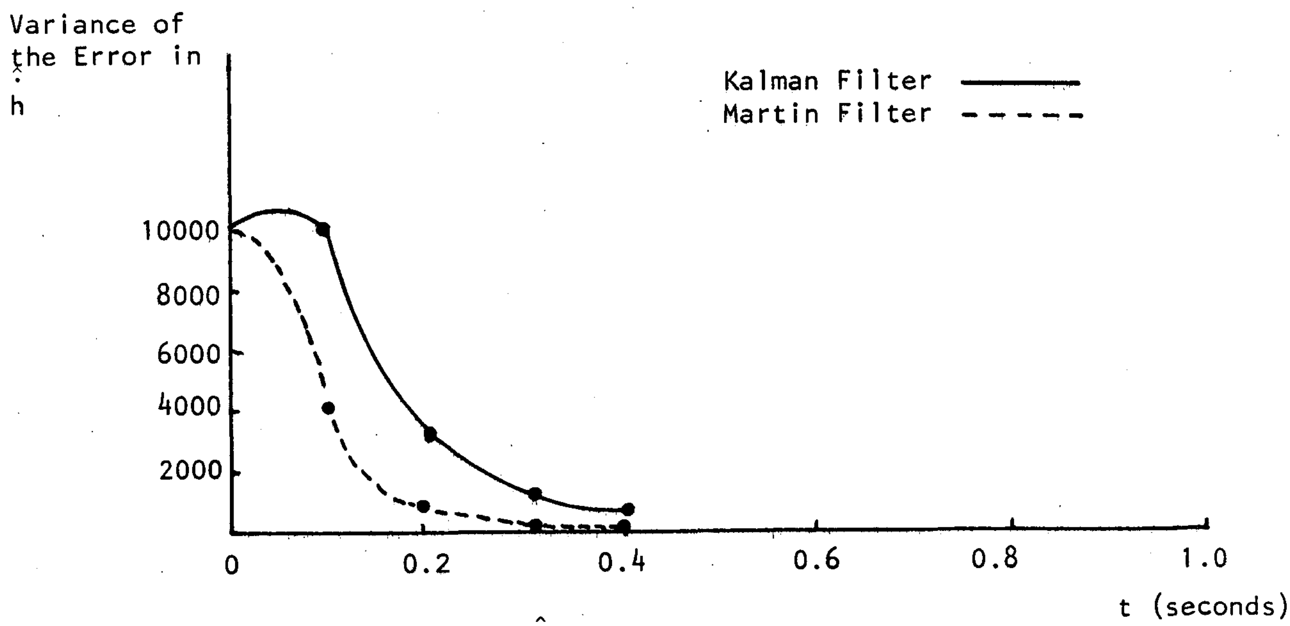

Figure 16. Variance of $\hat{\dot{h}}$ for High Turbulence Condition

At $t = 0.2$ seconds the variance of the error in the Martin filter is $\frac{1}{4}$ of the variance of the error in the Martin filter. Thus in extremely turbulent conditions, the Martin would have a steady state error variance of $\frac{1}{4}$ that of the Kalman filter.

To assure that the filters do converge when $P_1 \neq 0$ an arbitrary P_1 was chosen:

$$P_1 = \begin{pmatrix} 10^{-3} & 0 & 0 & 0 \\ 0 & 10^{-3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The system was simulated for 400 seconds. At the end of 10 seconds it was in steady state. The steady state variances in the error was 19.48 for the Kalman filter and 17.09 for the Martin filter. This is the same 12 percent improvement which appears in Figure 12 for low turbulence situations.

V. Summary:

The purpose of this grant was the development of a digital filter for the optimal estimation of the rate of descent of aircraft. A filter, called the Martin filter, was developed which gives the optimum estimate of the rate of change of the state of the system. In situations where the error variances are small the Martin filter will have an error variance of 88 percent of the Kalman filter. If the error variances are large, such as in very turbulent air, it will produce error variance of 25 percent of those produced by the Kalman filter. These error variances are approximate. More accurate results will not be possible until more data on the P matrix caused by various turbulence conditions is known.

VI. References:

1. Liebelt, P. B. and J. E. Bergeson, "The Generalized Least Square and Wiener Theories with Applications to Trajectory Prediction," Boeing Document No. DZ-90167, May 1962.

2. Martin, J. C., "Minimum Variance Estimates of Signal Derivatives," NASA Research Grant NGR-41-001-24, December 1970.